Thin Films Under Thermal Noise

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THIN FILMS UNDER THERMAL NOISE

Thin Films Under Thermal Noise

Proefschrift

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voorzitter van het College voor Promoties,
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door

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Contents

Summary

In this thesis the question of existence and uniqueness of non-negative solutions to the stochastic thin-film equation on the d-dimensional flat torus $\mathbb{T}^d$ is addressed. This refers to the initial value problem

$$\begin{cases} \partial_t u = -\mathrm{div}(u^n \nabla \Delta u) + \mathrm{div}(u^{n/2}\mathscr{W}), & \text{on } \Omega \times [0,T] \times \mathbb{T}^d, \\ u(0,\cdot) = u_0, & \text{on } \Omega \times \mathbb{T}^d, \end{cases}$$

where $\mathscr{W}\colon \Omega \times [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ is a temporally white Gaussian noise and $u_0\colon \Omega \times \mathbb{T}^d \to [0,\infty)$ the initial value. The solution $u\colon \Omega \times [0,T] \times \mathbb{T}^d \to [0,\infty)$ models the height of a thin liquid film driven by surface tension and thermal fluctuations.

In the first chapter we give a soft introduction to the subject. We review the physical motivation behind the equation and comment on relevant model assumptions which are imposed throughout the thesis. Moreover, we recall different notions of solutions to stochastic and partial differential equations to interpret the stochastic thin-film equation later on. Subsequently, we summarize the main results and techniques used in the thesis and give a concise literature review on other mathematical works concerning the stochastic thin-film equation.

In the second chapter we construct martingale solutions to the stochastic thin-film equation with quadratic mobility exponent $n = 2$ in the physically relevant two-dimensional setting $d = 2$. The conservative noise term becomes linear in this case and is well-behaved on its own. This allows for a decomposition of the deterministic and stochastic dynamics, which was previously employed by Gess and Gnann to construct solutions in the one-dimensional case by closing an energy estimate along the time-splitting scheme. We generalize their approach to the two-dimensional situation and overcome the analytical challenges due to the higher spatial dimension by invoking estimates on the α-entropy

$$\frac{1}{\alpha(\alpha+1)} \int_{\mathbb{T}^2} u^{\alpha+1}\,\mathrm{d}x, \qquad \alpha \in (-1,0),$$

and corresponding dissipation terms.

For the nonlinear noise case $n \neq 2$ not many existence results are known, even in one dimension $d = 1$. Specifically, existence of martingale solutions to the stochastic thin-film equation with nonlinear noise was shown by Dareiotis, Gess, Gnann and Grün for $n \in [8/3, 4)$ and initial profiles from the energy space $u_0 \in H^1(\mathbb{T})$ with finite entropy

$$\int_{\mathbb{T}} u_0^{2-n}\,\mathrm{d}x < \infty.$$

Their proof relies on a control of the energy production due to the noise by the entropy dissipation of the thin-film operator. In the third chapter of this thesis, we close the

resulting gap of mobility exponents by proving the existence of martingale solutions for $n \in (2,3)$ and initial values satisfying the slightly milder condition

$$-\int_{\mathbb{T}} \log(u_0)\,\mathrm{d}x < \infty.$$

Since the log-entropy functional works less well with non-negative approximations of the stochastic thin-film equation an additional approximation layer compared to the aforementioned result is used.

These results only answer the question of existence of solutions for $n \neq 2$ for initial values which are positive almost everywhere. In particular, the presence of a contact line, i.e., a triple junction of the three interfaces of liquid, solid and gas, is excluded. In the fourth chapter of this thesis we show existence of martingale solutions to the stochastic thin-film equation for $n \in (2,3)$ and initial values without full support by discarding the energy estimate and relying the compactness argument solely on α-entropy estimates for $\alpha > -1$. Since the functionals are lower order than the energy $\frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{T})}$ we can treat spatially less regular noise and allow for initial values from the space of Radon measures.

In the fifth chapter we turn our attention to the situation that the film-height is strictly positive. In this case, many difficulties of the stochastic thin-film equation do not occur, because the coefficient u^n does not degenerate and the equation remains effectively parabolic. Using recent results on quasilinear stochastic evolution equations, we show that unique, positive, probabilistically strong solutions to the stochastic thin-film equation exist in this case until some stopping time $\tau > 0$ for arbitrary n and $d \geq 1$. Moreover, we deduce that the solutions become as smooth as the spatial regularity of the noise $\mathcal{W}$ allows for. Lastly, we show that if repulsive interaction forces between the molecules of the fluid and the substrate are included in the equation it holds $\tau = \infty$ almost surely for $n \in (0,6)$ and $d = 1$ by closing first α-entropy estimates and subsequently an energy estimate.

Samenvatting

In dit proefschrift wordt de kwestie van de existentie en de uniciteit van niet-negatieve oplossingen van de stochastische dunne-filmvergelijking op de d-dimensionale vlakke torus $\mathbb{T}^d$ behandeld. Dit heeft betrekking op het beginwaardeprobleem

$$\begin{cases} \partial_t u = -\operatorname{div}(u^n \nabla \Delta u) + \operatorname{div}(u^{n/2}\mathscr{W}), & \text{op } \Omega \times [0,T] \times \mathbb{T}^d, \\ u(0,\cdot) = u_0 & \text{op } \Omega \times \mathbb{T}^d, \end{cases}$$

waarbij $\mathscr{W}\colon \Omega \times [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ een in de tijd witte Gaussische ruis is en $u_0\colon \Omega \times \mathbb{T}^d \to [0,\infty)$ de beginwaarde is. De oplossing $u\colon \Omega \times [0,T] \times \mathbb{T}^d \to [0,\infty)$ modelleert de hoogte van een dunne vloeistoffilm aangedreven door oppervlaktespanning en thermische fluctuaties.

In het eerste hoofdstuk geven we een zachte inleiding tot het onderwerp. We bespreken de fysische motivatie achter de vergelijking en relevante modelaannames die in het hele proefschrift worden opgelegd. Bovendien brengen we verschillende noties van oplossingen voor stochastische en partiële differentiaalvergelijkingen in herinnering om later de stochastische dunne-filmvergelijking te kunnen interpreteren. Vervolgens vatten we de belangrijkste resultaten en technieken samen die in het proefschrift zijn gebruikt en geven we een beknopt literatuuroverzicht van andere wiskundige werken met betrekking tot de stochastische dunne-filmvergelijking.

In het tweede hoofdstuk construeren we martingaaloplossingen voor de stochastische dunne-filmvergelijking met kwadratische mobiliteitsexponent $n = 2$ in de fysisch relevante tweedimensionale setting $d = 2$. De conservatieve ruisterm wordt in dit geval lineair en is op zichzelf goed gedragen. Dit maakt een decompositie van de deterministische en stochastische dynamica mogelijk, die eerder werd gebruikt door Gess en Gnann om oplossingen te construeren in het eendimensionale geval door een energieafschatting langs het tijd-splitsingsschema te sluiten. We veralgemenen hun benadering naar de tweedimensionale situatie en overwinnen de analytische uitdagingen die voortvloeien uit de hogere ruimtelijke dimensie door afschattingen van de α-entropie

$$\frac{1}{\alpha(\alpha+1)} \int_{\mathbb{T}^2} u^{\alpha+1}\,\mathrm{d}x, \qquad \alpha \in (-1,0),$$

en de bijbehorende dissipatietermen te gebruiken.

Voor het niet-lineaire ruisgeval $n \neq 2$ zijn niet veel existentieresultaten bekend, zelfs niet in één dimensie $d = 1$. Meer bepaald werd het bestaan van martingaaloplossingen voor de stochastische dunne-filmvergelijking met niet-lineaire ruis aangetoond door Dareiotis, Gess, Gnann en Grün voor $n \in [8/3, 4)$ en beginprofielen uit de energieruimte u_0 in $H^1(\mathbb{T})$ met eindige entropie

$$\int_{\mathbb{T}} u_0^{2-n}\,\mathrm{d}x < \infty.$$

Hun bewijs berust op het beheersen van de energieproductie als gevolg van de ruis door de entropiedissipatie van de dunne-filmoperator. In het derde hoofdstuk van dit proefschrift dichten we de resulterende kloof van mobiliteitsexponenten door het bestaan te bewijzen van martingaaloplossingen voor $n \in (2,3)$ en beginwaarden die voldoen aan de iets mildere voorwaarde

$$-\int_{\mathbb{T}} \log(u_0)\,\mathrm{d}x < \infty.$$

Omdat de log-entropiefunctie minder goed werkt met niet-negatieve benaderingen van de stochastische dunne-filmvergelijking wordt een extra benaderingslaag gebruikt in vergelijking met het bovenstaande resultaat.

Deze resultaten geven alleen antwoord op de vraag of er oplossingen zijn voor $n \neq 2$ voor beginwaarden die bijna overal positief zijn. In het bijzonder wordt de aanwezigheid van een contactlijn, d.w.z. een drievoudige kruising van de drie grensvlakken van vloeistof, vaste stof en gas, uitgesloten. In het vierde hoofdstuk van dit proefschrift tonen we het bestaan aan van martingaaloplossingen voor de stochastische dunne-filmvergelijking voor $n \in (2,3)$ en beginwaarden die niet volledig gedragen zijn door de energieafschatting weg te laten en het compactheidsargument alleen te baseren op α-entropieafschattingen voor $\alpha > -1$. Omdat de functies van lagere orde zijn dan de energie $\frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{T})}$ kunnen we ruimtelijk minder regelmatige ruis en beginwaarden uit de ruimte van Radonmaten behandelen.

In hoofdstuk vijf richten we onze aandacht op de situatie dat de filmhoogte strikt positief is. In dit geval treden veel problemen van de stochastische dunne-filmvergelijking niet op, omdat de coëfficiënt u^n niet ontaardt en de vergelijking in wezen parabolisch blijft. Met behulp van recente resultaten over quasilineaire stochastische evolutievergelijkingen laten we zien dat er in dit geval unieke, positieve, probabilistisch sterke oplossingen voor de stochastische dunne-filmvergelijking bestaan tot een bepaalde stoptijd $\tau > 0$ voor willekeurige n en $d \geq 1$. Bovendien leiden we af dat de oplossingen zo glad worden als de ruimtelijke regelmaat van de ruis $\mathscr{W}$ toelaat. Tot slot laten we zien dat als de afstotende interactiekrachten tussen de moleculen van de vloeistof en het substraat in de vergelijking worden meegenomen, het bijna zeker is dat $\tau = \infty$ geldt voor $n \in (0,6)$ en $d = 1$ door eerst α-entropieafschattingen en vervolgens een energieafschatting te sluiten.

PREFACE

This manuscript is based on the research which I conducted as a PhD candidate at the Delft University of Technology. Specifically, the contents of the preprints [3, 36] which I worked on in collaboration with Antonio Agresti and Konstantinos Dareiotis, Benjamin Gess and Manuel V. Gnann, respectively, and the single-authored article [116] and preprint [117] are almost verbatim contained in Chapters 2–5 of this thesis. They are accompanied by a soft introduction to the subject given in Chapter 1, as well as a concluding Chapter 6.

Regarding the writing of the aforementioned research works, I would like to thank Mark Veraar for discussions at the initial stage of [3] and a careful reading prior to publishing the preprint. I thank Günther Grün for communicating the proper approximation of the α-entropy functional used in [36]. Concerning [116], I would like thank Mark Veraar for pointing out the possibility to relax the integrability assumptions on the initial value and the anonymous referees for the careful reading of the manuscript and their valuable suggestions. I would also like to thank Konstantinos Dareiotis and Benjamin Gess for discussing the main idea worked out in [117]. Moreover, I would like to thank Manuel V. Gnann for carefully reading the manuscripts of [3, 116, 117] and the stimulating discussions during their making. Additionally, concerning the preparation of this thesis, I thank Antonio Agresti, Manuel V. Gnann, Mark Veraar and Joris van Winden for reading parts of it and Joshua Willems for helping to translate the summary to the Dutch language.

More generally, I would like to thank Manuel V. Gnann for giving me the opportunity to enter the exciting world of mathematical research and his guidance as a doctoral advisor. Along the same vein, I would like to thank my other collaborators Antonio Agresti, Konstantinos Dareiotis, Benjamin Gess and Martin Grothaus for the opportunity to participate in their research work and the mathematical community as a whole for its exchange of ideas during conferences and seminar talks. I also thank Benjamin Gess, Franco Flandoli, Marco Rehmeier and Günther Grün for enabeling me to visit Bielefeld University, the Scuola Normale Superiore in Pisa and the University of Erlangen–Nuremberg during my time as a PhD student.

I am very grateful for the privilege to have spent my time as a PhD candidate in the pleasant environment of the Analysis and Mathematical Physics groups of the Delft Institute of Applied Mathematics. My collegues there are the reason that I enter my workplace with a smile every morning. Last but not least, I would like to thank my loving friends, partner and family. Their kindness and support has always been a great source of strength and inspiration to me, also during the last four years.

Max Sauerbrey
Delft, June 2024

1

INTRODUCTION

In this thesis the question of existence and uniqueness of non-negative solutions to the *stochastic thin-film equation* on the d-dimensional flat torus $\mathbb{T}^d$ is addressed. This refers to the initial value problem

$$\begin{cases} \partial_t u = -\operatorname{div}(u^n \nabla \Delta u) + \operatorname{div}(u^{n/2} \mathcal{W}), & \text{on } \Omega \times [0,T] \times \mathbb{T}^d, \\ u(0,\cdot) = u_0, & \text{on } \Omega \times \mathbb{T}^d, \end{cases} \tag{STFE}$$

where $\mathcal{W}\colon \Omega \times [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ is a temporally white Gaussian noise and $u_0\colon \Omega \times \mathbb{T}^d \to [0,\infty)$ the initial value. The solution $u\colon \Omega \times [0,T] \times \mathbb{T}^d \to [0,\infty)$ models the height of a thin liquid film driven by *surface tension* and *thermal fluctuations* limited by viscous friction. The parameter n is called *mobility exponent* and is a positive real number depending on the boundary condition of the fluid velocity near the substrate. In particular, the quasilinear stochastic partial differential equation (STFE) is degenerate parabolic. The combination of the degenerate parabolic operator with the, generally, non-Lipschitz continuous noise term makes establishing a solution theory to (STFE) challenging.

The purpose of this chapter is to capture the main ideas employed in the subsequent parts of this manuscript. To this end, details are omitted with the hope to make the introduction more accessible, while a detailed analysis is postponed to later chapters.

In Section 1.1 we derive equation (STFE) from a physical model for a thin liquid film. On the one hand, this serves as a motivation to analyze (STFE) mathematically, while, on the other hand, it provides intuition about the expected behavior of a solution. In Section 1.2 we introduce some notions from stochastic calculus and the theory of partial differential equations to prepare ourselves to give a precise meaning to a solution to (STFE) later on. In Sections 1.3 and 1.4 we state the main results of this thesis in an informal fashion and sketch the underlying ideas of their proofs. In Section 1.5 we review the mathematical literature on (STFE) available at the moment and locate the results of this thesis among them.

1

1.1. PHYSICAL BACKGROUND

In this section, we motivate the stochastic partial differential equation (STFE). For simplicity, we restrict ourselves to the effective dimension $d = 1$ corresponding to a two-dimensional liquid film supported on a one-dimensional substrate.

1.1.1. LONG-WAVE APPROXIMATION

We follow the review article [111] in the derivation of the *deterministic thin-film equation* in effectively one spatial dimension

$$\begin{cases} \partial_t u = -\partial_x(u^n \partial_x^3 u), & \text{on } [0,T]\times\mathbb{R}, \\ u(0,\cdot) = u_0, & \text{on } \mathbb{R}, \end{cases} \tag{TFE}$$

using the *long-wave approximation* and refer additionally to [23, 56] for more information on the physics of wetting.

The shape of a liquid film at a given time t supported on a substrate is described by the free interface between the liquid and gas, which we assume to be given by the graph of a function $u(t,\cdot)$, see Figure 1.1. The velocity $v = (v^{(x)}, v^{(y)})^T$ and pressure p of the liquid are described by the incompressible *Navier–Stokes equations*

$$\begin{cases} \partial_t v + v\cdot\nabla v = \Delta v - \nabla p, & 0<y<u(t,x), \\ \operatorname{div}(v) = 0, & 0<y<u(t,x). \end{cases} \tag{1.1.1}$$

We supplement the system by a no-slip boundary condition near the substrate, i.e., we assume that

$$v = 0, \qquad \text{at } y = 0. \tag{1.1.2}$$

At the free boundary holds instead the kinematic boundary condition

$$v^{(y)} = \partial_t u + v^{(x)}\partial_x u, \qquad \text{at } y = u(t,x), \tag{1.1.3}$$

which prescribes an element of the free boundary to move according to the fluid velocity at that point.

We also impose the stress balance

$$\big(D_{(x,y)} v + (D_{(x,y)} v)^T - p\mathrm{I}_{\mathbb{R}^2}\big)\widehat{\mathbf{n}} = -\gamma\kappa\widehat{\mathbf{n}}, \qquad \text{at } y = u(t,x). \tag{1.1.4}$$

Here, $\gamma > 0$ is the *surface tension* which we will later assume to be large, and $\widehat{\mathbf{n}}$ and κ are the outward unit normal vector and the mean curvature of the graph of $u(t,\cdot)$, respectively, i.e., we set

$$\widehat{\mathbf{n}} = (1+(\partial_x u)^2)^{-1/2}(-\partial_x u, 1)^T, \qquad \kappa = (1+(\partial_x u)^2)^{-3/2}\partial_x^2 u.$$

Consequently, the equality (1.1.4) expresses that at the free interface the normal stresses due to viscosity and pressure have to be in balance with the stresses due to surface tension. For later purposes, we write out the normal and tangential component of (1.1.4) separately, resulting in

$$\begin{aligned} &2(\partial_x u)^2\partial_x v^{(x)} - 2(\partial_x u)\big(\partial_x v^{(y)} + \partial_y v^{(x)}\big) + 2\partial_y v^{(y)} - \big(1+(\partial_x u)^2\big)p \\ &\quad = \gamma\big(1+(\partial_x u)^2\big)^{-1/2}\partial_x^2 u, \\ &2(\partial_x u)\big(\partial_y v^{(y)} - \partial_x v^{(x)}\big) + \big(1-(\partial_x u)^2\big)\big(\partial_x v^{(y)} + \partial_y v^{(x)}\big) = 0. \end{aligned} \tag{1.1.5}$$

The idea of the long-wave approximation is to derive a model for the dynamics of the free boundary u alone by simplifying the described system. To this end, we integrate the incompressibility condition from (1.1.1) from 0 to u and use the boundary condition (1.1.2) resulting in

$$v^{(y)} = -\int_0^u \partial_x v^{(x)}\,\mathrm{d}y, \qquad \text{at } y = u(t,x).$$

By inserting this in (1.1.3) and using the integral rule of Leibniz, we arrive at the continuity equation

$$\partial_t u + \partial_x\left(\int_0^u v^{(x)}\,\mathrm{d}y\right) = 0 \tag{1.1.6}$$

for the film height with the flux given by the vertically averaged fluid velocity. As a consequence, we recover the conservation of mass $\partial_t \int u\,\mathrm{d}x = 0$ due to the conservative form of the equation.

To obtain from (1.1.6) a closed evolution equation for the film height u we impose the additional assumption that the film is thin, i.e., that the ratio

$$\varepsilon = \frac{\text{typical film heights}}{\text{typical lateral scales}}$$

is small. The resulting separation of scales is schematically depicted in Figure 1.1 in which the wave length of a droplet is significantly larger than the average film height. As a result, the film height u is essentially constant in space and varies only slowly over time.

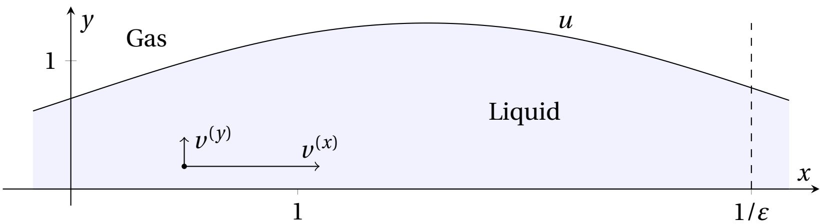

Figure 1.1: The thin-film model: We assume a separation between the vertical and lateral scales.

To nonetheless capture the dynamics of the free boundary, we zoom out of the lateral component by defining new coordinates $\widetilde{x} = \varepsilon x$, $\widetilde{y} = y$ and speed up the time $\widetilde{t} = \varepsilon t$, see Figure 1.2. Accordingly, we obtain the rescaled velocities $\widetilde{v}^{(\widetilde{x})} = v^{(x)}$, $\widetilde{v}^{(\widetilde{y})} = \varepsilon^{-1} v^{(y)}$ and we scale the pressure by $\widetilde{p} = \varepsilon p$ so that the Navier–Stokes system reads

$$\begin{cases} \varepsilon\left(\partial_{\widetilde{t}}\widetilde{v}^{(\widetilde{x})} + \widetilde{v}\cdot\nabla_{(\widetilde{x},\widetilde{y})}\widetilde{v}^{(\widetilde{x})}\right) = \varepsilon^2\partial^2_{\widetilde{x}}\widetilde{v}^{(x)} + \partial^2_{\widetilde{y}}\widetilde{v}^{(x)} - \partial_{\widetilde{x}}\widetilde{p}, & 0 < \widetilde{y} < \widetilde{u}(\widetilde{t},\widetilde{x}),\\ \varepsilon^3\left(\partial_{\widetilde{t}}\widetilde{v}^{(\widetilde{y})} + \widetilde{v}\cdot\nabla_{(\widetilde{x},\widetilde{y})}\widetilde{v}^{(\widetilde{y})}\right) = \varepsilon^2\left(\varepsilon^2\partial^2_{\widetilde{x}}\widetilde{v}^{(x)} + \partial^2_{\widetilde{y}}\widetilde{v}^{(x)}\right) - \partial_{\widetilde{y}}\widetilde{p}, & 0 < \widetilde{y} < \widetilde{u}(\widetilde{t},\widetilde{x}),\\ \operatorname{div}_{(\widetilde{x},\widetilde{y})}(\widetilde{v}) = 0, & 0 < \widetilde{y} < \widetilde{u}(\widetilde{t},\widetilde{x}). \end{cases}$$

1

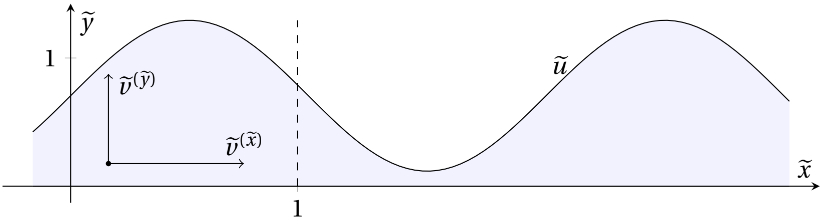

Figure 1.2: The thin-film model in rescaled variables, cf. Figure 1.1.

In particular, we observe that the dynamics of the rescaled velocity are dominated by the viscous forces and the inertial terms become negligible as $\varepsilon \searrow 0$. Consequently, the flow is laminar if ε is small and we obtain

$$\partial_{\widetilde{y}}^2 \widetilde{v}^{(\widetilde{x})} = \partial_{\widetilde{x}} \widetilde{p} \quad \text{and} \quad \partial_{\widetilde{y}} \widetilde{p} = 0, \qquad 0 < \widetilde{y} < \widetilde{u}(\widetilde{t}, \widetilde{x}), \tag{1.1.7}$$

as the leading order terms in ε. From the latter equality we deduce that the pressure varies only laterally, i.e., it can be written as a function solely depending on $(\widetilde{t}, \widetilde{x})$.

To rescale the stress balance, we assume that the surface tension is large in the sense that $\varepsilon^3\gamma = 1$, then (1.1.5) becomes

$$\begin{aligned} &2\varepsilon^4(\partial_{\widetilde{x}}\widetilde{u})^2\partial_{\widetilde{x}}\widetilde{v}^{(\widetilde{x})} - 2\varepsilon^2(\partial_{\widetilde{x}}\widetilde{u})\big(\varepsilon^2\partial_{\widetilde{x}}\widetilde{v}^{(\widetilde{y})} + \partial_{\widetilde{y}}\widetilde{v}^{(\widetilde{x})}\big) + 2\varepsilon^2\partial_{\widetilde{y}}\widetilde{v}^{(\widetilde{y})} - \big(1 + (\varepsilon\partial_{\widetilde{x}}\widetilde{u})^2\big)\widetilde{p} \\ &\quad = \big(1 + (\varepsilon\partial_{\widetilde{x}}\widetilde{u})^2\big)^{-1/2}\partial_{\widetilde{x}}^2\widetilde{u}, \\ &2\varepsilon^2(\partial_{\widetilde{x}}\widetilde{u})\big(\partial_{\widetilde{y}}\widetilde{v}^{(\widetilde{y})} - \partial_x\widetilde{v}^{(\widetilde{x})}\big) + \big(1 - (\varepsilon\partial_{\widetilde{x}}\widetilde{u})^2\big)\big(\varepsilon^2\partial_{\widetilde{x}}\widetilde{v}^{(\widetilde{y})} + \partial_{\widetilde{y}}\widetilde{v}^{(\widetilde{x})}\big) = 0. \end{aligned}$$

This reduces to the conditions

$$-\widetilde{p} = \partial_{\widetilde{x}}^2 \widetilde{u} \quad \text{and} \quad \partial_{\widetilde{y}} \widetilde{v}^{(\widetilde{x})} = 0, \qquad \text{at } \widetilde{y} = \widetilde{u}(\widetilde{t}, \widetilde{x}), \tag{1.1.8}$$

as $\varepsilon \searrow 0$. The boundary conditions (1.1.2) and (1.1.3) and the resulting continuity equation (1.1.6) on the other hand are invariant under the scaling and hold for the rescaled variables as well.

Using the latter equality of (1.1.8) as a boundary condition and integrating the first relation from (1.1.7) from $\widetilde{y}$ to $\widetilde{u}(\widetilde{t}, \widetilde{x})$ results in

$$\partial_{\widetilde{y}} \widetilde{v}^{(\widetilde{x})} = (\widetilde{y} - \widetilde{u})\partial_{\widetilde{x}} \widetilde{p}, \qquad 0 < \widetilde{y} < \widetilde{u}(\widetilde{t}, \widetilde{x}). \tag{1.1.9}$$

Integrating this again from 0 to $\widetilde{y}$ and using the no-slip condition (1.1.2) yields

$$\widetilde{v}^{(\widetilde{x})} = (\widetilde{y}^2/2 - \widetilde{u}\widetilde{y})\partial_{\widetilde{x}} \widetilde{p} \qquad 0 < \widetilde{y} < \widetilde{u}(\widetilde{t}, \widetilde{x}). \tag{1.1.10}$$

We conclude by inserting this into the continuity equation (1.1.6) and solving the integral with respect to $\widetilde{y}$ resulting in

$$\partial_{\widetilde{t}} \widetilde{u} - \partial_{\widetilde{x}}\big((\widetilde{u}^3/3)\partial_{\widetilde{x}} \widetilde{p}\big) = 0. \tag{1.1.11}$$

The pressure on the other hand is determined by the profile $\tilde{u}$ according to (1.1.8) so that, after *dropping the tilde-notation*, we arrive at the thin-film equation

$$\partial_t u = -\partial_x\big((u^3/3)\partial_x^3 u\big),$$

corresponding to (TFE) with $n = 3$, as desired.

We remark that, if we started from a Navier-slip condition

$$v^{(y)} = 0 \quad \text{and} \quad v^{(x)} - \lambda\partial_y v^{(x)} = 0, \qquad \text{at } y = 0, \tag{1.1.12}$$

instead of (1.1.2), we would have arrived at

$$\partial_t u = -\partial_x\big((u^3/3 + \lambda u^2)\partial_x^3 u\big). \tag{1.1.13}$$

Indeed, the no-slip condition was used as a boundary condition when integrating (1.1.9) from 0 to y. If we allow for slip according to (1.1.12) the term $\lambda\partial_y v^{(x)}$ evaluated at $y = 0$ enters the right-hand side of (1.1.10). However, by (1.1.9) this is nothing but $-\lambda u\partial_x p$. After the second integration, this leads to the additional term λu^2 in front of the pressure gradient in (1.1.11) and ultimately to (1.1.13). More generally, imposing the boundary condition (1.1.12) with a height-dependent slip length $\lambda = u^{n-2}$ results in

$$\partial_t u = -\partial_x\big((u^3/3 + u^n)\partial_x^3 u\big).$$

Since the qualitative properties of the above equation, like the dynamics of the contact line or positivity properties, are determined in regions where u is small, the u^3-term is less important for the dynamics, at least if $n < 3$. Hence, we recover the effective model (TFE) for the height of a thin liquid film with a variable exponent n reflecting the boundary condition of the fluid velocity near the substrate.

1.1.2. SURFACE ENERGY

A central assumption in the preceding derivation is that the effects due to surface tension govern the dynamics. Consequently, the energy of a configuration is determined by the total amount of surface energy, that is

$$\gamma_{\mathrm{gs}}\big|\{u=0\}\big| + \gamma_{\mathrm{ls}}\big|\{u>0\}\big| + \gamma_{\mathrm{gl}}\int_{\{u>0\}}\big(1+|\partial_x u|^2\big)^{1/2}\,\mathrm{d}x,$$

where γ_{gs}, γ_{ls} and γ_{gl} are the surface tension constants of the three interfaces gas-solid, liquid-solid and gas-liquid. Using the Taylor approximation of the square root around 1 and assuming that the lateral changes of the film height are small $|\partial_x u| \ll 1$, this can be simplified to

$$\begin{aligned} &\gamma_{\mathrm{gs}}\big|\{u=0\}\big| + \gamma_{\mathrm{ls}}\big|\{u>0\}\big| + \gamma_{\mathrm{gl}}\int_{\{u>0\}} 1 + \tfrac{1}{2}|\partial_x u|^2\,\mathrm{d}x \\ &\quad = \text{const.} + \underbrace{\big(\gamma_{\mathrm{gs}} - \gamma_{\mathrm{ls}} - \gamma_{\mathrm{gl}}\big)}_{S:=}\big|\{u=0\}\big| + \frac{\gamma_{\mathrm{gl}}}{2}\int_{\{u>0\}}|\partial_x u|^2\,\mathrm{d}x. \end{aligned} \tag{1.1.14}$$

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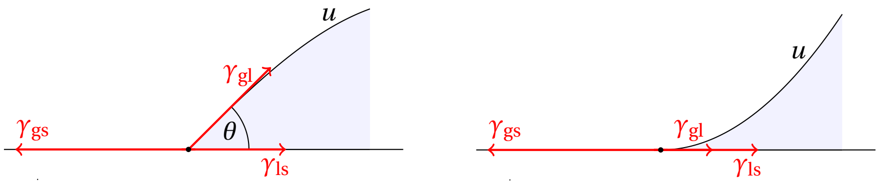
The emerging constant S is called *spreading coefficient* and determines the spreading behavior of the fluid film. Indeed, if $S \geq 0$, a completely wetted substrate is energetically advantageous so that this situation is referred to as the complete wetting regime. In contrast, the regime $S < 0$ is referred to as partial wetting.



(a) Forces due to surface tension at the contact point in the partial wetting regime.

(b) Forces due to surface tension at the contact point in the complete wetting regime.

Figure 1.3: In (a) an equilibrium of forces can be reached for some angle $\theta > 0$, in (b) it holds instead $\theta = 0$.

Young's law states that at the contact line, i.e., the triple junction of liquid, solid and gas, a balance of the lateral forces due to surface tension

$$\gamma_{\mathrm{gs}} = \gamma_{\mathrm{ls}} - \gamma_{\mathrm{gl}} \cos(\theta)$$

tries to form. Here, θ is the contact angle of the thin-film, see Figure 1.3. In particular, in the case of *complete wetting*, which *is the focus of this thesis*, we have that $\theta = 0$ at the contact line.

If specifically $S = 0$, the term involving $|\{u = 0\}|$ vanishes in (1.1.14), so that the dynamics is determined by

$$\mathscr{E} = \frac{1}{2} \int |\partial_x u|^2 \, \mathrm{d}x, \tag{1.1.15}$$

where we employed the normalizing assumption $\gamma_{\mathrm{gl}} = 1$. If instead $S < 0$, the energy of the system can be instantaneously reduced by an immediate wetting of the whole substrate, which is physically observed by a microscopic precursor film shooting ahead of the macroscopic film. Consequently, also in this case the term involving $|\{u = 0\}|$ disappears and, on the macroscopic level, the dynamics of the fluid film are determined by the simplified energy (1.1.15).

1.1.3. THE EFFECTS OF THERMAL NOISE

There exists a vast body of mathematical literature on the deterministic thin-film equation (TFE), including effective numerical algorithms to simulate the expected behavior of a thin fluid film described by the equation. Remarkably, a research group of mathematicians and physicists joined in [10] to compare the numerical predictions based on the mathematical model (TFE) to experiments involving fluid films with heights of several nanometers, i.e., films which consist of less than a hundred layers of atoms.

While their numerical simulations beautifully matched the experiments, an overlapping research group pointed out in the follow up work [78] that there was a small discrepancy between the numerical and experimental time scales, in which film rupture takes place, compared to the onset of droplet formation. They proceeded to incorporate the effects of *thermal noise* in the thin-film model, leading to the *stochastic thin-film equation* (STFE). To their satisfaction, they could reproduce the time scales observed in the experiments in numerical simulations of the stochastic equation, indicating that thermal fluctuations should be accounted for in the modeling and analysis of nanofilms.

We sketch the derivation of the stochastic thin-film equation from [78], and refer additionally to [43] for an elaborated version. The fundamental idea is to start the long-wave approximation from the Navier–Stokes equations of fluctuating hydrodynamics

$$\begin{cases} \partial_t v + v\cdot\nabla v = \Delta v - \nabla p + \operatorname{div}(\mathscr{S}), & 0<y<u(t,x),\\ \operatorname{div}(v) = 0, & 0<y<u(t,x),\end{cases}$$

in which the effects of thermal fluctuations on the fluid velocity are incorporated by the *fluctuating stress tensor* $\mathscr{S}$, see [99]. The latter is a mean-free $\mathbb{R}^{2\times 2}$-valued Gaussian noise whose entries $\mathscr{S}^{(x,x)}$, $\mathscr{S}^{(y,y)}$ and $\mathscr{S}^{(x,y)} = \mathscr{S}^{(y,x)}$ are white in time and space, where we refer to the forthcoming Subsection 1.2.1 for a description of Gaussian white noise. We supplement the system with the same boundary conditions

$$v=0 \quad \text{at } x=0, \qquad \text{and} \qquad v^{(y)} = \partial_t u + v^{(x)}\partial_x u \quad \text{at } y=u(t,x), \tag{1.1.16}$$

as in the deterministic case. However, we have to account for the fluctuating stresses in the stress balance (1.1.4) resulting in

$$\big(D_{(x,y)}v + (D_{(x,y)}v)^T - p\mathbb{I}_{\mathbb{R}^2} + \mathscr{S}\big)\hat{\mathbf{n}} = -\gamma\kappa\hat{\mathbf{n}}, \qquad \text{at } y=u(t,x).$$

To take again the long-wave limit, we rescale the fluctuating stress tensor in such a way that it has the same order in ε as the dominating terms in the viscous stress tensor, i.e., we set

$$\widetilde{\mathscr{S}}^{(\tilde{x},\tilde{x})} = \varepsilon^{-1}\mathscr{S}^{(x,x)}, \quad \widetilde{\mathscr{S}}^{(\tilde{y},\tilde{y})} = \varepsilon^{-1}\mathscr{S}^{(y,y)} \quad \text{and} \quad \widetilde{\mathscr{S}}^{(\tilde{x},\tilde{y})} = \mathscr{S}^{(x,y)},$$

corresponding to

$$\partial_{\tilde{x}}\tilde{v}^{(\tilde{x})} = \varepsilon^{-1}\partial_x v^{(x)}, \quad \partial_{\tilde{y}}\tilde{v}^{(\tilde{y})} = \varepsilon^{-1}\partial_y v^{(y)} \quad \text{and} \quad \partial_{\tilde{y}}\tilde{v}^{(\tilde{x})} = \partial_y v^{(x)}.$$

This ensures that also in the rescaled variables the viscous and stochastic stresses are comparable to each other.

Hence, as the contributions of $\partial_{\tilde{x}}\tilde{v}^{(\tilde{x})}$ and $\partial_{\tilde{y}}\tilde{v}^{(\tilde{y})}$ to the viscous stress disappeared in the governing equations (1.1.7) and (1.1.8) after letting $\varepsilon \searrow 0$, so do $\widetilde{\mathscr{S}}^{(\tilde{x},\tilde{x})}$ and $\widetilde{\mathscr{S}}^{(\tilde{y},\tilde{y})}$ in the stochastic case. On the other hand, whenever the contribution of $\partial_{\tilde{y}}\tilde{v}^{(\tilde{x})}$ is still visible as $\varepsilon \searrow 0$ so is $\widetilde{\mathscr{S}}^{(\tilde{x},\tilde{y})}$ resulting in the stochastic versions

$$\partial_{\tilde{y}}\big(\partial_{\tilde{y}}\tilde{v}^{(\tilde{x})} + \widetilde{\mathscr{S}}^{(\tilde{x},\tilde{y})}\big) = \partial_{\tilde{x}}\tilde{p} \quad \text{and} \quad \partial_{\tilde{y}}\tilde{p} = 0, \qquad 0<\tilde{y}<\tilde{u}(\tilde{t},\tilde{x}) \tag{1.1.17}$$

and

$$-\tilde{p} = \partial_{\tilde{x}}^2\tilde{u} \quad \text{and} \quad \partial_{\tilde{y}}\tilde{v}^{(\tilde{x})} + \widetilde{S}^{(\tilde{x},\tilde{y})} = 0, \qquad \text{at } \tilde{y} = \tilde{u}(\tilde{t},\tilde{x}) \tag{1.1.18}$$

of (1.1.7) and (1.1.8) for the rescaled variables.

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As in the deterministic case, it remains to solve the integral in the continuity equation

$$\partial_{\tilde{t}}\tilde{u} + \partial_{\tilde{x}}\left(\int_0^{\tilde{u}} \tilde{v}^{(\tilde{x})}\,\mathrm{d}\tilde{y}\right) = 0$$

by integrating the first equation of (1.1.17) and using (1.1.16) and (1.1.18) as boundary conditions. As result, one obtains the equation

$$\partial_t u = \partial_x\left((u^3/3)\partial_x^3 u + \int_0^u (u-y)\mathscr{S}^{(x,y)}\,\mathrm{d}y\right) \tag{1.1.19}$$

for the film height, where we dropped again the tilde-notation.

This is however not satisfactory, since the stochastic term still depends on the y-variable contradicting the idea of the long-wave approximation. Consequently, the authors of [78] show that the stochastic partial differential equation

$$\partial_t u = \partial_x\left((u^3/3)\partial_x^3 u + (u^3/3)^{1/2}\mathscr{W}\right) \tag{1.1.20}$$

for a spatio-temporal white noise $\mathscr{W}$ has the same statistical properties as (1.1.19) in the sense that their Fokker-Planck equations coincide, at least on a spatially discretized level. Equation (1.1.20) has indeed the form of (STFE) for $n = 3$. As in the deterministic case, other values of n can be obtained by imposing (film height-dependent) slip conditions of the fluid velocity near the substrate and we refer the interested reader to [126].

We conclude this subsection by remarking that, independently of [78], the same stochastic thin-film equation was derived in [37]. The latter group of researchers applied the fluctuation-dissipation relation to calculate the correct magnitude of thermal fluctuations around a flat film profile. Together with a spatial localization of a non-flat fluid film and requiring conservation of mass, this results in (STFE) as well.

1.2. INTERPRETATION OF THE EQUATION

In the present section we address the question what it means for u to be a solution to (STFE). Next to the classical problem of stochastic analysis to give meaning to the product $u^{n/2}\mathscr{W}$, we also discuss different notions of weak solutions from the theory of partial differential equations. Moreover, we comment on the subtle difference between probabilistically strong and martingale, or probabilistically weak, solutions. These topics are of course standard and we refer, e.g., to [46, 90, 97] for a more complete treatment.

1.2.1. GAUSSIAN WHITE NOISE

The aim of this subsection is to describe the properties of white noise, which we used in the previous section to model thermal fluctuations and which appears in (STFE). To this end, let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space and $X\colon \Omega \to \mathbb{R}$ be a random variable. Then, X is called a Gaussian random variable, if its law follows a normal distribution, i.e., if

$$\mathbb{P}\big(\{X \in A\}\big) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_A e^{-\frac{1}{2}((x-\mu)/\sigma)^2}\,\mathrm{d}x, \tag{1.2.1}$$

for any Borel measurable subset $A \subset \mathbb{R}$ and parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. In this case μ and σ^2 are nothing but the mean and the variance of the random variable X:

$$\mu = \mathbb{E}\big[X\big], \qquad \sigma^2 = \operatorname{Var}\big(X\big) = \mathbb{E}\big[(X-\mu)^2\big].$$

It is customary to also call X Gaussian if X is constant almost surely, meaning that

$$\mathbb{P}\big(\{X \in A\}\big) = \begin{cases} 1, & \mu \in A, \\ 0, & \text{else}, \end{cases} \tag{1.2.2}$$

for its mean μ and that its variance equals 0.

More generally, we call a stochastic process $\mathscr{X} : \Omega \times [0,T] \to \mathbb{R}$ with continuous paths $\mathscr{X}(\omega,\cdot) \in C([0,T])$ a *Gaussian process* if any linear combination

$$\sum_{i=1}^{n} \lambda_i \mathscr{X}_{t_i}, \qquad \lambda_i \in \mathbb{R},\ t_i \in [0,T], \tag{1.2.3}$$

of its marginals is a Gaussian random variable. As μ and σ^2 determine uniquely the distribution of a Gaussian random variable X via (1.2.1) or (1.2.2), also the probability distribution of X on the space $C([0,T])$ is uniquely determined by the mean and variance of the Gaussian random variables (1.2.3) for the various choices of λ_i and t_i.

These quantities, on the other hand, can be rewritten using the linearity of the expectation and the bilinearity of the covariance operator as

$$\mathbb{E}\Big[\sum_{i=1}^{n} \lambda_i \mathscr{X}_{t_i}\Big] = \sum_{i=1}^{n} \lambda_i \mathbb{E}[\mathscr{X}_{t_i}],$$

$$\operatorname{Var}\Big(\sum_{i=1}^{n} \lambda_i \mathscr{X}_{t_i}\Big) = \operatorname{Cov}\Big(\sum_{i=1}^{n} \lambda_i \mathscr{X}_{t_i}, \sum_{j=1}^{n} \lambda_j \mathscr{X}_{t_j}\Big) = \sum_{i,j=1}^{n} \lambda_i \lambda_j \operatorname{Cov}\big(\mathscr{X}_{t_i}, \mathscr{X}_{t_j}\big).$$

Consequently, we can uniquely determine the law of a Gaussian process $\mathscr{X}$ by specifying

$$\mathbb{E}\big[\mathscr{X}_t\big] \quad \text{and} \quad \operatorname{Cov}(\mathscr{X}_s, \mathscr{X}_t)$$

for all $s, t \in [0,T]$.

Probably the most famous Gaussian process is the *Brownian motion* appearing in many branches of probability theory. Its law is determined by

$$\mathbb{E}\big[B_t\big] = 0, \qquad \operatorname{Cov}\big(B_s, B_t\big) = \min(s,t) \tag{1.2.4}$$

for all $s, t \in [0,T]$. As a consequence, the paths of a Brownian motion $B(\omega,\cdot)$ lie for almost all ω in the Hölder class $C^{1/2-\varepsilon}([0,T])$ for each $\varepsilon \in (0,1/2)$ but not in $C^{1/2}([0,T])$.

Gaussian white noise can be thought of as the derivative of a Brownian motion. However, since the paths of a Brownian motion are not even 1/2-times Hölder continuous, their derivative cannot be defined in a classical way. Keeping in mind that we are dealing with a Gaussian process in a generalized sense we formally compute the limiting mean of the difference quotients

$$\mathbb{E}\big[\mathscr{W}_t\big] = \lim_{\varepsilon \searrow 0} \mathbb{E}\big[(B_{t+\varepsilon} - B_t)/\varepsilon\big] = 0, \qquad t \in [0,T),$$

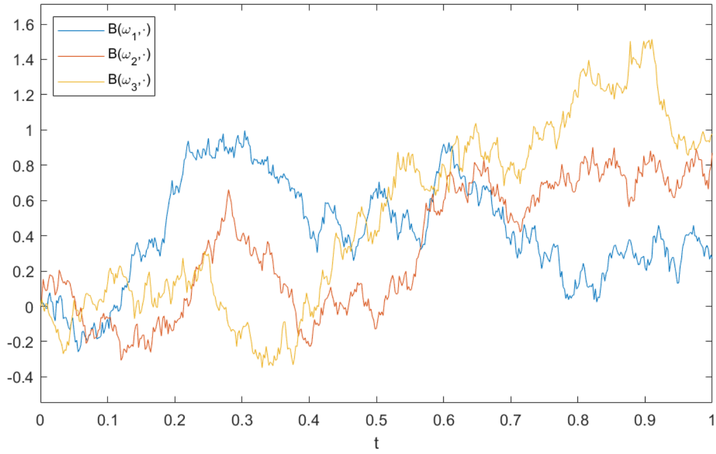

Figure 1.4: Paths of a Brownian motion generated in MATLAB.

and their covariance

$$\begin{aligned}\operatorname{Cov}\big(\mathcal{W}_s,\mathcal{W}_t\big) &= \lim_{\varepsilon\searrow 0}\varepsilon^{-2}\operatorname{Cov}\big((B_{s+\varepsilon}-B_s)(B_{t+\varepsilon}-B_t)\big)\\ &= \lim_{\varepsilon\searrow 0}\varepsilon^{-2}\big(\min(s+\varepsilon,t+\varepsilon)-\min(s+\varepsilon,t)-\min(s,t+\varepsilon)+\min(s,t)\big) = 0,\end{aligned}$$

for $0 \le s < t < T$. In the last step we have used that $\min(s+\varepsilon,t) = s+\varepsilon$ if ε is sufficiently small, so that the terms inside of the parentheses cancel each other. We conclude that Gaussian white noise is centered, i.e., its mean is 0, and that it is uncorrelated meaning that $\mathcal{W}_s$ and $\mathcal{W}_t$ are independent for $s \neq t$.

If we let however $s = t$, we obtain from (1.2.4) that

$$\begin{aligned}\operatorname{Cov}\big(\mathcal{W}_t,\mathcal{W}_t\big) &= \lim_{\varepsilon\searrow 0}\varepsilon^{-2}\operatorname{Cov}\big((B_{t+\varepsilon}-B_t)(B_{t+\varepsilon}-B_t)\big)\\ &= \lim_{\varepsilon\searrow 0}\varepsilon^{-2}\big(\min(t+\varepsilon,t+\varepsilon)-\min(t+\varepsilon,t)-\min(t,t+\varepsilon)+\min(t,t)\big)\\ &= \lim_{\varepsilon\searrow 0}\varepsilon^{-2}\big(t+\varepsilon-2t+t\big) = \infty,\end{aligned}$$

and we conclude that at a fixed time t the white noise $\mathcal{W}_t$ is a Gaussian random variable with zero mean and infinite variance. This is in a sense expected, because we have seen already that B does not admit a derivative in a classical sense.

To quantify the strength of $\mathcal{W}_t$, we consider the covariance $\operatorname{Cov}(\mathcal{W}_s,\mathcal{W}_t)$ as a function of s. Using as before that the covariation operator is bilinear, we obtain that

$$\begin{aligned}\operatorname{Cov}(\mathcal{W}_s,B_t) &= \lim_{\varepsilon\searrow 0}\varepsilon^{-1}\operatorname{Cov}(B_{s+\varepsilon}-B_s,B_t)\\ &= \lim_{\varepsilon\searrow 0}\varepsilon^{-1}\big(\min(s+\varepsilon,t)-\min(s,t)\big) = \tfrac{\mathrm{d}}{\mathrm{d}s}\min(s,t)\end{aligned}$$

and the latter can be evaluated as $\mathbf{1}_{[0,t)}(s)$ or equivalently $\mathbf{1}_{[s,T]}(t)$. By the same reasoning we calculate further

$$\mathrm{Cov}(\mathscr{W}_s, \mathscr{W}_t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Cov}(\mathscr{W}_s, B_t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{1}_{[s,T]}(t).$$

The latter is a Heaviside function, which is not classically differentiable, but admits the Dirac delta distribution $\delta(t-s)$ as a generalized derivative. Hence, we can think of the covariance function of $\mathscr{W}$ to be given by the distribution

$$\mathrm{Cov}(\mathscr{W}_s, \mathscr{W}_t) = \delta(t-s),$$

which acts on test functions $\Phi : [0,T]^2 \to \mathbb{R}$ via

$$\langle \delta(t-s), \Phi \rangle = \int_0^T \int_0^T \delta(t-s)\Phi(s,t)\,\mathrm{d}t\,\mathrm{d}s = \int_0^T \Phi(s,s)\,\mathrm{d}s.$$

Interestingly, this determines the statistics of the Fourier coefficients

$$w_k = \int_0^T \mathscr{W}_t f_k(t)\,\mathrm{d}t$$

of $\mathscr{W}$ along an orthonormal basis $(f_k)_{k\in\mathbb{N}}$ of $L^2([0,T])$. Indeed, they are centered Gaussian random variables with covariance structure

$$\begin{aligned}\mathrm{Cov}(w_k, w_l) &= \int_0^T \int_0^T \mathrm{Cov}(\mathscr{W}_s, \mathscr{W}_t) f_k(s) f_l(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_0^T \int_0^T \delta(t-s) f_k(s) f_l(t)\,\mathrm{d}t\,\mathrm{d}s = \int_0^T f_k(s) f_l(s)\,\mathrm{d}s = \begin{cases} 1, & k = l, \\ 0, & \text{else.}\end{cases}\end{aligned}$$

Consequently, $(w_k)_{k\in\mathbb{N}}$ is a sequence of independent, standard normally distributed random variables. Hence, we can either write

$$\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t}B \qquad \text{or} \qquad \mathscr{W} = \sum_{k\in\mathbb{N}} w_k f_k$$

for such a sequence, providing a second interpretation of white noise.

Combining both interpretations we can define *spatio-temporal white noise* on $[0,T]\times\mathscr{D}$ for a spatial domain $\mathscr{D}$ as the temporal derivative of

$$B_t = \sum_{k\in\mathbb{N}} e_k \beta_t^{(k)} \tag{1.2.5}$$

for a sequence of independent Brownian motions $(\beta^{(k)})_{k\in\mathbb{N}}$ and an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathscr{D})$. Using the presented ideas, one can compute that $\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t}B$ admits the covariance structure

$$\mathrm{Cov}(\mathscr{W}_s(x), \mathscr{W}_t(y)) = \delta(t-s)\delta(y-x)$$

in analogy to the temporal case.

While the roughness of a spatio-temporal white noise $\mathscr{W}$ in (STFE) in time can be dealt with using the classical ideas of stochastic integration, its roughness in space poses questions which are beyond the scope of this thesis. Therefore, we consider in the coming chapters (STFE) instead with a *spatially colored noise*, which can be obtained by replacing the orthonormal basis $(e_k)_{k\in\mathbb{N}}$ in (1.2.5) by a sequence of functions which decays in $L^2(\mathbb{T})$ as $k\to\infty$.

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1.2.2. STOCHASTIC INTEGRAL EQUATIONS

We use the ideas from the previous subsection to give meaning to (STFE) as a *stochastic integral equation*. To focus only on the challenges related to the stochasticity, we set $d = 1$ and $n = 2$ in (STFE) and disregard the thin-film operator, i.e., we consider the stochastic partial differential equation

$$\begin{cases} \partial_t u = \partial_x(u\mathscr{W}), & \text{on } \Omega \times [0,T] \times \mathbb{T}, \\ u(0,\cdot) = u_0, & \text{on } \Omega \times \mathbb{T}, \end{cases} \tag{1.2.6}$$

instead. We assume that the noise is white in time and colored in space, i.e., that

$$\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k\in\mathbb{N}} e_k \beta^{(k)}$$

for a sequence of independent Brownian motions $(\beta^{(k)})_{k\in\mathbb{N}}$ and a sequence of sufficiently smooth and decaying functions $e_k\colon \mathbb{T} \to \mathbb{R}$, for $k \in \mathbb{N}$.

As discussed before, the Brownian motions $\beta^{(k)}$ admit almost surely Hölder regularity $C^{1/2-\varepsilon}$ in time. Accordingly, the temporal regularity $\mathscr{W}$ is $C^{-1/2-\varepsilon}$ so that we can expect at most temporal regularity $C^{1/2-\varepsilon}$ of a solution u to (1.2.6). Hence, even in the best of all cases, the sum of the smoothness of u and the smoothness of $\mathscr{W}$ will be negative meaning that the product $u\mathscr{W}$ cannot be defined using classical tools from analysis.

The key idea of stochastic integration is to exploit the statistical properties of $\mathscr{W}$ to give a meaning to the temporal integral

$$\int u\mathscr{W}\,\mathrm{d}t = \int u \cdot \Big(\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\in\mathbb{N}} e_k\beta^{(k)}\Big)\mathrm{d}t = \sum_{k\in\mathbb{N}} \int u e_k \,\mathrm{d}\beta^{(k)} \tag{1.2.7}$$

instead. To this end, we equip the underlying probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with an increasing family of sub-σ-fields $\mathscr{F} = (\mathscr{F}_t)_{t\in[0,T]}$ encoding the information at each time. We assume that the resulting *filtration* $\mathscr{F}$ suffices the usual conditions, meaning that $\mathscr{F}_t$ contains all nullsets and that $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for all $t \in [0,T)$. A stochastic process $\mathscr{X}$ is called $\mathscr{F}$-adapted, if $\mathscr{X}_t$ is $\mathscr{F}_t$-measurable for each t, i.e., if $\mathscr{X}$ doesn't anticipate the future. We assume furthermore that each $\beta^{(k)}$ is an $\mathscr{F}$-Brownian motion, meaning that $\beta^{(k)}$ is $\mathscr{F}$-adapted and that the increments $\beta_t^{(k)} - \beta_s^{(k)}$ are independent of $\mathscr{F}_s$ for all $0 \le s < t \le T$.

Then, assuming that also the process u is $\mathscr{F}$-adapted, one calculates that the Riemann sums of the integral (1.2.7) satisfy

$$\begin{aligned} \mathbb{E}\Bigg[\Bigg|\sum_{i=0}^{n-1}(u(t_i,x)e_k(x))(\beta_{t_{i+1}}^{(k)} - \beta_{t_i}^{(k)})\Bigg|^2\Bigg] &= \sum_{i=0}^{n-1}\mathbb{E}\big[(u(t_i,x)e_k(x))^2\big]\cdot\mathbb{E}\big[\big(\beta_{t_{i+1}}^{(k)} - \beta_{t_i}^{(k)}\big)^2\big] \\ = \mathbb{E}\Bigg[\sum_{i=0}^{n-1}(u(t_i,x)e_k(x))^2(t_{i+1}-t_i)\Bigg] &\to \mathbb{E}\Bigg[\int_0^t (u(s,x)e_k(x))^2\,\mathrm{d}s\Bigg], \qquad n\to\infty, \end{aligned} \tag{1.2.8}$$

for any sequence of finer and finer partitions $0 = t_0 < \cdots < t_n = t$. Consequently, while for a fixed ω the Riemann sum can in general diverge, we expect it to converge as a random variable in $L^2(\Omega)$.

This observation allowed *Kiyoshi Itô* in [87] to define the integral (1.2.7) as an element of $L^2(\Omega)$ based on his famous isometry, which in our situation reads

$$\mathbb{E}\left[\left|\sum_{k\in\mathbb{N}}\int_0^t u(s,x)e_k(x)\,\mathrm{d}\beta_s^{(k)}\right|^2\right] = \mathbb{E}\left[\sum_{k\in\mathbb{N}}\int_0^t |u(s,x)e_k(x)|^2\,\mathrm{d}s\right].$$

We can use the resulting Itô integral to give a meaning to equation (1.2.6) by integrating in time resulting in the stochastic integral equation

$$u(t) - u_0 = \sum_{k\in\mathbb{N}}\int_0^t \partial_x(u(s,\cdot)e_k)\,\mathrm{d}\beta_s^{(k)}. \tag{1.2.9}$$

Moreover, by integrating the pointwise Itô isometry with respect to x, we deduce that

$$\mathbb{E}\left[\left\|\sum_{k\in\mathbb{N}}\int_0^t \partial_x(u(s,\cdot)e_k)\,\mathrm{d}\beta_s^{(k)}\,ds\right\|_{L^2(\mathbb{T})}^2\right] = \mathbb{E}\left[\sum_{k\in\mathbb{N}}\int_0^t \|\partial_x(u(s,\cdot)e_k)\|_{L^2(\mathbb{T})}^2\,\mathrm{d}s\right] \tag{1.2.10}$$

and that as soon as the right-hand side of the above is finite, the stochastic integral in (1.2.9) exists as an $L^2(\mathbb{T})$-valued random variable. In particular, since the finiteness of (1.2.10) does not require any smoothness of u in time, (1.2.9) is a mathematically tractable interpretation of (1.2.6).

However, the roughness of the Brownian motions leave some ambiguity in the definition of the stochastic integral. Indeed, the use of the left endpoints of the intervals $[t_i, t_i+1)$ in the Riemann sums from (1.2.8) is necessary to take full advantage of the adaptedness of the integrand. If one starts in contrast from the trapezoidal rule for integration

$$\sum_{i=0}^{n-1}\frac{u(t_i,x)+u(t_{i+1},x)}{2}e_k(x)\cdot\left(\beta_{t_{i+1}}^{(k)}-\beta_{t_i}^{(k)}\right),$$

one obtains instead the Stratonovich integral denoted by

$$\int_0^t ue_k \circ \mathrm{d}\beta^{(k)}$$

in the limit, as introduced independently in [52, 123]. While it falls short compared to the Itô integral in terms of mathematical properties, it has the advantage that it behaves like a deterministic integral under composition.

To illustrate this, we formally apply the chain rule to deduce that

$$\partial_t\phi(u) = \phi'(u)\partial_t u = \phi'(u)\partial_x(u\mathscr{W}) \tag{1.2.11}$$

for the composition of a function ϕ with a solution to (1.2.6). However, if u is a solution in the Itô sense (1.2.9), Itô's fomula dictates that

$$\begin{aligned}\phi(u(t)) - \phi(u_0) &= \frac{1}{2}\sum_{k\in\mathbb{N}}\int_0^t \phi''(u(s,\cdot))(\partial_x(u(s,\cdot)e_k))^2\,\mathrm{d}s\\ &\quad+\sum_{k\in\mathbb{N}}\int_0^t \phi'(u(s))\partial_x(u(s,\cdot)e_k)\,\mathrm{d}\beta_s^{(k)},\end{aligned}$$

1

which contains an additional term compared to the temporal integral of (1.2.11). If, on the other hand, $\widetilde{u}$ solves the Stratonovich version

$$\widetilde{u}(t) - u_0 = \sum_{k\in\mathbb{N}} \int_0^t \partial_x(\widetilde{u}(s,\cdot)e_k) \circ \mathrm{d}\beta_s^{(k)} \tag{1.2.12}$$

of (1.2.6), it holds instead that

$$\phi(\widetilde{u}(t)) - \phi(u_0) = \sum_{k\in\mathbb{N}} \int_0^t \phi'(\widetilde{u}(s))\partial_x(\widetilde{u}(s,\cdot)e_k) \circ \mathrm{d}\beta_s^{(k)}$$

in line with the rule from classical calculus (1.2.11).

That we followed the rules of classical calculus in the derivation of (STFE) given in Subsection 1.1.3 indicates that the Stratonovich integral is more suitable for our purposes, as pointed out in [58]. To still be able to use the various tools from Itô's stochastic calculus it is customary to exploit that the difference of the respective Riemann sums

$$\sum_{i=0}^{n-1} \frac{\widetilde{u}(t_{i+1},x) - \widetilde{u}(t_i,x)}{2} e_k(x) \cdot \left(\beta_{t_{i+1}}^{(k)} - \beta_{t_i}^{(k)}\right)$$

converges to half of the *covariation process* of $\widetilde{u}e_k$ and $\beta^{(k)}$ and therefore it holds

$$\int_0^t \widetilde{u}e_k \circ \mathrm{d}\beta^{(k)} = \tfrac{1}{2}\left\langle \widetilde{u}e_k, \beta^{(k)}\right\rangle_t + \int_0^t \widetilde{u}e_k \,\mathrm{d}\beta^{(k)}$$

in the limit. Inserting this conversion formula in the Stratonovich equation (1.2.12) results in the equivalent Itô formulation

$$\widetilde{u}(t) - u_0 = \frac{1}{2}\sum_{k\in\mathbb{N}} \partial_x\left\langle \widetilde{u}e_k, \beta^{(k)}\right\rangle_t + \sum_{k\in\mathbb{N}} \int_0^t \partial_x(\widetilde{u}(s,\cdot)e_k)\,\mathrm{d}\beta_s^{(k)}. \tag{1.2.13}$$

To further rewrite the covariation term, we multiply (1.2.13) with e_{k_0} for some $k_0 \in \mathbb{N}$ to deduce that

$$(\widetilde{u}e_{k_0})(t) - (u_0 e_{k_0}) = \frac{1}{2}\sum_{k\in\mathbb{N}} \left(\partial_x\left\langle \widetilde{u}e_k, \beta^{(k)}\right\rangle_t\right)e_{k_0} + \sum_{k\in\mathbb{N}} \int_0^t \partial_x(\widetilde{u}(s,\cdot)e_k)e_{k_0}\,\mathrm{d}\beta_s^{(k)}.$$

The covariation process between the right-hand side and $\beta^{(k_0)}$ can be evaluated using the rules of Itô's calculus resulting in

$$\left\langle \widetilde{u}e_{k_0}, \beta^{(k_0)}\right\rangle_t = \int_0^t \partial_x(\widetilde{u}(s,\cdot)e_{k_0})e_{k_0}\,\mathrm{d}s.$$

Consequently, the *Itô–Stratonovich correction* appearing in (1.2.13) takes the form

$$\frac{1}{2}\sum_{k_0\in\mathbb{N}} \partial_x\left\langle \widetilde{u}e_{k_0}, \beta^{(k_0)}\right\rangle_t = \frac{1}{2}\sum_{k_0\in\mathbb{N}} \int_0^t \partial_x\left(\partial_x(\widetilde{u}(s,\cdot)e_{k_0})e_{k_0}\right)\mathrm{d}s.$$

Inserting this in (1.2.13) results in the fully converted Itô formulation

$$\tilde{u}(t) - u_0 = \frac{1}{2}\sum_{k\in\mathbb{N}}\int_0^t \partial_x\big(\partial_x(\tilde{u}(s,\cdot)e_k)e_k\big)\,\mathrm{d}s + \sum_{k\in\mathbb{N}}\int_0^t \partial_x(\tilde{u}(s,\cdot)e_k)\,\mathrm{d}\beta_s^{(k)}$$

of the Stratonovich equation (1.2.12).

We conclude by remarking that we can analogously give meaning to the full equation (STFE) as an Itô or Stratonovich integral equation and that the Itô–Stratonovich correction term is not affected by the presence of the thin-film operator. Moreover, to ease notation, one commonly uses the differential notation for stochastic integral equations, which we will employ in the following chapters of this thesis. For example, one writes

$$\mathrm{d}u_t = \sum_{k\in\mathbb{N}} \partial_x(u_t e_k)\,\mathrm{d}\beta_t^{(k)} \qquad \text{and} \qquad \mathrm{d}u_t = \sum_{k\in\mathbb{N}} \partial_x(u_t e_k)\circ\mathrm{d}\beta_t^{(k)}$$

for (1.2.9) and (1.2.12), respectively.

1.2.3. ANALYTICALLY WEAK SOLUTIONS

So far, we have understood how to solve the problems related to the noise when defining solutions to (STFE). In this subsection on the contrary, we turn our attention to the deterministic thin-film equation

$$\begin{cases} \partial_t u = -\partial_x(u^2\partial_x^3 u), & \text{on } [0,T]\times\mathbb{T}, \\ u(0,\cdot) = u_0, & \text{on } \mathbb{T}, \end{cases} \tag{1.2.14}$$

with a quadratic mobility $n = 2$ in one spatial dimension, complementing the purely stochastic part (1.2.6). While interpreting (1.2.14) for a sufficiently smooth function u is not a problem, it can be advantageous for the construction of solutions to relax the regularity requirements for u by introducing a weaker solution concept. In fact, this can also be helpful in the construction of classical, smooth solutions to partial differential equations by first showing existence of a weak solution and as a second step additional regularity properties.

A fruitful tool to introduce weaker solution concepts is the weak derivative. The definition of the latter relies on the fact that whenever $\partial_x^l f = g$ for two continuous functions $f, g\colon \mathscr{D}\to\mathbb{R}$ on a domain $\mathscr{D}$ and $l\in\mathbb{N}$, it holds

$$\int_{\mathscr{D}} f\partial_x^l\varphi\,\mathrm{d}x = (-1)^l\int_{\mathscr{D}} g\varphi\,\mathrm{d}x \tag{1.2.15}$$

for any smooth and compactly supported test function $\varphi\colon\mathscr{D}\to\mathbb{R}$ by repeated integration by parts. Using the fact that (1.2.15) can be evaluated for merely locally integrable functions, we call $g\in L^1_{\mathrm{loc}}(\mathscr{D})$ the l-th *weak derivative* of $f\in L^1_{\mathrm{loc}}(\mathscr{D})$ as soon as (1.2.15) holds for each $\varphi\in C_c^\infty(\mathscr{D})$. Moreover, by the fundamental lemma of the calculus variation, (1.2.15) determines g dx-almost everywhere, so that, if it exists, the weak derivative is unique.

In the same spirit, we can remove derivatives from the thin-film operator by testing it with a smooth function $\varphi\colon\mathbb{T}\to\mathbb{R}$ resulting in

$$\int_{\mathbb{T}}\big(-\partial_x(u^2\partial_x^3 u)\big)\varphi\,\mathrm{d}x = \int_{\mathbb{T}} u^2\partial_x^3 u\partial_x\varphi\,\mathrm{d}x.$$

1

Additionally, we can assume that the appearing $\partial_x^3 u$ is the third derivative of u in the weak sense. As a result, we have already significantly reduced the smoothness assumptions on u compared to the classical notion of solution to (1.2.14).

However, this is not quite sufficient for our purposes since also the existence of the third weak derivative of a solution to (1.2.14) on $\mathbb{T}$ is hard to show due to the *degeneracy* of the problem, i.e., the possibility of the coefficient u^2 in the thin-film operator to become 0. In this case, the thin-film equation loses parabolicity and the corresponding smoothing effect so that we do not expect u to be three times weakly differentiable on the whole domain. However, we can use the trivial observation that

$$\int_{\mathbb{T}} u^2 \partial_x^3 u \partial_x \varphi \,\mathrm{d}x = \int_{\{u>0\}} u^2 \partial_x^3 u \partial_x \varphi \,\mathrm{d}x,$$

so that it suffices to require weak differentiability only on the positivity set $\{u > 0\}$, on which the equation remains effectively parabolic.

Hence, by integrating (1.2.14) against φ and subsequently in the time variable we obtain the *weak formulation* of the thin-film equation, namely that

$$\int_{\mathbb{T}} u(t)\varphi \,\mathrm{d}x - \int_{\mathbb{T}} u_0 \varphi \,\mathrm{d}x = \int_0^t \int_{\{u(s,\cdot)>0\}} u^2(s)\partial_x^3 u(s)\partial_x \varphi \,\mathrm{d}x\,\mathrm{d}s$$

for all $t \in [0, T]$ and $\varphi \in C^\infty(\mathbb{T})$. The weak derivative $\partial_x^3 u$ is only required to exist on $\{u > 0\}$, which, as demonstrated in [15], allows for the construction of a weak solution with a given initial profile u_0.

In other situations, the condition that u is three times weakly differentiable on its positivity set is still too restrictive. Then, one can further integrate by parts to deduce that

$$\begin{aligned}\int_{\{u>0\}} u^2 \partial_x^3 u \partial_x \varphi \,\mathrm{d}x &= -2\int_{\{u>0\}} u \partial_x u \partial_x^2 u \partial_x \varphi \,\mathrm{d}x - \int_{\{u>0\}} u^2 \partial_x^2 u \partial_x^2 \varphi \,\mathrm{d}x \\ &= \int_{\{u>0\}} (\partial_x u)^3 \partial_x \varphi \,\mathrm{d}x + 3\int_{\{u>0\}} u(\partial_x u)^2 \partial_x^2 \varphi \,\mathrm{d}x + \int_{\{u>0\}} u^2 \partial_x u \partial_x^3 \varphi \,\mathrm{d}x,\end{aligned}$$

where the boundary terms vanish because $u = 0$ on the boundary of the set $\{u > 0\}$. We obtain the *very weak formulation* of (1.2.14) by requiring that

$$\begin{aligned}\int_{\mathbb{T}} u(t)\varphi \,\mathrm{d}x - \int_{\mathbb{T}} u_0 \varphi \,\mathrm{d}x &= \int_0^t \int_{\{u(s,\cdot)>0\}} (\partial_x u(s))^3 \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}s \\ &+ 3\int_0^t \int_{\{u(s,\cdot)>0\}} u(s)(\partial_x u(s))^2 \partial_x^2 \varphi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\{u(s,\cdot)>0\}} u^2(s)\partial_x u(s)\partial_x^3 \varphi \,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

for any $\varphi \in C^\infty(\mathbb{T})$ and $t \in [0, T]$, as introduced in [33]. In particular, to evaluate the above u needs to be only one time weakly differentiable on its positivity set.

1.2.4. MARTINGALE SOLUTIONS

Combining the observations from the preceding subsections, we can give an exemplary definition of a solution to (STFE) with $n = 2$ in one dimension, namely

$$\begin{cases} \partial_t u = -\partial_x(u^2 \partial_x^3 u) + \partial_x(u\mathscr{W}), & \text{on } \Omega \times [0, T] \times \mathbb{T}, \\ u(0, \cdot) = u_0, & \text{on } \Omega \times \mathbb{T}. \end{cases} \tag{1.2.16}$$

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be again a probability space with a filtration $\mathscr{F}$ that satisfies the usual conditions and $(\beta^{(k)})_{k\in\mathbb{N}}$ a family of independent $\mathscr{F}$-Brownian motions. We assume that the noise in (1.2.16) admits the expansion

$$\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k\in\mathbb{N}} e_k \beta^{(k)} \tag{1.2.17}$$

for sufficiently smooth and decaying functions $(e_k)_{k\in\mathbb{N}}$ and that u_0 is an $\mathscr{F}_0$-measurable initial value. Then, an $\mathscr{F}$-adapted process u is a *very weak solution* to the *Stratonovich interpretation* of (1.2.16), if it satisfies

$$\begin{aligned}\int_{\mathbb{T}} u(t)\varphi \,\mathrm{d}x - \int_{\mathbb{T}} u_0\varphi \,\mathrm{d}x &= \int_0^t \int_{\{u(s,\cdot)>0\}} (\partial_x u(s))^3 \partial_x\varphi \,\mathrm{d}x\,\mathrm{d}s \\ &+ 3\int_0^t \int_{\{u(s,\cdot)>0\}} u(s)(\partial_x u(s))^2 \partial_x^2\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\{u(s,\cdot)>0\}} u^2(s)\partial_x u(s)\partial_x^3\varphi \,\mathrm{d}x\,\mathrm{d}s \\ &- \frac{1}{2}\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \big(\partial_x(u(s)e_k)e_k\big)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}s - \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} u(s)e_k\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}\end{aligned}$$

for all $t \in [0, T]$ and any test function $\varphi \in C^\infty(\mathbb{T})$. We remark that, analogously as for the thin-film operator, we transferred one spatial derivative of the Itô–Stratonovich correction term and the stochastic integrand to the test function.

Such a process constitutes a *probabilistically strong* solution to (1.2.16), expressing the fact that we considered the stochastic basis consisting of the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, the filtration $\mathscr{F}$ and the Brownian motions $(\beta^{(k)})_{k\in\mathbb{N}}$ as given. In contrast, a *probabilistically weak* or *martingale solution* does not only consist of the process, which solves the stochastic partial differential equation, but also the underlying stochastic basis.

For example, a very weak martingale solution to the Stratonovich interpretation of (1.2.16) consists of

- a complete probability space $(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathbb{P}})$,
- a filtration $\check{\mathscr{F}}$ on $(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathbb{P}})$ satisfying the usual conditions,
- a family $(\check{\beta}^{(k)})_{k\in\mathbb{N}}$ of independent $\check{\mathscr{F}}$-Brownian motions,
- an $\check{\mathscr{F}}_0$-measurable $\check{u}_0$ which has the same distribution as u_0,
- and an $\check{\mathscr{F}}$-adapted process $\check{u}$, such that

$$\begin{aligned}\int_{\mathbb{T}} \check{u}(t)\varphi \,\mathrm{d}x - \int_{\mathbb{T}} \check{u}_0\varphi \,\mathrm{d}x &= \int_0^t \int_{\{\check{u}(s,\cdot)>0\}} (\partial_x \check{u}(s))^3 \partial_x\varphi \,\mathrm{d}x\,\mathrm{d}s \\ &+ 3\int_0^t \int_{\{\check{u}(s,\cdot)>0\}} \check{u}(s)(\partial_x \check{u}(s))^2 \partial_x^2\varphi \,\mathrm{d}x\,\mathrm{d}s + \int_{\{\check{u}(s,\cdot)>0\}} \check{u}^2(s)\partial_x \check{u}(s)\partial_x^3\varphi \,\mathrm{d}x\,\mathrm{d}s \\ &- \frac{1}{2}\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \big(\partial_x(\check{u}(s)e_k)e_k\big)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}s - \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \check{u}(s)e_k\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}\check{\beta}_s^{(k)}\end{aligned}$$

  for all $t \in [0, T]$ and $\varphi \in C^\infty(\mathbb{T})$.

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Since the solution process $\check{u}$ is defined on the (not necessarily) different probability space $(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathbb{P}})$, we can unfortunately not prescribe its initial value as a random variable. However, as indicated in the fourth bullet point, we can specify the probability distribution of $\check{u}$ at time 0. Along the same vein, while $\check{u}$ is driven by a different sequence of Brownian motions $(\check{\beta}_k)_{k\in\mathbb{N}}$, the laws of

$$\check{\mathscr{W}} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k\in\mathbb{N}} e_k \check{\beta}^{(k)}$$

and (1.2.17) do coincide.

1.3. EXISTENCE OF MARTINGALE SOLUTIONS

The aim of this section is to summarize the results of Chapters 2–4 on the existence of (very) weak martingale solutions to (STFE). We start by illustrating the stochastic compactness method, a general construction scheme for martingale solutions to stochastic (partial) differential equations, on the example of the stochastic thin-film equation as considered in [58]. Subsequently, we review typical a-priori estimates for the deterministic thin-film equation which we will (partially) generalize to the stochastic setting. Finally, we discuss how a combination of these ideas culminates in the results of the aforementioned chapters.

1.3.1. THE STOCHASTIC COMPACTNESS METHOD

The stochastic compactness method is a general procedure to construct martingale solutions to a given stochastic (partial) differential equation. It is the probabilistic analogue to a compactness argument for deterministic equations dealing additionally with the ω-dependence of an approximating sequence.

To illustrate it, we consider the construction of *weak martingale solutions* to (STFE) with quadratic mobility exponent $n = 2$ in one dimension

$$\begin{cases} \partial_t u = -\partial_x(u^2\partial_x^3 u) + \partial_x(u\mathscr{W}), & \text{on } \Omega\times[0,T]\times\mathbb{T}, \\ u(0,\cdot) = u_0, & \text{on } \Omega\times\mathbb{T}, \end{cases} \tag{1.3.1}$$

from [58] and refer for a more general introduction to [25]. The starting point of the stochastic compactness method is the approximation of the equation by equations for which the question of existence is less delicate. Here, we use a Trotter–Kato scheme, i.e., a temporal decomposition of the deterministic and stochastic dynamics as illustrated in Figure 1.5.

For a given non-negative initial value u_0 from $H^1(\mathbb{T})$, there exists a non-negative weak solution $u_\varepsilon^{\text{det}}$ to the deterministic thin-film equation (1.2.14) on the time interval $[0,\varepsilon)$ by the aforementioned result of [15]. Here, $H^1(\mathbb{T})$ denotes the space of weakly differentiable $L^2(\mathbb{T})$-functions with weak derivative in $L^2(\mathbb{T})$ carrying the norm

$$\|f\|_{H^1(\mathbb{T})}^2 = \|f\|_{L^2(\mathbb{T})}^2 + \|\partial_x f\|_{L^2(\mathbb{T})}^2. \tag{1.3.2}$$

Using a viscous regularization, one constructs a non-negative weak solution $u_\varepsilon^{\text{stoch}}$ on $[0,\varepsilon)$ to the Stratonovich interpretation of the stochastic part (1.2.6) with the initial value

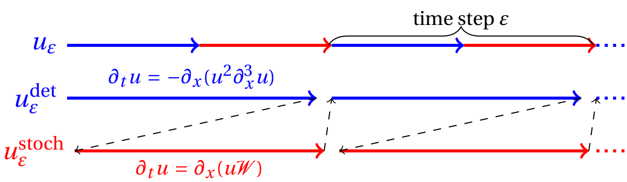

Figure 1.5: The splitting-up ansatz for the stochastic thin-film equation as used in [58].

$\lim_{t\nearrow\varepsilon} u_\varepsilon^{\text{stoch}}(t)$, which persists in $H^1(\mathbb{T})$. Hence, one can use $\lim_{t\nearrow\varepsilon} u_\varepsilon^{\text{stoch}}$ again as an initial value for the solution $u_\varepsilon^{\text{det}}$ to (1.2.14) on the time interval $[\varepsilon, 2\varepsilon)$ and iterate this procedure $\lceil T/\varepsilon\rceil$ times until one reaches the final time T. Concatenating the processes $u_\varepsilon^{\text{det}}$ and $u_\varepsilon^{\text{stoch}}$ yields an approximate solution u_ε to (1.3.1) and we expect a limit of u_ε as $\varepsilon \searrow 0$ to solve the original equation.

To extract a convergent subsequence, the authors of [58] show, among other, an estimate on

$$\mathbb{E}\Big[\sup_{0\le t\le T}\|u_\varepsilon\|_{H^1(\mathbb{T})}^q + \int_0^T \|u_\varepsilon(t)\|_{H^1(\mathbb{T})}^{q-2}\int_{\{u_\varepsilon(t,\cdot)>0\}} u_\varepsilon^2(t)\big(\partial_x^3 u_\varepsilon(t)\big)^2\,\mathrm{d}x\,\mathrm{d}t\Big], \qquad q\in[2,\infty), \tag{1.3.3}$$

which is uniform in ε. Such an estimate is very natural since, for $q = 2$, the leading order part $\|\partial_x u\|_{L^2(\mathbb{T})}^2$ of the $H^1(\mathbb{T})$-norm quantifies the surface energy of the system as elaborated in Subsection 1.1.2, while the second term is the energy dissipation due to the thin-film operator. Since the thermal fluctuations can lead to an increase in energy compared to the deterministic setting, the estimate on (1.3.3) expresses a control on the averaged energy production by the noise.

Using (1.3.3) and the Sobolev embedding theorem we can estimate the norms

$$\|u_\varepsilon\|_{L^q(\Omega;L^\infty(0,T;H^1(\mathbb{T})))}^q \qquad \text{and} \qquad \big\|u_\varepsilon \mathbf{1}_{\{u_\varepsilon>0\}}\partial_x^3 u_\varepsilon\big\|_{L^2(\Omega;L^2([0,T]\times\mathbb{T}))}^2 \tag{1.3.4}$$

uniformly in ε, so that by the Banach-Alaoglu theorem there exists a subsequence along which

$$\begin{aligned} u_{\varepsilon_l} &\rightharpoonup^* u && \text{in } L^q(\Omega;L^\infty(0,T;H^1(\mathbb{T}))),\\ u_{\varepsilon_l}^2\mathbf{1}_{\{u_{\varepsilon_l}>0\}}\partial_x^3 u_{\varepsilon_l} &\rightharpoonup J && \text{in } L^2(\Omega;L^2([0,T]\times\mathbb{T})). \end{aligned}$$

Of course, one wants the weak limit J to equal again $u^2\mathbf{1}_{\{u>0\}}\partial_x^3 u$, but weak convergences alone do usually not suffice to identify nonlinear terms. To demonstrate the latter, we consider the sequence $\sin(2\pi n\cdot)\in L^2(\mathbb{T})$, which converges weakly to 0 as the Fourier coefficients of any $L^2(\mathbb{T})$-function are square summable. The squared sequence $\sin^2(2\pi n\cdot)$ is bounded in $L^2(\mathbb{T})$ and hence converges weakly to some $g\in L^2(\mathbb{T})$ up to tasking a subsequence. Since however

$$\int_{\mathbb{T}} g\,\mathrm{d}x \leftarrow \int_{\mathbb{T}}\sin(2\pi n\cdot)^2\,\mathrm{d}x = \big\|\sin(2\pi n\cdot)\big\|_{L^2(\mathbb{T})}^2 = 1/2,$$

1

the limit g cannot coincide with (the square of) 0. Even after combining the estimate on (1.3.3) with the time splitting scheme to estimate the increments of u_ε resulting in a uniform bound on

$$\|u_\varepsilon\|^q_{L^q(\Omega;C^{1/16}((0,T);C^{1/8}(\mathbb{T})))}, \qquad q \in [2,\infty), \tag{1.3.5}$$

any convergence we can hope for would still be weak in ω.

The idea of the stochastic compactness method is to consider the probability laws of $(u_\varepsilon)_\varepsilon$ instead of dealing with the random variables themselves. Then, the celebrated *Theorem of Prokhorov* characterizes compactness of the family of probability distributions by uniform tightness, namely that for every $\delta > 0$ there exists a compact set K_δ with

$$\mathbb{P}\big(\{u_\varepsilon \in K_\delta\}\big) \geq 1-\delta \tag{1.3.6}$$

for all $\varepsilon > 0$. Hence, if we can verify (1.3.6) it follows that a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ converges in distribution. By the *Skorokhod representation theorem* there exists another probability space $(\check\Omega, \check{\mathfrak{A}}, \check{\mathbb{P}})$ supporting an equidistributed sequence which converges $\check{\mathbb{P}}$-almost surely to a random variable $\check u$. For example, by compactness of the embedding

$$C^{1/16}((0,T);C^{1/8}(\mathbb{T})) \hookrightarrow C([0,T]\times\mathbb{T})$$

and the Chebyshev–Markov inequality

$$\mathbb{P}\left(\left\{\|u_\varepsilon\|_{C^{1/16}(0,T;C^{1/8}(\mathbb{T}))} \geq \tfrac{1}{\delta}\right\}\right) \leq \delta^q \|u_\varepsilon\|^q_{L^q(\Omega;C^{1/16}(0,T;C^{1/8}(\mathbb{T})))},$$

uniform tightness of u_ε on $C([0,T]\times\mathbb{T})$ readily follows from the uniform bound on (1.3.5). Hence, by the aforementioned procedure, we obtain another probability space $(\check\Omega, \check{\mathfrak{A}}, \check{\mathbb{P}})$ and an equidistributed subsequence $\check u_{\varepsilon_l} \sim u_{\varepsilon_l}$ which converges in $C([0,T]\times\mathbb{T})$, $\check{\mathbb{P}}$-almost surely.

If we also want to exploit the bound on (1.3.4), we quickly run into the problem that the weak-* topology on $L^\infty(0,T;H^1(\mathbb{T}))$, and the weak toplogy on $L^2([0,T]\times\mathbb{T})$, are not metrizable and therefore the theorems of Prokhorov and Skorokhod are not applicable. However, this problem can be solved in an ad hoc way by applying the theorems instead to the sequence of norms $\|u_\varepsilon\|_{L^\infty(0,T;H^1(\mathbb{T}))}$. As a consequence, the previously constructed subsequence can be chosen in such a way that also $\|\check u_{\varepsilon_l}\|_{L^\infty(0,T;H^1(\mathbb{T}))}$ converges $\check{\mathbb{P}}$-almost surely and is in particular bounded. Since we know already that $\check u_{\varepsilon_l} \to \check u$ in $C([0,T]\times\mathbb{T})$ it follows by a subsequence-subsequence argument that this convergence holds also in the weak-* topology of $L^\infty(0,T;H^1(\mathbb{T}))$. We can apply the same idea to $u_\varepsilon^2 \mathbf{1}_{\{u_\varepsilon>0\}}\partial_x^3 u_\varepsilon$ by first showing convergence for an equidistributed subsequence in some ambient space and, by the additional convergence of $\|\check u_{\varepsilon_l}\mathbf{1}_{\{\check u_{\varepsilon_l}>0\}}\partial_x^3 \check u_{\varepsilon_l}\|_{L^2([0,T]\times\mathbb{T})}$, concluding the desired weak convergence $\check{\mathbb{P}}$-almost surely. Alternatively, one can also use the *Skorokhod–Jakubowski theorem*, which generalizes the described method to more general topological spaces, and yields an almost surely convergent, equidistributed subsequence under rather mild assumptions.

The resulting convergences are that

$$\check u_{\varepsilon_l} \to \check u \qquad \text{in } C([0,T]\times\mathbb{T}),$$

$$\begin{aligned} \check{u}_{\varepsilon_l} &\rightharpoonup^* \check{u} && \text{in } L^\infty(0,T;H^1(\mathbb{T})), \\ \check{u}_{\varepsilon_l}^2 \mathbf{1}_{\{\check{u}_{\varepsilon_l}>0\}} \partial_x^3 \check{u}_{\varepsilon_l} &\rightharpoonup \check{J} && \text{in } L^2([0,T]\times\mathbb{T}), \end{aligned}$$

$\check{\mathbb{P}}$-almost surely. Consequently, for any fixed $\check{\omega}$ the convergences are similiar to the deterministic case considered in [15] and therefore $\check{J} = \check{u}^2 \mathbf{1}_{\{\check{u}>0\}} \partial_x^3 \check{u}$ can be identified in an analogous fashion.

We still need to show that $\check{u}$ solves the stochastic partial differential equation (1.3.1) for a Gaussian noise $\check{\mathscr{W}}$. A natural candidate for the latter can be constructed by also applying the theorems of Prokhorov and Skorokhod to the Brownian motions resulting in equidistributed $\check{\beta}_{\varepsilon_l}^{(k)} \sim \beta^{(k)}$ such that

$$\check{\beta}_{\varepsilon_l}^{(k)} \to \check{\beta}^{(k)} \qquad \text{in } C([0,T]),$$

as $l \to \infty$. A corresponding filtration $\check{\mathscr{F}}$ is readily defined as the smallest filtration making the limiting processes $(\check{\beta}^{(k)})_{k\in\mathbb{N}}$ and $\check{u}$ adapted which also satisfies the usual conditions. To show that $(\check{\beta}^{(k)})_{k\in\mathbb{N}}$ is indeed a family of independent $\check{\mathscr{F}}$-Brownian motions, one shows first that each $\check{\beta}^{(k)}$ is an $\check{\mathscr{F}}$-martingale, meaning that the best prediction for the future value of $\check{\beta}^{(k)}$ is nothing but the current state:

$$\mathbb{E}\big[\check{\beta}_t^{(k)} \big| \check{\mathscr{F}}_s\big] = \check{\beta}_s^{(k)}, \qquad 0 \le s < t \le T. \tag{1.3.7}$$

Because this is a property determined by the probability laws, we can use the joint equidistribution

$$(\check{u}_{\varepsilon_l}, (\check{\beta}_{\varepsilon_l}^{(k)})_{k\in\mathbb{N}}) \sim (u_{\varepsilon_l}, (\beta^{(k)})_{k\in\mathbb{N}})$$

to verify (1.3.7). This can also be used to show that the covariation processes obey

$$\langle \check{\beta}^{(j)}, \check{\beta}^{(k)} \rangle_t = \begin{cases} t, & j = k, \\ 0, & \text{else}, \end{cases}$$

identifying $(\check{\beta}^{(k)})_{k\in\mathbb{N}}$ as a family of independent $\check{\mathscr{F}}$–Brownian motions by Levy's characterization theorem.

To deduce that the quadruple $(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathbb{P}})$, $\check{\mathscr{F}}$, $(\check{\beta}^{(k)})_{k\in\mathbb{N}}$ and $\check{u}$ constitutes a weak martingale solution to the Stratonovich interpretation of (1.3.1), it remains to verify that

$$\begin{aligned} &\int_{\mathbb{T}} \check{u}(t)\varphi \,\mathrm{d}x - \int_{\mathbb{T}} \check{u}(0)\varphi \,\mathrm{d}x - \int_0^t \int_{\{u(\check{s},\cdot)>0\}} \check{u}^2(s) \partial_x^3 \check{u} \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}s \\ &+ \frac{1}{2} \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} (\partial_x(\check{u}(s)e_k)e_k)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}s = -\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \check{u}(s) e_k \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}\check{\beta}_s^{(k)}, \quad t\in[0,T], \end{aligned}$$

for all test function $\varphi \in C^\infty(\mathbb{T})$. This can be achieved similarly as the identification of the Brownian motions and one starts again by showing that the left-hand side of the above equation constitutes an $\check{\mathscr{F}}$-martingale, which we denote by $\check{M}$. As a second step, we show that $\check{M}$ admits the expected (co-) variation processes, i.e., that

$$\begin{aligned} \langle \check{M} \rangle_t &= \sum_{k\in\mathbb{N}} \int_0^t \left(\int_{\mathbb{T}} \check{u}(s) e_k \partial_x \varphi \,\mathrm{d}x \right)^2 \mathrm{d}s, \\ \langle \check{M}, \check{\beta}^{(k)} \rangle_t &= -\int_0^t \int_{\mathbb{T}} \check{u}(s) e_k \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}s. \end{aligned} \tag{1.3.8}$$

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The rules of Itô's calculus dictate

$$\begin{aligned}&\left\langle \check{M}+\sum_{k\in\mathbb{N}}\int_0^{\cdot}\int_{\mathbb{T}}\check{u}(s)e_k\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}\check{\beta}^{(k)}_s\right\rangle_t\\ &\quad=\langle\check{M}\rangle_t+2\sum_{k\in\mathbb{N}}\int_0^t\left(\int_{\mathbb{T}}\check{u}(s)e_k\partial_x\varphi\,\mathrm{d}x\right)\mathrm{d}\langle\check{M},\check{\beta}^{(k)}\rangle_s+\sum_{k\in\mathbb{N}}\int_0^t\left(\int_{\mathbb{T}}\check{u}(s)e_k\partial_x\varphi\,\mathrm{d}x\right)^2\mathrm{d}s,\end{aligned}$$

and, by (1.3.8), we can evaluate the quadratic variation process as 0. This implies that the sum of martingales vanishes, completing the construction of a weak martingale solution.

1.3.2. A-PRIORI ESTIMATES FOR THE THIN-FILM EQUATION

While the machinery of the stochastic compactness method applies to various stochastic (partial) differential equations, the sequence of approximations needs to be carefully chosen in order to close uniform estimates, which are sufficient to identify the limit as a solution to the original problem. For example, the time-splitting ansatz employed in the previous section can be used for (STFE) with quadratic mobility, while for $n \neq 2$ it fails since the existence of a solution to the nonlinear stochastic conservation law

$$\begin{cases}\partial_t u=\partial_x(u^{n/2}\mathscr{W}), & \text{on } \Omega\times[0,T]\times\mathbb{T},\\ u(0,\cdot)=u_0, & \text{on } \Omega\times\mathbb{T},\end{cases}$$

in the energy space $H^1(\mathbb{T})$ is unclear.

To come up a with another, more suitable approximation of

$$\begin{cases}\partial_t u=-\partial_x(u^n\partial_x^3u)+\partial_x(u^{n/2}\mathscr{W}), & \text{on } \Omega\times[0,T]\times\mathbb{T}^d,\\ u(0,\cdot)=u_0, & \text{on } \Omega\times\mathbb{T}^d,\end{cases}\tag{1.3.9}$$

for $n\neq 2$ it is helpful to consider the fact that whichever uniform estimates we derive for the approximations will carry over to the constructed solution to (1.3.9). For example, in the situation of the previous subsection, we see by the lower semicontinuity of the norms with respect to weak(-*) convergence and Fatou's lemma that

$$\|\check{u}\|^q_{L^q(\check{\Omega};L^\infty(0,T;H^1(\mathbb{T})))}\qquad\text{and}\qquad\left\|\check{u}\mathbf{1}_{\{\check{u}>0\}}\partial_x^3\check{u}\right\|^2_{L^2(\check{\Omega};L^2([0,T]\times\mathbb{T}))}$$

admit the same upper bounds as (1.3.4). Since also any uniform estimate on approximations of (1.3.9) will result in a bound on the solution itself, it is fruitful to first show an *a-priori estimate* for (1.3.9) and then look for a compatible approximation scheme.

With this in mind, we review some a-priori estimates of the deterministic thin-film equation

$$\begin{cases}\partial_t u=-\partial_x(u^n\partial_x^3u), & \text{on } [0,T]\times\mathbb{T}^d,\\ u(0,\cdot)=u_0, & \text{on } \mathbb{T}^d,\end{cases}\tag{1.3.10}$$

which provide a natural starting point to derive estimates in the stochastic setting. To justify all the involved integrations by parts we assume that u is a strictly positive and smooth solution to (1.3.10). The assumption of strict positivity is not necessarily expected for the solutions that we construct, but can physically be justified in the complete wetting regime discussed in Subsection 1.1.2. Indeed, in this case the film height is strictly positive on a microscopic level and consequently the macroscopic film height can be obtained as a limit of strictly positive solutions to (1.3.10).

By integrating (1.3.10) against $-\partial_x^2 u$ we obtain

$$\frac{1}{2}\partial_t \|\partial_x u\|_{L^2(\mathbb{T})}^2 = -\int_{\mathbb{T}} u^n (\partial_x^3 u)^2 \,\mathrm{d}x,$$

which, after integration in time, results in the *energy estimate*

$$\frac{1}{2}\sup_{0\le t\le T} \|\partial_x u(t)\|_{L^2(\mathbb{T})}^2 + \int_0^T \int_{\mathbb{T}} u^n(t)(\partial_x^3 u(t))^2 \,\mathrm{d}x\,\mathrm{d}t \le \|\partial_x u_0\|_{L^2(\mathbb{T})}^2. \qquad (\mathscr{E}\text{-Est})$$

The name alludes to the fact that $\frac{1}{2}\|\partial_x u\|_{L^2(\mathbb{T})}^2$ is the excess surface energy of the profile u, see again Subsection 1.1.2, which is a very natural quantity to estimate. Correspondingly, the integral in ($\mathscr{E}$-Est) quantifies the dissipated surface energy until the final time T.

Integrating (1.3.10) instead against $g'(u)$ for

$$g(u) = \int_1^u \int_1^r \frac{1}{(r')^n} \,\mathrm{d}r'\,\mathrm{d}r, \qquad (1.3.11)$$

results in

$$\partial_t \int_{\mathbb{T}} g(u)\,\mathrm{d}x = -\int_{\mathbb{T}} (\partial_x^2 u)^2 \,\mathrm{d}x$$

and ultimately in the *entropy estimate*

$$\sup_{0\le t\le T} \int_{\mathbb{T}} g(u(t))\,\mathrm{d}x + \|\partial_x^2 u\|_{L^2([0,T]\times\mathbb{T})}^2 \le 2\int_{\mathbb{T}} g(u_0)\,\mathrm{d}x. \qquad (\mathscr{G}\text{-Est})$$

The choice of the lower integration bounds in (1.3.11) ensures that g is convex and attains its global minimum at $g(1) = 0$ and is therefore non-negative. Moreover, for $n \ge 2$, g admits a singularity of the order

$$g(r) \sim \begin{cases} -\log(r), & n = 2, \\ r^{2-n}, & n > 2, \end{cases}$$

near 0. Consequently, as soon as the left-hand side of ($\mathscr{G}$-Est) is finite for an arbitrary function $u\colon [0,T]\times\mathbb{T} \to [0,\infty)$, then $u(t,\cdot)$ is $\mathrm{d}x$-almost everywhere strictly positive for every $t \in [0,T]$. This preservation of positivity is special to (1.3.10), as the linear equation $\partial_t v = -\partial_x^4 v$ satisfies no comparison principle in contrast to second order equations.

The entropy estimate can be further generalized by introducing the α-entropy functions

$$g_\alpha(u) = \int_1^u \int_1^r (r')^{\alpha-1} \,\mathrm{d}r'\,\mathrm{d}r, \qquad \alpha \in [1/2-n, 2-n],$$

containing the entropy function as the special case g_{1-n}. Following [11], we calculate

$$\begin{aligned} \partial_t \int_{\mathbb{T}} g_\alpha(u)\,\mathrm{d}x &= \int_{\mathbb{T}} u^{\alpha+n-1}\partial_x^3 u \partial_x u \,\mathrm{d}x \\ &= -\int_{\mathbb{T}} u^{\alpha+n-1}(\partial_x^2 u)^2 \,\mathrm{d}x - (\alpha+n-1)\int_{\mathbb{T}} u^{\alpha+n-2}\partial_x^2 u(\partial_x u)^2 \,\mathrm{d}x \qquad (1.3.12) \\ &= -\int_{\mathbb{T}} u^{\alpha+n-1}(\partial_x^2 u)^2 \,\mathrm{d}x + \tfrac{(\alpha+n-1)(\alpha+n-2)}{3}\int_{\mathbb{T}} u^{\alpha+n-3}(\partial_x u)^4 \,\mathrm{d}x, \end{aligned}$$

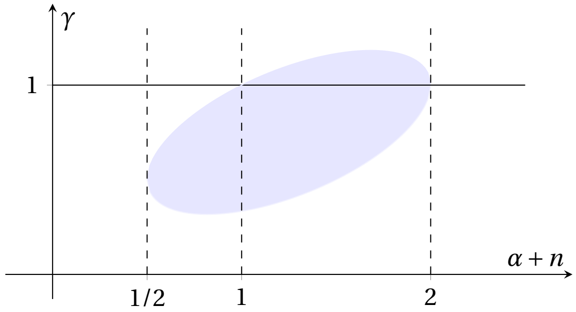

Figure 1.6: Admissible values of γ depending on $\alpha + n$.

by integrating (1.3.10) against $g'_\alpha(u)$ and repeated integration by parts. The prefactor of the last integral is non-positive if

$$C_{\alpha,n} = \tfrac{-(\alpha+n-1)(\alpha+n-2)}{3} \geq 0 \qquad \text{or equivalently} \qquad \alpha \in [1-n, 2-n],$$

in which case we recover the α*-entropy estimate*

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{\mathbb{T}} g_\alpha(u(t))\,\mathrm{d}x + \int_0^T \int_{\mathbb{T}} u^{\alpha+n-1}(\partial_x^2 u)^2\,\mathrm{d}x + C_{\alpha,n} \int_{\mathbb{T}} u^{\alpha+n-3}(\partial_x u)^4\,\mathrm{d}x\,\mathrm{d}t \\ &\leq 2\int_{\mathbb{T}} g_\alpha(u_0)\,\mathrm{d}x. \end{aligned} \tag{1.3.13}$$

To also obtain an a-priori estimate for smaller values of α, it is useful to substitute

$$(\partial_x^2 u)^2 = \frac{1}{\gamma^2} u^{2-2\gamma}(\partial_x^2 u^\gamma)^2 - (\gamma-1)^2 u^{-2}(\partial_x u)^4 - 2(\gamma-1)u^{-1}(\partial_x u)^2 \partial_x^2 u,$$

for a second parameter $\gamma \neq 0$ in the last line of (1.3.12), resulting in

$$\begin{aligned} \partial_t \int_{\mathbb{T}} g_\alpha(u)\,\mathrm{d}x &= -\frac{1}{\gamma^2}\int_{\mathbb{T}} u^{\alpha+n-2\gamma+1}(\partial_x^2 u^\gamma)^2\,\mathrm{d}x + 2(\gamma-1)\int_{\mathbb{T}} u^{\alpha+n-2}(\partial_x u)^2 \partial_x^2 u\,\mathrm{d}x \\ &+ \left(\tfrac{(\alpha+n-1)(\alpha+n-2)}{3} + (\gamma-1)^2\right)\int_{\mathbb{T}} u^{\alpha+n-3}(\partial_x u)^4\,\mathrm{d}x \\ &= -\frac{1}{\gamma^2}\int_{\mathbb{T}} u^{\alpha+n-2\gamma+1}(\partial_x^2 u^\gamma)^2\,\mathrm{d}x - C_{\alpha,\gamma,n}\int_{\mathbb{T}} u^{\alpha+n-3}(\partial_x u)^4\,\mathrm{d}x, \end{aligned}$$

with

$$C_{\alpha,\gamma,n} = -\left(\tfrac{(\alpha+n-1-2(\gamma-1))(\alpha+n-2)}{3} + (\gamma-1)^2\right).$$

A corresponding version of (1.3.13) can be established as soon as $C_{\alpha,\gamma,n} \geq 0$, which is equivalent to

$$\frac{\alpha+n+1-\sqrt{(2-(\alpha+n))(2(\alpha+n)-1)}}{3} \leq \gamma \leq \frac{\alpha+n+1+\sqrt{(2-(\alpha+n))(2(\alpha+n)-1)}}{3} \tag{1.3.14}$$

with the boundary cases corresponding to $C_{\alpha,\gamma,n} = 0$. Moreover, there exists an admissible γ if and only if α is in the previously specified range

$$\tfrac{1}{2} - n \leq \alpha \leq 2 - n. \tag{1.3.15}$$

We arrive at the general α*-entropy estimate*

$$\begin{aligned} \sup_{0\leq t\leq T} \int_{\mathbb{T}} g_\alpha(u(t))\,\mathrm{d}x + \frac{1}{\gamma^2} \int_0^T \int_{\mathbb{T}} u^{\alpha+n-2\gamma+1} (\partial_x^2 u^\gamma)^2 \,\mathrm{d}x\,\mathrm{d}t \\ + C_{\alpha,\gamma,n} \int_0^T \int_{\mathbb{T}} u^{\alpha+n-3} (\partial_x u)^4 \,\mathrm{d}x\,\mathrm{d}t \leq 2 \int_{\mathbb{T}} g_\alpha(u_0)\,\mathrm{d}x \end{aligned} \tag{$\mathscr{G}_\alpha$-Est}$$

for α and γ according to (1.3.14) and (1.3.15).

The lower values $\alpha < 1 - n$ improve the implications regarding the positivity of the solution compared to ($\mathscr{G}$-Est), while the higher values $\alpha > 1-n$ can be useful in situations in which the entropy of the initial value is infinite, for example if $\{u_0 = 0\}$ has positive measure.

1.3.3. The results of Chapters 2–4

After having reviewed the stochastic compactness method and relevant a-priori estimates for the deterministic thin-film equation, we summarize the results from this thesis on the existence of (very) weak martingale solutions to (STFE).

In Chapter 2 based on the article [116] we consider (STFE) in the physical situation of effective dimension $d = 2$ with a quadratic mobility function $n = 2$. We recall that in the higher-dimensional case the temporally white, spatially colored Gaussian noise $\mathscr{W}$ is vector-valued, i.e., that

$$\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k\in\mathbb{N}} e_k \beta^{(k)}, \qquad e_k\colon \mathbb{T}^2 \to \mathbb{R}^2,$$

for a sequence of independent Brownian motions $(\beta^{(k)})_{k\in\mathbb{N}}$. We assume that the smooth functions e_k decay sufficiently fast so that $\mathscr{W}$ is two times differentiable in space and impose a mild symmetry assumption on the e_k to close all the estimates involved. The latter essentially expresses that the vector components of $\mathscr{W}$ are independent and identically distributed.

The restriction to the case $n = 2$ is convenient since then the stochastic part is linear and therefore well-behaved on its own, at least when interpreted in the Stratonovich sense. Hence, analogously to [58] for the one-dimensional equation, we employ a time-splitting scheme in conjunction with a viscous regularization of the stochastic part leading to approximate solutions $u_{\delta,\varepsilon}$ as depicted in Figure 1.7.

Assuming that $u_0 \in H^1(\mathbb{T}^2)$ we show, as in the one-dimensional case, that a stochastic version of the energy estimate ($\mathscr{E}$-Est) holds with a right-hand side which is independent of δ and ε. Unlike in the one-dimensional situation however, the energy estimate itself does not suffice to identify a limit as $\delta, \varepsilon \to 0$ as a solution to (STFE). Indeed, already in the deterministic case the authors of [33] derive additionally uniform α-entropy estimates ($\mathscr{G}_\alpha$-Est) for approximations of the deterministic thin-film equation to show the

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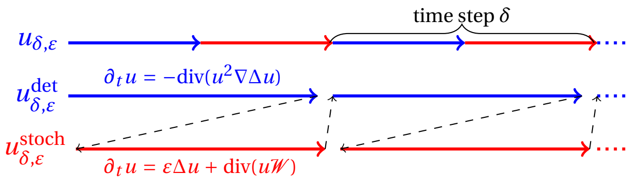

Figure 1.7: Time-splitting scheme for (STFE) with $d = n = 2$ and a regularized stochastic part as employed in Chapter 2.

existence of very weak solutions in higher dimensions. We follow their approach and derive a stochastic version of ($\mathscr{G}_\alpha$-Est) along the time-splitting scheme, which is uniform in δ and ε as well. This suffices to let us first pass $\varepsilon \to 0$ and subsequently $\delta \to 0$ to construct a very weak martingale solution to (STFE) with $d = n = 2$ based on the stochastic compactness method. Since it suffices to take some $\alpha > -1$ in line with (1.3.15), no additional positivity assumption on u_0 needs to be imposed.

The aim of the following Chapters 3 and 4 based on the preprints [36, 117] is to prove the existence of (very) weak martingale solutions to (STFE) also in the case of a nonlinear noise term $n \neq 2$. Since this is quite challenging by itself, we restrict ourselves there to the case of effective dimension $d = 1$.

Indeed, the only other existence result for (STFE) in the nonlinear noise case was obtained in [35]. Their result relies on the observation that, while the energy estimate ($\mathscr{E}$-Est) can even in the case of Stratonovich noise not be closed on its own, the energy production of the noise up to a time T can be quantified as

$$\int_0^T \int_{\mathbb{T}} u^{n-4}(\partial_x u)^4 \,\mathrm{d}x\,\mathrm{d}t. \tag{1.3.17}$$

After further integration by parts, (1.3.17) can be controlled using the terms from the entropy estimate ($\mathscr{G}$-Est), at least for $n \in [8/3, 4)$. Luckily, the entropy estimate ($\mathscr{G}$-Est) is in contrast compatible with the nonlinear Stratonovich noise and a combined energy-entropy estimate can be established. Since the resulting convergences suffice to identify a solution to the thin-film equation it remains to find a suitable approximation scheme. To this end, the authors of [35] use a stochastic thin-film equation with a non-degenerate mobility function

$$\partial_t u_\varepsilon = -\partial_x\big(m_\varepsilon(u_\varepsilon)\partial_x^3 u_\varepsilon\big) + \partial_x\big(m_\varepsilon^{1/2}(u_\varepsilon)\mathscr{W}\big), \qquad m_\varepsilon(r) = \big(r^2 + \varepsilon^2\big)^{n/2}, \tag{1.3.18}$$

for which the existence of solutions is less difficult to show. Using a corresponding approximate version of the energy-entropy estimate, they succeed in extracting a weak martingale solution to (STFE) with $d = 1$ and $n \in [8/3, 4)$ as $\varepsilon \to 0$. In particular, while u_ε may become negative, the limiting solution is positive almost everywhere due to the estimate on the entropy of the solution.

In Chapter 3 we resolve the resulting gap of mobility exponents $n \in (2, 8/3)$ for which no existence result for (STFE) was known. The underlying observation is that the energy production of the noise (1.3.17) coincides with the last integral on the left-hand side of ($\mathscr{G}_\alpha$-Est) for $\alpha = -1$ corresponding to the logarithmic α-entropy function

$$g_{-1}(u) = u - 1 - \log(u)$$

with a positive constant $C_{-1,1,n}$ for $n \in (2,3)$. Unfortunately, while the approximations (1.3.18) work well with ($\mathscr{G}$-Est), they are not compatible with the log-entropy estimate due to the possible negativity of u_ε.

We resolve this by adding another approximation layer and solving first

$$\begin{aligned} \partial_t u_{\delta,\varepsilon} &= -\partial_x\big(m_{\delta,\varepsilon}(u_{\delta,\varepsilon})\partial_x^3 u_{\delta,\varepsilon}\big) + \partial_x\big(m_{\delta,\varepsilon}^{1/2}(u_{\delta,\varepsilon})\mathscr{W}\big), \\ m_{\delta,\varepsilon}(r) &= m_\delta\big((r^2+\varepsilon^2)^{1/2}\big), \qquad m_\delta(r) = r^{n+\nu}\big/\big(r^{\nu/2} + \delta^{(\nu-n)/2} r^{n/2}\big)^2 \end{aligned}$$

for suitable $\nu \in (3,4)$. Since $m_\delta(u) \sim u^\nu$ near 0, we can adapt the proof from [35] to let $\varepsilon \to 0$ and obtain a non-negative weak martingale solution to

$$\partial_t u_\delta = -\partial_x\big(m_\delta(u_\delta)\partial_x^3 u_\delta\big) + \partial_x\big(m_\delta^{1/2}(u_\delta)\mathscr{W}\big).$$

The non-negativity of the solution allows us to prove a log-entropy estimate and finally establish a δ-uniform version of the energy-log-entropy estimate. Then we pass also to $\delta \to 0$ and deduce the existence of a weak martingale solution to (STFE) with $d = 1$ and $n \in (2,3)$ for initial values $u_0 \in H^1(\mathbb{T})$ satisfying

$$-\int_{\mathbb{T}} \log(u_0)\,\mathrm{d}x < \infty. \tag{1.3.19}$$

As before, we need to assume additionally that the noise $\mathscr{W}$ is two times differentiable in space to close all the aforementioned estimates.

We remark that the assumption (1.3.19) requires the initial profile to be strictly positive almost everywhere, and that the resulting solution remains positive almost everywhere by the log-entropy estimate, similarly to the result from [35]. This excludes the situation that $|\{u = 0\}| > 0$, and in particular that a contact line, i.e., a triple junction of gas, liquid and solid, is present. It is the aim of Chapter 4 to show existence of very weak martingale solutions to (STFE), again with $d = 1$ and $n \in (2,3)$, without any strict positivity assumption on u_0.

The main challenge is that, as suggested by the discussed results, the control of the energy production (1.3.17) by the noise requires an estimate on the smallness of the solution. Since such a control, e.g. by ($\mathscr{G}_\alpha$-Est), would require again for the initial value to be positive almost everywhere, our approach is to completely discard the energy estimate and instead rely the whole analysis on ($\mathscr{G}_\alpha$-Est) for $\alpha \in (-1, 2-n)$.

In particular, for a non-negative convolution kernel $(\eta_\varepsilon)_{\varepsilon>0}$, the problem

$$\begin{aligned} \partial_t u_\varepsilon &= -\partial_x(u_\varepsilon^n \partial_x^3 u_\varepsilon) + \partial_x(u_\varepsilon^{n/2}\mathscr{W}_\varepsilon), \\ u_\varepsilon(0) &= \eta_\varepsilon * u_0 + \varepsilon, \qquad \mathscr{W}_\varepsilon = \eta_\varepsilon * \mathscr{W}, \qquad \varepsilon \in (0,1), \end{aligned}$$

falls within the scope of Chapter 3 and the existence of weak martingale solutions follows. Since

$$\int_{\mathbb{T}} g_\alpha(\eta_\varepsilon * u_0 + \varepsilon)\,\mathrm{d}x \le c_{\alpha,n}\left(\int_{\mathbb{T}} \eta_\varepsilon * u_0\,\mathrm{d}x + 1\right) \le c_{\alpha,n}\left(\|u_0\|_{\mathscr{M}(\mathbb{T})} + 1\right)$$

1

for $\alpha \in (-1, 2-n)$, an ε-uniform version of ($\mathscr{G}_\alpha$-Est) can be established for non-negative initial values u_0 from the space of Radon measures $\mathscr{M}(\mathbb{T})$. In particular, since the α-entropy functional is lower order than the surface energy, we need only to assume that $\mathscr{W}$ is one time differentiable in space in contrast to the previously discussed results. As observed in [32], the resulting convergences suffice in one spatial dimension to identify a limit as a very weak solution to the thin-film equation. Consequently, we can take $\varepsilon \searrow 0$ and, after another application of the stochastic compactness method, obtain a very weak martingale solution to (STFE) with $d = 1$ and $n \in (2,3)$ attaining the initial value u_0 in distribution.

1.4. EXISTENCE, UNIQUENESS OF PROBABILISTICALLY STRONG SOLUTIONS

The purpose of this section is to present the results of Chapter 5 on the existence and uniqueness of probabilistically strong solutions to (STFE) as well as the methods used in their proofs. To this end, we demonstrate in a simple example how to reformulate the deterministic thin-film equation as a fixed-point problem and solve it using the Banach fixed-point theorem. We also comment on the advances in the theory on quasilinear stochastic evolution equations made in [5, 6]. Subsequently, we review how the effects of intermolecular forces can be incorporated in the thin-film model from Section 1.1, providing an interesting situation in which the aforementioned theory yields global in time well-posedness of the equation, at least in one spatial dimension. Finally, we give a summary of the results on the stochastic thin-film equation from Chapter 5.

1.4.1. QUASILINEAR STOCHASTIC EVOLUTION EQUATIONS

Next to the compactness method, another classical approach to address the question of existence (and uniqueness) of a partial differential equation is to reformulate it as a *fixed-point problem*. If one can show that for sufficiently small times the solution map is a *contraction*, existence and uniqueness of a solution follows. By consecutive extension, there is then a maximal time until which the unique solution exists, solving the problem locally in time.

To illustrate this idea, we consider once more the deterministic thin-film equation

$$\begin{cases} \partial_t u = -\partial_x(u^2 \partial_x^3 u), & \text{on } [0,T] \times \mathbb{T}, \\ u(0,\cdot) = u_0, & \text{on } \mathbb{T} \end{cases} \tag{1.4.1}$$

in one dimension with a quadratic mobility exponent $n = 2$ and refer for a more general exposition of the fixed-point method to [86, Chapter 18]. Concerning the initial value we assume that $u_0 \in H^1(\mathbb{T})$, but this time additionally that u_0 is strictly positive on whole $\mathbb{T}$. By adding $\partial_x(u_0^2 \partial_x^3 u)$ on both sides of (1.4.1), we can reformulate the equation as

$$\partial_t u + \partial_x(u_0^2 \partial_x^3 u) = \partial_x\big((u_0^2 - u^2)\partial_x^3 u\big),$$

i.e., as in a linearization around the initial profile. Now, the existence and uniqueness of a solution for small times follows, if we can show suitable estimates for the solution operator of the *linear problem*

$$\mathscr{S}\colon f \mapsto u \qquad \text{s.t.} \quad \partial_t u + \partial_x(u_0^2 \partial_x^3 u) = \partial_x f, \quad u(0) = u_0, \tag{1.4.2}$$

as well as a compatible estimate on the *nonlinearity*

$$\mathscr{N}\colon u \mapsto (u_0^2 - u^2)\partial_x^3 u.$$

The estimates being compatible means here that they should suffice to apply the Banach fixed point theorem to the composition $\mathscr{S} \circ \mathscr{N}$.

We start by deriving a Lipschitz estimate on the nonlinearity. A very natural estimate is given for the norm

$$\begin{aligned} \|\mathscr{N}(u) - \mathscr{N}(v)\|_{L^2(\mathbb{T})} &= \left\|(u_0^2 - u^2)\partial_x^3 u - (u_0^2 - v^2)\partial_x^3 v\right\|_{L^2(\mathbb{T})} \\ &\le \left\|(u_0^2 - u^2)\partial_x^3 u - (u_0^2 - u^2)\partial_x^3 v\right\|_{L^2(\mathbb{T})} + \left\|(u_0^2 - u^2)\partial_x^3 v - (u_0^2 - v^2)\partial_x^3 v\right\|_{L^2(\mathbb{T})} \\ &\le \left\|u_0^2 - u^2\right\|_{L^\infty(\mathbb{T})}\left\|\partial_x^3(u-v)\right\|_{L^2(\mathbb{T})} + \left\|u^2 - v^2\right\|_{L^\infty(\mathbb{T})}\left\|\partial_x^3 v\right\|_{L^2(\mathbb{T})}, \end{aligned} \tag{1.4.3}$$

since we can employ Hölder's inequality to obtain an upper bound on the products.

To obtain a corresponding estimate for (1.4.2) we consider the difference $u - v$ for two solutions $u = \mathscr{S}(f)$ and $v = \mathscr{S}(g)$, which satisfies

$$\partial_t(u-v) + \partial_x(u_0^2\partial_x^3(u-v)) = \partial_x(f-g), \quad (u-v)(0) = 0,$$

so that integration against $-\partial_x^2(u-v)$ yields

$$\begin{aligned} &\tfrac{1}{2}\partial_t\|\partial_x(u-v)\|_{L^2(\mathbb{T})}^2 + \int_{\mathbb{T}} u_0^n(\partial_x^3(u-v))^2\,\mathrm{d}x \le -\int_{\mathbb{T}} \partial_x(f-g)\partial_x^2(u-v)\,\mathrm{d}x \\ &\quad \le \|f-g\|_{L^2(\mathbb{T})}\|\partial_x^3(u-v)\|_{L^2(\mathbb{T})}. \end{aligned}$$

To proceed we use our assumption that the initial profile is strictly positive, and therefore that $u_0^n \ge \delta$ for some $\delta > 0$. By Young's inequality for products we deduce

$$\tfrac{1}{2}\partial_t\|\partial_x(u-v)\|_{L^2(\mathbb{T})}^2 + \delta\int_{\mathbb{T}}(\partial_x^3(u-v))^2\mathrm{d}x \le \tfrac{1}{2}\big(\delta^{-1}\|f-g\|_{L^2(\mathbb{T})}^2 + \delta\|\partial_x^3(u-v)\|_{L^2(\mathbb{T})}^2\big)$$

and consequently

$$\sup_{0\le s\le t}\|\partial_x(u-v)\|_{L^2(\mathbb{T})}^2 + \|\partial_x^3(u-v)\|_{L^2([0,t]\times\mathbb{T})}^2 \le C_\delta\|f-g\|_{L^2([0,t]\times\mathbb{T})}^2, \tag{1.4.4}$$

after absorbing the last term on the right-hand side and integrating in time.

We can combine this with the estimate on the nonlinearity (1.4.3) and obtain that

$$\begin{aligned} &\sup_{0\le s\le t}\|\partial_x(\mathscr{S}\circ\mathscr{N}(u) - \mathscr{S}\circ\mathscr{N}(v))\|_{L^2(\mathbb{T})}^2 + \|\partial_x^3(\mathscr{S}\circ\mathscr{N}(u) - \mathscr{S}\circ\mathscr{N}(v))\|_{L^2([0,t]\times\mathbb{T})}^2 \\ &\quad \le C_\delta\|\mathscr{N}(u) - \mathscr{N}(v)\|_{L^2([0,t]\times\mathbb{T})}^2 \\ &\quad \le C_\delta\left\|u_0 - u\right\|_{L^\infty([0,t]\times\mathbb{T})}^2\left\|u_0 + u\right\|_{L^\infty([0,t]\times\mathbb{T})}^2\left\|\partial_x^3(u-v)\right\|_{L^2([0,t]\times\mathbb{T})}^2 \\ &\qquad + C_\delta\left\|u - v\right\|_{L^\infty([0,t]\times\mathbb{T})}^2\left\|u + v\right\|_{L^\infty([0,t]\times\mathbb{T})}^2\left\|\partial_x^3 v\right\|_{L^2([0,t]\times\mathbb{T})}^2, \end{aligned} \tag{1.4.5}$$

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after enlarging the constant C_δ. Using the fact that the solutions $\mathscr{S}\circ\mathscr{N}(u)$ and $\mathscr{S}\circ\mathscr{N}(v)$ to the conservative equation (1.4.2) have the same mass, we see that the left-hand side is equivalent to the norm in the space

$$\mathfrak{X}_t = C([0,t];H^1(\mathbb{T}))\cap L^2(0,t;H^3(\mathbb{T})), \qquad t\in[0,T],$$

where $H^1(\mathbb{T})$ was defined in (1.3.2), and the space $H^3(\mathbb{T})$ is defined in an analogous fashion. We can estimate furthermore the right-hand side of (1.4.5) using the Sobolev embedding

$$H^1(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T}), \tag{1.4.6}$$

resulting in

$$\begin{aligned}&\big\|\mathscr{S}\circ\mathscr{N}(u)-\mathscr{S}\circ\mathscr{N}(v)\big\|_{\mathfrak{X}_t}^2\\ &\quad\le C_\delta\big\|u_0-u\big\|^2_{C([0,t];H^1(\mathbb{T}))}\big\|u_0+u\big\|^2_{C([0,t];H^1(\mathbb{T}))}\big\|u-v\big\|^2_{L^2(0,t;H^3(\mathbb{T}))}\\ &\qquad+C_\delta\big\|u-v\big\|^2_{C([0,t];H^1(\mathbb{T}))}\big\|u+v\big\|^2_{C([0,t];H^1(\mathbb{T}))}\big\|v\big\|^2_{L^2(0,t;H^3(\mathbb{T}))},\end{aligned} \tag{1.4.7}$$

after enlarging C_δ once more. To turn (1.4.7) into a contraction estimate, we need to make sure that the factors of the norms of $u-v$ are small and to this end, we define

$$\mathfrak{X}_{t,\varepsilon} = \left\{u\in\mathfrak{X}_t\,\middle|\,u(0)=u_0,\ \|u-u_0\|_{C([0,t];H^1(\mathbb{T}))}+\|u\|_{L^2(0,t;H^3(\mathbb{T}))}\le\varepsilon\right\},\qquad \varepsilon>0.$$

In particular, we can choose $\varepsilon>0$ small enough so that (1.4.7) implies a 1/4-Lipschitz estimate for the $\mathfrak{X}_t$-norm for any $t\in[0,T]$ under the additional assumption that $u,v\in\mathfrak{X}_{t,\varepsilon}$.

Since $(\mathfrak{X}_{t,\varepsilon},\|\cdot\|_{\mathfrak{X}_t})$ is a complete metric spaces, it remains to check that $\mathscr{S}\circ\mathscr{N}\colon\mathfrak{X}_{t,\varepsilon}\to\mathfrak{X}_{t,\varepsilon}$ for sufficiently small t. We first observe by inserting $v=0$ in (1.4.3) that $\mathscr{N}\colon\mathfrak{X}_T\to L^2([0,T]\times\mathbb{T})$. On the other hand, by integrating in (1.4.2) against $-\partial_x^2 u$, we see that

$$\begin{aligned}&\sup_{0\le s\le T}\|\partial_x\mathscr{S}(f)\|^2_{L^2(\mathbb{T})}+\|\partial_x^3\mathscr{S}(f)\|^2_{L^2([0,T]\times\mathbb{T})}\\ &\quad\le C_\delta\Big(\|\partial_x u_0\|^2_{L^2(\mathbb{T})}+\|f\|^2_{L^2([0,T]\times\mathbb{T})}\Big),\end{aligned} \tag{1.4.8}$$

analogously to (1.4.4) and therefore $\mathscr{S}\colon L^2([0,T]\times\mathbb{T})\to\mathfrak{X}_t$. Consequently, if we take any $\bar u\in\mathfrak{X}_{T,\varepsilon}$, we see that $\mathscr{S}\circ\mathscr{N}(\bar u)$ is an element of $\mathscr{X}_T$ starting at u_0. Hence, there exists a time $T^*\in(0,T]$ such that

$$\|\mathscr{S}\circ\mathscr{N}(\bar u)-u_0\|_{C([0,T^*];H^1(\mathbb{T}))}+\|\mathscr{S}\circ\mathscr{N}(\bar u)\|_{L^2(0,T^*;H^3(\mathbb{T}))}\le\varepsilon/2$$

and in particular $\mathscr{S}\circ\mathscr{N}(\bar u)\in\mathfrak{X}_{T^*,\varepsilon}$. Since any element $u\in\mathfrak{X}_{T^*,\varepsilon}$ satisfies

$$\big\|\mathscr{S}\circ\mathscr{N}(u)-\mathscr{S}\circ\mathscr{N}(\bar u)\big\|_{\mathfrak{X}_{T^*}}\le\tfrac14\|u-\bar u\|_{\mathfrak{X}_{T^*}}\le\varepsilon/2$$

by the choice of ε and $\bar u\in\mathfrak{X}_{T^*,\varepsilon}$, we deduce that also $\mathscr{S}\circ\mathscr{N}(u)\in\mathfrak{X}_{T^*,\varepsilon}$, as desired.

It follows by the Banach fixed-point theorem that there exists a unique solution on the time interval $[0,T^*]$ to equation (1.4.1). One can iterate this procedure as long as the solution stays positive and in $H^1(\mathbb{T})$ so that existence and uniqueness of a strictly positive, $H^1(\mathbb{T})$-valued solution follows. This solution exists either on the whole time interval $[0,T]$ or ceases to exist before the final time T. Since the surface energy is dissipated by ($\mathscr{E}$-Est), the latter can only happen if the film height touches down.

If we want to use a similar approach to solve the stochastic thin-film equation

$$\begin{cases} \partial_t u = -\partial_x(u^2\partial_x^3 u) + \partial_x(u\mathscr{W}), & \text{on } [0,T]\times\mathbb{T}, \\ u(0,\cdot) = u_0, & \text{on } \mathbb{T}, \end{cases} \tag{1.4.9}$$

for a Gaussian noise

$$\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\in\mathbb{N}} e_k\beta^{(k)},$$

we need to interpret the noise term as a comparable perturbation to the nonlinearity $\mathscr{N}$ due to the quasilinear operator.

To this end, we emphasize that the linear estimate (1.4.8) is the best we can expect for the solution $\mathscr{S}(f)$ to (1.4.2). Indeed, since

$$\mathscr{S}(f) = \left(\partial_t + \partial_x(u_0^2(\partial_x^3\cdot))\right)^{-1}\partial_x f, \tag{1.4.10}$$

$\mathscr{S}(f)$ admitting four derivatives more than $\partial_x f$ is optimal. Here we consider $\partial_x f$ as -1-times differentiable as it is the derivative of an $L^2(\mathbb{T})$-function and hence an element of the accordingly defined space $H^{-1}(\mathbb{T})$. For this reason, the estimate (1.4.8) is called *maximal regularity* and it implies moreover that

$$\begin{aligned} \|\partial_t\mathscr{S}(f)\|_{L^2(0,T;H^{-1}(\mathbb{T}))} &\le \left\|\partial_x(u_0^2\partial_x^3\mathscr{S}(f))\right\|_{L^2(0,T;H^{-1}(\mathbb{T}))} + \|\partial_x f\|_{L^2(0,T;H^{-1}(\mathbb{T}))} \\ &\le \|u_0\|^2_{L^\infty(\mathbb{T})}\|\partial_x^3\mathscr{S}(f)\|_{L^2([0,T]\times\mathbb{T})} + \|f\|_{L^2([0,T]\times\mathbb{T})} \\ &\le C_\delta\|u_0\|^2_{L^\infty(\mathbb{T})}\left(\|\partial_x u_0\|_{L^2(\mathbb{T})} + \|f\|_{L^2([0,t]\times\mathbb{T})}\right) + \|f\|_{L^2([0,T]\times\mathbb{T})} \end{aligned}$$

by inserting it in equation (1.4.2) satisfied by $\mathscr{S}(f)$. That means, not only gains the solution $\mathscr{S}(f)$ to (1.4.10) four derivatives in space compared to $\partial_x f$, but also one derivative in time.

By interpolating the corresponding estimates one can trade a fraction of the gained temporal regularity for four times as many spatial derivatives, meaning that $\mathscr{S}(f)$ can, for example, be shown to be $1/2$-times differentiable in time with values in $H^1(\mathbb{T})$. This gives a good indication about the right space for the stochastic nonlinearity in (1.4.9) to lie in. Indeed, since the temporally white noise $\mathscr{W}$ is $(-1/2-\varepsilon)$-times differentiable in time we need to use half of the regularizing effect of the solution operator $\mathscr{S}$ to turn $\partial_x(u\mathscr{W})$ into a function, as opposed to a distribution, in time. Then we can still use the other half of the regularizing effect to gain two derivatives in space. As a result, if we want solve (1.4.9) again in $L^2(0,T;H^3(\mathbb{T}))$ we need to treat $\partial_x(u(t)\mathscr{W}(t))$ as an $H^1(\mathbb{T})$-valued nonlinearity.

To make this precise, we consider again the linearized equation

$$u(t) - u_0 + \int_0^t \partial_x(u_0^2\partial_x^3 u)\,\mathrm{d}s = \int_0^t \partial_x f\,\mathrm{d}s + \sum_{k\in\mathbb{N}}\int_0^t g_k\,\mathrm{d}\beta_s^{(k)}, \tag{1.4.11}$$

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with a possibly random drift $\partial_x f \in L^2(\Omega \times [0,T]; H^{-1}(\mathbb{T}))$ and a generic Itô noise defined by $g_k \in L^2(\Omega \times [0,T]; H^1(\mathbb{T}))$, $k \in \mathbb{N}$. To derive a stochastic version of the maximal regularity estimate (1.4.8), we apply ∂_x to both sides of equation (1.4.11) and use Itô's formula to compute the square, resulting in

$$\begin{aligned}
&\tfrac{1}{2}(\partial_x u(t))^2 - \tfrac{1}{2}(\partial_x u_0)^2 + \int_0^t \partial_x u \partial_x^2 (u_0^2 \partial_x^3 u)\,\mathrm{d}s \\
&= \int_0^t \partial_x u \partial_x^2 f\,\mathrm{d}s + \frac{1}{2}\sum_{k\in\mathbb{N}} \int_0^t (\partial_x g_k)^2\,\mathrm{d}s + \sum_{k\in\mathbb{N}} \int_0^t \partial_x u \partial_x g_k\,\mathrm{d}\beta_s^{(k)}.
\end{aligned}$$

Integration in space, using integration by parts and again Young's inequality results in

$$\begin{aligned}
&\|\partial_x u(t)\|_{L^2(\mathbb{T})}^2 - \|\partial_x u_0\|_{L^2(\mathbb{T})}^2 + \delta \|\partial_x^3 u\|_{L^2([0,t]\times\mathbb{T})}^2 \\
&\le C_\delta \|f\|_{L^2([0,t]\times\mathbb{T})}^2 + \sum_{k\in\mathbb{N}} \|\partial_x g_k\|_{L^2([0,t]\times\mathbb{T})}^2 + 2\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \partial_x u \partial_x g_k\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}.
\end{aligned} \tag{1.4.12}$$

To estimate the supremum in time of the stochastic integral, we apply the Burkholder–Davis–Gundy inequality, which yields that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \partial_x u \partial_x g_k\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}\right|\right] \\
&\le C\mathbb{E}\left[\left(\sum_{k\in\mathbb{N}} \int_0^T \left(\int_{\mathbb{T}} \partial_x u \partial_x g_k\,\mathrm{d}x\right)^2 \mathrm{d}s\right)^{1/2}\right] \\
&\le C\mathbb{E}\left[\left(\sum_{k\in\mathbb{N}} \|\partial_x g_k\|_{L^2([0,T]\times\mathbb{T})}^2\right)^{1/2} \times \sup_{0\le t\le T} \|\partial_x u\|_{L^2(\mathbb{T})}\right] \\
&\le C^2 \mathbb{E}\left[\sum_{k\in\mathbb{N}} \|\partial_x g_k\|_{L^2([0,T]\times\mathbb{T})}^2\right] + \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le T} \|\partial_x u\|_{L^2(\mathbb{T})}^2\right].
\end{aligned}$$

Hence, after taking the supremum over $t \in [0,T]$ in (1.4.12) and then the expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T} \|u(t)\|_{L^2(\mathbb{T})}^2 + \|\partial_x^3 u\|_{L^2([0,T]\times\mathbb{T})}^2\right] \\
&\le C_\delta \mathbb{E}\left[\|\partial_x u_0\|_{L^2(\mathbb{T})}^2 + \|f\|_{L^2([0,T]\times\mathbb{T})}^2 + \sum_{k\in\mathbb{N}} \|\partial_x g_k\|_{L^2([0,T]\times\mathbb{T})}^2\right],
\end{aligned} \tag{1.4.13}$$

where we enlarged again the constant C_δ. This estimate is called *stochastic maximal regularity* and can be used to set up a fixed point argument to treat quasilinear stochastic partial differential equations as we have seen for the deterministic case. In particular, our intuition to require the stochastic term in (1.4.9) to be $H^1(\mathbb{T})$-valued is confirmed by the norm of g_k on the right-hand side of (1.4.13).

A recent contribution to the theory of quasilinear stochastic evolution equations was made in [5, 6]. While we presented in this subsection the concept of stochastic maximal regularity in the Hilbert space-valued setting and with an L^2-norm in time, this framework allows to treat stochastic evolution equations in Banach spaces and to use L^p-norms in time instead. The appropriate generalization of (1.4.13) is then called stochastic maximal L^p-regularity and taking $p > 2$ allows the authors of [5, 6] to additionally involve weights of the form t^κ, $\kappa > 0$. The latter is a key point of their theory, since, using the fact that on $[\varepsilon, T]$ for $\varepsilon > 0$ the weight t^κ becomes irrelevant, different regularity classes of solutions can be connected capturing the smoothing effect of parabolic equations. Additionally, the weighted setting allows for less regular initial values.

1.4.2. THE INTERFACE POTENTIAL

We have shown in the preceding subsection that the deterministic thin-film equation admits unique solutions on the whole time interval if the film height remains strictly positive, at least in one spatial dimension. While we have seen some positivity preserving mechanisms of the equation in Subsection 1.3.2, these do not suffice to conclude that the film height is bounded away from 0, as we would need to iterate the fixed-point argument. However, if we consider fluid films of the height of several nanometers only, the *interaction forces between the molecules* of the fluid and the substrate become prevalent and lead to a strictly positive film height on the microscopic level.

In the thin-film model discussed in Subsection 1.1.1, intermolecular forces can be modeled by a body force of the form $\nabla\Pi$ in the Navier–Stokes system, where Π is the disjoining pressure given by the negative derivative of the effective interface potential

$$\Pi(y) = -\phi'(y). \tag{1.4.14}$$

After rescaling $\widetilde{\Pi} = \varepsilon\Pi$ in the (deterministic) long-wave approximation, the leading order terms are

$$\partial^2_{\tilde{y}}\tilde{v}^{(\tilde{x})} = \partial_{\tilde{x}}(\tilde{p} - \widetilde{\Pi}) \quad \text{and} \quad \partial_{\tilde{y}}(\tilde{p} - \widetilde{\Pi}) = 0,$$

as opposed to (1.1.7). Assuming a no-slip condition on the liquid-solid interface, we obtain the closed equation

$$\partial_t u = -\partial_x\big((u^3/3)\partial_x\big(\partial_x^2 u + \Pi(u)\big)\big)$$

for the film height, see [111], and other mobility exponents result again from slip conditions near the substrate. Inserting (1.4.14) yields the equation

$$\partial_t u = -\partial_x\big((u^3/3)\partial_x\big(\partial_x^2 u - \phi'(u)\big)\big) \tag{1.4.15}$$

in terms of the effective interface potential. The latter is nothing but the surface average of the molecular pair-potentials, e.g., modeling the repulsive and attractive van der Waals forces by a 6-12 *Lennard–Jones potential* results in

$$\phi(u) = \frac{1}{u^8} - \frac{1}{u^2} + 1. \tag{1.4.16}$$

The same term as in (1.4.15) arises when including molecular interaction forces in the stochastic model. Indeed, in the derivation from [78] the interface potential ϕ is

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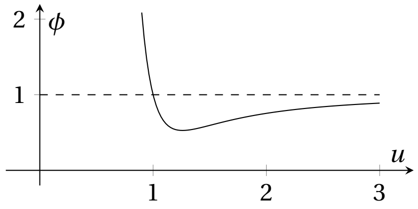

Figure 1.8: The effective interface potential (1.4.16).

actually present from the start, because at the length scales of several nanometers, in which thermal fluctuations are the most relevant, the van der Waals forces play also a role for the dynamics. The resulting version of (STFE) reads

$$\partial_t u = -\partial_x\big(u^n \partial_x(\partial_x^2 u - \phi'(u))\big) + \partial_x(u^{n/2}\mathscr{W})$$

in one spatial dimension.

The additional interaction forces need also to be accounted for when evaluating the energy of the system. Next to the surface energy considered in Subsection 1.1.2 the potential energy enters the energy of a film profile, which is then given by

$$\mathscr{E}_\phi = \int \tfrac{1}{2}|\partial_x u|^2 + \phi(u)\,\mathrm{d}x. \tag{1.4.17}$$

Concerning the positivity of the solution, we make the observation that if $\mathscr{E}_\phi < \infty$ and x_0 is a zero of the profile u, then

$$u(x) = u(x) - u(x_0) \le \int_{x_0}^{x} |\partial_x u|\,\mathrm{d}x \le |x - x_0|^{1/2}\|\partial_x u\|_{L^2(\mathbb{T})}$$

by Hölder's inequality. Hence, assuming that $\phi(u) \sim u^{-\vartheta}$ for u near 0, we deduce that $\phi(u)$ admits a non-integrable singularity at x_0 if $\vartheta \ge 2$ contradicting the finiteness of $\mathscr{E}_\phi$. As a consequence, $\mathscr{E}_\phi < \infty$ implies strict positivity of the film height for sufficiently repulsive interface potentials.

1.4.3. The results of Chapter 5

Finally, we summarize the results of Chapter 5, which is based on the prepint [3]. There, we apply the well-posedness theory from [5, 6] to stochastic partial differential equations of the form

$$\begin{cases} \partial_t u + \operatorname{div}(m(u)\nabla\Delta u) = \operatorname{div}(\Phi(u)\nabla u) + \operatorname{div}(g(u)\mathscr{W}), & \text{on } \Omega\times[0,T]\times\mathbb{T}^d, \\ u(0,\cdot) = u_0, & \text{on } \Omega\times\mathbb{T}^d, \end{cases} \tag{1.4.18}$$

in any spatial dimension $d \ge 1$ for smooth coefficient functions $m\colon (0,\infty)\to(0,\infty)$ and $g,\Phi\colon (0,\infty)\to\mathbb{R}$. The temporally white Gaussian noise is vector-valued, i.e.,

$$\mathscr{W} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k\in\mathbb{N}} e_k\beta^{(k)}, \qquad e_k\colon \mathbb{T}^d \to \mathbb{R}^d,$$

and u_0 is assumed to be positive and bounded away from 0. Unlike in the previously discussed results, we interpret (1.4.18) in the sense of Itô. Whence, the Itô interpretation of (STFE) falls into this class of equations, if we set $m(u) = u^n$, $g(u) = u^{n/2}$ and also the interface potential can be included by defining $\Phi(u) = m(u)\phi''(u)$. As pointed out in [75, 107], under reasonable symmetry assumptions on the noise, also the Itô–Stratonovich correction term of the stochastic thin-film equation can be incorporated in the Φ-term, so that also its Stratonovich interpretation admits the Itô form (1.4.18).

To apply the theory of [5, 6] we show stochastic maximal L^p-regularity of the linear part

$$u(t) - u_0 + \operatorname{div}(m(u_0)\nabla\Delta u) = \int_0^t f \,\mathrm{d}s + \sum_{k\in\mathbb{N}} \int_0^t g_k \mathrm{d}\beta_s^{(k)} \tag{1.4.19}$$

in suitable function spaces and corresponding estimates on the nonlinearities and the quasilinear operator. A natural scale of spaces to consider (1.4.19) in are the Bessel-potential spaces $H^{s,q}(\mathbb{T}^d)$ for $q \in [2,\infty)$ generalizing the Sobolev spaces to fractional smoothness indices $s \in \mathbb{R}$. In this case we consider the linearized thin-film operator $u \mapsto \operatorname{div}(m(u_0)\nabla\Delta u)$ as a mapping from $H^{s+2,q}(\mathbb{T}^d) \to H^{s-2,q}(\mathbb{T}^d)$, while f should lie in $H^{s-2,q}(\mathbb{T}^d)$ and the stochastic perturbation should satisfy $g_k \in H^{s,q}(\mathbb{T}^d)$ for each $k \in \mathbb{N}$, analogously to the case discussed in Subsection 1.4.1. To use the full strength of the abstract framework, we consider weighted L^p spaces in time, i.e., the solution u to (1.4.19) should satisfy

$$\int_0^T \|u\|^p_{H^{s+2,q}(\mathbb{T}^d)} t^\kappa \mathrm{d}t < \infty,$$

almost surely, for $p \in [2,\infty)$ and $\kappa > 0$.

To show these estimates we naturally need to require some assumption on the parameters (p,κ,s,q) since, for example, if s is too low then the products and nonlinearities may be ill-defined. A particular role is played by the stochastic nonlinearity, since the smoothness and decay of the family $(e_k)_{k\in\mathbb{N}}$ as $k \to \infty$ determines the spatial regularity of $\operatorname{div}(g(u)\mathscr{W})$ and ultimately of the solution. As part of Chapter 5, we show that whenever the noise $\mathscr{W}$ admits smoothness $(1/2+\varepsilon)$ in space for some $\varepsilon > 0$, then there is a suitable choice of function spaces (or equivalently parameters (p,κ,s,q)) for which a unique, positive solution to (1.4.18) exists locally in time. We remark that in the stochastic setting the life time of the solution may depend on ω, and is therefore given by a stopping time. Moreover since, in contrast to the previously discussed results, no compactness arguments are used, the solution is probabilistically strong, see Subsection 1.2.4 for a discussion.

As mentioned in Subsection 1.4.1, we can use that the weights t^κ become negligible after an arbitrarily small time to show that the solutions regularize instantaneously and therefore connect different regularity classes of solutions, if the noise is sufficiently regular in space. In particular, if $\mathscr{W}$ is smooth in space, the local solution to (1.4.18) is spatially smooth as well.

It remains to prove that the solution can be extended to the whole time interval for which we consider the setting $(p,\kappa,s,q) = (2,0,1,2)$ from Subsection 1.4.1, and we require $\mathscr{W}$ accordingly to be two times differentiable in space. Moreover, we consider

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specifically the stochastic thin-film equation with an interface potential

$$\begin{cases} \partial_t u = -\partial_x(u^n \partial_x^3 u) + \partial_x(u^n \partial_x \phi'(u)) + \partial_x(u^{n/2}\mathscr{W}), & \text{on } \Omega \times [0,T] \times \mathbb{T}, \\ u(0,\cdot) = u_0, & \text{on } \Omega \times \mathbb{T}, \end{cases} \tag{1.4.20}$$

and restrict ourselves to the one-dimensional situation due to the failure of the Sobolev embedding (1.4.6) for $d > 1$.

By the discussed result, we know that there exists a unique, probabilistically strong solution to (1.4.20) if the initial value is positive and lies in $H^1(\mathbb{T})$, and this argument can be iterated as long as u remains strictly positive and in the energy space. The latter follows if we can show that

$$\mathbb{E}\left[\sup_{0\le t<\tau} \mathscr{E}_\phi(u(t))\right] < \infty,$$

for the maximal life time τ of the solution and the energy $\mathscr{E}_\phi$ of (1.4.20) as defined in (1.4.17). Indeed, as laid out in the previous subsection $\mathscr{E}_\phi$ bounds $\|\partial_x u\|^2_{L^2(\mathbb{T})}$ and also yields strict positivity of the profile, at least if ϕ is sufficiently repulsive. We establish such an estimate by first closing a version of ($\mathscr{G}_\alpha$-Est) and then of ($\mathscr{E}$-Est) tailored to the energy $\mathscr{E}_\phi$. While in Chapters 2, 3 and 4 the Itô–Stratonovich correction term was indispensable to close such estimates, the estimates from Chapter 5 apply to both, the Itô and the Stratonovich interpretation of (1.4.20), since the ϕ-term also has a regularizing effect.

As a result, we obtain that (1.4.20) admits unique, strictly positive, probabilistically strong solutions for $n \in (0,6)$ and $\phi(r) \sim r^\vartheta$ for $r \sim 0$ and some $\vartheta > \max\{2, 6-2n\}$. The fact that we require the interface potential to be more repulsive near 0 for small values of n is due to the increased energy production by the noise coefficient $u \mapsto u^{n/2}$, which is not differentiable at 0 for $n < 2$. Moreover, while the solution is a-priori analytically weak, the discussed regularization results yield that

$$\begin{aligned} u(t,x) - u_0(x) &= \int_0^t \left[-\partial_x\big(u^n(s)\partial_x^3 u(s)\big) + \partial_x\big(u^n(s)\partial_x(\phi'(u(s)))\big)\right](x)\,\mathrm{d}s \\ &\quad + \sum_{k\in\mathbb{N}} \int_0^t \partial_x(u(s)e_k)(x)\,\mathrm{d}\beta_s^{(k)}, \end{aligned}$$

for all $x \in \mathbb{T}$ whenever u_0 and $\mathscr{W}$ are smooth enough, i.e., (1.4.20) is then satisfied in the analytically classical sense.

1.5. FURTHER LITERATURE

Concerning the existence of weak solutions to the deterministic thin-film equation, additionally to the aforementioned works [11, 15, 32, 33], we refer to [17, 70, 73] for the complete wetting regime and to [19, 106, 112] for partial wetting. Moreover, next to the reviewed a-priori estimates for the thin-film equation, also a modified energy is dissipated by the equation [41, 96]. A list of references on the properties of weak solutions to the thin-film equation and the strong solution theory can be found in Chapter 6.

Besides the already discussed results [35, 58] on the stochastic thin-film equation, we should mention the pioneering work [51] on the existence of weak martingale solutions to the stochastic thin-film equation. There, based on a spatial discretization of the equation, the existence of solutions to the Itô interpretation of (1.4.20) with $n = 2$ is proved. By letting the interface potential ϕ appearing in [51] tend to 0, a similiar result to the one of [58], was derived in [75] by closing additionally α-entropy estimates for the solutions. A version of [51] in the two-dimensional setting was obtained in [107]. Both of these works share some of their advances with Chapter 2, i.e., the stochastic α-entropy estimates and the physical dimension $d = 2$, and were developed independently of [116]. Moreover, a positive result on the finite speed of propagation of a regularized version of (STFE) is proved in a series of three works starting with [76, 77].

Comments on the stochastic gradient flow structure of (STFE) with spatio-temporal white noise $\mathscr{W}$ can be found in [59], where a corresponding numerical scheme is derived and analyzed. Also regarding the white noise case, an effort to treat (STFE) using regularity structures was recently initiated in [80], where appropriate counterterms are provided to give sense to the classically ill-defined nonlinearities.

2

Existence in the two-dimensional setting†

This chapter is concerned with the construction of very weak martingale solutions to the Stratonovich interpretation of (STFE) with a quadratic mobility $n = 2$ in the physically relevant two-dimensional setting $d = 2$. The proof is based on a generalization of the approach of [58] to the case $d = 2$, i.e., we use a decomposition of the deterministic and stochastic dynamics.

The higher spatial dimension leads to additional mathematical challenges due to the reduced gain of integrability after employing the Sobolev embedding theorem. Indeed, in [58] the control of the surface energy

$$\int_{\mathbb{T}} |u'|^2 \, \mathrm{d}x$$

suffices to show convergence of the nonlinear terms from the sequence of approximate solutions. As apparent from the deterministic setting [33], the additional control of the dissipation terms of the α-entropy

$$-\int_{\mathbb{T}^2} u^{\alpha+1} \, \mathrm{d}x, \quad \alpha \in (-1,0)$$

is necessary to deduce convergence of the nonlinear terms in the two-dimensional case. Hence, to adapt the time splitting approach from [58], we have to additionally control the α-entropy along the splitting scheme and use the more delicate limiting procedure from [33] compared to the one-dimensional case [15]. Combining this with the stochastic compactness method is the key challenge which we overcome in this chapter. Moreover,

†This chapter is based on the article [116]: M. Sauerbrey. "Martingale solutions to the stochastic thin-film equation in two dimensions". In: *Ann. Inst. Henri Poincaré Probab. Stat.* 60.1 (2024), pp. 373–412.

compared to the independently proven result in [107], where an existence result in the two-dimensional case based on the dissipation of the entropy

$$-\int_{\mathbb{T}^2} \log(u)\,\mathrm{d}x$$

is given, we do allow for solutions with a contact line between the fluid film and the solid.

2.1. Introduction to Chapter 2

In this section, we state and discuss the result of this chapter, we outline the strategy of its proof and collect the used notation.

2.1.1. Main result

We state the existence result which we will prove in the course of this chapter. Choosing $n = d = 2$ in (STFE) and interpreting the noise in the Stratonovich sense, we obtain the SPDE

$$\mathrm{d}u_t = -\operatorname{div}(u_t^2\nabla\Delta u_t)\,\mathrm{d}t + \operatorname{div}(u_t \circ \mathrm{d}W_t) \quad \text{on } \mathbb{T}^2, \tag{2.1.1}$$

where W_t is specified as follows. We let $(\psi_l)_{l\in\mathbb{N}}$ be the orthonormal basis in $H^2(\mathbb{T}^2;\mathbb{R}^2)$ consisting of the eigenfunctions to the periodic Laplacian in the first and second component respectively, i.e., every ψ_l is of the form $(\xi_k, 0)$ or $(0, \xi_k)$ for some $k \in \mathbb{Z}^2$, where

$$\xi_k(x,y) = \frac{\tilde{\xi}_{k_1}(x)\,\tilde{\xi}_{k_2}(y)}{\sqrt{1+(2\pi|k|)^2+(2\pi|k|)^4}} \tag{2.1.2}$$

and

$$\tilde{\xi}_j(x) = \begin{cases} \sqrt{2}\cos(2\pi jx), & j<0,\\ 1, & j=0,\\ \sqrt{2}\sin(2\pi jx), & j>0. \end{cases} \tag{2.1.3}$$

Moreover, we let $\Lambda = (\lambda_l)_{l\in\mathbb{N}} \in l^2(\mathbb{N})$ satisfy the symmetry relation

$$\lambda_l = \lambda_{\tilde{l}} \quad \text{whenever} \quad \psi_l = (\xi_k, 0) \wedge \psi_{\tilde{l}} = (0, \xi_k) \quad \text{for some } k\in\mathbb{Z}^2. \tag{2.1.4}$$

Then

$$W_\Lambda(t) = \sum_{l=1}^{\infty} \lambda_l \beta_t^{(l)} \psi_l \tag{2.1.5}$$

for independent Brownian motions $(\beta^{(l)})_{l\in\mathbb{N}}$ defines a centered Gaussian process on $H^2(\mathbb{T}^2;\mathbb{R}^2)$ with the covariance operator $Qf = \sum_{l=1}^\infty \lambda_l^2 (f,\psi_l)_{H^2(\mathbb{T}^2;\mathbb{R})}\psi_l$. Inserting W_Λ as the driving process in (2.1.1), writing the Stratonovich integral in Itô form, and writing $J = u^2\nabla\Delta u$ in the weak form from [33, Eq. (3.2)] yields the following notion of very weak martingale solutions to (2.1.1), see also Section 1.2.

Definition 2.1.1. *Let $T\in(0,\infty)$ and $q\in(2,\infty)$. A very weak martingale solution to (2.1.1) with q'-regular nonlinearity on $[0,T]$ consists out of a filtered probability space satisfying the usual conditions, a family of independent Brownian motions $(\beta^{(l)})_{l\in\mathbb{N}}$, a continuous process $(u(t))_{t\in[0,T]}$ in $H^1_w(\mathbb{T}^2)$ together with a random variable J with values in $L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$ such that*

(i) $u(t)$, $J|_{[0,t]}$ are $\mathfrak{F}_t$-measurable in $H^1(\mathbb{T}^2)$ and $L^2(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$, respectively, for every $t\in[0,T]$,

(ii) it holds almost surely $|\nabla u|\in L^3(\{u>0\})$ and

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} J\cdot\eta\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\{u(t)>0\}}|\nabla u|^2\nabla u\cdot\eta\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_{\{u(t)>0\}} u|\nabla u|^2\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t\\ &+2\int_0^T\int_{\{u(t)>0\}} u\nabla^T uD\eta\nabla u\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_0^T\int_{\mathbb{T}^2} u^2\nabla u\cdot\nabla\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{2.1.6}$$

for all $\eta\in L^\infty(0,T;W^{2,\infty}(\mathbb{T}^2;\mathbb{R}^2))$,

(iii) and for all $\varphi\in W^{1,q}(\mathbb{T}^2)$ we have

$$\begin{aligned}\langle u(t),\varphi\rangle-\langle u_0,\varphi\rangle &= \int_0^t -\langle\operatorname{div}(J),\varphi\rangle\,\mathrm{d}s+\frac{1}{2}\sum_{l=1}^\infty\int_0^t\lambda_l^2\langle\operatorname{div}(\operatorname{div}(u(s)\psi_l)\psi_l),\varphi\rangle\,\mathrm{d}s\\ &+\sum_{l=1}^\infty\lambda_l\int_0^t\langle\operatorname{div}(u(s)\psi_l),\varphi\rangle\,\mathrm{d}\beta_s^{(l)},\end{aligned}\tag{2.1.7}$$

almost surely for all $t\in[0,T]$.

Remark 2.1.2. (i) By the weak continuity in $H^1(\mathbb{T}^2)$ any solution u in the sense of Definition 2.1.1 satisfies

$$\sup_{0\le t\le T}\|u(t)\|_{H^1(\mathbb{T}^2)}<\infty,\tag{2.1.8}$$

almost surely.

(ii) The measurability assumption on J in item (i) ensures that all the terms on the right-hand side of (2.1.7) are adapted. Interpreting J as an element of the distribution space $\mathscr{D}'(\mathbb{R}\times\mathbb{T}^2;\mathbb{R}^2)$, one can equivalently demand that J is adapted to $\mathfrak{F}$ in the sense of distributions [25, Definition 2.2.13]. This follows by density of $C_c^\infty((0,t)\times\mathbb{T}^2;\mathbb{R}^2)$ in $L^2(0,t;L^q(\mathbb{T}^2;\mathbb{R}^2))$, separability of $L^2(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$, and the equivalence of weak and Borel measurability in separable Banach spaces [84, Proposition 1.1.1].

In the course of this chapter, we will derive the following existence result.

Theorem 2.1.3. *Let μ be a probability distribution on $H^1(\mathbb{T}^2)$ supported on the non-negative functions, $T\in(0,\infty)$, $q\in(2,\infty)$ and $\alpha\in(-1,0)$. Then there exists a very weak martingale solution to (2.1.1) on $[0,T]$ with q'-regular nonlinearity satisfying $u(0)\sim\mu$. Moreover,*

(i) $u(t)\ge 0$ *almost surely for all* $t\in[0,T]$,

2

(ii) we have for $p \in (0,\infty)$ the estimates

$$\mathbb{E}\left[\sup_{0\le t\le T}\|u\|^p_{H^1(\mathbb{T}^2)}\right] \lesssim_{\Lambda,p,T} \int \|\cdot\|^p_{H^1(\mathbb{T}^2)}\,\mathrm{d}\mu, \tag{2.1.9}$$

$$\mathbb{E}\left[\|J\|^{\frac{p}{2}}_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}\right] \lesssim_{\Lambda,p,q,T} \int \|\cdot\|^p_{H^1(\mathbb{T}^2)}\,\mathrm{d}\mu, \tag{2.1.10}$$

(iii) and it holds the additional spatio-temporal regularity

$$u^{\frac{\alpha+3}{2}} \in L^2(0,T;H^2(\mathbb{T}^2)) \quad \text{and} \quad u^{\frac{\alpha+3}{4}} \in L^4(0,T;W^{1,4}(\mathbb{T}^2)) \tag{2.1.11}$$

almost surely.

Remark 2.1.4. (i) We point out that we allow for the right-hand sides of (2.1.9) and (2.1.10) to be infinite, in which case the corresponding estimate trivializes.

(ii) We note that the differential identity

$$\nabla u = \frac{4}{\alpha+3} u^{\frac{1-\alpha}{4}} \nabla u^{\frac{\alpha+3}{4}},$$

the Sobolev embedding theorem, (2.1.11) and (2.1.8) imply that $|\nabla u| \in L^{4-}([0,T]\times\mathbb{T}^2)$ almost surely and hence the integrability condition from Definition 2.1.1 (ii).

2.1.2. DISCUSSION OF THE RESULT

Theorem 2.1.3 generalizes [58, Theorem 1.2] to the setting in two dimensions and is therefore together with the independently developed result [107, Theorem 3.5] the first existence result for the stochastic thin-film equation in higher dimension. As in the one-dimensional case, the time splitting approach is not only suitable to construct solutions to the stochastic thin-film equation, but suggests a numerical approach for their simulation as well. The assumption $\Lambda \in l^2(\mathbb{N})$ on the noise (2.1.5) is the same as in [35, 51, 58], where we refer the reader for an interpretation of the expansion (2.1.5) in terms of a spatial correlation function of the noise to the exposition in [20]. The additionally imposed symmetry condition (2.1.4) expresses that the coordinate-wise noise processes are distributed according to the same Gaussian law in $H^2(\mathbb{T}^2)$. This is a physically reasonable assumption since the noise is induced by thermal fluctuations and its distribution depends consequently on its position but not on its direction. The same symmetry condition appears in [107, Eq. (2.19)], where the use of Stratonovich noise is discussed, which indicates that it is an important assumption to treat the stochastic thin-film equation in higher dimensions. We point out that in [107], the expansion (2.1.5) in terms of eigenfunctions of the periodic Laplacian is relaxed to a slightly more general assumption.

In contrast to the existence results from the mentioned articles, there is no integrability assumption on the initial distribution required in Theorem 2.1.3. This is achieved by using a decomposition of the initial value in countably many parts which are each almost surely bounded in $H^1(\mathbb{T}^2)$. Then one can construct approximate solutions and apply tightness arguments for each of these parts separately and add them together afterwards. The only important feature of (2.1.1) for this to work is that $u(t) = 0$ is a solution to it. We remark that these kind of reductions to bounded or integrable initial values are well-known and can be achieved in the setting of probabilistically strong solutions via localization or changing the probability measure, see [6, Proposition 4.13] or [94, Theorem 6.9.2] for examples. We use the decomposition of the initial value instead, since it is more compatible with the stochastic compactness method as well as the estimates (2.1.9) and (2.1.10). The moment estimates (2.1.9) and (2.1.10) for $p < 2$ are also new for the stochastic thin-film equation and are obtained from the estimates for higher moments.

2.1.3. OUTLINE AND DISCUSSION OF THE PROOF

In Section 2.2 we review the existence result for very weak solutions to the deterministic thin-film equation in two dimensions from [33] and state properties of the obtained solutions which are immediate from their construction. Additionally, we show that there is a measurable solution operator using the measurable selection theorem, which is important to combine these results with the stochastic setting. This approach is to the author's knowledge new and might be of interest also for other situations, where a measurable solution operator is required.

In Section 2.3 we consider the regularized Stratonovich SPDE

$$du_t = \varepsilon \Delta u_t \, dt + \operatorname{div}(u_t \circ dW_t) \quad \text{on } \mathbb{T}^2 \tag{2.1.12}$$

and establish well-posedness in $H^1(\mathbb{T}^2)$ using the monotone operator theory for SPDEs. The coercivity estimates (2.3.7), (2.3.8) are obtained analogously to the one-dimensional case [58, Eq. (A.9)] and require only some adaptions to multivariable calculus, where the symmetry condition (2.1.4) is used. Their uniformity in ε is key to letting later on $\varepsilon \searrow 0$ and eliminating the regularization term $\varepsilon \Delta u_t$ from (2.1.12). We note that this procedure is well-known and refer the reader to the article [57] and the references therein for more information on degenerate parabolic SPDEs. However, the general result [57, Theorem 2.1] does not directly apply to

$$du_t = \operatorname{div}(u_t \circ dW_t) \quad \text{on } \mathbb{T}^2 \tag{2.1.13}$$

and the coercivity estimates are unique to our particular situation.

In Section 2.4, we start to construct approximate solutions to (2.1.1) by splitting the stochastic and deterministic dynamics along a time-splitting scheme with step length δ. Using the properties of the solutions to the deterministic thin-film equation and the solutions to (2.1.12), we derive estimates on the approximate solutions which are uniform in ε and δ. The procedure is analogous to the one-dimensional case, but we note that we take the slightly different approach to let $\varepsilon \searrow 0$ afterwards to be able to apply Itô's formula to the whole time splitting scheme. After these estimates are obtained, it is straightforward to deduce tightness statements on the approximating sequence in ε and employ the Skorokhod–Jakubowski theorem to obtain an almost surely convergent, equally distributed subsequence. Usually, the parabolic regularization procedure does not require to pass to another probability space, see again [57], but it is in our case convenient to ensure convergence of the solutions to the deterministic equation as well.

Finally, in Section 2.5, we derive additional estimates on the approximating sequence by controlling the α-entropy production along the stochastic dynamics by means of Itô's formula. Using the obtained estimates, we show additional tightness properties of powers of the solution by an adaption of the compactness argument in [33, Lemma 2.5], which is compatible with our splitting scheme. This line of arguments is unique to the higher-dimensional setting and distinguishes our approach from the one-dimensional case. We employ the Skorokhod–Jakubowski theorem once more to let $\delta \searrow 0$ and identify the limit as a solution to (2.1.1) combining the methods from [33, Theorem 3.2] and [58, Section 5.2]. As a result of the construction the additional estimates (2.1.9), (2.1.10), and the regularity properties (2.1.11) follow.

The reason to use the time-splitting approach instead of a linear parabolic regularization is that it directly yields non-negative solutions, because the deterministic result [33] provides non-negative solutions and the regularized stochastic part of the equation admits a maximum principle. Since we are dealing with a fourth order equation, a linear parabolic regularization of the whole equation would yield possibly negative solutions, which lack a reasonable physical interpretation. However, a more delicate, nonlinear regularization is possible as demonstrated in the one-dimensional case [35] or [75], but would require a longer proof.

2.1.4. NOTATION FOR CHAPTER 2

We use the notation $\lesssim$ to indicate that an inequality holds up to a universal constant and write $\lesssim_{p_1,\dots}$ if the constant depends on nothing but the parameters $p_1,\dots$. Similarly, we write C for a universal constant and $C_{p_1,\dots}$, if the constant depends on $p_1,\dots$. We write

$$G_\alpha(t) = \int_1^t \int_1^s \tau^{\alpha-1} \mathrm{d}\tau\, \mathrm{d}s, \quad \alpha \in \mathbb{R}$$

for the (mathematical) α-entropy, and point out for later reference that

$$G_\alpha(t) = \frac{t^{\alpha+1}}{\alpha(\alpha+1)} + r_\alpha(t), \quad t \geq 0, \tag{2.1.14}$$

if $\alpha \in (-1,0)$, where r_α is a first order polynomial. We use classical notation for differential operators, i.e., write ∇f, $\operatorname{div}(f)$, Δf for the gradient, divergence and Laplacian of a function or a vector field f, respectively. Moreover, we write Hf for the Hessian matrix and use the notational convention that a differential operator is only applied to the first function appearing afterwards so that, e.g.,

$$\nabla f g = g(\nabla f), \quad \text{but} \quad \nabla(fg) = f(\nabla g) + g(\nabla f).$$

We denote our domain, the 2-torus, by $\mathbb{T}^2$. We write $L^p(\mathbb{T}^2)$, $W^{k,p}(\mathbb{T}^2)$ and $H^k(\mathbb{T}^2)$ for the Lebesgue, Sobolev and Bessel-potential spaces on $\mathbb{T}^2$ with integrability and smoothness exponents p, k, where more information on periodic spaces can be found in [119, Section 3]. We note that if k is an integer, we equip $H^k(\mathbb{T}^2)$ with the equivalent $W^{k,2}(\mathbb{T}^2)$-inner product. We write $L^p(\mathbb{T}^2;\mathbb{R}^2)$, $W^{k,p}(\mathbb{T}^2;\mathbb{R}^2)$ and $H^k(\mathbb{T}^2;\mathbb{R}^2)$ for the corresponding spaces of vector fields and equip them with the direct sum norm and set for the special case $p = 2$

$$\|(f_1, f_2)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} = \|f_1\|^2_{L^2(\mathbb{T}^2)} + \|f_2\|^2_{L^2(\mathbb{T}^2)}, \quad \|(f_1, f_2)\|^2_{H^k(\mathbb{T}^2;\mathbb{R}^2)} = \|f_1\|^2_{H^k(\mathbb{T}^2)} + \|f_2\|^2_{H^k(\mathbb{T}^2)}$$

to preserve the Hilbert space structure. We write $\langle f, g\rangle$ for the dual pairing in $L^2(\mathbb{T}^2)$ and in $L^2(\mathbb{T}^2;\mathbb{R}^2)$ depending on f, g being functions or vector fields. If (S,ν) is a measure space and X is a Banach space, we write $L^p(S,\nu;X)$ for the Bochner space of strongly measurable, p-integrable, X-valued functions on (S,ν). For details we refer to [84, Section 1]. If it is clear which measure is considered, we use also the notation $L^p(S;X)$ and if $S=[s,t]$ and ν the Lebesgue measure $L^p(s,t;X)$. Moreover, we write $C([0,T];X)$, $H^1(0,T;X)$, $C^\gamma(0,T;X)$ and $W^{\gamma,p}(0,T;X)$, for the space of continuous functions, first-order Sobolev space, Hölder and Sobolev-Slobodeckij space on $[0,T]$ with values in X, where we will only consider fractional exponents $\gamma\in(0,1)$. The corresponding Hölder semi-norm is denoted by $[\cdot]_{\gamma,X}$ and for precise definitions of these spaces we refer to [9, Section 2]. If a Banach space X is considered with its weak or weak-* topology, we express this by writing X_w or X_{w*} respectively. Lastly we mention that we write $L(H_1,H_2)$ and $L_2(H_1,H_2)$ for the space of bounded linear operators and Hilbert–Schmidt operators between two Hilbert spaces H_1 and H_2, respectively.

2.2. THE DETERMINISTIC THIN-FILM EQUATION

In this section we summarize the existence result for very weak solutions to the deterministic thin-film equation in the special case of quadratic mobility

$$\partial_t v = -\operatorname{div}(v^2\nabla\Delta v) \tag{2.2.1}$$

from [33]. Moreover, we show that the solutions can be chosen in a measurable way, which will be important later. We remark that in [33] solutions to (2.2.1) are constructed on a domain with Neumann boundary conditions, but the arguments translate verbatim to the periodic setting. First, we recall the definition of very weak solutions to (2.2.1) from [33, Definition 3.1].

Definition 2.2.1. *Let* $q\in(2,\infty)$ *and* $T>0$. *A very weak solution to the (deterministic) thin-film equation on* $[0,T]$ *with* q'*-regular nonlinearity is a tuple*

$$(v,J)\in L^\infty(0,T;H^1(\mathbb{T}^2))\cap H^1(0,T;W^{-1,q'}(\mathbb{T}^2))\times L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2)),$$

such that $\partial_t v = -\operatorname{div} J$ *in* $L^2(0,T;W^{-1,q'}(\mathbb{T}^2))$, $|\nabla v|\in L^3(\{v>0\})$ *and*

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} J\cdot\eta\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\{v(t)>0\}}|\nabla v|^2\nabla v\cdot\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\int_{\{v(t)>0\}} v|\nabla v|^2\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+2\int_0^T\int_{\{v(t)>0\}} v\nabla^T v D\eta\nabla v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\int_{\mathbb{T}^2} v^2\nabla v\cdot\nabla\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.2.2}$$

for all $\eta\in L^\infty(0,T;W^{2,\infty}(\mathbb{T}^2;\mathbb{R}^2)))$.

Remark 2.2.2. By Rellich's theorem, see [1, Theorem 6.3, p.168], and the Aubin-Lions lemma [121, Corollary 5] there is a compact embedding

$$L^\infty(0,T;H^1(\mathbb{T}^2)) \cap H^1(0,T;W^{-1,q'}(\mathbb{T}^2)) \hookrightarrow C([0,T];L^r(\mathbb{T}^2))$$

for any $r \in [1,\infty)$. In the following, we will always identify a solution to the thin-film equation with its $L^r(\mathbb{T}^2)$-continuous version. By [24, Lemma II.5.9] this version is weakly continuous as a mapping with values in $H^1(\mathbb{T}^2)$.

The identity (2.2.2) is a weak formulation of $J = u^2\nabla\Delta u$. The following existence statement is given in [33, Theorem 3.2], where we add some quantitative estimates which follow from the construction in [33] and are proved in detail in Appendix 2.A.

Theorem 2.2.3. *Let $v_0 \in H^1(\mathbb{T}^2)$ be non-negative, $q \in (2,\infty)$, $T > 0$ and $\alpha \in (-1,0)$. Then there exists a very weak solution (v,J) to the thin-film equation on $[0,T]$ with q'-regular nonlinearity and $v(0) = v_0$, which satisfies the following properties for universal constants $0 < C_\alpha, C_q < \infty$.*

(i) We have for all $t \in [0,T]$ that

$$\int_{\mathbb{T}^2} v(t,\cdot)\,\mathrm{d}x = \int_{\mathbb{T}^2} v_0\,\mathrm{d}x \quad \textit{and} \quad v(t,\cdot) \geq 0.$$

(ii) It holds the energy estimate

$$\sup_{0\leq t\leq T} \|\nabla v(t)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \leq \|\nabla v_0\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}.$$

(iii) It holds that

$$\|J\|^2_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))} + C_q\|\nabla v(T)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\|\nabla v(T)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0\,\mathrm{d}x\right|^2\right)$$
$$\leq C_q\|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0\,\mathrm{d}x\right|^2\right).$$

(iv) We have the α-entropy estimate

$$\int_{\mathbb{T}^2} G_\alpha(v(T,\cdot))\,\mathrm{d}x + \frac{1}{C_\alpha}\int_0^T\int_{\mathbb{T}^2} |Hv^{\frac{\alpha+3}{2}}|^2 + |\nabla v^{\frac{\alpha+3}{4}}|^4\,\mathrm{d}x\,\mathrm{d}t \leq \int_{\mathbb{T}^2} G_\alpha(v_0)\,\mathrm{d}x.$$

The following result can be proved along the lines of [33, Lemma 2.5, Proposition 2.6, Corollary 2.7, Theorem 3.2].

Proposition 2.2.4. *Let $q \in (2,\infty)$, $T > 0$ and $(v_n, J_n)_{n\in\mathbb{N}}$ be a sequence of non-negative very weak solutions to the deterministic thin-film equation on $[0,T]$ with q'-regular nonlinearity. Assume that there is an $\alpha \in (-1,0)$ such that v_n, J_n, $v_n^{\frac{\alpha+3}{2}}$ and $v_n^{\frac{\alpha+3}{4}}$ are uniformly bounded in*

$$L^\infty(0,T;H^1(\mathbb{T}^2)),\ L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2)),\ L^2(0,T;H^2(\mathbb{T}^2)),\ L^4(0,T;W^{1,4}(\mathbb{T}^2)) \tag{2.2.3}$$

respectively. Then for a subsequence we have *(i)* $v_n \rightharpoonup^* v$ *in* $L^\infty(0,T;H^1(\mathbb{T}^2))$,

(ii) $J_n \rightharpoonup J$ *in* $L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$,

(iii) $v_n^{\frac{\alpha+3}{2}} \rightharpoonup v^{\frac{\alpha+3}{2}}$ *in* $L^2(0,T;H^2(\mathbb{T}^2))$,

(iv) $v_n^{\frac{\alpha+3}{4}} \rightharpoonup v^{\frac{\alpha+3}{4}}$ *in* $L^4(0,T;W^{1,4}(\mathbb{T}^2))$

and the limit (v,J) *is a non-negative very weak solution to the thin-film equation with* q'*-regular nonlinearity.*

Finally, we give proof to the existence of a measurable solution operator. To this end, we define the set $\mathscr{X}_{q,T}$ as the topological product of the spaces (2.2.3) equipped with the respective weak and weak-* topologies. Moreover, we write $B_X(r)$ for the ball in X centered at the origin with radius r, if X is a normed space.

Corollary 2.2.5. *Let* $q \in (2,\infty)$, $T > 0$ *and* $\alpha \in (-1,0)$. *There is a Borel-measurable mapping*

$$\mathscr{S}_{\alpha,q,T}\colon \{v_0 \in H^1(\mathbb{T}^2) | v_0 \geq 0\} \to \mathscr{X}_{q,T},\, v_0 \mapsto (v,J,v^{\frac{\alpha+3}{2}},v^{\frac{\alpha+3}{4}}), \tag{2.2.4}$$

which assigns to every initial value a very weak solution to the thin-film equation on $[0,T]$, *which satisfies the properties (i)–(iv) of Theorem 2.2.3.*

Proof. We define for v_0 in the domain of (2.2.4) the set of all very weak solutions to the stochastic thin-film equation with initial value v_0 and q'-regular nonlinearity satisfying (i)–(iv) from Theorem 2.2.3 together with its corresponding powers by $\mathrm{Sol}(v_0) \subset \mathscr{X}_{q,T}$. We write X_i for the i-th space in (2.2.3) and observe that if $\|v_0\|_{H^1(\mathbb{T}^2)} \leq n$ for some $n \in \mathbb{N}$ the a-priori bounds of Theorem 2.2.3 yield that

$$\mathrm{Sol}(v_0) \subset \mathscr{X}_{q,T}^{(n)} := \bigtimes_{i=1}^{4} B_{X_i}(r_{i,n})$$

for suitably chosen $r_{i,n}$. We equip each $B_{X_i}(r_{i,n})$ again with the weak (weak-*) topology of the respective space X_i and $\mathscr{X}_{q,T}^{(n)}$ with the resulting product topology. We note that each $B_{X_i}(r_{i,n})$ is metrizable by the separability of the (pre-) dual of X_i, see [84, Proposition 1.2.29, Corollary 1.3.22] and consequently also the topological product $\mathscr{X}_{q,T}^{(n)}$. Moreover, $\mathscr{X}_{q,T}^{(n)}$ is compact as a consequence of Tychonoff's and the Banach-Alaoglu theorem and therefore in particular a Polish space. Let $(v_{0,j})_{j\in\mathbb{N}}$ be a sequence in

$$\{v_0 \in H^1(\mathbb{T}^2) | v_0 \geq 0, \|v_0\|_{H^1(\mathbb{T}^2)} \leq n\}$$

converging to $v_{0,*}$ in $H^1(\mathbb{T}^2)$ and

$$(v_j, J_j, v_j^{\frac{\alpha+3}{2}}, v_j^{\frac{\alpha+3}{4}}) \in \mathrm{Sol}(v_{0,j}). \tag{2.2.5}$$

Then the measurable selection theorem as in [45, Corollary 103, p.506] yields a Borel-measurable solution map

$$\begin{aligned} \mathscr{S}_{\alpha,q,T}^{(n)}\colon \{v_0 \in H^1(\mathbb{T}^2) | v_0 \geq 0, \|v_0\|_{H^1(\mathbb{T}^2)} \leq n\} &\to \mathscr{X}_{q,T}^{(n)}, \\ v_0 \mapsto (v,J,v^{\frac{\alpha+3}{2}},v^{\frac{\alpha+3}{4}}) &\in \mathrm{Sol}(v_0), \end{aligned} \tag{2.2.6}$$

if we can verify that a subsequence of

$$(v_j, J_j, v_j^{\frac{\alpha+3}{2}}, v_j^{\frac{\alpha+3}{4}})_{j\in\mathbb{N}}$$

converges to an element of $\mathrm{Sol}(v_{0,*})$. Since (2.2.5) lies in $\mathscr{X}_{q,T}^{(n)}$, its components are uniformly bounded in (2.2.3). Therefore, we can apply Proposition 2.2.4 and obtain that

$$(v_j, J_j, v_j^{\frac{\alpha+3}{2}}, v_j^{\frac{\alpha+3}{4}}) \to (v, J, v^{\frac{\alpha+3}{2}}, v^{\frac{\alpha+3}{4}})$$

for a subsequence in $\mathscr{X}_{q,T}^{(n)}$, where (v, J) is a non-negative very weak solution to the thin-film equation with q'-regular nonlinearity. By [121, Corollary 5] we deduce that $v_j \to v$ in $C([0,T]; L^2(\mathbb{T}^2))$ and in particular $v_j(0) \to v(0)$ in $L^2(\mathbb{T}^2)$. Consequently we must have $v(0) = v_{0,*}$. By lower semi-continuity of the norm with respect to weak and weak-* convergence we deduce that (v, J) satisfies all the properties (i)–(iv) of Theorem 2.2.3 and therefore

$$(v, J, v^{\frac{\alpha+3}{2}}, v^{\frac{\alpha+3}{4}}) \in \mathrm{Sol}(v_{0,*}).$$

Hence, the measurable selection theorem indeed yields a Borel measurable map (2.2.6). Finally, we define $\mathscr{S}_{\alpha,q,T}\, v_0 = \mathscr{S}_{\alpha,q,T}^{(n)}\, v_0$ if $n-1 \le \|v_0\| < n$. Since balls in $H^1(\mathbb{T}^2)$ are Borel sets, $\mathscr{S}_{\alpha,q,T}$ has the desired properties. □

2.3. THE REGULARIZED LINEAR STRATONOVICH SPDE IN THE ENERGY SPACE

In this section we show that the regularized version of the stochastic part in (2.1.1)

$$\mathrm{d}w_t = \varepsilon\Delta w_t\,\mathrm{d}t + \mathrm{div}(w_t \circ \mathrm{d}W_t) \tag{2.3.1}$$

is well-posed using the variational approach to SPDEs [100, Chapter 4]. A key ingredient to checking the sufficient conditions for well-posedness is the spatial isotropy condition on the noise (2.1.4). Throughout this section, we fix a filtered probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ satisfying the usual conditions with a sequence of independent real-valued Brownian motions $(\beta^{(l)})_{l\in\mathbb{N}}$ and an $\varepsilon \in (0,1)$. The main statement of this section reads as follows.

Theorem 2.3.1. *Let $p \in [2,\infty)$, $T \in [0,\infty)$ and $w_0 \in L^p(\Omega; H^1(\mathbb{T}^2))$ be $\mathfrak{F}_0$-measurable. Then there exists a unique continuous, adapted $H^1(\mathbb{T}^2)$-valued process w such that $w \in L^2([0,T]\times\Omega; H^2(\mathbb{T}^2))$ and*

$$w(t) = w_0 + \int_0^t \varepsilon\Delta w(s) + \frac{1}{2}\sum_{l=1}^{\infty} \lambda_l^2\, \mathrm{div}(\mathrm{div}(w(s)\psi_l)\psi_l)\,\mathrm{d}s + \sum_{l=1}^{\infty} \lambda_l \int_0^t \mathrm{div}(w(s)\psi_l)\,\mathrm{d}\beta_s^{(l)} \tag{2.3.2}$$

for every $t \in [0,T]$. Moreover, w satisfies

$$\mathbb{E}\left[\sup_{0\le t\le T} \|w(t)\|_{H^1(\mathbb{T}^2)}^p\right] \lesssim_{p,T} \mathbb{E}\left[\|w_0\|_{H^1(\mathbb{T}^2)}^p\right], \tag{2.3.3}$$

almost surely we have

$$\int_{\mathbb{T}^2} w(t)\,\mathrm{d}x = \int_{\mathbb{T}^2} w_0\,\mathrm{d}x \tag{2.3.4}$$

and if $w_0 \ge 0$ also $w(t) \ge 0$ for all $t \in [0,T]$.

Remark 2.3.2. We convince ourselves that all the terms from (2.3.2) are well-defined. By (2.1.2) it holds

$$\sup_{|\alpha|\le 2} \sup_{k\in\mathbb{Z}^2} \|\partial_\alpha \xi_k\|_{L^\infty(\mathbb{T}^2)} < \infty. \tag{2.3.5}$$

and therefore we have

$$\|\operatorname{div}(\operatorname{div}(w\psi_l)\psi_l)\|_{L^2(\mathbb{T}^2)} \lesssim \|w\|_{H^2(\mathbb{T}^2)} \quad \text{and} \quad \|\operatorname{div}(w\psi_l)\|_{H^1(\mathbb{T}^2)} \lesssim \|w\|_{H^2(\mathbb{T}^2)}, \tag{2.3.6}$$

for every $w \in H^2(\mathbb{T}^2)$. Using the first estimate we derive that

$$\mathbb{E}\left[\int_0^T \left\|\varepsilon\Delta w(t) + \frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \operatorname{div}(\operatorname{div}(w\psi_l)\psi_l)\right\|_{L^2(\mathbb{T}^2)}^2 \mathrm{d}t\right] \lesssim_\Lambda \|w\|^2_{L^2([0,T]\times\Omega;H^2(\mathbb{T}^2))},$$

and consequently the deterministic integral in (2.3.2) exists almost surely as a Bochner integral in $L^2(\mathbb{T}^2)$. Using the second estimate from (2.3.6), one derives by the martingale moment inequality and Itô's isometry that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left\|\sum_{l=n}^m \lambda_l \int_0^t \operatorname{div}(w(s)\psi_l)\,\mathrm{d}\beta_s^{(l)}\right\|_{H^1(\mathbb{T}^2)}^2\right] \\
&\lesssim \mathbb{E}\left[\left\|\sum_{l=n}^m \lambda_l \int_0^T \operatorname{div}(w(t)\psi_l)\,\mathrm{d}\beta_t^{(l)}\right\|_{H^1(\mathbb{T}^2)}^2\right] \\
&= \mathbb{E}\left[\sum_{l=n}^m \lambda_l^2 \int_0^T \|\operatorname{div}(w(t)\psi_l)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}t\right] \\
&\lesssim \left(\sum_{l=n}^m \lambda_l^2\right)\|w\|^2_{L^2([0,T]\times\Omega;H^2(\mathbb{T}^2))}
\end{aligned}$$

and the latter part converges to 0 as $n, m \to \infty$. Therefore, the series of stochastic integrals in (2.3.2) converges to a continuous square-integrable martingale in $H^1(\mathbb{T}^2)$.

In order to treat the equation (2.3.2) within the variational setting [100, Chapter 4], we introduce the operators

$$\begin{aligned}
&A^\varepsilon\colon\ H^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2),\ w \mapsto \varepsilon\Delta w + \frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \operatorname{div}(\operatorname{div}(w\psi_l)\psi_l),\\
&B\colon\ H^2(\mathbb{T}^2) \to L_2(H^2(\mathbb{T}^2;\mathbb{R}^2), H^1(\mathbb{T}^2)),\ w \mapsto \left[v \mapsto \sum_{l=1}^\infty \lambda_l (v,\psi_l)_{H^2(\mathbb{T}^2;\mathbb{R}^2)} \operatorname{div}(w\psi_l)\right].
\end{aligned}$$

As in Remark 2.3.2 we conclude that the operators A^ε and B are well-defined, linear and bounded. In the following lemma we verify coercivity of (A^ε, B). Its proof is similar to [58, Lemma A.3], but nevertheless contained to stress the necessity of assumption (2.1.4).

Lemma 2.3.3. *There exists a constant $C_\Lambda < \infty$ such that*

$$2\langle A^\varepsilon w, w\rangle + \sum_{l=1}^\infty \|B(w)[\psi_l]\|^2_{L^2(\mathbb{T}^2)} \le C_\Lambda \|w\|^2_{L^2(\mathbb{T}^2)} - 2\varepsilon\|w\|^2_{H^1(\mathbb{T}^2)}, \tag{2.3.7}$$

$$2\langle \nabla A^\varepsilon w, \nabla w\rangle + \sum_{l=1}^{\infty} \left\| \nabla B(w)[\psi_l] \right\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \le C_\Lambda \|w\|^2_{H^1(\mathbb{T}^2)} - 2\varepsilon \|\nabla w\|^2_{H^1(\mathbb{T}^2;\mathbb{R}^2)} \tag{2.3.8}$$

for all $w \in H^2(\mathbb{T}^2)$.

2

Proof. By continuity of the involved operators, it suffices to verify (2.3.7) and (2.3.8) for $w \in C^\infty(\mathbb{T}^2)$. We first observe that

$$\begin{aligned}\langle A^\varepsilon w, w\rangle &= -\varepsilon \|\nabla w\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \frac{1}{2}\sum_{l=1}^{\infty} \lambda_l^2 \langle \operatorname{div}(w\psi_l)\psi_l, \nabla w\rangle \\ &= -\varepsilon \|\nabla w\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \frac{1}{2}\sum_{l=1}^{\infty} \lambda_l^2 \|\psi_l \cdot \nabla w\|^2_{L^2(\mathbb{T}^2)} + \frac{1}{4}\sum_{l=1}^{\infty} \lambda_l^2 \langle w^2, \operatorname{div}(\operatorname{div}(\psi_l)\psi_l)\rangle,\end{aligned}$$

where we have used the identity $\frac{1}{2}\nabla w^2 = w\nabla w$ in the second line. Utilizing the same identity again, we obtain

$$\begin{aligned}\left\| B(w)[\psi_l]\right\|^2_{L^2(\mathbb{T}^2)} &= \lambda_l^2 \|\operatorname{div}(w\psi_l)\|^2_{L^2(\mathbb{T}^2)} \\ &= \lambda_l^2 \left(\|\psi_l \cdot \nabla w\|^2_{L^2(\mathbb{T}^2)} + 2\langle w\nabla w, \operatorname{div}(\psi_l)\psi_l\rangle + \langle w^2, \operatorname{div}(\psi_l)^2\rangle \right) \\ &= \lambda_l^2 \left(\|\psi_l \cdot \nabla w\|^2_{L^2(\mathbb{T}^2)} - \langle w^2, \psi_l \cdot \nabla \operatorname{div}(\psi_l)\rangle \right).\end{aligned}$$

Considering the bound (2.3.5) we can calculate

$$\begin{aligned}&2\langle A^\varepsilon w, w\rangle + \sum_{l=1}^{\infty} \left\| B(w)[\psi_l]\right\|^2_{L^2(\mathbb{T}^2)} \\ &= -2\varepsilon \|\nabla w\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \frac{1}{2}\sum_{l=1}^{\infty} \lambda_l^2 \langle w^2, \operatorname{div}(\operatorname{div}(\psi_l)\psi_l) - \psi_l \cdot \nabla \operatorname{div}(\psi_l)\rangle \\ &\le C_\Lambda \|w\|^2_{L^2(\mathbb{T}^2)} - 2\varepsilon \|\nabla w\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\end{aligned}$$

for a suitable constant $C_\Lambda < \infty$. Enlarging C_Λ by 2 yields (2.3.7). For (2.3.8) we observe that

$$\langle \nabla A^\varepsilon w, \nabla w\rangle = -\frac{1}{2}\sum_{l=1}^{\infty} \lambda_l^2 \langle \operatorname{div}(\operatorname{div}(w\psi_l)\psi_l), \Delta w\rangle - \varepsilon \|\Delta w\|^2_{L^2(\mathbb{T}^2)}.$$

To further analyze the involved series, we set $\mu_k = \lambda_l$ in the situation of (2.1.4) and rewrite

$$-\frac{1}{2}\sum_{l=1}^{\infty} \lambda_l^2 \langle \operatorname{div}\cdot(\operatorname{div}(w\psi_l)\psi_l), \Delta w\rangle = -\frac{1}{2}\sum_{k\in\mathbb{Z}^2} \mu_k^2 \langle \operatorname{div}(\xi_k \nabla(w\xi_k)), \Delta w\rangle.$$

Before moving on, we notice that

$$\tilde{\xi}_j^2(x) \overset{(2.1.3)}{=} \begin{cases} 1+\cos(4\pi j x), & j<0, \\ 1, & j=0, \\ 1-\cos(4\pi j x), & j>0 \end{cases} \tag{2.3.9}$$

and therefore

$$\xi_k^2(x,y) = \frac{\tilde{\xi}_{k_1}^2(x)\,\tilde{\xi}_{k_2}^2(y)}{1+(2\pi|k|)^2+(2\pi|k|)^4}$$

yields the bound

$$\sup_{|\alpha|\le 4}\sup_{k\in\mathbb{Z}^2}\|\partial_\alpha \xi_k^2\|_{L^\infty(\mathbb{T}^2)}<\infty.$$

Using this estimate, the product rule for Δ, integration by parts, as well as the differential identities

$$\nabla(\nabla f\cdot\nabla g)=Hf\nabla g+Hg\nabla f \quad \text{and} \quad Hf\nabla f=\frac{1}{2}\nabla|\nabla f|^2, \tag{2.3.10}$$

we calculate

$$\begin{aligned}
\langle \operatorname{div}(\xi_k\nabla(w\xi_k)),\Delta w\rangle &= \langle \nabla\xi_k\cdot\nabla(w\xi_k)+\xi_k\Delta(w\xi_k),\Delta w\rangle\\
&= \langle \xi_k^2\Delta w+\frac{3}{2}\nabla w\cdot\nabla\xi_k^2+\frac{1}{2}w\Delta\xi_k^2,\Delta w\rangle\\
&= \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2-\frac{3}{2}\langle Hw\nabla\xi_k^2+H\xi_k^2\nabla w,\nabla w\rangle+\frac{1}{2}\langle \Delta\xi_k^2,\frac{1}{2}\Delta w^2-\nabla w\cdot\nabla w\rangle\\
&\ge \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2+\frac{3}{4}\langle \Delta\xi_k^2,|\nabla w|^2\rangle+\frac{1}{2}\langle \Delta^2\xi_k^2,w^2\rangle-C\|w\|_{H^1(\mathbb{T}^2)}^2\\
&\ge \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2-C\|w\|_{H^1(\mathbb{T}^2)}^2.
\end{aligned}$$

Here, we have enlarged the constant $C<\infty$ from the second last to the last line. Concerning the other summand in (2.3.8), we observe that by integration by parts

$$\|\nabla B(w)[(\xi_k,0)]\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2=\mu_k^2\|\nabla\partial_1(w\xi_k)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2=\mu_k^2\langle\Delta(w\xi_k),\partial_{11}(w\xi_k)\rangle.$$

Rewriting the expression $\|\nabla B(w)[(0,\xi_k)]\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2$ analogously yields that

$$\|\nabla B(w)[(\xi_k,0)]\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2+\|\nabla B(w)[(0,\xi_k)]\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2=\mu_k^2\|\Delta(w\xi_k)\|_{L^2(\mathbb{T}^2)}^2.$$

Using again the product rule for Δ, the bound (2.3.5), integration by parts and the formulas from (2.3.10), we can estimate the latter term by

$$\begin{aligned}
\|\Delta(w\xi_k)\|_{L^2(\mathbb{T}^2)}^2 &= \|\xi_k\Delta w+2\nabla w\cdot\nabla\xi_k+w\Delta\xi_k\|_{L^2(\mathbb{T}^2)}^2\\
&= \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2+\|2\nabla w\cdot\nabla\xi_k+w\Delta\xi_k\|_{L^2(\mathbb{T}^2)}^2+2\langle\xi_k\Delta w,2\nabla w\cdot\nabla\xi_k+w\Delta\xi_k\rangle\\
&\le \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2+C\|w\|_{H^1(\mathbb{T}^2)}^2-2\langle\nabla w,\nabla[\xi_k w\Delta\xi_k+2\xi_k\nabla w\cdot\nabla\xi_k]\rangle\\
&= \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2+C\|w\|_{H^1(\mathbb{T}^2)}^2-2\langle\nabla w,\xi_k\Delta\xi_k\nabla w+w\nabla[\xi_k\Delta\xi_k]\rangle\\
&\quad -4\langle\nabla w,(\nabla\xi_k\otimes\nabla\xi_k)\nabla w+\xi_k Hw\nabla\xi_k+\xi_k H\xi_k\nabla w\rangle\\
&\le \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2+C\|w\|_{H^1(\mathbb{T}^2)}^2+\langle w^2,\Delta(\xi_k\Delta\xi_k)\rangle+2\langle|\nabla w|^2,\operatorname{div}(\xi_k\nabla\xi_k)\rangle\\
&\le \|\xi_k\Delta w\|_{L^2(\mathbb{T}^2)}^2+C\|w\|_{H^1(\mathbb{T}^2)}^2.
\end{aligned}$$

We enlarged again the constant $C<\infty$ from line to line. Moreover, in the last line we have employed that

$$\|\Delta(\xi_k\Delta\xi_k)\|_{L^\infty(\mathbb{T}^2)}=(2\pi|k|)^2\|\Delta\xi_k^2\|_{L^\infty(\mathbb{T}^2)}\le\frac{2(2\pi|k|)^2(4\pi|k|)^2}{1+(2\pi|k|)^2+(2\pi|k|)^4}\le 8$$

by $\Delta\xi_k = -(2\pi|k|^2)\xi_k$ and (2.3.9). Combining all the previous estimates we finally obtain that

$$\begin{aligned}
&2\langle \nabla A^\varepsilon w, \nabla w\rangle + \sum_{l=1}^{\infty} \|\nabla B(w)[\psi_l]\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \\
&\quad\le -2\varepsilon\|\Delta w\|^2_{L^2(\mathbb{T}^2)} - \sum_{k\in\mathbb{Z}^2} \mu_k^2 \left[\|\xi_k \Delta w\|^2_{L^2(\mathbb{T}^2)} - C\|w\|^2_{H^1(\mathbb{T}^2)}\right] \\
&\qquad + \sum_{k\in\mathbb{Z}^2} \mu_k^2 \left[\|\xi_k \Delta w\|^2_{L^2(\mathbb{T}^2)} + C\|w\|^2_{H^1(\mathbb{T}^2)}\right] \\
&\quad\le C_\Lambda \|w\|^2_{H^1(\mathbb{T}^2)} - 2\varepsilon\|\Delta w\|^2_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

We arrive at (2.3.8) by enlarging C_Λ by 2. □

Proof of Theorem 2.3.1. The existence and uniqueness assertion follows, if we verify the assumptions of [100, Theorem 4.2.4] on the couple (A^ε, B) considered on the Gelfand triple

$$H^2(\mathbb{T}^2) \subset H^1(\mathbb{T}^2) \subset L^2(\mathbb{T}^2).$$

Here we equip $H^2(\mathbb{T}^2)$ with the equivalent Bessel-potential norm to ensure that the usual norm in $L^2(\mathbb{T}^2)$ coincides with the norm of the dual of $H^2(\mathbb{T}^2)$ under the pairing in $H^1(\mathbb{T}^2)$, for details see Appendix 2.B. Hemicontinuity and boundedness of A^ε follow from $A^\varepsilon \in L(H^2(\mathbb{T}^2), L^2(\mathbb{T}^2))$. Coercivity is obtained by adding (2.3.7) and (2.3.8) together. By linearity, coercivity implies weak monotonicity. The proof of (2.3.3) translates verbatim from the one-dimensional case [58, Proposition A.2] and (2.3.4) follows from testing (2.3.2) with $\mathbf{1}_{\mathbb{T}^2}$. The claim regarding non-negativity of w is a consequence of the maximum principle for second-order parabolic SPDEs [95, Theorem 4.3], which holds by analogous reasoning also on $\mathbb{T}^2$. □

2.4. TIME DISCRETIZATION SCHEME WITH DEGENERATE LIMIT

In this section we fix $N \in \mathbb{N}$. The goal of this section is to construct for a given end time T and an initial value u_0 a (very) weak martingale solution to the split-up problem

$$\begin{cases}
u(t) = v(2(t-j\delta)+j\delta), & j\delta \le t < (j+\frac{1}{2})\delta, \\
u(t) = w(2(t-(j+\frac{1}{2})\delta)+j\delta), & (j+\frac{1}{2}) \le t < (j+1)\delta, \\
\partial_t v = -\operatorname{div}(v^2\nabla\Delta v), & \text{on } [j\delta, (j+1)\delta), \\
\mathrm{d}w_t = \operatorname{div}(w_t \circ \mathrm{d}W_t), & \text{on } [j\delta, (j+1)\delta),
\end{cases}$$

where $\delta = \frac{T}{N+1}$ and $j \in \{0,\dots,N\}$. Starting at the initial value u_0 the process $u(t)$ satisfies alternately the deterministic thin-film equation and the purely stochastic equation (2.1.13) on time intervals of length $\frac{\delta}{2}$ and yields thus a time splitting scheme for the stochastic thin-film equation (2.1.1). During the construction we derive bounds which are uniform in N, and will be important in the final section, where we take the time step limit $N \to \infty$ to construct a solution to the original problem. We refer the interested reader for more information on the time-splitting procedure to [81]. The main statement of this section is the following.

Theorem 2.4.1. *Let $T \in (0,\infty)$, $q \in (2,\infty)$ and $\alpha \in (-1,0)$. We assume that u_0 is a non-negative random variable in $H^1(\mathbb{T}^2)$ and set $R^{(k)} = \{k-1 \le \|u_0\|_{H^1(\mathbb{T}^2)} < k\}$ and $u_0^{(k)} = \mathbf{1}_{R^{(k)}} u_0$ for every $k \in \mathbb{N}$. Then there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}}, \widetilde{\mathbb{P}})$ with a filtration $\widetilde{\mathfrak{F}}$ satisfying the usual conditions, a family of independent Brownian motions $(\widetilde{\beta}^{(l)})_{l\in\mathbb{N}}$, random variables $\mathbf{1}_{\widetilde{R}^{(k)}}$, $H^1_w(\mathbb{T}^2)$-continuous processes $\widetilde{u}^{(k)}$ and $L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$-valued random variables $\widetilde{J}^{(k)}$ for $k \in \mathbb{N}$, such that $\widetilde{u}^{(k)}$, $\widetilde{J}^{(k)}$ and the processes $\widetilde{v}^{(k)}$ and $\widetilde{w}^{(k)}$ defined by*

$$\begin{cases} \widetilde{u}^{(k)}(t) = \widetilde{v}^{(k)}(2(t-j\delta)+j\delta), & j\delta \le t < (j+\frac{1}{2})\delta, \\ \widetilde{u}^{(k)}(t) = \widetilde{w}^{(k)}(2(t-(j+\frac{1}{2})\delta)+j\delta), & (j+\frac{1}{2}) \le t < (j+1)\delta \end{cases}$$

satisfy the following.

(i) The sequence $(\mathbf{1}_{\widetilde{R}^{(k)}}, \widetilde{u}^{(k)}(0))_{k\in\mathbb{N}}$ has the same distribution as $(\mathbf{1}_{R^{(k)}}, u_0^{(k)})_{k\in\mathbb{N}}$, in particular we have $\sum_{k=1}^{\infty} \widetilde{u}^{(k)}(0) \sim u_0$. Moreover, $\widetilde{u}^{(k)}$ and $\widetilde{J}^{(k)}$ are $\widetilde{\mathbb{P}}$-almost surely zero outside of the set $\widetilde{R}^{(k)}$.

(ii) $\widetilde{u}^{(k)}(t)$ and $\widetilde{J}^{(k)}|_{[0,t]}$ are $\widetilde{\mathfrak{F}}_t$-measurable in $H^1(\mathbb{T}^2)$ and $L^2(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$ for every $t \in [0,T]$ and $k \in \mathbb{N}$.

(iii) The tuples $(\widetilde{v}^{(k)}, \widetilde{J}^{(k)})$ are $\widetilde{\mathbb{P}}$-almost surely solutions to the deterministic thin-film equation on $[j\delta,(j+1)\delta)$ satisfying property (iv) from Theorem 2.2.3 with initial value $\widetilde{u}^{(k)}(j\delta)$ for every $j = 0,\dots,N$.

(iv) For $k \in \mathbb{N}$, $\varphi \in H^1(\mathbb{T}^2)$ and $t \in [j\delta,(j+1)\delta)$ we have that

$$\begin{aligned} \langle \widetilde{w}^{(k)}(t),\varphi\rangle - \langle \widetilde{w}^{(k)}(j\delta),\varphi\rangle = &\frac{1}{2}\sum_{l=1}^{\infty}\lambda_l^2 \int_{j\delta}^{t} \langle \operatorname{div}(\operatorname{div}(\widetilde{w}^{(k)}(s)\psi_l)\psi_l),\varphi\rangle \,\mathrm{d}s \\ &+ \sum_{l=1}^{\infty}\lambda_l \int_{j\delta}^{t} \langle \operatorname{div}(\widetilde{w}^{(k)}(s)\psi_l),\varphi\rangle \,\mathrm{d}\beta_s^{(l)}. \end{aligned}$$

(v) For every $k \in \mathbb{N}$, $p \in (0,\infty)$ we have

$$\widetilde{\mathbb{E}}\left[\sup_{0\le t\le T} \|\widetilde{u}^{(k)}\|_{H^1(\mathbb{T}^2)}^p\right] \lesssim_{\Lambda,p,T} \mathbb{E}\left[\|u_0^{(k)}\|_{H^1(\mathbb{T}^2)}^p\right],$$
$$\widetilde{\mathbb{E}}\left[\|\widetilde{J}^{(k)}\|_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}^{\frac{p}{2}}\right] \lesssim_{\Lambda,p,q,T} \mathbb{E}\left[\|u_0^{(k)}\|_{H^1(\mathbb{T}^2)}^p\right].$$

(vi) Moreover, for any $\gamma \in (0,\frac{1}{2})$ and $K \in (1,\infty)$ it holds

$$\widetilde{\mathbb{P}}\left(\left\{\left[\widetilde{u}^{(k)}\right]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \lesssim_{\Lambda,q,\gamma,T} \frac{1+\mathbb{E}\left[\|u_0^{(k)}\|_{H^1(\mathbb{T}^2)}^2\right]}{K}.$$

2.4.1. CONSTRUCTION AND ANALYSIS OF A REGULARIZED SCHEME

Let $u_0 \in L^\infty(\Omega; H^1(\mathbb{T}^2))$ be non-negative. Up to extension and completion of the probability space we can assume that there exists a filtration $\mathfrak{F}$ satisfying the usual conditions with a family of independent Brownian motions $(\beta^{(l)})_{l\in\mathbb{N}}$ such that u_0 is $\mathfrak{F}_0$-measurable.

Remark 2.4.2. The construction with initial value u_0 within this subsection will in Subsection 2.4.2 be applied to each of the cut-off parts $u_0^{(k)}$ from Theorem 2.4.1. This justifies the strong assumption $u_0 \in L^\infty(\Omega; H^1(\mathbb{T}^2))$ here.

We fix for the rest of this subsection also $T \in (0,\infty)$, $q \in (2,\infty)$, $\alpha \in (-1,0)$, $\varepsilon \in (0,1)$ and apply the operator $\mathscr{S}_{\alpha,q,\delta}$ from Corollary 2.2.5 to the initial value u_0. We define $v_\varepsilon|_{[0,\delta)}, J_\varepsilon|_{[0,\delta)}$ as the version of the solution which is in $C([0,\delta]; L^2(\mathbb{T}^2))$ and in particular continuous in $H^1_w(\mathbb{T}^2)$, see Remark 2.2.2. Moreover, we define $w_\varepsilon|_{[0,\delta)}$ as the solution to (2.3.2) with initial value $\lim_{t\nearrow\delta} v_\varepsilon(t)$. Notice that since $v_\varepsilon|_{[0,\delta)}$ fulfills the properties (i) and (ii) of Theorem 2.2.3, we have

$$\mathbb{E}\left[\|\lim_{t\nearrow\delta} v_\varepsilon\|^p_{H^1(\mathbb{T}^2)}\right] \lesssim \mathbb{E}\left[\|u_0\|^p_{H^1(\mathbb{T}^2)}\right]$$

for any $p \in [2,\infty)$, and therefore Theorem 2.3.1 is indeed applicable and yields a non-negative solution $w_\varepsilon|_{[0,\delta)}$. In particular, the terminal value $\lim_{t\nearrow\delta} w_\varepsilon(t)$ lies again in $L^p(\Omega; H^1(\mathbb{T}^2))$. We repeat this and obtain inductively very weak solutions $v|_{[j\delta,(j+1)\delta)}$ to (2.2.1) and variational solutions $w_\varepsilon|_{[j\delta,(j+1)\delta)}$ to (2.3.1) for $j \in \{1,\dots,N\}$. Finally, we define the $H^1_w(\mathbb{T}^2)$-continuous, adapted process

$$u_\varepsilon(t) = \begin{cases} v_\varepsilon(2(t-j\delta)+j\delta), & j\delta \le t < (j+\frac{1}{2})\delta, \\ w_\varepsilon(2(t-(j+\frac{1}{2})\delta)+j\delta), & (j+\frac{1}{2}) \le t < (j+1)\delta. \end{cases}$$

for $t \in [0,T)$. We note that we set for the final time $u_\varepsilon(T) = \lim_{t\nearrow\delta} w_\varepsilon(t)$. The divergence form of (2.2.1), (2.3.1), and an application of Itô's formula yield the following estimates along the whole time-splitting scheme.

Lemma 2.4.3. *It holds almost surely that*

$$\int_{\mathbb{T}^2} u_\varepsilon(t)\,\mathrm{d}x = \int_{\mathbb{T}^2} u_0\,\mathrm{d}x. \tag{2.4.1}$$

for all $t \in [0,T]$. Moreover, we have additionally

$$\mathbb{E}\left[\sup_{0\le t\le T}\|u_\varepsilon\|^p_{H^1(\mathbb{T}^2)}\right] \lesssim_{\Lambda,p,T} \mathbb{E}\left[\|u_0\|^p_{H^1(\mathbb{T}^2)}\right] \tag{2.4.2}$$

for $p \in (0,\infty)$.

Proof. The equality (2.4.1) follows from its respective counterparts from Theorem 2.2.3 (i) and (2.3.4). Next, we apply Itô's formula to the composition of $\|\nabla\cdot\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}$ with the process w_ε, which yields that

$$\begin{aligned} \|\nabla w_\varepsilon(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} &= \|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + 2\int_{j\delta}^t \langle \nabla w_\varepsilon(s), \nabla A^\varepsilon(w_\varepsilon(s))\rangle\,\mathrm{d}s \\ &+ \sum_{l=1}^\infty \lambda_l \int_{j\delta}^t 2\langle \nabla\operatorname{div}(w_\varepsilon(s)\psi_l), \nabla w_\varepsilon(s)\rangle\,\mathrm{d}\beta_s^l + \sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \|\nabla\operatorname{div}(w_\varepsilon(s)\psi_l)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\,\mathrm{d}s. \end{aligned} \tag{2.4.3}$$

for $t \in [j\delta,(j+1)\delta)$. A justification of the applicability of Itô's formula is given in Appendix 2.C. As pointed out in (2.C.6), the martingale given by the series of stochastic integrals, which we denote by $M_{2,j}$, has quadratic variation

$$4\sum_{l=1}^{\infty}\lambda_l^2\int_{j\delta}^{t}\langle\nabla\operatorname{div}(w_\varepsilon(s)\psi_l),\nabla w_\varepsilon(s)\rangle^2\,\mathrm{d}s.$$

Combining (2.4.3) with (2.3.8) we conclude that

$$\|\nabla w_\varepsilon(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}-\|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}-M_{2,j}(t)\lesssim_\Lambda\int_{j\delta}^{t}\|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s$$

and for the endpoint $t=(j+1)\delta$

$$\begin{aligned}&\|\nabla v_\varepsilon((j+1)\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}-\|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}-M_{2,j}((j+1)\delta))\\&\lesssim_\Lambda\int_{j\delta}^{(j+1)\delta}\|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s.\end{aligned}\tag{2.4.4}$$

By Theorem 2.2.3 (ii) we have

$$\|\nabla w_{N,\varepsilon}(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\le\|\nabla v_{N,\varepsilon}(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)},$$

so that a telescoping sum argument yields

$$\|\nabla w_\varepsilon(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}-\|\nabla u_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}-M_2(t)\lesssim_\Lambda\int_0^t\|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s\tag{2.4.5}$$

for $t\in[0,T]$. The appearing process M_2 is defined by the sum of orthogonal martingales

$$M_2(t)=\sum_{k=0}^{j-1}M_{2,k}((k+1)\delta)+M_{2,j}(t),\quad t\in[j\delta,(j+1)\delta)$$

and has therefore quadratic variation

$$4\sum_{l=1}^{\infty}\lambda_l^2\int_0^t\langle\nabla\operatorname{div}(w_\varepsilon(s)\psi_l),\nabla w_\varepsilon(s)\rangle^2\,\mathrm{d}s.$$

For $p\ge 2$ we deduce from (2.4.5) with help of the inequality $(a+b+c)^{\frac{p}{2}}\lesssim_p a^{\frac{p}{2}}+b^{\frac{p}{2}}+c^{\frac{p}{2}}$ and the Burkholder–Davis–Gundy inequality that

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le s\le t}\|\nabla w_\varepsilon(s)\|^p_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right]-C_p\mathbb{E}\left[\|\nabla u_0\|^p_{L^2(\mathbb{T}^2;\mathbb{R})}\right]\\&\lesssim_{\Lambda,p}\mathbb{E}\left[\left(\int_0^t\|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_0^t\langle\nabla\operatorname{div}(w_\varepsilon(s)\psi_l),\nabla w_\varepsilon(s)\rangle^2\,\mathrm{d}s\right)^{\frac{p}{4}}\right].\end{aligned}\tag{2.4.6}$$

To estimate the latter expression we observe that

$$\nabla\operatorname{div}(w\psi_l)=Hw\psi_l+D\psi_l\nabla w+w\nabla\operatorname{div}\psi_l+\operatorname{div}(\psi_l)\nabla w$$

and due to (2.3.5) and (2.3.10) consequently

$$\big|\langle \nabla \operatorname{div}(w\psi_l), \nabla w\rangle\big| \lesssim \|\nabla w\|_{L^2(\mathbb{T}^2;\mathbb{R})}\|w\|_{H^1(\mathbb{T}^2)}, \tag{2.4.7}$$

for $w \in H^2(\mathbb{T}^2)$. We conclude with help of Young's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \langle \nabla \operatorname{div}(w_\varepsilon(s)\psi_l), \nabla w_\varepsilon(s)\rangle^2 \,\mathrm{d}s\right)^{\frac{p}{4}}\right] \\
&\lesssim \mathbb{E}\left[\sup_{0\le s\le t} \|\nabla w_\varepsilon(s)\|^{\frac{p}{2}}_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\int_0^t \|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s\right)^{\frac{p}{4}}\right] \\
&\le \frac{\kappa}{2}\mathbb{E}\left[\sup_{0\le s\le t} \|\nabla w_\varepsilon(s)\|^{p}_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right] + \frac{1}{2\kappa}\mathbb{E}\left[\left(\int_0^t \|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s\right)^{\frac{p}{2}}\right]
\end{aligned}$$

for any $\kappa > 0$. An appropriate choice of κ and (2.4.6) yield that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le t} \|\nabla w_\varepsilon(s)\|^{p}_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right] - C_p\mathbb{E}\left[\|\nabla u_0\|^p_{L^2(\mathbb{T}^2;\mathbb{R})}\right] \\
&\lesssim_{\Lambda,p} \mathbb{E}\left[\left(\int_0^t \|w_\varepsilon(s)\|^2_{H^1(\mathbb{T}^2)}\,\mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\lesssim_{p,T} \mathbb{E}\left[\int_0^t \|w_\varepsilon(s)\|^p_{H^1(\mathbb{T}^2)}\,\mathrm{d}s\right] \lesssim_p \mathbb{E}\left[\left(\int_0^t \left(\int_{\mathbb{T}^2} u_0 \,\mathrm{d}x\right)^p + \|\nabla w_\varepsilon(s)\|^p_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\,\mathrm{d}s\right)\right] \\
&\lesssim_T \mathbb{E}\left[\|u_0\|^p_{L^2(\mathbb{T}^2)}\right] + \int_0^t \mathbb{E}\left[\sup_{0\le \tau\le s} \|\nabla w_\varepsilon(\tau)\|^p_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right] \mathrm{d}s.
\end{aligned}$$

We additionally employed Jensen's and the Poincaré inequality here. Because of $u_0 \in L^p(\Omega; H^1(\mathbb{T}^2))$, the monotone function

$$t \longmapsto \mathbb{E}\left[\sup_{0\le s\le t} \|\nabla w_\varepsilon(s)\|^p_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right]$$

takes finite values by (2.3.3) and Theorem 2.2.3 (i), (ii) and therefore an application of Grönwall's inequality yields

$$\mathbb{E}\left[\sup_{0\le s\le T} \|\nabla w_\varepsilon(s)\|^p_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right] \lesssim_{\Lambda,p,T} \mathbb{E}\left[\|u_0\|^p_{H^1(\mathbb{T}^2)}\right] \tag{2.4.8}$$

In order to obtain the above inequality also for $p \in (0,2)$ we observe that $\mathbf{1}_R u_\varepsilon$ coincides with the process $u_{\varepsilon,R}$ obtained by constructing the splitting scheme with initial value $\mathbf{1}_R u_0$ for $R \in \mathfrak{F}_0$. Indeed, from the properties in Theorem 2.2.3 (i) we conclude that $\mathscr{S}_{\alpha,q,\delta}$ maps 0 to the solution which is 0 for all times. Consequently, we have

$$v_{\varepsilon,R}|_{[0,\delta)} = \mathbf{1}_R v_\varepsilon|_{[0,\delta)} \quad \text{and} \quad w_{\varepsilon,R}(0) = \mathbf{1}_R w_\varepsilon(0).$$

Therefore $w_{\varepsilon,R}|_{[0,\delta)}$ and $\mathbf{1}_R w_\varepsilon|_{[0,\delta)}$ are both solutions to (2.3.2) and have the same initial value so that $w_\varepsilon^{(R)}|_{[0,\delta)} = \mathbf{1}_R w_\varepsilon|_{[0,\delta)}$. It is left to apply the uniqueness statement from Theorem 2.3.1 and repeat these arguments on $[j\delta, (j+1)\delta)$ for $j = 1, \dots N$. Hence applying

(2.4.8) to $w_{\varepsilon,R}$ with exponent $p = 2$ yields

$$\mathbb{E}\left[\mathbf{1}_R \sup_{0\le s\le T} \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right] \lesssim_{\Lambda,T} \mathbb{E}\left[\mathbf{1}_R\|u_0\|^2_{H^1(\mathbb{T}^2)}\right].$$

Since $R \in \mathfrak{F}_0$ was arbitrary, it follows that

$$\mathbb{E}\left[\sup_{0\le s\le T} \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\middle|\mathfrak{F}_0\right] \lesssim_{\Lambda,T} \|u_0\|^2_{H^1(\mathbb{T}^2)}.$$

For $p \in (0,2)$ we can use Jensen's inequality to deduce that

$$\mathbb{E}\left[\sup_{0\le s\le T} \|\nabla w_\varepsilon(s)\|^p_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\middle|\mathfrak{F}_0\right] \le \mathbb{E}\left[\sup_{0\le s\le T} \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\middle|\mathfrak{F}_0\right]^{\frac{p}{2}} \lesssim_{\Lambda,T} \|u_0\|^p_{H^1(\mathbb{T}^2)}$$

and it is left to take the expectation. Finally, we use Theorem 2.2.3 (ii) to obtain (2.4.8) with w_ε replaced by u_ε which together with (2.4.1) implies (2.4.2). □

Lemma 2.4.4. *We have*

$$\mathbb{E}\left[\|J_\varepsilon\|^{\frac{p}{2}}_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}\right] \lesssim_{\Lambda,p,q,T} \mathbb{E}\left[\|u_0\|^p_{H^1(\mathbb{T}^2)}\right]$$

for $p \in (0,\infty)$.

Proof. We observe that as a consequence of Theorem 2.2.3 (iii) and (2.4.1)

$$\begin{aligned}\|J_\varepsilon\|^2_{L^2(j\delta,(j+1)\delta;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))} \lesssim_q{}& \|\nabla v_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \\ &+ \left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 \left(\|\nabla v_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right).\end{aligned}$$

Using that

$$\|J_\varepsilon\|^2_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))} = \sum_{j=0}^N \|J_\varepsilon\|^2_{L^2(j\delta,(j+1)\delta;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}$$

we obtain the bound

$$\begin{aligned}\|J_\varepsilon\|^2_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))} \lesssim_q{}& \sum_{j=0}^N \|\nabla v_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \\ &+ \sum_{j=0}^N \left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 \left(\|\nabla v_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right).\end{aligned}$$

Applying the $\frac{p}{4}$-th power and using that $(a_1+\dots+a_4)^{\frac{p}{4}} \lesssim_p a_1^{\frac{p}{4}}+\dots+a_4^{\frac{p}{4}}$ we conclude

$$
\begin{aligned}
&\|J_\varepsilon\|^{\frac{p}{2}}_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}\\
&\lesssim_{p,q} \left|0 \vee \left(\|\nabla u_0\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(T-\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right)\right|^{\frac{p}{4}}\\
&\quad + \left|\sum_{j=0}^{N-1} \|\nabla v_\varepsilon((j+1)\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right|^{\frac{p}{4}}\\
&\quad + \left(\int_{\mathbb{T}^2} u_0 \,\mathrm{d}x\right)^{\frac{p}{2}} \left|0 \vee \left(\|\nabla u_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(T-\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right)\right|^{\frac{p}{4}}\\
&\quad + \left|\left(\int_{\mathbb{T}^2} u_0 \,\mathrm{d}x\right)^2 \sum_{j=0}^{N-1} \|\nabla v_\varepsilon((j+1)\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right|^{\frac{p}{4}}.
\end{aligned} \tag{2.4.9}
$$

The expectation of the first and the third summand of the right-hand side of (2.4.9) can be each estimated by $\mathbb{E}\left[\|u_0\|^p_{H^1(\mathbb{T}^2)}\right]$. To control also the second term we apply Itô's formula, see, e.g., [90, Theorem 15.19], to the composition of $(\cdot)^2$ with the real-valued semimartingale (2.4.3) and obtain that

$$
\begin{aligned}
\|\nabla w_\varepsilon(t)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} &= \|\nabla w_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R})} + 2\int_{j\delta}^t \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \,\mathrm{d}M_{2,j}(s)\\
&\quad + 4\int_{j\delta}^t \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \langle \nabla w_\varepsilon(s), \nabla A^\varepsilon(w_\varepsilon(s))\rangle \,\mathrm{d}s\\
&\quad + 2\int_{j\delta}^t \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \|\nabla \operatorname{div}(w_\varepsilon(s)\psi_l)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \,\mathrm{d}s\\
&\quad + 4\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \langle \nabla \operatorname{div}(w_\varepsilon(s)\psi_l), \nabla w_\varepsilon(s)\rangle^2 \,\mathrm{d}s.
\end{aligned}
$$

Using (2.3.8), (2.4.7) we conclude for the endpoint $t=(j+1)\delta$ that

$$
\begin{aligned}
&\|\nabla v_\varepsilon((j+1)\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R})} - 2\int_{j\delta}^{(j+1)\delta} \|\nabla w_\varepsilon(s)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \,\mathrm{d}M_{2,j}(s)\\
&\lesssim_\Lambda \int_{j\delta}^{(j+1)\delta} \|w_\varepsilon(s)\|^4_{H^1(\mathbb{T}^2)} \,\mathrm{d}s
\end{aligned}
$$

Summing up over j, taking the $\frac{p}{4}$-power, applying the Burkholder–Davis–Gundy inequality, as well as (2.4.2) and (2.4.7) yields

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{j=0}^{N-1} \|\nabla v_\varepsilon((j+1)\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)} - \|\nabla w_\varepsilon(j\delta)\|^4_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\right|^{\frac{p}{4}}\right]\\
&\quad \lesssim_{\Lambda,p} \mathbb{E}\left[\left(\int_0^{T-\delta} \|w_\varepsilon(s)\|^8_{H^1(\mathbb{T}^2)} \,\mathrm{d}s\right)^{\frac{p}{8}} + \left(\int_0^{T-\delta} \|w_\varepsilon(s)\|^4_{H^1(\mathbb{T}^2)} \,\mathrm{d}s\right)^{\frac{p}{4}}\right]
\end{aligned}
$$

$$\lesssim_{p,T} \mathbb{E}\left[\sup_{0\le t\le T} \|w_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^p\right] \lesssim_{\Lambda,p,T} \mathbb{E}\left[\|u_0\|_{H^1(\mathbb{T}^2)}^p\right].$$

Similarly, by summing up over j in (2.4.4) and taking the power $\frac{p}{4}$ we obtain

$$\begin{aligned}
&\left|\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 \sum_{j=0}^{N-1} \|\nabla v_\varepsilon((j+1)\delta)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2 - \|\nabla w_\varepsilon(j\delta)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2\right|^{\frac{p}{4}} \\
&\lesssim_{\Lambda,p} \left|\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 M_2(T-\delta)\right|^{\frac{p}{4}} + \left(\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 \int_0^{T-\delta} \|w_\varepsilon(s)\|_{H^1(\mathbb{T}^2)}^2\,\mathrm{d}s\right)^{\frac{p}{4}}
\end{aligned}$$

Taking the expectation, using the Burkholder–Davis–Gundy inequality, (2.4.2) and (2.4.7) yields the estimate

$$\begin{aligned}
&\mathbb{E}\left[\left|\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 \sum_{j=0}^{N-1} \|\nabla v_\varepsilon((j+1)\delta)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2 - \|\nabla w_\varepsilon(j\delta)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}^2\right|^{\frac{p}{4}}\right] \\
&\lesssim_{\Lambda,p} \mathbb{E}\left[\left(\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^4 \int_0^{T-\delta} \|w_\varepsilon(s)\|_{H^1(\mathbb{T}^2)}^4\,\mathrm{d}s\right)^{\frac{p}{8}}\right. \\
&\qquad \left. + \left(\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^2 \int_0^{T-\delta} \|w_\varepsilon(s)\|_{H^1(\mathbb{T}^2)}^2\,\mathrm{d}s\right)^{\frac{p}{4}}\right] \\
&\lesssim_{p,T} \sqrt{\mathbb{E}\left[\left(\int_{\mathbb{T}^2} u_0\,\mathrm{d}x\right)^p\right] \mathbb{E}\left[\sup_{0\le t\le T} \|w_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^p\right]} \lesssim_{\Lambda,p,T} \mathbb{E}\left[\|u_0\|_{H^1(\mathbb{T}^2)}^p\right].
\end{aligned}$$

Finally, taking the expectation of (2.4.9) and using the estimates on the individual summands yields the claim. □

We also show tail estimates of the powers of v_ε in their respective space. We note that the obtained bound depends on N and will therefore be improved to a bound, which is uniform in N after letting $\varepsilon \searrow 0$.

Lemma 2.4.5. *We have for $K \in (1,\infty)$ the estimate*

$$\begin{aligned}
&\mathbb{P}\left(\left\{\left\|v_\varepsilon^{\frac{\alpha+3}{2}}\right\|_{L^2(0,T;H^2(\mathbb{T}^2))}^2 + \left\|v_\varepsilon^{\frac{\alpha+3}{4}}\right\|_{L^4(0,T;W^{1,4}(\mathbb{T}^2))}^4 > K\right\}\right) \\
&\lesssim_{\Lambda,\alpha,T,N} \frac{\mathbb{E}\left[\|u_0\|_{H^1(\mathbb{T}^2)}^{\alpha+1}\right] + \mathbb{E}\left[\|u_0\|_{H^1(\mathbb{T}^2)}^2\right]}{K^{\frac{2}{\alpha+3}}}.
\end{aligned}$$

Proof. As a consequence of Theorem 2.2.3 (iv), (2.1.14) and Hölder's inequality we have

$$\begin{aligned}
&\int_{j\delta}^{(j+1)\delta} \int_{\mathbb{T}^2} |Hv_\varepsilon^{\frac{\alpha+3}{2}}|^2 + |\nabla v_\varepsilon^{\frac{\alpha+3}{4}}|^4\,\mathrm{d}x\,\mathrm{d}t \lesssim \int_{\mathbb{T}^2} G_\alpha(v_\varepsilon(j\delta)) - G_\alpha(w_\varepsilon(j\delta))\,\mathrm{d}x \\
&\lesssim_\alpha \|v_\varepsilon(j\delta)\|_{L^2(\mathbb{T}^2)}^{\alpha+1} + \|v_\varepsilon(j\delta)\|_{L^2(\mathbb{T}^2)} + \|w_\varepsilon(j\delta)\|_{L^2(\mathbb{T}^2)}^{\alpha+1} + \|w_\varepsilon(j\delta)\|_{L^2(\mathbb{T}^2)}.
\end{aligned}$$

2

Summing up over j and taking the expectation yields that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^2}|Hv_\varepsilon^{\frac{\alpha+3}{2}}|^2+|\nabla v_\varepsilon^{\frac{\alpha+3}{4}}|^4\,\mathrm{d}x\,\mathrm{d}t\right]\\&\quad\lesssim_{\alpha,N}\mathbb{E}\left[\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^{\alpha+1}+\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}\right].\end{aligned}\tag{2.4.10}$$

Moreover, by the Sobolev embedding theorem we have the estimate

$$\int_0^T\int_{\mathbb{T}^2}\left|v_\varepsilon^{\frac{\alpha+3}{2}}(t)\right|^2\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_{\mathbb{T}^2}\left|v_\varepsilon^{\frac{\alpha+3}{4}}(t)\right|^4\mathrm{d}x\,\mathrm{d}t\lesssim_\alpha\int_0^T\|v_\varepsilon\|_{H^1(\mathbb{T}^2)}^{\alpha+3}\,\mathrm{d}t$$

which implies by taking the $\frac{2}{\alpha+3}$-th power and the expectation that

$$\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{T}^2}\left|v_\varepsilon^{\frac{\alpha+3}{2}}(t)\right|^2+\left|v_\varepsilon^{\frac{\alpha+3}{4}}(t)\right|^4\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{\alpha+3}}\right]\lesssim_{\alpha,T}\mathbb{E}\left[\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^2\right].\tag{2.4.11}$$

Combining (2.4.10) and (2.4.11) with Chebyshev's inequality yields respectively that

$$\begin{aligned}&\mathbb{P}\left(\left\{\int_0^T\int_{\mathbb{T}^2}|Hv_\varepsilon^{\frac{\alpha+3}{2}}|^2+|\nabla v_\varepsilon^{\frac{\alpha+3}{4}}|^4\,\mathrm{d}x\,\mathrm{d}t>K\right\}\right)\\&\quad\lesssim_{\alpha,N}\frac{1}{K}\mathbb{E}\left[\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^{\alpha+1}+\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}\right],\\&\mathbb{P}\left(\left\{\int_0^T\int_{\mathbb{T}^2}\left|v_\varepsilon^{\frac{\alpha+3}{2}}(t)\right|^2+\left|v_\varepsilon^{\frac{\alpha+3}{4}}(t)\right|^4\mathrm{d}x\,\mathrm{d}t>K\right\}\right)\\&\quad\lesssim_{\alpha,T}\frac{1}{K^{\frac{2}{\alpha+3}}}\mathbb{E}\left[\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^2\right].\end{aligned}$$

Combining these estimates, the assumption $K\in(1,\infty)$ and the interpolation inequality

$$\|f\|_{H^2(\mathbb{T}^2)}^2\lesssim\int_{\mathbb{T}^2}|f|^2+|Hf|^2\,\mathrm{d}x,\quad f\in H^2(\mathbb{T}^2),\tag{2.4.12}$$

we obtain that

$$\begin{aligned}&\mathbb{P}\left(\left\{\left\|v_\varepsilon^{\frac{\alpha+3}{2}}\right\|_{L^2(0,T;H^2(\mathbb{T}^2))}^2+\left\|v_\varepsilon^{\frac{\alpha+3}{4}}\right\|_{L^4(0,T;W^{1,4}(\mathbb{T}^2))}^4>K\right\}\right)\\&\quad\lesssim_{\alpha,T,N}\frac{1}{K^{\frac{2}{\alpha+3}}}\mathbb{E}\left[\sup_{0\le t\le T}\|w_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^{\alpha+1}+\sup_{0\le t\le T}\|w_\varepsilon(t)\|_{H^1(\mathbb{T}^2)}^2\right].\end{aligned}$$

It is left to apply (2.4.2) to conclude the claim. □

In the final part of the analysis of the approximate scheme, we show Hölder regularity in time of u_ε.

Lemma 2.4.6. *Let* $\gamma \in (0, \frac{1}{2})$ *and* $K \in (1, \infty)$, *then*

$$\mathbb{P}\left(\left\{[u_\varepsilon]_{\gamma, W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \lesssim_{\Lambda,q,\gamma,T} \frac{1 + \mathbb{E}\left[\|u_0\|^2_{H^1(\mathbb{T}^2)}\right]}{K}. \tag{2.4.13}$$

Proof. We divide the proof into three steps.

Step 1 (Deterministic integrals). By Hölder's inequality we have

$$\begin{aligned}\left\|\int_s^t \operatorname{div} J_\varepsilon(\tau)\,\mathrm{d}\tau\right\|_{W^{-1,q'}(\mathbb{T}^2)} &\le \int_s^t \|\operatorname{div} J_\varepsilon(\tau)\|_{W^{-1,q'}(\mathbb{T}^2)}\,\mathrm{d}\tau \\ &\lesssim |t-s|^{\frac{1}{2}} \|J_\varepsilon\|_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}\end{aligned}$$

for any $s, t \in [0, T]$. Analogously, using that A^ε maps $H^1(\mathbb{T}^2)$ continuously to $H^{-1}(\mathbb{T}^2)$ due to (2.3.5) (with operator norm depending solely on Λ) we obtain that

$$\begin{aligned}\left\|\int_s^t A^\varepsilon w_\varepsilon(\tau)\,\mathrm{d}\tau\right\|_{H^{-1}(\mathbb{T}^2)} &\le \int_s^t \left\|A^\varepsilon w_\varepsilon(\tau)\right\|_{H^{-1}(\mathbb{T}^2)}\,\mathrm{d}\tau \\ &\lesssim_\Lambda |t-s| \sup_{0\le\tau\le T} \|w_\varepsilon(\tau)\|_{H^1(\mathbb{T}^2)}.\end{aligned}$$

From the above inequalities, Lemma 2.4.4, and (2.4.2), we deduce that

$$\begin{aligned}\mathbb{P}\left(\left\{\left[\int_0^\cdot \operatorname{div} J_\varepsilon(s)\,\mathrm{d}s\right]_{\frac{1}{2}, W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) &\lesssim_{\Lambda,q,T} \frac{\mathbb{E}\left[\|u_0\|^2_{H^1(\mathbb{T}^2)}\right]}{K}, \\ \mathbb{P}\left(\left\{\left[\int_0^\cdot A^\varepsilon w_\varepsilon(s)\,\mathrm{d}s\right]_{1, H^{-1}(\mathbb{T}^2)} > K\right\}\right) &\lesssim_{\Lambda,T} \frac{\mathbb{E}\left[\|u_0\|^2_{H^1(\mathbb{T}^2)}\right]}{K^2}.\end{aligned} \tag{2.4.14}$$

Step 2 (Stochastic integral). By [110, Theorem 3.2] we conclude that for

$$\begin{aligned}\mathbb{P}\Bigg(&\left\{\left[\sum_{l=1}^\infty \lambda_l \int_0^\cdot \operatorname{div}(w_\varepsilon(s)\psi_l)\,\mathrm{d}\beta_s^{(l)}\right]_{\gamma, L^2(\mathbb{T}^2)} > K\right\} \\ &\bigcap \left\{\sup_{0\le t\le T} \|B(w_\varepsilon(t))\|_{L_2(H^2(\mathbb{T}^2;\mathbb{R}^2), L^2(\mathbb{T}^2))} \le \sqrt{K}\right\}\Bigg)\end{aligned}$$

is dominated by $2e^{-C_{\gamma,T}K}$, where $C_{\gamma,T} \in (0, \infty)$ is a suitable constant. We observe that due to (2.3.5), B maps continuously from $H^1(\mathbb{T}^2)$ to $L_2(H^2(\mathbb{T}^2;\mathbb{R}^2), L^2(\mathbb{T}^2))$ with operator norm only depending on Λ. Using additionally (2.4.2), we obtain the estimate

$$\begin{aligned}&\mathbb{P}\left(\left\{\left[\sum_{l=1}^\infty \lambda_l \int_0^\cdot \operatorname{div}(w_\varepsilon(s)\psi_l)\,\mathrm{d}\beta_s^{(l)}\right]_{\gamma, L^2(\mathbb{T}^2)} > K\right\}\right) \\ &\le 2e^{-C_{\gamma,T}K} + \frac{\mathbb{E}\left[\sup_{0\le t\le T} \|B(w_\varepsilon(t))\|^2_{H^1(\mathbb{T}^2)}\right]}{K} \lesssim_{\Lambda,\gamma,T} \frac{1 + \mathbb{E}\left[\|u_0\|^2_{H^1(\mathbb{T}^2)}\right]}{K}.\end{aligned} \tag{2.4.15}$$

Step 3 (Combination of the estimates). Let $0 \le s < t \le T$. Splitting the process u_ε in its increments corresponding to v_ε and w_ε we obtain that

$$u_\varepsilon(t) - u_\varepsilon(s) = \int_{s'}^{t'} \operatorname{div} J_\varepsilon(\tau)\,\mathrm{d}\tau + \int_{s''}^{t''} A^\varepsilon(w_\varepsilon(\tau))\,\mathrm{d}\tau + \sum_{l=1}^{\infty} \lambda_l \int_{s''}^{t''} \operatorname{div}(w(\tau)\psi_l)\,\mathrm{d}\beta_\tau^{(l)}$$

with an appropriate choice of $s', s'', t', t'' \in [0,T]$ satisfying in particular $|t'-s'|, |t''-s''| < 2|t-s|$. Therefore, we can estimate $[u_\varepsilon(t) - u_\varepsilon(s)]_{\gamma, W^{-1,q'}(\mathbb{T}^2)}$ by

$$C_{q,T}\Bigg(\Bigg[\int_0^{\cdot} \operatorname{div} J_\varepsilon(s)\,\mathrm{d}s\Bigg]_{\frac{1}{2}, W^{-1,q'}(\mathbb{T}^2)} + \Bigg[\int_0^{\cdot} A^\varepsilon w_\varepsilon(s)\,\mathrm{d}s\Bigg]_{1, H^{-1}(\mathbb{T}^2)} + \Bigg[\sum_{l=1}^{\infty} \lambda_l \int_0^{\cdot} \operatorname{div}(w_\varepsilon(s)\psi_l)\,\mathrm{d}\beta_s^{(l)}\Bigg]_{\gamma, L^2(\mathbb{T}^2)}\Bigg),$$

where $C_{q,T} < \infty$ is an appropriate constant. Invoking the estimates (2.4.14), (2.4.15) as well as the assumption $K \in (1,\infty)$ we conclude (2.4.13). $\square$

2.4.2. THE VANISHING VISCOSITY LIMIT

In this subsection we let T, q, α, u_0 and $R^{(k)}$ as in Theorem 2.4.1 and assume, as in the previous subsection, that u_0 is an $\mathfrak{F}_0$-measurable random variable on a filtered probability space subject to the usual conditions with a family of independent Brownian motions $(\beta^{(l)})_{l\in\mathbb{N}}$. We let $\mathscr{I}$ be a sequence converging to zero and apply for every $k \in \mathbb{N}$ and $\varepsilon \in \mathscr{I}$ the construction from the previous subsection to the initial value $u_0^{(k)} = \mathbf{1}_{R^{(k)}} u_0$ and obtain a regularized splitting scheme consisting of $u_\varepsilon^{(k)}, v_\varepsilon^{(k)}, w_\varepsilon^{(k)}, J_\varepsilon^{(k)}$. We consider the sequence

$$\left(\left(\mathbf{1}_{R^{(l)}}, \beta^{(l)}, u_\varepsilon^{(l)}, v_\varepsilon^{(l)}, w_\varepsilon^{(l)}, J_\varepsilon^{(l)}, (v_\varepsilon^{(l)})^{\frac{\alpha+3}{2}}, (v_\varepsilon^{(l)})^{\frac{\alpha+3}{4}}\right)_{l\in\mathbb{N}}\right)_{\varepsilon\in\mathscr{I}} \tag{2.4.16}$$

in the topological product space

$$\prod_{l=1}^{\infty} \mathbb{R} \times C([0,T]) \times C([0,T]; L^2(\mathbb{T}^2)) \times L^\infty_{w*}(0,T; H^1(\mathbb{T}^2)) \times L^\infty_{w*}(0,T; H^1(\mathbb{T}^2)) \times L^2_w(0,T; L^{q'}(\mathbb{T}^2;\mathbb{R}^2)) \times L^2_w(0,T; H^2(\mathbb{T}^2)) \times L^4_w(0,T; W^{1,4}(\mathbb{T}^2)). \tag{2.4.17}$$

Proposition 2.4.7. *The sequence* (2.4.16) *is tight on* (2.4.17)

Proof. By Tychonoff's theorem it is sufficient to show tightness of every component of (2.4.16) separately, so we fix an $l \in \mathbb{N}$. The distribution of $\mathbf{1}_{R^{(l)}}$ and $\beta^{(l)}$ is independent of ε and since the corresponding space is a Radon space, the sequences $(\mathbf{1}_{R^{(l)}})_{\varepsilon\in\mathscr{I}}$, $(\beta^{(l)})_{\varepsilon\in\mathscr{I}}$ are tight. Using (2.4.2) we deduce that

$$\mathbb{P}\left(\left\{\|v_\varepsilon^{(l)}\|_{L^\infty(0,T;H^1(\mathbb{T}^2))} > K\right\}\right) \lesssim_{\Lambda,T} \frac{\mathbb{E}\left[\|u_0^{(l)}\|^2_{H^1(\mathbb{T}^2)}\right]}{K^2} \to 0$$

as $K \to \infty$ uniformly in ε so that tightness is a consequence of the Banach-Alaoglu theorem. The components $w_\varepsilon^{(l)}$, $J_\varepsilon^{(l)}$, $(v_\varepsilon^{(l)})^{\frac{\alpha+3}{2}}$, $(v_\varepsilon^{(l)})^{\frac{\alpha+3}{4}}$ can be treated analogously using

(2.4.2) and Lemmas 2.4.4 and 2.4.5. Lastly, we obtain from (2.4.2) and (2.4.13) that

$$\mathbb{P}\left(\left\{\max\left\{\|u_\varepsilon^{(l)}\|_{L^\infty(0,T;H^1(\mathbb{T}^2))}, \left[u_\varepsilon^{(l)}\right]_{\gamma,W^{-1,q'}(\mathbb{T}^2)}\right\} > K\right\}\right) \lesssim_{\Lambda,q,\gamma,T} \frac{1+\mathbb{E}\left[\|u_0^{(l)}\|_{H^1(\mathbb{T}^2)}^2\right]}{K}$$

for $K \in (1,\infty)$ so that tightness of $u_\varepsilon^{(l)}$ follows by [121, Theorem 5]. □

2

An application of [88, Theorem 2] yields that there exists for a subsequence, which we index again by $\mathscr{I}$, a complete probability space $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$ and a sequence of Borel-measurable random variables

$$\left(\left(\mathbf{1}_{\widetilde{R}_\varepsilon^{(l)}}, \widetilde{\beta}_\varepsilon^{(l)}, \widetilde{u}_\varepsilon^{(l)}, \widetilde{v}_\varepsilon^{(l)}, \widetilde{w}_\varepsilon^{(l)}, \widetilde{J}_\varepsilon^{(l)}, \widetilde{f}_\varepsilon^{(l)}, \widetilde{g}_\varepsilon^{(l)}\right)_{l\in\mathbb{N}}\right)_{\varepsilon\in\mathscr{I}}$$

with values in (2.4.17) such that

$$\left(\mathbf{1}_{\widetilde{R}_\varepsilon^{(l)}}, \widetilde{\beta}_\varepsilon^{(l)}, \widetilde{u}_\varepsilon^{(l)}, \widetilde{v}_\varepsilon^{(l)}, \widetilde{w}_\varepsilon^{(l)}, \widetilde{J}_\varepsilon^{(l)}, \widetilde{f}_\varepsilon^{(l)}, \widetilde{g}_\varepsilon^{(l)}\right)_{l\in\mathbb{N}} \tag{2.4.18}$$

has the same distribution as

$$\left(\mathbf{1}_{R^{(l)}}, \beta^{(l)}, u_\varepsilon^{(l)}, v_\varepsilon^{(l)}, w_\varepsilon^{(l)}, J_\varepsilon^{(l)}, (v_\varepsilon^{(l)})^{\frac{\alpha+3}{2}}, (v_\varepsilon^{(l)})^{\frac{\alpha+3}{4}}\right)_{l\in\mathbb{N}}$$

for every $\varepsilon \in \mathscr{I}$. Moreover, as $\varepsilon \searrow 0$, (2.4.18) converges to a $\mathfrak{B}$-measurable random variable

$$\left(\mathbf{1}_{\widetilde{R}^{(l)}}, \widetilde{\beta}^{(l)}, \widetilde{u}^{(l)}, \widetilde{v}^{(l)}, \widetilde{w}^{(l)}, \widetilde{J}^{(l)}, \widetilde{f}^{(l)}, \widetilde{g}^{(l)}\right)_{l\in\mathbb{N}}$$

in (2.4.17).

Remark 2.4.8. In order to apply [88, Theorem 2] one needs to check that there exists a countable sequence of $[-1,1]$-valued continuous functions on (2.4.17) which separate the points. Such a sequence is straightforward to construct using point-evaluations for the spaces of continuous functions and separability of the respective (pre-) dual for the spaces equipped with weak (weak-*) topology, see [84, Proposition 1.2.29, Corollary 1.3.22].

Lemma 2.4.9. *The sets $(\widetilde{R}^{(k)})_{k\in\mathbb{N}}$ form up to $\widetilde{\mathbb{P}}$-null sets a disjoint partition of $\widetilde{\Omega}$. Moreover, the following holds $\widetilde{\mathbb{P}}$-almost surely for every $k \in \mathbb{N}$.*

(i) *The random variables*

$$\widetilde{u}^{(k)},\ \widetilde{v}^{(k)},\ \widetilde{w}^{(k)},\ \widetilde{J}^{(k)},\ \widetilde{f}^{(k)} \ \textit{and} \ \ \widetilde{g}^{(k)}$$

vanish outside of of $\widetilde{R}^{(k)}$.

(ii) $\widetilde{u}^{(k)}(t) \ge 0$ *for all* $t \in [0,T]$.

(iii) *For almost all $t \in [0,T]$ we have*

$$\widetilde{u}^{(k)}(t) = \begin{cases} \widetilde{v}^{(k)}(2(t-j\delta)+j\delta), & j\delta \le t < (j+\frac{1}{2})\delta, \\ \widetilde{w}^{(k)}(2(t-(j+\frac{1}{2})\delta)+j\delta), & (j+\frac{1}{2}) \le t < (j+1)\delta. \end{cases} \tag{2.4.19}$$

(iv) *The tuples* $(\tilde{v}^{(k)}, \tilde{J}^{(k)})$ *are solutions to the deterministic thin-film equation on the intervals* $[j\delta, (j+1)\delta)$ *satisfying property (iv) from Theorem 2.2.3 with initial value* $\tilde{u}^{(k)}(j\delta)$ *for every* $j = 0, \ldots, N$.

(v) $\tilde{f}^{(k)} = (\tilde{v}^{(k)})^{\frac{\alpha+3}{2}}$ *and* $\tilde{g}^{(k)} = (\tilde{v}^{(k)})^{\frac{\alpha+3}{4}}$.

2

Proof. For every $\varepsilon \in \mathcal{I}$ we have

$$\tilde{\mathbb{E}}\left[\mathbf{1}_{\tilde{R}_\varepsilon^{(k_1)}} \mathbf{1}_{\tilde{R}_\varepsilon^{(k_2)}}\right] = \delta_{k_1,k_2} \mathbb{P}\left(R^{(k_1)}\right),$$

so that by letting $\varepsilon \searrow 0$ we conclude the first part of the claim. Part (i) follows by letting $\varepsilon \searrow 0$ in $\mathbf{1}_{\tilde{R}_\varepsilon^{(k)}} \|\tilde{u}_\varepsilon^{(k)}\|_{C([0,T];L^2(\mathbb{T}^2))} = 0$ and the same argument for the other random variables. Part (ii) is a consequence of $\tilde{u}_\varepsilon^{(k)}(t) \geq 0$ together with conservation of this property under limits in $C([0,T];L^2(\mathbb{T}^2))$. Analogously, we deduce (iii) from the respective property in of $\tilde{u}_\varepsilon^{(k)}$, $\tilde{v}_\varepsilon^{(k)}$ and $\tilde{w}_\varepsilon^{(k)}$. For (iv) and (v) we observe first that by measurability of $\mathcal{S}_{\alpha,q,\delta}$ we have

$$\mathcal{S}_{\alpha,q,\delta} \tilde{u}_\varepsilon^{(k)}(j\delta) = \left(\tilde{v}_\varepsilon^{(l)}|_{[j\delta,(j+1)\delta)}, \tilde{J}_\varepsilon^{(k)}|_{[j\delta,(j+1)\delta)}, \tilde{f}_\varepsilon^{(k)}|_{[j\delta,(j+1)\delta)}, \tilde{g}_\varepsilon^{(k)}|_{[j\delta,(j+1)\delta)}\right). \tag{2.4.20}$$

In particular, we have $\tilde{f}_\varepsilon^{(k)} = (\tilde{f}_\varepsilon^{(k)})^{\frac{\alpha+3}{2}}$, $\tilde{g}_\varepsilon^{(k)} = (\tilde{f}_\varepsilon^{(k)})^{\frac{\alpha+3}{4}}$ and the right-hand side of (2.4.20) fulfills the properties stated in Theorem 2.2.3. If we let $\varepsilon \searrow 0$ we deduce that the limit

$$\left(\tilde{v}^{(k)}|_{[j\delta,(j+1)\delta)}, \tilde{J}^{(k)}|_{[j\delta,(j+1)\delta)}\right)$$

is a solution to the thin-film equation and that (v) holds true by Proposition 2.2.4. In light of Remark 2.2.2 the initial value of (2.4.20) is indeed $\tilde{u}_\varepsilon^{(k)}(j\delta)$. It is left to observe that property (iv) of Theorem 2.2.3 is preserved due to lower semi-continuity of the norm with respect to weak convergence. □

By (2.4.19) we deduce that $\tilde{u}_\varepsilon^{(k)}$ converges to $\tilde{u}^{(k)}$ also in $L^\infty_{w*}(0,T;H^1(\mathbb{T}^2))$ and that $\tilde{u}^{(k)}$ is weakly continuous in $H^1(\mathbb{T}^2)$ again. Moreover, we identify in the following $\tilde{v}^{(k)}$ and $\tilde{w}^{(k)}$ with their versions such that (2.4.19) holds for all $t \in [0,T]$. We define $\tilde{\mathfrak{F}}$ as the augmentation of the filtration $\tilde{\mathfrak{G}}$ given by

$$\tilde{\mathfrak{G}}_t = \sigma\left(\left\{\mathbf{1}_{\tilde{R}^{(l)}}, \tilde{J}^{(l)}|_{[0,t]} \,\middle|\, l \in \mathbb{N}\right\} \cup \left\{\tilde{u}^{(l)}(s), \tilde{\beta}^{(l)}(s) \,\middle|\, 0 \leq s \leq t,\, l \in \mathbb{N}\right\}\right),$$

where we consider $\tilde{J}^{(l)}|_{[0,t]}$ as a $\mathfrak{B}$-random variable in $L^2(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$.

Lemma 2.4.10. *The processes* $(\tilde{\beta}^{(l)})_{l\in\mathbb{N}}$ *are a family of independent* $\tilde{\mathfrak{F}}$*-Brownian motions. Moreover, we have for every* $k \in \mathbb{N}$, $j \in \{0,\ldots,N\}$ *and* $\varphi \in H^1(\mathbb{T}^2)$ *that* $\tilde{\mathbb{P}}$*-almost surely*

$$\begin{aligned}\langle \tilde{w}^{(k)}(t), \varphi\rangle - \langle \tilde{w}^{(k)}(j\delta), \varphi\rangle = &\frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \langle \operatorname{div}(\operatorname{div}(\tilde{w}^{(k)}(s)\psi_l)\psi_l), \varphi\rangle \,\mathrm{d}s \\ &+ \sum_{l=1}^\infty \lambda_l \int_{j\delta}^t \langle \operatorname{div}(\tilde{w}^{(k)}(s)\psi_l), \varphi\rangle \,\mathrm{d}\beta_s^{(l)}\end{aligned}$$

for all $t \in [j\delta, (j+1)\delta)$.

The proof of the lemma above is a simpler version of the proof of Theorem 2.5.12 and is therefore omitted. Finally, we observe that many of the estimates from the previous subsection carry over to their limit.

Proposition 2.4.11. *For every $k \in \mathbb{N}$ and $p \in (0,\infty)$ we have*

$$\begin{aligned} \widetilde{\mathbb{E}}\left[\sup_{0\le t\le T} \|\widetilde{u}^{(k)}\|^p_{H^1(\mathbb{T}^2)}\right] &\lesssim_{\Lambda,p,T} \mathbb{E}\left[\|u_0^{(k)}\|^p_{H^1(\mathbb{T}^2)}\right], \\ \widetilde{\mathbb{E}}\left[\|\widetilde{J}^{(k)}\|^{\frac{p}{2}}_{L^2(0,T;L^{q'}(\mathbb{T}^2))}\right] &\lesssim_{\Lambda,p,q,T} \mathbb{E}\left[\|u_0^{(k)}\|^p_{H^1(\mathbb{T}^2)}\right]. \end{aligned} \tag{2.4.21}$$

Moreover, for any $\gamma \in (0,\frac{1}{2})$ and $K \in (1,\infty)$ it holds

$$\widetilde{\mathbb{P}}\left(\left\{\left[\widetilde{u}^{(k)}\right]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \lesssim_{\Lambda,q,\gamma,T} \frac{1+\mathbb{E}\left[\|u_0^{(k)}\|^2_{H^1(\mathbb{T}^2)}\right]}{K}.$$

Proof. The estimates (2.4.21) follow from lower semi-continuity of the norm with respect to weak (weak-*) convergence, (2.4.2), Lemma 2.4.4 and Fatou's lemma. Moreover, we observe that

$$[\widetilde{u}^{(l)}]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} \le \liminf_{\varepsilon\searrow 0}[\widetilde{u}^{(l)}_\varepsilon]_{\gamma,W^{-1,q'}(\mathbb{T}^2)}$$

by convergence in $C([0,T];L^2(\mathbb{T}^2))$. Therefore, we have

$$\begin{aligned} \widetilde{\mathbb{P}}\left(\left\{[\widetilde{u}^{(l)}]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) &\le \widetilde{\mathbb{P}}\left(\left\{\liminf_{\varepsilon\searrow 0}[\widetilde{u}^{(l)}_\varepsilon]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \\ &\le \widetilde{\mathbb{P}}\left(\liminf_{\varepsilon\searrow 0}\left\{[\widetilde{u}^{(l)}_\varepsilon]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \le \liminf_{\varepsilon\searrow 0}\widetilde{\mathbb{P}}\left(\left\{[\widetilde{u}^{(l)}_\varepsilon]_{\gamma,W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \end{aligned}$$

and it is left to apply (2.4.13). ☐

Proof of Theorem 2.4.1. The limiting random variables $(\beta^{(l)})_{l\in\mathbb{N}}$, $\mathbf{1}_{\widetilde{R}^{(k)}}$, $\widetilde{u}^{(k)}$ and $\widetilde{J}^{(k)}$ have the desired properties. Indeed, (i) is a consequence of

$$(\mathbf{1}_{\widetilde{R}^{(k)}_\varepsilon}, \widetilde{u}^{(k)}_\varepsilon(0))_{k\in\mathbb{N}} \sim (\mathbf{1}_{R^{(k)}}, u_0^{(k)})_{k\in\mathbb{N}}, \quad \varepsilon \in \mathscr{I},$$

the convergence

$$(\mathbf{1}_{\widetilde{R}^{(k)}_\varepsilon}, \widetilde{u}^{(k)}_\varepsilon(0))_{k\in\mathbb{N}} \to (\mathbf{1}_{\widetilde{R}^{(k)}}, \widetilde{u}^{(k)}(0))_{k\in\mathbb{N}}$$

in $(\mathbb{R}\times L^2(\mathbb{T}^2))^\infty$ and Lemma 2.4.9 (i). Part (ii) follows by the choice of $\widetilde{\mathfrak{F}}$. Parts (iii), (iv), (v) and (vi) are the content of Lemma 2.4.9 (iv), Lemma 2.4.10 and Proposition 2.4.11. ☐

2.5. CONSTRUCTION OF SOLUTIONS

Let finally μ, T, q, α as in Theorem 2.1.3. We apply Theorem 2.4.1 for every $N \in \mathbb{N}$ to a random variable which is distributed according to μ and obtain processes $u_N^{(k)}$, families of Brownian motions $\beta_N^{(l)}$ and random variables $\mathbf{1}_{R_N^{(k)}}$, $J_N^{(k)}$ for $l,k \in \mathbb{N}$ satisfying the stated properties. We assume that these random variables are defined on the same probability space $(\Omega,\mathfrak{A},\mathbb{P})$ with filtration $\mathfrak{F}$ and moreover, that $\beta^{(l)} = \beta_N^{(l)}$ is independent of N. This does not influence the mathematical analysis since we analyze the solutions for each N separately and serves only notational convenience.

Remark 2.5.1. Alternatively, one can also apply the limiting procedure from Subsection 2.4.2 for all step numbers $N \in N$ simultaneously to end up in the assumed situation. This, however, would lead to a notational overload in the previous section.

We remark also that we have dropped the ~-notation since we want to pass to another probability space once more. For $k, N \in \mathbb{N}$ and we define $v_N^{(k)}$ and $w_N^{(k)}$ by

$$\begin{cases} u_N^{(k)}(t) = v_N^{(k)}(2(t-j\delta)+j\delta), & j\delta \le t < (j+\frac{1}{2})\delta, \\ u_N^{(k)}(t) = w_N^{(k)}(2(t-(j+\frac{1}{2})\delta)+j\delta), & (j+\frac{1}{2}) \le t < (j+1)\delta, \end{cases} \tag{2.5.1}$$

where again $\delta = \delta(N) = \frac{T}{N+1}$.

2.5.1. ADDITIONAL TIGHTNESS PROPERTIES

The approximate solutions $u_N^{(k)}$, $J_N^{(k)}$ satisfy the bounds from Theorem 2.4.1 (v), (vi), which can as in Proposition 2.4.7 be used to derive tightness in suitable spaces.

Remark 2.5.2. In light of Theorem 2.4.1 (i), the right-hand sides of the aforementioned bounds can be expressed in terms of the cut-off moments

$$\nu_{k,p} = \int \mathbf{1}_{\left\{k-1 \le \|\cdot\|_{H^1(\mathbb{T}^2)} < k\right\}} \|\cdot\|_{H^1(\mathbb{T}^2)}^p \,\mathrm{d}\mu \tag{2.5.2}$$

of the initial distribution μ. We remark that the notation (2.5.2) will be used during the remainder of this section.

In this subsection we provide an additional tightness property, which can be seen as the adaption of the compactness statement [33, Lemma 2.5] to our setting. Its proof relies on deriving a version of Lemma 2.4.5 with a uniform estimate in N and a simplified proof of [33, Lemma 2.5]. The former is based on a control of the α-entropy production along the stochastic dynamics. We point out that the simplification of the compactness proof is only possible due to the assumption $\alpha \in (0,1)$, which is less general than the situation in [33, Lemma 2.5].

Lemma 2.5.3. *It holds for every $k, N \in \mathbb{N}$ that*

$$\mathbb{E}\left[\left\|\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right\|_{L^2(0,T;H^2(\mathbb{T}^2))}^2 + \left\|\left(v_N^{(k)}\right)^{\frac{\alpha+3}{4}}\right\|_{L^4(0,T;W^{1,4}(\mathbb{T}^2))}^4\right] \lesssim_{\Lambda,\alpha,T} 1 + \nu_{k,\alpha+3}.$$

Proof. An application of Theorem 2.2.3 (iv) yields the estimate

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^2} \left|H\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right|^2 + \left|\nabla\left(v_N^{(k)}\right)^{\frac{\alpha+3}{4}}\right|^4 \mathrm{d}x\,\mathrm{d}t &\lesssim_\alpha \sum_{j=0}^{N} \int_{\mathbb{T}^2} G_\alpha(v_N^{(k)}(j\delta)) - G_\alpha(w_N^{(k)}(j\delta))\,\mathrm{d}x \\ = \int_{\mathbb{T}^2} G_\alpha(v_N^{(k)}(0)) - G_\alpha(w_N^{(k)}(N\delta))\,\mathrm{d}x &+ \sum_{j=0}^{N-1} \int_{\mathbb{T}^2} G_\alpha(v_N^{(k)}((j+1)\delta)) - G_\alpha(w_N^{(k)}(j\delta))\,\mathrm{d}x. \end{aligned} \tag{2.5.3}$$

The first summand can be estimated directly using the expression (2.1.14) by

$$C_\alpha\left(\|v_N^{(k)}(0)\|_{L^2(\mathbb{T}^2)}^{\alpha+1} + \|v_N^{(k)}(0)\|_{L^2(\mathbb{T}^2)} + \|w_N^{(k)}(0)\|_{L^2(\mathbb{T}^2)}^{\alpha+1} + \|w_N^{(k)}(0)\|_{L^2(\mathbb{T}^2)}\right)$$

$$\lesssim_\alpha \sup_{0\le t\le T} \|u_N^{(k)}(t)\|_{L^2(\mathbb{T}^2)} + 1.$$

Taking the expectation and employing Theorem 2.4.1 (v) we obtain the estimate

$$\mathbb{E}\left[\int_{\mathbb{T}^2} G_\alpha(v_N^{(k)}(0)) - G_\alpha(w_N^{(k)}(N\delta))\,\mathrm{d}x\right] \lesssim_{\Lambda,\alpha,T} 1 + \nu_{k,1}. \tag{2.5.4}$$

To estimate the second summand of the right-hand side of (2.5.3), we fix a function $\eta \in C^\infty(\mathbb{R})$ such that $0 \le \eta \le 1$, $\eta = 1$ on $[2,\infty)$ and $\eta = 0$ on $(-\infty, 1]$. We define smooth functions

$$\eta_\kappa(x) = \eta\left(\frac{x}{\kappa}\right), \quad G_{\alpha,\kappa}(x) = G_\alpha(x)\eta_\kappa(x).$$

for $\kappa > 0$. Correspondingly, we define the regularized functional

$$\phi_\kappa\colon L^2(\mathbb{T}^2) \to \mathbb{R},\ w \mapsto \int_{\mathbb{T}^2} G_{\alpha,\kappa}(w)\,\mathrm{d}x \tag{2.5.5}$$

We observe that there is a constant $C_{\alpha,\kappa} < \infty$ such that

$$|G_{\alpha,\kappa}(x)| \le C_{\alpha,\kappa}|x|^2, \quad |G'_{\alpha,\kappa}(x)| \le C_{\alpha,\kappa}|x| \quad \text{and} \quad |G''_{\alpha,\kappa}(x)| \le C_{\alpha,\kappa}$$

for each $x \in \mathbb{R}$. An application of Itô's formula to the composition of the functional ϕ_κ with the process $w_N^{(k)}$ satisfying the SPDE from Theorem 2.4.1 (iv) yields that

$$\begin{aligned}
\phi_\kappa(w_N^{(k)}(t)) = {}&\phi_\kappa(w_N^{(k)}(j\delta)) + \sum_{l=1}^\infty \lambda_l \int_{j\delta}^t \int_{\mathbb{T}^2} G'_{\alpha,\kappa}(w_N^{(k)}(s))\,\mathrm{div}(w_N^{(k)}(s)\psi_l)\,\mathrm{d}x\,\mathrm{d}\beta^{(l)}(s)\\
&- \frac{1}{2}\sum_{l=1}^\infty \int_{j\delta}^t \int_{\mathbb{T}^2} \lambda_l^2 G''_{\alpha,\kappa}(w_N^{(k)}(s))[\mathrm{div}(w_N^{(k)}(s)\psi_l)][\psi_l\cdot\nabla w_N^{(k)}(s)]\,\mathrm{d}x\,\mathrm{d}s\\
&+ \frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \int_{\mathbb{T}^2} G''_{\alpha,\kappa}(w_N^{(k)}(s))\left[\mathrm{div}(w_N^{(k)}(s)\psi_l)\right]^2 \mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{2.5.6}$$

for $t \in [j\delta, (j+1)\delta)$. For a justification of the applicability of Itô's formula see Appendix 2.C. We note that as pointed out there, the above formula is also valid for the end-point $t = (j+1)\delta$, but then the term on the left-hand side has to be replaced by $\phi_\kappa(v_N^{(k)}((j+1)\delta))$. Due to the smoothness of ψ_l it holds

$$\begin{aligned}
[\mathrm{div}(w\psi_l)][\psi_l\cdot\nabla w] &= [\mathrm{div}(w\psi_l)]^2 - [\mathrm{div}(w\psi_l)][w\,\mathrm{div}\,\psi_l]\\
&= [\mathrm{div}(w\psi_l)]^2 - \left[w\,\mathrm{div}\,\psi_l\right]^2 - w\nabla w\cdot\left[\psi_l\,\mathrm{div}\,\psi_l\right].
\end{aligned}$$

The derivative of $\zeta_\kappa(x) = \int_0^x yG''_{\alpha,\kappa}(y)\,\mathrm{d}y$ is bounded so that an application of [26, Proposition 9.5] yields

$$\int_{\mathbb{T}^2} G''_{\alpha,\kappa}(w)w\nabla w\cdot\left[\psi_l\,\mathrm{div}\,\psi_l\right]\mathrm{d}x = -\int_{\mathbb{T}^2} \zeta_\kappa(w)\,\mathrm{div}\left[\psi_l\,\mathrm{div}\,\psi_l\right]\mathrm{d}x,$$

for $w \in H^1(\mathbb{T}^2)$. We also introduce the function $\theta_\kappa(x) = x^2 G''_{\alpha,\kappa}(x)$ and rewrite (2.5.6) using the previous identities as

$$\begin{aligned}\phi_\kappa(w_N^{(k)}(t)) =& \phi_\kappa(w_N^{(k)}(j\delta)) + \sum_{l=1}^\infty \lambda_l \int_{j\delta}^t \int_{\mathbb{T}^2} G'_{\alpha,\kappa}(w_N^{(k)}(s))\,\mathrm{div}(w_N^{(k)}(s)\psi_l)\,\mathrm{d}x\,\mathrm{d}\beta^{(l)}(s) \\ &+ \frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \int_{\mathbb{T}^2} \theta_\kappa(w_N^{(k)}(s))(\mathrm{div}\,\psi_l)^2\,\mathrm{d}x\,\mathrm{d}s \\ &- \frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \int_{\mathbb{T}^2} \zeta_\kappa(w_N^{(k)}(s))\,\mathrm{div}\left[\psi_l\,\mathrm{div}\,\psi_l\right]\,\mathrm{d}x\,\mathrm{d}s.\end{aligned} \tag{2.5.7}$$

Using that

$$G''_{\alpha,\kappa}(x) = \frac{1}{\kappa^2}\eta''\left(\frac{x}{\kappa}\right)G_\alpha(x) + \frac{2}{\kappa}\eta'\left(\frac{x}{\kappa}\right)\left[\frac{x^\alpha}{\alpha} + r'_\alpha(x)\right] + \eta\left(\frac{x}{\kappa}\right)x^{\alpha-1}$$

and that η' and η'' vanish outside of $[1,2]$ we deduce that $|\theta_\kappa(x)| \le C_\alpha(1+|x|)$. The same argument yields that $|\zeta_\kappa(x)| \le C_\alpha(1+|x|)$, indeed we can estimate for example

$$\left|\int_0^x \frac{y}{\kappa^2}\eta''\left(\frac{y}{\kappa}\right)G_\alpha(y)\,\mathrm{d}y\right| \lesssim \frac{1}{\kappa}\int_0^{|x|\wedge 2\kappa} |G_\alpha(y)|\,\mathrm{d}y \lesssim_\alpha \frac{|x|\wedge 2\kappa + (|x|\wedge 2\kappa)^2}{\kappa} \lesssim 1+|x|.$$

Using (2.3.5) we obtain the estimates

$$\begin{aligned}&\left|\frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^{(j+1)\delta} \int_{\mathbb{T}^2} \theta_\kappa(w_N^{(k)}(s))(\mathrm{div}\,\psi_l)^2\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\quad\lesssim_{\Lambda,\alpha} \delta\left(1 + \operatorname*{ess\,sup}_{0\le t\le T}\|w_N^{(k)}(t)\|_{L^2(\mathbb{T}^2)}\right), \\ &\left|\frac{1}{2}\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^{(j+1)\delta} \int_{\mathbb{T}^2} \zeta_\kappa(w_N^{(k)}(s))\,\mathrm{div}\left[\psi_l\,\mathrm{div}\,\psi_l\right]\,\mathrm{d}x\,\mathrm{d}s\right| \\ &\quad\lesssim_{\Lambda,\alpha} \delta\left(1 + \operatorname*{ess\,sup}_{0\le t\le T}\|w_N^{(k)}(t)\|_{L^2(\mathbb{T}^2)}\right).\end{aligned}$$

For the series of stochastic integrals in (2.5.7) we observe that

$$\begin{aligned}&\mathbb{E}\left[\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^{(j+1)\delta} \left(\int_{\mathbb{T}^2} G'_{\alpha,\kappa}(w_N^{(k)}(s))\,\mathrm{div}(w_N^{(k)}(s)\psi_l)\,\mathrm{d}x\right)^2\mathrm{d}s\right] \\ &\quad\lesssim_{\alpha,\kappa} \mathbb{E}\left[\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^{(j+1)\delta}\left(\|w_N^{(k)}(s)\|_{H^1(\mathbb{T}^2)}\right)^2\mathrm{d}s\right] < \infty\end{aligned}$$

We used (2.3.5) and that the function $G'_{\alpha,\kappa}$ is bounded in the first inequality and Theorem 2.4.1 (v) for the second one. Hence, the series of stochastic integrals has integrable quadratic variation and is therefore a martingale. Therefore, taking the expectation of (2.5.7) with $t = (j+1)\delta$, using the previous estimates, as well as Theorem 2.4.1 (v) once more, yields that

$$\mathbb{E}\left[\phi_\kappa(v_N^{(k)}((j+1)\delta)) - \phi_\kappa(w_N^{(k)}(j\delta))\right] \lesssim_{\Lambda,\alpha} \delta\left(1 + \mathbb{E}\left[\operatorname*{ess\,sup}_{0\le t\le T}\|w_N^{(k)}(t)\|_{L^2(\mathbb{T}^2)}\right]\right)$$

$$\lesssim_{\Lambda,T} \delta\left(1+\nu_{k,1}\right)$$

Finally, taking the expectation of (2.5.3) and using additionally (2.5.4), we end up with

$$\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^2}\left|H\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right|^2+\left|\nabla\left(v_N^{(k)}\right)^{\frac{\alpha+3}{4}}\right|^4\mathrm{d}x\,\mathrm{d}t\right]\lesssim_{\Lambda,\alpha,T} 1+\nu_{k,1}.$$

As in the proof of Lemma 2.4.5, we use that

$$\begin{aligned}\mathbb{E}\left[\int_0^T\int_{\mathbb{T}^2}\left|\left(v_N^{(k)}(t)\right)^{\frac{\alpha+3}{2}}\right|^2+\left|\left(v_N^{(k)}(t)\right)^{\frac{\alpha+3}{4}}\right|^4\mathrm{d}x\,\mathrm{d}t\right]&\lesssim_{\alpha,T}\mathbb{E}\left[\sup_{0\le t\le T}\|u_N^{(k)}(t)\|_{H^1(\mathbb{T}^2)}^{\alpha+3}\right]\\&\lesssim_{\Lambda,\alpha,T}\nu_{k,\alpha+3}\end{aligned}$$

as a consequence of Theorem 2.4.1 (v) and therefore

$$\mathbb{E}\left[\left\|\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right\|^2_{L^2(0,T;H^2(\mathbb{T}^2))}+\left\|\left(v_N^{(k)}\right)^{\frac{\alpha+3}{4}}\right\|^4_{L^4(0,T;W^{1,4}(\mathbb{T}^2))}\right]\lesssim_{\Lambda,\alpha,T} 1+\nu_{k,\alpha+3}$$

by (2.4.12). □

Lemma 2.5.4. *For every $k\in\mathbb{N}$, the laws of $\left(\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right)_{N\in\mathbb{N}}$ are tight on $L^2(0,T;H^1(\mathbb{T}^2))$.*

Proof. We divide the proof into three steps.

Step 1 (Hölder regularity of $u_N^{(k)}$ in $L^2(\mathbb{T}^2)$). First, we observe that the paths of $u_N^{(k)}$ are weakly continuous in $H^1(\mathbb{T}^2)$ and in particular $\|u_N^{(k)}(t)\|_{H^1(\mathbb{T}^2)}\le\|u_N^{(k)}\|_{L^\infty(0,T;H^1(\mathbb{T}^2))}$ for every $t\in[0,T]$. The Sobolev embedding theorem, see [119, Section 3.5.5], states that $W^{-1,q'}(\mathbb{T}^2)\hookrightarrow H^{\frac{-2}{q'}}(\mathbb{T}^2)$ and therefore we can estimate the seminorm $[u_N^{(k)}]_{\gamma,H^{\frac{-2}{q'}}(\mathbb{T}^2)}$ by $C_q[u_N^{(k)}]_{\gamma,W^{-1,q'}(\mathbb{T}^2)}$ for $\gamma\in(0,1)$. The interpolation inequality

$$\|f\|_{L^2(\mathbb{T}^2)}\le\|f\|^\theta_{H^1(\mathbb{T}^2)}\|f\|^{1-\theta}_{H^{\frac{-2}{q'}}(\mathbb{T}^2)},\qquad f\in H^1(\mathbb{T}^2),$$

with $\theta=\frac{\frac{2}{q'}}{1+\frac{2}{q'}}$ can be derived using Fourier methods. We obtain the estimate

$$\begin{aligned}\|u_N^{(k)}(t)-u_N^{(k)}(s)\|_{L^2(\mathbb{T}^2)}&\le\|u_N^{(k)}(t)-u_N^{(k)}(s)\|^\theta_{H^1(\mathbb{T}^2)}\|u_N^{(k)}(t)-u_N^{(k)}(s)\|^{1-\theta}_{H^{\frac{-2}{q'}}(\mathbb{T}^2)}\\&\lesssim_q[u_N^{(k)}]^{1-\theta}_{\gamma,W^{-1,q'}(\mathbb{T}^2)}\|u_N^{(k)}\|^\theta_{L^\infty(0,T;H^1(\mathbb{T}^2))}|t-s|^{(1-\theta)\gamma}\end{aligned}$$

on the increments, which yields that

$$[u_N^{(k)}]_{(1-\theta)\gamma,L^2(\mathbb{T}^2)}\lesssim_q[u_N^{(k)}]^{1-\theta}_{\gamma,W^{-1,q'}(\mathbb{T}^2)}\|u_N^{(k)}\|^\theta_{L^\infty(0,T;H^1(\mathbb{T}^2))}. \tag{2.5.8}$$

Step 2 (Integral estimate on the increments of $v_N^{(k)}$). In this step, we deduce from the first step an estimate on

$$\|\tau_h v_N^{(k)}-v_N^{(k)}\|_{L^4(0,T-h;L^2(\mathbb{T}^2))},$$

in similar terms, where τ_h denotes the translation operator by $h > 0$ in the time variable. To quantify the jumps in the paths of of $v_N^{(k)}$ we introduce the function

$$\phi_{N,h}\colon [0,T] \to \mathbb{N},\, N \mapsto \lfloor t+h \rfloor_\delta - \lfloor t \rfloor_\delta,$$

which counts how many discretization points lie between t and $t+h$. The function $\lfloor \cdot \rfloor_\delta$ denotes here the biggest integer multiple of δ which is less or equal to its input value. Then we have

$$\| v_N^{(k)}(t+h) - v_N^{(k)}(t) \|_{L^2(\mathbb{T}^2)} \le [u_N^{(k)}]_{(1-\theta)\gamma, L^2(\mathbb{T}^2)} \left(h + \phi_{N,h}(t)\delta\right)^{(1-\theta)\gamma} \tag{2.5.9}$$

for $t \in [0, T-h]$. We introduce the sets

$$C_{N,h} = \left\{ t \in [0, T-h] | \phi_{N,h}(t) \neq 1 \right\} \text{ and } D_{N,h} = \left\{ t \in [0, T-h] | \phi_{N,h}(t) = 1 \right\},$$

distinguishing between points t, where one can estimate the right-hand side of (2.5.9) in terms of h or not. Indeed, if $t \in C_{N,h}$ it holds either $\phi_{N,h}(t) = 0$ or $\phi_{N,h}(t) \ge 2$ so that in any case $\phi_{N,h}(t)\delta \le 2h$ and therefore

$$\left(h + \phi_{N,h}(t)\delta\right)^{(1-\theta)\gamma} \le (3h)^{(1-\theta)\gamma}.$$

We deduce that

$$\begin{aligned} \|\tau_h v_N^{(k)} - v_N^{(k)}\|^4_{L^4(0,T-h;L^2(\mathbb{T}^2))} &= \int_0^{T-h} \| v_N^{(k)}(t+h) - v_N^{(k)}(t) \|^4_{L^2(\mathbb{T}^2)} \,\mathrm{d}t \\ &\le [u_N^{(k)}]^4_{(1-\theta)\gamma, L^2(\mathbb{T}^2)} \left((3h)^{4(1-\theta)\gamma} |C_{N,h}| + (h+\delta)^{4(1-\theta)\gamma} |D_{N,h}| \right). \end{aligned} \tag{2.5.10}$$

If $h \ge \delta$, we use the trivial estimate $|C_{N,h}| + |D_{N,h}| \le T$ to conclude

$$(3h)^{4(1-\theta)\gamma} |C_{N,h}| + (h+\delta)^{4(1-\theta)\gamma} |D_{N,h}| \le (3h)^{4(1-\theta)\gamma} T. \tag{2.5.11}$$

For $h < \delta$ we use instead that

$$t \in [j\delta, (j+1)\delta - h) \quad \Rightarrow \quad \phi_{N,h}(t) = 0 \quad \Rightarrow \quad t \in C_{N,h}$$

and consequently $|D_{N,h}| \le (N+1)h$. We define the function

$$f_h(x) = \left(h + \frac{T}{x+1} \right)^{4(1-\theta)\gamma} (x+1)h, \quad x \in \left[1, \frac{T}{h} - 1 \right]$$

so that

$$(h+\delta)^{4(1-\theta)\gamma} |D_{N,h}| \le f_h(N)$$

and it suffices to estimate the maximum of f_h. Its derivative is given by

$$f_h'(x) = h\left(h + \frac{T}{x+1} \right)^{4(1-\theta)\gamma} - \frac{4(1-\theta)\gamma T \left(h + \frac{T}{x+1}\right)^{4(1-\theta)\gamma - 1} h(x+1)}{(x+1)^2},$$

which can vanish only if

$$\frac{4(1-\theta)\gamma T}{(x+1)\left(h+\frac{T}{x+1}\right)} = 1 \quad \Rightarrow \quad h(x+1) = (4(1-\theta)\gamma - 1)T.$$

We choose for the rest of the proof that $\gamma = \frac{1}{4}$ so that the above is not feasible for $x \in \left[1, \frac{T}{h} - 1\right]$. Hence f_h can attain its maximum only at the boundary points 1 and $\frac{T}{h} - 1$. Evaluating f_h gives

$$f_h(1) = \left(h + \frac{T}{2}\right)^{1-\theta} 2h \le 2T^{1-\theta} h, \quad f_h\left(\frac{T}{h} - 1\right) = (2h)^{1-\theta} T.$$

We end up with the estimate

$$(3h)^{1-\theta}|C_{N,h}| + (h+\delta)^{1-\theta}|D_{N,h}| \le 2T(3h)^{1-\theta} + 2T^{1-\theta}h.$$

We define the right-hand side as $g_{\theta,T}(h)$ and obtain from (2.5.8) and (2.5.10) that

$$\|\tau_h v_N^{(k)} - v_N^{(k)}\|_{L^4(0,T-h;L^2(\mathbb{T}^2))} \lesssim_q [u_N^{(k)}]^{1-\theta}_{\frac{1}{4},W^{-1,q'}(\mathbb{T}^2)} \|u_N^{(k)}\|^{\theta}_{L^\infty(0,T;H^1(\mathbb{T}^2))} (g_{\theta,T}(h))^{\frac{1}{4}}. \tag{2.5.12}$$

By (2.5.11), this holds also if $h \ge \delta$.

Step 3 (Proof of tightness). Due to Theorem 2.4.1 (v), (vi) and Lemma 2.5.3 we have

$$\mathbb{P}\left(\left\{\sup_{0\le t\le T} \|u_N^{(k)}\|_{H^1(\mathbb{T}^2)} > K\right\}\right) \lesssim_{\Lambda,T} \frac{\nu_{k,2}}{K^2},$$

$$\mathbb{P}\left(\left\{\left[u_N^{(k)}\right]_{\frac{1}{4},W^{-1,q'}(\mathbb{T}^2)} > K\right\}\right) \lesssim_{\Lambda,q,T} \frac{1+\nu_{k,2}}{K},$$

$$\mathbb{P}\left\{\left\|\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right\|_{L^2(0,T;H^2(\mathbb{T}^2))} > K\right\} \lesssim_{\Lambda,\alpha,T} \frac{1+\nu_{k,\alpha+3}}{K^2}$$

for $K \in (1,\infty)$. In particular,

$$\mathbb{P}\left(\left(F_{N,K}^{(k)}\right)^c\right) \lesssim_{\Lambda,\alpha,q,T} \frac{1+\nu_{k,\alpha+3}}{K}, \tag{2.5.13}$$

where we define

$$F_{N,K}^{(k)} = \left\{\max\left\{\sup_{0\le t\le T}\|u_N^{(k)}\|_{H^1(\mathbb{T}^2)}, \left[u_N^{(k)}\right]_{\frac{1}{4},W^{-1,q'}(\mathbb{T}^2)}, \left\|\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right\|_{L^2(0,T;H^2(\mathbb{T}^2))}\right\} \le K\right\}.$$

Moreover, using that for $a, b \ge 0$ we have

$$|a^{\frac{\alpha+3}{2}} - b^{\frac{\alpha+3}{2}}| \le \frac{\alpha+3}{2}|a-b|\max(a,b)^{\frac{\alpha+1}{2}} \lesssim |a-b|\left[a^{\frac{\alpha+1}{2}} + b^{\frac{\alpha+1}{2}}\right]$$

as a consequence of the fundamental theorem of calculus, we deduce that

$$\left\|\tau_h\left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}} - \left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}}\right\|_{L^2\left(0,T-h;L^{\frac{4}{\alpha+3}}(\mathbb{T}^2)\right)}$$

2

$$\lesssim \left\| \left(\tau_h v_N^{(k)} - v_N^{(k)}\right)\left(\tau_h \left(v_N^{(k)}\right)^{\frac{\alpha+1}{2}} + \left(v_N^{(k)}\right)^{\frac{\alpha+1}{2}}\right)\right\|_{L^2\left(0,T-h;L^{\frac{4}{\alpha+3}}(\mathbb{T}^2)\right)}$$
$$\lesssim \|\tau_h v_N^{(k)} - \tilde{v}_N\|_{L^4(0,T-h;L^2(\mathbb{T}^2))} \left\| \left(v_N^{(k)}\right)^{\frac{\alpha+1}{2}} \right\|_{L^4\left(0,T;L^{\frac{4}{\alpha+1}}(\mathbb{T}^2)\right)}$$

from Hölder's inequality. We estimate the latter term by

$$\left\| \left(v_N^{(k)}\right)^{\frac{\alpha+1}{2}} \right\|_{L^4\left(0,T;L^{\frac{4}{\alpha+1}}(\mathbb{T}^2)\right)} = \left(\int_0^T \left(\int_{\mathbb{T}^2} \left(v_N^{(k)}\right)^2 \mathrm{d}x\right)^{\alpha+1} \mathrm{d}t\right)^{\frac{1}{4}} \lesssim_T \sup_{0\le t\le T} \|v_N^{(k)}\|_{L^2(\mathbb{T}^2)}^{\frac{\alpha+1}{2}}.$$

We conclude by (2.5.12) that

$$\left\|\tau_h \left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}} - \left(v_N^{(k)}\right)^{\frac{\alpha+3}{2}} \right\|_{L^2\left(0,T-h;L^{\frac{4}{\alpha+3}}(\mathbb{T}^2)\right)}$$
$$\lesssim_{q,T} [u_N^{(k)}]_{\frac{1}{4},W^{-1,q'}(\mathbb{T}^2)}^{1-\theta} \|u_N^{(k)}\|_{L^\infty(0,T;H^1(\mathbb{T}^2))}^{\theta+\frac{\alpha+1}{2}} (g_{\theta,T}(h))^{\frac{1}{4}}.$$

Hence, for $\omega \in F_{N,K}^{(k)}$ we have that

$$\left\|\tau_h \left(v_N^{(k)}(\omega)\right)^{\frac{\alpha+3}{2}} - \left(v_N^{(k)}(\omega)\right)^{\frac{\alpha+3}{2}} \right\|_{L^2\left(0,T-h;L^{\frac{4}{\alpha+3}}(\mathbb{T}^2)\right)} \lesssim_{q,T} K^{\frac{\alpha+3}{2}} (g_{\theta,T}(h))^{\frac{1}{4}}.$$

and

$$\left\| \left(v_N^{(k)}(\omega)\right)^{\frac{\alpha+3}{2}} \right\|_{L^2(0,T;H^2(\mathbb{T}^2))} \le K$$

and therefore $v_N^{(k)}(\omega)$ lies in a compact subset of $L^2(0,T;H^1(\mathbb{T}^2))$ by [121, Theorem 5, p.84], which we denote by $\chi_{q,\alpha,T,K}$. From (2.5.13) we deduce that

$$\mathbb{P}\left(\left\{v_N^{(k)} \notin \chi_{q,\alpha,T,K}\right\}\right) \lesssim_{\Lambda,\alpha,q,T} \frac{1+\nu_{k,\alpha+3}}{K}.$$

The tightness assertion follows since the right-hand side goes uniformly in N to 0 as $K \to \infty$. □

2.5.2. THE TIME-STEP LIMIT

In this last subsection, we finally let $N \to \infty$ and show that the limit satisfies the assertions of Theorem 2.1.3. This time we consider the sequence

$$\left(\left(\mathbf{1}_{R_N^{(l)}}, \beta^{(l)}, u_N^{(l)}, v_N^{(l)}, w_N^{(l)}, J_N^{(l)}, (v_N^{(l)})^{\frac{\alpha+3}{2}}, (v_N^{(l)})^{\frac{\alpha+3}{4}}, (v_N^{(l)})^{\frac{\alpha+3}{2}}\right)_{l\in\mathbb{N}}\right)_{N\in\mathbb{N}} \tag{2.5.14}$$

in the topological space

$$\prod_{l=1}^{\infty} \mathbb{R} \times C([0,T]) \times C([0,T];L^2(\mathbb{T}^2)) \times L^\infty_{w*}(0,T;H^1(\mathbb{T}^2)) \times L^\infty_{w*}(0,T;H^1(\mathbb{T}^2))$$
$$\times L^2_w(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2)) \times L^2_w(0,T;H^2(\mathbb{T}^2)) \times L^4_w(0,T;W^{1,4}(\mathbb{T}^2)) \times L^2(0,T;H^1(\mathbb{T}^2)). \tag{2.5.15}$$

Notice that this differs from (2.4.17) by the additional appearance of $L^2(0,T;H^1(\mathbb{T}^2))$.

Corollary 2.5.5. *The sequence* (2.5.14) *is tight on* (2.5.15).

Proof. This can be shown analogously to Proposition 2.4.7, using Theorem 2.4.1 (v) and (vi) and invoking additionally the findings from Lemma 2.5.3 and Lemma 2.5.4. □

As in Subsection 2.4.2, we obtain that for a subsequence indexed by $\mathcal{N} \subset \mathbb{N}$, there exists a sequence of $\mathfrak{B}$-measurable random variables

$$\left(\left(\mathbf{1}_{\widetilde{R}_N^{(l)}}, \widetilde{\beta}_N^{(l)}, \widetilde{u}_N^{(l)}, \widetilde{v}_N^{(l)}, \widetilde{w}_N^{(l)}, \widetilde{J}_N^{(l)}, \widetilde{f}_N^{(l)}, \widetilde{g}_N^{(l)}, \widetilde{h}_N^{(l)}\right)_{l\in\mathbb{N}}\right)_{N\in\mathcal{N}}$$

defined on a complete probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{A}}, \widetilde{\mathbb{P}})$, such that

$$\left(\mathbf{1}_{\widetilde{R}_N^{(l)}}, \widetilde{\beta}_N^{(l)}, \widetilde{u}_N^{(l)}, \widetilde{v}_N^{(l)}, \widetilde{w}_N^{(l)}, \widetilde{J}_N^{(l)}, \widetilde{f}_N^{(l)}, \widetilde{g}_N^{(l)}, \widetilde{h}_N^{(l)}\right)_{l\in\mathbb{N}} \tag{2.5.16}$$

has the same distribution on (2.5.15) as

$$\left(\mathbf{1}_{R_N^{(l)}}, \beta^{(l)}, u_N^{(l)}, v_N^{(l)}, w_N^{(l)}, J_N^{(l)}, (v_N^{(l)})^{\frac{\alpha+3}{2}}, (v_N^{(l)})^{\frac{\alpha+3}{4}}, (v_N^{(l)})^{\frac{\alpha+3}{2}}\right)_{l\in\mathbb{N}} \tag{2.5.17}$$

for every $N \in \mathcal{N}$. Moreover, as $N \to \infty$, (2.5.16) converges to a $\mathfrak{B}$-measurable random variable

$$\left(\mathbf{1}_{\widetilde{R}^{(l)}}, \widetilde{\beta}^{(l)}, \widetilde{u}^{(l)}, \widetilde{v}^{(l)}, \widetilde{w}^{(l)}, \widetilde{J}^{(l)}, \widetilde{f}^{(l)}, \widetilde{g}^{(l)}, \widetilde{h}^{(l)}\right)_{l\in\mathbb{N}} \tag{2.5.18}$$

in (2.5.15). The following properties are inherited from the approximating sequence.

Lemma 2.5.6. *The sets* $(\widetilde{R}^{(k)})_{k\in\mathbb{N}}$ *form, up to* $\widetilde{\mathbb{P}}$*-null sets, a disjoint partition of* $\widetilde{\Omega}$*. Moreover, the following holds* $\widetilde{\mathbb{P}}$*-almost surely for every* $k \in \mathbb{N}$*.*

(i) The random variables

$$\widetilde{u}^{(k)},\ \widetilde{v}^{(k)},\ \widetilde{w}^{(k)},\ \widetilde{J}^{(k)},\ \widetilde{f}^{(k)},\ \widetilde{g}^{(k)}\ \textit{and}\ \ \widetilde{h}^{(k)}$$

vanish outside of of $\widetilde{R}^{(k)}$*.*

(ii) $\widetilde{u}^{(k)}(t) \geq 0$ *for all* $t \in [0, T]$*.*

(iii) $\widetilde{u}^{(k)} = \widetilde{v}^{(k)} = \widetilde{w}^{(k)}$*.*

(iv) $\widetilde{f}^{(k)} = \widetilde{h}^{(k)} = (\widetilde{u}^{(k)})^{\frac{\alpha+3}{2}}$ *and* $\widetilde{g}^{(k)} = (\widetilde{u}^{(k)})^{\frac{\alpha+3}{4}}$*.*

Proof. The claim regarding the sets $\widetilde{R}^{(k)}$, as well as part (i) and (ii) follow as in the proof of Lemma 2.4.9. For part (iii) we conclude first form (2.5.1) that $\widetilde{\mathbb{P}}$-almost surely

$$\widetilde{v}_N^{(k)}(t) = \widetilde{u}_N^{(k)}\left(j\delta + \tfrac{t-j\delta}{2}\right), \qquad t \in [j\delta, (j+1)\delta), \tag{2.5.19}$$

for almost all $t \in [0, T]$. Fixing such t that (2.5.19) holds for all $N \in \mathcal{N}$ and using that $\widetilde{u}_N^{(k)}$ converges uniformly to an $L^2(\mathbb{T}^2)$-continuous function we conclude that

$$\|\widetilde{v}_N^{(k)}(t) - \widetilde{u}^{(k)}(t)\|_{L^2(\mathbb{T}^2)} < \varepsilon \tag{2.5.20}$$

for sufficiently large N. It follows that $\widetilde{v}_N^{(k)} \to \widetilde{u}^{(k)}$ in $L^\infty(0,T;L^2(\mathbb{T}^2))$ and therefore the limit has to coincide with $\widetilde{v}^{(k)}$. The proof of $\widetilde{u}^{(k)} = \widetilde{w}^{(k)}$ works analogously. Since in contrast to the proof of Lemma 2.4.9 we cannot just rely on Proposition 2.2.4 for the identification of powers in (iv), we carry out the argument by hand. Since (2.5.16) and (2.5.17) have the same distribution, it holds

$$\widetilde{f}_N^{(k)} = (\widetilde{v}_N^{(k)})^{\frac{\alpha+3}{2}} \tag{2.5.21}$$

for every $N \in \mathcal{N}$. Due to the verified convergence $\widetilde{v}_N^{(k)} \to \widetilde{u}^{(k)}$ in $L^\infty(0,T;L^2(\mathbb{T}^2))$ it follows that the same convergence holds $[0,T]\times\mathbb{T}^2$-almost everywhere up to taking a subsequence. Moreover, since $\widetilde{v}_N^{(k)}$ is also weakly convergent in $L^\infty(0,T;H^1(\mathbb{T}^2))$, we conclude that it is uniformly in N bounded in $L^r([0,T]\times\mathbb{T}^2)$ for every $r>0$ by the Sobolev embedding theorem. Vitali's convergence theorem yields that

$$(\widetilde{v}_N^{(k)})^{\frac{\alpha+3}{2}} \to (\widetilde{u}^{(k)})^{\frac{\alpha+3}{2}}$$

in $L^r([0,T]\times\mathbb{T}^2)$ for every $r>0$. Invoking (2.5.21) and that $\widetilde{f}_N^{(k)} \rightharpoonup f^{(k)}$ in $L^2(0,T;H^2(\mathbb{T}^2))$ the identification $\widetilde{f}^{(k)} = (\widetilde{u}^{(k)})^{\frac{\alpha+3}{2}}$ follows. The remaining part of (iv) can be shown analogously. □

Proposition 2.5.7. *For all $\eta \in L^\infty(0,T;W^{2,\infty}(\mathbb{T}^2;\mathbb{R}^2))$ it holds*

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} \widetilde{J}^{(k)}\cdot\eta\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\{\widetilde{u}^{(k)}(s)>0\}} |\nabla\widetilde{u}^{(k)}|^2\nabla\widetilde{u}^{(k)}\cdot\eta\,\mathrm{d}x\,\mathrm{d}s\\ &+\int_0^T\int_{\{\widetilde{u}^{(k)}(s)>0\}} \widetilde{u}^{(k)}|\nabla\widetilde{u}^{(k)}|^2\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}s\\ &+2\int_0^T\int_{\{\widetilde{u}^{(k)}(s)>0\}} \widetilde{u}^{(k)}\,\nabla^T\widetilde{u}^{(k)}D\eta\nabla\widetilde{u}^{(k)}\,\mathrm{d}x\,\mathrm{d}s\\ &+\int_0^T\int_{\mathbb{T}^2}(\widetilde{u}^{(k)})^2\nabla\widetilde{u}^{(k)}\cdot\nabla\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

$\widetilde{\mathbb{P}}$-almost surely.

Proof. Since (2.5.16) and (2.5.17) have the same distribution, we conclude that

$$\begin{aligned}\int_{j\delta}^{(j+1)\delta}\int_{\mathbb{T}^2} \widetilde{J}_N^{(k)}\cdot\eta\,\mathrm{d}x\,\mathrm{d}t &= \int_{j\delta}^{(j+1)\delta}\int_{\{\widetilde{v}_N^{(k)}(s)>0\}} |\nabla\widetilde{v}_N^{(k)}|^2\nabla\widetilde{v}_N^{(k)}\cdot\eta\,\mathrm{d}x\,\mathrm{d}s\\ &+\int_{j\delta}^{(j+1)\delta}\int_{\{\widetilde{v}_N^{(k)}(s)>0\}} \widetilde{v}_N^{(k)}|\nabla\widetilde{v}_N^{(k)}|^2\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}s\\ &+2\int_{j\delta}^{(j+1)\delta}\int_{\{\widetilde{v}_N^{(k)}(s)>0\}} \widetilde{v}_N^{(k)}\,\nabla^T\widetilde{v}_N^{(k)}D\eta\nabla\widetilde{v}_N^{(k)}\,\mathrm{d}x\,\mathrm{d}s\\ &+\int_{j\delta}^{(j+1)\delta}\int_{\mathbb{T}^2}(\widetilde{v}_N^{(k)})^2\nabla\widetilde{v}_N^{(k)}\cdot\nabla\operatorname{div}\eta\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

by Theorem 2.4.1 (iii) for every $N \in \mathcal{N}$ and $j \in \{0,\dots,N\}$. Summing up over j yields that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^2} \widetilde{J}_N^{(k)} \cdot \eta \,\mathrm{d}x\,\mathrm{d}t = &\int_0^T \int_{\{\widetilde{v}_N^{(k)}(s)>0\}} |\nabla \widetilde{v}_N^{(k)}|^2 \nabla \widetilde{v}_N^{(k)} \cdot \eta \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^T \int_{\{\widetilde{v}_N^{(k)}(s)>0\}} \widetilde{v}_N^{(k)} |\nabla \widetilde{v}_N^{(k)}|^2 \operatorname{div}\eta \,\mathrm{d}x\,\mathrm{d}s \\ &+ 2\int_0^T \int_{\{\widetilde{v}_N^{(k)}(s)>0\}} \widetilde{v}_N^{(k)} \nabla^T \widetilde{v}_N^{(k)} D\eta \nabla \widetilde{v}_N^{(k)} \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^T \int_{\mathbb{T}^2} \widetilde{v}_N^{(k)} \nabla \widetilde{v}_N^{(k)} \cdot \nabla \operatorname{div}\eta \,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

It is left to take the limit $N \to \infty$ in the above equality, which works exactly as in the deterministic case [33, Corollary 2.7, Theorem 3.2]. □

Remark 2.5.8. We stress the role of the additional convergence $(\widetilde{v}_N^{(k)})^{\frac{\alpha+3}{2}} \to (\widetilde{u}^{(k)})^{\frac{\alpha+3}{2}}$ in $L^2(0,T;H^1(\mathbb{T}^2))$ for the limiting argument [33, Corollary 2.7, Theorem 3.2].

The previous statement shows that the weak formulation of $\widetilde{J}^{(k)} = (\widetilde{u}^{(k)})^2 \nabla\Delta(\widetilde{u}^{(k)})$ as in (2.1.6) is satisfied. We gather some more convergence and integrability results before we recover (2.1.7) as well. We note that we use again the convention to identify $\widetilde{v}_N^{(k)} \in L^\infty(0,T;H^1(\mathbb{T}^2))$ with its version defined by (2.5.19) as well as the rounding function $\lfloor\cdot\rfloor_\delta$ from the proof of Lemma 2.5.4.

Lemma 2.5.9. *For every $\varphi \in W^{1,q}(\mathbb{T}^2)$, $k \in \mathbb{N}$ and $t \in [0,T]$ it holds*

$$\langle \widetilde{v}_N^{(k)}(t), \varphi\rangle \to \langle \widetilde{u}^{(k)}(t), \varphi\rangle, \tag{2.5.22}$$

$$\int_0^t \langle \operatorname{div}(\widetilde{J}_N^{(k)}), \varphi\rangle \,\mathrm{d}s \to \int_0^t \langle \operatorname{div}(\widetilde{J}_N^{(k)}), \varphi\rangle \,\mathrm{d}s, \tag{2.5.23}$$

$$\begin{aligned}&\sum_{l=1}^\infty \int_0^{\lfloor t\rfloor_\delta} \lambda_l^2 \langle \operatorname{div}(\operatorname{div}(\widetilde{w}_N^{(k)}(s)\psi_l)\psi_l), \varphi\rangle \,\mathrm{d}s \\ &\quad \to \sum_{l=1}^\infty \int_0^t \lambda_l^2 \langle \operatorname{div}(\operatorname{div}(\widetilde{u}^{(k)}(s)\psi_l)\psi_l), \varphi\rangle \,\mathrm{d}s,\end{aligned} \tag{2.5.24}$$

$$\int_0^{\lfloor t\rfloor_\delta} \sum_{l=1}^\infty \lambda_l^2 \langle \operatorname{div}(\psi_l \widetilde{w}_N^{(k)}), \varphi\rangle^2 \,\mathrm{d}s \to \int_0^t \sum_{l=1}^\infty \lambda_l^2 \langle \operatorname{div}(\psi_l \widetilde{u}^{(k)}), \varphi\rangle^2 \,\mathrm{d}s, \tag{2.5.25}$$

$$\int_0^{\lfloor t\rfloor_\delta} \lambda_l \langle \operatorname{div}(\psi_l \widetilde{w}_N^{(k)}), \varphi\rangle \,\mathrm{d}\tau \to \int_0^t \lambda_l \langle \operatorname{div}(\psi_l \widetilde{u}^{(k)}), \varphi\rangle \,\mathrm{d}\tau, \tag{2.5.26}$$

$$\widetilde{\beta}_N^{(l)}(\lfloor t\rfloor_\delta) \to \widetilde{\beta}^{(l)}(t) \tag{2.5.27}$$

$\widetilde{\mathbb{P}}$-almost surely as $N \to \infty$.

Proof. The convergence (2.5.22) follows by (2.5.20). Part (2.5.23) is a direct consequence of $\widetilde{J}_N^{(k)} \rightharpoonup \widetilde{J}^{(k)}$ in $L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$. Next, we observe that

$$\|\psi_l \cdot \nabla(\psi_l \cdot \nabla\varphi)\|_{H^{-1}(\mathbb{T}^2)} \lesssim \|\varphi\|_{H^1(\mathbb{T}^2)}$$

2

due to (2.3.5). Using $\widetilde{w}_N^{(k)} \rightharpoonup^* \widetilde{u}^{(k)}$ in $L^\infty(0,T;H^1(\mathbb{T}^2))$ we deduce that

$$\sum_{l=1}^{\infty}\int_0^t \lambda_l^2 \langle \widetilde{w}_N^{(k)}(s), \psi_l\cdot\nabla(\psi_l\cdot\nabla\varphi)\rangle \,\mathrm{d}s \to \sum_{l=1}^{\infty}\int_0^t \lambda_l^2 \langle \widetilde{u}^{(k)}(s), \psi_l\cdot\nabla(\psi_l\cdot\nabla\varphi)\rangle \,\mathrm{d}s. \qquad (2.5.28)$$

Since weak-* convergent sequences are norm bounded, we obtain (2.5.24) by combining

$$\begin{aligned}&\left|\sum_{l=1}^{\infty}\lambda_l^2\int_{\lfloor t\rfloor_\delta}^t \langle \widetilde{w}_N^{(k)}(s), \psi_l\cdot\nabla(\psi_l\cdot\nabla\varphi)\rangle\,\mathrm{d}s\right|\\ &\quad\le \delta\sum_{l=1}^{\infty}\lambda_l^2\|\widetilde{w}_N^{(k)}\|_{L^\infty(0,T;H^1(\mathbb{T}^2))}\|\psi_l\cdot\nabla(\psi_l\cdot\nabla\varphi)\|_{H^{-1}(\mathbb{T}^2)}\end{aligned}$$

with (2.5.28). For (2.5.25), we estimate

$$\begin{aligned}&\left|\int_0^t\sum_{l=1}^{\infty}\lambda_l^2\langle \operatorname{div}(\psi_l\widetilde{w}_N^{(k)}),\varphi\rangle^2\,\mathrm{d}s - \int_0^t\sum_{l=1}^{\infty}\lambda_l^2\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle^2\,\mathrm{d}s\right|\\ &\le \sum_{l=1}^{\infty}\lambda_l^2\int_0^T\left|\langle\widetilde{w}_N^{(k)},\psi_l\cdot\nabla\varphi\rangle^2 - \langle\widetilde{u}^{(k)},\psi_l\cdot\nabla\varphi\rangle^2\right|\mathrm{d}s\\ &\lesssim \sum_{l=1}^{\infty}\lambda_l^2\int_0^T\left|\langle\widetilde{w}_N^{(k)},\psi_l\cdot\nabla\varphi\rangle - \langle\widetilde{u}^{(k)},\psi_l\cdot\nabla\varphi\rangle\right|\left(\left|\langle\widetilde{w}_N^{(k)},\psi_l\cdot\nabla\varphi\rangle\right| + \left|\langle\widetilde{u}^{(k)},\psi_l\cdot\nabla\varphi\rangle\right|\right)\mathrm{d}s,\end{aligned}$$

where we employed that

$$a^2 - b^2 \le 2|a-b|\max(|a|,|b|) \le 2|a-b|(|a|+|b|)$$

for $a, b \in \mathbb{R}$. Since $\|\psi_l\cdot\nabla\varphi\|_{L^2(\mathbb{T}^2)}$ is uniformly bounded in l, we obtain further that

$$\begin{aligned}&\left|\int_0^t\sum_{l=1}^{\infty}\lambda_l^2\langle \operatorname{div}(\psi_l\widetilde{w}_N^{(k)}),\varphi\rangle^2\,\mathrm{d}s - \int_0^t\sum_{l=1}^{\infty}\lambda_l^2\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle^2\,\mathrm{d}s\right|\\ &\lesssim_{\Lambda,\varphi}\int_0^T \|\widetilde{w}_N^{(k)}-\widetilde{u}^{(k)}\|_{L^2(\mathbb{T}^2)}\left(\|\widetilde{u}^{(k)}\|_{L^2(\mathbb{T}^2)} + \|\widetilde{w}_N^{(k)}\|_{L^2(\mathbb{T}^2)}\right)\mathrm{d}t \qquad (2.5.29)\\ &\lesssim_T \|\widetilde{w}_N^{(k)}-\widetilde{u}^{(k)}\|_{L^\infty(0,T;L^2(\mathbb{T}^2))}\left(\|\widetilde{u}^{(k)}\|_{L^\infty(0,T;L^2(\mathbb{T}^2))} + \sup_{N\in\mathcal{N}}\|\widetilde{w}_N^{(k)}\|_{L^\infty(0,T;L^2(\mathbb{T}^2))}\right).\end{aligned}$$

As a consequence of the proof of Lemma 2.5.6 (iii) we have $\widetilde{w}_N^{(k)} \to \widetilde{u}^{(k)}$ in $L^\infty(0,T;L^2(\mathbb{T}^2))$ as $N\to\infty$ and therefore the right-hand side of (2.5.29) tends to 0. Using that by the same arguments

$$\left|\int_{\lfloor t\rfloor_\delta}^t\sum_{l=1}^{\infty}\lambda_l^2\langle \operatorname{div}(\psi_l\widetilde{w}_N^{(k)}),\varphi\rangle^2\,\mathrm{d}s\right| \lesssim_{\Lambda,\varphi}\delta\|\widetilde{w}_N^{(k)}\|^2_{L^\infty(0,T;L^2(\mathbb{T}^2))},$$

we obtain indeed (2.5.25). The assertion (2.5.26) can be shown analogously to (2.5.24). The last convergence (2.5.27) is a consequence of $\widetilde{\beta}_N^{(l)} \to \widetilde{\beta}^{(l)}$ in $C([0,T])$. □

Lemma 2.5.10. *It holds for every $k \in \mathbb{N}$ that*

$$\widetilde{\mathbb{E}}\left[\sup_{0\le t\le T} \|\widetilde{u}^{(k)}\|_{H^1(\mathbb{T}^2)}^p\right] \lesssim_{\Lambda,p,T} \nu_{k,p},$$

$$\widetilde{\mathbb{E}}\left[\|\widetilde{J}^{(k)}\|_{L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))}^{\frac{p}{2}}\right] \lesssim_{\Lambda,p,q,T} \nu_{k,p},$$

$$\widetilde{\mathbb{E}}\left[\left\|\left(\widetilde{u}^{(k)}\right)^{\frac{\alpha+3}{2}}\right\|_{L^2(0,T;H^2(\mathbb{T}^2))}^2 + \left\|\left(\widetilde{u}^{(k)}\right)^{\frac{\alpha+3}{4}}\right\|_{L^4(0,T;W^{1,4}(\mathbb{T}^2))}^4\right] \lesssim_{\Lambda,\alpha,T} 1+\nu_{k,\alpha+3}.$$

Proof. This follows since (2.5.16) and (2.5.17) have the same distribution, the lower semi-continuity of the norm with respect to weak and weak-* convergence, as well as the bounds from Theorem 2.4.1 (v) and Lemma 2.5.3. □

Finally, we define, as in Subsection 2.4.2, $\widetilde{\mathfrak{F}}$ as the augmentation of the filtration $\widetilde{\mathfrak{G}}$ given by

$$\widetilde{\mathfrak{G}}_t = \sigma\left(\left\{\mathbf{1}_{\widetilde{R}^{(l)}}, \widetilde{J}^{(l)}|_{[0,t]} \,\middle|\, l\in\mathbb{N}\right\} \cup \left\{\widetilde{u}^{(l)}(s), \widetilde{\beta}^{(l)}(s) \,\middle|\, 0\le s\le t,\, l\in\mathbb{N}\right\}\right),$$

where we consider $J|_{[0,t]}$ again as a $\mathfrak{B}$-random variable in $L^2(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$.

Remark 2.5.11. The smallest σ-field $\widetilde{\mathfrak{H}}_t$ on $\widetilde{\Omega}$ such that $\phi(X)$ with

$$X = \left(\mathbf{1}_{\widetilde{R}^{(l)}}, \widetilde{\beta}^{(l)}|_{[0,t]}, \widetilde{u}^{(l)}|_{[0,t]}, \widetilde{J}^{(l)}|_{[0,t]}\right)_{l\in\mathbb{N}}$$

is measurable for every bounded and continuous function

$$\phi\colon \prod_{l=1}^{\infty} \mathbb{R}\times C([0,t])\times C([0,t];L^2(\mathbb{T}^2))\times L^2_w(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2)) \to \mathbb{R} \tag{2.5.30}$$

coincides with $\widetilde{\mathfrak{G}}_t$. Indeed, the inclusion $\widetilde{\mathfrak{G}}_t \subset \widetilde{\mathfrak{H}}_t$ follows since one can choose ϕ as a function depending only on one of the components of

$$\prod_{l=1}^{\infty} \mathbb{R}\times C([0,t])\times C([0,t];L^2(\mathbb{T}^2))\times L^2_w(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2)).$$

For the reverse inclusion $\widetilde{\mathfrak{H}}_t \subset \widetilde{\mathfrak{G}}_t$, we assume that ϕ as in (2.5.30) is bounded and continuous, so that it suffices to show that $\phi(X)$ is measurable with respect to $\widetilde{\mathfrak{G}}_t$. In particular, ϕ is continuous as mapping from

$$\prod_{l=1}^{\infty} \mathbb{R}\times C([0,t])\times C([0,t];L^2(\mathbb{T}^2))\times L^2(0,t;L^{q'}(\mathbb{T}^2;\mathbb{R}^2)) \tag{2.5.31}$$

into $\mathbb{R}$. But (2.5.31) is a complete separable metric space so that $\widetilde{\mathfrak{G}}_t$-$\mathfrak{B}$ measurability of the (2.5.31)-valued random variable X can be checked using a suitable family of functions separating the points by [22, Theorem 6.8.9].

Theorem 2.5.12. *The processes $(\widetilde{\beta}^{(l)})_{l\in\mathbb{N}}$ are a family of independent $\widetilde{\mathfrak{F}}$-Brownian motions. Moreover, we have for every $k\in\mathbb{N}$, $\varphi\in W^{1,q'}(\mathbb{T}^2)$ and $t\in[0,T]$*

$$\langle\widetilde{u}^{(k)}(t),\varphi\rangle-\langle\widetilde{u}^{(k)}(0),\varphi\rangle=\int_0^t-\langle\operatorname{div}(\widetilde{J}^{(k)}),\varphi\rangle\,\mathrm{d}s+\sum_{l=1}^{\infty}\int_0^t\lambda_l^2\langle\operatorname{div}(\operatorname{div}(\widetilde{u}^{(k)}(s)\psi_l)\psi_l),\varphi\rangle\,\mathrm{d}s$$
$$+\sum_{l=1}^{\infty}\lambda_l\int_0^t\langle\operatorname{div}(\widetilde{u}^{(k)}(s)\psi_l),\varphi\rangle\,\mathrm{d}\widetilde{\beta}_s^l,$$

$\widetilde{\mathbb{P}}$-almost surely.

Proof. For the claim regarding the family $(\widetilde{\beta}^{(l)})_{l\in\mathbb{N}}$ we refer to [51]. For the remainder of the proof we fix $k\in\mathbb{N}$ and $\varphi\in W^{1,q}(\mathbb{T}^2)$ and define the $\widetilde{\mathfrak{F}}$-adapted process

$$M(t)=\langle\widetilde{u}^{(k)}(t),\varphi\rangle-\langle\widetilde{u}^{(k)}(0),\varphi\rangle+\int_0^t\langle\operatorname{div}(\widetilde{J}^{(k)}),\varphi\rangle\,\mathrm{d}s$$
$$-\sum_{l=1}^{\infty}\int_0^t\lambda_l^2\langle\operatorname{div}(\operatorname{div}(\widetilde{u}^{(k)}(s)\psi_l)\psi_l),\varphi\rangle\,\mathrm{d}s$$

and the approximating processes

$$M_N(t)=\langle\widetilde{v}_N^{(k)}(t),\varphi\rangle-\langle\widetilde{u}_N^{(k)}(0),\varphi\rangle+\int_0^t\langle\operatorname{div}(\widetilde{J}^{(k)}),\varphi\rangle\,\mathrm{d}s$$
$$-\sum_{l=1}^{\infty}\int_0^{\lfloor t\rfloor_\delta}\lambda_l^2\langle\operatorname{div}(\operatorname{div}(\widetilde{u}^{(k)}(s)\psi_l)\psi_l),\varphi\rangle\,\mathrm{d}s$$

As a consequence of (2.5.22), (2.5.23) and (2.5.24), $M_N(t)$ converges to $M(t)$ for every $t\in[0,T]$ as $N\to\infty$. Moreover, we let

$$\phi\colon\prod_{l=1}^{\infty}\mathbb{R}\times C([0,s])\times C([0,s];L^2(\mathbb{T}^2))\times L_w^2(0,s;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))\longrightarrow\mathbb{R}\tag{2.5.32}$$

be bounded and continuous and consider the random variables

$$\rho=\phi\Big(\Big(\mathbf{1}_{\widetilde{R}^{(l)}},\widetilde{\beta}^{(l)}|_{[0,s]},\widetilde{u}^{(l)}|_{[0,s]},\widetilde{J}^{(l)}|_{[0,s]}\Big)_{l\in\mathbb{N}}\Big),$$
$$\rho_N=\phi\Big(\Big(\mathbf{1}_{\widetilde{R}_N^{(l)}},\widetilde{\beta}_N^{(l)}|_{[0,s]},\widetilde{u}_N^{(l)}|_{[0,s]},\widetilde{J}_N^{(l)}|_{[0,s]}\Big)_{l\in\mathbb{N}}\Big).$$

As consequence of the convergence of (2.5.16) to (2.5.18) in (2.5.15) we have $\rho_N\to\rho$ as $N\to\infty$. Using that as a consequence of Theorem 2.4.1 (iii) and (iv)

$$\langle v_N^{(k)}(t),\varphi\rangle-\langle u_N^{(k)}(0),\varphi\rangle+\int_0^t\langle\operatorname{div}(J_N^{(k)}),\varphi\rangle\,\mathrm{d}s$$
$$-\sum_{l=1}^{\infty}\int_0^{\lfloor t\rfloor_\delta}\lambda_l^2\langle\operatorname{div}(\operatorname{div}(w_N^{(k)}(s)\psi_l)\psi_l),\varphi\rangle\,\mathrm{d}s$$
$$=\sum_{l=1}^{\infty}\lambda_l\int_0^{\lfloor t\rfloor_\delta}\langle\operatorname{div}(w_N^{(k)}\psi_l),\varphi\rangle\,\mathrm{d}\beta_s^l,$$

we conclude by additionally invoking Theorem 2.4.1 (ii) that

$$\begin{aligned}
&\widetilde{\mathbb{E}}\big[(M_N(t)-M_N(s+\kappa))\rho_N\big] = 0,\\
&\widetilde{\mathbb{E}}\left[\left(M_N^2(t)-M_N^2(s+\kappa)-\int_{\lfloor s+\kappa\rfloor_\delta}^{\lfloor t\rfloor_\delta}\sum_{l=1}^{\infty}\lambda_l^2\,\langle \operatorname{div}(\psi_l\widetilde{w}_N^{(k)}),\varphi\rangle^2\,\mathrm{d}\tau\right)\rho_N\right] = 0,\\
&\widetilde{\mathbb{E}}\left[\left(\widetilde{\beta}_N^{(l)}(\lfloor t\rfloor_\delta)M_N(t)-\widetilde{\beta}_N^{(l)}(\lfloor s+\kappa\rfloor_\delta)M_N(s+\kappa)-\int_{\lfloor s+\kappa\rfloor_\delta}^{\lfloor t\rfloor_\delta}\lambda_l\,\langle \operatorname{div}(\psi_l\widetilde{w}_N^{(k)}),\varphi\rangle\,\mathrm{d}\tau\right)\rho_N\right] = 0,
\end{aligned}\tag{2.5.33}$$

for $s,t\in[0,T]$, $\kappa>0$ such that $s+\kappa\le t$, and N large enough so that $\lfloor s+\kappa\rfloor_\delta\ge s$. Due to Theorem 2.4.1 (v) and the Burkholder–Davis–Gundy inequality we have

$$\sup_{N\in\mathbb{N}}\widetilde{\mathbb{E}}\left[\|\widetilde{w}_N^{(k)}\|^p_{L^\infty(0,T;H^1(\mathbb{T}^2))}+\sup_{\tau\in[0,T]}|M_N(\tau)|^p+|\widetilde{\beta}_N^{(l)}(\tau)|^p\right]<\infty$$

for every $p\in(0,\infty)$ so that Vitali's convergence theorem and (2.5.25), (2.5.26), (2.5.27) yield

$$\begin{aligned}
&\widetilde{\mathbb{E}}\big[(M(t)-M(s+\kappa))\rho\big] = 0,\\
&\widetilde{\mathbb{E}}\left[\left(M^2(t)-M^2(s+\kappa)-\int_{s+\kappa}^{t}\sum_{l=1}^{\infty}\lambda_l^2\,\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle^2\,\mathrm{d}\tau\right)\rho\right] = 0,\\
&\widetilde{\mathbb{E}}\left[\left(\widetilde{\beta}^{(l)}(t)M(t)-\widetilde{\beta}^{(l)}(s+\kappa)M(s+\kappa)-\int_{s+\kappa}^{t}\lambda_l\,\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle\,\mathrm{d}\tau\right)\rho\right] = 0,
\end{aligned}\tag{2.5.34}$$

by letting $N\to\infty$ in (2.5.33). Using that

$$\left\{\left\{\phi\left(\left(1_{\widetilde{R}_N^{(l)}},\widetilde{\beta}_N^{(l)}|_{[0,s]},\widetilde{u}_N^{(l)}|_{[0,s]},\widetilde{J}_N^{(l)}|_{[0,s]}\right)_{l\in\mathbb{N}}\right)\in B\right\}\,\middle|\,\phi \text{ as in (2.5.32) cont. bdd., } B\in\mathfrak{B}(\mathbb{R})\right\}$$

is an intersection stable generator of $\widetilde{\mathfrak{G}}_t$ by Remark 2.5.11, we conclude that (2.5.34) holds for every $\widetilde{\mathfrak{G}}_t$-measurable and bounded random variable ρ. Finally, we let $0\le s'\le t'\le T$ and ρ be a $\widetilde{\mathfrak{F}}_{s'}$-measurable and bounded random variable. If we can show that

$$\begin{aligned}
&\widetilde{\mathbb{E}}\big[\big(M(t')-M(s')\big)\rho\big] = 0,\\
&\widetilde{\mathbb{E}}\left[\left(M^2(t')-M^2(s')-\int_{s'}^{t'}\sum_{l=1}^{\infty}\lambda_l^2\,\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle^2\,\mathrm{d}\tau\right)\rho\right] = 0,\\
&\widetilde{\mathbb{E}}\left[\left(\widetilde{\beta}^{(l)}(t)M(t')-\widetilde{\beta}^{(l)}(s')M(s')-\int_{s'}^{t'}\lambda_l\,\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle\,\mathrm{d}\tau\right)\rho\right] = 0,
\end{aligned}\tag{2.5.35}$$

the claim follows by [82, Proposition A.1], because M is $\widetilde{\mathfrak{F}}$-adapted and square-integrable due to Lemma 2.5.10. To this end, we let $\kappa'>0$ and define $\kappa=\frac{\kappa'}{2}$, $s=s'+\kappa$ and $t=t'+2\kappa$. By definition of the augmented filtration, there exists a $\widetilde{\mathbb{P}}$-version of ρ which is $\widetilde{\mathfrak{G}}_s$-measurable and moreover we have $s+\kappa\le t$. Therefore, we can apply (2.5.34) and rephrase in terms of s', t', κ' to deduce that

$$\begin{aligned}
&\widetilde{\mathbb{E}}\big[(M(t'+\kappa')-M(s'+\kappa'))\rho\big]=0,\\
&\widetilde{\mathbb{E}}\left[\left(M^2(t'+\kappa')-M^2(s'+\kappa')-\int_{s'+\kappa'}^{t'+\kappa'}\sum_{l=1}^{\infty}\lambda_l^2\,\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle^2\,\mathrm{d}\tau\right)\rho\right]=0,\\
&\widetilde{\mathbb{E}}\left[\left(\widetilde{\beta}^{(l)}(t'+\kappa')M(t'+\kappa')-\widetilde{\beta}^{(l)}(s'+\kappa')M(s'+\kappa')-\int_{s'+\kappa'}^{t'+\kappa'}\lambda_l\,\langle \operatorname{div}(\psi_l\widetilde{u}^{(k)}),\varphi\rangle\,\mathrm{d}\tau\right)\rho\right]=0.
\end{aligned}$$

Since $\kappa'>0$ was arbitrary, we can use continuity of $\widetilde{\beta}^{(l)}$ and M, Vitali's theorem and the consequence

$$\widetilde{\mathbb{E}}\left[\sup_{\tau\in[0,T]}\|\widetilde{u}^{(k)}(\tau)\|_{H^1(\mathbb{T}^2)}^p+|M(\tau)|^p+|\widetilde{\beta}^{(l)}(\tau)|^p\right]<\infty$$

of Lemma 2.5.10 to let $\kappa'\searrow 0$ and obtain (2.5.35). □

Finally, we put

$$\widetilde{u}=\sum_{k=0}^{\infty}\widetilde{u}^{(k)},\quad \widetilde{J}=\sum_{k=0}^{\infty}\widetilde{J}^{(k)},\tag{2.5.36}$$

which is in light of Lemma 2.5.6 equivalent to require

$$\widetilde{u}=\widetilde{u}^{(k)}\quad\text{and}\quad \widetilde{J}=\widetilde{J}^{(k)}\quad\text{on }\widetilde{R}^{(k)}.$$

Proof of Theorem 2.1.3. We first show that $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$, $\widetilde{\mathfrak{F}}$, $(\widetilde{\beta}^{(l)})_{l\in\mathbb{N}}$, $\widetilde{u}$ together with $\widetilde{J}$ constitute a solution to the stochastic thin-film equation with q'-regular nonlinearity in the sense of Definition 2.1.1. By definition, $\widetilde{\mathfrak{F}}$ fulfills the usual conditions and $(\widetilde{\beta}^{(l)})_{l\in\mathbb{N}}$ is a family of independent Brownian motions by Theorem 2.5.12. Furthermore, $\widetilde{u}$ and $\widetilde{J}$ are, as each of their summands, an $H^1_w(\mathbb{T}^2)$-continuous, $\widetilde{\mathfrak{F}}$-adapted process and a random variable in $L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$, respectively. Moreover, $\widetilde{J}|_{[0,t]}$ is $\widetilde{\mathfrak{F}}_t$-measurable by definition of $\widetilde{\mathfrak{F}}$ and we have $\sup_{0\le t\le T}\|\widetilde{u}(t)\|_{H^1(\mathbb{T}^2)}<\infty$ because of Lemma 2.5.10. Proposition 2.5.7 together with (2.5.36) yield that (2.1.6) is indeed fulfilled. Similarly, we obtain (2.1.7) from Theorem 2.5.12, (2.5.36) and the fact that one can pull the $\widetilde{\mathfrak{F}}_0$-measurable random variable $\mathbf{1}_{\widetilde{R}^{(k)}}$ outside of the stochastic integrals in (2.1.7). For the initial condition, we observe that

$$(\mathbf{1}_{\widetilde{R}_N^{(k)}},\widetilde{u}_N^{(k)}(0))_{k\in\mathbb{N}}\sim(\mathbf{1}_{R_N^{(k)}},u_N^{(k)}(0))_{k\in\mathbb{N}},\quad N\in\mathcal{N},$$

and

$$(\mathbf{1}_{\widetilde{R}_N^{(k)}},\widetilde{u}_N^{(k)}(0))_{k\in\mathbb{N}}\to(\mathbf{1}_{\widetilde{R}^{(k)}},\widetilde{u}^{(k)}(0))_{k\in\mathbb{N}}$$

as $N\to\infty$ in $(\mathbb{R}\times L^2(\mathbb{T}^2))^\infty$. Hence, we have

$$(\mathbf{1}_{\widetilde{R}^{(k)}},\widetilde{u}^{(k)}(0))_{k\in\mathbb{N}}\sim(\mathbf{1}_{R_N^{(k)}},u_N^{(k)}(0))_{k\in\mathbb{N}}$$

and therefore

$$\widetilde{u}(0)=\sum_{k=1}^{\infty}\widetilde{u}^{(k)}(0)\sim\sum_{k=1}^{\infty}u_N^{(k)}(0)\sim\mu$$

by Theorem 2.4.1 (i). The non-negativity of $\tilde{u}(t)$ follows from Lemma 2.5.6 (ii). From Lemma 2.5.10 together with the monotone convergence theorem we deduce the energy estimates (2.1.9) and (2.1.10). Finally due to Lemma 2.5.6 (iv) we conclude that the additional spatial regularity property (2.1.11) is by construction fulfilled. □

APPENDIX TO CHAPTER 2

2.A. PROPERTIES OF THE SOLUTIONS TO THE DETERMINISTIC THIN-FILM EQUATION

Proof of Theorem 2.2.3. Since $\alpha \in (-1,0)$ and $v \in H^1(\mathbb{T}^2)$ we have due to (2.1.14) that

$$\int_{\mathbb{T}^2} G_\alpha(v_0)\,\mathrm{d}x < \infty.$$

Therefore, [33, Theorem 3.2] applies and yields that a very weak solution (v, J) with q'-regular nonlinearity and initial value v_0 exists. The first part of (i) follows by testing the equation $\partial_t u = -\operatorname{div} J$ with $\varphi \otimes \mathbf{1}_{\mathbb{T}^2}$ for arbitrary $\varphi \in C_c^\infty((0,T))$. For the other properties we consider the approximation procedure in [33], which takes place in two steps. First problems of the form

$$\begin{cases} \partial_t(v_{\delta\varepsilon}) + \operatorname{div}(J_{\delta\varepsilon}) = 0, & \text{in } L^2(0,T;H^{-1}(\mathbb{T}^2)), \\ J_{\delta\varepsilon} = m_{\delta\varepsilon}(v_{\delta\varepsilon})\nabla\Delta v_{\delta\varepsilon}, & \text{weakly}, \\ \operatorname{esslim}_{t\to 0} v_{\delta\varepsilon}(t,\cdot) = v_0 + \delta + \varepsilon^\theta, & \text{in } H^1(\mathbb{T}^2), \end{cases} \tag{$P_{\delta\varepsilon}$}$$

are solved by [70, Theorem 1.1]. Letting $\varepsilon \searrow 0$ yields solutions to

$$\begin{cases} \partial_t(u_\delta) + \operatorname{div}(J_{\delta\varepsilon}) = 0, \\ J_\delta = m_\delta(v_\delta)\nabla\Delta v_\delta, \\ v_\delta(0,\cdot) = v_0 + \delta, \end{cases}$$

in the sense of [33, Definition 3.1] which again are used to construct v. The functions m_δ and $m_{\delta\varepsilon}$ are auxiliary mobilities, which take the form $m_\delta(\tau) = \frac{\tau^2}{1+\delta\tau^2}$ and $m_{\delta\varepsilon} = \frac{\tau^s m_\delta(\tau)}{\varepsilon m_\delta(\tau)+\tau^s}$ for some $s > 4$, see [33, p.323, p.331], so we can choose for example $s = 5$. The number θ from ($P_{\delta\varepsilon}$) is a sufficiently small constant. By [33, Lemma 2.1] it follows that

$$\|\nabla v_{\delta\varepsilon}\|_{L^\infty(0,T;L^2(\mathbb{T}^2;\mathbb{R}^2))} \le \|\nabla v_0\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)}. \tag{2.A.1}$$

Since $v_{\delta\varepsilon} \rightharpoonup^* v_\delta$ and $v_\delta \rightharpoonup^* v$ in $L^\infty(0,T;H^1(\mathbb{T}^2))$, see [33, Proposition 2.6], we conclude under additional consideration of Remark 2.2.2 that part (ii) holds true. Moreover, since also $v_{\delta\varepsilon} \rightharpoonup v_\delta$ and $v_\delta \rightharpoonup v$ in $H^1(0,T;W^{-1,q'}(\mathbb{T}^2))$, it follows that strong convergence takes place in $C([0,T];L^2(\mathbb{T}^2))$ by Remark 2.2.2. Hence non-negativity is preserved and the second part of (i) follows by the non-negativity of $v_{\delta\varepsilon}$, see [33, Lemma 2.1]. Furthermore, in [33, Equation (2.26)] one finds the estimate (iv). Finally, to convince ourselves also of part (iii), we conclude that as consequence of [33, Lemma 2.1] it holds

$$\operatorname*{ess\,sup}_{T-\rho\le t\le T} \|\nabla v_{\delta\varepsilon}(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + 2\int_0^{T-\rho} \|\sqrt{m_{\delta\varepsilon}(v_{\delta\varepsilon}(t))}\nabla\Delta v_{\delta\varepsilon}(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\,\mathrm{d}t \le \|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \tag{2.A.2}$$

for any $\rho > 0$. Since by definition

$$m_{\delta\varepsilon}(\tau) \le m_\delta(\tau) \le \tau^2$$

we obtain by Sobolev's inequality, see [1, Theorem 4.51], the periodic Poincaré inequality and (2.A.1) that

$$\begin{aligned}\|\sqrt{m_{\delta\varepsilon}(v_{\delta\varepsilon}(t))}\|^2_{L^r(\mathbb{T}^2)} &\le \|v_{\delta\varepsilon}(t)\|^2_{L^r(\mathbb{T}^2)} \lesssim_r \|v_{\delta\varepsilon}(t)\|^2_{H^1(\mathbb{T}^2)}\\ &\lesssim \|\nabla v_{\delta\varepsilon}(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_{\delta\varepsilon}(t)\,\mathrm{d}x\right|^2\\ &\le \|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0 + \delta + \varepsilon^\theta\,\mathrm{d}x\right|^2\end{aligned}$$

for any $0 \le t \le T$ and $r \in [1,\infty)$. Because we have $J_{\delta\varepsilon}(t) = m_{\delta\varepsilon}(v_{\delta\varepsilon}(t))\nabla\Delta v_{\delta\varepsilon}(t)$ for almost all $0 \le t \le T$, see [33, Lemma 2.1], we obtain by (2.A.1), (2.A.2), Hölder's inequality and the choice $\frac{1}{2} + \frac{1}{r} = \frac{1}{q'}$ that

$$\begin{aligned}&C_q \operatorname*{ess\,sup}_{T-\rho\le t\le T}\left[\|\nabla v_{\delta\varepsilon}(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\|\nabla v_{\delta\varepsilon}(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0 + \delta + \varepsilon^\theta\,\mathrm{d}x\right|^2\right)\right]\\ &+ \int_0^{T-\rho} \|J_{\delta\varepsilon}(t)\|^2_{L^{q'}(\mathbb{T}^2;\mathbb{R}^2)}\,\mathrm{d}t \le C_q\|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0 + \delta + \varepsilon^\theta\,\mathrm{d}x\right|^2\right).\end{aligned}$$

Using that $J_{\delta\varepsilon} \rightharpoonup J_\delta$ and $J_\delta \rightharpoonup J$ in $L^2(0,T;L^{q'}(\mathbb{T}^2;\mathbb{R}^2))$ as well as that $\nabla v_{\delta\varepsilon} \rightharpoonup^* \nabla v_\delta$ and $\nabla v_\delta \rightharpoonup^* \nabla v$ in $L^\infty(0,T;L^2(\mathbb{T}^2;\mathbb{R}^2))$, see [33, Proposition 2.6], we infer that

$$\begin{aligned}&C_q \operatorname*{ess\,sup}_{T-\rho\le t\le T}\left[\|\nabla v(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\|\nabla v(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0\,\mathrm{d}x\right|^2\right)\right]\\ &+ \int_0^{T-\rho} \|J(t)\|^2_{L^{q'}(\mathbb{T}^2;\mathbb{R}^2)}\,\mathrm{d}t \le C_q\|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\left(\|\nabla v_0\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + \left|\int_{\mathbb{T}^2} v_0\,\mathrm{d}x\right|^2\right)\end{aligned}$$

The claimed estimate follows by letting $\rho \searrow 0$ together with the weak continuity of v in $H^1(\mathbb{T}^2)$. □

2.B. GELFAND TRIPLE OF BESSEL-POTENTIAL SPACES

The purpose of this section is to verify that $H^2(\mathbb{T}^2) \subset H^1(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$ is a Gelfand triple, when equipping $H^2(\mathbb{T}^2)$ with the Bessel-potential norm, as claimed in the proof of Theorem 2.3.1. We recall that the Bessel-potential norm on $H^2(\mathbb{T}^2)$ is defined by

$$|||f|||^2_{H^2(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2} (1 + |2\pi k|^2)^2 |\widehat{f}(k)|^2,$$

where

$$\widehat{f}(k) = \int_{\mathbb{T}^2} f(x) e^{-2\pi i k\cdot x}\, \mathrm{d}x, \quad k \in \mathbb{Z}^2, \tag{2.B.1}$$

is the k-th Fourier coefficient of a function $f \in L^2(\mathbb{T}^2)$. Moreover, by definition of the Bessel-potential spaces

$$H^2(\mathbb{T}^2) = \left\{ f \in L^2(\mathbb{T}^2) \mid |||f|||_{H^2(\mathbb{T}^2)} < \infty \right\}.$$

The pairing of two functions $f \in H^1(\mathbb{T}^2)$, $g \in H^2(\mathbb{T}^2)$ in $H^1(\mathbb{T}^2)$ can be rewritten by Parseval's relation [69, Proposition 3.2.7] as

$$(f,g)_{H^1(\mathbb{T}^2)} = (f,g)_{L^2(\mathbb{T}^2)} + (\nabla f, \nabla g)_{L^2(\mathbb{T}^2;\mathbb{R}^2)} = \sum_{k\in\mathbb{Z}^2} (1+|2\pi k|^2) \widehat{f}(k) \overline{\widehat{g}(k)} \tag{2.B.2}$$

and therefore

$$|(f,g)_{H^1(\mathbb{T}^2)}| \le \left(\sum_{k\in\mathbb{Z}^2} |\widehat{f}(k)|^2 \right) \left(\sum_{k\in\mathbb{Z}^2} (1+|2\pi k|^2)^2 |\widehat{g}(k)|^2 \right) = \|f\|_{L^2(\mathbb{T}^2)} |||g|||_{H^2(\mathbb{T}^2)}$$

by the Cauchy-Schwarz inequality. Hence,

$$\|(f,\cdot)_{H^1(\mathbb{T}^2)}\|_{(H^2(\mathbb{T}^2))'} \le \|f\|_{L^2(\mathbb{T}^2)}. \tag{2.B.3}$$

Moreover, since the coefficients are square summable, the series

$$\sum_{k\in\mathbb{Z}} \frac{\widehat{f}(k)}{1+|2\pi k|^2} e^{2\pi i k\cdot x}$$

converges to an element $g_f \in L^2(\mathbb{T}^2)$. Since

$$|||g_f|||^2_{H^2(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2} (1+|2\pi k|^2)^2 \left(\frac{|\widehat{f}(k)|}{1+|2\pi k|^2} \right)^2 = \|f\|^2_{L^2(\mathbb{T}^2)} < \infty, \tag{2.B.4}$$

it satisfies $g_f \in H^2(\mathbb{T}^2)$. Using (2.B.2), we obtain that

$$(f, g_f)_{H^1(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2} (1+|2\pi k|^2) \widehat{f}(k) \frac{\overline{\widehat{f}(k)}}{1+|2\pi k|^2} = \|f\|^2_{L^2(\mathbb{T}^2)},$$

so that

$$\|(f,\cdot)_{H^1(\mathbb{T}^2)}\|_{(H^2(\mathbb{T}^2))'} \ge \|f\|_{L^2(\mathbb{T}^2)}$$

by (2.B.4). Due to (2.B.3), the above inequality is an equality. Consequently, identifying $f \in H^1(\mathbb{T}^2)$ with $(f,\cdot)_{H^1(\mathbb{T}^2)}$ and taking the completion of these functions with respect to

$$\|(f,\cdot)_{H^1(\mathbb{T}^2)}\|_{(H^2(\mathbb{T}^2))'}$$

yields the space $L^2(\mathbb{T}^2)$. With this identification, the dual pairing between a function $g \in H^2(\mathbb{T}^2)$ and a function $f \in L^2(\mathbb{T}^2)$ is given by

$$\langle\langle f, g\rangle\rangle_{H^1(\mathbb{T}^2)} = \langle f, g\rangle + \langle \nabla f, \nabla g\rangle,$$

where we recall that $\langle\cdot,\cdot\rangle$ is the dual pairing in $L^2(\mathbb{T}^2)$. Indeed, for $f \in H^1(\mathbb{T}^2)$ this follows since it was identified with $(f,\cdot)_{H^1(\mathbb{T}^2)}$. For $f \in L^2(\mathbb{T}^2)$ we take a sequence $(f_n)_{n\in\mathbb{N}}$ from $H^1(\mathbb{T}^2)$ converging to f in $L^2(\mathbb{T}^2)$. Then also $\nabla f_n \to \nabla f$ in $H^{-1}(\mathbb{T}^2;\mathbb{R}^2)$ and hence

$$\langle\langle f,g\rangle\rangle_{H^1(\mathbb{T}^2)} \leftarrow \langle\langle f_n,g\rangle\rangle_{H^1(\mathbb{T}^2)} = \langle f_n,g\rangle + \langle\nabla f_n,\nabla g\rangle \to \langle f,g\rangle + \langle\nabla f,\nabla g\rangle$$

for all $g \in H^2(\mathbb{T}^2)$.

2.C. JUSTIFICATIONS OF ITÔ'S FORMULA

First, we justify the use of Itô's formula in the proof of Lemma 2.4.3. To this end, we introduce the equivalence relation

$$f \sim g \quad \Longleftrightarrow \quad \exists c \in \mathbb{R}:\ f = g + c$$

for $f,g \in H^s(\mathbb{T}^2)$, $s \geq 0$ and write $\dot f$ for the respective equivalence class in $H^s(\mathbb{T}^2)$. We recall that the Bessel-potential space is given by

$$H^s(\mathbb{T}^2) = \left\{ f \in L^2(\mathbb{T}^2) \,|\, |||f|||_{H^s(\mathbb{T}^2)} < \infty \right\},$$

where the appearing Bessel-potential norm is defined as

$$|||f|||^2_{H^s(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2} (1+|2\pi k|^2)^s |\widehat{f}(k)|^2$$

with $\widehat{f}(k)$ being the k-th Fourier coefficient (2.B.1) of a function $f \in L^2(\mathbb{T}^2)$. Under this norm, the quotient space $H^s(\mathbb{T}^2)/\sim$ is equipped with

$$\begin{aligned}|||\dot f|||^2_{H^s(\mathbb{T}^2)/\sim} = \inf_{g\in\dot f} \sum_{k\in\mathbb{Z}^2} (1+|2\pi k|^2)^s |\widehat{g}(k)|^2 &= \inf_{g\in\dot f} |\widehat{g}(0)|^2 + \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} (1+|2\pi k|^2)^s |\widehat{f}(k)|^2 \\ &= \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} (1+|2\pi k|^2)^s |\widehat{f}(k)|^2.\end{aligned}$$

Here, we used in the second equality that

$$\int_{\mathbb{T}^2} e^{-2\pi k i\cdot x}\,\mathrm{d}x = 0$$

for $k \in \mathbb{Z}^2\setminus\{(0,0)\}$ and therefore

$$\widehat{f}(k) = \widehat{g}(k)$$

for all $g \in \dot f$. In the following, we write $\dot H^s(\mathbb{T}^2)$ for $H^s(\mathbb{T}^2)/\sim$ and equip it with the equivalent norm

$$\|\dot f\|^2_{\dot H^s(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} |2\pi k|^{2s} |\widehat{f}(k)|^2.$$

Analogously to Appendix 2.B, one verifies that $\dot H^0(\mathbb{T}^2)$ can be identified with the dual of $\dot H^2(\mathbb{T}^2)$ under the pairing in $\dot H^1(\mathbb{T}^2)$. Moreover, the dual pairing is given by

$$\langle\langle \dot f,\dot g\rangle\rangle_{\dot H^1(\mathbb{T}^2)} = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} |2\pi k|^2 \widehat{f}(k)\overline{\widehat{g}(k)} = \langle\nabla f,\nabla g\rangle \tag{2.C.1}$$

for $\dot{f} \in \dot{H}^1(\mathbb{T}^2)$ and $\dot{g} \in \dot{H}^2(\mathbb{T}^2)$ by Parseval's relation [69, Proposition 3.2.7]. For general $\dot{f} \in \dot{H}^0(\mathbb{T}^2)$ the equality (2.C.1) holds by an approximation argument as in Appendix 2.B.

Next, we denote by P_{hom} the operator mapping a function $f \in H^s(\mathbb{T}^2)$ to its equivalence class in $\dot{H}^s(\mathbb{T}^2)$, i.e., $P_{hom} f = \dot{f}$. Then

$$P_{hom} \in L(L^2(\mathbb{T}^2), \dot{H}^0(\mathbb{T}^2)) \cap L(H^1(\mathbb{T}^2), \dot{H}^1(\mathbb{T}^2))$$

and applying P_{hom} to equation (2.3.2) satisfied by w_ε on $[j\delta,(j+1)\delta)$ yields that

$$\begin{aligned} P_{hom} w_\varepsilon(t) = P_{hom} w_\varepsilon(j\delta) &+ \int_{j\delta}^t P_{hom} A^\varepsilon(w_\varepsilon(s))\,\mathrm{d}s \\ &+ \int_{j\delta}^t P_{hom} B(w_\varepsilon(s))\,\mathrm{d}V_t, \qquad t \in [j\delta,(j+1)\delta), \end{aligned} \tag{2.C.2}$$

where V is the cylindrical Wiener process in $H^2(\mathbb{T}^2;\mathbb{R}^2)$ given by

$$V_t = \sum_{l=1}^\infty \beta_t^{(l)} \psi_l. \tag{2.C.3}$$

Because of $P_{hom} \in L(H^2(\mathbb{T}^2), \dot{H}^2(\mathbb{T}^2))$, $w_\varepsilon \in L^2([j\delta,(j+1)\delta] \times \Omega; H^2(\mathbb{T}^2))$ by Theorem 2.3.1, and the boundedness of the operators A^ε and B, it holds

$$\begin{aligned} &P_{hom} w_\varepsilon \in L^2([j\delta,(j+1)\delta] \times \Omega; \dot{H}^2(\mathbb{T}^2)), \\ &P_{hom} A^\varepsilon(w_\varepsilon) \in L^2([j\delta,(j+1)\delta] \times \Omega; \dot{H}^0(\mathbb{T}^2)), && (2.C.4) \\ &P_{hom} B(w_\varepsilon) \in L^2([j\delta,(j+1)\delta] \times \Omega; L_2(H^2(\mathbb{T}^2;\mathbb{R}^2), \dot{H}^0(\mathbb{T}^2))). && (2.C.5) \end{aligned}$$

Moreover, because of right-continuity in $H^1(\mathbb{T}^2)$, w_ε admits a progressively measurable, $H^2(\mathbb{T}^2)$-valued $\mathrm{d}t \otimes \mathbb{P}$-version by [100, Exercise 4.2.3]. Since later in the proof of Lemma 2.4.3 we integrate in time and take the expectation anyways, we denote this progressive version again by w_ε to ease the notation. By continuity of the involved operators, also the processes (2.C.4) and (2.C.5) are progressive when choosing this $\mathrm{d}t \otimes \mathbb{P}$-version of w_ε, so that Itô's formula for the squared norm in $\dot{H}^1(\mathbb{T}^2)$ from [100, Theorem 4.2.5] is applicable to (2.C.2). Noting that by Parseval's relation, the norm in $\dot{H}^1(\mathbb{T}^2)$ can be written as

$$\|\dot{f}\|^2_{\dot{H}^1(\mathbb{T}^2)} = \|\nabla f\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)},$$

we obtain that

$$\begin{aligned} \|\nabla w_\varepsilon(t)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} &= \|\nabla w_\varepsilon(j\delta)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)} + 2\int_{j\delta}^t \langle \nabla w_\varepsilon(s), \nabla A^\varepsilon(w_\varepsilon(s))\rangle\,\mathrm{d}s \\ &+ 2\int_{j\delta}^t \langle \nabla w_\varepsilon(s), \nabla B(w_\varepsilon(s))\,\mathrm{d}V_s\rangle + \sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \|\nabla \operatorname{div}(w_\varepsilon(s)\psi_l)\|^2_{L^2(\mathbb{T}^2;\mathbb{R}^2)}\,\mathrm{d}s \end{aligned}$$

for $t \in [j\delta,(j+1)\delta)$. Writing the stochastic integral with respect to V as its series representation results in (2.4.3). Moreover, its quadratic variation is given by

$$4\int_{j\delta}^t \|\langle \nabla w_\varepsilon(s), \nabla B(w_\varepsilon(s))\,\cdot\rangle\|^2_{L_2(H^2(\mathbb{T}^2;\mathbb{R}^2),\mathbb{R})}\,\mathrm{d}s = 4\sum_{l=1}^\infty \lambda_l^2 \int_{j\delta}^t \langle \nabla \operatorname{div}(w_\varepsilon(s)\psi_l), \nabla w_\varepsilon(s)\rangle^2\,\mathrm{d}s. \tag{2.C.6}$$

2

Secondly, we justify the use of Itô's formula in the proof of Lemma 2.5.3, where we use instead [39, Proposition A.1]. Choosing $\psi = \mathbf{1}_{\mathbb{T}^2}$, $\varphi = G_{\alpha,\kappa}$ in the notation of this proposition, we see that the functional (2.5.5) is of the required form. Next, we observe that as a consequence of Theorem 2.4.1 (iv), the process $w_N^{(k)}$ satisfies

$$\mathrm{d}w_N^{(k)} = \operatorname{div}(G(t))\,\mathrm{d}t + H(t)\,\mathrm{d}V_t$$

on $[j\delta,(j+1)\delta)$, where

$$G(t) = \frac{1}{2}\sum_{l=1}^{\infty}\lambda_l^2 \operatorname{div}(w_N^{(k)}(t)\psi_l)\psi_l,$$
$$H(t)[v] = \sum_{l=1}^{\infty}\lambda_l (v,\psi_l)_{H^2(\mathbb{T}^2;\mathbb{R}^2)} \operatorname{div}(w_N^{(k)}(t)\psi_l), \quad v \in H^2(\mathbb{T}^2;\mathbb{R}^2)$$

and V_t as in (2.C.3). By Theorem 2.4.1 (v), we have

$$w_N^{(k)} \in L^2(\Omega; L^2(j\delta,(j+1)\delta; H^1(\mathbb{T}^2))) \tag{2.C.7}$$

and

$$w_N^{(k)} \in L^2(\Omega; C([j\delta,(j+1)\delta]; L^2(\mathbb{T}^2))),$$

if we replace its terminal value $w_N^{(k)}((j+1)\delta)$ by $v_N^{(k)}((j+1)\delta)$. By (2.3.5) and $\Lambda \in l^2(\mathbb{N})$ we have that

$$\begin{aligned}\|G(t)\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} &\le \frac{1}{2}\sum_{l=1}^{\infty}\lambda_l^2 \|\operatorname{div}(w_N^{(k)}(t)\psi_l)\psi_l\|_{L^2(\mathbb{T}^2;\mathbb{R}^2)} \\ &\lesssim \sum_{l=1}^{\infty}\lambda_l^2 \|(w_N^{(k)}(t)\|_{H^1(\mathbb{T}^2)} \lesssim_\Lambda \|(w_N^{(k)}(t)\|_{H^1(\mathbb{T}^2)}\end{aligned}$$

and consequently (2.C.7) implies that

$$G \in L^2(\Omega; L^2(j\delta,(j+1)\delta; L^2(\mathbb{T}^2;\mathbb{R}^2))).$$

Similarly, we estimate

$$\begin{aligned}\|H(t)\|^2_{L_2(H^2(\mathbb{T}^2;\mathbb{R}^2),L^2(\mathbb{T}^2))} &= \sum_{l=1}^{\infty}\lambda_l^2 \|\operatorname{div}(w_N^{(k)}(t)\psi_l)\|^2_{L^2(\mathbb{T}^2)} \\ &\lesssim \sum_{l=1}^{\infty}\lambda_l^2 \|w_N^{(k)}(t)\|^2_{H^1(\mathbb{T}^2)} \lesssim_\Lambda \|w_N^{(k)}(t)\|^2_{H^1(\mathbb{T}^2)},\end{aligned}$$

so that

$$H \in L^2(\Omega; L^2(j\delta,(j+1)\delta; L_2(H^2(\mathbb{T}^2;\mathbb{R}^2), L^2(\mathbb{T}^2)))).$$

Hence, all the assumptions of [39, Proposition A.1] are satisfied, which results in (2.5.6).

3

Existence with nonlinear noise: positive initial data†

In this chapter we construct weak martingale solutions to the Stratonovich interpretation of (STFE) with $n \in (2,3)$ and $d = 1$ for initial values which are positive almost everywhere. This can be seen as an extension of [35, Theorem 2.2], which is limited to $n \in [\frac{8}{3}, 4)$, to the case $n \in (2, \frac{8}{3})$. Indeed, the authors of [35] write 'We expect that the limitations $n \geq \frac{8}{3}$ and $n < 4$ are due to technical reasons and that these restrictions can be potentially removed in future work by making use of so-called α-entropies'. We employ this suggested strategy as follows: At the core of previous existence results for (STFE) are suitable approximations, which are compatible with the energy estimate. Namely, one formally derives for the deterministic thin-film equation

$$\partial_t u = -\partial_x(u^n \partial_x^3 u) \tag{3.0.1}$$

using integration by parts that

$$\tfrac{1}{2}\partial_t \|\partial_x u\|_{L_x^2}^2 = -\|u^{\frac{n}{2}} \partial_x^3 u\|_{L_x^2}^2. \tag{3.0.2}$$

If the energy estimate is fulfilled also by a sequence of approximations of (3.0.1), it provides sufficient compactness to show that a limit solves (3.0.1) in a weak sense, as demonstrated in [15]. For (STFE) with Stratonovich noise a formal application of Itô's formula shows that

$$\tfrac{1}{2}\mathrm{d}\|\partial_x u\|_{L_x^2}^2 = -\|u^{\frac{n}{2}} \partial_x^3 u\|_{L_x^2}^2 \,\mathrm{d}t + \mathrm{d}M + \mathrm{d}R,$$

where M is a local martingale and R is the remainder after canceling the energy production due to the noise with the energy dissipation of the Stratonovich correction. As

†This chapter is based on the preprint [117]: M. Sauerbrey. "Solutions to the stochastic thin-film equation for the range of mobility exponents $n \in (2,3)$". In: *arXiv preprint arXiv:2310.02765* (2024).

calculated in [35, Eq. (4.10)], this remainder term takes the form

$$\mathrm{d}R = \left(\ldots + \int u^{n-4}(\partial_x u)^4 \,\mathrm{d}x\right)\mathrm{d}t, \tag{3.0.3}$$

where the left out terms are less difficult to estimate than the one made explicit. In this chapter we use that this term appears in the log-entropy dissipation

$$\partial_t \int (u-1) - \log(u)\,\mathrm{d}x = -\int u^{n-2}(\partial_x^2 u)^2\,\mathrm{d}x - \tfrac{(n-2)(3-n)}{3}\int u^{n-4}(\partial_x u)^4\,\mathrm{d}x \tag{3.0.4}$$

for a solution to (3.0.1), which is indeed a special case of the α-entropy estimates as introduced in [11, Proposition 2.1]. This allows us to close an energy estimate and set up a compactness argument.

3.1. INTRODUCTION TO CHAPTER 3

We state the results of this chapter and outline the strategy of their proof.

3.1.1. MAIN RESULT

We consider the Stratonovich interpretation of (STFE) on the one-dimensional torus $\mathbb{T}$. To state our result, we first specify our assumption on the spatial smoothness of the noise. Following the notation of [35], we write

$$e_k(x) = \begin{cases} \sqrt{2}\cos(2\pi kx), & k<0, \\ 1, & k=0, \\ \sqrt{2}\sin(2\pi kx), & k>0, \end{cases} \qquad k\in\mathbb{Z},$$

for the eigenfunctions of the periodic Laplace operator and define $\sigma_k = \lambda_k e_k$ for a sequence $(\lambda_k)_{k\in\mathbb{Z}}$. We let $(\beta^{(k)})_{k\in\mathbb{Z}}$ be a family of independent Brownian motions and define the Wiener process

$$B(t,x) = \sum_{k\in\mathbb{Z}} \sigma_k(x)\beta^{(k)}(t).$$

We insert $\mathscr{W} = \frac{\mathrm{d}B}{\mathrm{d}t}$ in (STFE), set $F_0(r) = r^{\frac{n}{2}}$ and, as in [35, Eq. (2.4)], obtain the following Itô formulation of the Stratonovich interpretation of (STFE)

$$\mathrm{d}u = -\partial_x(F_0^2(u)\partial_x^3 u)\,\mathrm{d}t + \frac{1}{2}\sum_{k\in\mathbb{Z}}\partial_x(\sigma_k F_0'(u)\partial_x(\sigma_k F_0(u)))\,\mathrm{d}t + \sum_{k\in\mathbb{Z}}\partial_x(\sigma_k F_0(u))\,\mathrm{d}\beta^{(k)}. \tag{3.1.1}$$

We use the notion of weak martingale solutions to (3.1.1) from [35, Definition 2.1]. We point out that, in contrast to [35, Definition 2.1], we restrict our definition to the relevant case of non-negative solutions.

Definition 3.1.1. *A weak martingale solution to the SPDE (3.1.1) with initial value $u_0 \in L^2(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$ consists of a probability space $(\tilde{\Omega},\tilde{\mathfrak{A}},\tilde{\mathbb{P}})$ with a filtration $\tilde{\mathfrak{F}}$ satisfying the usual conditions, a family $(\tilde{\beta}^{(k)})_{k\in\mathbb{Z}}$ of independent $\tilde{\mathfrak{F}}$-Brownian motions and a weakly continuous, $\tilde{\mathfrak{F}}$-adapted, non-negative, $H^1(\mathbb{T})$-valued process $\tilde{u}$ such that* *(i)* $\widetilde{u}(0)$ *has the same distribution as* u_0,

(ii) $\widetilde{\mathbb{E}}[\sup_{t\in[0,T]}\|\widetilde{u}(t)\|^2_{H^1(\mathbb{T})}] < \infty$,

(iii) for $\widetilde{\mathbb{P}}\otimes \mathrm{d}t$*-almost all* $(\widetilde{\omega},t)\in\widetilde{\Omega}\times[0,T]$ *the weak derivative of third order* $\partial_x^3\widetilde{u}$ *exists on* $\{\widetilde{u}>0\}$ *and satisfies* $\widetilde{\mathbb{E}}[\|\mathbf{1}_{\{\widetilde{u}>0\}}F_0(\widetilde{u})\partial_x^3\widetilde{u}\|^2_{L^2([0,T]\times\mathbb{T})}] < \infty$,

(iv) for all $\varphi\in C^\infty(\mathbb{T})$, $\widetilde{\mathbb{P}}$*-almost surely, we have*

$$\begin{aligned}(\widetilde{u}(t),\varphi)_{L^2(\mathbb{T})} &= (\widetilde{u}(0),\varphi)_{L^2(\mathbb{T})} + \int_0^t\int_{\{\widetilde{u}(s)>0\}} F_0^2(\widetilde{u})\partial_x^3\widetilde{u}\,\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}s\\ &\quad -\frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t(\sigma_kF_0'(\widetilde{u})\partial_x(\sigma_kF_0(\widetilde{u})),\partial_x\varphi)_{L^2(\mathbb{T})}\,\mathrm{d}s\\ &\quad -\sum_{k\in\mathbb{Z}}\int_0^t(\sigma_kF_0(\widetilde{u}),\partial_x\varphi)_{L^2(\mathbb{T})}\,\mathrm{d}\widetilde{\beta}^{(k)},\end{aligned} \tag{3.1.2}$$

for all $t\in[0,T]$.

The main result of this chapter reads as follows. We remark that for $n\in[\frac{8}{3},3)$, a comparable existence result is already available, see [35, Theorem 2.2], and that $\mathscr{A}(u_0)$ denotes the spatial average of u_0.

Theorem 3.1.2. *Let* $T\in(0,\infty)$, $n\in(2,3)$, $p>n+2$ *and* $u_0\in L^p(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$ *non-negative with*

$$\mathbb{E}\Big[|\mathscr{A}(u_0)|^{\frac{p(n+2)}{8-2n}}\Big] + \mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big] < \infty \tag{3.1.3}$$

and moreover

$$\sum_{k\in\mathbb{Z}}\lambda_k^2k^4 < \infty. \tag{3.1.4}$$

Then (3.1.1) *admits a weak martingale solution*

$$\{(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathfrak{F}},\check{\mathbb{P}}),\,(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}},\,\widetilde{u}\}$$

in the sense of Definition 3.1.1 with initial value u_0 *satisfying* $\widetilde{u}>0$, $\widetilde{\mathbb{P}}\otimes\mathrm{d}t\otimes\mathrm{d}x$*-almost everywhere. Moreover, this solution satisfies the estimate*

$$\begin{aligned}&\widetilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\|\partial_x\widetilde{u}(t)\|^p_{L^2(\mathbb{T})} + \sup_{t\in[0,T]}\Big(\int_{\mathbb{T}}(\widetilde{u}(t)-1)-\log(\widetilde{u}(t))\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\ &\quad+\widetilde{\mathbb{E}}\Big[\|\mathbf{1}_{\{\widetilde{u}>0\}}F_0(\widetilde{u})\partial_x^3\widetilde{u}\|^p_{L^2([0,T]\times\mathbb{T})}\Big]\\ &\lesssim_{n,\sigma,p,T}\mathbb{E}\Big[\|\partial_xu_0\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u_0)|^{\frac{p(n+2)}{8-2n}} + \Big(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]+1\end{aligned} \tag{3.1.5}$$

and

$$\widetilde{u}\in L^q(\widetilde{\Omega};C^{\frac{\gamma}{4},\gamma}([0,T]\times\mathbb{T}))\quad \textit{for all}\quad \gamma\in\big(0,\tfrac{1}{2}\big)\quad\textit{and}\quad q\in\big[1,\tfrac{2p}{n+2}\big). \tag{3.1.6}$$

Moreover, in case that the initial value has finite entropy, we also recover an entropy estimate in the spirit of [35, Eq. (2.6)].

Proposition 3.1.3. *Under the assumptions of Theorem 3.1.2, the constructed solution $\tilde{u}$ satisfies*

$$\begin{aligned}\tilde{\mathbb{E}}\Big[\sup_{t\in[0,T]}\big\|(\tilde{u}(t))^{2-n}\big\|^{q}_{L^1(\mathbb{T})}+\|\partial_x^2\tilde{u}\|^{2q}_{L^2([0,T]\times\mathbb{T})}\Big]\\ \lesssim_{n,q,\sigma,T}\mathbb{E}\Big[\big\|u_0^{2-n}\big\|^{q}_{L^1(\mathbb{T})}+|\mathscr{A}(u_0)|^{2q}\Big]+1\end{aligned}\tag{3.1.7}$$

for each $q\geq 1$.

3.1.2. Strategy of the proof

As laid out in the beginning of this chapter, our idea is to use a stochastic version of the log-entropy estimate (3.0.4) to bound the energy production (3.0.3) due to the noise for suitable approximations of (3.1.1). While the entropy estimate used in [35] is compatible with non-negative approximations, the log-entropy or more generally α-entropy estimates require usually non-negative approximations. This is also evident from the deterministic case [11], where to find an approximating sequence of (3.0.1) which is in line with (3.0.4) the authors use a special nonlinear regularization, which ensures that the approximations are non-negative. We will follow this idea and first construct non-negative solutions to

$$\begin{aligned}\mathrm{d}u_\delta &= -\partial_x(F_\delta^2(u_\delta)\partial_x^3u_\delta)\,\mathrm{d}t\\ &\quad+\frac{1}{2}\sum_{k\in\mathbb{Z}}\partial_x(\sigma_kF_\delta'(u_\delta)\partial_x(\sigma_kF_\delta(u_\delta)))\,\mathrm{d}t+\sum_{k\in\mathbb{Z}}\partial_x(\sigma_kF_\delta(u_\delta))\mathrm{d}\beta^{(k)},\end{aligned}\tag{3.1.8}$$

where

$$F_\delta(r)=\frac{r^{\frac{n+\nu}{2}}}{r^{\frac{\nu}{2}}+\delta^{\frac{\nu-n}{2}}r^{\frac{n}{2}}},$$

for appropriate $\nu\in(3,4)$. In particular $F_\delta(r)\sim r^{\frac{\nu}{2}}$ for r close to 0, so that up to minor modifications the strategy from [35] can be employed. Subsequently, we use the aforementioned idea and show a version of the log-entropy estimate (3.0.4) and then of the energy estimate (3.0.2), which are uniform in δ. This again enables us to apply the stochastic compactness method to extract a weak martingale solution to (3.1.1) as $\delta\searrow 0$.

To implement the laid out idea, we need to carefully introduce approximate versions of the functions involved in the desired a-priori estimates, which is the content of Section 3.2. Further conventions regarding notation can be found in Section 3.3. The proof of existence of solutions to (3.1.8) is given in Section 3.4. Since this is mainly analogous to the proof of [35, Theorem 2.2], this section is kept as short as possible, but also detailed enough to explain how to treat the slightly more delicate situation of the inhomogeneous mobility function F_δ. Finally, in Section 3.5 we obtain a δ-uniform version of the energy and log-entropy estimate and based on this extract a solution to the original equation (3.1.1). Since this is the main innovation of this chapter, Section 3.5 is as self-contained as possible. This comes at the expense that some estimates derived in Section 3.5 are used in the preceding Section 3.4, instead of the other way around, which the author would like to excuse.

3.2. APPROXIMATE MOBILITIES AND FUNCTIONALS

For the rest of this chapter we fix $n \in (2,3)$ and a corresponding $\nu \in (3,4)$ subject to the conditions

$$\nu < 6 - n, \tag{3.2.1}$$

$$\nu^2 + \nu(2-4n) + n(n+2) \leq 0. \tag{3.2.2}$$

We convince ourselves that these conditions are compatible as follows. Since (3.2.1) is satisfied as soon as we choose ν close to 3, it suffices to check that

$$15 - 10n + n^2 = 3^2 + 3(2-4n) + n(n+2) < 0$$

for all $n \in (2,3)$. This, however, is an easy exercise. The assumption that $\nu < 4$ is necessary to ensure that the strategy of [35] carries over to Section 3.4, while the reasons for (3.2.1) and (3.2.2) are subtle, see, e.g., (3.5.17) and (3.5.44), where these conditions are used. We moreover use throughout the chapter the regularization parameters

$$\delta, \varepsilon \in (0,1) \tag{3.2.3}$$

and $R \in (0,\infty)$. Furthermore, to make this chapter more readable, we introduce the following notation:

$$l = \tfrac{\nu-n}{2},$$

$$F_0(r) = r^{\frac{n}{2}}, \quad r \geq 0,$$

$$F_\delta(r) = \frac{r^{\frac{n+\nu}{2}}}{r^{\frac{\nu}{2}} + \delta^l r^{\frac{n}{2}}}, \quad r \geq 0, \tag{3.2.4}$$

$$K_\varepsilon(r) = (r^2+\varepsilon^2)^{\frac{1}{2}},$$

$$F_{\delta,\varepsilon}(r) = F_\delta(K_\varepsilon(r)),$$

$$J_\delta(r) = \int_0^r \int_{r'}^\infty (F_\delta''(r''))^2 \,\mathrm{d}r''\,\mathrm{d}r', \quad r \geq 0, \tag{3.2.5}$$

$$J_{\delta,\varepsilon}(r) = \int_0^r \int_{r'}^\infty (F_{\delta,\varepsilon}''(r''))^2 \,\mathrm{d}r''\,\mathrm{d}r', \tag{3.2.6}$$

$$L_\delta(r) = \int_1^r \int_1^{r'} \frac{J_\delta(r'')}{F_\delta^2(r'')} \,\mathrm{d}r''\,\mathrm{d}r', \quad r > 0, \tag{3.2.7}$$

$$L_{\delta,\varepsilon}(r) = \int_1^r \int_1^{r'} \frac{J_{\delta,\varepsilon}(r'')}{F_{\delta,\varepsilon}^2(r'')} \,\mathrm{d}r''\,\mathrm{d}r', \tag{3.2.8}$$

$$G_\delta(r) = \int_r^\infty \int_{r'}^\infty \frac{1}{F_\delta^2(r'')} \,\mathrm{d}r''\,\mathrm{d}r', \quad r > 0,$$

$$G_{\delta,\varepsilon}(r) = \int_r^\infty \int_{r'}^\infty \frac{1}{F_{\delta,\varepsilon}^2(r'')} \,\mathrm{d}r''\,\mathrm{d}r', \tag{3.2.9}$$

$$H_{\delta,\varepsilon}(r) = \int_r^\infty \frac{1}{F_{\delta,\varepsilon}(r')} \,\mathrm{d}r'. \tag{3.2.10}$$

3

The parameter l is chosen in this way to ensure that the denominator of F_δ scales appropriately. While F_0 is the square-root of the original mobility u^n, F_δ is its approximation in the spirit of [11, Eq. (1.6)]. The smooth approximations of the modulus function K_ε and the corresponding mobility function $F_{\delta,\varepsilon}$ appear in the regularization procedure from [35]. To interpret J_δ and L_δ, we calculate that, when inserting F_0 instead of its modifications, one obtains

$$J_0(r) = \int_0^r \int_{r'}^\infty (F_0''(r''))^2 \,\mathrm{d}r''\,\mathrm{d}r' \eqsim_n \int_0^r \int_{r'}^\infty (r'')^{n-4} \,\mathrm{d}r''\,\mathrm{d}r' \eqsim_n r^{n-2}, \quad r > 0,$$

and consequently

$$L_0(r) = \int_1^r \int_1^{r'} \frac{J_0(r'')}{F_0^2(r'')} \,\mathrm{d}r''\,\mathrm{d}r' \eqsim_n \int_1^r \int_1^{r'} \frac{(r'')^{n-2}}{(r'')^n} \,\mathrm{d}r''\,\mathrm{d}r' = (r-1) - \log(r), \quad r > 0.$$

Hence, L_δ and $L_{\delta,\varepsilon}$ are approximations of the log-entropy functional.

Also a remark concerning the existence of the integrals in the definitions of J_δ and $J_{\delta,\varepsilon}$ is in order. We calculate explicitly that

$$F_\delta'(r) = r^{\frac{n+\nu}{2}-1} \frac{n r^{\frac{\nu}{2}} + \delta^l \nu r^{\frac{n}{2}}}{2(r^{\frac{\nu}{2}} + \delta^l r^{\frac{n}{2}})^2}, \quad r > 0, \tag{3.2.11}$$

$$F_\delta''(r) = r^{\frac{n+\nu}{2}-2} \frac{\delta^{2l}(\nu-2)\nu r^n - \delta^l(\nu^2+\nu(2-4n)+n(n+2)) r^{\frac{n+\nu}{2}} + n(n-2) r^\nu}{4(r^{\frac{\nu}{2}} + \delta^l r^{\frac{n}{2}})^3}, \quad r > 0, \tag{3.2.12}$$

from which we conclude

$$|F_\delta'(r)| \lesssim_{n,\nu} r^{\frac{n}{2}-1}, \quad r > 0, \tag{3.2.13}$$

$$|F_\delta''(r)| \lesssim_{n,\nu} r^{\frac{n}{2}-2}, \quad r > 0. \tag{3.2.14}$$

In particular (3.2.14) implies that

$$(F_\delta''(r))^2 \lesssim_{n,\nu} r^{n-4}, \quad r > 0,$$

is integrable at infinity with

$$\int_r^\infty (F_\delta''(r'))^2 \,\mathrm{d}r' \lesssim_{n,\nu} r^{n-3}, \quad r > 0.$$

This again is integrable at 0 so that (3.2.5) is well-defined.

To argue in the same way for the definition of $J_{\delta,\varepsilon}$, we observe first that by the chain rule

$$F_{\delta,\varepsilon}''(r) = F_\delta''(K_\varepsilon(r))(K_\varepsilon'(r))^2 + F_\delta'(K_\varepsilon(r))K_\varepsilon''(r) \tag{3.2.15}$$

and furthermore

$$K_\varepsilon'(r) = \frac{r}{(r^2+\varepsilon^2)^{\frac{1}{2}}} \in (-1,1), \tag{3.2.16}$$

$$K_\varepsilon''(r) = \frac{\varepsilon^2}{(r^2+\varepsilon^2)^{\frac{3}{2}}} \le \frac{1}{K_\varepsilon(r)}. \tag{3.2.17}$$

Combining (3.2.13)–(3.2.17), we conclude that

$$|F''_{\delta,\varepsilon}(r)| \lesssim_{n,\nu} K_\varepsilon^{\frac{n}{2}-2}(r) + \frac{K_\varepsilon^{\frac{n}{2}-1}(r)}{K_\varepsilon(r)} \approx K_\varepsilon^{\frac{n}{2}-2}(r). \tag{3.2.18}$$

Taking the square results in

$$(F''_{\delta,\varepsilon}(r))^2 \lesssim_{n,\nu} K_\varepsilon^{n-4}(r). \tag{3.2.19}$$

The right-hand side is integrable at infinity and since $F_{\delta,\varepsilon}$ is smooth also

$$\int_r^\infty (F''_{\delta,\varepsilon}(r'))^2 \, \mathrm{d}r'$$

defines a smooth function and in particular (3.2.6) is well-defined.

Since, by the previous considerations, the integrands in (3.2.7) and (3.2.8) are smooth functions on the domains of integration, also the definitions of L_δ and $L_{\delta,\varepsilon}$ are meaningful. Lastly, the functions G_δ, $G_{\delta,\varepsilon}$ and $H_{\delta,\varepsilon}$ are defined analogously to [35, Eq. (4.1)] and G_δ, $G_{\delta,\varepsilon}$ are approximations of the entropy function for (3.0.1).

3

3.3. FURTHER NOTATION FOR CHAPTER 3

For the remainder of this chapter we fix the finite time horizon $T \in (0,\infty)$ and a sequence $(\lambda_k)_{k\in\mathbb{Z}}$ subject to the condition (3.1.4). Moreover, we write $\mathbb{T}$ for the flat torus, i.e. for the interval $[0,1]$ with its endpoints identified. For a function f, we write f_+ and f_- for its positive and negative part, respectively. To not overload the subscript, we may also use the notations f^+ and f^-.

Let $\mathscr{X}$ be a Banach space and ν a non-negative measure on a measurable space S. Then we write $L^p(S;\mathscr{X})$, $p \in [1,\infty]$, for the Bochner space on S with values in $\mathscr{X}$, equipped with the norm

$$\|f\|^p_{L^p(S;\mathscr{X})} = \int_S \|f\|^p_{\mathscr{X}} \, \mathrm{d}\nu, \quad p \in [1,\infty),$$

and the usual modification for $p = \infty$. For a sub-σ-field $\mathfrak{H}$, we denote the subspace of $\mathfrak{H}$-measurable functions in $L^p(S;\mathscr{X})$ by $L^p(S,\mathfrak{H};\mathscr{X})$. If $[0,T]$ is an interval, we use $L^p(0,T;\mathscr{X})$ to denote $L^p([0,T];\mathscr{X})$. In the case $\mathscr{X} = \mathbb{R}$, we simply write $L^p(S)$. For a function $f \in L^1(\mathbb{T})$ we write

$$\mathscr{A}(f) = \int_{\mathbb{T}} f \, \mathrm{d}x$$

for the averaging operator.

If $\mathscr{T}$ is a compact topological space, we write $C(\mathscr{T};\mathscr{X})$ for the space of continuous, $\mathscr{X}$-valued functions equipped with the norm

$$\|f\|_{C(\mathscr{T};\mathscr{X})} = \sup_{y\in\mathscr{T}} \|f(y)\|_{\mathscr{X}}.$$

In the case $\mathscr{X} = \mathbb{R}$, we simply write $C(\mathscr{T})$. For $\beta \in (0,1)$, we write $C^\beta([0,T];\mathscr{X})$ for the Hölder space of $\mathscr{X}$-valued functions, which carries the norm

$$\|f\|_{C^\beta([0,T];\mathscr{X})} = \|f\|_{C([0,T];\mathscr{X})} + [f]_{C^\beta([0,T];\mathscr{X})},$$

where

$$[f]_{C^\beta([0,T];\mathscr{X})} = \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{\|f(t)-f(s)\|_{\mathscr{X}}}{|t-s|^\beta}.$$

For $l \in \mathbb{N}$, we denote the space of l-times continuously differentiable functions on $\mathbb{T}$ by $C^l(\mathbb{T})$ and equip it with the norm

$$\|f\|_{C^l(\mathbb{T})} = \sum_{j=0}^{l} \|\partial_x^j f\|_{C(\mathbb{T})}.$$

For the smooth functions on $\mathbb{T}$ we write $C^\infty(\mathbb{T})$. We also write $H^l(\mathbb{T})$ for the Sobolev space of order l equipped with the norm

$$\|f\|_{H^l(\mathbb{T})}^2 = \sum_{j=0}^{l} \|\partial_x^j f\|_{L^2(\mathbb{T})}^2.$$

For two exponents $\beta,\gamma \in (0,1)$, we write $C^{\beta,\gamma}([0,T]\times\mathbb{T})$ for the mixed exponent Hölder space carrying the norm

$$\|f\|_{C^{\beta,\gamma}([0,T]\times\mathbb{T})} = \|f\|_{C([0,T]\times\mathbb{T})} + [f]_{C^{\beta,\gamma}([0,T]\times\mathbb{T})},$$

where

$$[f]_{C^{\beta,\gamma}([0,T]\times\mathbb{T})} = \sup_{x\in\mathbb{T}} \sup_{\substack{s,t\in[0,T]\\ s\neq t}} \frac{|f(t,x)-f(s,x)|}{|t-s|^\beta} + \sup_{t\in[0,T]} \sup_{\substack{x,y\in\mathbb{T}\\ x\neq y}} \frac{|f(t,x)-f(t,y)|}{|x-y|^\gamma}.$$

If $(\Omega,\mathfrak{A},\mathbb{P})$ is a probability space, we write $\mathbb{E}[\cdot]$ for the expectation. For two quantities A and B, we write $A \lesssim B$, if there exists a universal constant C such that $A \le CB$. If this constant depends on parameters $p_1,\ldots$, we write $A \lesssim_{p_1,\ldots} B$ instead. We write $A \eqsim_{p_1,\ldots} B$, whenever $A \lesssim_{p_1,\ldots} B$ and $B \lesssim_{p_1,\ldots} A$.

3.4. SOLUTIONS TO THE STFE WITH INHOMOGENEOUS MOBILITY FUNCTION

The aim of this section is to construct weak martingale solutions to (3.1.8) and follows essentially along the lines of the proof [35, Theorem 2.2]. Since, unlike in the situation of [35, Theorem 2.2], F_δ is not homogeneous, the proof has to be adapted to our situation nevertheless and we include an appropriate amount of details.

The starting point is a Galerkin approximation to construct weak martingale solutions to the non-degenerate SPDE

$$\begin{aligned} \mathrm{d}u_{\delta,\varepsilon,R} &= -\partial_x(F_{\delta,\varepsilon}^2(u_{\delta,\varepsilon,R})\partial_x^3 u_\delta)\,\mathrm{d}t \\ &\quad + \frac{1}{2} g_R^2(\|u_{\delta,\varepsilon,R}\|_{C(\mathbb{T})}) \sum_{k\in\mathbb{Z}} \partial_x(\sigma_k F_{\delta,\varepsilon}'(u_{\delta,\varepsilon,R})\partial_x(\sigma_k F_{\delta,\varepsilon}(u_{\delta,\varepsilon,R})))\,\mathrm{d}t \\ &\quad + g_R(\|u_{\delta,\varepsilon,R}\|_{C(\mathbb{T})}) \sum_{k\in\mathbb{Z}} \partial_x(\sigma_k F_{\delta,\varepsilon}(u_{\delta,\varepsilon,R}))\,\mathrm{d}\beta^{(k)}. \end{aligned} \tag{3.4.1}$$

Here, $g\colon [0,\infty) \to [0,1]$ is a smooth function with $g(r) = 1$ for $r \in [0,1]$ and $g(r) = 0$ for $r \in [2,\infty)$ and $g_R(r) = g(\frac{r}{R})$. We use the following notion of weak martingale solutions to (3.4.1) from [35, Definition 3.2].

Definition 3.4.1. *A weak martingale solution to the SPDE* (3.4.1) *with initial value* $u_0 \in L^2(\Omega,\mathfrak{F}_0; H^1(\mathbb{T}))$ *consists of a probability space* $(\widehat{\Omega},\widehat{\mathfrak{A}},\widehat{\mathbb{P}})$ *with a filtration* $\widehat{\mathfrak{F}}$ *satisfying the usual conditions, a family* $(\widehat{\beta}^{(k)})_{k\in\mathbb{Z}}$ *of independent* $\widehat{\mathfrak{F}}$*-Brownian motions and a continuous,* $\widehat{\mathfrak{F}}$*-adapted,* $H^1(\mathbb{T})$*-valued process* $\widehat{u}_{\delta,\varepsilon,R}$ *such that*

(i) $\widehat{u}_{\delta,\varepsilon,R}(0)$ *has the same distribution as* u_0*,*

(ii) $\widehat{\mathbb{E}}\big[\sup_{t\in[0,T]} \|\widehat{u}_{\delta,\varepsilon,R}(t)\|^2_{H^1(\mathbb{T})}\big] < \infty$*,*

(iii) for $\widehat{\mathbb{P}}\otimes \mathrm{d}t$*-almost all* $(\widehat{\omega},t) \in \widehat{\Omega}\times[0,T]$ *the weak derivative of third order* $\partial_x^3 \widehat{u}_{\delta,\varepsilon,R}$ *exists and satisfies* $\widehat{\mathbb{E}}\big[\|F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x^3\widehat{u}_{\delta,\varepsilon,R}\|^2_{L^2([0,T]\times\mathbb{T})}\big] < \infty$*,*

(iv) for all $\varphi \in C^\infty(\mathbb{T})$*,* $\widehat{\mathbb{P}}$*-almost surely, we have*

$$
\begin{aligned}
(\widehat{u}_{\delta,\varepsilon,R}(t),\varphi)_{L^2(\mathbb{T})} &= (\widehat{u}_{\delta,\varepsilon,R}(0),\varphi)_{L^2(\mathbb{T})} + \int_0^t\int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x^3\widehat{u}_{\delta,\varepsilon,R}\,\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t g_R^2(\|\widehat{u}_{\delta,\varepsilon,R}\|_{C(\mathbb{T})})(\sigma_k F'_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x(\sigma_k F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})),\partial_x\varphi)_{L^2(\mathbb{T})}\,\mathrm{d}s\\
&\quad-\sum_{k\in\mathbb{Z}}\int_0^t g_R(\|\widehat{u}_{\delta,\varepsilon,R}\|_{C(\mathbb{T})})(\sigma_k F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}),\partial_x\varphi)_{L^2(\mathbb{T})}\,\mathrm{d}\widehat{\beta}^{(k)},
\end{aligned}
$$

for all $t\in[0,T]$.

The proof of existence of solutions to (3.4.1) is completely analogous to the one of [35, Proposition 3.4] with mobility exponent n.

Lemma 3.4.2. *Let* $p \geq n+2$ *and* $u_0 \in L^p(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$*, then there exists a weak martingale solution*

$$\{(\widehat{\Omega},\widehat{\mathfrak{A}},\widehat{\mathfrak{F}},\widehat{\mathbb{P}}),\,(\widehat{\beta}^{(k)})_{k\in\mathbb{Z}},\,\widehat{u}_{\delta,\varepsilon,R}\}$$

to (3.4.1) *in the sense of Definition 3.4.1 with initial value* u_0*. Moreover, this solution satisfies the estimate*

$$
\begin{aligned}
&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\|\partial_x\widehat{u}_{\delta,\varepsilon,R}(t)\|^p_{L^2(\mathbb{T})} + \|F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x^3\widehat{u}_{\delta,\varepsilon,R}\|^p_{L^2([0,T]\times\mathbb{T})}\Big]\\
&\lesssim_{\delta,\varepsilon,n,\nu,p,R,\sigma,T}\mathbb{E}\Big[\|\partial_x u_0\|^p_{L^2(\mathbb{T})}\Big]+1.
\end{aligned}
\tag{3.4.2}
$$

Proof. The proof of [35, Proposition 3.4] is based on a Galerkin scheme, which, by [35, Lemma 3.1], is compatible with the energy inequality for (3.4.1). With the energy estimate [35, Eq. (3.6)] at hand, it suffices to use that the function $F_{\delta,\varepsilon}$ is bounded away from zero and $F_{\delta,\varepsilon}(r) \sim |r|^{\frac{n}{2}}$ for $|r|$ large to take the limit in the Galerkin scheme. The estimate (3.4.2) follows by using lower semicontinuity of the norm with respect to weak-* convergence and Fatou's lemma to take $N\to\infty$ in [35, Eq. (3.6)]. □

We provide a version of the entropy estimate [35, Lemma 4.3], which is uniform in δ, ε and R.

Lemma 3.4.3. *Let $p \geq 1$ and $u_0 \in L^{n+2}(\Omega, \mathfrak{F}_0; H^1(\mathbb{T}))$. Then any weak martingale solution*

$$\{(\widehat{\Omega}, \widehat{\mathfrak{A}}, \widehat{\mathfrak{F}}, \widehat{\mathbb{P}}), (\widehat{\beta}^{(k)})_{k\in\mathbb{Z}}, \widehat{u}_{\delta,\varepsilon,R}\}$$

to (3.4.1) in the sense of Definition 3.4.1 with initial value u_0 satisfies

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \|G_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}(t))\|_{L^1(\mathbb{T})}^p + \|\partial_x^2 \widehat{u}_{\delta,\varepsilon,R}\|_{L^2([0,T]\times\mathbb{T})}^{2p}\Big] \lesssim_{n,\nu,p,\sigma,T} \mathbb{E}\Big[\|G_{\delta,\varepsilon}(u_0)\|_{L^1(\mathbb{T})}^p + |\mathscr{A}(u_0)|^{2p}\Big] + 1. \tag{3.4.3}$$

Proof. The proof follows along the lines of [35, Lemma 4.3] using the analogs of [35, Lemma 4.1, Lemma 4.2] provided in the following Lemma 3.4.4 and Lemma 3.4.5. □

Lemma 3.4.4. *It holds that*

$$H_{\delta,\varepsilon}^2(r) \lesssim_{n,\nu} G_{\delta,\varepsilon}(r). \tag{3.4.4}$$

Proof. We use that F_δ is increasing on $(0,\infty)$ together with (3.5.26) to conclude that

$$H_{\delta,\varepsilon}(r) \overset{(3.2.10),(3.5.26)}{\leq} \int_r^\infty \frac{1}{F_\delta\big(\frac{1}{\sqrt{2}}(r'+\varepsilon)\big)}\,\mathrm{d}r' \overset{(3.2.4)}{=} \int_r^\infty 2^{\frac{n}{4}}(r'+\varepsilon)^{-\frac{n}{2}} + 2^{\frac{\nu}{4}}\delta^l (r'+\varepsilon)^{-\frac{\nu}{2}}\,\mathrm{d}r' \eqsim_{n,\nu} (r+\varepsilon)^{1-\frac{n}{2}} + \delta^l (r+\varepsilon)^{1-\frac{\nu}{2}}, \quad r \geq 0. \tag{3.4.5}$$

Next, we use that

$$F_\delta(r) \leq r^{\frac{n}{2}}, \quad r > 0, \tag{3.4.6}$$

$$F_\delta(r) \leq \delta^{-l} r^{\frac{\nu}{2}}, \quad r > 0, \tag{3.4.7}$$

by (3.2.4) to derive

$$G_{\delta,\varepsilon}(r) \overset{(3.2.9),(3.5.26)}{\geq} \int_r^\infty \int_{r'}^\infty \frac{1}{F_\delta^2(r''+\varepsilon)}\,\mathrm{d}r''\,\mathrm{d}r' \overset{(3.4.6)}{\geq} \int_r^\infty \int_{r'}^\infty \frac{1}{(r''+\varepsilon)^n}\,\mathrm{d}r''\,\mathrm{d}r' \eqsim_n (r+\varepsilon)^{2-n}, \quad r \geq 0, \tag{3.4.8}$$

and

$$G_{\delta,\varepsilon}(r) \overset{(3.2.9),(3.4.7),(3.5.26)}{\gtrsim_\nu} \delta^{2l}(r+\varepsilon)^{2-\nu}, \quad r \geq 0.$$

Inserting these estimates in (3.4.5), we arrive at (3.4.4) for $r \geq 0$. For $r < 0$, we use that $F_{\delta,\varepsilon}$ is an even function together with $G_{\delta,\varepsilon}$ being decreasing to conclude that

$$H_{\delta,\varepsilon}^2(r) \overset{(3.2.10)}{\leq} (2H_{\delta,\varepsilon}(0))^2 \lesssim_{n,\nu} G_{\delta,\varepsilon}(0) \leq G_{\delta,\varepsilon}(r), \quad r < 0,$$

finishing the proof. □

Lemma 3.4.5. *It holds that*

$$|\log(F_{\delta,\varepsilon}(r))| \lesssim_{n,\nu} G_{\delta,\varepsilon}(r) + |r| + 1. \tag{3.4.9}$$

Proof. We distinguish several cases of r. Firstly we assume that $r \geq 0$ and $K_\varepsilon(r) \geq \delta$, in which case we deduce from (3.4.6), (3.5.15) and (3.5.26) that

$$\frac{1}{2^{\frac{n+4}{4}}}(r+\varepsilon)^{\frac{n}{2}} \leq \frac{1}{2}K_\varepsilon^{\frac{n}{2}}(r) \leq F_{\delta,\varepsilon}(r) \leq K_\varepsilon^{\frac{n}{2}}(r) \leq (r+\varepsilon)^{\frac{n}{2}}, \quad r \geq 0 \wedge K_\varepsilon(r) \geq \delta.$$

Thus, we have

$$\tfrac{n}{2}\log(r+\varepsilon) - \tfrac{n+4}{4}\log(2) \leq \log(F_{\delta,\varepsilon}(r)) \leq \tfrac{n}{2}\log(r+\varepsilon), \quad r \geq 0 \wedge K_\varepsilon(r) \geq \delta$$

and therefore

$$|\log(F_{\delta,\varepsilon}(r))| \lesssim_n |\log(r+\varepsilon)| + 1, \quad r \geq 0 \wedge K_\varepsilon(r) \geq \delta. \tag{3.4.10}$$

Next, we consider the case that $r \geq 0$ and $K_\varepsilon(r) < \delta$ and use (3.2.3), (3.4.6), (3.5.18) and (3.5.26) to estimate

$$\frac{1}{2^{\frac{\nu+4}{4}}}(r+\varepsilon)^{\frac{\nu}{2}} \leq \frac{1}{2\delta^l}K_\varepsilon^{\frac{\nu}{2}}(r) \leq F_{\delta,\varepsilon}(r) \leq K_\varepsilon^{\frac{n}{2}}(r) \leq 1, \quad r \geq 0 \wedge K_\varepsilon(r) < \delta$$

Hence,

$$\tfrac{\nu}{2}\log(r+\varepsilon) - \tfrac{\nu+4}{4}\log(2) \leq \log(F_{\delta,\varepsilon}(r)) \leq 0, \quad r \geq 0 \wedge K_\varepsilon(r) < \delta.$$

Combining this with (3.2.3), (3.4.8) and (3.4.10), we obtain

$$\begin{aligned} |\log(F_{\delta,\varepsilon}(r))| &\lesssim_{n,\nu} |\log(r+\varepsilon)| + 1 \\ &\lesssim_n (r+\varepsilon)^{2-n} + (r+\varepsilon) + 1 \lesssim_n G_{\delta,\varepsilon}(r) + r + 1, \quad r \geq 0. \end{aligned}$$

For $r < 0$, (3.4.9) follows as well, because $F_{\delta,\varepsilon}$ is even and $G_{\delta,\varepsilon}$ is decreasing. □

The following version of the energy estimate corresponding to [35, Lemma 4.6] is uniform in R and ε, but not in δ.

Lemma 3.4.6. *Let $p \geq 1$ and $u_0 \in L^{n+2}(\Omega, \mathfrak{F}_0; H^1(\mathbb{T}))$. Then any weak martingale solution*

$$\{(\widehat{\Omega}, \widehat{\mathfrak{A}}, \widehat{\mathfrak{F}}, \widehat{\mathbb{P}}), (\widehat{\beta}^{(k)})_{k\in\mathbb{Z}}, \widehat{u}_{\delta,\varepsilon,R}\}$$

to (3.4.1) in the sense of Definition 3.4.1 with initial value u_0 satisfies

$$\begin{aligned} &\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \|\partial_x \widehat{u}_{\delta,\varepsilon,R}(t))\|^p_{L^2(\mathbb{T})} + \Big(\int_0^T\int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p}{2}}\Big] \\ &\lesssim_{\delta,n,\nu,\sigma,p,T} \mathbb{E}\Big[\|\partial_x u_0\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u_0)|^{\frac{pn}{2}}\Big] + \widehat{\mathbb{E}}\Big[\|\partial_x^2\widehat{u}_{\delta,\varepsilon,R}\|^{2p}_{L^2([0,T]\times\mathbb{T})}\Big] + 1. \end{aligned} \tag{3.4.11}$$

Proof. To ease notation, we drop the hat notation and write u for $u_{\delta,\varepsilon,R}$ and abbreviate $\gamma_u = g_R(\|u\|_{C(\mathbb{T})})$ during this proof. Analogously to the proof of [35, Lemma 4.6], we start from (3.5.30) and estimate the terms on the right-hand side. We postpone again the technical parts to Lemmas 3.4.7–3.4.9, which can be seen as analogs of [35, Lemma 4.4, Lemma 4.5].

$(\partial_x u)^4$**-Term.** Using Lemma 3.4.7 and the interpolation inequality

$$\|\partial_x u\|_{L^\infty(\mathbb{T})} \lesssim \|\partial_x u\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \|\partial_x^2 u\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \tag{3.4.12}$$

from [35, Eq. (4.12)], we see that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} \sigma_k^2 (F''_{\delta,\varepsilon}(u))^2 (\partial_x u)^4 \,\mathrm{d}x\,\mathrm{d}s \\
&\overset{(3.5.4)}{\lesssim_\sigma} \int_0^t \int_{\mathbb{T}} (F''_{\delta,\varepsilon}(u))^2 (\partial_x u)^4 \,\mathrm{d}x\,\mathrm{d}s \\
&= -3 \int_0^t \int_{\mathbb{T}} \left(\int_1^u (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \right) (\partial_x u)^2 (\partial_x^2 u) \,\mathrm{d}x\,\mathrm{d}s \\
&\overset{(3.4.13)}{\lesssim_{\delta,n,\nu}} \int_0^t \|\partial_x u\|_{L^\infty(\mathbb{T})}^2 \|\partial_x^2 u\|_{L^2(\mathbb{T})} \,\mathrm{d}s \\
&\overset{(3.4.12)}{\lesssim} \int_0^t \|\partial_x u\|_{L^2(\mathbb{T})} \|\partial_x^2 u\|_{L^2(\mathbb{T})}^2 \,\mathrm{d}s \\
&\le \sup_{s\in[0,t]} \|\partial_x u(s)\|_{L^2(\mathbb{T})} \times \|\partial_x^2 u\|_{L^2([0,t]\times\mathbb{T})}^2 \\
&\le \kappa \sup_{s\in[0,t]} \|\partial_x u(s)\|_{L^2(\mathbb{T})}^2 + \tfrac{1}{\kappa} \|\partial_x^2 u\|_{L^2([0,T]\times\mathbb{T})}^4,
\end{aligned}$$

for each $\kappa > 0$.

$(\partial_x u)^3$**-Term.** Using Lemma 3.4.8 we retrieve moreover that

$$\begin{aligned}
&\left| \sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} \partial_x(\sigma_k^2) \big((F_{\delta,\varepsilon}^2)'''(u) + 4((F'_{\delta,\varepsilon})^2)'(u) \big) (\partial_x u)^3 \,\mathrm{d}x\,\mathrm{d}s \right| \\
&\overset{(3.4.14),(3.4.15),(3.5.4)}{\lesssim_{\delta,n,\nu,\sigma}} \int_0^t \int_{\mathbb{T}} |\partial_x u|^3 \,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^t \|\partial_x u\|_{L^\infty(\mathbb{T})}^3 \,\mathrm{d}s \\
&\overset{(3.4.12)}{\lesssim} \int_0^t \|\partial_x u\|_{L^2(\mathbb{T})} \|\partial_x^2 u\|_{L^2(\mathbb{T})}^2 \,\mathrm{d}s \\
&\le \sup_{s\in[0,t]} \|\partial_x u(s)\|_{L^2(\mathbb{T})} \|\partial_x^2 u\|_{L^2([0,t]\times\mathbb{T})}^2 \\
&\le \kappa \sup_{s\in[0,t]} \|\partial_x u(s)\|_{L^2(\mathbb{T})}^2 + \tfrac{1}{\kappa} \|\partial_x^2 u\|_{L^2([0,T]\times\mathbb{T})}^4,
\end{aligned}$$

for each $\kappa > 0$.

$(\partial_x u)^2$-Term. We use Lemma 3.4.9, the embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, Poincaré's inequality and the conservation of mass of u to conclude furthermore

$$
\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\int_0^t \gamma_u^2 \int_{\mathbb{T}} \big(8((\partial_x\sigma_k)^2 - \sigma_k(\partial_x^2\sigma_k))(F'_{\delta,\varepsilon}(u))^2 + (\partial_x^2(\sigma_k^2))(F^2_{\delta,\varepsilon})''(u)\big)(\partial_x u)^2 \,\mathrm{d}x\,\mathrm{d}s\right| \\
&\overset{(3.5.4)}{\lesssim_\sigma} \int_0^t \int_{\mathbb{T}} \big((F'_{\delta,\varepsilon}(u))^2 + \big|(F^2_{\delta,\varepsilon})''(u)\big|\big)(\partial_x u)^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\overset{(3.4.17),(3.4.18)}{\lesssim_{n,\nu}} \int_0^t \int_{\mathbb{T}} (|u|^{n-2}+1)(\partial_x u)^2 \,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^t (\|u\|^{n-2}_{L^\infty(\mathbb{T})}+1)\|\partial_x u\|^2_{L^2(\mathbb{T})}\,\mathrm{d}s \\
&\lesssim \int_0^t \big(\|\partial_x u\|^{n-2}_{L^2(\mathbb{T})} + |\mathscr{A}(u(0))|^{n-2} + 1\big)\|\partial_x^2 u\|^2_{L^2(\mathbb{T})}\,\mathrm{d}s \\
&\le \Big(\sup_{s\in[0,t]} \|\partial_x u(s)\|^{n-2}_{L^2(\mathbb{T})} + |\mathscr{A}(u(0))|^{n-2} + 1\Big) \times \|\partial_x^2 u\|^2_{L^2([0,t]\times\mathbb{T})} \\
&\overset{\frac{n-2}{2}+\frac{4-n}{2}=1}{\lesssim_n} \kappa \sup_{s\in[0,t]} \|\partial_x u(s)\|^2_{L^2(\mathbb{T})} + \kappa|\mathscr{A}(u(0))|^2 + \kappa + \left(\tfrac{1}{\kappa}\right)^{\frac{n-2}{4-n}} \|\partial_x^2 u\|^{\frac{4}{4-n}}_{L^2([0,T]\times\mathbb{T})},
\end{aligned}
$$

where again $\kappa > 0$.

$(\partial_x u)^0$-Term. We use once more Lemma 3.4.9, the embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, Poincaré's inequality and the conservation of mass of u to obtain

$$
\begin{aligned}
&\left|\sum_{k\in\mathbb{Z}}\int_0^t \gamma_u^2 \int_{\mathbb{T}} (4\sigma_k\partial_x^4\sigma_k - \partial_x^4(\sigma_k^2))F^2_{\delta,\varepsilon}(u)\,\mathrm{d}x\,\mathrm{d}s\right| \\
&\overset{(3.5.4),(3.5.31)}{\lesssim_\sigma} \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(u)\,\mathrm{d}x\,\mathrm{d}s \\
&\overset{(3.4.19)}{\lesssim_n} \int_0^t \int_{\mathbb{T}} |u|^n + 1\,\mathrm{d}x\,\mathrm{d}s \\
&\le \int_0^t \|u\|^n_{L^\infty(\mathbb{T})} + 1\,\mathrm{d}s \\
&\lesssim \int_0^t \big(\|\partial_x u\|_{L^2(\mathbb{T})} + |\mathscr{A}(u(0))|\big)^n + 1\,\mathrm{d}s \\
&\lesssim_{n,T} \int_0^t \|\partial_x u\|^{n-2}_{L^2(\mathbb{T})}\|\partial_x^2 u\|^2_{L^2(\mathbb{T})}\,\mathrm{d}s + |\mathscr{A}(u(0))|^n + 1 \\
&\le \sup_{s\in[0,t]} \|\partial_x u(s)\|^{n-2}_{L^2(\mathbb{T})}\|\partial_x^2 u\|^2_{L^2([0,t]\times\mathbb{T})} + |\mathscr{A}(u(0))|^n + 1 \\
&\overset{\frac{n-2}{2}+\frac{4-n}{2}=1}{\le} \kappa \sup_{t'\in[0,t]} \|\partial_x u\|^2_{L^2(\mathbb{T})} + |\mathscr{A}(u(0))|^n + 1 + \left(\tfrac{1}{\kappa}\right)^{\frac{n-2}{4-n}} \|\partial_x^2 u\|^{\frac{4}{4-n}}_{L^2([0,T]\times\mathbb{T})}
\end{aligned}
$$

for $\kappa > 0$.

Closing the estimate. We insert the previous estimates in the Itô expansion (3.5.30) and conclude that

$$\begin{aligned}&\tfrac{1}{2}\|\partial_x u(t)\|^2_{L^2(\mathbb{T})} + \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(u)(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le \tfrac{1}{2}\|\partial_x u(0)\|^2_{L^2(\mathbb{T})} + |M(t)| \\ &\qquad + C_{\delta,n,\nu,\sigma,T}\Big[\kappa \sup_{s\in[0,t]} \|\partial_x u(s)\|^2_{L^2(\mathbb{T})} + \tfrac{1}{\kappa}\|\partial_x^2 u\|^4_{L^2([0,T]\times\mathbb{T})} + (\tfrac{1}{\kappa})^{\frac{n-2}{4-n}}\|\partial_x^2 u\|^{\frac{4}{4-n}}_{L^2([0,T]\times\mathbb{T})}\Big] \\ &\qquad + C_{\delta,n,\nu,\sigma,T}\Big[|\mathscr{A}(u(0))|^n + \kappa|\mathscr{A}(u(0))|^2 + \kappa + 1\Big],\end{aligned}$$

where $M(t)$ is given by (3.5.32). Choosing κ small enough, taking the supremum in t until the stopping time

$$\tau_m = \inf\Big\{t\in[0,T]:\ \sup_{s\in[0,t]}\|\partial_x u(s)\|^2_{L^2(\mathbb{T})} + \int_0^t\int_{\mathbb{T}} (F_{\delta,\varepsilon}(u))^2(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s \ge m\Big\}$$

and absorbing the intermediate powers, we proceed to

$$\begin{aligned}&\tfrac{1}{4}\sup_{t\in[0,\tau_m]}\|\partial_x u(t)\|^2_{L^2(\mathbb{T})} + \int_0^{\tau_m}\int_{\mathbb{T}} F^2_{\delta,\varepsilon}(u)(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad\le \|\partial_x u(0)\|^2_{L^2(\mathbb{T})} + 2\sup_{t\in[0,\tau_m]}|M(t)| \\ &\qquad + C_{\delta,n,\nu,\sigma,T}\Big[|\mathscr{A}(u(0))|^n + \|\partial_x^2 u\|^4_{L^2([0,T]\times\mathbb{T})} + 1\Big].\end{aligned}$$

We raise both sides to the power $\frac{p}{2}$ and use the Burkholder–Davis–Gundy inequality to conclude

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,\tau_m]}\|\partial_x u(t)\|^p_{L^2(\mathbb{T})} + \Big(\int_0^{\tau_m}\int_{\mathbb{T}} F^2_{\delta,\varepsilon}(u)(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &\quad\lesssim_{\delta,n,\nu,\sigma,p,T} \mathbb{E}\Big[\|\partial_x u(0)\|^p_{L^2(\mathbb{T})} + \langle M\rangle^{\frac{p}{4}}_{\tau_m}\Big] \\ &\qquad + \mathbb{E}\Big[|\mathscr{A}(u(0))|^{\frac{pn}{2}} + \|\partial_x^2 u\|^{2p}_{L^2([0,T]\times\mathbb{T})}\Big] + 1.\end{aligned}$$

From (3.5.34), we deduce that

$$\begin{aligned}\mathbb{E}\Big[\langle M\rangle^{\frac{p}{4}}_{\tau_m}\Big] &\lesssim_{\sigma,p} \mathbb{E}\Big[\Big(\int_0^{\tau_m}\int_{\mathbb{T}}(F_{\delta,\varepsilon}(u))^2(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{4}}\Big] \\ &\le \kappa\,\mathbb{E}\Big[\Big(\int_0^{\tau_m}\int_{\mathbb{T}}(F_{\delta,\varepsilon}(u))^2(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big] + \frac{1}{\kappa}\end{aligned}$$

and therefore

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,\tau_m]}\|\partial_x u(t)\|^p_{L^2(\mathbb{T})} + \Big(\int_0^{\tau_m}\int_{\mathbb{T}} F^2_{\delta,\varepsilon}(u)(\partial_x^3 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big] \\ &\quad\lesssim_{\delta,n,\nu,\sigma,p,T} \mathbb{E}\Big[\|\partial_x u(0)\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u(0))|^{\frac{pn}{2}} + \|\partial_x^2 u\|^{2p}_{L^2([0,T]\times\mathbb{T})}\Big] + 1\end{aligned}$$

by choosing $\kappa > 0$ again sufficiently small. Letting $m\to\infty$, applying Fatou's lemma and using that $u(0)\sim u_0$, we obtain (3.4.11). □

Lemma 3.4.7. *It holds that*

$$\int_{-\infty}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \lesssim_{\delta,n,\nu} 1. \tag{3.4.13}$$

Proof. We use that $F_{\delta,\varepsilon}$ is an even function together with (3.2.18) and (3.5.13), to estimate

$$\begin{aligned}\int_{-\infty}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r &= 2\int_{0}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \\ &\lesssim_{\delta,n,\nu} \int_0^1 K_\varepsilon^{\nu-4}(r)\,\mathrm{d}r + \int_1^\infty K_\varepsilon^{n-4}(r)\,\mathrm{d}r \\ &\le \int_0^1 r^{\nu-4}\,\mathrm{d}r + \int_1^\infty r^{n-4}\,\mathrm{d}r \lesssim_{n,\nu} 1.\end{aligned}$$

□

Lemma 3.4.8. *It holds that*

$$\left|(F^2_{\delta,\varepsilon})'''(r)\right| \lesssim_{\delta,n,\nu} 1, \tag{3.4.14}$$

$$\left|((F'_{\delta,\varepsilon})^2)'(r)\right| \lesssim_{\delta,n,\nu} 1. \tag{3.4.15}$$

Proof. We observe that by (3.5.36) and (3.5.37), it suffices to show

$$(F_{\delta,\varepsilon}(r))^{\frac{1}{2}} (F''_{\delta,\varepsilon}(r))^{\frac{3}{2}} \lesssim_{\delta,n,\nu} 1 \tag{3.4.16}$$

to conclude (3.4.14) and (3.4.15). To this end, we combine (3.2.18) and (3.5.2) to conclude that

$$(F_{\delta,\varepsilon}(r))^{\frac{1}{2}} (F''_{\delta,\varepsilon}(r))^{\frac{3}{2}} \lesssim_{n,\nu} K_\varepsilon^{n-3}(r) \le |r|^{n-3} \le 1, \quad |r| \ge 1.$$

On the other hand, (3.5.13) and (3.5.28) yield that

$$(F_{\delta,\varepsilon}(r))^{\frac{1}{2}} (F''_{\delta,\varepsilon}(r))^{\frac{3}{2}} \lesssim_{\delta,n,\nu} K_\varepsilon^{\nu-3}(r) \overset{(3.2.3)}{\le} (1^2+1^2)^{\frac{\nu-3}{2}} \lesssim_\nu 1, \quad |r| < 1.$$

Consequently, (3.4.16) holds and the proof is complete. □

Lemma 3.4.9. *It holds that*

$$|F'_{\delta,\varepsilon}(r)|^2 \lesssim_{n,\nu} |r|^{n-2} + 1, \tag{3.4.17}$$

$$\left|(F^2_{\delta,\varepsilon})''(r)\right| \lesssim_{n,\nu} |r|^{n-2} + 1, \tag{3.4.18}$$

$$\left|(F^2_{\delta,\varepsilon}(r))\right| \lesssim_{n} |r|^{n} + 1. \tag{3.4.19}$$

Proof. As a consequence of (3.5.38) and (3.5.39), it suffices to verify

$$|F_{\delta,\varepsilon}(r)F''_{\delta,\varepsilon}(r)| \lesssim_{n,\nu} |r|^{n-2} + 1$$

to conclude (3.4.17) and (3.4.18). Combining (3.2.18) and (3.5.2), we estimate

$$|F_{\delta,\varepsilon}(r)F''_{\delta,\varepsilon}(r)| \lesssim_n K_\varepsilon^{n-2}(r) \le |r|^{n-2} + \varepsilon^{n-2} \overset{(3.2.3)}{\le} |r|^{n-2} + 1,$$

as desired. The remaining (3.4.19) follows from (3.5.2). □

For the sake of completeness, we repeat Definition 3.1.1 for the approximate equation (3.1.8). We remark that we restrict ourselves also here to the case of non-negative solutions.

Definition 3.4.10. *A weak martingale solution to the SPDE* (3.1.8) *with initial value* $u_0 \in L^2(\Omega, \mathfrak{F}_0; H^1(\mathbb{T}))$ *consists of a probability space* $(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathbb{P}})$ *with a filtration* $\check{\mathfrak{F}}$ *satisfying the usual conditions, a family* $(\check{\beta}^{(k)})_{k\in\mathbb{Z}}$ *of independent* $\check{\mathfrak{F}}$*-Brownian motions and a weakly continuous,* $\check{\mathfrak{F}}$*-adapted, non-negative,* $H^1(\mathbb{T})$*-valued process* $\check{u}_\delta$ *such that*

(i) $\check{u}_\delta(0)$ *has the same distribution as* u_0*,*

(ii) $\check{\mathbb{E}}\big[\sup_{t\in[0,T]} \|\check{u}_\delta(t)\|^2_{H^1(\mathbb{T})}\big] < \infty$*,*

(iii) *for* $\check{\mathbb{P}} \otimes \mathrm{d}t$*-almost all* $(\check{\omega}, t) \in \check{\Omega} \times [0, T]$ *the weak derivative of third order* $\partial_x^3 \check{u}_\delta$ *exists on* $\{\check{u}_\delta > 0\}$ *and satisfies* $\check{\mathbb{E}}\big[\|\mathbf{1}_{\{\check{u}_\delta>0\}} F_\delta(\check{u}_\delta)\partial_x^3 \check{u}_\delta\|^2_{L^2([0,T]\times\mathbb{T})}\big] < \infty$*,*

(iv) *for all* $\varphi \in C^\infty(\mathbb{T})$*,* $\check{\mathbb{P}}$*-almost surely, we have*

$$\begin{aligned}(\check{u}_\delta(t), \varphi)_{L^2(\mathbb{T})} &= (\check{u}_\delta(0), \varphi)_{L^2(\mathbb{T})} + \int_0^t \int_{\{\check{u}_\delta(s)>0\}} F_\delta^2(\check{u}_\delta)\partial_x^3 \check{u}_\delta\, \partial_x\varphi \,\mathrm{d}x\,\mathrm{d}s\\ &- \frac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t (\sigma_k F_\delta'(\check{u}_\delta)\partial_x(\sigma_k F_\delta(\check{u}_\delta)), \partial_x\varphi)_{L^2(\mathbb{T})}\,\mathrm{d}s\\ &- \sum_{k\in\mathbb{Z}} \int_0^t (\sigma_k F_\delta(\check{u}_\delta), \partial_x\varphi)_{L^2(\mathbb{T})}\,\mathrm{d}\check{\beta}^{(k)},\end{aligned}$$

for all $t \in [0, T]$*.*

Since, as in [35], we have obtained an (R,ε)-uniform estimate on the entropy and energy of the approximate solutions, we can take the limit $R \to \infty$ and then $\varepsilon \searrow 0$ to extract a weak martingale solution to (3.1.8) as in [35, Proposition 4.7, Section 5].

Lemma 3.4.11. *Let* $p > n + 2$ *and* $u_0 \in L^p(\Omega, \mathfrak{F}_0, H^1(\mathbb{T}))$ *with* $u_0 \geq 0$*,* $\mathbb{E}[|\mathscr{A}(u_0)|^{2p}] < \infty$ *and* $\mathbb{E}[\|G_\delta(u_0)\|^p_{L^1(\mathbb{T})}] < \infty$*. Then there exists a weak martingale solution*

$$\{(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathfrak{F}}, \check{\mathbb{P}}), (\check{\beta}^{(k)})_{k\in\mathbb{Z}}, \check{u}_\delta\}$$

to (3.1.8) *in the sense of Definition 3.4.10 with initial value* u_0*. Moreover, this solution satisfies the estimate*

$$\begin{aligned}&\check{\mathbb{E}}\Big[\sup_{t\in[0,T]} \|\partial_x \check{u}_\delta(t))\|^p_{L^2(\mathbb{T})} + \sup_{t\in[0,T]} \|G_\delta(\check{u}_\delta(t))\|^p_{L^1(\mathbb{T})}\Big]\\ &\quad + \check{\mathbb{E}}\Big[\|\mathbf{1}_{\{\check{u}_\delta>0\}} F_\delta(\check{u}_\delta)\partial_x^3(\check{u}_\delta)\|^p_{L^2([0,T]\times\mathbb{T})} + \|\partial_x^2 \check{u}_\delta\|^{2p}_{L^2([0,T]\times\mathbb{T})}\Big]\\ &\lesssim_{\delta,n,\nu,\sigma,p,T} \mathbb{E}\Big[\|\partial_x u_0\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u_0)|^{2p} + \|G_\delta(u_0)\|^p_{L^1(\mathbb{T})}\Big] + 1\end{aligned} \tag{3.4.20}$$

and in particular $\check{u}_\delta > 0$*,* $\check{\mathbb{P}} \otimes \mathrm{d}t \otimes \mathrm{d}x$*-almost everywhere.*

Proof. By Lemma 3.4.2, there exists for each R and ε a weak martingale solution

$$\{(\widehat{\Omega},\widehat{\mathfrak{A}},\widehat{\mathfrak{F}},\widehat{\mathbb{P}}),\ (\widehat{\beta}^{(k)})_{k\in\mathbb{Z}},\ \widehat{u}_{\delta,\varepsilon,R}\}$$

to (3.4.1) in the sense of Definition 3.4.1 with initial value u_0, where we notationally disregard the (R,ε)-dependence of the stochastic basis. Due Lemma 3.4.3 and Lemma 3.4.6, $\widehat{u}_{\delta,\varepsilon,R}$ satisfies the estimate

$$\begin{aligned}&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\|\partial_x\widehat{u}_{\delta,\varepsilon,R}(t))\|^p_{L^2(\mathbb{T})}+\sup_{t\in[0,T]}\|G_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}(t))\|^p_{L^1(\mathbb{T})}\Big]\\&\quad+\widehat{\mathbb{E}}\Big[\Big(\int_0^T\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{p}{2}}+\|\partial_x^2\widehat{u}_{\delta,\varepsilon,R}\|^{2p}_{L^2([0,T]\times\mathbb{T})}\Big]\\&\lesssim_{\delta,n,\nu,\sigma,p,T}\mathbb{E}\Big[\|\partial_x u_0\|^p_{L^2(\mathbb{T})}+|\mathscr{A}(u_0)|^{2p}+\|G_{\delta,\varepsilon}(u_0)\|^p_{L^1(\mathbb{T})}\Big]+1.\end{aligned}\tag{3.4.21}$$

The right-hand side is uniformly in R and ε bounded by [35, Remark 4.8] and we can take the limits $R\to\infty$ and $\varepsilon\searrow 0$ along the lines of [35, Proposition 4.7, Section 5]. Indeed, while our mobility function F_δ is not homogeneous, the approximations $F_{\delta,\varepsilon}$ still satisfy the growth bounds (3.5.2) and (3.5.3), which is sufficient to estimate the nonlinear terms appearing in (3.4.1) and thereby identify their limits. The estimate (3.4.20) follows by lower semicontinuity of the norm with respect to weak-* convergence and Fatou's lemma from (3.4.21). □

3.5. SOLUTIONS TO THE STFE WITH THE ORIGINAL MOBILITY FUNCTION

In this section we take the limit $\delta\searrow 0$ following the strategy explained in Section 3.1.2, i.e., we first derive a stochastic version of (3.0.4) and subsequently of (3.0.2). From a technical viewpoint, the main obstacle is the application of Itô's formula for the energy of $\check{u}_\delta$ for which one would require $\check{u}_\delta\in L^2(0,T;H^3(\mathbb{T}))$. We solve this issue by applying Itô's formula on the level of the approximations, namely for $\widehat{u}_{\delta,\varepsilon,R}$, instead. The possible negativity of $\widehat{u}_{\delta,\varepsilon,R}$ leads to problems when closing the estimate, which is reflected by the right hand side of (3.5.1) containing constants which depend on δ. As we will show in Lemma 3.5.5 and Lemma 3.5.7, the terms containing these constants disappear when $\varepsilon\searrow 0$. Before proceeding, we recall the notation $f^\pm$ and $f_\pm$ for the positive and negative part of a function f.

Lemma 3.5.1. *Let $p>n+2$ and $u_0\in L^\infty(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$ such that $u_0\geq\delta$. Then any weak martingale solution*

$$\{(\widehat{\Omega},\widehat{\mathfrak{A}},\widehat{\mathfrak{F}},\widehat{\mathbb{P}}),\ (\widehat{\beta}^{(k)})_{k\in\mathbb{Z}},\ \widehat{u}_{\delta,\varepsilon,R}\}$$

to (3.4.1) in the sense of Definition 3.4.1 with initial value u_0 constructed in Lemma 3.4.2 satisfies

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\Big(\int_{\mathbb{T}}L^+_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}(t))\,\mathrm{d}x\Big)^{\frac{p}{2}}+\Big(\int_0^T\int_{\mathbb{T}}J^+_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^2\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]$$

3

$$
\begin{aligned}
&+\widehat{\mathbb{E}}\bigg[\bigg(\int_0^T\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s\bigg)^{\frac{p}{2}}\bigg]\\
&\lesssim_{n,\nu,p,\sigma,T} C_{\delta,n,\nu,p,T}\widehat{\mathbb{E}}\Big[\|\widehat{u}^-_{\delta,\varepsilon,R}\|^p_{C([0,T]\times\mathbb{T})}\Big]^{\frac12}\Big(\mathbb{E}\Big[\|G_{\delta,\varepsilon}(u_0)\|^p_{L^1(\mathbb{T})}+|\mathscr{A}(u_0)|^{2p}\Big]+1\Big)^{\frac12}\\
&\quad+\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\|\widehat{u}^+_{\delta,\varepsilon,R}(t)\|^{(n-2)\frac{p}{2}}_{L^1(\mathbb{T})}\Big]+C_{\delta,n,\nu,p}\widehat{\mathbb{E}}\Big[\|\widehat{u}^-_{\delta,\varepsilon,R}\|^{(4-n)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}+\|\widehat{u}^-_{\delta,\varepsilon,R}\|^{(4-\nu)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}\Big]\\
&\quad+\mathbb{E}\bigg[\bigg(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\bigg)^{\frac{p}{2}}\bigg]. \qquad (3.5.1)
\end{aligned}
$$

Proof. Since we assumed $u_0\in L^\infty(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$, Lemma 3.4.2 is applicable. To ease notation we write u for $\widehat{u}_{\delta,\varepsilon,R}$ and drop the hat notation during this proof. We apply Itô's formula as in [39, Proposition A.1] to the composition of $L_{\delta,\varepsilon}$ with u and check its assumptions. Firstly, by (3.2.8), $L_{\delta,\varepsilon}$ admits $\frac{J_{\delta,\varepsilon}}{F^2_{\delta,\varepsilon}}$ as its second derivative, which needs to be bounded. Since $F_{\delta,\varepsilon}$ is bounded away from 0, we have

$$
\frac{|J_{\delta,\varepsilon}(r)|}{F^2_{\delta,\varepsilon}(r)}\lesssim_{\delta,\varepsilon}\frac{|J_{\delta,\varepsilon}(r)|}{F_{\delta,\varepsilon}(r)},
$$

which is seen to be bounded by the estimates (3.5.16) and (3.5.21) which we prove later. Furthermore, we observe that $u\in L^2(\Omega;C([0,T];H^1(\mathbb{T})))$ and additionally that

$$
F_{\delta,\varepsilon}(r)\le K_\varepsilon^{\frac{n}{2}}(r)\lesssim_n |r|^{\frac{n}{2}}+1 \qquad (3.5.2)
$$

by (3.2.3) and (3.2.4). Thus, also the quantity

$$
\begin{aligned}
&\mathbb{E}\Big[\|F^2_{\delta,\varepsilon}(u)\partial_x^3u\|^2_{L^2([0,T]\times\mathbb{T})}\Big]\\
&\le\mathbb{E}\Big[\|F_{\delta,\varepsilon}(u)\|^2_{C([0,T]\times\mathbb{T})}\|F_{\delta,\varepsilon}(u)\partial_x^3u\|^2_{L^2([0,T]\times\mathbb{T})}\Big]\\
&\le\mathbb{E}\Big[\|F_{\delta,\varepsilon}(u)\|^{\frac{2(n+2)}{n}}_{C([0,T]\times\mathbb{T})}\Big]^{\frac{n}{n+2}}\mathbb{E}\Big[\|F_{\delta,\varepsilon}(u)\partial_x^3u\|^{n+2}_{L^2([0,T]\times\mathbb{T})}\Big]^{\frac{2}{n+2}}\\
&\overset{(3.5.2)}{\lesssim_n}\mathbb{E}\Big[\|u\|^{n+2}_{C([0,T]\times\mathbb{T})}+1\Big]^{\frac{n}{n+2}}\mathbb{E}\Big[\|F_{\delta,\varepsilon}(u)\partial_x^3u\|^{n+2}_{L^2([0,T]\times\mathbb{T})}\Big]^{\frac{2}{n+2}}\\
&\le\Big(\mathbb{E}\Big[\|u\|^{n+2}_{C([0,T]\times\mathbb{T})}\Big]^{\frac{n}{n+2}}+1\Big)\times\mathbb{E}\Big[\|F_{\delta,\varepsilon}(u)\partial_x^3u\|^{n+2}_{L^2([0,T]\times\mathbb{T})}\Big]^{\frac{2}{n+2}}
\end{aligned}
$$

is finite due to (3.4.2). We observe that

$$
|F'_{\delta,\varepsilon}(r)|=|F'_\delta(K_\varepsilon(r))K'_\varepsilon(r)|\lesssim_{n,\nu}K_\varepsilon^{\frac{n}{2}-1}(r)\le|r|^{\frac{n}{2}-1}+1 \qquad (3.5.3)
$$

by the chain rule, (3.2.3), (3.2.13) and (3.2.16) and recall the consequence

$$
\sum_{k\in\mathbb{Z}}\|\sigma_k\|^2_{C^2(\mathbb{T})}<\infty \qquad (3.5.4)
$$

of condition (3.1.4), see [35, Eq. (2.2f)]. Hence, we can follow the calculations of Proposition 4.2.2 from Chapter 4 to obtain that

$$
\mathbb{E}\bigg[\int_0^T\int_{\mathbb{T}}\bigg|\sum_{k\in\mathbb{Z}}\sigma_kF'_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x(\sigma_kF_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))\bigg|^2\,\mathrm{d}x\,\mathrm{d}t\bigg]
$$

$$\overset{(3.5.2),(3.5.3),(3.5.4)}{\lesssim_{n,\nu,\sigma,T}} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u\|_{H^1(\mathbb{T})}^{2n-2}\Big] + 1$$

and

$$\mathbb{E}\Big[\sum_{k\in\mathbb{Z}} \int_0^T \|\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\|_{L^2(\mathbb{T})}^2 \,\mathrm{d}t\Big] \overset{(3.5.2),(3.5.3),(3.5.4)}{\lesssim_{n,\nu,\sigma,T}} \mathbb{E}\Big[\sup_{t\in[0,T]} \|u\|_{H^1(\mathbb{T})}^{n}\Big] + 1$$

are finite by (3.4.2). Hence, the assumptions of [39, Proposition A.1] are verified and it follows

$$\begin{aligned}
&\int_{\mathbb{T}} L_{\delta,\varepsilon}(u(t))\,\mathrm{d}x - \int_{\mathbb{T}} L_{\delta,\varepsilon}(u(0))\,\mathrm{d}x \\
&= \int_0^t \int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)\partial_x u\big(F^2_{\delta,\varepsilon}(u)\partial_x^3 u\big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)\partial_x u\big(\sigma_k F'_{\delta,\varepsilon}(u)\big)\partial_x(\sigma_k F_{\delta,\varepsilon}(u)\big)\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)(\partial_x(\sigma_k F_{\delta,\varepsilon}(u)))^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \sum_{k\in\mathbb{Z}} \int_0^t \gamma_u \int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)},
\end{aligned}$$

where we use again the notation $\gamma_u = g_R(\|u\|_{C(\mathbb{T})})$. Integrating by parts several times leads to

$$\begin{aligned}
&\int_{\mathbb{T}} L_{\delta,\varepsilon}(u(t))\,\mathrm{d}x - \int_{\mathbb{T}} L_{\delta,\varepsilon}(u(0))\,\mathrm{d}x \\
&\overset{(3.2.8)}{=} \int_0^t \int_{\mathbb{T}} J_{\delta,\varepsilon}(u)\partial_x u \partial_x^3 u\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)\partial_x\sigma_k F_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \sum_{k\in\mathbb{Z}} \int_0^t \gamma_u \int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)} \\
&= -\int_0^t \int_{\mathbb{T}} J_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^t \int_{\mathbb{T}} J'_{\delta,\varepsilon}(u)(\partial_x u)^2\partial_x^2 u\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)(\partial_x\sigma_k)^2(F_{\delta,\varepsilon}(u))^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t \gamma_u^2 \int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)(\partial_x\sigma_k)\sigma_k F_{\delta,\varepsilon}(u)F'_{\delta,\varepsilon}(u)\partial_x u\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \sum_{k\in\mathbb{Z}} \int_0^t \gamma_u \int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}
\end{aligned}$$

$$\overset{(3.2.8)}{=} -\int_0^t \int_{\mathbb{T}} J_{\delta,\varepsilon}(u)(\partial_x^2 u)^2 \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \tfrac{1}{3}\int_0^t \int_{\mathbb{T}} J''_{\delta,\varepsilon}(u)(\partial_x u)^4 \,\mathrm{d}x\,\mathrm{d}s$$
$$+ \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t \gamma_u^2 \int_{\mathbb{T}} J_{\delta,\varepsilon}(u)(\partial_x\sigma_k)^2 \,\mathrm{d}x\,\mathrm{d}s$$
$$- \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t \gamma_u^2 \int_{\mathbb{T}}\int_0^u L''_{\delta,\varepsilon}(r)F_{\delta,\varepsilon}(r)F'_{\delta,\varepsilon}(r)\,\mathrm{d}r\,\partial_x((\partial_x\sigma_k)\sigma_k)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \sum_{k\in\mathbb{Z}}\int_0^t \gamma_u \int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}.$$

We introduce the function

$$I_{\delta,\varepsilon}(r) = \int_0^r L''_{\delta,\varepsilon}(r')F_{\delta,\varepsilon}(r')F'_{\delta,\varepsilon}(r')\,\mathrm{d}r' \tag{3.5.5}$$

and use (3.2.6) to rearrange this to

$$\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(t))\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}} J^+_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s + \tfrac{1}{3}\int_0^t\int_{\mathbb{T}} (F''_{\delta,\varepsilon}(u))^2(\partial_x u)^4\,\mathrm{d}x\,\mathrm{d}s$$
$$= \int_0^t\int_{\mathbb{T}} J^-_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{T}} L^-_{\delta,\varepsilon}(u(t))\,\mathrm{d}x + \int_{\mathbb{T}} L_{\delta,\varepsilon}(u(0))\,\mathrm{d}x$$
$$+ \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t \gamma_u^2\int_{\mathbb{T}} J_{\delta,\varepsilon}(u)(\partial_x\sigma_k)^2\,\mathrm{d}x\,\mathrm{d}s - \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t \gamma_u^2\int_{\mathbb{T}} I_{\delta,\varepsilon}(u)\partial_x((\partial_x\sigma_k)\sigma_k)\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \sum_{k\in\mathbb{Z}}\int_0^t\int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}.$$

Taking absolute values, the $\frac{p}{2}$-th power and the supremum in time on both sides, this results in

$$\sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\right)^{\frac{p}{2}} + \left(\int_0^T\int_{\mathbb{T}} J^+_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}$$
$$+ \left(\int_0^T\int_{\mathbb{T}} (F''_{\delta,\varepsilon}(u))^2(\partial_x u)^4\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\lesssim_p \left(\int_0^T\int_{\mathbb{T}} J^-_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}} + \sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L^-_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\right)^{\frac{p}{2}} + \left(\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(0))\,\mathrm{d}x\right)^{\frac{p}{2}}$$
$$+ \left(\sum_{k\in\mathbb{Z}}\int_0^T\int_{\mathbb{T}} |J_{\delta,\varepsilon}(u)|(\partial_x\sigma_k)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}} + \left(\sum_{k\in\mathbb{Z}}\int_0^T\int_{\mathbb{T}} |I_{\delta,\varepsilon}(u)\partial_x((\partial_x\sigma_k)\sigma_k)|\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}$$
$$+ \sup_{t\in[0,T]}\left|\sum_{k\in\mathbb{Z}}\int_0^t\int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\,\mathrm{d}\beta_s^{(k)}\right|^{\frac{p}{2}}.$$

Taking the expectation and applying the Burkholder–Davis–Gundy inequality, we conclude

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\right)^{\frac{p}{2}}+\left(\int_0^T\int_{\mathbb{T}} J^+_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(u))^2(\partial_x u)^4\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\lesssim_p \mathbb{E}\left[\left(\int_0^T\int_{\mathbb{T}} J^-_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L^-_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\right)^{\frac{p}{2}}+\left(\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(0))\,\mathrm{d}x\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}}\int_0^T\int_{\mathbb{T}}|J_{\delta,\varepsilon}(u)|(\partial_x\sigma_k)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}}\int_0^T\int_{\mathbb{T}}|I_{\delta,\varepsilon}(u)\partial_x((\partial_x\sigma_k)\sigma_k)|\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left(\sum_{k\in\mathbb{Z}}\int_0^T\left(\int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x\right)^2\mathrm{d}s\right)^{\frac{p}{4}}\right].
\end{aligned}$$

To simplify the last term, we obtain through integration by parts that

$$\begin{aligned}
\int_{\mathbb{T}} L'_{\delta,\varepsilon}(u)\partial_x(\sigma_k F_{\delta,\varepsilon}(u))\,\mathrm{d}x &= -\int_{\mathbb{T}} L''_{\delta,\varepsilon}(u)\partial_x u\sigma_k F_{\delta,\varepsilon}(u)\,\mathrm{d}x\\
&=\int_{\mathbb{T}}\int_0^u L''_{\delta,\varepsilon}(r)F_{\delta,\varepsilon}(r)\,\mathrm{d}r\,\partial_x\sigma_k\,\mathrm{d}x.
\end{aligned}$$

Using again (3.5.4), it follows

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\right)^{\frac{p}{2}}+\left(\int_0^T\int_{\mathbb{T}} J^+_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(u))^2(\partial_x u)^4\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\lesssim_{p,\sigma} \mathbb{E}\left[\left|\int_0^T\int_{\mathbb{T}} J^-_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_{\mathbb{T}} L^-_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\right|^{\frac{p}{2}}+\left|\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(u(0))\,\mathrm{d}x\right|^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left|\int_0^T\int_{\mathbb{T}}|J_{\delta,\varepsilon}(u)|\,\mathrm{d}x\,\mathrm{d}s\right|^{\frac{p}{2}}+\left|\int_0^T\int_{\mathbb{T}}|I_{\delta,\varepsilon}(u)|\,\mathrm{d}x\,\mathrm{d}s\right|^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left|\int_0^T\left(\int_{\mathbb{T}}\left|\int_0^u L''_{\delta,\varepsilon}(r)F_{\delta,\varepsilon}(r)\,\mathrm{d}r\right|\mathrm{d}x\right)^2\mathrm{d}s\right|^{\frac{p}{4}}\right].
\end{aligned} \tag{3.5.6}$$

We use the properties of the approximate functions appearing in the right-hand side of (3.5.6), whose proof is postponed to the following Lemmas 3.5.2–3.5.4, to estimate

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}\Big(\int_{\mathbb{T}} L^{+}_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\Big)^{\frac{p}{2}}+\Big(\int_0^T\int_{\mathbb{T}} J^{+}_{\delta,\varepsilon}(u)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_0^T\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(u))^2(\partial_x u)^4\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\lesssim_{n,\nu,p,\sigma} C_{\delta,n,\nu,p}\,\mathbb{E}\Big[\|u_-\|^{\frac{p}{2}}_{C([0,T]\times\mathbb{T})}\Big(\int_0^T\int_{\mathbb{T}}(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\quad+C_{\delta,n,\nu,p}\,\mathbb{E}\Big[\|u_-\|^{\frac{p}{2}}_{C([0,T]\times\mathbb{T})}\Big(1+\sup_{t\in[0,T]}\int_{\mathbb{T}} G_{\delta,\varepsilon}(u(t))\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u(0)-1)-\log(u(0))\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_0^T\int_{\mathbb{T}} u_+^{n-2}\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]+C_{\delta,n,\nu,p}\,\mathbb{E}\Big[\Big(\int_0^T\int_{\mathbb{T}} u_-\,\mathrm{d}x\,\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\quad+\mathbb{E}\Big[\Big(\int_0^T\Big(\int_{\mathbb{T}} u_+^{\frac{n}{2}-1}+C_{\delta,n,\nu}\big(u_-^{2-\frac{n}{2}}+u_-^{2-\frac{\nu}{2}}\big)\,\mathrm{d}x\Big)^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\\
&\lesssim_{n,p,T} C_{\delta,n,\nu,p}\,\mathbb{E}\Big[\|u_-\|^{p}_{C([0,T]\times\mathbb{T})}\Big]^{\frac12}\,\mathbb{E}\Big[\Big(\int_0^T\int_{\mathbb{T}}(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}s\Big)^{p}\Big]^{\frac12}\\
&\quad+C_{\delta,n,\nu,p}\,\mathbb{E}\Big[\|u_-\|^{p}_{C([0,T]\times\mathbb{T})}\Big]^{\frac12}\Big(1+\mathbb{E}\Big[\Big(\sup_{t\in[0,T]}\int_{\mathbb{T}} G_{\delta,\varepsilon}(u)\,\mathrm{d}x\Big)^{p}\Big]\Big)^{\frac12}\\
&\quad+\mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u(0)-1)-\log(u(0))\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\quad+\mathbb{E}\Big[\sup_{t\in[0,T]}\|u_+(t)\|^{(n-2)\frac{p}{2}}_{L^1(\mathbb{T})}\Big]+C_{\delta,n,\nu,p}\,\mathbb{E}\Big[\|u_-\|^{(4-n)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}+\|u_-\|^{(4-\nu)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}\Big].
\end{aligned}
$$

We note that we have used the assumption $u_0 \geq \delta$, when applying (3.5.22). Employing the entropy estimate (3.4.3) from Lemma 3.4.3 we arrive at (3.5.1). □

Lemma 3.5.2. *It holds that*

$$J^{+}_{\delta,\varepsilon}(r) \lesssim_{n,\nu} r_+^{n-2}, \tag{3.5.7}$$

$$J^{-}_{\delta,\varepsilon}(r) \lesssim_{\delta,n,\nu} r_-, \tag{3.5.8}$$

$$|I_{\delta,\varepsilon}(r)| \lesssim_{n,\nu} r_+^{n-2} + C_{\delta,n,\nu} r_-. \tag{3.5.9}$$

Proof. Ad (3.5.7). From (3.2.6) and (3.2.19), we conclude that

$$J_{\delta,\varepsilon}(r) \lesssim_{n,\nu} \int_0^r\int_{r'}^{\infty}(K_\varepsilon(r''))^{n-4}\,\mathrm{d}r''\,\mathrm{d}r' \leq \int_0^r\int_{r'}^{\infty}(r'')^{n-4}\,\mathrm{d}r''\,\mathrm{d}r' \eqsim_n r^{n-2}, \quad r\geq 0.$$

Ad (3.5.8). We observe that

$$0 \geq J_{\delta,\varepsilon}(r) \geq -\int_r^0 \int_{-\infty}^{\infty} (F''_{\delta,\varepsilon}(r''))^2 \,\mathrm{d}r''\,\mathrm{d}r' \gtrsim_{\delta,n,\nu} r, \quad r \leq 0$$

follows, as soon as we can verify

$$\int_{-\infty}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \lesssim_{\delta,n,\nu} 1. \tag{3.5.10}$$

To this end, we use that $F_{\delta,\varepsilon}$ is an even function to obtain

$$\int_{-\infty}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \eqsim \int_0^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r = \int_0^{\frac{\delta}{2}} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r + \int_{\frac{\delta}{2}}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r. \tag{3.5.11}$$

By (3.2.19) we can estimate the latter integral by

$$\int_{\frac{\delta}{2}}^{\infty} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \lesssim_{n,\nu} \int_{\frac{\delta}{2}}^{\infty} (K_\varepsilon(r))^{n-4} \,\mathrm{d}r \leq \int_{\frac{\delta}{2}}^{\infty} r^{n-4} \,\mathrm{d}r \eqsim \left(\tfrac{\delta}{2}\right)^{n-3}. \tag{3.5.12}$$

From (3.2.11) and (3.2.12), we conclude furthermore that

$$|F'_\delta(r)| \lesssim_{n,\nu} \frac{r^{\frac{\nu}{2}-1}}{\delta^l}, \quad r > 0,$$
$$|F''_\delta(r)| \lesssim_{n,\nu} \frac{r^{\frac{\nu}{2}-2}}{\delta^l}, \quad r > 0.$$

Using (3.2.15)–(3.2.17), we proceed to

$$|F''_{\delta,\varepsilon}(r)| \lesssim_{n,\nu} \delta^{-l}\left((K_\varepsilon(r))^{\frac{\nu}{2}-2} + \frac{(K_\varepsilon(r))^{\frac{\nu}{2}-1}}{K_\varepsilon(r)}\right) \lesssim_{\delta,n,\nu} (K_\varepsilon(r))^{\frac{\nu}{2}-2}. \tag{3.5.13}$$

Using this estimate, we derive that

$$\int_0^{\frac{\delta}{2}} (F''_{\delta,\varepsilon}(r))^2 \,\mathrm{d}r \lesssim_{\delta,n,\nu} \int_0^{\frac{\delta}{2}} (K_\varepsilon(r))^{\nu-4} \,\mathrm{d}r \leq \int_0^{\frac{\delta}{2}} r^{\nu-4} \,\mathrm{d}r \eqsim_\nu \left(\tfrac{\delta}{2}\right)^{\nu-3}.$$

Combining this with (3.5.11) and (3.5.12), we conclude (3.5.10).

Ad (3.5.9). By (3.2.8) and (3.5.5), it holds

$$I_{\delta,\varepsilon}(r) = \int_0^r J_{\delta,\varepsilon}(r') \frac{F'_{\delta,\varepsilon}(r')}{F_{\delta,\varepsilon}(r')} \,\mathrm{d}r',$$

where we can express the fraction as

$$\frac{F'_{\delta,\varepsilon}(r)}{F_{\delta,\varepsilon}(r)} = \frac{F'_\delta(K_\varepsilon(r))}{F_\delta(K_\varepsilon(r))} K'_\varepsilon(r),$$

3

using the chain rule. From (3.2.4) and (3.2.11) we deduce

$$\frac{F_\delta'(r)}{F_\delta(r)} \lesssim_{n,\nu} r^{-1}, \quad r > 0,$$

so that

$$\left|\frac{F_{\delta,\varepsilon}'(r)}{F_{\delta,\varepsilon}(r)}\right| \lesssim_{n,\nu} \frac{1}{K_\varepsilon(r)}$$

by (3.2.16). Using (3.5.7), we observe that

$$\begin{aligned} |I_{\delta,\varepsilon}(r)| &\lesssim_{n,\nu} \int_0^r \frac{J_{\delta,\varepsilon}^+(r')}{K_\varepsilon(r')}\,\mathrm{d}r' \lesssim_{n,\nu} \int_0^r \frac{(r')^{n-2}}{K_\varepsilon(r')}\,\mathrm{d}r' \\ &\le \int_0^r (r')^{n-3}\,\mathrm{d}r' \eqsim_n r^{n-2}, \quad r \ge 0. \end{aligned}$$

For $r < 0$, we use instead (3.5.8) to conclude

$$|I_{\delta,\varepsilon}(r)| \lesssim_{n,\nu} \int_r^0 \frac{J_{\delta,\varepsilon}^-(r')}{K_\varepsilon(r')}\,\mathrm{d}r' \lesssim_{\delta,n,\nu} \int_r^0 \frac{r'_-}{K_\varepsilon(r')}\,\mathrm{d}r' \le r_-, \quad r < 0.$$

□

Lemma 3.5.3. *It holds that*

$$\left|\int_0^r L_{\delta,\varepsilon}''(r')F_{\delta,\varepsilon}(r')\,\mathrm{d}r'\right| \lesssim_{n,\nu} r_+^{\frac{n}{2}-1} + C_{\delta,n,\nu}\left(r_-^{2-\frac{n}{2}} + r_-^{2-\frac{\nu}{2}}\right).$$

Proof. We notice that

$$L_{\delta,\varepsilon}''(r)F_{\delta,\varepsilon}(r) = \frac{J_{\delta,\varepsilon}(r)}{F_{\delta,\varepsilon}(r)} \tag{3.5.14}$$

by (3.2.8) and estimate this term by distinguishing different cases of r.

For $r \ge \delta$ we have

$$F_{\delta,\varepsilon}(r) \ge F_\delta(r) \ge \frac{r^{\frac{n+\nu}{2}}}{2r^{\frac{\nu}{2}}} \eqsim r^{\frac{n}{2}}, \quad r \ge \delta, \tag{3.5.15}$$

by (3.2.4). Hence, using (3.5.7), we conclude that

$$\frac{J_{\delta,\varepsilon}(r)}{F_{\delta,\varepsilon}(r)} \lesssim_{n,\nu} \frac{r^{n-2}}{r^{\frac{n}{2}}} \le r^{\frac{n}{2}-2}, \quad r \ge \delta. \tag{3.5.16}$$

For $r \in (0,\delta)$, we provide another estimate on $J_{\delta,\varepsilon}(r)$, namely, we recall that in (3.5.13) we proved that

$$|F_{\delta,\varepsilon}''(r)| \lesssim_{n,\nu} \frac{(K_\varepsilon(r))^{\frac{\nu}{2}-2}}{\delta^l}.$$

Hence, using (3.2.1), (3.2.6) and (3.2.18), we compute

$$\begin{aligned} J_{\delta,\varepsilon}(r) &\lesssim_{n,\nu} \delta^{-l} \int_0^r \int_{r'}^{\infty} (K_\varepsilon(r''))^{\frac{\nu+n}{2}-4} \,\mathrm{d}r''\,\mathrm{d}r' \\ &\le \delta^{-l} \int_0^r \int_{r'}^{\infty} (r'')^{\frac{\nu+n}{2}-4} \,\mathrm{d}r''\,\mathrm{d}r' \eqsim_{n,\nu} \delta^{-l} r^{\frac{\nu+n}{2}-2}, \quad r \ge 0. \end{aligned} \tag{3.5.17}$$

We moreover have that

$$F_{\delta,\varepsilon}(r) \ge F_\delta(r) \ge \frac{r^{\frac{n+\nu}{2}}}{2\delta^l r^{\frac{n}{2}}} \eqsim \delta^{-l} r^{\frac{\nu}{2}}, \quad r \in (0,\delta), \tag{3.5.18}$$

and consequently we arrive also in this case at

$$\frac{J_{\delta,\varepsilon}(r)}{F_{\delta,\varepsilon}(r)} \lesssim_{n,\nu} \frac{\delta^{-l} r^{\frac{\nu+n}{2}-2}}{\delta^{-l} r^{\frac{\nu}{2}}} \le r^{\frac{n}{2}-2}, \quad r \in (0,\delta). \tag{3.5.19}$$

Combining (3.5.14), (3.5.16) and (3.5.19), we deduce

$$\left| \int_0^r L''_{\delta,\varepsilon}(r') F_{\delta,\varepsilon}(r') \,\mathrm{d}r' \right| \lesssim_{n,\nu} \int_0^r (r')^{\frac{n}{2}-2} \,\mathrm{d}r' \eqsim_n r^{\frac{n}{2}-1}, \quad r \ge 0.$$

For $r<0$, we use that $F_{\delta,\varepsilon}$ is an even function to conclude that

$$\begin{aligned} F_{\delta,\varepsilon}(r) &\gtrsim r_-^{\frac{n}{2}}, \quad r \le -\delta, \\ F_{\delta,\varepsilon}(r) &\gtrsim \delta^{-l} r_-^{\frac{\nu}{2}}, \quad r \in (-\delta,0) \end{aligned} \tag{3.5.20}$$

from (3.5.15) and (3.5.18). Invoking (3.5.8), we arrive at

$$\frac{|J_{\delta,\varepsilon}(r)|}{F_{\delta,\varepsilon}(r)} \lesssim_{\delta,n,\nu} r_-^{1-\frac{n}{2}} + r_-^{1-\frac{\nu}{2}}, \quad r<0. \tag{3.5.21}$$

Hence, integration together with (3.5.14) yields

$$\left| \int_r^0 L''_{\delta,\varepsilon}(r') F_{\delta,\varepsilon}(r') \,\mathrm{d}r' \right| \lesssim_{\delta,n,\nu} \int_r^0 (r'_-)^{1-\frac{n}{2}} + (r'_-)^{1-\frac{\nu}{2}} \,\mathrm{d}r' \eqsim_{n,\nu} r_-^{2-\frac{n}{2}} + r_-^{2-\frac{\nu}{2}}, \quad r<0.$$

□

Lemma 3.5.4. *It holds that*

$$L^+_{\delta,\varepsilon}(r) \le (r-1) - \log(r), \quad r \ge \delta, \tag{3.5.22}$$

$$L^-_{\delta,\varepsilon}(r) \lesssim_{\delta,n,\nu} \big(G_{\delta,\varepsilon}(r)+1\big) r_-. \tag{3.5.23}$$

Proof. Ad (3.5.22). We let $r \ge \delta$ and use (3.5.7) and (3.5.15) to conclude

$$L_{\delta,\varepsilon}(r) = \int_1^r \int_1^{r'} \frac{J_{\delta,\varepsilon}(r'')}{F^2_{\delta,\varepsilon}(r'')} \,\mathrm{d}r''\,\mathrm{d}r' \lesssim_{n,\nu} \int_1^r \int_1^{r'} (r'')^{-2} \,\mathrm{d}r''\,\mathrm{d}r' = (r-1) - \log(r), \quad r \ge \delta.$$

Ad (3.5.23). We point out that $L_{\delta,\varepsilon}(r) \geq 0$ for $r \geq 0$, since it is convex on $(0,\infty)$ by (3.2.6) and (3.2.8) with a minimum at $r = 1$. For $r < 0$, we use that

$$L_{\delta,\varepsilon}(r) = \int_r^1 \int_{r'}^1 \frac{J_{\delta,\varepsilon}(r'')}{F_{\delta,\varepsilon}^2(r'')}\, \mathrm{d}r''\, \mathrm{d}r' \geq -\int_r^1 \int_{r'}^1 \frac{J_{\delta,\varepsilon}^-(r'')}{F_{\delta,\varepsilon}^2(r'')}\, \mathrm{d}r''\, \mathrm{d}r'$$
$$= -\int_r^0 \int_{r'}^0 \frac{J_{\delta,\varepsilon}^-(r'')}{F_{\delta,\varepsilon}^2(r'')}\, \mathrm{d}r''\, \mathrm{d}r' \geq r \int_{-\infty}^0 \frac{J_{\delta,\varepsilon}^-(r'')}{F_{\delta,\varepsilon}^2(r'')}\, \mathrm{d}r'', \quad r < 0,$$

and consequently

$$\begin{aligned} L_{\delta,\varepsilon}^-(r) &\leq r_- \int_{-\infty}^0 \frac{J_{\delta,\varepsilon}^-(r')}{F_{\delta,\varepsilon}^2(r')}\, \mathrm{d}r' \\ &= r_- \bigg(\int_{-\infty}^{-\delta} \frac{J_{\delta,\varepsilon}^-(r')}{F_{\delta,\varepsilon}^2(r')}\, \mathrm{d}r' + \int_{-\delta}^0 \frac{J_{\delta,\varepsilon}^-(r')}{F_{\delta,\varepsilon}^2(r')}\, \mathrm{d}r' \bigg), \quad r < 0. \end{aligned} \tag{3.5.24}$$

We use (3.5.8) and (3.5.20) to estimate

$$\int_{-\infty}^{-\delta} \frac{J_{\delta,\varepsilon}^-(r')}{F_{\delta,\varepsilon}^2(r')}\, \mathrm{d}r' \lesssim_{\delta,n,\nu} \int_{-\infty}^{-\delta} \frac{r'_-}{(r'_-)^n}\, \mathrm{d}r' \eqsim_{\delta,n} 1. \tag{3.5.25}$$

For the remaining part, we deduce form (3.5.18) and the fact that $F_{\delta,\varepsilon}$ is even

$$F_{\delta,\varepsilon}(r) \gtrsim \delta^{-l} (K_\varepsilon(r))^{\frac{\nu}{2}}, \quad r \in (-\delta, 0).$$

Thus, using that

$$\tfrac{1}{\sqrt{2}}(r+\varepsilon) \leq K_\varepsilon(r) \leq r + \varepsilon, \quad r > 0 \tag{3.5.26}$$

and again (3.5.8), we obtain

$$\int_{-\delta}^0 \frac{J_{\delta,\varepsilon}^-(r')}{F_{\delta,\varepsilon}^2(r')}\, \mathrm{d}r' \lesssim_{\delta,n,\nu} \int_{-\delta}^0 \frac{r'_-}{(K_\varepsilon(r'))^\nu}\, \mathrm{d}r' \leq \int_0^\delta (K_\varepsilon(r'))^{1-\nu}\, \mathrm{d}r'$$
$$\eqsim_\nu \int_\varepsilon^{\delta+\varepsilon} (r')^{1-\nu}\, \mathrm{d}r' \lesssim_\nu \varepsilon^{2-\nu}.$$

Combining this with (3.5.25) we obtain that

$$\int_{-\infty}^0 \frac{J_{\delta,\varepsilon}^-(r')}{F_{\delta,\varepsilon}^2(r')}\, \mathrm{d}r' \lesssim_{\delta,n,\nu} 1 + \varepsilon^{2-\nu}. \tag{3.5.27}$$

Next, from (3.2.4), we deduce that

$$F_{\delta,\varepsilon}(r) \leq \delta^{-l} (K_\varepsilon(r))^{\frac{\nu}{2}}, \tag{3.5.28}$$

which together with (3.2.9) and (3.5.26) leads to

$$G_{\delta,\varepsilon}(0) \geq \delta^{2l} \int_0^\infty \int_{r'}^\infty (K_\varepsilon(r''))^{-\nu}\, \mathrm{d}r''\, \mathrm{d}r' \eqsim_\nu \delta^{2l} \int_0^\infty \int_{r'}^\infty (r''+\varepsilon)^{-\nu}\, \mathrm{d}r''\, \mathrm{d}r' \eqsim_{\delta,n,\nu} \varepsilon^{2-\nu}.$$

Since $G_{\delta,\varepsilon}(r)$ is decreasing, we conclude

$$G_{\delta,\varepsilon}(r) \geq G_{\delta,\varepsilon}(0) \gtrsim_{\delta,n,\nu} \varepsilon^{2-\nu}, \quad r \leq 0.$$

In combination with (3.5.24) and (3.5.27), we arrive at (3.5.23). □

Lemma 3.5.5. *Let $p > n+2$, $u_0 \in L^\infty(\Omega, \mathfrak{F}_0; H^1(\mathbb{T}))$ such that $u_0 \geq \delta$. Then any weak martingale solution*

$$\{(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathfrak{F}}, \check{\mathbb{P}}), (\check{\beta}^{(k)})_{k\in\mathbb{Z}}, \check{u}_\delta\}$$

to (3.1.8) in the sense of Definition 3.4.10 with initial value u_0 constructed in Lemma 3.4.11 satisfies

$$\begin{aligned}
&\check{\mathbb{E}}\left[\sup_{t\in[0,T]} \|\partial_x \check{u}_\delta(t)\|^p_{L^2(\mathbb{T})} + \left(\int_0^T \int_{\mathbb{T}} F_\delta^2(\check{u}_\delta)(\partial_x^3 \check{u}_\delta)^2 \, \mathrm{d}x \, \mathrm{d}s\right)^{\frac{p}{2}}\right] \\
&\quad \lesssim_{n,\nu,\sigma,p,T} \mathbb{E}\left[\|\partial_x u_0\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u_0)|^{\frac{p(n+2)}{8-2n}} + \mathbb{E}\left(\int_{\mathbb{T}} (u_0 - 1) - \log(u_0) \, \mathrm{d}x\right)^{\frac{p}{2}}\right] + 1.
\end{aligned} \tag{3.5.29}$$

Proof. Since we assumed $u_0 \in L^\infty(\Omega, \mathfrak{F}_0; H^1(\mathbb{T}))$ and $u_0 \geq \delta$, we have in particular that $\mathbb{E}[|\mathscr{A}(u_0)|^{2p}] < \infty$ and $\mathbb{E}[\|G_\delta(u_0)\|^p_{L^1(\mathbb{T})}] < \infty$ so that Lemma 3.4.11 is indeed applicable. We verify a version of (3.5.29) on the level of the approximations $\hat{u}_{\delta,\varepsilon,R}$ of $\check{u}_\delta$. To this end, we use the Itô expansion of the energy functional

$$\begin{aligned}
\tfrac{1}{2}\|\partial_x \hat{u}_{\delta,\varepsilon,R}(t)\|^2_{L^2(\mathbb{T})} &= \tfrac{1}{2}\|\partial_x \hat{u}_{\delta,\varepsilon,R}(0)\|^2_{L^2(\mathbb{T})} - \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\hat{u}_{\delta,\varepsilon,R})(\partial_x^3 \hat{u}_{\delta,\varepsilon,R})^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tfrac{1}{6} \sum_{k\in\mathbb{Z}} \int_0^t \gamma^2_{\hat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} \sigma_k^2 (F''_{\delta,\varepsilon}(\hat{u}_{\delta,\varepsilon,R}))^2 (\partial_x \hat{u}_{\delta,\varepsilon,R})^4 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tfrac{1}{16} \sum_{k\in\mathbb{Z}} \int_0^t \gamma^2_{\hat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} (\partial_x(\sigma_k^2))\big((F^2_{\delta,\varepsilon})'''(\hat{u}_{\delta,\varepsilon,R}) + 4((F'_{\delta,\varepsilon})^2)'(\hat{u}_{\delta,\varepsilon,R})\big)(\partial_x \hat{u}_{\delta,\varepsilon,R})^3 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tfrac{3}{2} \sum_{k\in\mathbb{Z}} \int_0^t \gamma^2_{\hat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} ((\partial_x \sigma_k)^2 - \sigma_k(\partial_x^2 \sigma_k))(F'_{\delta,\varepsilon}(\hat{u}_{\delta,\varepsilon,R}))^2 (\partial_x \hat{u}_{\delta,\varepsilon,R})^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tfrac{3}{16} \sum_{k\in\mathbb{Z}} \int_0^t \gamma^2_{\hat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} (\partial_x^2(\sigma_k^2))(F^2_{\delta,\varepsilon})''(\hat{u}_{\delta,\varepsilon,R})(\partial_x \hat{u}_{\delta,\varepsilon,R})^2 \, \mathrm{d}x \, \mathrm{d}s \\
&+ \tfrac{1}{8} \sum_{k\in\mathbb{Z}} \int_0^t \gamma^2_{\hat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} (4\sigma_k \partial_x^4 \sigma_k - \partial_x^4(\sigma_k^2)) F^2_{\delta,\varepsilon}(\hat{u}_{\delta,\varepsilon,R}) \, \mathrm{d}x \, \mathrm{d}s \\
&+ \sum_{k\in\mathbb{Z}} \int_0^t \gamma_{\hat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} \sigma_k F_{\delta,\varepsilon}(\hat{u}_{\delta,\varepsilon,R}) \partial_x^3 \hat{u}_{\delta,\varepsilon,R} \, \mathrm{d}x \, \mathrm{d}\beta^{(k)},
\end{aligned} \tag{3.5.30}$$

see [35, Eq. (4.10)], where we use again the notation $\gamma_{\hat{u}_{\delta,\varepsilon,R}} = g_R(\|\hat{u}_{\delta,\varepsilon,R}\|_{C(\mathbb{T})})$. We proceed as in the proof of [35, Lemma 4.6] and estimate the deterministic terms on the right-hand side separately. Here we use in particular (3.5.4) together with the relation

$$\partial_x \sigma_k = 2\pi k \sigma_{-k}, \quad k \in \mathbb{Z} \tag{3.5.31}$$

3

from [35, Eq. (2.2b)] to estimate the terms involving σ_k. Technical estimates on the approximating functions are postponed to Lemma 3.5.6.

$(\partial_x u)^4$**-Term.** We readily see that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\int_0^t \gamma^2_{\widehat{u}_{\delta,\varepsilon,R}}\int_{\mathbb{T}}\sigma_k^2(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s\\ &\overset{(3.5.4)}{\lesssim_\sigma}\int_0^t\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

$(\partial_x u)^3$**-Term.** We estimate this term by absorbing it into the highest and lowest order term. To this end, we use the properties of the approximate function $F_{\delta,\varepsilon}$, which we prove in Lemma 3.5.6, to conclude that

$$\begin{aligned}&\left|\sum_{k\in\mathbb{Z}}\int_0^t \gamma^2_{\widehat{u}_{\delta,\varepsilon,R}}\int_{\mathbb{T}}(\partial_x(\sigma_k^2))\big((F^2_{\delta,\varepsilon})'''(\widehat{u}_{\delta,\varepsilon,R})+4((F'_{\delta,\varepsilon})^2)'(\widehat{u}_{\delta,\varepsilon,R})\big)(\partial_x\widehat{u}_{\delta,\varepsilon,R})^3\,\mathrm{d}x\,\mathrm{d}s\right|\\ &\overset{(3.5.4)}{\lesssim_\sigma}\int_0^t\int_{\mathbb{T}}\left|\big((F^2_{\delta,\varepsilon})'''(\widehat{u}_{\delta,\varepsilon,R})+4((F'_{\delta,\varepsilon})^2)'(\widehat{u}_{\delta,\varepsilon,R})\big)(\partial_x\widehat{u}_{\delta,\varepsilon,R})^3\right|\mathrm{d}x\,\mathrm{d}s\\ &\overset{(3.5.36),(3.5.37)}{\lesssim_{n,\nu}}\int_0^t\int_{\mathbb{T}}(F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^{\frac12}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^{\frac32}|\partial_x\widehat{u}_{\delta,\varepsilon,R}|^3\,\mathrm{d}x\,\mathrm{d}s\\ &\le\left(\int_0^t\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac14}+\left(\int_0^t\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac34}\\ &\le 2+\int_0^t\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

$(\partial_x u)^2$**-Term.** Similarly, we conclude that

$$\begin{aligned}&\left|\sum_{k\in\mathbb{Z}}\int_0^t \gamma^2_{\widehat{u}_{\delta,\varepsilon,R}}\int_{\mathbb{T}}8((\partial_x\sigma_k)^2-\sigma_k(\partial_x^2\sigma_k))(F'_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s\right|\\ &+\left|\sum_{k\in\mathbb{Z}}\int_0^t \gamma^2_{\widehat{u}_{\delta,\varepsilon,R}}\int_{\mathbb{T}}(\partial_x^2(\sigma_k^2))(F^2_{\delta,\varepsilon})''(\widehat{u}_{\delta,\varepsilon,R})(\partial_x\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s\right|\\ &\overset{(3.5.4)}{\lesssim_\sigma}\int_0^t\int_{\mathbb{T}}\Big((F'_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2+\left|(F^2_{\delta,\varepsilon})''(\widehat{u}_{\delta,\varepsilon,R})\right|\Big)(\partial_x\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s\\ &\overset{(3.5.38),(3.5.39)}{\lesssim_{n,\nu}}\int_0^t\int_{\mathbb{T}}F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s\\ &\le\left(\int_0^t\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}+\left(\int_0^t\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac12}\\ &\le 2+\int_0^t\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

$(\partial_x u)^0$**-Term.** Lastly, we also have that

$$\left|\sum_{k\in\mathbb{Z}}\int_0^t \gamma^2_{\widehat{u}_{\delta,\varepsilon,R}}\int_{\mathbb{T}}(4\sigma_k\partial_x^4\sigma_k-\partial_x^4(\sigma_k^2))F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s\right|$$

$$\overset{(3.5.4),(3.5.31)}{\lesssim_\sigma} \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s.$$

Inserting all this in (3.5.30), we arrive at

$$\begin{aligned}
&\tfrac{1}{2}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}(t)\|^2_{L^2(\mathbb{T})} + \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3 \widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\le \tfrac{1}{2}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}(0)\|^2_{L^2(\mathbb{T})} + |M(t)| \\
&\qquad + C_{n,\nu,\sigma}\Big[1 + \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\mathbb{T}} (F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x \widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s\Big],
\end{aligned}$$

with

$$M(t) = \sum_{k\in\mathbb{Z}} \int_0^t \gamma_{\widehat{u}_{\delta,\varepsilon,R}} \int_{\mathbb{T}} \sigma_k F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x^3 \widehat{u}_{\delta,\varepsilon,R}\,\mathrm{d}x\,\mathrm{d}\beta^{(k)}. \tag{3.5.32}$$

We apply the Gagliardo Nirenberg inequality with the choice of exponents

$$\tfrac{-1}{n} = \theta(\tfrac{-1}{1}) + (1-\theta)(1-\tfrac{1}{2}) \qquad \Longleftrightarrow \qquad \theta = \tfrac{n+2}{3n}$$

and Young's inequality with $\frac{4-n}{3} + \frac{n-1}{3} = 1$ to estimate further

$$\begin{aligned}
\int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\,\mathrm{d}x &\overset{(3.5.2)}{\lesssim_n} 1 + \|\widehat{u}_{\delta,\varepsilon,R}\|^n_{L^n(\mathbb{T})} \\
&\lesssim_n 1 + \|\widehat{u}_{\delta,\varepsilon,R}\|^n_{L^1(\mathbb{T})} + \|\widehat{u}_{\delta,\varepsilon,R}\|^{\theta n}_{L^1(\mathbb{T})}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}\|^{(1-\theta)n}_{L^2(\mathbb{T})} \\
&\le 1 + \|\widehat{u}_{\delta,\varepsilon,R}\|^n_{L^1(\mathbb{T})} + \left(\tfrac{1}{\kappa}\right)^{\frac{n-1}{4-n}}\|\widehat{u}_{\delta,\varepsilon,R}\|^{\frac{3\theta n}{4-n}}_{L^1(\mathbb{T})} + \kappa\|\partial_x \widehat{u}_{\delta,\varepsilon,R}\|^{\frac{3(1-\theta)n}{n-1}}_{L^2(\mathbb{T})} \\
&= 1 + \|\widehat{u}_{\delta,\varepsilon,R}\|^n_{L^1(\mathbb{T})} + \left(\tfrac{1}{\kappa}\right)^{\frac{n-1}{4-n}}\|\widehat{u}_{\delta,\varepsilon,R}\|^{\frac{n+2}{4-n}}_{L^1(\mathbb{T})} + \kappa\|\partial_x \widehat{u}_{\delta,\varepsilon,R}\|^2_{L^2(\mathbb{T})} \\
&\le 2 + \left(\left(\tfrac{1}{\kappa}\right)^{\frac{n-1}{4-n}} + 1\right)\|\widehat{u}_{\delta,\varepsilon,R}\|^{\frac{n+2}{4-n}}_{L^1(\mathbb{T})} + \kappa\|\partial_x \widehat{u}_{\delta,\varepsilon,R}\|^2_{L^2(\mathbb{T})}.
\end{aligned}$$

In the last step we used that $n > 2$. Choosing κ sufficiently small, we obtain that

$$\begin{aligned}
&\tfrac{1}{2}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}(t)\|^2_{L^2(\mathbb{T})} + \int_0^t \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3 \widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\le \tfrac{1}{2}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}(0)\|^2_{L^2(\mathbb{T})} + |M(t)| + \tfrac{1}{4}\sup_{s\in[0,t]}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}(s)\|^2_{L^2(\mathbb{T})} \\
&\qquad + C_{n,\nu,\sigma,T}\Big[1 + \sup_{t\in[0,T]}\|\widehat{u}_{\delta,\varepsilon,R}(t)\|^{\frac{n+2}{4-n}}_{L^1(\mathbb{T})} + \int_0^T \int_{\mathbb{T}} (F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x \widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}s\Big].
\end{aligned}$$

We take the supremum in time and take the $\frac{p}{2}$-th power on both sides to obtain

$$\begin{aligned}
&\tfrac{1}{4}\sup_{t\in[0,T]}\|\partial_x \widehat{u}_{\delta,\varepsilon,R}(t)\|^p_{L^2(\mathbb{T})} + \left(\int_0^T \int_{\mathbb{T}} F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3 \widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{2}} \\
&\quad\lesssim_p \|\partial_x \widehat{u}_{\delta,\varepsilon,R}(0)\|^p_{L^2(\mathbb{T})} + \sup_{t\in[0,T]}|M(t)|^{\frac{p}{2}}
\end{aligned}$$

$$+ C_{n,\nu,p,\sigma,T}\left[1+\sup_{t\in[0,T]}\|\widehat{u}_{\delta,\varepsilon,R}(t)\|_{L^1(\mathbb{T})}^{\frac{p(n+2)}{8-2n}}+\left(\int_0^T\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{2}}\right].$$

Taking the expectation and employing the Burkholder–Davis–Gundy inequality leads to

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\partial_x\widehat{u}_{\delta,\varepsilon,R}(t)\|_{L^2(\mathbb{T})}^p+\left(\int_0^T\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{2}}\right]\\
&\lesssim_{n,\nu,\sigma,p,T}\widehat{\mathbb{E}}\left[\|\partial_x\widehat{u}_{\delta,\varepsilon,R}(0)\|_{L^2(\mathbb{T})}^p+\langle M\rangle_T^{\frac{p}{4}}\right]+1\\
&\quad+\widehat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\widehat{u}_{\delta,\varepsilon,R}(t)\|_{L^1(\mathbb{T})}^{\frac{p(n+2)}{8-2n}}+\left(\int_0^T\int_{\mathbb{T}}(F''_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x\widehat{u}_{\delta,\varepsilon,R})^4\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{2}}\right].
\end{aligned}\tag{3.5.33}$$

Since

$$\begin{aligned}
\langle M\rangle_t&=\sum_{k\in\mathbb{Z}}\int_0^t\left(\gamma_{\widehat{u}_{\delta,\varepsilon,R}}\int_{\mathbb{T}}\sigma_kF_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})\partial_x^3\widehat{u}_{\delta,\varepsilon,R}\,\mathrm{d}x\right)^2\mathrm{d}s\\
&\overset{(3.5.4)}{\lesssim_\sigma}\int_0^t\int_{\mathbb{T}}(F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}s,
\end{aligned}\tag{3.5.34}$$

we can estimate

$$\begin{aligned}
\widehat{\mathbb{E}}[\langle M\rangle_T^{\frac{p}{4}}]&\lesssim_{\sigma,p}\widehat{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{T}}(F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{4}}\right]\\
&\le\kappa\widehat{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{T}}(F_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}))^2(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{2}}\right]+\frac{1}{\kappa}.
\end{aligned}$$

Inserting this as well as the approximate log-entropy estimate (3.5.1) from Lemma 3.5.1 in (3.5.33), and choosing κ sufficiently small, we conclude

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\partial_x\widehat{u}_{\delta,\varepsilon,R}(t)\|_{L^2(\mathbb{T})}^p\right]+\widehat{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{T}}F^2_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R})(\partial_x^3\widehat{u}_{\delta,\varepsilon,R})^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{2}}\right]\\
&\lesssim_{n,\nu,\sigma,p,T}\mathbb{E}\left[\|\partial_xu_0\|_{L^2(\mathbb{T})}^p+\left(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\right)^{\frac{p}{2}}\right]\\
&\quad+\widehat{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\widehat{u}_{\delta,\varepsilon,R}(t)\|_{L^1(\mathbb{T})}^{\frac{p(n+2)}{8-2n}}\right]+1\\
&\quad+C_{\delta,n,\nu,p,T}\widehat{\mathbb{E}}\left[\|\widehat{u}^-_{\delta,\varepsilon,R}\|^p_{C([0,T]\times\mathbb{T})}\right]^{\frac{1}{2}}\left(\mathbb{E}\left[\|G_{\delta,\varepsilon}(u_0)\|^p_{L^1(\mathbb{T})}+|\mathscr{A}(u_0)|^{2p}\right]+1\right)^{\frac{1}{2}}\\
&\quad+C_{\delta,n,\nu,p}\widehat{\mathbb{E}}\left[\|\widehat{u}^-_{\delta,\varepsilon,R}\|^{(4-n)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}+\|\widehat{u}^-_{\delta,\varepsilon,R}\|^{(4-\nu)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}\right],
\end{aligned}\tag{3.5.35}$$

where we also used

$$\frac{p(n+2)}{8-2n}>\frac{p(n-2)}{2}.$$

When the limit $R\to\infty$ is taken in the proof of Lemma 3.4.11, which follows along the lines of [35, Proposition 4.7], the estimate (3.5.35) is preserved. By taking the limit $\varepsilon\searrow$

0, we derive (3.5.29) as follows. We denote the sequence from [35, Eq. (5.14a)] which converges $\check{\mathbb{P}}$-almost surely to $\check{u}_\delta$ in $C([0,T]\times\mathbb{T})$ by $\check{u}_{\delta,\varepsilon}$, so that in particular

$$\begin{aligned}&\sup_{t\in[0,T]}\|\check{u}_{\delta,\varepsilon}(t)\|_{L^1(\mathbb{T})}\to\sup_{t\in[0,T]}\|\check{u}_\delta(t)\|_{L^1(\mathbb{T})},\\&\|\check{u}^-_{\delta,\varepsilon}\|_{C([0,T]\times\mathbb{T})}\to 0,\end{aligned}$$

$\check{\mathbb{P}}$-almost surely. By [35, Eq. (4.24)] the sequence $\check{u}_{\delta,\varepsilon}$ is uniformly ε bounded in the space $L^q(\check{\Omega},C([0,T]\times\mathbb{T}))$ for all $q\in(1,\infty)$, and we obtain by Vitali's convergence theorem that

$$\check{\mathbb{E}}\Big[\|\check{u}^-_{\delta,\varepsilon}\|^p_{C([0,T]\times\mathbb{T})}\Big]^{\frac12}+\check{\mathbb{E}}\Big[\|\check{u}^-_{\delta,\varepsilon}\|^{(4-n)\frac p2}_{C([0,T]\times\mathbb{T})}+\|\check{u}^-_{\delta,\varepsilon}\|^{(4-\nu)\frac p2}_{C([0,T]\times\mathbb{T})}\Big]\to 0$$

as $\varepsilon\searrow 0$. Since $\check{u}_\delta$ is non-negative and preserves mass, (3.5.29) follows. □

We provide the technical estimates on the nonlinearities from the $(\partial_x u)^3$- and $(\partial_x u)^2$-term in (3.5.30). They correspond to the trivial estimates

$$\begin{aligned}u^{n-3}&\le(u^{\frac n2})^{\frac12}(u^{\frac n2-2})^{\frac32},\\u^{n-2}&\le u^{\frac n2}u^{\frac n2-2}\end{aligned}$$

for $u>0$ on the unregularized level.

Lemma 3.5.6. *It holds that*

$$\Big|((F'_{\delta,\varepsilon})^2)'(r)\Big|\lesssim_{n,\nu}(F_{\delta,\varepsilon}(r))^{\frac12}(F''_{\delta,\varepsilon}(r))^{\frac32},\tag{3.5.36}$$

$$\Big|(F^2_{\delta,\varepsilon})'''(r)\Big|\lesssim_{n,\nu}(F_{\delta,\varepsilon}(r))^{\frac12}(F''_{\delta,\varepsilon}(r))^{\frac32},\tag{3.5.37}$$

$$(F'_{\delta,\varepsilon}(r))^2\lesssim_{n,\nu}F_{\delta,\varepsilon}(r)F''_{\delta,\varepsilon}(r),\tag{3.5.38}$$

$$\Big|(F^2_{\delta,\varepsilon}(r))''\Big|\lesssim_{n,\nu}F_{\delta,\varepsilon}(r)F''_{\delta,\varepsilon}(r).\tag{3.5.39}$$

Proof. Ad (3.5.36). We have that

$$(F'_{\delta,\varepsilon}(r))^2=(F'_\delta(K_\varepsilon(r)))^2(K'_\varepsilon(r))^2\tag{3.5.40}$$

by the chain rule and hence

$$\begin{aligned}((F'_{\delta,\varepsilon})^2)'(r)&=2(F'_\delta(K_\varepsilon(r)))^2K'_\varepsilon(r)K''_\varepsilon(r)+2F'_\delta(K_\varepsilon(r))F''_\delta(K_\varepsilon(r))(K'_\varepsilon(r))^3\\&=2F'_\delta(K_\varepsilon(r))K'_\varepsilon(r)F''_{\delta,\varepsilon}(r)\end{aligned}\tag{3.5.41}$$

by (3.2.15). Next, we convince ourselves that

$$|F'_\delta(r)|^2\lesssim_{n,\nu}F_\delta(r)F''_\delta(r),\quad r>0.\tag{3.5.42}$$

Using (3.2.4), (3.2.11) and (3.2.12) we observe that

$$\begin{aligned}(F'_\delta(r))^2&=r^{n+\nu-2}\frac{(nr^{\frac\nu2}+\delta^l\nu r^{\frac n2})^2}{4(r^{\frac\nu2}+\delta^l r^{\frac n2})^4}\lesssim r^{n+\nu-2}\frac{n^2r^\nu+\delta^{2l}\nu^2r^n}{(r^{\frac\nu2}+\delta^lr^{\frac n2})^4},\quad r>0,\\F_\delta(r)F''_\delta(r)&=r^{n+\nu-2}\frac{\delta^{2l}(\nu-2)\nu r^n-\delta^l(\nu^2+\nu(2-4n)+n(n+2))r^{\frac{n+\nu}{2}}+n(n-2)r^\nu}{4(r^{\frac\nu2}+\delta^lr^{\frac n2})^4},\quad r>0,\end{aligned}\tag{3.5.43}$$

and hence it suffices to show

$$n^2 r^\nu + \delta^{2l}\nu^2 r^n \lesssim_{n,\nu} \delta^{2l}(\nu-2)\nu r^n - \delta^l(\nu^2+\nu(2-4n)+n(n+2))r^{\frac{n+\nu}{2}} + n(n-2)r^\nu, \quad r>0. \tag{3.5.44}$$

However, this follows from the assumption (3.2.2), so that (3.5.42) indeed holds. Together with (3.5.41), we proceed to estimate

$$\begin{aligned}\left|((F'_{\delta,\varepsilon})^2)'(r)\right| &\lesssim_{n,\nu} \left(F_\delta(K_\varepsilon(r))F''_\delta(K_\varepsilon(r))\right)^{\frac12} K'_\varepsilon(r)F''_{\delta,\varepsilon}(r)\\ &= (F_{\delta,\varepsilon}(r))^{\frac12} F''_{\delta,\varepsilon}(r)\left(F''_\delta(K_\varepsilon(r))(K'_\varepsilon(r))^2\right)^{\frac12}.\end{aligned}$$

By combining (3.2.11), (3.2.15) and (3.2.17), we see that

$$F''_\delta(K_\varepsilon(r))(K'_\varepsilon(r))^2 \le F''_{\delta,\varepsilon}(r) \tag{3.5.45}$$

and (3.5.36) follows.

Ad (3.5.37). An algebraic computation shows that

$$\begin{aligned}(F^2_{\delta,\varepsilon})'''(r) &= (F^2_\delta)'''(K_\varepsilon(r))(K'_\varepsilon(r))^3 + 3(F^2_\delta)''(K_\varepsilon(r))K'_\varepsilon(r)K''_\varepsilon(r) + (F^2_\delta)'(K_\varepsilon(r))K'''_\varepsilon(r)\\ &= 6(F'_\delta F''_\delta)(K_\varepsilon(r))(K'_\varepsilon(r))^3 + 2(F_\delta F'''_\delta)(K_\varepsilon(r))(K'_\varepsilon(r))^3 + 6(F_\delta F''_\delta)(K_\varepsilon(r))K'_\varepsilon(r)K''_\varepsilon(r)\\ &\quad + 6(F'_\delta)^2(K_\varepsilon(r))K'_\varepsilon(r)K''_\varepsilon(r) + 2(F'_\delta F_\delta)(K_\varepsilon(r))K'''_\varepsilon(r)\\ &= T_1 + \ldots + T_5.\end{aligned}$$

We notice that

$$|T_1 + T_4| = 3\left|((F'_{\delta,\varepsilon})^2)'(r)\right| \lesssim_{n,\nu} (F_{\delta,\varepsilon}(r))^{\frac12}(F''_{\delta,\varepsilon}(r))^{\frac32} \tag{3.5.46}$$

by (3.5.36) and (3.5.41). For T_3, we use (3.5.43) to conclude that

$$|F_\delta(r)F''_\delta(r)| \lesssim_{n,\nu} (F'_\delta(r))^2, \quad r>0,$$

and thus $|T_3| \lesssim_{n,\nu} |T_4|$. By (3.2.11), (3.2.12), (3.2.16), (3.2.17) and (3.5.44), we have that $\operatorname{sgn}(T_1) = \operatorname{sgn}(T_4) = \operatorname{sgn}(r)$ and therefore

$$|T_1| + |T_4| = |T_1 + T_4|, \tag{3.5.47}$$

which was estimated appropriately in (3.5.46). For T_2, we use that

$$\begin{aligned}F'''_\delta(r) = \frac{r^{\frac{n+\nu}{2}-3}}{8(r^{\frac{\nu}{2}}+\delta^l r^{\frac{n}{2}})^4} \times \Big(&\delta^{3l}\nu(\nu^2-6\nu+8)r^{\frac{3n}{2}}\\ &- \delta^{2l}(4\nu^3 + n(-12\nu^2 - n^2 - 6n - 8) + 2\nu(3n^2+12n-8))r^{\frac{2n+\nu}{2}}\\ &+ \delta^l(\nu^3 - 6\nu^2(n-1) + 4\nu(3n^2-6n+2) - 4n(n^2-4))r^{\frac{n+2\nu}{2}}\\ &+ n(n^2-6n+8)r^{\frac{3\nu}{2}}\Big)\end{aligned}$$

so that

$$|F_\delta(r)F'''_\delta(r)| \lesssim_{n,\nu} r^{n+\nu-3}\frac{\delta^{3l}r^{\frac{3n}{2}}+r^{\frac{3\nu}{2}}}{(r^{\frac{\nu}{2}}+\delta^l r^{\frac{n}{2}})^5}, \quad r>0,$$

by Young's inequality. Using (3.2.11),(3.2.12) and (3.5.44) we observe that

$$F'_\delta(r)F''_\delta(r) \gtrsim_{n,\nu} r^{n+\nu-3}\frac{(nr^{\frac{\nu}{2}}+\delta^l\nu r^{\frac{n}{2}})(n^2r^\nu+\delta^{2l}\nu^2r^n)}{8(r^{\frac{\nu}{2}}+\delta^l r^{\frac{n}{2}})^5}$$
$$\gtrsim_{n,\nu} r^{n+\nu-3}\frac{\delta^{3l}r^{\frac{3n}{2}}+r^{\frac{3\nu}{2}}}{(r^{\frac{\nu}{2}}+\delta^l r^{\frac{n}{2}})^5}, \quad r>0.$$

The bound $|T_2| \lesssim_{n,\nu} |T_1|$ follows, which again is bounded by (3.5.46) and (3.5.47). It is left to estimate T_5 and to this end we compute

$$K'''_\varepsilon(r) = \frac{-3\varepsilon^2 r}{(\varepsilon^2+r^2)^{\frac{5}{2}}} = -3\frac{K''_\varepsilon(r)K'_\varepsilon(r)}{K_\varepsilon(r)}$$

using (3.2.16) and (3.2.17). Hence,

$$|T_5| = |2(F'_\delta F_\delta)(K_\varepsilon(r))K'''_\varepsilon(r)| \lesssim \left(\frac{F'_\delta F_\delta}{(\cdot)}\right)(K_\varepsilon(r))K''_\varepsilon(r)|K'_\varepsilon(r)|$$

and we obtain the estimate $|T_5| \lesssim |T_4|$ as soon as we can show that

$$\frac{F_\delta(r)}{r} \lesssim_{n,\nu} F'_\delta(r), \quad r>0.$$

But this follows by

$$\frac{F_\delta(r)}{rF'_\delta(r)} \overset{(3.2.4),(3.2.11)}{=} \frac{2(r^{\frac{\nu}{2}}+\delta^l r^{\frac{n}{2}})^2}{(r^{\frac{\nu}{2}}+\delta^l r^{\frac{n}{2}})(nr^{\frac{\nu}{2}}+\delta^l\nu r^{\frac{n}{2}})} \lesssim_{n,\nu} 1, \quad r>0,$$

so that (3.5.37) is a consequence of (3.5.46) and (3.5.47).

Ad (3.5.38). From (3.5.40) and (3.5.42), we conclude that

$$(F'_{\delta,\varepsilon}(r))^2 \lesssim_{n,\nu} F_\delta(K_\varepsilon(r))F''_\delta(K_\varepsilon(r))(K'_\varepsilon(r))^2.$$

Inserting (3.5.45), we obtain

$$(F'_{\delta,\varepsilon}(r))^2 \lesssim_{n,\nu} F_{\delta,\varepsilon}(r)F''_{\delta,\varepsilon}(r)$$

as desired.

Ad (3.5.39). We have that

$$(F^2_{\delta,\varepsilon})''(r) = 2(F'_{\delta,\varepsilon}(r))^2 + 2F_{\delta,\varepsilon}(r)F''_{\delta,\varepsilon}(r),$$

and thus (3.5.39) is a consequence of (3.5.38). □

Lemma 3.5.7. *Let $p > n+2$, $u_0 \in L^\infty(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$ such that $u_0 \geq \delta$. Then any weak martingale solution*

$$\{(\check{\Omega},\check{\mathfrak{A}},\check{\mathfrak{F}},\check{\mathbb{P}}),(\check{\beta}^{(k)})_{k\in\mathbb{Z}},\check{u}_\delta\}$$

to (3.1.8) in the sense of Definition 3.4.10 with initial value u_0 constructed in Lemma 3.4.11 satisfies

$$\check{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L_\delta(\check{u}_\delta(t))\,\mathrm{d}x\right)^{\frac{p}{2}}\right] \lesssim_{n,\nu,\sigma,p,T} \mathbb{E}\left[|\mathscr{A}(u_0)|^{(n-2)\frac{p}{2}} + \left(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\right)^{\frac{p}{2}}\right].$$

Proof. As in the proof of Lemma 3.5.5 we first take $R \to \infty$ in (3.5.1) and obtain that

$$\begin{aligned}
&\check{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L^+_{\delta,\varepsilon}(\check{u}_{\delta,\varepsilon}(t))\,\mathrm{d}x\right)^{\frac{p}{2}}\right]\\
&\quad\lesssim_{n,\nu,p,\sigma,T} C_{\delta,n,\nu,p,T}\check{\mathbb{E}}\left[\|\check{u}^-_{\delta,\varepsilon}\|^p_{C([0,T]\times\mathbb{T})}\right]^{\frac{1}{2}}\left(\mathbb{E}\left[\|G_{\delta,\varepsilon}(u_0)\|^p_{L^1(\mathbb{T})}+|\mathscr{A}(u_0)|^{2p}\right]+1\right)^{\frac{1}{2}}\\
&\qquad+\check{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\check{u}^+_{\delta,\varepsilon}(t)\|^{(n-2)\frac{p}{2}}_{L^1(\mathbb{T})}\right]+C_{\delta,n,\nu,p}\check{\mathbb{E}}\left[\|\check{u}^-_{\delta,\varepsilon}\|^{(4-n)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}+\|\check{u}^-_{\delta,\varepsilon}\|^{(4-\nu)\frac{p}{2}}_{C([0,T]\times\mathbb{T})}\right]\\
&\qquad+\mathbb{E}\left[\left(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\right)^{\frac{p}{2}}\right]
\end{aligned}$$

holds for the sequence $\check{u}_{\delta,\varepsilon}$ converging $\check{\mathbb{P}}$-almost surely to $\check{u}_\delta$ from [35, Eq. (5.14)]. Continuing as in the proof of Lemma 3.5.5, we obtain

$$\begin{aligned}
&\check{\mathbb{E}}\left[\sup_{t\in[0,T]}\left(\int_{\mathbb{T}}\liminf_{\varepsilon\searrow 0} L^+_{\delta,\varepsilon}(\check{u}_{\delta,\varepsilon}(t))\,\mathrm{d}x\right)^{\frac{p}{2}}\right]\\
&\quad\lesssim_{n,\nu,p,\sigma,T}\mathbb{E}\left[|\mathscr{A}(u_0)|^{(n-2)\frac{p}{2}}+\left(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\right)^{\frac{p}{2}}\right]
\end{aligned}$$

by letting $\varepsilon\searrow 0$ and additionally employing Fatou's lemma. It is left to argue that

$$\liminf_{\varepsilon\searrow 0} L^+_{\delta,\varepsilon}(\check{u}_{\delta,\varepsilon}) \ge L_\delta(\check{u}_\delta). \tag{3.5.48}$$

To this end, we use that $\check{u}_{\delta,\varepsilon}$ becomes eventually positive $\check{\mathbb{P}}\otimes\mathrm{d}t\otimes\mathrm{d}x$-almost everywhere by Lemma 3.4.11 and the notation

$$\mathscr{S}_r=\begin{cases}(1,r), & r>1,\\ \varnothing, & r=1,\\ (r,1), & r<1\end{cases}$$

to deduce

$$\begin{aligned}
\liminf_{\varepsilon\searrow 0} L^+_{\delta,\varepsilon}(\check{u}_{\delta,\varepsilon}(t)) &= \liminf_{\varepsilon\searrow 0}\int_0^\infty \mathbf{1}_{\mathscr{S}_{\check{u}_{\delta,\varepsilon}}}(r')\int_0^\infty \mathbf{1}_{\mathscr{S}_{r'}}(r'')\frac{J_{\delta,\varepsilon}(r'')}{F^2_{\delta,\varepsilon}(r'')}\,\mathrm{d}r''\,\mathrm{d}r'\\
&\ge \int_0^\infty\left(\liminf_{\varepsilon\searrow 0}\mathbf{1}_{\mathscr{S}_{\check{u}_{\delta,\varepsilon}}}(r')\right)\times\left(\liminf_{\varepsilon\searrow 0}\int_0^\infty \mathbf{1}_{\mathscr{S}_{r'}}(r'')\frac{J_{\delta,\varepsilon}(r'')}{F^2_{\delta,\varepsilon}(r'')}\,\mathrm{d}r''\right)\mathrm{d}r'\\
&\ge \int_0^\infty \mathbf{1}_{\mathscr{S}_{\check{u}_\delta}}(r')\int_0^\infty \mathbf{1}_{\mathscr{S}_{r'}}(r'')\times\left(\liminf_{\varepsilon\searrow 0}\frac{J_{\delta,\varepsilon}(r'')}{F^2_{\delta,\varepsilon}(r'')}\right)\mathrm{d}r''\,\mathrm{d}r'\\
&\overset{(3.2.15)\text{–}(3.2.17)}{\ge}\int_0^\infty \mathbf{1}_{\mathscr{S}_{\check{u}_\delta}}(r')\int_0^\infty \mathbf{1}_{\mathscr{S}_{r'}}(r'')\frac{J_\delta(r'')}{F^2_\delta(r'')}\,\mathrm{d}r''\,\mathrm{d}r' \overset{(3.2.7)}{=} L_\delta(\check{u}_\delta).
\end{aligned}$$

Here, we repeatedly applied Fatou's lemma together with the the properties of the limes inferior. This shows (3.5.48) and finishes the proof. □

With a δ-uniform energy estimate at hand, the proof of Theorem 3.1.2 follows along the lines of the proof of [35, Theorem 2.2].

Proof of Theorem 3.1.2. Let $p > n+2$ and $u_0 \in L^p(\Omega, \mathfrak{F}_0; H^1(\mathbb{T}))$ in accordance with (3.1.3). Moreover, for the given n, we fix one feasible ν, so that we can drop the ν-dependence in the following estimates. We define

$$u_{0,\delta} = \mathbf{1}_{\{\|u_0\|_{H^1(\mathbb{T})} < e^{\frac{1}{\delta}}\}} u_0 + \delta. \tag{3.5.49}$$

In particular, Lemma 3.4.11 is applicable and yields existence of weak martingale solutions $(\check{u}_\delta)_{\delta>0}$ to (3.1.8) in the sense of Definition 3.4.10 with initial value $u_{0,\delta}$. For ease of notation, we assume these solutions to be defined with respect to the same stochastic basis

$$\{(\check{\Omega}, \check{\mathfrak{A}}, \check{\mathfrak{F}}, \check{\mathbb{P}}), (\check{\beta}^{(k)})_{k\in\mathbb{Z}}\}.$$

Also Lemma 3.5.5 and Lemma 3.5.7 are applicable by (3.5.49) and yield that the solutions satisfy the uniform estimate

$$\begin{aligned}&\check{\mathbb{E}}\Bigg[\sup_{t\in[0,T]} \|\partial_x \check{u}_\delta(t)\|^p_{L^2(\mathbb{T})} + \sup_{t\in[0,T]}\left(\int_{\mathbb{T}} L_\delta(\check{u}_\delta(t))\,\mathrm{d}x\right)^{\frac{p}{2}} + \left(\int_0^T\int_{\mathbb{T}} F_\delta^2(\check{u}_\delta)(\partial_x^3\check{u}_\delta)^2\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{p}{2}}\Bigg]\\ &\lesssim_{n,p,\sigma,T} \mathbb{E}\left[\|\partial_x u_{0,\delta}\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u_{0,\delta})|^{\frac{p(n+2)}{8-2n}} + \left(\int_{\mathbb{T}} (u_{0,\delta}-1) - \log(u_{0,\delta})\,\mathrm{d}x\right)^{\frac{p}{2}}\right] + 1. \end{aligned}\tag{3.5.50}$$

Proceeding as in [35, Lemma 5.1], we obtain the estimate

$$\begin{aligned}\check{\mathbb{E}}\left[\|\check{u}_\delta\|^q_{C^{\frac14}([0,T];L^2(\mathbb{T}))}\right] &\lesssim_{n,p,q,\sigma,T} \left(\check{\mathbb{E}}\left[\sup_{t\in[0,T]}\|\partial_x\check{u}_\delta\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(\check{u}_\delta(0))|^p\right]\right)^{\frac{(n+2)q}{2p}}\\ &+\left(\check{\mathbb{E}}\left[\left(\int_0^T\int_{\mathbb{T}} F_\delta^2(\check{u}_\delta)(\partial_x^3\check{u}_\delta)^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p}{n+2}}\right]\right)^{\frac{(n+2)q}{2p}} + 1,\end{aligned}$$

for any $q \in [1, \frac{2p}{n+2})$ from the last equation of the proof of [35, Lemma 5.1]. Indeed, to achieve this one has to estimate the nonlinearities from (3.1.8), which satisfy the same bounds

$$F_\delta(r) \le r^{\frac{n}{2}}, \quad r > 0,$$

and (3.2.13) as in the case of a homogeneous mobility function. Moreover we remark that our parameters (p,q) are labeled $((n+2)p, p')$ in [35, Lemma 5.1]. Inserting (3.5.50), we conclude

$$\begin{aligned}\check{\mathbb{E}}\left[\|\check{u}_\delta\|^q_{C^{\frac14}([0,T];L^2(\mathbb{T}))}\right] &\lesssim_{n,p,q,\sigma,T} \left(\mathbb{E}\left[\|\partial_x u_{0,\delta}\|^p_{L^2(\mathbb{T})} + |\mathscr{A}(u_{0,\delta})|^{\frac{p(n+2)}{8-2n}}\right]\right)^{\frac{(n+2)q}{2p}}\\ &+\left(\mathbb{E}\left[\left(\int_{\mathbb{T}}(u_{0,\delta}-1)-\log(u_{0,\delta})\,\mathrm{d}x\right)^{\frac{p}{2}}\right]\right)^{\frac{(n+2)q}{2p}} + 1.\end{aligned}\tag{3.5.51}$$

Next, we verify that

$$\limsup_{\delta\searrow 0}\mathbb{E}\left[\left(\int_{\mathbb{T}}(u_{0,\delta}-1)-\log(u_{0,\delta})\,\mathrm{d}x\right)^{\frac{p}{2}}\right] \le \mathbb{E}\left[\left(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\right)^{\frac{p}{2}}\right] \tag{3.5.52}$$

to ensure that (3.5.50) and (3.5.51) have a uniformly bounded right-hand side in δ. To this end, we calculate

$$
\begin{aligned}
&\limsup_{\delta\searrow 0}\mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u_{0,\delta}-1)-\log(u_{0,\delta})\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\overset{(3.5.49)}{\le}\limsup_{\delta\searrow 0}\mathbb{E}\Big[\mathbf{1}_{\{\|u_0\|_{H^1(\mathbb{T})}<e^{\frac{1}{\delta}}\}}\Big(\int_{\mathbb{T}}(u_0+\delta-1)-\log(u_0+\delta)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\quad+\limsup_{\delta\searrow 0}\mathbb{E}\Big[\mathbf{1}_{\{\|u_0\|_{H^1(\mathbb{T})}\ge e^{\frac{1}{\delta}}\}}\Big(\int_{\mathbb{T}}(\delta-1)-\log(\delta)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\le\limsup_{\delta\searrow 0}\mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u_0+\delta-1)-\log(u_0)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\quad+\limsup_{\delta\searrow 0}\mathbb{E}\Big[\mathbf{1}_{\{\|u_0\|_{H^1(\mathbb{T})}\ge e^{\frac{1}{\delta}}\}}\Big(\int_{\mathbb{T}}\log(\tfrac{1}{\delta})\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\le\mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big]\\
&\quad+\limsup_{\delta\searrow 0}\mathbb{E}\Big[\mathbf{1}_{\{\|u_0\|_{H^1(\mathbb{T})}\ge e^{\frac{1}{\delta}}\}}\big(\log(\log(\|u_0\|_{H^1(\mathbb{T})}))\big)^{\frac{p}{2}}\Big]\\
&=\mathbb{E}\Big[\Big(\int_{\mathbb{T}}(u_0-1)-\log(u_0)\,\mathrm{d}x\Big)^{\frac{p}{2}}\Big],
\end{aligned}
\tag{3.5.53}
$$

where in the last step we used dominated convergence and $u_0\in L^p(\Omega,\mathfrak{F}_0;H^1(\mathbb{T}))$. This shows (3.5.52) and following [35, Corollary 5.2, Proposition 5.4, Lemma 5.5], we obtain a new filtered probability space $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathfrak{F}},\widetilde{\mathbb{P}})$ with a family of independent $\widetilde{\mathfrak{F}}$-Brownian motions $(\widetilde{\beta}_k)_{k\in\mathbb{Z}}$ and an equidistributed subsequence $\widetilde{u}_\delta\sim\check{u}_\delta$ converging to an $\widetilde{\mathfrak{F}}$-adapted process $\widetilde{u}$ in $C^{\frac{1}{8}-,\frac{1}{2}-}([0,T]\times\mathbb{T})$, $\widetilde{\mathbb{P}}$-almost everywhere. As in [58, Proposition 5.6], we conclude that

$$\mathbf{1}_{\{\widetilde{u}_\delta>0\}}F_\delta(\widetilde{u}_\delta)\partial_x^3\widetilde{u}_\delta\rightharpoonup\mathbf{1}_{\{\widetilde{u}>0\}}F_0(\widetilde{u})\partial_x^3\widetilde{u}$$

in $L^2([0,T]\times\mathbb{T})$, $\widetilde{\mathbb{P}}$-almost surely. Estimate (3.1.5) follows as in [35, Proposition 5.6] from (3.5.50) and (3.5.52) and consequently also that $\widetilde{u}>0$ $\widetilde{\mathbb{P}}\otimes\mathrm{d}t\otimes\mathrm{d}x$-almost everywhere. As in [35, Proof of Theorem 2.2], we show that equation (3.1.2) holds. Lastly, the temporal regularity statement (3.1.6) can be deduced in the same way as [35, Eq. (5.15)]. □

Proof of Proposition 3.1.3. We remark again, that, since we chose one particular ν in the proof of Theorem 3.1.2, we can drop the ν-dependence in the following estimates. Let $\widehat{u}_{\delta,\varepsilon,R}$ be the sequence of solutions to (3.4.1) used in Lemma 3.4.11 to construct the $\check{u}_\delta$ from the proof of Theorem 3.1.2. By Lemma 3.4.3, $\widehat{u}_{\delta,\varepsilon,R}$ suffices the estimate

$$
\begin{aligned}
&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\|G_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}(t))\|_{L^1(\mathbb{T})}^q+\|\partial_x^2\widehat{u}_{\delta,\varepsilon,R}\|_{L^2([0,T]\times\mathbb{T})}^{2q}\Big]\\
&\quad\lesssim_{n,q,\sigma,T}\mathbb{E}\Big[\|G_{\delta,\varepsilon}(u_{0,\delta})\|_{L^1(\mathbb{T})}^q+|\mathscr{A}(u_{0,\delta})|^{2q}\Big]+1.
\end{aligned}
\tag{3.5.54}
$$

We use (3.5.15) to estimate

$$G_{\delta,\varepsilon}(r) \lesssim \int_r^\infty \int_{r'}^\infty \frac{1}{(r'')^n} \,\mathrm{d}r'' \,\mathrm{d}r' \eqsim_n r^{2-n}, \quad r \geq \delta$$

and since $u_{0,\delta} \geq \delta$ by (3.5.49), we can use (3.5.54) to obtain

$$\begin{aligned} &\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} \|G_{\delta,\varepsilon}(\widehat{u}_{\delta,\varepsilon,R}(t))\|_{L^1(\mathbb{T})}^q + \|\partial_x^2 \widehat{u}_{\delta,\varepsilon,R}\|_{L^2([0,T]\times\mathbb{T})}^{2q}\Big] \\ &\lesssim_{n,q,\sigma,T} \mathbb{E}\Big[\|u_{0,\delta}^{2-n}\|_{L^1(\mathbb{T})}^q + |\mathscr{A}(u_{0,\delta})|^{2q}\Big] + 1. \end{aligned}$$

3

Estimate (3.1.7) follows by Fatou's lemma and

$$\limsup_{\delta\searrow 0} \mathbb{E}\Big[\|u_{0,\delta}^{2-n}\|_{L^1(\mathbb{T})}^q\Big] \leq \mathbb{E}\Big[\|u_0^{2-n}\|_{L^1(\mathbb{T})}^q\Big],$$

which can be derived analogously to (3.5.53). □

4

EXISTENCE WITH NONLINEAR NOISE: THE GENERAL CASE[†]

The aim of this chapter is to prove the existence of very weak martingale solutions to (STFE) in the nonlinear noise case also for non-fully supported initial values, complementing [35] and Chapter 3. More precisely, we consider again the Stratonovich interpretation of (STFE) with $n \in (2,3)$ and $d = 1$, but require u_0 only to be non-negative.

Constructions of martingale solutions to (STFE) rely on closing a-priori estimates, known for the deterministic thin-film equation

$$\partial_t u = -\partial_x(u^n \partial_x^3 u), \tag{4.0.1}$$

for suitable approximations of (STFE). In the deterministic setting, these a-priori estimates are given by the energy estimate

$$\tfrac{1}{2}\partial_t \int (\partial_x u)^2 \,\mathrm{d}x \leq -\int (u^n \partial_x^3 u)^2 \,\mathrm{d}x, \tag{4.0.2}$$

the entropy estimate

$$\partial_t \int G(u)\,\mathrm{d}x \leq -\int (\partial_x^2 u)^2 \,\mathrm{d}x, \tag{4.0.3}$$

and the α-entropy estimate

$$\partial_t \int G_\alpha(u)\,\mathrm{d}x \eqsim_{\alpha,n,\theta} -\int u^{\alpha+n-2\theta+1}(\partial_x^2 u^\theta)^2 \,\mathrm{d}x - \int u^{\alpha+n-3}(\partial_x u)^4 \,\mathrm{d}x, \tag{4.0.4}$$

where $G(u) = \int^u \int^t s^{-n}\,\mathrm{d}s\,\mathrm{d}t$ and more generally $G_\alpha(u) = \int^u \int^t s^{\alpha-1}\,\mathrm{d}s\,\mathrm{d}t$, see Subsection 1.3.2. We point out that $G_\alpha = G$ for $\alpha = 1-n$. Moreover, (4.0.4) holds if

$$\tfrac{1}{3}(\alpha+n-2)(2\theta-1-(\alpha+n)) - (\theta-1)^2 > 0, \tag{4.0.5}$$

[†]This chapter is based on the preprint [36]: K. Dareiotis, B. Gess, M. V. Gnann, and M. Sauerbrey. "Solutions to the stochastic thin-film equation for initial values with non-full support". In: *arXiv preprint arXiv:2305.06017* (2023).

and a parameter $\theta \in (0,\infty)$ subject to (4.0.5) exists if $\alpha \in (\frac{1}{2}-n, 2-n)$. If $\alpha \in (1-n, 2-n)$, the particular choice $\theta = 1$ satisfies (4.0.5), while for the boundary cases $\alpha \in \{\frac{1}{2}-n, 2-n\}$ a version of (4.0.4) applies too.

By Itô's formula, the time increments of these quantities consist in the stochastic setting next to the negative dissipation terms from the thin-film operator and the martingale part also of possibly positive terms arising from the Itô correction of the multiplicative noise term. The main challenge in the nonlinear noise case is that the Itô expansion of the energy contains terms, which can explode for $u = 0$, see [35, Eq. (4.10)] and (3.0.3), and hence a control on the smallness of the solution is required. The strategy in this chapter is to discard the energy estimate and to base the whole analysis on the α-entropy estimates as well as the conservation of mass $\partial_t \int u \,\mathrm{d}x = 0$ as performed successfully in [32, Section 6] for the deterministic case. By restricting ourselves to the range of mobility exponents $n \in (2,3)$, we can take $\alpha \in (-1, 2-n)$, such that (4.0.4) holds true with $\theta = 1$ and $\int G_\alpha(u)\,\mathrm{d}x < \infty$ also for functions u without full support. Since the α-entropy estimates yield less control on the spatial derivatives of the solution, we use the very weak form of the thin-film operator in accordance with [32]. In particular, in contrast to [35] and Chapter 3, we treat measure-valued initial data and noise with less spatial regularity.

4.1. Introduction to Chapter 4

In this section, we state and discuss the results of this chapter, outline their proof and review the used notation.

4.1.1. Main result

In this chapter, we write

$$m(u) = |u|^n \qquad \text{and} \qquad q(u) = \sqrt{m(u)} = |u|^{\frac{n}{2}}$$

for the mobility function and its square-root, respectively, and restrict ourselves to mobility exponents from the following range.

Assumption 4.1.1. *We assume that $n \in (2,3)$.*

Moreover, we assume $\mathscr{W}$ in (STFE) to be spatio-temporal Gaussian noise, which is white in time and colored in space. Specifically, we assume that the noise $\mathscr{W}$ is given by the time derivative of the Wiener process B, defined by

$$B(t) = \sum_{k\in\mathbb{Z}} \sigma_k \beta_t^{(k)} \tag{4.1.1}$$

for a family of independent $\mathfrak{F}$-Brownian motions $(\beta^{(k)})_{k\in\mathbb{Z}}$ defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with filtration $\mathfrak{F}$. Here, we assume that

$$\sigma_k = \lambda_k f_k, \tag{4.1.2}$$

where $\Lambda = (\lambda_k)_{k\in\mathbb{Z}}$ is a sequence of real numbers and

$$f_k(x) = \begin{cases} \sqrt{2}\cos(2\pi k x), & k < 0, \\ 1, & k = 0, \\ \sqrt{2}\sin(2\pi k x), & k > 0 \end{cases} \tag{4.1.3}$$

are the eigenfunctions of the periodic Laplace operator. We impose the following condition on the smoothness of the process B in terms of the sequence Λ.

Assumption 4.1.2. *It holds* $\sum_{k\in\mathbb{Z}}(k\lambda_k)^2 < \infty$.

As initial values to (STFE), we allow for non-negative Borel measure-valued random variables. To be precise, we introduce the σ-field $\mathcal{Z}$ on $\mathcal{M}(\mathbb{T})$ as the σ-field generated by the pre-dual space $C(\mathbb{T})$, i.e. an $\mathcal{M}(\mathbb{T})$-valued random variable X is $\mathcal{Z}$-measurable, iff $\langle X,\varphi\rangle$ is measurable for each $\varphi \in C(\mathbb{T})$.

Assumption 4.1.3. *The initial value* $u_0\colon \Omega \to \mathcal{M}(\mathbb{T})$ *is* $\mathfrak{F}_0$*-*$\mathcal{Z}$ *measurable and* $u_0 \geq 0$ *almost surely.*

Interpreting (STFE) in Stratonovich form, using the notation $q(u) = |u|^{\frac{n}{2}}$ and the description of the noise (4.1.1), we obtain the equivalent Itô formulation

$$\begin{aligned} \mathrm{d}u = &-\partial_x(u^n\partial_x^3 u)\,\mathrm{d}t + \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\partial_x(\sigma_k q'(u)\partial_x(\sigma_k q(u)))\,\mathrm{d}t \\ &+ \sum_{k\in\mathbb{Z}}\partial_x(\sigma_k q(u))\,\mathrm{d}\beta^{(k)} \end{aligned} \tag{4.1.4}$$

of the stochastic thin-film equation. In order to obtain a sufficiently weak formulation for the case of a possibly compactly supported initial value u_0, we test (4.1.4) with a smooth function $\varphi \in C^\infty(\mathbb{T})$ in the dual pairing $\langle\cdot,\cdot\rangle$ on $\mathbb{T}$ and rewrite the thin-film operator in the very weak form introduced in [33, Eq. (3.2)], see also [32, Definition 1]. We obtain the formulation

$$\begin{aligned} \mathrm{d}\langle u,\varphi\rangle = &\Big[\tfrac{n(n-1)}{2}\langle u^{n-2}(\partial_x u)^3,\partial_x\varphi\rangle + \tfrac{3n}{2}\langle u^{n-1}(\partial_x u)^2,\partial_x^2\varphi\rangle + \langle u^n\partial_x u,\partial_x^3\varphi\rangle\Big]\,\mathrm{d}t \\ &- \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\langle\sigma_k q'(u)\partial_x(\sigma_k q(u)),\partial_x\varphi\rangle\,\mathrm{d}t - \sum_{k\in\mathbb{Z}}\langle\sigma_k q(u),\partial_x\varphi\rangle\,\mathrm{d}\beta^{(k)}, \end{aligned} \tag{4.1.5}$$

which gives rise to the following notion of very weak martingale solutions to the stochastic thin-film equation up to a fixed time horizon $T \in (0,\infty)$.

Definition 4.1.4. *A very weak martingale solution to* (4.1.5) *consists out of a probability space* $(\tilde{\Omega},\tilde{\mathfrak{A}},\tilde{\mathbb{P}})$ *with a filtration* $\tilde{\mathfrak{F}}$ *satisfying the usual conditions, a family of independent* $\tilde{\mathfrak{F}}$*-Brownian motions* $(\tilde{\beta}^{(k)})_{k\in\mathbb{Z}}$ *and a non-negative, vaguely continuous,* $\tilde{\mathfrak{F}}$*-adapted* $(\mathcal{M}(\mathbb{T}),\mathcal{Z})$*-valued process* $\tilde{u}$ *defined on* $[0,T]$ *such that* $\tilde{\mathbb{P}}\otimes\mathrm{d}t$*-almost everywhere* $\tilde{u} \in W^{1,1}(\mathbb{T})$ *with*

$$\tilde{u}^{n-2}(\partial_x\tilde{u})^3,\ \tilde{u}^{n-1}(\partial_x\tilde{u})^2,\ \tilde{u}^n\partial_x\tilde{u},\ \tilde{u}^{n-2}\partial_x\tilde{u},\ \tilde{u}^n \in L^1([0,T]\times\mathbb{T}) \tag{4.1.6}$$

almost surely and for every $\varphi \in C^\infty(\mathbb{T})$, $t \in [0,T]$ *we have*

$$\begin{aligned} \langle\tilde{u}(t),\varphi\rangle - \langle\tilde{u}(0),\varphi\rangle = &\tfrac{n(n-1)}{2}\int_0^t\langle\tilde{u}^{n-2}(\partial_x\tilde{u})^3,\partial_x\varphi\rangle\,\mathrm{d}s + \tfrac{3n}{2}\int_0^t\langle\tilde{u}^{n-1}(\partial_x\tilde{u})^2,\partial_x^2\varphi\rangle\,\mathrm{d}s \\ &+ \int_0^t\langle\tilde{u}^n\partial_x\tilde{u},\partial_x^3\varphi\rangle\,\mathrm{d}s - \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t\langle\sigma_k q'(\tilde{u})\partial_x(\sigma_k q(\tilde{u})),\partial_x\varphi\rangle\,\mathrm{d}s \\ &- \sum_{k\in\mathbb{Z}}\int_0^t\langle\sigma_k q(\tilde{u}),\partial_x\varphi\rangle\,\mathrm{d}\tilde{\beta}_s^{(k)}. \end{aligned} \tag{4.1.7}$$

Remark 4.1.5. (i) The requirement (4.1.6) ensures that the deterministic and stochastic integrals in (4.1.7) converge.

(ii) We demand in Definition 4.1.4 that for every $\varphi \in C^\infty(\mathbb{T})$, $t \in [0,T]$ the identity (4.1.7) holds outside of some $\widetilde{\mathbb{P}}$-nullset. Since the vague continuity of $\tilde{u}$ implies that all the processes in (4.1.7) are continuous in time, this nullset can be chosen independently of t.

In the course of this chapter, we prove the existence of very weak martingale solutions to the stochastic thin-film equation in the sense of Definition 4.1.4 under the previous assumptions.

Theorem 4.1.6. *Under the Assumptions 4.1.1–4.1.3 and σ_k given by (4.1.2), there exists a very weak martingale solution $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$, $\widetilde{\mathfrak{F}}$, $(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}}$, $\tilde{u}$ to (4.1.5) in the sense of Definition 4.1.4 such that $\tilde{u}(0)$ has the same distribution as u_0 on $(\mathcal{M}(\mathbb{T}),\mathcal{Z})$. Moreover, $\tilde{u}$ admits the following properties.*

(i) Mass is conserved, i.e., almost surely $\langle \tilde{u}(t), \mathbf{1}_{\mathbb{T}}\rangle = \langle \tilde{u}(0), \mathbf{1}_{\mathbb{T}}\rangle$ for all $t \in [0,T]$.

(ii) Almost surely, $\tilde{u} \in L^p([0,T]\times\mathbb{T}) \cap L^r(0,T;W^{1,r}(\mathbb{T}))$ for each $p \in (n+4,7)$ and $r \in (\frac{n+4}{2},\frac{7}{2})$ with

$$\widetilde{\mathbb{E}}\big[\|\tilde{u}\|^p_{L^p([0,T]\times\mathbb{T})}\big] \lesssim_{n,p,\Lambda,T} \mathbb{E}\big[\|u_0\|^{p-n}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^p_{\mathcal{M}(\mathbb{T})}\big], \tag{4.1.8}$$

$$\widetilde{\mathbb{E}}\big[\|\partial_x\tilde{u}\|^r_{L^r([0,T]\times\mathbb{T})}\big] \lesssim_{n,r,\Lambda,T} \mathbb{E}\big[\|u_0\|^{r-n}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^r_{\mathcal{M}(\mathbb{T})}\big], \tag{4.1.9}$$

whenever the respective right-hand side is finite.

(iii) Almost surely, we have $\tilde{u} \in C([0,T];H^\kappa(\mathbb{T}))$ for $\kappa \in (-\infty,\frac{-1}{2})$ and if $\gamma \in (0,\frac{1}{2})$, $\mu \in (\frac{n+4}{n+2},\frac{7}{n+2})$ and $\nu \in (1,\frac{7}{n+4})$, it holds

$$\begin{aligned}&\widetilde{\mathbb{E}}\Big[\|\tilde{u}\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big]\\ &\lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \mathbb{E}\Big[\|u_0\|^{(n-1-\frac{n^2}{p_{\mu,\nu}})\nu}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^{(n+1)\nu}_{\mathcal{M}(\mathbb{T})}\Big],\end{aligned} \tag{4.1.10}$$

where $p_{\mu,\nu} = \max\{\mu(n+2),\nu(n+4)\}$, whenever the right-hand side is finite.

(iv) For $\alpha \in (-1,2-n)$ we have almost surely

$$\tilde{u}^{\frac{\alpha+n+1}{4}} \in L^4(0,T;W^{1,4}(\mathbb{T})), \quad \tilde{u}^{\frac{\alpha+n+1}{2}} \in L^2(0,T;H^2(\mathbb{T}))$$

and it holds the α-entropy type estimate

$$\widetilde{\mathbb{E}}\Big[\big\|\partial_x\tilde{u}^{\frac{\alpha+n+1}{4}}\big\|^4_{L^4([0,T]\times\mathbb{T})} + \big\|\partial_x^2\tilde{u}^{\frac{\alpha+n+1}{2}}\big\|^2_{L^2([0,T]\times\mathbb{T})}\Big] \lesssim_{\alpha,n,\Lambda,T} \mathbb{E}\big[\|u_0\|^{\alpha+1}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^{\alpha+n-1}_{\mathcal{M}(\mathbb{T})}\big], \tag{4.1.11}$$

if the right-hand side is finite.

Remark 4.1.7. We convince ourselves, that the regularity statements from Theorem 4.1.6 suffice to deduce the most restrictive integrability assumption from Definition 4.1.4, namely that $\tilde{u}^{n-2}(\partial_x \tilde{u})^3 \in L^1([0,T]\times\mathbb{T})$. As a consequence of Theorem 4.1.6 (ii), we have that almost surely

$$\tilde{u} \in L^{n+4}([0,T]\times\mathbb{T}), \quad \partial_x \tilde{u} \in L^{\frac{n+4}{2}}([0,T]\times\mathbb{T}).$$

Hence, using Hölder's inequality, we can indeed conclude that

$$\|\tilde{u}^{n-2}(\partial_x \tilde{u})^3\|_{L^1([0,T]\times\mathbb{T})} \le \|\tilde{u}^{n-2}\|_{L^{\frac{n+4}{n-2}}([0,T]\times\mathbb{T})} \|(\partial_x \tilde{u})^3\|_{L^{\frac{n+4}{6}}([0,T]\times\mathbb{T})}$$

is almost surely finite.

Remark 4.1.8. Let $\tilde{u}$ be the very weak martingale solution to (4.1.5) obtained from Theorem 4.1.6. For $\alpha \in (-1, 2-n)$, the α-entropy type estimate from Theorem 4.1.6 (iv) yields the following additional properties of $\tilde{u}$.

(i) We have that $\tilde{\mathbb{P}} \otimes \mathrm{d}t$-almost everywhere $\tilde{u}^{\frac{\alpha+n+1}{2}} \in H^2(\mathbb{T})$. Hence, by the Sobolev-embedding $\tilde{u} \in C(\mathbb{T})$ and therefore $\tilde{u}$ is uniformly supported away from zero and bounded on compact subsets of $\{\tilde{u} > 0\}$. Thus, $\tilde{u} \in H^2_{\mathrm{loc}}(\{\tilde{u} > 0\})$ and we have

$$\begin{aligned} \partial_x \tilde{u}^{\frac{\alpha+n+1}{4}} &= \frac{\alpha+n+1}{4} \tilde{u}^{\frac{\alpha+n-3}{4}} \partial_x \tilde{u}, \\ \partial_x^2 \tilde{u}^{\frac{\alpha+n+1}{2}} &= \frac{\alpha+n+1}{2} \tilde{u}^{\frac{\alpha+n-1}{2}} \partial_x^2 \tilde{u} + \frac{(\alpha+n+1)(\alpha+n-1)}{4} \tilde{u}^{\frac{\alpha+n-3}{2}} (\partial_x \tilde{u})^2, \end{aligned}$$

on $\{\tilde{u} > 0\}$. We conclude that

$$\int_0^T \int_{\{\tilde{u}>0\}} \tilde{u}^{\alpha+n-3} (\partial_x \tilde{u})^4 \,\mathrm{d}x\,\mathrm{d}t \lesssim_{\alpha,n} \|\partial_x \tilde{u}^{\frac{\alpha+n+1}{4}}\|^4_{L^4([0,T]\times\mathbb{T})}$$

and consequently

$$\begin{aligned} &\int_0^T \int_{\{\tilde{u}>0\}} \tilde{u}^{\alpha+n-1} (\partial_x^2 \tilde{u})^2 \,\mathrm{d}x\,\mathrm{d}t \lesssim_{\alpha,n} \int_0^T \int_{\{\tilde{u}>0\}} (\partial_x^2 \tilde{u}^{\frac{\alpha+n+1}{2}})^2 + \tilde{u}^{\alpha+n-3} (\partial_x \tilde{u})^4 \,\mathrm{d}t \\ &\lesssim_{\alpha,n} \|\partial_x^2 \tilde{u}^{\frac{\alpha+n+1}{2}}\|^2_{L^2([0,T]\times\mathbb{T})} + \|\partial_x \tilde{u}^{\frac{\alpha+n+1}{4}}\|^4_{L^4([0,T]\times\mathbb{T})}. \end{aligned}$$

As a result, (4.1.11) implies that

$$\begin{aligned} &\tilde{\mathbb{E}}\Big[\int_0^T \int_{\{\tilde{u}>0\}} \tilde{u}^{\alpha+n-3} (\partial_x \tilde{u})^4 + \tilde{u}^{\alpha+n-1} (\partial_x^2 \tilde{u})^2 \,\mathrm{d}x\,\mathrm{d}t\Big] \\ &\quad \lesssim_{\alpha,n,\Lambda,T} \mathbb{E}\big[\|u_0\|^{\alpha+1}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^{\alpha+n-1}_{\mathcal{M}(\mathbb{T})}\big] \end{aligned} \tag{4.1.12}$$

appealing to the classical form of the α-entropy estimate (4.0.4) with $\theta = 1$.

(ii) We demonstrate, that as a consequence of (4.1.11) one can also recover an estimate in the spirit of (4.0.4) for $\theta \ne 1$. Arguing as in (i), we conclude that $\tilde{\mathbb{P}} \otimes \mathrm{d}t$-almost everywhere $\tilde{u}^\theta \in H^2_{\mathrm{loc}}(\{\tilde{u} > 0\})$ for any $\theta > 0$, with

$$\partial_x^2 \tilde{u}^\theta = \theta \tilde{u}^{\theta-1} \partial_x^2 \tilde{u} + \theta(\theta-1) \tilde{u}^{\theta-2} (\partial_x \tilde{u})^2.$$

By taking the square on both sides, we obtain that

$$(\partial_x^2 \tilde{u}^\theta)^2 \lesssim_\theta \tilde{u}^{2\theta-2}(\partial_x^2 \tilde{u})^2 + \tilde{u}^{2\theta-4}(\partial_x \tilde{u})^4.$$

Hence, using (4.1.12) we infer

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{T}} \tilde{u}^{\alpha+n-2\theta+1}(\partial_x^2 \tilde{u}^\theta)^2 \,\mathrm{d}x\,\mathrm{d}t\right] \lesssim_{\alpha,n,\theta,\Lambda,T} \mathbb{E}\left[\|u_0\|_{\mathcal{M}(\mathbb{T})}^{\alpha+1} + \|u_0\|_{\mathcal{M}(\mathbb{T})}^{\alpha+n-1}\right].$$

(iii) In [75, Corollary 3.2] it is shown that the solutions to the stochastic thin-film equation constructed in [75] admit a zero contact angle almost everywhere based on the finiteness of the α-entropy dissipation. Following the proof of [75, Corllary 3.2] we obtain the same statement as a consequence of Theorem 4.1.6 (iv), namely that $\tilde{\mathbb{P}} \otimes \mathrm{d}t$-almost everywhere, $\tilde{u}$ admits 0 as its classical derivative at every point from its zero set.

4

4.1.2. OUTLINE AND DISCUSSION OF THE PROOF

As pointed out in the introduction, the main innovation of this chapter is to provide solutions to (STFE) for initial data without full support in the case of a non-quadratic mobility $n \neq 2$. A difficulty in the analysis of (STFE) is to close the deterministic a-priori estimates (4.0.2)–(4.0.4) in the stochastic setting, where at least in the case of Stratonovich noise, the entropy and α-entropy estimate seem to hold for a wide range of n, see [35, Lemma 4.3], [75, Eq. (3.2)], Lemma 2.5.3 and Proposition 3.1.3. However, the additional energy production due to the stochastic term in (STFE) seems to require a control on the smallness of the solution, which we are unable to provide in the case of initial data without full support. Since at least for the case $n \geq 2$, the entropy estimate also fails for such initial values, we rely the whole analysis on the remaining α-entropy estimate (4.0.4) for $\alpha > -1$ as well as the conservation of mass. As observed in [32, Section 6] for the deterministic case, this still suffices to extract a limit which solves the equation in the very weak sense.

To review the argument in the deterministic setting, we set $w = u^{\frac{\alpha+n+1}{4}}$, where u is a solution to (4.0.1) and $\alpha \in (-1, 2-n)$. Then, by the chain rule, integration of (4.0.4) provides an estimate on

$$\int_0^T \|\partial_x w\|_{L^4(\mathbb{T})}^4 \,\mathrm{d}t \eqsim_{\alpha,n} \int_0^T \int_{\mathbb{T}} \left(u^{\frac{\alpha+n-3}{4}} \partial_x u\right)^4 \mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{T}} u^{\alpha+n-3}(\partial_x u)^4 \,\mathrm{d}x\,\mathrm{d}t \tag{4.1.13}$$

and the conservation of mass on

$$\sup_{t\in[0,T]} \|w\|_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})}^{\frac{4}{\alpha+n+1}} = \sup_{t\in[0,T]} \|u\|_{L^1(\mathbb{T})},$$

for non-negative u. An application of the Gagliardo-Nirenberg interpolation inequality allows to find between this estimate in $L^4(0,T;\dot{W}^{1,4}(\mathbb{T}))$ and $L^\infty(0,T;L^{\frac{4}{\alpha+n+1}}(\mathbb{T}))$ an estimate on w, which has the same integrability in space and time, see (4.2.38). Using the identity

$$\|u\|_{L^p([0,T]\times\mathbb{T})} = \|w\|_{L^{\frac{4p}{\alpha+n+1}}([0,T]\times\mathbb{T})}^{\frac{4}{\alpha+n+1}}$$

and that the admissible range for α is $(-1,2-n)$, this can be translated to an estimate on u for $p<7$. Together with the estimate on (4.1.13) and the application of Hölder's inequality

$$\begin{aligned}\|\partial_x u\|_{L^r([0,T]\times\mathbb{T})} &\eqsim_{\alpha,n} \left\| u^{\frac{3-(\alpha+n)}{4}}\partial_x w\right\|_{L^r([0,T]\times\mathbb{T})}\\ &\le \left\| u^{\frac{3-(\alpha+n)}{4}}\right\|_{L^{\frac{4p}{3-(\alpha+n)}}([0,T]\times\mathbb{T})}\|\partial_x w\|_{L^4([0,T]\times\mathbb{T})}\end{aligned}$$

a space-time integral estimate on $\partial_x u$ for $r<\frac{7}{2}$ is obtained in Lemma 4.2.4. Then, for example, the first term on the right-hand side of (4.1.5) can be estimated via Hölder's inequality

$$\|u^{n-2}(\partial_x u)^3\|_{L^{\frac{7}{n+4}-}([0,T]\times\mathbb{T})} \le \|u^{n-2}\|_{L^{\frac{7}{n-2}-}([0,T]\times\mathbb{T})}\|(\partial_x u)^3\|_{L^{\frac{7}{6}-}([0,T]\times\mathbb{T})},$$

where $\frac{7}{n+4}>1$ by Assumption 4.1.1. This turns out to be enough to conclude some temporal regularity and also identify the term $u^{n-2}(\partial_x u)^3$ in the limit using Vitali's convergence theorem.

We generalize this to the stochastic setting by estimating moments of the involved quantities and accounting additionally for the nonlinear conservative noise term from (STFE). For optimal control in ω, we express these estimates conditioned on $\mathfrak{F}_0$ reducing effectively to the situation of a deterministic initial value. Subsequently, we carry out the limiting procedure and identify the stochastic integrals using the stochastic compactness method.

4.1.3. DISCUSSION OF THE RESULT

The fact that the approach from [32] can be applied also in the stochastic setting is essentially due to the use of Stratonovich noise, which is compatible with the α-entropy estimates. This allows us to construct solutions to (STFE) for non-negative initial values from the space of measures, including the interesting case of the Dirac distribution. Moreover, only closing the α-entropy estimates requires less spatial regularity of the noise compared to cases in which also the energy estimate is used. Indeed, Assumption 4.1.2 essentially expresses that B is a Q-Wiener process in $H^1(\mathbb{T})$, while the results from [35, 51, 58, 75, 107] and Chapters 2 and 3 require that B lies in $H^2(\mathbb{T}^d)$, $d\in\{1,2\}$. Although it would be preferable to have solutions satisfying also the energy estimate, our result is the only one so far providing solutions to the stochastic thin-film equation, which allows for initial values without full support in the nonlinear noise case.

4.1.4. NOTATION FOR CHAPTER 4

Let $\mathscr{X}$ be a Banach space and ν a non-negative measure on a measurable space S. Then we write $L^p(S;\mathscr{X})$, $p\in[1,\infty]$, for the Bochner space on S with values in $\mathscr{X}$, equipped with the norm

$$\|f\|^p_{L^p(S;\mathscr{X})}=\int_S \|f\|^p_{\mathscr{X}}\,\mathrm{d}\nu,\quad p\in[1,\infty),$$

and the usual modification for $p=\infty$. In the case $\mathscr{X}=\mathbb{R}$, we simply write $L^p(S)$ or, if S is equipped with the counting measure, $l^p(S)$.

If $[0,T]$ is an interval, we use the notation $L^p(0,T;\mathscr{X})$ for $L^p([0,T];\mathscr{X})$. Moreover, we write $C([0,T];\mathscr{X})$ for the space of continuous, $\mathscr{X}$-valued functions equipped with the norm

$$\|f\|_{C([0,T];\mathscr{X})} = \sup_{0\le t\le T} \|f(t)\|_{\mathscr{X}}.$$

We write $W^{\kappa,p}(0,T;\mathscr{X})$, $\kappa\in(0,1)$, $p\in[1,\infty)$, for the Sobolev–Slobodeckij space equipped with the norm

$$\|f\|^p_{W^{\kappa,p}(0,T;\mathscr{X})} = \|f\|^p_{L^p(0,T;\mathscr{X})} + \int_0^T\int_0^T \frac{\|f(t)-f(s)\|^p_{\mathscr{X}}}{|t-s|^{1+\kappa p}}\,\mathrm{d}t\,\mathrm{d}s.$$

For $l\in\mathbb{N}$ and $p\in[1,\infty]$, $W^{l,p}(0,T;\mathscr{X})$ denotes the usual Sobolev space with norm

$$\|f\|^p_{W^{k,p}(0,T;\mathscr{X})} = \sum_{j=0}^{l} \|\partial_t^l f\|^p_{L^p(0,T;\mathscr{X})}.$$

We write $\mathbb{T}$ for the flat torus, i.e. the interval $[0,1]$ with its endpoints identified. We write $C(\mathbb{T})$ and $C^l(\mathbb{T})$, $l\in\mathbb{N}$, for the continuous and l-times continuously differentiable functions on $\mathbb{T}$, respectively, equipped with the norms

$$\|f\|_{C(\mathbb{T})} = \sup_{x\in\mathbb{T}}|f(x)|, \quad \|f\|_{C^l(\mathbb{T})} = \sum_{j=0}^{l}\|\partial_x^j f\|_{C(\mathbb{T})}.$$

The smooth functions on $\mathbb{T}$ we denote by $C^\infty(\mathbb{T})$ and we write $W^{l,p}(\mathbb{T})$, $l\in\mathbb{N}$, $p\in[1,\infty]$, for the Sobolev space equipped with the norm

$$\|f\|^p_{W^{l,p}(\mathbb{T})} = \sum_{j=0}^{l}\|\partial_x^j f\|^p_{L^p(\mathbb{T})}.$$

For the case $p=2$ we use the notation $H^l(\mathbb{T}) = W^{l,2}(\mathbb{T})$. If $p\in(1,\infty)$, we write $W^{-l,p}(\mathbb{T})$ for the dual space of $W^{l,p'}(\mathbb{T})$ under the duality pairing $\langle\cdot,\cdot\rangle$ in $L^2(\mathbb{T})$, where p' is the Hölder conjugate of p, and equip it with the norm

$$\|f\|_{W^{-l,p}(\mathbb{T})} = \sup_{\|g\|_{W^{l,p'}(\mathbb{T})}\le 1} |\langle f,g\rangle|.$$

Denoting by $\widehat{f}(k) = \langle f, e^{2\pi i k\cdot}\rangle$, $k\in\mathbb{Z}$, the k-th Fourier coefficient of a distribution f on $\mathbb{T}$, we denote by $H^\kappa(\mathbb{T})$ for $\kappa\in\mathbb{R}\setminus\mathbb{N}$ the Bessel-potential space consisting of all distributions with

$$\|f\|^2_{H^\kappa(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2(1+|2\pi k|^2)^\kappa$$

being finite. Moreover, we write $\mathscr{M}(\mathbb{T})$ for the space of Radon measures on $\mathbb{T}$ and equip it with the total variation norm $\|\nu\|_{\mathscr{M}(\mathbb{T})} = |\nu|(\mathbb{T})$.

Lastly, for separable Hilbert spaces G and H we denote the space of Hilbert–Schmidt operators between G and H by $L_2(G,H)$, which carries the norm

$$\|\Psi\|^2_{L_2(G,H)} = \sum_{k\in\mathbb{N}} \|\Psi g_k\|^2_H$$

for an orthonormal basis $(g_k)_{k\in\mathbb{N}}$ of G. If $(\Omega,\mathfrak{A},\mathbb{P})$ is a probability space, we write $\mathbb{E}[\cdot]$ and $\mathbb{E}[\cdot|\mathfrak{H}]$ for the expectation and conditional expectation with respect to a sub-σ-field $\mathfrak{H}\subset\mathfrak{A}$, respectively. For two quantities A and B, we write $A\lesssim B$, if there exists a universal constant C such that $A\le CB$. If this constant depends on parameters $p_1,\dots$, we write $A\lesssim_{p_1,\dots} B$ instead. We write $A\eqsim_{p_1,\dots} B$, whenever $A\lesssim_{p_1,\dots} B$ and $B\lesssim_{p_1,\dots} A$.

4.2. APPROXIMATE SOLUTIONS

In the course of this chapter, we prove the existence of solutions to (4.1.5) under Assumptions 4.1.1–4.1.3. For this purpose, we use solutions to the stochastic thin-film equation with a strictly positive and regularized initial value and spatially smooth noise as approximations, which exist by the results of Chapter 3. To regularize u_0, we let $(\eta_\varepsilon)_{\varepsilon>0}$ be a family of non-negative, smooth functions $\eta_\varepsilon\colon\ \mathbb{T}\to\mathbb{R}$ sufficing $\|\eta_\varepsilon\|_{L^1(\mathbb{T})}=1$ and

4

$$\int_{\mathbb{T}\setminus B_\delta(0)}|\eta_\varepsilon|\,\mathrm{d}x\to 0$$

as $\varepsilon\searrow 0$ for every $\delta>0$. Then, we define

$$u_{0,\varepsilon}=\mathbf{1}_{\{\|u_0\|_{\mathcal{M}(\mathbb{T})}<\frac{1}{\varepsilon}\}}(u_0*\eta_\varepsilon)+\varepsilon,$$

which is strictly positive and smooth by [69, Theorem 2.3.20] with

$$\partial_x^l u_{0,\varepsilon}=\mathbf{1}_{\{\|u_0\|_{\mathcal{M}(\mathbb{T})}<\frac{1}{\varepsilon}\}}(u_0*\partial_x^l\eta_\varepsilon)\tag{4.2.1}$$

for $l\in\mathbb{N}$. By the convolution inequality

$$\|\nu*\eta\|_{L^1(\mathbb{T})}\le\|\eta\|_{L^1(\mathbb{T})}\|\nu\|_{\mathcal{M}(\mathbb{T})}\tag{4.2.2}$$

for general $\nu\in\mathcal{M}(\mathbb{T})$, $\eta\in C^\infty(\mathbb{T})$, see [54, Proposition 8.49], we conclude with (4.2.1) that the $\mathfrak{F}_0$-measurable $u_{0,\varepsilon}$ lies in $L^\infty(\Omega;H^1(\mathbb{T}))$. Moreover, since $\varphi*(\eta_\varepsilon(-\cdot))$ converges uniformly to φ for $\varphi\in C(\mathbb{T})$ by [69, Theorem 1.2.21], we have that

$$\langle u_0*\eta_\varepsilon,\varphi\rangle=\int_{\mathbb{T}}\int_{\mathbb{T}}\eta_\varepsilon(x-y)\,\mathrm{d}u_0(y)\varphi(x)\,\mathrm{d}x=\int_{\mathbb{T}}\int_{\mathbb{T}}\eta_\varepsilon(x-y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}u_0(y)\to\langle u_0,\varphi\rangle,$$

so that $u_{0,\varepsilon}$ converges almost surely to u_0 in the vague topology of $\mathcal{M}(\mathbb{T})$ as $\varepsilon\searrow 0$. Moreover, we introduce the cut-off weights

$$\lambda_{k,\varepsilon}=\begin{cases}\lambda_k, & |k|<\frac{1}{\varepsilon},\\ 0, & \text{else},\end{cases}\tag{4.2.3}$$

and define correspondingly

$$\sigma_{k,\varepsilon}=\lambda_{k,\varepsilon}f_k.\tag{4.2.4}$$

Assumption 4.1.1 in conjunction with Theorem 3.1.2 from Chapter 3 implies that for each $\varepsilon\in(0,1)$ there exists a weak martingale solution in the sense of Definition 3.1.1 to

$$\mathrm{d}u=-\partial_x(u^n\partial_x^3u)\,\mathrm{d}t+\frac{1}{2}\sum_{k\in\mathbb{Z}}\partial_x(\sigma_{k,\varepsilon}q'(u)\partial_x(\sigma_{k,\varepsilon}q(u)))\,\mathrm{d}t+\sum_{k\in\mathbb{Z}}\partial_x(\sigma_{k,\varepsilon}q(u))\,\mathrm{d}\beta^{(k)}$$

with initial value $u_{0,\varepsilon}$. Moreover, by Proposition 3.1.3 this solution also satisfies an entropy estimate. More precisely, the consequences of Assumption 4.1.1 and Chapter 3, which we use in the proof of Theorem 4.1.6, are listed below.

Consequence 4.2.1. *Assumption 4.1.1 implies the following.*

(i) We have $n \in (2,3)$.

(ii) Let $\varepsilon > 0$. *Then there exists a probability space* $(\widetilde{\Omega}_\varepsilon, \widetilde{\mathfrak{A}}_\varepsilon, \widetilde{\mathbb{P}}_\varepsilon)$ *with a filtration* $\widetilde{\mathfrak{F}}_\varepsilon$ *satisfying the usual conditions, a family of independent* $\widetilde{\mathfrak{F}}_\varepsilon$*-Brownian motions* $(\widetilde{\beta}_\varepsilon^{(k)})_{k\in\mathbb{Z}}$, *an* $\widetilde{\mathfrak{F}}_\varepsilon$*-adapted, weakly continuous* $H^1(\mathbb{T})$*-valued process* $\widetilde{u}_\varepsilon$ *together with an* $\widetilde{\mathfrak{F}}_{\varepsilon,0}$*-measurable random variable* $\widetilde{\xi}_\varepsilon$, *subject to the following properties.*

(iii) $(\widetilde{u}_\varepsilon(0), \widetilde{\xi}_\varepsilon)$ *has the same distribution as* $\big(u_{0,\varepsilon}, \|u_0\|_{\mathcal{M}(\mathbb{T})}\big)$.

(iv) We have $\widetilde{\mathbb{P}}_\varepsilon \otimes \mathrm{d}t \otimes \mathrm{d}x$*-almost everywhere* $\widetilde{u}_\varepsilon > 0$.

(v) It holds

$$\widetilde{\mathbb{E}}_\varepsilon\Big[\sup_{0\le t\le T} \|\widetilde{u}_\varepsilon(t)\|_{H^1(\mathbb{T})}^{2n} + \|\mathbf{1}_{\{\widetilde{u}_\varepsilon>0\}} q(\widetilde{u}_\varepsilon)\partial_x^3 \widetilde{u}_\varepsilon\|_{L^2([0,T]\times\mathbb{T})}^4 + \|\widetilde{u}_\varepsilon\|_{L^2(0,T;H^2(\mathbb{T}))}^4\Big] < \infty.$$

(vi) For all $\varphi \in C^\infty(\mathbb{T})$ *and* $t \in [0,T]$, *it holds*

$$\begin{aligned}\langle \widetilde{u}_\varepsilon(t), \varphi\rangle - \langle \widetilde{u}_\varepsilon(0), \varphi\rangle &= \int_0^t \int_{\{\widetilde{u}_\varepsilon>0\}} m(\widetilde{u}_\varepsilon)\partial_x^3 \widetilde{u}_\varepsilon \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \tfrac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^t \langle \sigma_{k,\varepsilon} q'(\widetilde{u}_\varepsilon)\partial_x(\sigma_{k,\varepsilon} q(\widetilde{u}_\varepsilon)), \partial_x\varphi\rangle\,\mathrm{d}s \\ &\quad - \sum_{k\in\mathbb{Z}} \int_0^t \langle \sigma_{k,\varepsilon} q(\widetilde{u}_\varepsilon), \partial_x\varphi\rangle\,\mathrm{d}\widetilde{\beta}_{\varepsilon,s}^{(k)}.\end{aligned} \tag{4.2.5}$$

For technical reasons, we also demanded existence of the random variable $\widetilde{\xi}_\varepsilon$, which can be obtained from Theorem 3.1.2 by including the random variable $\|u_0\|_{\mathcal{M}(\mathbb{T})}$ in the application of the stochastic compactness method. This will be important later, since we will define the decomposition (4.2.60) on the new probability space, which cannot be recovered from the regularized initial value $u_{0,\varepsilon}$.

To simplify the notation, we assume that all martingale solutions are defined on the original filtered probability space $(\Omega, \mathfrak{A}, \mathfrak{F}, \mathbb{P})$ with respect to the given family of Brownian motions $(\beta^{(k)})_{k\in\mathbb{Z}}$, attain the original initial value $u_{0,\varepsilon}$, and denote them by u_ε. Moreover, by equidistribution, we can assume that the auxiliary random variables $\widetilde{\xi}_\varepsilon$ are given by $\|u_0\|_{\mathcal{M}(\mathbb{T})}$. This simplification is possible, since the forthcoming estimates only depend on the distribution of the solutions.

4.2.1. APPLICATION OF ITÔ'S FORMULA

In this section, we establish estimates on the dissipation terms of the α-entropy

$$\int_{\mathbb{T}} G_\alpha(u_\varepsilon)\,\mathrm{d}x, \tag{4.2.6}$$

which are uniform in ε, where

$$G_\alpha(u) = \tfrac{1}{\alpha(\alpha+1)} u^{\alpha+1} \tag{4.2.7}$$

and

$$-1 < \alpha < 2-n. \tag{4.2.8}$$

These estimates are expressed conditionally on $\mathfrak{F}_0$, where we remark that the conditional expectation is also well-defined for non-negative random variables which do not lie in $L^1(\Omega)$, for details see [90, Chapter 5].

Proposition 4.2.2. *Assume* (4.2.8). *Then for every* $\varepsilon \in (0,1)$ *we have*

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \int_{\mathbb{T}} u_\varepsilon^{\alpha+n-1} (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}} u_\varepsilon^{\alpha+n-3} (\partial_x u_\varepsilon)^4 \,\mathrm{d}x\,\mathrm{d}t \Big| \mathfrak{F}_0\Big] \\ \lesssim_{\alpha,n,\Lambda,T} \|u_0\|_{\mathcal{M}(\mathbb{T})}^{\alpha+1} + \|u_0\|_{\mathcal{M}(\mathbb{T})}^{\alpha+n-1} + \varepsilon^{\alpha+1}.\end{aligned} \tag{4.2.9}$$

4

Proof. Throughout this proof we fix an $\varepsilon \in (0,1)$ and introduce the functions

$$\begin{aligned} f &= -\mathbf{1}_{\{u_\varepsilon>0\}} m(u_\varepsilon) \partial_x^3 u_\varepsilon, \\ g &= \tfrac{1}{2} \sum_{k\in\mathbb{Z}} \sigma_{k,\varepsilon} q'(u_\varepsilon) \partial_x (\sigma_{k,\varepsilon} q(u_\varepsilon)), \end{aligned}$$

and the $L_2(l^2(\mathbb{Z}), L^2(\mathbb{T}))$-valued process Ψ, defined by

$$\Psi e_k = \partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon)),$$

where e_k denotes the k-th unit vector in $l^2(\mathbb{Z})$. We convince ourselves that

$$f, g \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}))) \tag{4.2.10}$$

and

$$\Psi \in L^2(\Omega, L^2(0,T; L_2(l^2(\mathbb{Z}), L^2(\mathbb{T})))). \tag{4.2.11}$$

To this end, we observe that

$$\begin{aligned} \mathbb{E}\big[\|f\|^2_{L^2(0,T;L^2(\mathbb{T}))}\big] &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{T}} |\mathbf{1}_{\{u_\varepsilon>0\}} m(u_\varepsilon)\partial_x^3 u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t\Big] \\ &\le \mathbb{E}\Big[\int_0^T \|q(u_\varepsilon)\|^2_{L^\infty(\mathbb{T})} \|\mathbf{1}_{\{u_\varepsilon>0\}} q(u_\varepsilon)\partial_x^3 u_\varepsilon\|^2_{L^2(\mathbb{T})} \,\mathrm{d}t\Big] \\ &\le \mathbb{E}\Big[\sup_{0\le t\le T} \|u_\varepsilon(t)\|^{2n}_{H^1(\mathbb{T})}\Big]^{\frac{1}{2}} \mathbb{E}\big[\|\mathbf{1}_{\{u_\varepsilon>0\}} q(u_\varepsilon)\partial_x^3 u_\varepsilon\|^4_{L^2([0,T]\times\mathbb{T})}\big]^{\frac{1}{2}} < \infty \end{aligned}$$

by Consequence 4.2.1 (v). By Assumption 4.1.2 we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|\sigma_k\|^2_{C^1(\mathbb{T})} &= \sum_{k\in\mathbb{Z}} (\lambda_k(1+|k|))^2 < \infty, \\ \sum_{k\in\mathbb{Z}} \|\sigma_k\|_{C^2(\mathbb{T})} \|\sigma_k\|_{C(\mathbb{T})} &= \sum_{k\in\mathbb{Z}} \lambda_k^2 (1+|k|+|k|^2) < \infty, \end{aligned} \tag{4.2.12}$$

and hence

$$\begin{aligned}
\mathbb{E}\big[\|g\|^2_{L^2(0,T;L^2(\mathbb{T}))}\big] &\le \mathbb{E}\bigg[\int_0^T\int_{\mathbb{T}}\Big|\sum_{k\in\mathbb{Z}}\sigma_{k,\varepsilon}q'(u_\varepsilon)\partial_x(\sigma_{k,\varepsilon}q(u_\varepsilon))\Big|^2\,\mathrm{d}x\,\mathrm{d}t\bigg]\\
&\lesssim \mathbb{E}\bigg[\int_0^T\Big(\sum_{k\in\mathbb{Z}}\|\sigma_{k,\varepsilon}\|^2_{L^\infty(\mathbb{T})}\Big)^2\int_{\mathbb{T}}((q'(u_\varepsilon))^2\partial_x u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\bigg]\\
&\quad+\mathbb{E}\bigg[\int_0^T\Big(\sum_{k\in\mathbb{Z}}\|\sigma_{k,\varepsilon}\|_{L^\infty(\mathbb{T})}\|\partial_x\sigma_{k,\varepsilon}\|_{L^\infty(\mathbb{T})}\Big)^2\int_{\mathbb{T}}(q'(u_\varepsilon)q(u_\varepsilon))^2\,\mathrm{d}x\,\mathrm{d}t\bigg]\\
&\lesssim_{n,\Lambda}\mathbb{E}\bigg[\int_0^T\int_{\mathbb{T}}u_\varepsilon^{2n-4}(\partial_x u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{T}}u_\varepsilon^{2n-2}\,\mathrm{d}x\,\mathrm{d}t\bigg]\\
&\le \mathbb{E}\bigg[\int_0^T\|u_\varepsilon\|^{2n-4}_{L^\infty(\mathbb{T})}\|\partial_x u_\varepsilon\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t+\int_0^T\|u_\varepsilon\|^{2n-2}_{L^\infty(\mathbb{T})}\,\mathrm{d}t\bigg]\\
&\lesssim_T \mathbb{E}\Big[\sup_{0\le t\le T}\|u_\varepsilon(t)\|^{2n-2}_{H^1(\mathbb{T})}\Big]<\infty,
\end{aligned}$$

where we have used again Consequence 4.2.1 (v). By the same arguments, we also have

$$\begin{aligned}
\mathbb{E}\big[\|\Psi\|^2_{L^2(0,T;L_2(l^2(\mathbb{Z}),L^2(\mathbb{T})))}\big] &= \mathbb{E}\bigg[\int_0^T\sum_{k\in\mathbb{Z}}\|\partial_x(\sigma_{k,\varepsilon}q(u_\varepsilon))\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t\bigg]\\
&\lesssim \mathbb{E}\bigg[\int_0^T\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}(\partial_x\sigma_{k,\varepsilon})^2 m(u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\bigg]\\
&\quad+\mathbb{E}\bigg[\int_0^T\sum_{k\in\mathbb{Z}}\int_{\mathbb{T}}\sigma^2_{k,\varepsilon}(q'(u_\varepsilon)\partial_x u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\bigg]\\
&\lesssim_n \mathbb{E}\bigg[\int_0^T\sum_{k\in\mathbb{Z}}\|\partial_x\sigma_{k,\varepsilon}\|^2_{L^\infty(\mathbb{T})}\|u_\varepsilon\|^n_{L^\infty(\mathbb{T})}\,\mathrm{d}t\bigg]\\
&\quad+\mathbb{E}\bigg[\int_0^T\sum_{k\in\mathbb{Z}}\|\sigma_{k,\varepsilon}\|^2_{L^\infty(\mathbb{T})}\|u_\varepsilon\|^{n-2}_{L^\infty(\mathbb{T})}\|\partial_x u_\varepsilon\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t\bigg]\\
&\lesssim_{\Lambda,T}\mathbb{E}\Big[\sup_{0\le t\le T}\|u_\varepsilon(t)\|^n_{H^1(\mathbb{T})}\Big]<\infty,
\end{aligned} \tag{4.2.13}$$

so that (4.2.10) and (4.2.11) indeed hold. Denoting the cylindrical Wiener process $e_k\mapsto\beta^{(k)}$ in $l^2(\mathbb{Z})$ by β, we see that

$$u_\varepsilon(t)=u_{0,\varepsilon}+\int_0^t\partial_x(f+g)\,\mathrm{d}s+\int_0^t\Psi\,\mathrm{d}\beta_s$$

suffices all the assumptions of Itô's formula [39, Proposition A.1] on the involved processes by Consequence 4.2.1 (v) and the integrability statements (4.2.10) and (4.2.11). We would like to apply [39, Proposition A.1] to calculate the Itô expansion of the functional (4.2.6). Since, however, the function (4.2.7) is not twice continuously differentiable, we instead use the shifted version of (4.2.7) as introduced in [33, Proposition 2.2]. Specifically, we let $G_{\alpha,\delta}\in C^\infty(\mathbb{R})$ such that

$$G_{\alpha,\delta}(u)=\begin{cases}G_\alpha(u+\delta), & u\ge 0,\\ 0, & u\le -1,\end{cases} \tag{4.2.14}$$

for $\delta \in (0,1)$. In particular, $G_{\alpha,\delta}$ has bounded second derivative and hence evaluating Itô's formula [39, Proposition A.1] at time T yields

$$\begin{aligned}
&\int_{\mathbb{T}} G_{\alpha,\delta}(u_\varepsilon(T))\,\mathrm{d}x - \int_{\mathbb{T}} G_{\alpha,\delta}(u_{0,\varepsilon})\,\mathrm{d}x \\
&= \int_0^T \int_{\{u_\varepsilon>0\}} G''_{\alpha,\delta}(u_\varepsilon)\partial_x u_\varepsilon m(u_\varepsilon)\partial_x^3 u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^T \int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon)\partial_x u_\varepsilon \sigma_{k,\varepsilon} q'(u_\varepsilon)\partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \langle G'_{\alpha,\delta}(u_\varepsilon)\Psi\,\mathrm{d}\beta_t, \mathbf{1}_{\mathbb{T}}\rangle + \frac{1}{2}\sum_{k\in\mathbb{Z}} \int_0^T \int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon)(\partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon)))^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.2.15}$$

To simplify the second and fourth term appearing on the right-hand side, we observe that

$$\begin{aligned}
\mathscr{I}_{2,4}(k,t) &:= -\int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon)\partial_x u_\varepsilon \sigma_{k,\varepsilon} q'(u_\varepsilon)\partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon))\,\mathrm{d}x \\
&\quad + \int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon)(\partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon)))^2\,\mathrm{d}x \\
&= \int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon)\partial_x \sigma_{k,\varepsilon} q(u_\varepsilon)\partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon))\,\mathrm{d}x \\
&= \int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon) m(u_\varepsilon)(\partial_x \sigma_{k,\varepsilon})^2\,\mathrm{d}x \\
&\quad + \int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon) q(u_\varepsilon) q'(u_\varepsilon)\partial_x u_\varepsilon \sigma_{k,\varepsilon}\partial_x \sigma_{k,\varepsilon}\,\mathrm{d}x.
\end{aligned} \tag{4.2.16}$$

We define

$$\zeta_{\alpha,\delta}(u) = \int_0^u G''_{\alpha,\delta}(r) q(r) q'(r)\,\mathrm{d}r = \frac{n}{2}\int_0^u (r+\delta)^{\alpha-1} r^{n-1}\,\mathrm{d}r$$

for $u \geq 0$, so that

$$\int_{\mathbb{T}} G''_{\alpha,\delta}(u_\varepsilon) q(u_\varepsilon) q'(u_\varepsilon)\partial_x u_\varepsilon \sigma_{k,\varepsilon}\partial_x \sigma_{k,\varepsilon}\,\mathrm{d}x = -\int_{\mathbb{T}} \zeta_{\alpha,\delta}(u_\varepsilon)\partial_x(\sigma_{k,\varepsilon}\partial_x \sigma_{k,\varepsilon})\,\mathrm{d}x. \tag{4.2.17}$$

By the definition of $\zeta_{\alpha,\delta}$ we have

$$\zeta_{\alpha,\delta}(u) \leq \frac{n}{2}\int_0^u (r+\delta)^{\alpha+n-2}\,\mathrm{d}r,$$

and consequently

$$|\zeta_{\alpha,\delta}(u)| \lesssim_{\alpha,n} (u+\delta)^{\alpha+n-1}. \tag{4.2.18}$$

Moreover, it holds

$$|G''_{\alpha,\delta}(u) m(u)| \leq (u+\delta)^{\alpha+n-1} \tag{4.2.19}$$

for $u \geq 0$ by (4.2.14) and

$$\|u_\varepsilon(t)\|_{L^1(\mathbb{T})} = \|u_{0,\varepsilon}\|_{L^1(\mathbb{T})} \leq \|u_0\|_{\mathcal{M}(\mathbb{T})}\|\eta_\varepsilon\|_{L^1(\mathbb{T})} + \varepsilon = \|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon \tag{4.2.20}$$

by the divergence form of (4.2.5), non-negativity of u_ε and (4.2.2). Inserting (4.2.17)–(4.2.19) in (4.2.16) and using (4.2.8), (4.2.12) and (4.2.20), we conclude

$$\begin{aligned} &\left|\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^T \mathscr{I}_{2,4}(k,t)\,\mathrm{d}t\right| \\ &\lesssim_{\alpha,n} \sum_{k\in\mathbb{Z}}\int_0^T\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha+n-1}\big((\partial_x\sigma_{k,\varepsilon})^2 + |\partial_x(\sigma_{k,\varepsilon}\partial_x\sigma_{k,\varepsilon})|\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim_{\Lambda,T}\Big(\sup_{0\leq t\leq T}\|u_\varepsilon\|_{L^1(\mathbb{T})} + \delta\Big)^{\alpha+n-1} \leq \big(\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon + \delta\big)^{\alpha+n-1}. \end{aligned} \tag{4.2.21}$$

Next, we estimate the first term on the right-hand side of (4.2.15). The calculations are the same as in the proof of [33, Proposition 2.2] and only contained for convenience of the reader. Namely by integration by parts, which can be justified using a cut-off function near 0, we can rewrite

$$\begin{aligned} &\int_{\{u_\varepsilon>0\}} G''_{\alpha,\delta}(u_\varepsilon)\partial_x u_\varepsilon m(u_\varepsilon)\partial_x^3 u_\varepsilon\,\mathrm{d}x \\ &= \int_{\{u_\varepsilon>0\}}(u_\varepsilon+\delta)^{\alpha+n-1}\Big(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}\Big)\partial_x u_\varepsilon\partial_x^3 u_\varepsilon\,\mathrm{d}x \\ &= -\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha-1}m(u_\varepsilon)(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x \\ &\quad -(\alpha+n-1)\int_{\mathbb{T}}\Big(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}\Big)(u_\varepsilon+\delta)^{\alpha+n-2}(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon\,\mathrm{d}x \\ &\quad -\int_{\mathbb{T}}\Big(\tfrac{m'(u_\varepsilon)}{m(u_\varepsilon+\delta)} - \tfrac{m(u_\varepsilon)m'(u_\varepsilon+\delta)}{(m(u_\varepsilon+\delta))^2}\Big)(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon\,\mathrm{d}x \\ &= \mathscr{K}_1 + \mathscr{K}_2 + \mathscr{R}_1, \end{aligned} \tag{4.2.22}$$

$\mathbb{P}\otimes \mathrm{d}t$-almost everywhere. Integrating by parts and employing Young's inequality, we retrieve

$$\begin{aligned} \mathscr{K}_2 &= -(\alpha+n-1)\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha+n-2}(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon\,\mathrm{d}x \\ &\quad -(\alpha+n-1)\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\frac{\alpha+n-3}{2}}(\partial_x u_\varepsilon)^2\Big(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}-1\Big)(u_\varepsilon+\delta)^{\frac{\alpha+n-1}{2}}\partial_x^2 u_\varepsilon\,\mathrm{d}x \\ &\leq \Big(\tfrac{(\alpha+n-1)(\alpha+n-2)}{3} + \kappa_1\Big)\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha+n-3}(\partial_x u_\varepsilon)^4\,\mathrm{d}x \\ &\quad + \tfrac{(\alpha+n-1)^2}{\kappa_1}\int_{\mathbb{T}}\Big(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}-1\Big)^2(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2 u_\varepsilon)^2\,\mathrm{d}x, \end{aligned} \tag{4.2.23}$$

for every $\kappa_1 > 0$. Moreover, by inserting $m(u) = u^n$, we obtain the estimate

$$\left|\tfrac{m'(u)}{m(u+\delta)} - \tfrac{m(u)m'(u+\delta)}{(m(u+\delta))^2}\right| = n\big(\tfrac{u}{u+\delta}\big)^{n-1}\tfrac{\delta}{(u+\delta)^2} \leq \tfrac{n\delta}{(u+\delta)^2},$$

for $u \geq 0$. Hence for each $\kappa_2 > 0$, we have

$$\begin{aligned} |\mathscr{R}_1| &\leq \int_{\mathbb{T}} (u_\varepsilon+\delta)^{\frac{\alpha+n-3}{2}} (\partial_x u_\varepsilon)^2 \left(\tfrac{n\delta}{u_\varepsilon+\delta}\right)(u_\varepsilon+\delta)^{\frac{\alpha+n-1}{2}} |\partial_x^2 u_\varepsilon| \,\mathrm{d}x \\ &\leq \kappa_2 \int_{\mathbb{T}} (u_\varepsilon+\delta)^{\alpha+n-3}(\partial_x u_\varepsilon)^4 \,\mathrm{d}x + \tfrac{1}{\kappa_2}\int_{\mathbb{T}} \left(\tfrac{n\delta}{u_\varepsilon+\delta}\right)^2 (u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x, \end{aligned} \tag{4.2.24}$$

again by Young's inequality. Choosing

$$\kappa_1 = \kappa_2 = \tfrac{-(\alpha+n-1)(\alpha+n-2)}{12},$$

which is positive by (4.2.8), and inserting (4.2.23) and (4.2.24) in (4.2.22) yields

$$\begin{aligned} &\int_{\{u_\varepsilon>0\}} G''_{\alpha,\delta}(u_\varepsilon)\partial_x u_\varepsilon m(u_\varepsilon)\partial_x^3 u_\varepsilon \,\mathrm{d}x \\ &\leq -\int_{\mathbb{T}} (u_\varepsilon+\delta)^{\alpha-1} m(u_\varepsilon)(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x \\ &\quad + \tfrac{(\alpha+n-1)(\alpha+n-2)}{6}\int_{\mathbb{T}} (u_\varepsilon+\delta)^{\alpha+n-3}(\partial_x u_\varepsilon)^4 \,\mathrm{d}x \\ &\quad - \tfrac{12(\alpha+n-1)}{\alpha+n-2}\int_{\mathbb{T}} \left(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)} - 1\right)^2 (u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x \\ &\quad - \tfrac{12n^2}{(\alpha+n-1)(\alpha+n-2)}\int_{\mathbb{T}} \left(\tfrac{\delta}{u_\varepsilon+\delta}\right)^2 (u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x. \end{aligned}$$

By using the preceding estimate and (4.2.21) in (4.2.15), we obtain moreover

$$\begin{aligned} &\int_{\mathbb{T}} G_{\alpha,\delta}(u_\varepsilon(T))\,\mathrm{d}x - \int_{\mathbb{T}} G_{\alpha,\delta}(u_{0,\varepsilon})\,\mathrm{d}x + \int_0^T\int_{\mathbb{T}} (u_\varepsilon+\delta)^{\alpha-1} m(u_\varepsilon)(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &- \tfrac{(\alpha+n-1)(\alpha+n-2)}{6}\int_0^T\int_{\mathbb{T}} (u_\varepsilon+\delta)^{\alpha+n-3}(\partial_x u_\varepsilon)^4 \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \langle G'_{\alpha,\delta}(u_\varepsilon)\Psi\,\mathrm{d}\beta_t, \mathbf{1}_{\mathbb{T}}\rangle \\ &\quad \lesssim_{\alpha,n,\Lambda,T} \left(\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon + \delta\right)^{\alpha+n-1} \\ &\qquad + \left|\int_0^T\int_{\mathbb{T}} \left(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)} - 1\right)^2 (u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad + \left|\int_0^T\int_{\mathbb{T}} \left(\tfrac{\delta}{u_\varepsilon+\delta}\right)^2 (u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t\right|. \end{aligned} \tag{4.2.25}$$

We have, by (4.2.8) and (4.2.14),

$$-\int_{\mathbb{T}} G_{\alpha,\delta}(u_{0,\varepsilon})\,\mathrm{d}x \geq 0. \tag{4.2.26}$$

Moreover, it holds $|G_{\alpha,\delta}(u)| \lesssim_\alpha (u+\delta)^{\alpha+1}$ for $u \geq 0$, so that

$$\left|\int_{\mathbb{T}} G_{\alpha,\delta}(u_\varepsilon(T))\,\mathrm{d}x\right| \lesssim_\alpha \left(\|u_\varepsilon(T)\|_{L^1(\mathbb{T})} + \delta\right)^{\alpha+1} \leq \left(\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon + \delta\right)^{\alpha+1}, \tag{4.2.27}$$

by further invoking (4.2.20). Inserting (4.2.26) and (4.2.27) in (4.2.25) yields

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha-1}m(u_\varepsilon)(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\\&-\tfrac{(\alpha+n-1)(\alpha+n-2)}{6}\int_0^T\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha+n-3}(\partial_xu_\varepsilon)^4\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\langle G'_{\alpha,\delta}(u_\varepsilon)\Psi_l\,\mathrm{d}\beta_t,\mathbf{1}_{\mathbb{T}}\rangle\\&\quad\lesssim_{\alpha,n,\Lambda,T}\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon+\delta\big)^{\alpha+1}+\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon+\delta\big)^{\alpha+n-1}\\&\qquad+\left|\int_0^T\int_{\mathbb{T}}\Big(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}-1\Big)^2(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\right|\\&\qquad+\left|\int_0^T\int_{\mathbb{T}}\big(\tfrac{\delta}{u_\varepsilon+\delta}\big)^2(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\right|.\end{aligned}\tag{4.2.28}$$

We point out that the prefactor

$$\tfrac{(\alpha+n-1)(\alpha+n-2)}{6}$$

of the second term on the left-hand side of (4.2.28) is negative by (4.2.8). Due to (4.2.8) and (4.2.14) the function $G'_{\alpha,\delta}$ is bounded and thus

$$\begin{aligned}\mathbb{E}\big[\|\langle G'_{\alpha,\delta}(u_\varepsilon)\Psi\cdot,\mathbf{1}_{\mathbb{T}}\rangle\|^2_{L^2(0,T;L_2(l^2(\mathbb{Z}),\mathbb{R}))}\big]&=\mathbb{E}\left[\int_0^T\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{T}}G'_{\alpha,\delta}(u_\varepsilon)\partial_x(\sigma_{k,\varepsilon}q(u_\varepsilon))\,\mathrm{d}x\right)^2\mathrm{d}t\right]\\&\lesssim_{\alpha,\delta}\mathbb{E}\left[\int_0^T\sum_{k\in\mathbb{Z}}\|\partial_x(\sigma_{k,\varepsilon}q(u_\varepsilon))\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t\right],\end{aligned}$$

which is finite by (4.2.13). Consequently, the stochastic integral in (4.2.28) is a square integrable martingale. In order to estimate the conditional expectation on the left-hand side of (4.2.9), we let $A\in\mathfrak{F}_0$. Multiplying both sides in (4.2.28) with $\mathbf{1}_A$ and taking the expectation, we conclude that

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_A\int_0^T\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha-1}m(u_\varepsilon)(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\right]+\mathbb{E}\left[\mathbf{1}_A\int_0^T\int_{\mathbb{T}}(u_\varepsilon+\delta)^{\alpha+n-3}(\partial_xu_\varepsilon)^4\,\mathrm{d}x\,\mathrm{d}t\right]\\&\quad\lesssim_{\alpha,n,\Lambda,T}\mathbb{E}\big[\mathbf{1}_A\big(\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon+\delta\big)^{\alpha+1}+\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon+\delta\big)^{\alpha+n-1}\big)\big]\\&\qquad+\mathbb{E}\left[\left|\int_0^T\int_{\mathbb{T}}\Big(\tfrac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}-1\Big)^2(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\right|\right]\\&\qquad+\mathbb{E}\left[\left|\int_0^T\int_{\mathbb{T}}\big(\tfrac{\delta}{u_\varepsilon+\delta}\big)^2(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\right|\right]=\mathcal{E}_1+\mathcal{R}_2+\mathcal{R}_3.\end{aligned}\tag{4.2.29}$$

Due to Fatou's lemma and Consequence 4.2.1 (iv), we can deduce that

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_A\int_0^T\int_{\mathbb{T}}u_\varepsilon^{\alpha+n-1}(\partial_x^2u_\varepsilon)^2\,\mathrm{d}x\,\mathrm{d}t\right]+\mathbb{E}\left[\mathbf{1}_A\int_0^T\int_{\mathbb{T}}u_\varepsilon^{\alpha+n-3}(\partial_xu_\varepsilon)^4\,\mathrm{d}x\,\mathrm{d}t\right]\\&\quad\lesssim_{\alpha,n,\Lambda,T}\mathbb{E}\big[\mathbf{1}_A\big(\|u_0\|^{\alpha+1}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{\alpha+n-1}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{\alpha+1}\big)\big]\end{aligned}\tag{4.2.30}$$

by letting $\delta\searrow0$ in (4.2.29), if we can argue that $\mathcal{R}_2+\mathcal{R}_3\to0$ as $\delta\searrow0$. To this end, we observe first that

$$(u_\varepsilon+\delta)^{\alpha+n-1}(\partial_x^2u)^2\le(1+|u_\varepsilon|)(\partial_x^2u)^2,$$

$$\left(\frac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}-1\right)^2 \le 1,$$
$$\left(\frac{\delta}{u_\varepsilon+\delta}\right)^2 \le 1,$$

by (4.2.8). Moreover,

$$\left(\frac{m(u_\varepsilon)}{m(u_\varepsilon+\delta)}-1\right)^2 \to 0,$$
$$\left(\frac{\delta}{u_\varepsilon+\delta}\right)^2 \to 0,$$

$\mathbb{P}\otimes \mathrm{d}t\otimes \mathrm{d}x$-almost everywhere as $\delta\searrow 0$, by Conseuqence 4.2.1 (iv). Due to Consequence 4.2.1 (v), we have that

$$\begin{aligned}\mathbb{E}\left[\int_0^T\int_{\mathbb{T}}(1+|u_\varepsilon|)(\partial_x^2 u)^2\,\mathrm{d}x\,\mathrm{d}t\right] &\le \mathbb{E}\left[\int_0^T (1+\|u_\varepsilon\|_{H^1(\mathbb{T})})\|\partial_x^2 u\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t\right]\\ &\le \mathbb{E}\left[\left(1+\sup_{0\le t\le T}\|u_\varepsilon(t)\|_{H^1(\mathbb{T})}\right)\|u_\varepsilon\|^2_{L^2(0,T;H^2(\mathbb{T}))}\right]\\ &\le \left(1+\mathbb{E}\left[\sup_{0\le t\le T}\|u_\varepsilon(t)\|^2_{H^1(\mathbb{T})}\right]^{\frac12}\right)\mathbb{E}\left[\|u_\varepsilon\|^4_{L^2(0,T;H^2(\mathbb{T}))}\right]^{\frac12},\end{aligned}$$

and thus $\mathscr{R}_2+\mathscr{R}_3\to 0$ by the dominated convergence theorem. Therefore, (4.2.30) holds true, and (4.2.9) follows, since $A\in\mathfrak{F}_0$ was arbitrary. □

4.2.2. SPATIAL REGULARITY

In this section, we proceed as explained in Section 4.1.2 and use the Gagliardo–Nirenberg interpolation inequality in conjunction with the α-entropy estimate (4.2.9) and conservation of mass (4.2.20) to obtain estimates on u_ε in suitable Lebesgue and Sobolev norms.

Lemma 4.2.3. *Let $p\in(n+4,7)$, then*

$$\mathbb{E}\left[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\,\middle|\,\mathfrak{F}_0\right]\lesssim_{n,p,\Lambda,T}(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^4\left(\|u_0\|^{p-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{p-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{p-n-4}\right). \quad (4.2.31)$$

Proof. We choose α in accordance with

$$p=\alpha+n+5 \quad (4.2.32)$$

so that in particular (4.2.8) is satisfied and thus Proposition 4.2.2 applies. We define the random function $w_\varepsilon=u_\varepsilon^{\frac{\alpha+n+1}{4}}$ and claim that $w_\varepsilon\in W^{1,4}(\mathbb{T})$, $\mathbb{P}\otimes\mathrm{d}t$-almost everywhere, and that the chain rule holds for it. To verify this, we observe that $w_{\varepsilon,\kappa}=(u_\varepsilon+\kappa)^{\frac{\alpha+n+1}{4}}$ has the weak derivative $\frac{\alpha+n+1}{4}(u_\varepsilon+\kappa)^{\frac{\alpha+n-3}{4}}\partial_x u_\varepsilon$ by the chain rule [26, Corollary 8.11] for each $\kappa>0$. Hence,

$$\left(\frac{4}{\alpha+n+1}\right)^4\|\partial_x w_{\varepsilon,\kappa}\|^4_{L^4(\mathbb{T})}=\int_{\mathbb{T}}(u_\varepsilon+\kappa)^{\alpha+n-3}(\partial_x u_\varepsilon)^4\,\mathrm{d}x\le\int_{\mathbb{T}}u_\varepsilon^{\alpha+n-3}(\partial_x u_\varepsilon)^4\,\mathrm{d}x, \quad (4.2.33)$$

which is $\mathbb{P}\otimes\mathrm{d}t$-almost everywhere finite by (4.2.9) so that taking $\kappa\searrow 0$, $w_{\varepsilon,\kappa}$ admits a subsequence converging weakly in $W^{1,4}(\mathbb{T})$. This limit coincides with w_ε, since $w_{\varepsilon,\kappa}\to$

4

w_ε almost everywhere. Moreover, using the weak convergence $\partial_x w_{\varepsilon,\kappa} \rightharpoonup \partial_x w_\varepsilon$ in $L^4(\mathbb{T})$ and the dominated convergence theorem, we conclude that

$$\langle \partial_x w_\varepsilon, \varphi\rangle \leftarrow \langle \partial_x w_{\varepsilon,\kappa}, \varphi\rangle = \tfrac{\alpha+n+1}{4}\big\langle (u_\varepsilon+\kappa)^{\frac{\alpha+n-3}{4}}\partial_x u_\varepsilon, \varphi\big\rangle \to \tfrac{\alpha+n+1}{4}\big\langle u_\varepsilon^{\frac{\alpha+n-3}{4}}\partial_x u_\varepsilon, \varphi\big\rangle, \tag{4.2.34}$$

for a subsequence $\kappa \searrow 0$ and every $\varphi \in C^\infty(\mathbb{T})$, and therefore the chain rule applies to w_ε, too.

By the Gagliardo-Nirenberg interpolation inequality [26, Eq. (42), p.233] it holds

$$\|w_\varepsilon\|_{L^s(\mathbb{T})} \lesssim_{\alpha,n} \|w_\varepsilon\|^{\nu}_{W^{1,4}(\mathbb{T})}\|w_\varepsilon\|^{1-\nu}_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})} \tag{4.2.35}$$

for

$$s = \tfrac{4(\alpha+n+4)+4}{\alpha+n+1}, \quad \nu = \tfrac{\alpha+n+1-\frac{4}{s}}{\alpha+n+4}.$$

Moreover, by the Poincaré–Wirtinger inequality

$$\left\| w_\varepsilon - \fint_{\mathbb{T}} w_\varepsilon \,\mathrm{d}x \right\|_{L^4(\mathbb{T})} \lesssim \|\partial_x w_\varepsilon\|_{L^4(\mathbb{T})},$$

we conclude that

$$\|w_\varepsilon\|_{W^{1,4}(\mathbb{T})} \lesssim \|w_\varepsilon\|_{L^1(\mathbb{T})} + \|\partial_x w_\varepsilon\|_{L^4(\mathbb{T})}.$$

Inserting this in (4.2.35) and using that $L^{\frac{4}{\alpha+n+1}}(\mathbb{T}) \hookrightarrow L^1(\mathbb{T})$ due to (4.2.8) yields

$$\|w_\varepsilon\|_{L^s(\mathbb{T})} \lesssim_{\alpha,n} \|\partial_x w_\varepsilon\|^{\nu}_{L^4(\mathbb{T})}\|w_\varepsilon\|^{1-\nu}_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})} + \|w_\varepsilon\|_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})}. \tag{4.2.36}$$

Since

$$s\nu = \tfrac{s(\alpha+n+1)-4}{\alpha+n+4} = 4, \tag{4.2.37}$$

we obtain by integrating the s-th power of (4.2.36) in time

$$\begin{aligned} \|w_\varepsilon\|^s_{L^s([0,T]\times\mathbb{T})} &\lesssim_{\alpha,n} \int_0^T \|\partial_x w_\varepsilon\|^{s\nu}_{L^4(\mathbb{T})}\|w_\varepsilon\|^{s(1-\nu)}_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})}\,\mathrm{d}t + \int_0^T \|w_\varepsilon\|^s_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})}\,\mathrm{d}t \\ &\lesssim_T \|\partial_x w_\varepsilon\|^4_{L^4([0,T]\times\mathbb{T})}\|w_\varepsilon\|^{s(1-\nu)}_{L^\infty(0,T;L^{\frac{4}{\alpha+n+1}}(\mathbb{T}))} + \|w_\varepsilon\|^s_{L^\infty(0,T;L^{\frac{4}{\alpha+n+1}}(\mathbb{T}))}. \end{aligned} \tag{4.2.38}$$

By (4.2.32) we have $\frac{4p}{\alpha+n+1} = s$ and consequently

$$\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})} = \left\|u_\varepsilon^{\frac{\alpha+n+1}{4}}\right\|^{\frac{4p}{\alpha+n+1}}_{L^{\frac{4p}{\alpha+n+1}}([0,T]\times\mathbb{T})} = \|w_\varepsilon\|^s_{L^s([0,T]\times\mathbb{T})},$$

and moreover

$$\|u_\varepsilon\|_{L^1(\mathbb{T})} = \left\|u_\varepsilon^{\frac{\alpha+n+1}{4}}\right\|^{\frac{4}{\alpha+n+1}}_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})} = \|w_\varepsilon\|^{\frac{4}{\alpha+n+1}}_{L^{\frac{4}{\alpha+n+1}}(\mathbb{T})}.$$

Using these two identities in (4.2.38), taking the conditional expectation with respect to $\mathfrak{F}_0$ and applying estimate (4.2.20), we conclude that

$$\mathbb{E}\left[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\,\middle|\,\mathfrak{F}_0\right]$$

$$\begin{aligned}
&\lesssim_{\alpha,n,T} \mathbb{E}\Big[\|\partial_x w_\varepsilon\|^4_{L^4([0,T]\times\mathbb{T})}\|u_\varepsilon\|^{\frac{s(1-\nu)(\alpha+n+1)}{4}}_{L^\infty(0,T;L^1(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] + \mathbb{E}\Big[\|u_\varepsilon\|^{\frac{s(\alpha+n+1)}{4}}_{L^\infty(0,T;L^1(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] \\
&\le \mathbb{E}\Big[\|\partial_x w_\varepsilon\|^4_{L^4([0,T]\times\mathbb{T})}\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{\frac{s(1-\nu)(\alpha+n+1)}{4}}\Big|\mathfrak{F}_0\Big] + \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{\frac{s(\alpha+n+1)}{4}}.
\end{aligned}$$

Using that by (4.2.32) and (4.2.37)

$$\begin{aligned}
\frac{s(\alpha+n+1)}{4} &= p, \\
\frac{s(1-\nu)(\alpha+n+1)}{4} &= p - \frac{s\nu(\alpha+n+1)}{4} = 4,
\end{aligned}$$

and estimates (4.2.9), (4.2.33), we obtain

$$\begin{aligned}
&\mathbb{E}\big[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big] \\
&\quad\lesssim_{\alpha,n,T} \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^4\mathbb{E}\big[\|\partial_x w_\varepsilon\|^4_{L^4([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big] + \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^p \\
&\quad\lesssim_{\alpha,n,p,\Lambda,T} \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^4\big(\|u_0\|^{\alpha+1}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{\alpha+n-1}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{\alpha+1}+\|u_0\|^{p-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{p-4}\big).
\end{aligned}$$

We use Consequence 4.2.1 (i), (4.2.32) and that $\varepsilon \in (0,1)$ to simplify the right-hand side to

$$\mathbb{E}\big[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big] \lesssim_{\alpha,n,p,\Lambda,T} \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^4\big(\|u_0\|^{p-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{p-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{p-n-4}\big).$$

Finally, (4.2.31) follows by observing that α depends only on n and p. □

Lemma 4.2.4. *Let $r \in (\frac{n+4}{2}, \frac{7}{2})$, then*

$$\mathbb{E}\big[\|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big] \lesssim_{n,r,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4-r}\big(\|u_0\|^{2r-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{2r-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{2r-n-4}\big). \tag{4.2.39}$$

Proof. We define $p = 2r$ and α according to (4.2.32), so that in particular the assumptions of Proposition 4.2.2 and Lemma 4.2.3 are satisfied. We consider again the function $w_\varepsilon = u_\varepsilon^{\frac{\alpha+n+1}{4}}$, which satisfies the chain rule by (4.2.34). Hence, using Hölder's inequality and that

$$\frac{1}{r} = \frac{1}{4} + \frac{3-(\alpha+n)}{4p}, \tag{4.2.40}$$

we can estimate

$$\begin{aligned}
\|\partial_x u_\varepsilon\|_{L^r([0,T]\times\mathbb{T})} &\eqsim_{\alpha,n} \Big\|u_\varepsilon^{\frac{3-(\alpha+n)}{4}}\partial_x w_\varepsilon\Big\|_{L^r([0,T]\times\mathbb{T})} \\
&\le \|\partial_x w_\varepsilon\|_{L^4([0,T]\times\mathbb{T})}\Big\|u_\varepsilon^{\frac{3-(\alpha+n)}{4}}\Big\|_{L^{\frac{4p}{3-(\alpha+n)}}([0,T]\times\mathbb{T})} \\
&= \|\partial_x w_\varepsilon\|_{L^4([0,T]\times\mathbb{T})}\|u_\varepsilon\|^{\frac{3-(\alpha+n)}{4}}_{L^p([0,T]\times\mathbb{T})}.
\end{aligned}$$

Taking the r-th power on both sides, taking the conditional expectation with respect to $\mathfrak{F}_0$, and employing the conditional Hölder inequality yields

$$\begin{aligned}
\mathbb{E}\big[\|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big] &\lesssim_{\alpha,n} \mathbb{E}\Big[\|\partial_x w_\varepsilon\|^r_{L^4([0,T]\times\mathbb{T})}\|u_\varepsilon\|^{\frac{r(3-(\alpha+n))}{4}}_{L^p([0,T]\times\mathbb{T})}\Big|\mathfrak{F}_0\Big] \\
&\le \mathbb{E}\big[\|\partial_x w_\varepsilon\|^4_{L^4([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big]^{\frac{r}{4}}\mathbb{E}\big[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big]^{\frac{r(3-(\alpha+n))}{4p}}.
\end{aligned}$$

By (4.2.40) we have $\frac{r(3-(\alpha+n))}{4p} = 1-\frac{r}{4}$, so that inserting (4.2.9), (4.2.31) and the definitions of α, p results in

$$\mathbb{E}\big[\|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}|\mathfrak{F}_0\big] \lesssim_{\alpha,n,p,\Lambda,T} \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{4-r}\big(\|u_0\|^{2r-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{2r-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{2r-n-4}\big).$$

Finally, using that α, p only depend on n and r, we infer that (4.2.39) holds. □

4.2.3. Temporal regularity

In what follows, we use the estimates derived in Subsection 4.2.2 to deduce uniform estimates on the time increments of u_ε with values in a suitable negative Sobolev space on $\mathbb{T}$. Since the estimates from Subsection 4.2.2 only give estimates on $\partial_x u_\varepsilon$ and u_ε, we need to rewrite the thin-film operator in the very weak form [33, Eq. (3.2)] discussed in Subsection 1.2.3. Specifically, by integrating by parts we obtain that

$$\begin{aligned}\int_{\{u_\varepsilon>0\}} & m(u_\varepsilon)\partial_x^3 u_\varepsilon \eta \,\mathrm{d}x \\ &= \tfrac{n(n-1)}{2}\langle u_\varepsilon^{n-2}(\partial_x u_\varepsilon)^3,\eta\rangle + \tfrac{3n}{2}\langle u_\varepsilon^{n-1}(\partial_x u_\varepsilon)^2,\partial_x\eta\rangle + \langle u_\varepsilon^n\partial_x u_\varepsilon,\partial_x^2\eta\rangle\end{aligned} \tag{4.2.41}$$

$\mathbb{P}\otimes\mathrm{d}t$-almost everywhere for every $\eta\in C^\infty(\mathbb{T})$. The integration by parts is justified by the regularity of u_ε stated in Consequence 4.2.1 (v). In the subsequent lemma, we deduce estimates on the terms on the right-hand side of (4.2.41).

Lemma 4.2.5. *Let $l\in\{0,1,2\}$ and $\nu_l\in(\frac{n+4}{n+4-l},\frac{7}{n+4-l})$, then*

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg\|\int_0^\cdot & u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t\Bigg\|^{\nu_l}_{W^{1,\nu_l}(0,T;L^{\nu_l}(\mathbb{T}))}\Bigg|\mathfrak{F}_0\Bigg] \\ &\lesssim_{l,n,\nu_l,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4-\nu_l(3-l)} \\ &\quad\times\big(\|u_0\|^{\nu_l(n+4-l)-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{\nu_l(n+4-l)-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{\nu_l(n+4-l)-n-4}\big).\end{aligned} \tag{4.2.42}$$

Proof. We choose $p=2r=\nu_l(n+4-l)$, so that in particular $p\in(n+4,7)$ and $r\in(\frac{n+4}{2},\frac{7}{2})$, meaning that the assumptions of Lemma 4.2.3 and Lemma 4.2.4 are satisfied. Moreover, we have

$$\frac{n-2+l}{p}+\frac{3-l}{r}=\frac{n+4-l}{2r}=\frac{1}{\nu_l}. \tag{4.2.43}$$

Hence, using that $\int_0^\cdot u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t$ is starting at 0 and admits its integrand as weak derivative, as well as Hölder's inequality, we can estimate

$$\begin{aligned}\Bigg\|\int_0^\cdot u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t\Bigg\|_{W^{1,\nu_l}(0,T;L^{\nu_l}(\mathbb{T}))} &\lesssim_T \|u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\|_{L^{\nu_l}([0,T]\times\mathbb{T})} \\ \le \|u_\varepsilon^{n-2+l}\|_{L^{\frac{p}{n-2+l}}([0,T]\times\mathbb{T})}\|(\partial_x u_\varepsilon)^{3-l}\|_{L^{\frac{r}{3-l}}([0,T]\times\mathbb{T})} &= \|u_\varepsilon\|^{n-2+l}_{L^p([0,T]\times\mathbb{T})}\|\partial_x u_\varepsilon\|^{3-l}_{L^r([0,T]\times\mathbb{T})}.\end{aligned}$$

Taking the ν_l-th power on both sides and the conditional expectation with respect to $\mathfrak{F}_0$, using (4.2.31) and (4.2.39), and employing the conditional Hölder's inequality we conclude

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\int_0^\cdot u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t\Big\|^{\nu_l}_{W^{1,\nu_l}(0,T;L^{\nu_l}(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] \\
&\lesssim_{\nu_l,T} \mathbb{E}\big[\|u_\varepsilon\|^{\nu_l(n-2+l)}_{L^p([0,T]\times\mathbb{T})}\|\partial_x u_\varepsilon\|^{\nu_l(3-l)}_{L^r([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big] \\
&\le \mathbb{E}\big[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big]^{\frac{\nu_l(n-2+l)}{p}}\mathbb{E}\big[\|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big]^{\frac{\nu_l(3-l)}{r}} \\
&\lesssim_{n,p,r,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{\frac{4\nu_l(n-2+l)}{p}}\big(\|u_0\|^{p-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{p-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{p-n-4}\big)^{\frac{\nu_l(n-2+l)}{p}} \\
&\quad\times(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{\frac{(4-r)\nu_l(3-l)}{r}}\big(\|u_0\|^{2r-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{2r-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{2r-n-4}\big)^{\frac{\nu_l(3-l)}{r}}.
\end{aligned} \tag{4.2.44}$$

The claim follows by using (4.2.43) and inserting the definitions of p and r. □

We derive similar estimates on terms appearing in the Stratonovich correction term in (4.2.5).

Lemma 4.2.6. *Let $l\in\{3,4\}$ and $\nu_l\in(\frac{n+4}{n+3-l},\frac{7}{n+3-l})$, then*

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon\,\mathrm{d}t\Big\|^{\nu_3}_{W^{1,\nu_3}(0,T;L^{\nu_3}(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] \\
&\quad\lesssim_{n,\nu_3,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4-\nu_3}\big(\|u_0\|^{n\nu_3-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{n\nu_3-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{n\nu_3-n-4}\big)
\end{aligned} \tag{4.2.45}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon}\partial_x\sigma_{k,\varepsilon}q(u_\varepsilon)q'(u_\varepsilon)\,\mathrm{d}t\Big\|^{\nu_4}_{W^{1,\nu_4}(0,T;L^{\nu_4}(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] \\
&\lesssim_{n,\nu_4,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^4\big(\|u_0\|^{(n-1)\nu_4-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{(n-1)\nu_4-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{(n-1)\nu_4-n-4}\big).
\end{aligned} \tag{4.2.46}$$

Proof. We first consider (4.2.45) and define p,r by $p=2r=n\nu_3$, so that the assumptions of Lemma 4.2.3 and Lemma 4.2.4 are satisfied. Then

$$\frac{n-2}{p}+\frac{1}{r}=\frac{1}{\nu_3}. \tag{4.2.47}$$

We use that $\sum_{k\in\mathbb{Z}}\int_0^\cdot\sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon\,\mathrm{d}t$ starts at 0 and (4.2.12) to estimate

$$\begin{aligned}
&\Big\|\sum_{k\in\mathbb{Z}}\int_0^\cdot\sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon\,\mathrm{d}t\Big\|_{W^{1,\nu_3}(0,T;L^{\nu_3}(\mathbb{T}))} \lesssim_T \Big\|\sum_{k\in\mathbb{Z}}\sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon\Big\|_{L^{\nu_3}([0,T]\times\mathbb{T})} \\
&\lesssim_n \sum_{k\in\mathbb{Z}}\|\sigma_{k,\varepsilon}\|^2_{C(\mathbb{T})}\|u_\varepsilon^{n-2}\partial_x u_\varepsilon\|_{L^{\nu_3}([0,T]\times\mathbb{T})} \lesssim_\Lambda \|u_\varepsilon^{n-2}\partial_x u_\varepsilon\|_{L^{\nu_3}([0,T]\times\mathbb{T})}.
\end{aligned}$$

Proceeding as in (4.2.44), we obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\Big\|\sum_{k\in\mathbb{Z}}\int_0^\cdot\sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon\,\mathrm{d}t\Big\|^{\nu_3}_{W^{1,\nu_3}(0,T;L^{\nu_3}(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] \\
&\lesssim_{n,\Lambda,T} \mathbb{E}\big[\|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big]^{\frac{\nu_3(n-2)}{p}}\mathbb{E}\big[\|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\big|\mathfrak{F}_0\big]^{\frac{\nu_3}{r}}
\end{aligned}$$

$$\lesssim_{n,p,r,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{\frac{4\nu_3(n-2)}{p}} \left(\|u_0\|_{\mathcal{M}(\mathbb{T})}^{p-n-4}+\|u_0\|_{\mathcal{M}(\mathbb{T})}^{p-4}+\varepsilon^{p-n-4}\right)^{\frac{\nu_3(n-2)}{p}}$$
$$\times (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{\frac{(4-r)\nu_3}{r}} \left(\|u_0\|_{\mathcal{M}(\mathbb{T})}^{2r-n-4}+\|u_0\|_{\mathcal{M}(\mathbb{T})}^{2r-4}+\varepsilon^{2r-n-4}\right)^{\frac{\nu_3}{r}}.$$

The claimed estimate (4.2.45) follows by using (4.2.47) and inserting the definitions of p and r. The second estimate (4.2.46) can be derived analogously with the choice $p=2r=(n-1)\nu_3$. □

Lastly, we obtain temporal regularity of the stochastic integral in (4.2.5).

Lemma 4.2.7. *Let* $\nu_5 \in (\frac{2(n+4)}{n}, \frac{14}{n})$, $\gamma_5 \in (0,\frac{1}{2})$. *Then*

$$\mathbb{E}\left[\left\|\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon} q(u_\varepsilon)\,\mathrm{d}\beta_t^{(k)}\right\|_{W^{\gamma_5,\nu_5}(0,T;L^2(\mathbb{T}))}^{\nu_5}\middle|\mathfrak{F}_0\right]$$
$$\lesssim_{\gamma_5,n,\nu_5,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^4\left(\|u_0\|_{\mathcal{M}(\mathbb{T})}^{\frac{n\nu_5}{2}-n-4}+\|u_0\|_{\mathcal{M}(\mathbb{T})}^{\frac{n\nu_5}{2}-4}+\varepsilon^{\frac{n\nu_5}{2}-n-4}\right). \tag{4.2.48}$$

Proof. We define the linear operator $\Phi_\varepsilon\colon l^2(\mathbb{Z})\to L^2(\mathbb{T})$ by setting $\Phi_\varepsilon e_k=\sigma_{k,\varepsilon}q(u_\varepsilon)$ so that we can write in what follows

$$\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon}q(u_\varepsilon)\,\mathrm{d}\beta_t^{(k)} = \int_0^\cdot \Phi_\varepsilon\,\mathrm{d}\beta_t,$$

where β is the cylindrical Wiener process in $l^2(\mathbb{Z})$ given by $e_k\mapsto\beta^{(k)}$. We let $A\in\mathfrak{F}_0$. Then, using [53, Lemma 2.1], we calculate

$$\mathbb{E}\left[\mathbf{1}_A\left\|\int_0^\cdot \Phi_\varepsilon\,\mathrm{d}\beta_t\right\|_{W^{\gamma_5,\nu_5}(0,T;L^2(\mathbb{T}))}^{\nu_5}\right] = \mathbb{E}\left[\left\|\int_0^\cdot \mathbf{1}_A\Phi_\varepsilon\,\mathrm{d}\beta_t\right\|_{W^{\gamma_5,\nu_5}(0,T;L^2(\mathbb{T}))}^{\nu_5}\right]$$
$$\lesssim_{\gamma_5,\nu_5} \mathbb{E}\left[\int_0^T \|\mathbf{1}_A\Phi_\varepsilon\|_{L_2(l^2(\mathbb{Z}),L^2(\mathbb{T}))}^{\nu_5}\,\mathrm{d}t\right] = \mathbb{E}\left[\mathbf{1}_A\int_0^T \|\Phi_\varepsilon\|_{L_2(l^2(\mathbb{Z}),L^2(\mathbb{T}))}^{\nu_5}\,\mathrm{d}t\right].$$

To further estimate the latter, we use (4.2.12) and that $\nu_5\geq 2$ to infer

$$\int_0^T \|\Phi_\varepsilon\|_{L_2(l^2(\mathbb{Z}),L^2(\mathbb{T}))}^{\nu_5}\,\mathrm{d}t = \int_0^T\left(\sum_{k\in\mathbb{Z}}\|\sigma_{k,\varepsilon}q(u_\varepsilon)\|_{L^2(\mathbb{T})}^2\right)^{\frac{\nu_5}{2}}\mathrm{d}t$$
$$\leq \int_0^T\left(\sum_{k\in\mathbb{Z}}\|\sigma_{k,\varepsilon}\|_{C(\mathbb{T})}^2\|u_\varepsilon^{\frac{n}{2}}\|_{L^2(\mathbb{T})}^2\right)^{\frac{\nu_5}{2}}\mathrm{d}t \lesssim_\Lambda \int_0^T \|u_\varepsilon\|_{L^n(\mathbb{T})}^{\frac{\nu_5 n}{2}}\,\mathrm{d}t \leq \|u_\varepsilon\|_{L^{\frac{\nu_5 n}{2}}([0,T]\times\mathbb{T})}^{\frac{\nu_5 n}{2}}.$$

Finally, we set $p=\frac{\nu_5 n}{2}$ in accordance with the assumption of Lemma 4.2.3 and consequently we can use (4.2.31) to conclude that

$$\mathbb{E}\left[\mathbf{1}_A\left\|\int_0^\cdot \Phi_\varepsilon\,\mathrm{d}\beta_t\right\|_{W^{\gamma_5,\nu_5}(0,T;L^2(\mathbb{T}))}^{\nu_5}\right] \lesssim_{\gamma_5,\nu_5,\Lambda} \mathbb{E}\left[\mathbf{1}_A\|u_\varepsilon\|_{L^p([0,T]\times\mathbb{T})}^p\right]$$
$$= \mathbb{E}\left[\mathbf{1}_A\mathbb{E}\left[\|u_\varepsilon\|_{L^p([0,T]\times\mathbb{T})}^p\middle|\mathfrak{F}_0\right]\right]$$
$$\lesssim_{n,p,\Lambda,T} \mathbb{E}\left[\mathbf{1}_A(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^4\left(\|u_0\|_{\mathcal{M}(\mathbb{T})}^{p-n-4}+\|u_0\|_{\mathcal{M}(\mathbb{T})}^{p-4}+\varepsilon^{p-n-4}\right)\right].$$

It remains to use that $A\in\mathfrak{F}_0$ was arbitrary and to insert the definition of p. □

Finally, we combine the previous results from this subsection to deduce a uniform estimate on the temporal increments of u_ε in terms of its Sobolev–Slobodeckij norm.

Lemma 4.2.8. *Let $\gamma \in (0, \frac{1}{2})$, $\mu \in (\frac{n+4}{n+2}, \frac{7}{n+2})$ and $\nu \in (1, \frac{7}{n+4})$, then*

$$\mathbb{E}\Big[\|u_\varepsilon\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big|\mathfrak{F}_0\Big] \lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{(n-1-\frac{n^2}{p})\nu} + \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{(n+1)\nu}, \tag{4.2.49}$$

where $p = \max\{\mu(n+2), \nu(n+4)\}$.

Proof. By Consequence 4.2.1 (vi) and (4.2.41), the equality

$$\begin{aligned} u_\varepsilon = u_{0,\varepsilon} &- \tfrac{n(n-1)}{2}\int_0^\cdot \partial_x(u_\varepsilon^{n-2}(\partial_x u_\varepsilon)^3)\,\mathrm{d}t + \tfrac{3n}{2}\int_0^\cdot \partial_x^2(u_\varepsilon^{n-1}(\partial_x u_\varepsilon)^2)\,\mathrm{d}t \\ &- \int_0^\cdot \partial_x^3(u_\varepsilon^n\partial_x u_\varepsilon)\,\mathrm{d}t + \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^\cdot \partial_x(\sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon)\,\mathrm{d}t \\ &+ \tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^\cdot \partial_x(\sigma_{k,\varepsilon}\partial_x\sigma_{k,\varepsilon}q(u_\varepsilon)q'(u_\varepsilon))\,\mathrm{d}t + \sum_{k\in\mathbb{Z}}\int_0^\cdot \partial_x(\sigma_{k,\varepsilon}q(u_\varepsilon))\,\mathrm{d}\beta_t^{(k)} \end{aligned} \tag{4.2.50}$$

holds almost surely, where the integrals on the right-hand side converge in suitable negative Sobolev spaces by Lemmas 4.2.5–4.2.7. We proceed by separately estimating the $W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))$-norm of each of the terms on the right-hand side of (4.2.50). Since $u_{0,\varepsilon}$ is constant in time, we can estimate by the Sobolev embedding theorem and (4.2.20)

$$\|u_{0,\varepsilon}\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \lesssim_\mu \|u_{0,\varepsilon}\|_{L^{\frac{2\nu}{2-\nu}}(0,T;L^1(\mathbb{T}))} \lesssim_{\nu,T} \|u_{0,\varepsilon}\|_{L^1(\mathbb{T})} \le \|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon. \tag{4.2.51}$$

For the remaining terms, we choose

$$\begin{aligned} \nu_l &= \tfrac{p}{n+4-l}, \quad l\in\{0,1,2\}, \\ \nu_l &= \tfrac{p}{n+3-l}, \quad l\in\{3,4\}, \\ \nu_5 &= \tfrac{2p}{n}, \quad \gamma_5=\gamma, \end{aligned} \tag{4.2.52}$$

where p is defined in the claim. In particular, we have $\nu_l \ge \nu$ and therefore

$$1-\tfrac{1}{\nu_l} \ge 1-\tfrac{1}{\nu} = \tfrac{1}{2}-\tfrac{2-\nu}{2\nu} > \gamma - \tfrac{2-\nu}{2\nu}, \quad l\in\{0,\dots,4\}. \tag{4.2.53}$$

Using additionally that $\nu_2 \ge \mu$ and employing the Sobolev embedding theorem in time and space, we obtain that

$$\begin{aligned} &\Big\|\int_0^\cdot \partial_x^{l+1}(u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l})\,\mathrm{d}t\Big\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \\ &\le \Big\|\int_0^\cdot u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t\Big\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{l-2,\mu}(\mathbb{T}))} \\ &\lesssim_{l,\gamma,\mu,\nu,\nu_l,T} \Big\|\int_0^\cdot u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t\Big\|_{W^{1,\nu_l}(0,T;L^{\nu_l}(\mathbb{T}))} \end{aligned} \tag{4.2.54}$$

for $l \in \{0,1,2\}$. For $l \in \{3,4\}$, we proceed similarly and use again (4.2.53) and the Sobolev embedding to conclude

$$\begin{aligned}&\Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \partial_x(\sigma_{k,\varepsilon}^2 (q'(u_\varepsilon))^2 \partial_x u_\varepsilon)\,\mathrm{d}t \Big\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \\ &\quad \lesssim_{\gamma,\mu,\nu,\nu_3,T} \Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \sigma_{k,\varepsilon}^2 (q'(u_\varepsilon))^2 \partial_x u_\varepsilon \,\mathrm{d}t \Big\|_{W^{1,\nu_3}(0,T;L^{\nu_3}(\mathbb{T}))}\end{aligned} \tag{4.2.55}$$

and

$$\begin{aligned}&\Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \partial_x(\sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} q(u_\varepsilon) q'(u_\varepsilon))\,\mathrm{d}t \Big\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \\ &\quad \lesssim_{\gamma,\mu,\nu,\nu_4,T} \Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} q(u_\varepsilon) q'(u_\varepsilon)\,\mathrm{d}t \Big\|_{W^{1,\nu_4}(0,T;L^{\nu_4}(\mathbb{T}))}.\end{aligned} \tag{4.2.56}$$

Lastly, we observe that by Consequence 4.2.1 (i) and $\nu < \frac{7}{n+4}$, it also holds $\nu_5 \geq \frac{2(n+4)\nu}{n} \geq \frac{2\nu}{2-\nu}$. Hence, by the Sobolev embedding theorem, we infer

$$\begin{aligned}&\Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon))\,\mathrm{d}\beta_t^{(k)} \Big\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \\ &\quad \lesssim_{\gamma,\mu,\nu,\nu_5} \Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \sigma_{k,\varepsilon} q(u_\varepsilon)\,\mathrm{d}\beta_t^{(k)} \Big\|_{W^{\gamma_5,\nu_5}(0,T;L^2(\mathbb{T}))}.\end{aligned} \tag{4.2.57}$$

Employing the triangle inequality in $W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))$ as well as the conditional Minkowski inequality in (4.2.50) yields

$$\begin{aligned}\mathbb{E}\Big[\|u_\varepsilon\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} &\lesssim_n \mathbb{E}\Big[\|u_{0,\varepsilon}\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} \\ &+ \sum_{l=0}^{2} \mathbb{E}\Big[\Big\| \int_0^{\cdot} \partial_x^{l+1}(u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l})\,\mathrm{d}t \Big\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} \\ &+ \mathbb{E}\Big[\Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \partial_x(\sigma_{k,\varepsilon}^2 (q'(u_\varepsilon))^2 \partial_x u_\varepsilon)\,\mathrm{d}t \Big\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} \\ &+ \mathbb{E}\Big[\Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \partial_x(\sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} q(u_\varepsilon) q'(u_\varepsilon))\,\mathrm{d}t \Big\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} \\ &+ \mathbb{E}\Big[\Big\| \sum_{k\in\mathbb{Z}} \int_0^{\cdot} \partial_x(\sigma_{k,\varepsilon} q(u_\varepsilon))\,\mathrm{d}\beta_t^{(k)} \Big\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}}.\end{aligned}$$

The estimates (4.2.51), (4.2.54)–(4.2.57) and the conditional Jensen inequality lead to

$$\begin{aligned}\mathbb{E}\Big[\|u_\varepsilon\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} &\lesssim_{\gamma,n,\mu,\nu,T} \mathbb{E}\Big[\big(\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon\big)^{\nu} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu}} \\ &+ \sum_{l=0}^{2} \mathbb{E}\Big[\Big\| \int_0^{\cdot} u_\varepsilon^{n-2+l}(\partial_x u_\varepsilon)^{3-l}\,\mathrm{d}t \Big\|^{\nu_l}_{W^{1,\nu_l}(0,T;L^{\nu_l}(\mathbb{T}))} \Big| \mathfrak{F}_0\Big]^{\frac{1}{\nu_l}}\end{aligned}$$

$$+\mathbb{E}\left[\left\|\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon}^2(q'(u_\varepsilon))^2\partial_x u_\varepsilon\,\mathrm{d}t\right\|^{\nu_3}_{W^{1,\nu_3}(0,T;L^{\nu_3}(\mathbb{T}))}\middle|\mathfrak{F}_0\right]^{\frac{1}{\nu_3}}$$
$$+\mathbb{E}\left[\left\|\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon}\partial_x\sigma_{k,\varepsilon}q(u_\varepsilon)q'(u_\varepsilon)\,\mathrm{d}t\right\|^{\nu_4}_{W^{1,\nu_4}(0,T;L^{\nu_4}(\mathbb{T}))}\middle|\mathfrak{F}_0\right]^{\frac{1}{\nu_4}}$$
$$+\mathbb{E}\left[\left\|\sum_{k\in\mathbb{Z}}\int_0^\cdot \sigma_{k,\varepsilon}q(u_\varepsilon)\,\mathrm{d}\beta_t^{(k)}\right\|^{\nu_5}_{W^{\gamma_5,\nu_5}(0,T;L^2(\mathbb{T}))}\middle|\mathfrak{F}_0\right]^{\frac{1}{\nu_5}},$$

where we also used that the ν_l only depend on n, ν, μ. Since $p\in(n+4,7)$, the parameters from (4.2.52) satisfy the assumptions of Lemmas 4.2.5–4.2.7. Hence, using (4.2.42), (4.2.45), (4.2.46) and (4.2.48), we obtain that

$$\begin{aligned}
\mathbb{E}\left[\|u_\varepsilon\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\middle|\mathfrak{F}_0\right]^{\frac{1}{\nu}} &\lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \left(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\right)\\
&+\sum_{l=0}^{2}\Big[(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4-\nu_l(3-l)}\\
&\quad\times\left(\|u_0\|^{\nu_l(n+4-l)-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{\nu_l(n+4-l)-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{\nu_l(n+4-l)-n-4}\right)\Big]^{\frac{1}{\nu_l}}\\
&+\left[(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4-\nu_3}\left(\|u_0\|^{n\nu_3-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{n\nu_3-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{n\nu_3-n-4}\right)\right]^{\frac{1}{\nu_3}}\\
&+\left[(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4}\left(\|u_0\|^{(n-1)\nu_4-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{(n-1)\nu_4-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{(n-1)\nu_4-n-4}\right)\right]^{\frac{1}{\nu_4}}\\
&+\left[(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4}\left(\|u_0\|^{\frac{n\nu_5}{2}-n-4}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{\frac{n\nu_5}{2}-4}_{\mathcal{M}(\mathbb{T})}+\varepsilon^{\frac{n\nu_5}{2}-n-4}\right)\right]^{\frac{1}{\nu_5}}.
\end{aligned}\tag{4.2.58}$$

To simplify the right-hand side, we estimate ε and $\|u_0\|_{\mathcal{M}(\mathbb{T})}$ in (4.2.58) by $(\varepsilon+\|u_0\|_{\mathcal{M}(\mathbb{T})})$ to conclude

$$\begin{aligned}
\mathbb{E}\left[\|u_\varepsilon\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\middle|\mathfrak{F}_0\right]^{\frac{1}{\nu}} &\lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \left(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\right)\\
&+\sum_{l=0}^{2}(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{n+1-\frac{n}{\nu_l}}+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{n+1}\\
&+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{n-1-\frac{n}{\nu_3}}+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{n-1}\\
&+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{n-1-\frac{n}{\nu_4}}+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{n-1}\\
&+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{\frac{n}{2}-\frac{n}{\nu_5}}+(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{\frac{n}{2}}.
\end{aligned}\tag{4.2.59}$$

We notice that the largest power on the right-hand side of (4.2.59) is $n+1$. By Consequence 4.2.1 (i), the smallest power is either 1, $n-1-\frac{n}{\nu_3}$ or $\frac{n}{2}-\frac{n}{\nu_5}$. To find the smallest one, we insert (4.2.52) to rewrite the powers to

$$1,\qquad n-1-\tfrac{n}{\nu_3}=n-1-\tfrac{n^2}{p},\qquad \tfrac{n}{2}-\tfrac{n}{\nu_5}=\tfrac{n}{2}-\tfrac{n^2}{2p}$$

and consider the respective parabolas

$$g_1(s)=1,\qquad g_2(s)=s-1-\tfrac{s^2}{p},\qquad g_3(s)=\tfrac{s}{2}-\tfrac{s^2}{2p}.$$

We notice that all three parabolas attain their maximum value at $\frac{p}{2}$, with

$$g_1\left(\tfrac{p}{2}\right) = 1 \geq g_3\left(\tfrac{p}{2}\right) = \tfrac{p}{8} \geq g_2\left(\tfrac{p}{2}\right) = \tfrac{p}{4} - 1,$$

since $p \leq 8$. Because the second derivatives of the parabolas obey the same ordering, we conclude $g_1(s) \geq g_3(s) \geq g_2(s)$ for all $s \in \mathbb{R}$ and in particular that $n - 1 - \frac{n^2}{p}$ is the smallest power in (4.2.59). Whence,

$$\mathbb{E}\left[\|u_\varepsilon\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \,\middle|\, \mathfrak{F}_0\right]^{\frac{1}{\nu}} \lesssim_{\gamma,n,\mu,\nu,\Lambda,T} (\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon)^{n-1-\frac{n^2}{p}} + (\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon)^{n+1}$$

and raising both sides of the preceding inequality to the ν-th power yields (4.2.49). □

4

4.2.4. SIMPLIFIED ESTIMATES

In the previous subsections, we derived uniform estimates on the conditional expectations of the approximate solutions $(u_\varepsilon)_{\varepsilon \in (0,1)}$. To work with these estimates efficiently in the preceding section, we derive corresponding moment estimates with a simpler right-hand side. To this end, we introduce the sets

$$A_j = \left\{\|u_0\|_{\mathcal{M}(\mathbb{T})} \in [j-1, j)\right\}, \quad j \in \mathbb{N}, \tag{4.2.60}$$

providing an $\mathfrak{F}_0$-measurable partition of the probability space Ω and point out that it suffices to show tightness on each of the sets A_j separately in light of Lemma 4.B.3. Using that

$$\|u_0\|_{\mathcal{M}(\mathbb{T})} + \varepsilon \leq j + 1 \tag{4.2.61}$$

on A_j, we obtain by multiplying (4.2.9) with $\mathbf{1}_{A_j}$ and taking the expectation

$$\begin{aligned} &\forall \alpha \in (-1, 2-n): \\ &\mathbb{E}\left[\mathbf{1}_{A_j} \int_0^T \int_{\mathbb{T}} u_\varepsilon^{\alpha+n-1} (\partial_x^2 u_\varepsilon)^2 \,\mathrm{d}x\,\mathrm{d}t + \mathbf{1}_{A_j} \int_0^T \int_{\mathbb{T}} u_\varepsilon^{\alpha+n-3} (\partial_x u_\varepsilon)^4 \,\mathrm{d}x\,\mathrm{d}t\right] \\ &\quad \lesssim_{\alpha,n,\Lambda,T} (j+1)^{\alpha+n-1}. \end{aligned} \tag{4.2.62}$$

In the same way, we conclude from Lemma 4.2.3 that

$$\forall p \in [1,7): \quad \mathbb{E}\left[\mathbf{1}_{A_j} \|u_\varepsilon\|^p_{L^p([0,T]\times\mathbb{T})}\right] \lesssim_{n,p,\Lambda,T} (j+1)^p, \tag{4.2.63}$$

where the fact that the inequality also holds for $p \in [1, n+4]$ follows by Hölder's inequality. Analogously, we conclude from Lemma 4.2.4 that

$$\forall r \in [1, \tfrac{7}{2}): \quad \mathbb{E}\left[\mathbf{1}_{A_j} \|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\right] \lesssim_{n,r,\Lambda,T} (j+1)^r. \tag{4.2.64}$$

Using additionally the Sobolev embedding theorem in space, we conclude from Lemma 4.2.8

$$\begin{aligned} &\forall \gamma \in (0, \tfrac{1}{2})\, \forall \mu \in (1, \tfrac{7}{n+2})\, \forall \nu \in [1, \tfrac{7}{n+4}): \\ &\mathbb{E}\left[\mathbf{1}_{A_j} \|u_\varepsilon\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\right] \lesssim_{\gamma,n,\mu,\nu,\Lambda,T} (j+1)^{\nu(n+1)}. \end{aligned} \tag{4.2.65}$$

4.3. LIMITING PROCEDURE

In this section, we construct a very weak martingale solution in the sense of Definition 4.1.4 to the stochastic thin-film equation with initial value u_0. To this end, we show tightness of the approximating family $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in suitable spaces in Subsection 4.3.1 and extract an equidistributed convergent subsequence converging to a solution in the following Subsection 4.3.2.

4.3.1. TIGHTNESS PROPERTIES

We define $\mathfrak{X}_{\mathrm{ind}} = \mathbb{R}$, $\mathfrak{X}_{\mathrm{BM}} = C([0,T])$, $\mathfrak{X}_{\mathrm{power}} = L^2(0,T;H^2(\mathbb{T}))$, where we equip the latter space with its weak topology. Moreover, we choose sequences $-1 \le \kappa_l \nearrow \frac{-1}{2}$, $1 < p_l \nearrow 7$ and $1 < r_l \nearrow \frac{7}{2}$. Then, we define $\mathfrak{X}_{\mathrm{cont}}$, $\mathfrak{X}_{\mathrm{Lebesgue}}$ and $\mathfrak{X}_{\mathrm{Sobolev}}$ as the projective limit of the sequences $(C([0,T];H^{\kappa_l}(\mathbb{T})))_{l\in\mathbb{N}}$, $(L^{p_l}([0,T]\times\mathbb{T}))_{l\in\mathbb{N}}$ and $(L^{r_l}(0,T;W^{1,r_l}(\mathbb{T})))_{l\in\mathbb{N}}$, where we consider the latter sequence of spaces with their weak typologies, for details see Appendix 4.A. Finally, we define the space

$$\mathfrak{X} = \mathfrak{X}_{\mathrm{ind}}^\infty \times \mathfrak{X}_{\mathrm{BM}}^\infty \times \mathfrak{X}_{\mathrm{cont}} \times \mathfrak{X}_{\mathrm{Lebesgue}} \times \mathfrak{X}_{\mathrm{Sobolev}} \times \mathfrak{X}_{\mathrm{power}}^\infty, \tag{4.3.1}$$

and equip it with the product topology.

Lemma 4.3.1. *Let $\alpha_l \nearrow 2-n$ be a sequence satisfying* (4.2.8). *Then the family*

$$\Big(\Big((\mathbf{1}_{A_j})_{j\in\mathbb{N}}, (\beta^{(k)})_{k\in\mathbb{Z}}, u_\varepsilon, u_\varepsilon, u_\varepsilon, \Big(u_\varepsilon^{\frac{\alpha_l+n+1}{2}}\Big)_{l\in\mathbb{N}}\Big)\Big)_{\varepsilon\in(0,1)} \tag{4.3.2}$$

lies tight on $\mathfrak{X}$.

Proof. By Lemma 4.B.2 it suffices to show tightness of each of the components of (4.3.2) in their respective space separately.

Tightness of the indicator functions and Brownian motions. The set of real numbers $\mathfrak{X}_{\mathrm{ind}}$ is a Radon space and thus the law of $\mathbf{1}_{A_j}$ is inner regular for every $j\in\mathbb{N}$. Consequently, the family $(\mathbf{1}_{A_j})_{\varepsilon\in(0,1)}$ lies tight on $\mathfrak{X}_{\mathrm{ind}}$. Tightness of the sequence $((\mathbf{1}_{A_j})_{j\in\mathbb{N}})_{\varepsilon\in(0,1)}$ on $\mathfrak{X}_{\mathrm{ind}}^\infty$ follows by Lemma 4.B.2. Similarly, since $\mathfrak{X}_{\mathrm{BM}}$ is a Radon space, the law of $\beta^{(k)}$ is inner regular, and consequently the family $(\beta^{(k)})_{\varepsilon\in(0,1)}$ lies tight on it. Another application of Lemma 4.B.2 yields tightness of $((\beta^{(k)})_{k\in\mathbb{Z}})_{\varepsilon\in(0,1)}$ on $\mathfrak{X}_{\mathrm{BM}}^\infty$.

Tightness on $\mathfrak{X}_{cont}$. By Lemma 4.B.1 and Lemma 4.B.3 it suffices to show tightness of $(\mathbf{1}_{A_j}u_\varepsilon)_{\varepsilon\in(0,1)}$ on $C([0,T];H^{\kappa_l}(\mathbb{T}))$ for each $j,l\in\mathbb{N}$. To this end, we choose $\gamma\in(0,\frac{1}{2})$, $\mu\in(1,\frac{7}{n+2})$ and $\nu\in[1,\frac{7}{n+4})$ so that $\gamma-\frac{2-\nu}{2\nu}>0$ and in particular the embedding

$$L^\infty(0,T;L^1(\mathbb{T}))\cap W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T})) \hookrightarrow C([0,T];H^{\kappa_l}(\mathbb{T}))$$

is compact by [121, Corollary 5] and the Rellich–Kondrachov theorem. Consequently, the set

$$K_\delta = \left\{u \,\middle|\, \|u\|_{L^\infty(0,T;L^1(\mathbb{T}))} \le \tfrac{1}{\delta},\ \|u\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} \le \tfrac{1}{\delta}\right\}$$

lies compact in $C([0,T];H^{\kappa_l}(\mathbb{T}))$ for $\delta>0$. By (4.2.20) and (4.2.61), we have that

$$\mathbb{E}\big[\mathbf{1}_{A_j}\|u_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}))}\big] \le j+1$$

and hence together with (4.2.65) and Chebychev's inequality, we conclude that

$$\begin{aligned}\mathbb{P}(\{\mathbf{1}_{A_j}u_\varepsilon \notin K_\delta\}) &\le \mathbb{P}\Big(\big\{\mathbf{1}_{A_j}\|u_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}))} > \tfrac{1}{\delta}\big\}\Big) + \mathbb{P}\Big(\Big\{\mathbf{1}_{A_j}\|u_\varepsilon\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))} > \tfrac{1}{\delta}\Big\}\Big)\\ &\le \delta\,\mathbb{E}\big[\mathbf{1}_{A_j}\|u_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{T}))}\big] + \delta^\nu\,\mathbb{E}\Big[\mathbf{1}_{A_j}\|u_\varepsilon\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big]\\ &\lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \delta(j+1) + \delta^\nu(j+1)^{\nu(n+1)}.\end{aligned} \tag{4.3.3}$$

Tightness follows, since the right-hand side tends to 0 as $\delta \searrow 0$.

Tightness on $\mathfrak{X}_{Lebesgue}$. By Lemma 4.B.1 and Lemma 4.B.3 it is enough to show tightness of $(\mathbf{1}_{A_j}u_\varepsilon)_{\varepsilon\in(0,1)}$ on $L^{p_l}([0,T]\times\mathbb{T})$ for each $j,l\in\mathbb{N}$. We let again $\gamma\in(0,\frac{1}{2})$, $\mu\in(1,\frac{7}{n+2})$, $\nu\in[1,\frac{7}{n+4})$ with $\gamma-\frac{2-\nu}{2\nu}>0$ and define the set

$$K_\delta = \Big\{u\,\Big|\,\max\Big\{\|u\|_{L^{p_{l+1}}([0,T]\times\mathbb{T})}, \|\partial_x u\|_{L^1([0,T]\times\mathbb{T})}, \|u\|_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big\} \le \tfrac{1}{\delta}\Big\}.$$

The set K_δ is bounded in $L^1(0,T;W^{1,1}(\mathbb{T}))$ and thus compact in $L^1([0,T]\times\mathbb{T})$ by compactness of the embedding

$$L^1(0,T;W^{1,1}(\mathbb{T}))\cap W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T})) \hookrightarrow L^1(0,T;L^1(\mathbb{T})),$$

see again [121, Corollary 5]. Since K_δ is moreover bounded in $L^{p_{l+1}}([0,T]\times\mathbb{T})$, it is compact in $L^{p_l}([0,T]\times\mathbb{T})$ by interpolation. Lastly, with the help of (4.2.63)–(4.2.65), we can conclude that $\mathbb{P}(\{\mathbf{1}_{A_j}u_\varepsilon\notin K_\delta\})\to 0$ uniformly in ε as $\delta\searrow 0$, analogously to (4.3.3).

Tightness on $\mathfrak{X}_{Sobolev}$. By Lemma 4.B.1 and Lemma 4.B.3 it is again sufficient to verify that $(\mathbf{1}_{A_j}u_\varepsilon)_{\varepsilon\in(0,1)}$ lies tight on $L^{r_l}(0,T;W^{1,r_l}(\mathbb{T}))$, equipped with its weak topology for each $j,l\in\mathbb{N}$. To this end, we define the set

$$K_\delta = \big\{u\,\big|\,\|u\|_{L^{r_l}([0,T]\times\mathbb{T})}\le\tfrac{1}{\delta},\ \|\partial_x u\|_{L^{r_l}([0,T]\times\mathbb{T})}\le\tfrac{1}{\delta}\big\},$$

which is bounded in $L^{r_l}(0,T;W^{1,r_l}(\mathbb{T}))$ and consequently compact with respect to the weak topology by the Banach–Alaoglu theorem. Following the lines of (4.3.3), we conclude that $\mathbb{P}(\{\mathbf{1}_{A_j}u_\varepsilon\notin K_\delta\})\to 0$ uniformly in ε as $\delta\searrow 0$ by (4.2.63) and (4.2.64), which implies tightness.

Tightness of the powers. A last application of Lemma 4.B.2 and Lemma 4.B.3 yields that it suffices to show tightness of $\big(\mathbf{1}_{A_j}u_\varepsilon^{\frac{\alpha_l+n+1}{2}}\big)_{\varepsilon\in(0,1)}$ on $\mathfrak{X}_{\mathrm{power}}$ for all $j,l\in\mathbb{N}$. To this end, we define $v_\varepsilon = u_\varepsilon^{\frac{\alpha_l+n+1}{2}}$ and arguing as for w_ε at the beginning of the proof of Lemma 4.2.3, we conclude that v_ε admits

$$\frac{(\alpha_l+n+1)(\alpha_l+n-1)}{4}u_\varepsilon^{\frac{\alpha_l+n-3}{2}}(\partial_x u_\varepsilon)^2 + \frac{\alpha_l+n+1}{2}u_\varepsilon^{\frac{\alpha_l+n-1}{2}}\partial_x^2 u_\varepsilon$$

as a second weak derivative. In particular, since Parseval's relation yields

$$\begin{aligned}\|v_\varepsilon\|^2_{H^2(\mathbb{T})} &= \sum_{k\in\mathbb{Z}}(1+(2\pi k)^2+(2\pi k)^4)|\widehat{v_\varepsilon}(k)|^2\\ &\lesssim |\widehat{v_\varepsilon}(0)|^2 + \sum_{k\in\mathbb{Z}}(2\pi k)^4|\widehat{v_\varepsilon}(k)|^2 \le \|v_\varepsilon\|^2_{L^2(\mathbb{T})} + \|\partial_x^2 v_\varepsilon\|^2_{L^2(\mathbb{T})},\end{aligned}$$

we can estimate

$$\begin{aligned}
&\|v_\varepsilon\|^2_{L^2(0,T;H^2(\mathbb{T}))}\\
&\lesssim \int_0^T \|v_\varepsilon\|^2_{L^2(\mathbb{T})} + \|\partial_x^2 v_\varepsilon\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t\\
&\lesssim_{\alpha_l,n} \int_0^T \|u_\varepsilon^{\frac{\alpha_l+n+1}{2}}\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}} u_\varepsilon^{\alpha_l+n-1}(\partial_x^2 u_\varepsilon)^2 + u_\varepsilon^{\alpha_l+n-3}(\partial_x u_\varepsilon)^4\,\mathrm{d}x\,\mathrm{d}t\\
&= \|u_\varepsilon\|^{\alpha_l+n+1}_{L^{\alpha_l+n+1}([0,T]\times\mathbb{T})} + \int_0^T\int_{\mathbb{T}} u_\varepsilon^{\alpha_l+n-1}(\partial_x^2 u_\varepsilon)^2 + u_\varepsilon^{\alpha_l+n-3}(\partial_x u_\varepsilon)^4\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Hence, by invoking (4.2.62) and (4.2.63) we obtain

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{A_j}\|v_\varepsilon\|^2_{L^2(0,T;H^2(\mathbb{T}))}\big]\\
&\quad\lesssim_{\alpha_l,n} \mathbb{E}\big[\mathbf{1}_{A_j}\|u_\varepsilon\|^{\alpha_l+n+1}_{L^{\alpha_l+n+1}([0,T]\times\mathbb{T})}\big]\\
&\qquad + \mathbb{E}\Big[\mathbf{1}_{A_j}\int_0^T\int_{\mathbb{T}} u_\varepsilon^{\alpha_l+n-1}(\partial_x^2 u_\varepsilon)^2 + u_\varepsilon^{\alpha_l+n-3}(\partial_x u_\varepsilon)^4\,\mathrm{d}x\,\mathrm{d}t\Big]\\
&\quad\lesssim_{\alpha_l,n,\Lambda,T} (j+1)^{\alpha_l+n+1}.
\end{aligned}$$

Tightness of $(\mathbf{1}_{A_j} v_\varepsilon)_{\varepsilon\in(0,1)}$ in $\mathfrak{X}_{\text{power}}$ follows by Chebychev's inequality and the Banach–Alaoglu theorem. □

Since the sequence (4.3.2) lies tight on the space (4.3.1), we can extract an equidistributed convergent subsequence.

Corollary 4.3.2. *There exists a complete probability space* $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$ *and an equidistributed subsequence*

$$\Big(\Big((\widetilde{\chi}_\varepsilon^{(j)})_{j\in\mathbb{N}},(\widetilde{\beta}_\varepsilon^{(k)})_{k\in\mathbb{Z}},\widetilde{u}_\varepsilon,\widetilde{u}_\varepsilon,\widetilde{u}_\varepsilon,\big(\widetilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}}\big)_{l\in\mathbb{N}}\Big)\Big)_\varepsilon \tag{4.3.4}$$

of (4.3.2)*, consisting of* $\mathfrak{B}(\mathfrak{X})$*-measurable,* $\mathfrak{X}$*-valued random variables as well as a* $\mathfrak{B}(\mathfrak{X})$*-measurable,* $\mathfrak{X}$*-valued random variable*

$$\Big((\widetilde{\chi}^{(j)})_{j\in\mathbb{N}},(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}},\widetilde{u},\widetilde{u},\widetilde{u},\big(\widetilde{u}^{\frac{\alpha_l+n+1}{2}}\big)_{l\in\mathbb{N}}\Big), \tag{4.3.5}$$

defined on $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$ *such that*

$$\Big((\widetilde{\chi}_\varepsilon^{(j)})_{j\in\mathbb{N}},(\widetilde{\beta}_\varepsilon^{(k)})_{k\in\mathbb{Z}},\widetilde{u}_\varepsilon,\widetilde{u}_\varepsilon,\widetilde{u}_\varepsilon,\big(\widetilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}}\big)_{l\in\mathbb{N}}\Big)\to\Big((\widetilde{\chi}^{(j)})_{j\in\mathbb{N}},(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}},\widetilde{u},\widetilde{u},\widetilde{u},\big(\widetilde{u}^{\frac{\alpha_l+n+1}{2}}\big)_{l\in\mathbb{N}}\Big) \tag{4.3.6}$$

$\widetilde{\mathbb{P}}$*-almost surely in* $\mathfrak{X}$*, as* $\varepsilon\searrow 0$.

Proof. The existence of an equidistributed subsequence follows from Lemma 4.3.1, if we can verify the technical assumption of [88, Theorem 2], i.e. that $\mathfrak{X}$ admits a countable family of continuous functions separating its points and $\mathfrak{A}$-measurability of these functions, when composed with (4.3.2). To construct such a family of functions, separating the points in $\mathfrak{X}$, we can project on a component of (4.3.1) and then apply functions separating the points in this component, so that we can consider the spaces $\mathfrak{X}_{\text{ind}}$,

$\mathfrak{X}_{\mathrm{BM}}$, $\mathfrak{X}_{\mathrm{cont}}$, $\mathfrak{X}_{\mathrm{Lebesgue}}$, $\mathfrak{X}_{\mathrm{Sobolev}}$ and $\mathfrak{X}_{\mathrm{power}}$ individually. For $\mathfrak{X}_{\mathrm{cont}}$, we let $\rho\colon \mathbb{R} \to [-1,1]$ be a continuous injection and take the functions $\mathfrak{X}_{\mathrm{cont}} \to [-1,1], u \mapsto \rho(\langle u(t), f_k\rangle)$ for $t \in [0,T] \cap \mathbb{Q}$ and $k \in \mathbb{Z}$, where we recall that f_k was defined by (4.1.3). For $\mathfrak{X}_{\mathrm{BM}}$ the same construction applies and for $\mathfrak{X}_{\mathrm{ind}}$ the function ρ itself separates the points. Since the spaces $\mathfrak{X}_{\mathrm{Lebesgue}}$, $\mathfrak{X}_{\mathrm{Sobolev}}$ and $\mathfrak{X}_{\mathrm{power}}$ embed into $L^2([0,T]\times\mathbb{T})$ with its weak topology, the family $u \mapsto \rho(\langle u, f_j(T^{-1}\cdot)\otimes f_k\rangle)$, $j,k \in \mathbb{Z}$ separates the points in them.

To also check $\mathfrak{A}$-measurability of these functions composed with (4.3.2), we first observe that $\rho(\mathbf{1}_{A_j})$ and $\rho(\beta^{(k)}(t))$ are random variables in $\mathbb{R}$ for every $j \in \mathbb{N}$ and $k \in \mathbb{Z}$. For the other cases, we note that u_ε is adapted in $H^1(\mathbb{T})$ and hence $u_\varepsilon(t,x)$ is $\mathfrak{A}$-measurable for each $(t,x) \in [0,T]\times\mathbb{T}$ and thus also u_ε as a random variable in $C([0,T]\times\mathbb{T})$. In particular, the compositions

$$\rho(\langle u_\varepsilon(t), f_k\rangle), \quad \rho(\langle u_\varepsilon, f_j(T^{-1}\cdot)\otimes f_k\rangle)$$

are $\mathfrak{A}$-measurable, too. Lastly, we use that

$$C([0,T]\times\mathbb{T}) \to C([0,T]\times\mathbb{T}),\ u \mapsto u^{\frac{\alpha_l+n+1}{2}}$$

is continuous, to conclude that

$$\rho\big(\big\langle u_\varepsilon^{\frac{\alpha_l+n+1}{2}}, f_j(T^{-1}\cdot)\otimes f_k\big\rangle\big)$$

is $\mathfrak{A}$-measurable.

Hence, [88, Theorem 2] is indeed applicable and there exists an equidistributed, convergent subsequence (4.3.4) of (4.3.2), which converges almost surely to a random variable

$$\big((\tilde{\chi}^{(j)})_{j\in\mathbb{N}}, (\tilde{\beta}^{(k)})_{k\in\mathbb{Z}}, \tilde{u}, \tilde{f}, \tilde{g}, (\tilde{v}_l)_{l\in\mathbb{N}}\big)$$

in $\mathfrak{X}$. Since $\tilde{u}_\varepsilon$ converges almost surely to $\tilde{u}$, $\tilde{f}$ and $\tilde{g}$ in the space of distributions on $(0,T)\times\mathbb{T}$, it holds $\tilde{u} = \tilde{f} = \tilde{g}$. Moreover, since $\tilde{u}_\varepsilon \to \tilde{u}$ in $\mathfrak{X}_{\mathrm{Lebesgue}}$ we have that $\tilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \tilde{u}^{\frac{\alpha_l+n+1}{2}}$ in $L^2([0,T]\times\mathbb{T})$ by Vitali's convergence theorem. With the help of $\tilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \tilde{v}_l$ in $\mathfrak{X}_{\mathrm{power}}$, we conclude $\tilde{v}_l = \tilde{u}^{\frac{\alpha_l+n+1}{2}}$, which finishes the proof. □

4.3.2. CONVERGENCE TO A SOLUTION AND A-PRIORI ESTIMATES

For the rest of this chapter, we consider the complete probability space $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mathbb{P}})$ with the random variables (4.3.4) and (4.3.5) obtained in Corollary 4.3.2. We introduce the filtration $\tilde{\mathfrak{F}}$ on $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mathbb{P}})$ as the augmentation of $\tilde{\mathfrak{G}}$, defined by

$$\tilde{\mathfrak{G}}_t = \sigma\big\{\tilde{\chi}^{(j)}, \tilde{u}(s), \tilde{\beta}^{(k)}(s) \mid j \in \mathbb{N}, 0 \le s \le t, k \in \mathbb{Z}\big\}.$$

Here, we consider $\tilde{u}(s)$ as a random element in $H^{-1}(\mathbb{T})$.

Lemma 4.3.3. *The family $(\tilde{\beta}^{(k)})_{k\in\mathbb{Z}}$ is a family of independent $\tilde{\mathfrak{F}}$-Brownian motions.*

Proof. The proof is standard and we refer to [51, Lemma 5.7]. □

Lemma 4.3.4. *The family $(\widetilde{\chi}^{(j)})_{j\in\mathbb{N}}$ consists of indicator functions of a measurable partition $(\widetilde{A}_j)_{j\in\mathbb{N}}$ of $\widetilde{\Omega}$.*

Proof. The family $(\widetilde{\chi}^{(j)}_\varepsilon)_{j\in\mathbb{N}} \sim (\mathbf{1}_{A_j})_{j\in\mathbb{N}}$ converges to $(\widetilde{\chi}_j)_{j\in\mathbb{N}}$ almost surely and thus in law by [42, Proposition 9.3.5]. Hence, the law of $(\widetilde{\chi}_j)_{j\in\mathbb{N}}$ coincides with the law of $(\mathbf{1}_{A_j})_{j\in\mathbb{N}}$ and the claim follows. □

Additionally to the convergences (4.3.6), we obtain also strong convergence for the powers of $\widetilde{u}_\varepsilon$.

Lemma 4.3.5. *For each $l\in\mathbb{N}$ one has*

$$\widetilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \widetilde{u}^{\frac{\alpha_l+n+1}{2}} \tag{4.3.7}$$

almost surely in $L^2(0,T;H^1(\mathbb{T}))$, as $\varepsilon\searrow 0$.

Proof. At the end of the proof of Corollary 4.3.2, we showed that $\widetilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \widetilde{u}^{\frac{\alpha_l+n+1}{2}}$ almost surely in $L^2([0,T]\times\mathbb{T})$. Hence, the claim follows by the weak convergence $\widetilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \widetilde{u}^{\frac{\alpha_l+n+1}{2}}$ in $\mathfrak{X}_{\text{power}}$ and interpolation. □

Lemma 4.3.6. *For each $\varphi\in C^\infty(\mathbb{T})$ and $t\in[0,T]$, the following holds almost surely as $\varepsilon\searrow 0$.*

(i)

$$\langle \widetilde{u}_\varepsilon(t),\varphi\rangle \to \langle \widetilde{u}(t),\varphi\rangle,$$

(ii)

$$\int_0^t \langle \widetilde{u}_\varepsilon^{n-2}(\partial_x\widetilde{u}_\varepsilon)^3,\partial_x\varphi\rangle\,\mathrm{d}s \to \int_0^t \langle \widetilde{u}^{n-2}(\partial_x\widetilde{u})^3,\partial_x\varphi\rangle\,\mathrm{d}s,$$

(iii)

$$\int_0^t \langle \widetilde{u}_\varepsilon^{n-1}(\partial_x\widetilde{u}_\varepsilon)^2,\partial_x^2\varphi\rangle\,\mathrm{d}s \to \int_0^t \langle \widetilde{u}^{n-1}(\partial_x\widetilde{u})^2,\partial_x\varphi\rangle\,\mathrm{d}s,$$

(iv)

$$\int_0^t \langle \widetilde{u}_\varepsilon^{n}\partial_x\widetilde{u}_\varepsilon,\partial_x^3\varphi\rangle\,\mathrm{d}s \to \int_0^t \langle \widetilde{u}^{n}\partial_x\widetilde{u},\partial_x^3\varphi\rangle\,\mathrm{d}s,$$

(v)

$$\sum_{k\in\mathbb{Z}}\int_0^t \langle \sigma_{k,\varepsilon}q'(\widetilde{u}_\varepsilon)\partial_x(\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon)),\partial_x\varphi\rangle\,\mathrm{d}s \to \sum_{k\in\mathbb{Z}}\int_0^t \langle \sigma_{k}q'(\widetilde{u})\partial_x(\sigma_{k}q(\widetilde{u})),\partial_x\varphi\rangle\,\mathrm{d}s,$$

(vi)

$$\sum_{k\in\mathbb{Z}}\int_0^t \langle \sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle^2\,\mathrm{d}s \to \sum_{k\in\mathbb{Z}}\int_0^t \langle \sigma_{k}q(\widetilde{u}),\partial_x\varphi\rangle^2\,\mathrm{d}s,$$

(vii)

$$\int_0^t \langle \sigma_{k,\varepsilon} q(\tilde{u}_\varepsilon), \partial_x \varphi \rangle \,\mathrm{d}s \to \int_0^t \langle \sigma_k q(\tilde{u}), \partial_x \varphi \rangle \,\mathrm{d}s, \quad k \in \mathbb{Z}.$$

Proof. We fix $\tilde{\omega} \in \tilde{\Omega}$ such that (4.3.6) holds in $\mathfrak{X}$ and (4.3.7) holds in $L^2(0,T;H^1(\mathbb{T}))$ for each $l \in \mathbb{N}$. The convergence (i) follows from $\tilde{u}_\varepsilon \to \tilde{u}$ in $\mathfrak{X}_{\mathrm{cont}}$. For (ii)–(v), we consider an arbitrary subsequence so that it suffices to show that for another subsequence the respective convergence holds. Because $\tilde{u}_\varepsilon \to \tilde{u}$ in $\mathfrak{X}_{\mathrm{Lebesgue}}$ and $\tilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \tilde{u}^{\frac{\alpha_l+n+1}{2}}$ in $L^2(0,T;H^1(\mathbb{T}))$, we can choose this subsequence such that $\tilde{u}_\varepsilon \to \tilde{u}$ and $\partial_x \tilde{u}_\varepsilon^{\frac{\alpha_l+n+1}{2}} \to \partial_x \tilde{u}^{\frac{\alpha_l+n+1}{2}}$ almost everywhere on $[0,T] \times \mathbb{T}$ and therefore $\partial_x \tilde{u}_\varepsilon \to \partial_x \tilde{u}$ on the set $\{\tilde{u} > 0\}$.

We verify (ii) by showing separately that

$$\int_0^t \int_{\{\tilde{u}=0\}} \tilde{u}_\varepsilon^{n-2} (\partial_x \tilde{u}_\varepsilon)^3 \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}s \to 0, \tag{4.3.8}$$

$$\int_0^t \int_{\{\tilde{u}>0\}} \tilde{u}_\varepsilon^{n-2} (\partial_x \tilde{u}_\varepsilon)^3 \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_{\{\tilde{u}>0\}} \tilde{u}^{n-2} (\partial_x \tilde{u})^3 \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}s. \tag{4.3.9}$$

Using Consequence 4.2.1 (i) together with $\tilde{u}_\varepsilon \to \tilde{u}$ in $\mathfrak{X}_{\mathrm{Lebesgue}}$ and $\mathfrak{X}_{\mathrm{Sobolev}}$, (4.3.8) follows by Hölder's inequality and (4.3.9) by Vitali's convergence theorem. The claims (iii) and (iv) can be shown analogously.

For (v) we rewrite

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \int_0^t \langle \sigma_{k,\varepsilon} q'(\tilde{u}_\varepsilon) \partial_x (\sigma_{k,\varepsilon} q(\tilde{u}_\varepsilon)), \partial_x \varphi \rangle \,\mathrm{d}s \\
&\quad = \sum_{k\in\mathbb{Z}} \int_0^t \langle \sigma_{k,\varepsilon}^2 (q'(\tilde{u}_\varepsilon))^2 \partial_x \tilde{u}_\varepsilon, \partial_x \varphi \rangle + \langle \sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} q(\tilde{u}_\varepsilon) q'(\tilde{u}_\varepsilon), \partial_x \varphi \rangle \,\mathrm{d}s
\end{aligned}$$

and observe that for the individual summands

$$\begin{aligned}
\tfrac{n^2}{4} \int_0^t \langle \sigma_{k,\varepsilon}^2 \tilde{u}_\varepsilon^{n-2} \partial_x \tilde{u}_\varepsilon, \partial_x \varphi \rangle \,\mathrm{d}s &\to \tfrac{n^2}{4} \int_0^t \langle \sigma_k^2 \tilde{u}^{n-2} \partial_x \tilde{u}, \partial_x \varphi \rangle \,\mathrm{d}s, \\
\tfrac{n}{2} \int_0^t \langle \sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} \tilde{u}_\varepsilon^{n-1}, \partial_x \varphi \rangle \,\mathrm{d}s &\to \tfrac{n}{2} \int_0^t \langle \sigma_k \partial_x \sigma_k \tilde{u}^{n-1}, \partial_x \varphi \rangle \,\mathrm{d}s,
\end{aligned}$$

as $\varepsilon \searrow 0$, since $\tilde{u}_\varepsilon \to \tilde{u}$ in $\mathfrak{X}_{\mathrm{Lebesgue}}$, $\mathfrak{X}_{\mathrm{Sobolev}}$ and $\sigma_{k,\varepsilon} \to \sigma_k$ in $C^1(\mathbb{T})$ for fixed $k \in \mathbb{Z}$ by (4.1.2), (4.2.3) and (4.2.4). Hence, by the dominated convergence theorem, it suffices to find a summable, dominating sequence of

$$\left(\int_0^t |\langle \sigma_{k,\varepsilon}^2 \tilde{u}_\varepsilon^{n-2} \partial_x \tilde{u}_\varepsilon, \partial_x \varphi \rangle| + |\langle \sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} \tilde{u}_\varepsilon^{n-1}, \partial_x \varphi \rangle| \,\mathrm{d}s \right)_{k\in\mathbb{Z}}$$

independent of ε to conclude (v). To this end, we estimate using Hölder's inequality

$$\int_0^t |\langle \sigma_{k,\varepsilon}^2 \tilde{u}_\varepsilon^{n-2} \partial_x \tilde{u}_\varepsilon, \partial_x \varphi \rangle| + |\langle \sigma_{k,\varepsilon} \partial_x \sigma_{k,\varepsilon} \tilde{u}_\varepsilon^{n-1}, \partial_x \varphi \rangle| \,\mathrm{d}s$$

$$
\begin{aligned}
&\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{T})}\int_0^t \|\sigma_{k,\varepsilon}\|^2_{L^\infty(\mathbb{T})}\|\widetilde{u}_\varepsilon\|^{n-2}_{L^{2(n-2)}(\mathbb{T})}\|\partial_x\widetilde{u}_\varepsilon\|_{L^2(\mathbb{T})}\,\mathrm{d}s\\
&\quad+\|\partial_x\varphi\|_{L^\infty(\mathbb{T})}\int_0^t \|\sigma_{k,\varepsilon}\|_{L^\infty(\mathbb{T})}\|\partial_x\sigma_{k,\varepsilon}\|_{L^\infty(\mathbb{T})}\|\widetilde{u}_\varepsilon\|^{n-1}_{L^{n-1}(\mathbb{T})}\,\mathrm{d}s\\
&\leq \|\partial_x\varphi\|_{L^\infty(\mathbb{T})}\big(\|\widetilde{u}_\varepsilon\|^{n-2}_{L^{2(n-2)}([0,T]\times\mathbb{T})}\|\partial_x\widetilde{u}_\varepsilon\|_{L^2([0,T]\times\mathbb{T})}+\|\widetilde{u}_\varepsilon\|^{n-1}_{L^{n-1}([0,T]\times\mathbb{T})}\big)\|\sigma_k\|^2_{C^1(\mathbb{T})}.
\end{aligned}
$$

The prefactor on the right-hand side is uniformly bounded in ε, since $\widetilde{u}_\varepsilon\to\widetilde{u}$ in $\mathfrak{X}_{\text{Lebesgue}}$, $\mathfrak{X}_{\text{Sobolev}}$ and the sequence $(\|\sigma_k\|^2_{C^1(\mathbb{T})})_{k\in\mathbb{Z}}$ is summable by (4.2.12), which finishes the proof of (v). For (vi), we proceed analogously and observe first that for each summand

$$
\begin{aligned}
&\left|\int_0^t \langle\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle^2-\langle\sigma_k q(\widetilde{u}),\partial_x\varphi\rangle^2\,\mathrm{d}s\right|\\
&\quad\leq\int_0^t\big|\langle\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle-\langle\sigma_k q(\widetilde{u}),\partial_x\varphi\rangle\big|\big|\langle\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle+\langle\sigma_k q(\widetilde{u}),\partial_x\varphi\rangle\big|\,\mathrm{d}s\\
&\quad\leq\|\partial_x\varphi\|^2_{L^2(\mathbb{T})}\int_0^t\|\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon)-\sigma_k q(\widetilde{u})\|_{L^2(\mathbb{T})}\|\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon)+\sigma_k q(\widetilde{u})\|_{L^2(\mathbb{T})}\,\mathrm{d}s\\
&\quad\leq\|\partial_x\varphi\|^2_{L^2(\mathbb{T})}\|\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon)-\sigma_k q(\widetilde{u})\|_{L^2([0,T]\times\mathbb{T})}\|\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon)+\sigma_k q(\widetilde{u})\|_{L^2([0,T]\times\mathbb{T})},
\end{aligned}
$$

which tends to 0 as $\varepsilon\searrow 0$, since $\widetilde{u}_\varepsilon\to\widetilde{u}$ in $\mathfrak{X}_{\text{Lebesgue}}$ and $\sigma_{k,\varepsilon}\to\sigma_k$ in $C(\mathbb{T})$. We use again the dominated convergence theorem together with

$$
\begin{aligned}
\int_0^t\langle\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle^2\,\mathrm{d}s&\leq\int_0^t\|\sigma_{k,\varepsilon}\|^2_{L^\infty(\mathbb{T})}\|\widetilde{u}_\varepsilon\|^n_{L^n(\mathbb{T})}\|\partial_x\varphi\|^2_{L^2(\mathbb{T})}\,\mathrm{d}s\\
&\leq\|\partial_x\varphi\|^2_{L^2(\mathbb{T})}\|\widetilde{u}_\varepsilon\|^n_{L^n([0,T]\times\mathbb{T})}\|\sigma_k\|^2_{C(\mathbb{T})},
\end{aligned}
$$

the convergence $\widetilde{u}_\varepsilon\to\widetilde{u}$ in $\mathfrak{X}_{\text{Lebesgue}}$ and (4.2.12) to conclude (vi). The convergence (vii) follows from $\widetilde{u}_\varepsilon\to\widetilde{u}$ in $\mathfrak{X}_{\text{Lebesgue}}$ and $\sigma_{k,\varepsilon}\to\sigma_k$ in $C(\mathbb{T})$. □

Lemma 4.3.7. *For every $\varphi\in C^\infty(\mathbb{T})$, $j\in\mathbb{N}$ and $t\in[0,T]$ we have that*

$$
\begin{aligned}
\mathbf{1}_{\widetilde{A}_j}\big[\langle\widetilde{u}(t),\varphi\rangle-\langle\widetilde{u}(0),\varphi\rangle\big]=\mathbf{1}_{\widetilde{A}_j}\Bigg[&\frac{n(n-1)}{2}\int_0^t\langle\widetilde{u}^{n-2}(\partial_x\widetilde{u})^3,\partial_x\varphi\rangle\,\mathrm{d}s\\
&+\frac{3n}{2}\int_0^t\langle\widetilde{u}^{n-1}(\partial_x\widetilde{u})^2,\partial_x^2\varphi\rangle\,\mathrm{d}s+\int_0^t\langle\widetilde{u}^n\partial_x\widetilde{u},\partial_x^3\varphi\rangle\,\mathrm{d}s\\
&-\frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t\langle\sigma_k q'(\widetilde{u})\partial_x(\sigma_k q(\widetilde{u})),\partial_x\varphi\rangle\,\mathrm{d}s\\
&-\sum_{k\in\mathbb{Z}}\int_0^t\langle\sigma_k q(\widetilde{u}),\partial_x\varphi\rangle\,\mathrm{d}\widetilde{\beta}_s^{(k)}\Bigg].
\end{aligned}
\tag{4.3.10}
$$

Proof. Throughout this proof, we fix $j\in\mathbb{N}$ and $\varphi\in C^\infty(\mathbb{T})$ and define the process

$$
\begin{aligned}
\widetilde{M}(t)=\mathbf{1}_{\widetilde{A}_j}\Bigg[&\langle\widetilde{u}(t),\varphi\rangle-\langle\widetilde{u}(0),\varphi\rangle-\frac{n(n-1)}{2}\int_0^t\langle\widetilde{u}^{n-2}(\partial_x\widetilde{u})^3,\partial_x\varphi\rangle\,\mathrm{d}s\\
&-\frac{3n}{2}\int_0^t\langle\widetilde{u}^{n-1}(\partial_x\widetilde{u})^2,\partial_x^2\varphi\rangle\,\mathrm{d}s-\int_0^t\langle\widetilde{u}^n\partial_x\widetilde{u},\partial_x^3\varphi\rangle\,\mathrm{d}s
\end{aligned}
$$

$$+\frac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t \langle\sigma_k q'(\tilde{u})\partial_x(\sigma_k q(\tilde{u})),\partial_x\varphi\rangle\,\mathrm{d}s\Big]$$

and the approximating processes

$$\begin{aligned}\widetilde{M}_\varepsilon(t) = \tilde{\chi}_\varepsilon^{(j)}\Big[&\langle\tilde{u}_\varepsilon(t),\varphi\rangle - \langle\tilde{u}_\varepsilon(0),\varphi\rangle - \tfrac{n(n-1)}{2}\int_0^t\langle\tilde{u}_\varepsilon^{n-2}(\partial_x\tilde{u}_\varepsilon)^3,\partial_x\varphi\rangle\,\mathrm{d}s\\ &-\tfrac{3n}{2}\int_0^t\langle\tilde{u}_\varepsilon^{n-1}(\partial_x\tilde{u}_\varepsilon)^2,\partial_x^2\varphi\rangle\,\mathrm{d}s - \int_0^t\langle\tilde{u}_\varepsilon^{n}\partial_x\tilde{u}_\varepsilon,\partial_x^3\varphi\rangle\,\mathrm{d}s\\ &+\tfrac{1}{2}\sum_{k\in\mathbb{Z}}\int_0^t\langle\sigma_{k,\varepsilon}q'(\tilde{u}_\varepsilon)\partial_x(\sigma_{k,\varepsilon}q(\tilde{u}_\varepsilon)),\partial_x\varphi\rangle\,\mathrm{d}s\Big].\end{aligned}\tag{4.3.11}$$

As a consequence of Lemma 4.3.6 we have indeed $\widetilde{M}_\varepsilon(t)\to\widetilde{M}(t)$ as $\varepsilon\searrow 0$. Now let

$$\phi\colon \prod_{j=1}^{\infty}\mathbb{R}\times C([0,s];H^{-1}(\mathbb{T}))\times\prod_{k\in\mathbb{Z}}C([0,s])\to\mathbb{R}\tag{4.3.12}$$

be continuous and bounded and define

$$\begin{aligned}\tilde{\rho} &= \phi\big((\tilde{\chi}^{(j)})_{j\in\mathbb{N}},\tilde{u},(\tilde{\beta}^{(k)})_{k\in\mathbb{Z}}\big),\\ \tilde{\rho}_\varepsilon &= \phi\big((\tilde{\chi}_\varepsilon^{(j)})_{j\in\mathbb{N}},\tilde{u}_\varepsilon,(\tilde{\beta}_\varepsilon^{(k)})_{k\in\mathbb{Z}}\big),\end{aligned}$$

so that $\tilde{\rho}_\varepsilon\to\tilde{\rho}$ as $\varepsilon\searrow 0$ by (4.3.6). Defining M_ε on the original probability space $(\Omega,\mathfrak{A},\mathbb{P})$ as the right-hand side of (4.3.11) with $\tilde{u}_\varepsilon$ replaced by u_ε, we find that

$$M_\varepsilon(t) = -\mathbf{1}_{A_j}\int_0^t\langle\Phi_\varepsilon\mathrm{d}\beta_s,\partial_x\varphi\rangle,$$

because of Consequence 4.2.1 (vi) and (4.2.41), where Φ_ε and β are defined as in the proof of Lemma 4.2.7. The quadratic variation process of M_ε is given by

$$\begin{aligned}\mathbf{1}_{A_j}\sum_{k\in\mathbb{Z}}\int_0^t\langle\sigma_{k,\varepsilon}q(u_\varepsilon),\partial_x\varphi\rangle^2\,\mathrm{d}s &\le \mathbf{1}_{A_j}\sum_{k\in\mathbb{Z}}\int_0^t\|\sigma_{k,\varepsilon}\|_{L^\infty(\mathbb{T})}^2\|u_\varepsilon\|_{L^n(\mathbb{T})}^n\|\partial_x\varphi\|_{L^2(\mathbb{T})}^2\,\mathrm{d}s\\ &\lesssim_\Lambda \mathbf{1}_{A_j}\|\partial_x\varphi\|_{L^2(\mathbb{T})}^2\|u_\varepsilon\|_{L^n([0,T]\times\mathbb{T})}^n,\end{aligned}\tag{4.3.13}$$

where we used (4.2.12) in the second inequality, which is integrable by Consequence 4.2.1 (v) and thus M_ε is a square integrable martingale. Since $(\widetilde{M}_\varepsilon,(\tilde{\chi}_\varepsilon^{(j)})_{j\in\mathbb{N}},\tilde{u}_\varepsilon,(\tilde{\beta}_\varepsilon^{(k)})_{k\in\mathbb{Z}})$ has the same distribution as $(M_\varepsilon,(\mathbf{1}_{A_j})_{j\in\mathbb{N}},u_\varepsilon,(\beta^{(k)})_{k\in\mathbb{Z}})$ by Corollary 4.3.2, we obtain that

$$\begin{aligned}&\tilde{\mathbb{E}}\big[(\widetilde{M}_\varepsilon(t)-\widetilde{M}_\varepsilon(s))\tilde{\rho}_\varepsilon\big] = 0,\\ &\tilde{\mathbb{E}}\Big[\Big(\widetilde{M}_\varepsilon^2(t)-\widetilde{M}_\varepsilon^2(s)-\tilde{\chi}_\varepsilon^{(j)}\sum_{k\in\mathbb{Z}}\int_s^t\langle\sigma_{k,\varepsilon}q(\tilde{u}_\varepsilon),\partial_x\varphi\rangle^2\,\mathrm{d}\tau\Big)\tilde{\rho}_\varepsilon\Big] = 0,\\ &\tilde{\mathbb{E}}\Big[\Big(\widetilde{M}_\varepsilon(t)\tilde{\beta}_\varepsilon^{(k)}(t)-\widetilde{M}_\varepsilon(s)\tilde{\beta}_\varepsilon^{(k)}(s)-\tilde{\chi}_\varepsilon^{(j)}\int_s^t\langle\sigma_{k,\varepsilon}q(\tilde{u}_\varepsilon),\partial_x\varphi\rangle\,\mathrm{d}\tau\Big)\tilde{\rho}_\varepsilon\Big] = 0.\end{aligned}\tag{4.3.14}$$

Next, we note that by (4.3.13) and the Burkholder–Davis–Gundy inequality

$$\begin{aligned}\widetilde{\mathbb{E}}\Big[\sup_{0\le t\le T}|\widetilde{M}_\varepsilon(t)|^\rho\Big] &= \mathbb{E}\Big[\sup_{0\le t\le T}|M_\varepsilon(t)|^\rho\Big]\\ &\lesssim_{\rho,\Lambda}\|\partial_x\varphi\|^\rho_{L^2(\mathbb{T})}\,\mathbb{E}\Big[\mathbf{1}_{A_j}\|u_\varepsilon\|^{\frac{\rho n}{2}}_{L^n([0,T]\times\mathbb{T})}\Big],\end{aligned}\tag{4.3.15}$$

which is uniformly in ε bounded by (4.2.63) for $\rho\in(2,\frac{14}{n})$. Using (4.3.13) and (4.2.63) again, we also obtain that

$$\widetilde{\mathbb{E}}\bigg[\bigg(\widetilde{\chi}^{(j)}_\varepsilon\sum_{k\in\mathbb{Z}}\int_0^T\langle\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle^2\,\mathrm{d}t\bigg)^{\frac{\rho}{2}}\bigg]\tag{4.3.16}$$

and more simply

$$\begin{aligned}&\widetilde{\mathbb{E}}\bigg[\bigg(\widetilde{\chi}^{(j)}_\varepsilon\int_0^T|\langle\sigma_{k,\varepsilon}q(\widetilde{u}_\varepsilon),\partial_x\varphi\rangle|\,\mathrm{d}t\bigg)^\rho\bigg]\\ &\le\big(\|\sigma_{k,\varepsilon}\|_{L^\infty(\mathbb{T})}\|\partial_x\varphi\|_{L^\infty(\mathbb{T})}\big)^\rho\,\mathbb{E}\Big[\mathbf{1}_{A_j}\|u_\varepsilon\|^{\frac{\rho n}{2}}_{L^{\frac{n}{2}}([0,T]\times\mathbb{T})}\Big]\end{aligned}\tag{4.3.17}$$

are uniformly in ε bounded for these values of ρ. Since also

$$\widetilde{\mathbb{E}}\Big[\sup_{0\le t\le T}|\widetilde{\beta}^{(k)}_\varepsilon(t)|^\upsilon\Big]=\mathbb{E}\Big[\sup_{0\le t\le T}|\beta^{(0)}(t)|^\upsilon\Big]<\infty\tag{4.3.18}$$

for all ε and $\upsilon\in(1,\infty)$, we can use uniform integrability of the random variables in (4.3.14) and the almost sure convergences $\widetilde{M}_\varepsilon(t)\to\widetilde{M}(t)$, $\widetilde{\rho}_\varepsilon\to\widetilde{\rho}$, Lemma 4.3.6 (vi) and (vii), $\widetilde{\beta}^{(k)}_\varepsilon\to\widetilde{\beta}^{(k)}$ in $\mathfrak{X}_{\mathrm{BM}}$ as well as $\widetilde{\chi}^{(j)}_\varepsilon\to\mathbf{1}_{\widetilde{A}_j}$ in $\mathfrak{X}_{\mathrm{ind}}$ as $\varepsilon\searrow0$, to conclude

$$\begin{aligned}&\widetilde{\mathbb{E}}\big[(\widetilde{M}(t)-\widetilde{M}(s))\widetilde{\rho}\big]=0,\\ &\widetilde{\mathbb{E}}\bigg[\bigg(\widetilde{M}^2(t)-\widetilde{M}^2(s)-\mathbf{1}_{\widetilde{A}_j}\sum_{k\in\mathbb{Z}}\int_s^t\langle\sigma_kq(\widetilde{u}),\partial_x\varphi\rangle^2\,\mathrm{d}\tau\bigg)\widetilde{\rho}\bigg]=0,\\ &\widetilde{\mathbb{E}}\bigg[\bigg(\widetilde{M}(t)\widetilde{\beta}^{(k)}(t)-\widetilde{M}(s)\widetilde{\beta}^{(k)}(s)-\mathbf{1}_{\widetilde{A}_j}\int_s^t\langle\sigma_kq(\widetilde{u}),\partial_x\varphi\rangle\,\mathrm{d}\tau\bigg)\widetilde{\rho}\bigg]=0.\end{aligned}\tag{4.3.19}$$

An application of the monotone class theorem [22, Theorem 2.12.9] yields that (4.3.19) holds for any $\widetilde{\rho}$, which is bounded and measurable with respect to the σ-field generated by random variables of the form

$$\phi\big((\widetilde{\chi}^{(j)})_{j\in\mathbb{N}},\widetilde{u},(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}}\big)$$

with ϕ as in (4.3.12) continuous and bounded. Arguing as in Remark 2.5.11 from Chapter 2 one finds that this σ-field coincides with $\widetilde{\mathfrak{G}}_s$. Another application of Vitali's convergence theorem, using continuity in time of the random variables in (4.3.19) and the moment estimates (4.3.15)–(4.3.18) once more yields that (4.3.19) holds also for bounded, $\widetilde{\mathfrak{F}}_s$-measurable $\widetilde{\rho}$. Consequently, an application of [82, Proposition A.1] leads to (4.3.10).

□

Before completing the proof of Theorem 4.1.6, we deduce versions of the a-priori estimates, which we derived for u_ε, also for $\widetilde{u}$.

Lemma 4.3.8. *We assume that* $p \in (n+4, 7)$, $r \in (\frac{n+4}{2}, \frac{7}{2})$, $\gamma \in (0, \frac{1}{2})$, $\mu \in (\frac{n+4}{n+2}, \frac{7}{n+2})$, $\nu \in (1, \frac{7}{n+4})$ *and* $\alpha \in (-1, 2-n)$.

(i) *The estimates* (4.1.8) *and* (4.1.9) *hold, whenever their right-hand side is finite.*

(ii) *The estimate* (4.1.10) *holds with* $p_{\mu,\nu} = \max\{\mu(n+2), \nu(n+4)\}$, *if the right-hand side is finite.*

(iii) *We have almost surely* $\widetilde{u}^{\frac{\alpha+n+1}{4}} \in L^4(0,T;W^{1,4}(\mathbb{T}))$, $\widetilde{u}^{\frac{\alpha+n+1}{2}} \in L^2(0,T;H^2(\mathbb{T}))$ *and it holds* (4.1.11), *whenever its right-hand side is finite.*

4

Proof. For (i), we only verify (4.1.9), because (4.1.8) can be derived analogously. Since almost surely $\widetilde{u}_\varepsilon \to \widetilde{u}$ in $\mathfrak{X}_{\text{Sobolev}}$, we conclude

$$\begin{aligned}
&\widetilde{\mathbb{E}}\big[\|\partial_x \widetilde{u}\|^r_{L^r([0,T]\times\mathbb{T})}\big] \\
&\quad\le \widetilde{\mathbb{E}}\Big[\liminf_{\varepsilon\searrow 0} \|\partial_x \widetilde{u}_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\Big] \\
&\quad\le \liminf_{\varepsilon\searrow 0} \widetilde{\mathbb{E}}\big[\|\partial_x \widetilde{u}_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\big] \\
&\quad= \liminf_{\varepsilon\searrow 0} \mathbb{E}\big[\mathbb{E}\big[\|\partial_x u_\varepsilon\|^r_{L^r([0,T]\times\mathbb{T})}\,\big|\,\mathfrak{F}_0\big]\big] \\
&\quad\lesssim_{n,r,\Lambda,T} \liminf_{\varepsilon\searrow 0} \mathbb{E}\big[(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon)^{4-r}\big(\|u_0\|^{2r-n-4}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^{2r-4}_{\mathcal{M}(\mathbb{T})} + \varepsilon^{2r-n-4}\big)\big] \\
&\quad= \mathbb{E}\big[\|u_0\|^{r-n}_{\mathcal{M}(\mathbb{T})} + \|u_0\|^r_{\mathcal{M}(\mathbb{T})}\big],
\end{aligned}$$

using Fatou's lemma and Lemma 4.2.4.

For (ii), we use Lemma 4.2.8 to conclude

$$\begin{aligned}
&\widetilde{\mathbb{E}}\Big[\big\|\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon\big\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big] \\
&\quad= \mathbb{E}\Big[\mathbf{1}_{A_j}\mathbb{E}\Big[\|u_\varepsilon\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\,\Big|\,\mathfrak{F}_0\Big]\Big] \qquad (4.3.20)\\
&\quad\lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \mathbb{E}\Big[\mathbf{1}_{A_j}\Big(\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{(n-1-\frac{n^2}{p_{\mu,\nu}})\nu} + \big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{(n+1)\nu}\Big)\Big],
\end{aligned}$$

yielding a uniform bound on $\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon$ in $L^\nu(\widetilde{\Omega}; W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T})))$. Hence, up to taking another subsequence, we can assume that $\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon$ admits a weak limit in the space $L^\nu(\Omega; W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T})))$. Since almost surely $\widetilde{\chi_\varepsilon}^{(j)} \to \mathbf{1}_{\widetilde{A}_j}$ in $\mathfrak{X}_{\text{ind}}$ and $\widetilde{u}_\varepsilon \to \widetilde{u}$ in $\mathfrak{X}_{\text{power}}$, we also have $\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon \to \mathbf{1}_{\widetilde{A}_j}\widetilde{u}$ in $L^p([0,T]\times\mathbb{T})$. Hence, using Vitali's convergence theorem and (4.2.63), we deduce that

$$\mathbb{E}\big[\big\|\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon - \mathbf{1}_{\widetilde{A}_j}\widetilde{u}\big\|^p_{L^p([0,T]\times\mathbb{T})}\big] \to 0,$$

and therefore the weak limit of $\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon$ has to be $\mathbf{1}_{\widetilde{A}_j}\widetilde{u}$. By lower semicontinuity of the norm with respect to weak convergence, Fatou's lemma and (4.3.20), we conclude that

$$\widetilde{\mathbb{E}}\Big[\mathbf{1}_{\widetilde{A}_j}\|\widetilde{u}\|^\nu_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big]$$

$$\leq \liminf_{\varepsilon\searrow 0}\widetilde{\mathbb{E}}\Big[\big\|\widetilde{\chi_\varepsilon}^{(j)}\widetilde{u}_\varepsilon\big\|^{\nu}_{W^{\gamma,\frac{2\nu}{2-\nu}}(0,T;W^{-3,\mu}(\mathbb{T}))}\Big]$$
$$\lesssim_{\gamma,n,\mu,\nu,\Lambda,T} \liminf_{\varepsilon\searrow 0}\mathbb{E}\Big[\mathbf{1}_{A_j}\Big(\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{(n-1-\frac{n^2}{p_{\mu,\nu}})\nu}+\big(\|u_0\|_{\mathcal{M}(\mathbb{T})}+\varepsilon\big)^{(n+1)\nu}\Big)\Big]$$
$$=\mathbb{E}\Big[\mathbf{1}_{A_j}\Big(\|u_0\|^{(n-1-\frac{n^2}{p_{\mu,\nu}})\nu}_{\mathcal{M}(\mathbb{T})}+\|u_0\|^{(n+1)\nu}_{\mathcal{M}(\mathbb{T})}\Big)\Big].$$

It remains to sum over $j\in\mathbb{N}$ to obtain (4.1.10). Analogously, (iii) follows from Proposition 4.2.2. □

Proof of Theorem 4.1.6. We first show that $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$, $\widetilde{\mathfrak{F}}$, $(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}}$ and $\widetilde{u}$ indeed constitute a very weak martingale solution to (4.1.5) in the sense of Definition 4.1.4. To this end, we observe that almost surely $\widetilde{u}_\varepsilon(t)\geq 0$ for all $t\in[0,T]$ since it is equidistributed to u_ε. Consequently, its limit $\widetilde{u}$ in $\mathfrak{X}_{\mathrm{cont}}$ satisfies almost surely $\widetilde{u}(t)\geq 0$ as well and in particular $\widetilde{u}(t)$ is a non-negative measure on $\mathbb{T}$ for all $t\in[0,T]$ by [98, Theorem 6.22]. Moreover, since $\widetilde{u}(t)$ is $\widetilde{\mathfrak{F}}_t$-$\mathfrak{B}(H^{-1}(\mathbb{T}))$-measurable by the definition of $\widetilde{\mathfrak{F}}$, the real valued random variables $\langle\widetilde{u}(t),\varphi\rangle$ for $\varphi\in C^\infty(\mathbb{T})$ are $\widetilde{\mathfrak{F}}_t$-measurable. By approximation, the same holds when $\varphi\in C(\mathbb{T})$, so that $\widetilde{u}$ is $\widetilde{\mathfrak{F}}$-adapted with values in $(\mathcal{M}(\mathbb{T}),\mathcal{Z})$. Moreover, by the divergence form of (4.3.10), the total variation norm $\|\widetilde{u}(t)\|_{\mathcal{M}(\mathbb{T})}=\langle\widetilde{u}(t),\mathbf{1}_{\mathbb{T}}\rangle$ is constant in time. Using this, we show vague continuity of $\widetilde{u}$ in $\mathcal{M}(\mathbb{T})$ and assume for contradiction that $\widetilde{u}$ is not continuous at some time t. Then, there is a vaguely open neighborhood $\mathcal{O}\subset\mathcal{M}(\mathbb{T})$ of $\widetilde{u}(t)$ and a sequence $t_n\to t$ with $\widetilde{u}(t_n)\notin\mathcal{O}$ for all $n\in\mathbb{N}$. However, since $\widetilde{u}$ is bounded in $\mathcal{M}(\mathbb{T})$, a subsequence of $\widetilde{u}(t_n)$ converges vaguely to some $\nu\in\mathcal{M}(\mathbb{T})$ by the Banach–Alaoglu theorem and since $\widetilde{u}\in\mathfrak{X}_{\mathrm{cont}}$ it must hold $\nu=\widetilde{u}(t)$, contradicting $\widetilde{u}(t_n)\notin\mathcal{O}$. As demonstrated in Remark 4.1.7, the integrability conditions (4.1.6) follow from $\widetilde{u}$ lying in $\mathfrak{X}_{\mathrm{Lebesgue}}$ and $\mathfrak{X}_{\mathrm{Sobolev}}$. Lemma 4.3.3 states that $(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}}$ is a family of independent $\widetilde{\mathfrak{F}}$-Brownian motions. Since Lemma 4.3.4 and Lemma 4.3.7 imply that $\widetilde{u}$ satisfies (4.1.7), we showed that the quadruple $(\widetilde{\Omega},\widetilde{\mathfrak{A}},\widetilde{\mathbb{P}})$, $\widetilde{\mathfrak{F}}$, $(\widetilde{\beta}^{(k)})_{k\in\mathbb{Z}}$, $\widetilde{u}$ suffices Definition 4.1.4.

To verify that $\widetilde{u}(0)\sim u_0$, we observe that the sets of the form

$$\big\{\nu\in\mathcal{M}(\mathbb{T})\,\big|\,(\langle\nu,\varphi_1\rangle,\dots,\langle\nu,\varphi_l\rangle)\in A\big\}$$

for $\varphi_1,\dots,\varphi_l\in C^\infty(\mathbb{T})$ and Borel sets $A\subset\mathbb{R}^l$ form an intersection stable generator of $\mathcal{Z}$ by density of $C^\infty(\mathbb{T})$ in $C(\mathbb{T})$. Since $\widetilde{u}_\varepsilon(0)\sim u_{0,\varepsilon}$ as $H^{-1}(\mathbb{T})$-valued random variables, almost surely $u_{0,\varepsilon}\to u_0$ vaguely and $\widetilde{u}_\varepsilon(0)\to\widetilde{u}(0)$ in $H^{-1}(\mathbb{T})$ as $\varepsilon\searrow 0$, it holds

$$(\langle u_0,\varphi_1\rangle,\dots,\langle u_0,\varphi_l\rangle)\sim(\langle\widetilde{u}(0),\varphi_1\rangle,\dots,\langle\widetilde{u}(0),\varphi_l\rangle),$$

yielding that the laws of u_0 and $\widetilde{u}(0)$ on $(\mathcal{M}(\mathbb{T}),\mathcal{Z})$ coincide. Due to Corollary 4.3.2, we have $\widetilde{u}\in\mathfrak{X}_{\mathrm{cont}}\cap\mathfrak{X}_{\mathrm{Lebesgue}}\cap\mathfrak{X}_{\mathrm{Sobolev}}$. Together with Lemma 4.3.8 (i), (ii) this leads to (ii) and (iii). The claim in (i) was already checked at the beginning of this proof. Part (iv) is the content of Lemma 4.3.8 (iii). □

APPENDIX TO CHAPTER 4

4.A. PROJECTIVE LIMITS OF LOCALLY CONVEX VECTOR SPACES

We give a short summary of useful facts on topological vector spaces following [118, Section II.4, Section II.5]. In the following, we consider vector spaces over $\mathbb{R}$ equipped with a topology. Such a tuple is a topological vector space, if addition and scalar multiplication are continuous mappings. A Hausdorff topological vector space $\mathscr{X}$ is called locally convex, if every neighborhood of a point $x \in \mathscr{X}$ contains a convex neighborhood of x. Since balls are convex, every normed vector space is locally convex. Moreover, if $\mathscr{X}$ is a normed vector space, and we denote its topological dual by $\mathscr{X}^*$, the weak topology on $\mathscr{X}$ admits the collection of sets

4

$$\left\{y \in \mathscr{X} \,\middle|\, |\langle y-x, x_i^*\rangle| < \delta,\ i \in \{1,\dots,j\}\right\} \tag{4.A.1}$$

for $x_i^* \in \mathscr{X}^*$, $\delta > 0$ as a neighborhood basis at $x \in \mathscr{X}$. Since the set (4.A.1) is convex, $\mathscr{X}$ with its weak topology is locally convex as well. Now, let $\mathscr{X}_l$ for $l \in \mathbb{N}$ be a Hausdorff, locally convex space, such that $\mathscr{X}_{l+1} \hookrightarrow \mathscr{X}_l$ continuously. Then the projective limit of $(\mathscr{X}_l)_{l\in\mathbb{N}}$ is the space $\mathscr{X} = \bigcap_{l\in\mathbb{N}} \mathscr{X}_l$ equipped with the coarsest topology, such that each of the embeddings $\mathscr{X} \hookrightarrow \mathscr{X}_l$ is continuous, and is itself a Hausdorff, locally convex topological vector space again.

Lemma 4.A.1. *Let K be a subset of the projective limit $\mathscr{X}$ of a sequence of Hausdorff, locally convex spaces $(\mathscr{X}_l)_{l\in\mathbb{N}}$. Then K is compact in the topology of $\mathscr{X}$, iff $K = \bigcap_{l\in\mathbb{N}} K_l$ for compact subsets K_l of $\mathscr{X}_l$.*

Proof. If K is compact with the topology of $\mathscr{X}$, it is also compact with the topology of $\mathscr{X}_l$ since the embedding $\mathscr{X} \hookrightarrow \mathscr{X}_l$ is continuous. The claim follows since $K = \bigcap_{l\in\mathbb{N}} K$, trivially. For the reverse implication, we note that $\mathscr{X}$ is homeomorphic to the subset

$$D = \left\{(x_l)_{l\in\mathbb{N}} \in \prod_{l\in\mathbb{N}} \mathscr{X}_l \,\middle|\, \forall i,j: x_i = x_j\right\}$$

of the topological product space $\prod_{l\in\mathbb{N}} \mathscr{X}_l$ by [118, p.52]. Denoting the homeomorphism by $f\colon D \to \mathscr{X}$, we notice that $K = f(D \cap \prod_{l\in\mathbb{N}} K_l)$. Because $\prod_{l\in\mathbb{N}} K_l$ is compact by Tychonoff's theorem, it suffices to show that $D \subset \prod_{l\in\mathbb{N}} \mathscr{X}_l$ is closed, since then K is compact as it is the image of a compact set under a continuous mapping. To do so, let $(x_l)_{l\in\mathbb{N}} \in \prod_{l\in\mathbb{N}} \mathscr{X}_l$ not in D, i.e. we assume that there are indices $i<j$ with $x_i \neq x_j$. Since $\mathscr{X}_j \subset \mathscr{X}_i$ and $\mathscr{X}_i$ is Hausdorff, there exist disjoint open neighborhoods $B_i, B_j \subset \mathscr{X}_i$ of x_i and x_j, respectively. By the continuity of the embedding $\mathscr{X}_j \hookrightarrow \mathscr{X}_i$, B_j is also open in $\mathscr{X}_j$. Hence, denoting by p_i, p_j the continuous projection from $\prod_{l\in\mathbb{N}} \mathscr{X}_l$ onto the i-th and j-th component, we have constructed the open neighborhood $p_i^{-1}(B_i) \cap p_j^{-1}(B_j)$ of $(x_l)_{l\in\mathbb{N}}$ which is disjoint from D. Hence, D is closed and the proof is finished. □

4.B. TIGHTNESS CRITERIA

Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space. We recall that a family $(Y_i)_{i\in\mathscr{I}}$ of mappings defined on Ω with values in a topological space $\mathscr{X}$ is called tight, if for every δ there is a compact set $K_\delta \subset \mathscr{X}$ such that

$$\mathbb{P}(\{Y_i \notin K_\delta\}) < \delta$$

for all $i \in \mathscr{I}$. For this definition to make sense, we only require $\{Y_i \notin K_\delta\} \in \mathfrak{A}$ for the compact sets K_δ, which is in line with the setting in [88]. This is the case when $(Y_i)_{i\in\mathscr{I}}$ is a family of random variables and $\mathscr{X}$ is Hausdorff, since then the compact sets K_δ are closed and in particular Borel measurable.

Lemma 4.B.1. *Let $\mathscr{X}$ be the projective limit of a sequence of Hausdorff, locally convex spaces $(\mathscr{X}_l)_{l\in\mathbb{N}}$ and $Y_i\colon \Omega \to \mathscr{X}$ for $i \in \mathscr{I}$ a random variable in each of the spaces $\mathscr{X}_l$. The family $(Y_i)_{i\in\mathscr{I}}$ is tight on $\mathscr{X}$ iff it is tight on $\mathscr{X}_l$ for each $l \in \mathbb{N}$.*

Proof. If $(Y_i)_{i\in\mathscr{I}}$ is tight on $\mathscr{X}$ it is also tight on $\mathscr{X}_l$ by continuity of the embedding $\mathscr{X} \hookrightarrow \mathscr{X}_l$. Conversely, if $(Y_i)_{i\in\mathscr{I}}$ is tight on $\mathscr{X}_l$ for each $l \in \mathbb{N}$, we can for $\delta > 0$ choose compact subsets $K_{\delta,l} \subset \mathscr{X}_l$ such that

$$\mathbb{P}(\{Y_i \notin K_{\delta,l}\}) < \frac{\delta}{2^l}$$

for all $i \in \mathscr{I}$. The set $K_\delta = \bigcap_{l\in\mathbb{N}} K_{\delta,l}$ is compact with the topology of $\mathscr{X}$ by Lemma 4.A.1 and since

$$\mathbb{P}(\{Y_i \notin K_\delta\}) \le \sum_{l\in\mathbb{N}} \mathbb{P}(\{Y_i \notin K_{\delta,l}\}) < \delta,$$

the family $(Y_i)_{i\in\mathscr{I}}$ is tight on $\mathscr{X}$. □

Using the same argument, one can also reduce tightness in a countable product of topological spaces to tightness in each of the separate spaces.

Lemma 4.B.2. *Let $\mathscr{X}^{(l)}$ be a topological space and $(Y_i^{(l)})_{i\in\mathscr{I}}$ be a family of $\mathscr{X}^{(l)}$-valued mappings defined on Ω for each $l \in \mathbb{N}$. If $(Y_i^{(l)})_{i\in\mathscr{I}}$ is tight on $\mathscr{X}^{(l)}$ for each $l \in \mathbb{N}$, then also the family $((Y_i^{(l)})_{l\in\mathbb{N}})_{i\in\mathscr{I}}$ lies tight on the topological product $\prod_{l\in\mathbb{N}} \mathscr{X}^{(l)}$.*

Proof. For $\delta > 0$, $l \in \mathbb{N}$ there are compact sets $K_{\delta,l} \subset \mathscr{X}^{(l)}$ such that

$$\mathbb{P}(\{Y_i^{(l)} \notin K_{\delta,l}\}) < \frac{\delta}{2^l}$$

for all $i \in \mathscr{I}$. The set $K_\delta = \prod_{l\in\mathbb{N}} K_{\delta,l}$ is compact by Tychonoff's theorem and

$$\mathbb{P}(\{(Y_i^{(l)})_{l\in\mathbb{N}} \notin K_\delta\}) \le \sum_{l\in\mathbb{N}} \mathbb{P}(\{Y_i^{(l)} \notin K_{\delta,l}\}) < \delta$$

yields the claim. □

Lastly, we also show that it suffices to show tightness locally on Ω.

Lemma 4.B.3. *Let $\mathscr{X}$ be a Hausdorff topological vector space, $(Y_i)_{i\in\mathscr{I}}$ a family of $\mathscr{X}$-valued random variables and $(A_j)_{j\in\mathbb{N}}$ a measurable partition of Ω. If $(\mathbf{1}_{A_j} Y_i)_{i\in\mathscr{I}}$ lies tight on $\mathscr{X}$ for every $j \in \mathbb{N}$, then $(Y_i)_{i\in\mathscr{I}}$ lies also tight on $\mathscr{X}$.*

Proof. For a given $\delta > 0$ we choose $J_0 \in \mathbb{N}$ such that

$$\sum_{j=J_0+1}^{\infty} \mathbb{P}\left(A_j\right) < \tfrac{\delta}{2}.$$

Since $(\mathbf{1}_{A_j} Y_i)_{i\in\mathscr{I}}$ lies tight on $\mathscr{X}$ for every $j \in \{1,\dots,J_0\}$, there exist compact sets $K_\delta^{(j)} \subset \mathscr{X}$ such that

$$\mathbb{P}\big(\{\mathbf{1}_{A_j} Y_i \notin K_\delta^{(j)}\}\big) < \tfrac{\delta}{2J_0}$$

for all $i \in \mathscr{I}$. Then, defining the compact set

$$K_\delta = \bigcup_{j=1}^{J_0} K_\delta^{(j)} \cup \{0\},$$

we can calculate that

$$\begin{aligned}\mathbb{P}\left(\{Y_i \notin K_\delta\}\right) &= \mathbb{P}\Big(\bigcup_{j\in\mathbb{N}} \{\mathbf{1}_{A_j} Y_i \notin K_\delta\}\Big) \le \sum_{j\in\mathbb{N}} \mathbb{P}\big(\{\mathbf{1}_{A_j} Y_i \notin K_\delta\}\big) \\ &\le \sum_{j=1}^{J_0} \mathbb{P}\big(\{\mathbf{1}_{A_j} Y_i \notin K_\delta^{(j)}\}\big) + \sum_{j=J_0+1}^{\infty} \mathbb{P}(A_j) < \delta\end{aligned}$$

for every $i \in \mathscr{I}$. $\square$

5

Well-posedness with an interface potential†

In this chapter, we prove existence and uniqueness of probabilistically strong solutions to (STFE) in the situation that u_0 is positive and bounded away from 0 until some stopping time $\sigma > 0$. This holds for any mobility exponent n and spatial dimension $d \geq 1$ as long as the initial value and the noise are sufficiently regular in space. Moreover, in the one-dimensional case and for $n \in [0,6)$ we show that the solution persists in the trace space and remains strictly positive if the effects of repulsive intermolecular forces are included in (STFE), so that the unique solution exits globally in time. In contrast to the previous chapters, the results of this chapter apply to the Itô and Stratonovich interpretation of the equation.

5.1. Introduction to Chapter 5

More generally, we consider fourth-order quasilinear stochastic PDEs of the form

$$\begin{cases} \mathrm{d}u + \operatorname{div}(m(u)\nabla\Delta u)\,\mathrm{d}t = \operatorname{div}(\Phi(u)\nabla u)\,\mathrm{d}t + \sum_{k\in\mathbb{N}} \operatorname{div}(g(u)\psi_k)\,\mathrm{d}\beta^{(k)}, \\ u(0) = u_0, \end{cases} \tag{5.1.1}$$

on the d-dimensional torus $\mathbb{T}^d$. Throughout this chapter, we assume that

$$m\colon (0,\infty) \to (0,\infty) \quad \text{and} \quad g,\Phi\colon (0,\infty) \to \mathbb{R} \quad \text{are smooth functions,}$$

and we only specify them for positive values of u, since we are interested in the situation in which (5.1.1) preserves positivity, and the initial value u_0 is a strictly positive function. Moreover, $(\psi_k)_{k\in\mathbb{N}}$ is a family of vector fields $\psi_k\colon \mathbb{T}^d \to \mathbb{R}^d$ and $(\beta^{(k)})_{k\in\mathbb{N}}$ a corresponding

†This chapter is based on the preprint [3]: A. Agresti, and M. Sauerbrey. "Well-posedness of the stochastic thin-film equation with an interface potential". In: *arXiv preprint arXiv:2403.12652* (2024).

family of independent standard Brownian motions. We remark that the stochastic PDE (5.1.1) is understood in the Itô sense.

The class of equations of the form (5.1.1) contains, in particular, the stochastic thin-film equation with an interface potential

$$\partial_t u = -\operatorname{div}(m(u)\nabla(\Delta u - \phi'(u))) + \operatorname{div}(m^{1/2}(u)\mathscr{W}) \tag{5.1.2}$$

as discussed in Subsection 1.4.2, which results from setting $g(u) = m^{1/2}(u)$ as well as $\Phi(u) = m(u)\phi''(u)$ in (5.1.1) and defining the sequences $(\psi_k)_{k\in\mathbb{N}}$ and $(\beta^{(k)})_{k\in\mathbb{N}}$ in a way such that the time derivative of $\sum_{k\in\mathbb{N}} \psi_k \beta^{(k)}$ is the temporally white Gaussian noise $\mathscr{W}$. Then (5.1.1) becomes indeed the Itô interpretation of (5.1.2), but we stress that under reasonable symmetry conditions on the noise $\mathscr{W}$ also the Stratonovich interpretation of (5.1.2) can be cast into the form (5.1.1) by adjusting the coefficient Φ, see [107, Remark 2.1], [75, Appendix A] and the comments below (5.1.30).

The main results of this chapter can be summarized as follows:

5

- Local well-posedness and blow-up criteria for (5.1.1) in any space dimension—Theorem 5.1.6 and Proposition 5.1.8.
- Global well-posedness for (5.1.2) with a repulsive potential in one dimension—Theorems 5.1.12 and 5.1.13 (Itô & Stratonovich noise).

Let us stress that, since the derivations of the stochastic thin-film equation in [37] and [78], the results of the current chapter are the first ones on the *well-posedness* of (5.1.2). Indeed, only existence and *no* uniqueness results for solutions to (5.1.2) are available at the moment. In particular, at least if the interface potential is sufficiently singular near 0 as in [51] we can prove the uniqueness of global solutions, as conjectured in [51, Section 6] for (5.1.2) with $m(u) = u^2$. Interestingly, our results cover the typical example

$$\phi(u) = u^{-8} - u^{-2} + 1, \tag{5.1.3}$$

corresponding to conjoining and disjoining van der Waals forces modeled by the 6-12 *Lennard–Jones potential*, even in presence of a non-quadratic mobility, cf., Assumption 5.1.9. Of course, the main obstacle in obtaining well-posedness for the stochastic thin-film equation is the degeneracy of the leading order operator. However, in the presence of a repulsive potential, the solutions are strictly positive for all times and therefore the thin-film operator remains parabolic a posteriori. As a consequence, pathwise uniqueness is amenable to be proven.

Further novelties of our approach are:

- Global well-posedness for various mobility functions—see Assumption 5.1.9.
- Reduced regularity of the initial data—Theorem 5.1.6 and Remark 5.1.2.
- Instantaneous high-order regularization—Proposition 5.1.7.

The global well-posedness results of Theorems 5.1.12 and 5.1.13 hold for a wide range of mobility functions including power laws of the form $m(u) = u^n$ with $n \in [0, 6)$, but also

$m(u) = u^3 + \lambda^{3-n} u^n$ for some $\lambda > 0$. While the former is often assumed in the mathematical literature on (5.1.2) it usually serves as a simplification of the latter mobility function obtained from the lubrication approximation, see Subsection 1.1.1. Moreover, the literature on martingale solutions to the stochastic thin-film equation, including Chapters 2–4, imposes more restrictive assumptions on the exponent n.

Since our global well-posedness result relies on the positivity preserving mechanism of the effective interface potential ϕ, it comes at the expense of excluding the interesting case in which a contact line, i.e., a triple junction of liquid-solid and gas, is present. Nevertheless, if one is interested in the situation of a non-fully supported fluid film, Theorems 5.1.12 and 5.1.13 can be useful to construct solutions by taking $\phi \searrow 0$, as performed successfully in [75] for $m(u) = u^2$. This limit is of particular interest for future research since positive approximations of the thin-film equation are often compatible with formal a-priori estimates of the equation. Additionally, the uniqueness part of Theorems 5.1.12 and 5.1.13 has also numerical implications—for example in the case $m(u) = u^2$ in which a subsequence of a finite difference discretization of (5.1.2) was shown to converge in law to a solution in [51, Theorem 3.2]. Indeed, the pathwise uniqueness implies by the Gyöngy–Krylov lemma [25, Theorem 2.10.3] that the finite difference scheme converges in probability to the unique solution to (5.1.2) on the original probability space, at least for a subsequence. Since this solution is unique, the convergence holds also for the full sequence of approximations.

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Concerning the regularity of the initial data u_0, in all dimensions, we can allow $u_0 \in H^{1/2+\varepsilon,q}(\mathbb{T}^d)$ with $\varepsilon > 0$ arbitrary and $q \geq 2$ large, and thus below the energy level $H^1(\mathbb{T}^d)$. Moreover, in the case $d = 1$, we can choose $u_0 \in H^{1/2+\varepsilon}(\mathbb{T})$ for $\varepsilon > 0$. Proposition 5.1.7 shows that the regularity of the initial data only affects the regularity of u at times $t \sim 0$, while for $t > 0$ the solutions become smooth. More precisely, if $(\psi)_{k\in\mathbb{N}}$ are regular enough, then u becomes smooth in space regardless the regularity of u_0:

$$u \in C^{\theta,\infty}_{\text{loc}}((0,\sigma) \times \mathbb{T}^d) \text{ a.s. where } \sigma \text{ is the explosion time of } u. \tag{5.1.4}$$

Let us remark that $\sigma = \infty$ a.s. if $d = 1$ and ϕ is sufficiently singular, cf., Theorems 5.1.12 and 5.1.13.

The proof of these results relies on the following three advances. Firstly, we show *stochastic maximal regularity* estimates for thin film-type operators with strictly positive coefficients which depend only measurably on time. The latter is central in the derivation of suitable blow-up criteria for the quasilinear stochastic PDE (5.1.1). Secondly, we adapt the theory [5, 6] on quasilinear stochastic evolution equations to stochastic PDEs which are *a-priori* only degenerate parabolic. Thirdly, we estimate the *energy production* of (5.1.2) by the *α-entropy dissipation* for different α leading to new a-priori estimates for the stochastic thin-film equation with an interface potential. In particular, this allows us to deduce that the equation remains a.s. parabolic, *a-posteriori*.

The rest of this section is organized as follows. In Subsection 5.1.1 we discuss the local well-posedness of (5.1.1) in all dimensions $d \geq 1$, while in Subsection 5.1.2 we state the global well-posedness result for (5.1.2) in $d = 1$. Finally, we review the unexplained notation in Subsection 5.1.3.

5.1.1. Local well-posedness, regularity, blow-up criteria in any dimension

For the local well-posedness of (5.1.1), we use the well-posedness theory for quasilinear stochastic evolution equations developed by Agresti and Veraar in [5, 6]. To consider (5.1.1) as a quasilinear stochastic evolution equation we introduce the operators

$$A[u](f) = \operatorname{div}(m(u)\nabla\Delta f), \qquad F(u) = \operatorname{div}(\Phi(u)\nabla u), \qquad G_k(u) = \operatorname{div}(g(u)\psi_k), \quad (5.1.5)$$

where the latter gives rise to the operator on $\ell^2(\mathbb{N})$ defined by $G[u](e_k) = G_k(u)$ for the k-th unit vector $e_k \in \ell^2(\mathbb{N})$. If we introduce the cylindrical Brownian motion

$$W(t) = \sum_{k\in\mathbb{N}} e_k\beta_t^{(k)}$$

on $\ell^2(\mathbb{N})$, (5.1.1) takes the form of a quasilinear stochastic evolution equation [5, Eq. (1.1)]:

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$$\mathrm{d}u + A[u](u)\,\mathrm{d}t = F(u)\,\mathrm{d}t + G[u]\,\mathrm{d}W, \qquad u(0) = u_0. \quad (5.1.6)$$

We let the solution u lie for almost all times in the Bessel potential space $H^{s+2,q}(\mathbb{T}^d)$ while the deterministic and stochastic nonlinearities take values in the spaces $H^{s-2,q}(\mathbb{T}^d)$ and $H^{s,q}(\mathbb{T}^d;\ell^2(\mathbb{N}))$, respectively. The trajectory of the solution u is continuous in the *trace space* $B_{q,p}^{s+2-4\frac{1+\kappa}{p}}(\mathbb{T}^d)$ depending on the temporal integrability of u, which is described by the parameters p and κ. More precisely, we require u itself and the nonlinearities in (5.1.6) to be p-integrable in time with respect to the power weight $w_\kappa(t) = |t|^\kappa$. By the trace theory of anisotropic spaces, see e.g. [2] or [114, Section 3.4], this determines the above trace space as the optimal space for the initial value u_0. The use of temporal weights plays a central role in the proof of high-order regularity and of blow-up criteria, see Propositions 5.1.7 and 5.1.8 below.

In what follows, we employ the following condition on the parameters (p,κ,s,q).

Assumption 5.1.1 (Admissible parameters)**.** *The parameters $s \in (\frac{-1}{2},\infty)$, $p,q \in [2,\infty)$ and $\kappa \in [0,\infty)$ satisfy the following conditions:*

$$p \in (2,\infty), \kappa \in \left[0, \tfrac{p}{2}-1\right) \quad \text{or} \quad q = p = 2, \kappa = 0, \quad (5.1.7)$$

$$s + 2 - 4\tfrac{1+\kappa}{p} - \tfrac{d}{q} > 0, \quad (5.1.8)$$

$$s + 2 - 4\tfrac{1+\kappa}{p} > 1 - s. \quad (5.1.9)$$

We also say that (p,κ,s,q) are *admissible parameters* if they satisfy Assumption 5.1.1.

The restriction of the smoothness parameter $s > \frac{-1}{2}$ is made explicit in Assumption 5.1.1 because it is anyways implied by (5.1.9). We impose the condition $q \in [2,\infty)$ to make sure that the spaces $H^{s\pm2,q}(\mathbb{T}^d)$ are UMD-Banach spaces of type 2, see [84, Ex. 3.6.13, Prop. 4.2.15, Prop. 4.2.17 (1)]. Together with $p \in [2,\infty)$ and (5.1.7) this ensures that [5, Assumption 3.1] is satisfied. Condition (5.1.8) on the other hand implies that the trace space $B_{q,p}^{s+2-4\frac{1+\kappa}{p}}(\mathbb{T}^d)$ embeds into a space of Hölder continuous functions by

Sobolev embeddings. Therefore, we can control the oscillations of the coefficient $m(u)$ and the operator $A[u]$ behaves locally like the Bi-Laplacian, which is a key ingredient in proving stochastic maximal regularity of $A[u]$ required to apply [5, 6]. On the other hand, condition (5.1.9) is imposed to give sense to the product

$$m(u)\nabla\Delta f \tag{5.1.10}$$

in the definition of $A[u]$ using Sobolev pointwise multipliers. Indeed, for u in the trace space $B_{q,p}^{s+2-4\frac{1+\kappa}{p}}(\mathbb{T}^d)$ we expect smoothness $s+2-4\frac{1+\kappa}{p}$ from $m(u)$ and $s-1$ from $\nabla\Delta f$ where $f\in H^{s+2,q}(\mathbb{T}^d)$. If $s-1\geq 0$, there is no problem in giving sense to the pointwise multiplication (5.1.10) and the condition (5.1.9) is implied by (5.1.8). However if $s-1<0$, the function $\nabla\Delta f$ becomes a distribution and $m(u)$ must admit smoothness $1-s$ to define the product (5.1.10), which is expressed in (5.1.9).

Remark 5.1.2. We investigate which choices of parameters are compatible with Assumption 5.1.1. Firstly, we observe that for any $s\in(\frac{-1}{2},\infty)$ choosing $p,q\in[2,\infty)$ large and $\kappa=0$ guarantees that (5.1.7), (5.1.8) and (5.1.9) are satisfied. Therefore, for each $s\in(\frac{-1}{2},\infty)$ there is a feasible choice of parameters (p,κ,q) subject to Assumption 5.1.1. Secondly, we analyze how low we can choose the smoothness $s+2-4\frac{1+\kappa}{p}$ of the trace space, determining the roughness of initial values we can allow for (5.1.6). Condition (5.1.7) implies that

$$s+2-4\frac{1+\kappa}{p}\geq s,$$

which together with (5.1.9) yields

$$s+2-4\frac{1+\kappa}{p}>\frac{1}{2}.$$

We convince ourselves that it is possible to choose admissible parameters (p,κ,s,q) such that $s+2-4\frac{1+\kappa}{p}$ becomes arbitrarily close to $\frac{1}{2}$. If $d=1$, we can choose simply $p=q=2$, $\kappa=0$ for any $s>\frac{1}{2}$, resulting in the trace space $B_{2,2}^{s}(\mathbb{T})=H^{s}(\mathbb{T})$. If however $d\geq 2$, we choose $q>\frac{d}{s}$ for $s>\frac{1}{2}$, $p\in(2,\infty)$ and κ close to $\frac{p}{2}-1$. Then the smoothness of the trace space is close to s which can be chosen arbitrarily close to $\frac{1}{2}$.

We can now define local solutions to (5.1.1). Recall that we are interested in the situation where the solution remains positive for all times due to the possible loss of parabolicity of $A[u]$ for non-positive u.

Definition 5.1.3 (Local solution)**.** *Let (p,κ,s,q) be admissible parameters as in Assumption 5.1.1. Let $\sigma\colon\Omega\to[0,\infty]$ be a stopping time and $u\colon[\![0,\sigma)\!)\to H^{s+2,q}(\mathbb{T}^d)$ be a progressively measurable process. Then the tuple (u,σ) is called a* positive local (p,κ,s,q)-solution *to* (5.1.1)*, if there exists a sequence of stopping times $(\sigma_l)_{l\in\mathbb{N}}$ such that $0\leq\sigma_l\nearrow\sigma$ and a.s. for all $l\in\mathbb{N}$ we have*

$$u\in L^p(0,\sigma_l,w_\kappa;H^{s+2,q}(\mathbb{T}^d))\cap C([0,\sigma_l];B_{q,p}^{s+2-4\frac{1+\kappa}{p}}(\mathbb{T}^d)), \tag{5.1.11}$$

$$F(u)\in L^p(0,\sigma_l,w_\kappa;H^{s-2,q}(\mathbb{T}^d)),\quad G[u]\in L^p(0,\sigma_l,w_\kappa;H^{s,q}(\mathbb{T}^d;\ell^2(\mathbb{N}))), \tag{5.1.12}$$

$$\inf_{[0,\sigma_l]\times\mathbb{T}^d}u>0\quad(\textit{local positivity}), \tag{5.1.13}$$

and a.s. for all $t \in [0,\sigma_l]$:

$$u(t) - u(0) + \int_0^t A[u(r)](u(r))\,\mathrm{d}r = \int_0^t F(u(r))\,\mathrm{d}r + \int_0^t G[u(r)]\,\mathrm{d}W_r. \quad (5.1.14)$$

Here, the term local refers to the fact that the requirements (5.1.11)–(5.1.13) are demanded only away from the stopping time σ. Note that, due to the latter conditions and $L^p(w_\kappa) \hookrightarrow L^2$ for $\kappa < \frac{p}{2} - 1$, the deterministic and the stochastic integrals in (5.1.14) are well-defined as $H^{s-2,q}$- and $H^{s,q}$-valued Bochner and Itô integral, respectively (see, e.g., [109, Theorem 4.7 and Proposition 5.3] for Itô integration in type 2 spaces). We sometimes refer to a sequence $(\sigma_l)_{l\in\mathbb{N}}$ of stopping times as in Definition 5.1.3 as *localizing sequence* for (u,σ). Finally, we define positive maximal unique (p,κ,s,q)-solutions.

Definition 5.1.4 (Maximal unique positive solution)**.** *A positive local* (p,κ,s,q)*-solution* (u,σ) *to* (5.1.1) *is called* positive maximal unique (p,κ,s,q)-solution*, if for every positive local* (p,κ,s,q)*-solution* (v,τ) *to* (5.1.1)*, one has* $\tau \le \sigma$ *a.s. and* $u = v$ *a.s. on* $[0,\tau)$.

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It remains to specify the regularity of the noise coefficients needed for the local well-posedness of (5.1.1). In what follows, we use the parameters $s_\psi > -1/2$ and $q_\psi \in [2,\infty)$ to capture the smoothness of the noise.

Assumption 5.1.5(s_ψ, q_ψ) (Noise regularity—Local well-posedness)**.** *For* $s_\psi \in \mathbb{R}$ *and* $q_\psi \in [2,\infty)$*, we have*

$$\Big\| \Big(\sum_{k\in\mathbb{N}} \big|(1-\Delta)^{(1+s_\psi)/2}\psi_k\big|^2 \Big)^{1/2} \Big\|_{L^{q_\psi}(\mathbb{T}^d)} < \infty. \quad (5.1.15)$$

The above condition expresses that $(\psi_k)_{k\in\mathbb{N}} \in H^{1+s_\psi,q_\psi}(\mathbb{T}^d;\ell^2(\mathbb{N};\mathbb{R}^d))$ and holds in particular if $(\psi_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N};H^{1+s_\psi,q_\psi}(\mathbb{T}^d;\mathbb{R}^d))$, see [85, Theorem 9.2.10] and (5.2.4) below. For the local well-posedness of (5.1.1) formulated below, we only need Assumption 5.1.5(s_ψ, q_ψ) for some $s_\psi > -1/2$ and q_ψ large depending on d, see the discussion in Remark 5.1.2. In particular, this includes less regular noise than assumed in the literature on (global in time) martingale solutions to (5.1.2). Indeed, the most general condition treated so far, Assumption 4.1.2 from Chapter 4, implies that Assumption 5.1.5(s_ψ, q_ψ) holds for $s_\psi = 0$ and any $q_\psi < \infty$. A more restrictive condition on $(\psi_k)_{k\in\mathbb{N}}$ for the global well-posedness of (5.1.2) with $d = 1$ is given below in Assumption 5.1.11.

We are ready to state our first result on local well-posedness of (5.1.1) in all dimensions $d \ge 1$.

Theorem 5.1.6 (Local well-posedness)**.** *Let Assumptions 5.1.1 and 5.1.5*(s_ψ, q_ψ) *be satisfied with* $(s_\psi, q_\psi) = (s,q)$*, and*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d)) \quad \text{satisfies} \quad \inf_{\mathbb{T}^d} u_0 > 0 \text{ a.s.}$$

Then there exists a positive maximal unique (p,κ,s,q)*-solution* (u,σ) *to* (5.1.1) *as defined in Definitions 5.1.3 and 5.1.4 such that a.s.* $\sigma > 0$ *and*

$$u \in H^{\theta,p}_{\mathrm{loc}}([0,\sigma), w_\kappa; H^{s+2-4\theta,q}(\mathbb{T}^d)) \cap C((0,\sigma); B^{s+2-\frac{4}{p}}_{q,p}(\mathbb{T}^d))$$

for all $\theta \in [0,\frac{1}{2})$*, if* $p > 2$.

Next, we discuss how the regularity of the noise coefficients affects the regularity of solutions. Let us stress that the following result is independent of the regularity of the initial data u_0.

Proposition 5.1.7 (Instantaneous regularization)**.** *Let the assumptions of Theorem 5.1.6 be satisfied, and (u,σ) be the corresponding positive maximal unique (p,κ,s,q)-solution to (5.1.1). Assume that Assumption 5.1.5(s_ψ,q_ψ) holds for some $s_\psi \geq s$ and all $q_\psi \in [2,\infty)$. Then*

$$u \in H^{\theta,r}_{\mathrm{loc}}(0,\sigma;H^{2+s_\psi-4\theta,\zeta}(\mathbb{T}^d)) \text{ a.s. for all } \theta \in [0,\tfrac{1}{2}),\ r,\zeta \in (2,\infty).$$

In particular $u \in C^{\theta_1,\theta_2}_{\mathrm{loc}}((0,\sigma)\times\mathbb{T}^d)$ a.s. for all $\theta_1 \in [0,\frac{1}{2})$ and $\theta_2 \in (0,2+s_\psi)$.

The above shows that, if Assumption 5.1.5(s_ψ,q_ψ) is satisfied for all $s_\psi > 0$ and $q_\psi \in [2,\infty)$, then the solution (u,σ) provided by Theorem 5.1.6 is smooth in space as claimed in (5.1.4). Moreover, the above regularization result for $s_\psi = 1$ will play an important role in the study of global well-posedness as it allows us to justify the integrations by parts when working on intervals (t_0,T) with $t_0 > 0$ even if $u_0 \notin H^1(\mathbb{T}^d)$.

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Now, we turn to the question of how to determine whether $\sigma = \infty$ a.s. or not. Usually, one needs to obtain a-priori estimates for solutions to the corresponding stochastic PDE in a sufficiently regular norm. In practice, the smoothness needed in the blow-up criteria reflects the one used for the initial data in local well-posedness results. In particular, the lower the regularity allowed from the initial data, the better the corresponding blow-up criteria. Since we allow for a leading order operator which degenerates near 0, we also need to assume a positivity condition in the following blow-up criterion.

Proposition 5.1.8 (Blow-up criteria)**.** *Let the assumptions of Theorem 5.1.6 be satisfied, and let (u,σ) be the corresponding positive maximal unique (p,κ,s,q)-solution to (5.1.1). Assume that Assumption 5.1.5(s_ψ,q_ψ) holds for some $s_\psi \geq s$ and all $q_\psi \in [2,\infty)$. Moreover, let (p_0,κ_0,s_0,q_0) be admissible exponents (cf., the comments below Assumption 5.1.1) satisfying $s_0 \leq s_\psi$ and set $\gamma_0 := s_0 + 2 - 4\frac{1+\kappa_0}{p_0}$. Then, for all $0 < \varepsilon < T < \infty$,*

$$\mathbb{P}\Big(\varepsilon < \sigma < T,\ \sup_{t\in[\varepsilon,\sigma\wedge T)} \|u(t)\|_{B^{\gamma_0}_{q_0,p_0}(\mathbb{T}^d)} < \infty,\ \inf_{[\varepsilon,\sigma\wedge T)\times\mathbb{T}^d} u > 0\Big) = 0.$$

The norm in the above blow-up criterion is well-defined even if $s_0 \gg s$ by Proposition 5.1.7. Moreover, due to (5.1.8) in the admissibility condition, $B^{\gamma_0}_{q_0,p_0} \hookrightarrow C^{\alpha_0}$ for some $\alpha_0 > 0$. Finally, if $d = 1$ and Assumption 5.1.5(s_ψ,q_ψ) holds with $s_\psi = 1$ and for all $q_\psi < \infty$, then one can choose $p_0 = q_0 = 2$, $\kappa_0 = 0$ and $s_0 = 1$ corresponding to the usual energy space for the thin-film equation.

The key point in the above result is the independence of the blow-up criteria on the original set (p,κ,s,q) of admissible parameters. Such independence is essentially a consequence of the instantaneous regularization of solutions as given in Proposition 5.1.7. Indeed, at any time $t > \varepsilon$, the regularity of $u|_{[\varepsilon,\sigma)}$ does not depend on the original admissible parameters, and thus one can restart (5.1.1) with any new set of admissible parameters (p_0,κ_0,s_0,q_0) as long they are compatible with the noise, i.e., $s_0 \leq s_\psi$ (see the proof of [7, Theorem 2.10]).

5.1.2. GLOBAL WELL-POSEDNESS IN ONE DIMENSION

We turn our attention to the global in time well-posedness of the stochastic thin-film equation (5.1.2) which is of the form (5.1.1) with $g(u) = m^{1/2}(u)$. This stochastic PDE admits several dissipated quantities, see Subsection 1.3.2. This allow us in one dimension to extend the local well-posedness and regularization results globally in time by means of the blow-up criteria of Proposition 5.1.8. In particular, as in many works on the stochastic thin-film equation, our analysis is centered around the *energy functional*

$$\mathscr{E}(u) = \int_{\mathbb{T}} \left[\tfrac{1}{2}|u_x|^2 + \phi(u)\right] \mathrm{d}x \tag{5.1.16}$$

of the solution u to (5.1.2). The key point is that the functional $\mathscr{E}$ estimates at the same time the norm $\|u\|^2_{H^1(\mathbb{T})}$ by virtue of mass conservation and the smallness of u due to the singularity of ϕ which we demand in Assumption 5.1.10 to obtain global well-posedness, see (5.1.22) and Subsection 1.4.2.

In general, it is challenging to close an a-priori estimate on (5.1.16), especially in the *nonlinear noise* case, where $m(u)$ is not proportional to u^2 and therefore $g(u) = m^{1/2}(u)$ is nonlinear. In [35] and Chapter 3 this is achieved for the Stratonovich interpretation of (5.1.2) with $\phi = 0$ by controlling the smallness of the solution leading to a film height which is positive a.e. for all times. In [51] however, a repulsive interface potential ϕ was used instead of the Stratonovich correction to close an a-priori estimate on (5.1.16) together with the *entropy functional*

$$\mathscr{H}_0(u) = \int_{\mathbb{T}} h_0(u)\,\mathrm{d}x, \qquad h_0(r) = \int_1^r \int_1^{r'} \frac{1}{m(r'')}\,\mathrm{d}r''\,\mathrm{d}r'$$

in the linear noise case $m(u) = u^2$. We extrapolate this approach using so-called α-*entropy functionals*

$$\mathscr{H}_\beta(u) = \int_{\mathbb{T}} h_\beta(u)\,\mathrm{d}x, \qquad h_\beta(r) = \int_1^r \int_1^{r'} \frac{(r'')^\beta}{m(r'')}\,\mathrm{d}r''\,\mathrm{d}r' \tag{5.1.17}$$

for $\beta \in (-1/2, 1)$ originally defined in [11] for $\beta = \alpha + n - 1$. Notably, while a key ingredient in [51] is the spatial discretization of (5.1.2) developed in [79, 127], which is compatible with the entropy estimate, an α-entropy consistent discretization of (5.1.2) is to the authors' knowledge not available and may depend on the specific choice of α. Thus, working directly with the maximal local solutions provided by Theorem 5.1.6 enables us to use a wider class of a-priori estimates and to cover many different cases of m. Specifically, we impose the following assumptions on the smooth function $m\colon (0,\infty) \to (0,\infty)$.

Assumption 5.1.9 (Mobility coefficient)**.** *There exist* $n \in \mathbb{R}$ *and* $\nu \in [0,6)$ *such that, for all* $r \in (0,\infty)$,

$$\lim_{r\searrow 0} m(r)/r^n \in (0,\infty), \tag{5.1.18}$$

$$\limsup_{r\to\infty} m(r)/r^\nu < \infty, \qquad \liminf_{r\to\infty} m(r) > 0, \tag{5.1.19}$$

$$|m'(r)| \lesssim m(r)/r, \qquad |m''(r)| \lesssim m(r)/r^2. \tag{5.1.20}$$

In this case, we call n and ν the *exponent of degeneracy* and *growth exponent* of m, respectively.

We remark that the exponent of degeneracy is uniquely determined by m, but if ν is a growth exponent any other $\tilde{\nu} \in (\nu, 6)$ is a growth exponent of m as well. While (5.1.18) expresses that $m(r)$ behaves like the power r^n near 0, the condition (5.1.19) bounds the growth of m and assumes non-degeneracy near ∞. The technical condition (5.1.20) assumes that m behaves under differentiation like a power-law.

One can readily check that the following examples of mobility coefficients satisfy Assumption 5.1.9.

(i) (Power laws) $m(r) = r^n$ for $n \in [0,6)$.

(ii) (Mixed powers) $m(r) = c_1 r^{n_1} + \ldots + c_J r^{n_J}$, $c_j \in (0,\infty)$, $n_j \in (-\infty, 6)$ provided that $\max_{j\in\{1,\ldots,J\}} n_j \geq 0$.

(iii) (Nonlinear Interpolation) Let m and $\tilde{m}$ be two mobility functions satisfying Assumption 5.1.9 with respective exponents of degeneracy $n, \tilde{n}$ and growth exponents $\nu, \tilde{\nu}$. Then, for all $\delta > 0$

$$m_\delta(r) = \frac{m(r)\tilde{m}(r)}{\delta m(r) + \tilde{m}(r)}$$

suffices Assumption 5.1.9 with exponent of degeneracy $\max\{n, \tilde{n}\}$ and growth exponent $\min\{\nu, \tilde{\nu}\}$.

As mentioned before, when deriving the stochastic thin-film equation using a lubrication approximation, one obtains the mobility function $m(u) = u^3 + \lambda^{3-n} u^n$ with $n \in [1,3]$ and $\lambda \geq 0$ depending on the boundary condition of the fluid velocity near the substrate. Since the physically relevant regime is $u \ll 1$ this is usually approximated by u^n in the mathematical literature which is covered by (i), but we can also cover the former mobility function with (ii). The class (iii) is of mathematical interest since these nonlinear interpolations of two mobility functions can be used to construct solutions to the (stochastic) thin-film equation as a limit of strictly positive solutions to regularized equations, see [15, Section 6] and Chapter 3. We impose a corresponding assumption on the effective interface potential given by a smooth function $\phi\colon (0,\infty) \to (0,\infty)$.

Assumption 5.1.10 (Interface potential)**.** *Let n be the exponent of degeneracy of the mobility function m as in Assumption 5.1.9. There exists $\vartheta > \max\{2, 6-2n\}$ and $c_0 \in (0,\infty)$ such that, for all $r \in (0,\infty)$,*

$$r^{-\vartheta} \lesssim \phi(r) \qquad \textit{and} \qquad r^{-\vartheta-2} - c_0 \lesssim \phi''(r) \lesssim r^{-\vartheta-2}. \tag{5.1.21}$$

The above assumption includes effective interface potentials of the form $\phi(u) = u^{-\vartheta} - u^{-2} + c_\vartheta$ with $\vartheta > \max\{2, 6-2n\}$ as long as c_ϑ is large enough to ensure that $\phi(r) > 0$ for all $r > 0$. For $\vartheta = 8$, this becomes (5.1.3) corresponding to the 6-12 *Lennard–Jones pair potential* for the van der Waals forces between the fluid and solid molecules. For $n \geq 2$

5

Assumption 5.1.10 reduces precisely to the assumption on the interface potential imposed in [51, Hypothesis (H2)]. In particular, the continuous version

$$\sup_{x\in\mathbb{T}} u^{(2-\vartheta)/2}(x) \lesssim \mathscr{E}(u) + \left(\int_{\mathbb{T}} u\,\mathrm{d}x\right)^{(2-\vartheta)/2} \tag{5.1.22}$$

of [51, Lemma 4.1] implies that a profile u is strictly positive, if its energy (5.1.16) is finite. Moreover, for $n<2$ the coefficient $m^{1/2}(u)$ is not differentiable at 0 anymore leading to increased production of energy by the noise for small film heights. Thus an improved control of the smallness of u is required which is reflected by the additional condition $\vartheta > 6-2n$. Finally, we state the main assumption on the noise which allows us to obtain global well-posedness of the stochastic thin-film equation in dimension one.

Assumption 5.1.11 (Noise regularity—Global well-posedness)**.** *We assume that* $(\psi_k)_{k\in\mathbb{N}}$ *satisfies*

$$\sum_{k\in\mathbb{N}} \|\psi_k\|^2_{W^{2,\infty}(\mathbb{T}^d;\mathbb{R}^d)} < \infty.$$

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Note that the above implies that Assumption 5.1.5(s_ψ, q_ψ) holds with $s_\psi = 1$ and for all $q_\psi < \infty$. We are ready to state our result on global well-posedness in one spatial dimension.

Theorem 5.1.12 (Global well-posedness in one dimension—Itô)**.** *Fix $d=1$ as well as $s\in(1/2,1]$. Suppose that Assumptions 5.1.9–5.1.11 are satisfied and*

$$u_0 \in L^0_{\mathscr{F}_0}(\Omega; H^s(\mathbb{T})) \quad \textit{satisfies} \quad \inf_{\mathbb{T}} u_0 > 0 \text{ a.s.} \tag{5.1.23}$$

Then there exists a unique progressively measurable process $u\colon [\![0,\infty)\!) \to H^{s+2}(\mathbb{T})$, such that, a.s.,

$$\inf_{[0,t)\times\mathbb{T}} u > 0 \quad \textit{for all } t<\infty, \tag{5.1.24}$$

$$u \in L^2_{\mathrm{loc}}([0,\infty); H^{s+2}(\mathbb{T})) \cap C([0,\infty); H^s(\mathbb{T})), \tag{5.1.25}$$

and, a.s. for all $t>0$ and $\varphi\in C^\infty(\mathbb{T})$,

$$\begin{aligned}\int_{\mathbb{T}} (u(t)-u_0)\varphi\,\mathrm{d}x = \int_0^t \langle \varphi_x m(u), (u_{xx}-\phi'(u))_x\rangle_{H^{1-s}(\mathbb{T})\times H^{s-1}(\mathbb{T})}\,\mathrm{d}r \\ - \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \varphi_x m^{1/2}(u)\psi_k\,\mathrm{d}x\,\mathrm{d}\beta^{(k)}.\end{aligned} \tag{5.1.26}$$

Finally, the solution u instantaneously regularizes in time and space:

$$u \in H^{\theta,r}_{\mathrm{loc}}(0,\infty; H^{3-4\theta,\zeta}(\mathbb{T})) \quad \textit{for all } \theta\in[0,\tfrac{1}{2}),\ r,\zeta\in(2,\infty), \tag{5.1.27}$$

$$u \in C^{\theta_1,\theta_2}_{\mathrm{loc}}((0,\infty)\times\mathbb{T}) \quad \textit{for all } \theta_1\in[0,\tfrac{1}{2}),\ \theta_2\in(0,3). \tag{5.1.28}$$

By Proposition 5.1.7, in case of more regular noise, the assertions (5.1.27)–(5.1.28) can be improved. It follows from the instantaneous regularization result of (5.1.27)–(5.1.28) that, in the PDE formulation of the stochastic thin-film equation (5.1.26), we can replace the term $\langle \varphi_x m(u), (u_{xx} - \phi'(u))_x \rangle_{H^{1-s}(\mathbb{T})\times H^{s-1}(\mathbb{T})}$ by

$$\int_{\mathbb{T}} \varphi_x m(u)(u_{xx} - \phi'(u))_x \, \mathrm{d}x \tag{5.1.29}$$

for the solution u, recovering the usual weak formulation.

Next, we turn our attention to the stochastic thin-film equation with noise in the Stratonovich form in one dimension:

$$\mathrm{d}u = -(m(u)(u_{xx} - \phi'(u))_x)_x \, \mathrm{d}t + \sum_{k\in\mathbb{N}} (m^{1/2}(u)\psi_k)_x \circ \mathrm{d}\beta^{(k)}, \quad u(0) = u_0. \tag{5.1.30}$$

Note that, as $d = 1$, the Stratonovich correction takes the form

$$\begin{aligned} &\tfrac{1}{4} \sum_{k\in\mathbb{N}} (m'(u) m^{-1/2}(u)\psi_k (m^{1/2}(u)\psi_k)_x)_x \\ &\quad = \tfrac{1}{8} \sum_{k\in\mathbb{N}} ((m'(u))^2 m^{-1}(u) u_x \psi_k^2)_x + \tfrac{1}{4} \sum_{k\in\mathbb{N}} (m'(u)\psi_k \psi_k')_x. \end{aligned}$$

Whenever $\sum_{k\in\mathbb{N}} \psi_k^2$ sums up to a constant C independent of x the above simplifies to

$$\tfrac{C}{8}((m'(u))^2 m^{-1}(u) u_x)_x,$$

which can be included in the second order term of (5.1.1) if one modifies Φ appropriately. Therefore, the local well-posedness result Theorem 5.1.6 holds also for the Stratonovich interpretation of (5.1.2) in this case. Since the aforementioned a-priori estimates on the α-entropy (5.1.17) and the energy (5.1.16) can be carried out analogously as for the Itô interpretation of (5.1.2), we obtain also the following result.

Theorem 5.1.13 (Global well-posedness in one dimension—Stratonovich)**.** *Fix $d = 1$ as well as $s \in (1/2, 1]$. Let u_0 be as in (5.1.23). Suppose that Assumptions 5.1.9–5.1.11 are satisfied, and that there exists $C > 0$ such that*

$$\sum_{k\in\mathbb{N}} \psi_k^2(x) \equiv C \quad \textit{for all } x \in \mathbb{T}. \tag{5.1.31}$$

Then there exists a unique progressively measurable process $u\colon [\![0,\infty)\!) \to H^{s+2}(\mathbb{T})$ satisfying (5.1.24), (5.1.25) and, for all $t > 0$ and $\varphi \in C^\infty(\mathbb{T})$,

$$\begin{aligned} \int_{\mathbb{T}} (u(t) - u_0)\varphi \, \mathrm{d}x = &\int_0^t \langle \varphi_x m(u), (u_{xx} - \phi'(u))_x \rangle_{H^{1-s}(\mathbb{T})\times H^{s-1}(\mathbb{T})} \, \mathrm{d}r \\ &- \tfrac{C}{8} \int_0^t \int_{\mathbb{T}} \varphi_x (m'(u))^2 m^{-1}(u) u_x \, \mathrm{d}x \, \mathrm{d}r \\ &- \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \varphi_x m^{1/2}(u)\psi_k \, \mathrm{d}x \, \mathrm{d}\beta^{(k)}. \end{aligned} \tag{5.1.32}$$

Moreover, the solution u enjoys the additional regularity (5.1.27) and (5.1.28).

The assumption (5.1.31) is for instance satisfied by the noise specified in [75, (H3)]. As for the Itô formulation of the stochastic thin-film equation, we can replace the integrand of the deterministic integral on the right-hand side of (5.1.32) by (5.1.29) for the solution u.

We moreover remark that in the case of Stratonovich noise additional cancellations occur, which allow for closing the (α-) entropy estimate and subsequently the energy estimate for (5.1.2) even if no interface potential is present, as carried out in [35] and Chapter 3. Moreover, the α-entropy function $h_\beta(r)$ behaves like $r^{\beta-n+2}$ near 0 and the latter can become as singular as the interface potential $\phi(r)$ for mobility functions with an exponent of degeneracy $n > 7/2$. Therefore, we expect preservation of positivity in the spirit of [11, Theorem 4.1 (iii)] and consequently also global well-posedness of (5.1.30) without the interface potential for sufficiently degenerating mobility functions. However, this lies beyond the scope of this chapter.

Lastly, we remark that global well-posedness in the physical dimension $d = 2$ seems out of reach using our approach. The main obstacle is that the blow-up criteria in Proposition 5.1.8 require an a-priori estimate at least in C^α for some $\alpha > 0$. However, the energy functional only provides an estimate in $H^1(\mathbb{T}^d)$, and the latter embeds in a space of Hölder continuous functions precisely if $d = 1$.

5

5.1.3. NOTATION FOR CHAPTER 5

We write $\mathbb{T}^d$ for the flat d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$. The usual Sobolev space on an open subset $\mathcal{O}$ of $\mathbb{R}^d$ or $\mathbb{T}^d$ is denoted by $W^{s,q}(\mathcal{O})$ for an integer smoothness index $s \in \mathbb{N}_0$. For the definition of the Besov spaces $B^s_{q,p}(\mathbb{R}^d)$ and Bessel-potential spaces $H^{s,q}(\mathbb{R}^d)$ the reader is referred to [115, Section 2.1.2] and for the periodic spaces $B^s_{q,p}(\mathbb{T}^d)$ and $H^{s,q}(\mathbb{T}^d)$ to [119, Section 3.5.4]. The Besov and Bessel-potential space on an open subset $\mathcal{O}$ of $\mathbb{R}^d$ or $\mathbb{T}^d$ is defined as the set of restrictions of functions from the Besov and Bessel-potential space on the whole space and equipped with the induced quotient norm. The corresponding spaces of vector fields $W^{s,q}(\mathcal{O};\mathbb{R}^d)$, $B^s_{q,p}(\mathcal{O};\mathbb{R}^d)$ and $H^{s,q}(\mathcal{O};\mathbb{R}^d)$ are defined as the d-fold direct sum of these spaces. In any of these situations the symbol H^s stands for $H^{s,2}$.

We write $L^p(S,\mu;\mathcal{X})$ for the Bochner space of strongly measurable, p-integrable $\mathcal{X}$-valued functions for a measure space (S,μ) and a Banach space $\mathcal{X}$ as defined in [84, Section 1.2b]. If $\mathcal{X} = \mathbb{R}$, we write $L^p(S,\mu)$ and if it is clear which measure we refer to we also leave out μ. Moreover, if S is countable and equipped with the counting measure we write $\ell^p(S)$ instead of $L^p(S)$. If on the other hand I is an open interval and w a density, we write $L^p(I,w;\mathcal{X})$ for $L^p(I,w\,\mathrm{d}t;\mathcal{X})$. In particular, we are interested in power weights of the form $w^s_\kappa(t) = |t-s|^\kappa$. The corresponding fractional Sobolev space with weight w^s_κ as defined in [5, Definition 2.2] is denoted by $H^{\theta,p}(I,w^s_\kappa;\mathcal{X})$ and $H^{\theta,p}_{\mathrm{loc}}(I,w^s_\kappa;\mathcal{X})$ as the intersection of $H^{\theta,p}(J,w^s_\kappa;\mathcal{X})$ for all intervals J, which are compactly contained in I. Whenever we write 'a,b' instead of an interval in the above spaces we mean the open interval $I=(a,b)$, e.g., $H^{\theta,p}(a,b,w^s_\kappa;\mathcal{X})$ stands for $H^{\theta,p}((a,b),w^s_\kappa;\mathcal{X})$.

If (S,d) is a metric space we write $C(S;\mathcal{X})$ for the continuous function and $C^\theta(S;\mathcal{X})$ for the subset of θ-Hölder continuous functions for $\theta \in (0,\infty)\setminus\mathbb{N}$, where we leave $\mathcal{X}$ out again if $\mathscr{X} = \mathbb{R}$. If I is an open interval, we define the anisotropic Hölder space

$$C^{\theta_1,\theta_2}(I \times \mathbb{T}^d) = C^{\theta_1}(I; C(\mathbb{T}^d)) \cap C(I; C^{\theta_2}(\mathbb{T}^d))$$

for $\theta_i \in (0,\infty) \setminus \mathbb{N}$, $i \in \{1,2\}$ and accordingly $C^{\rho_1-,\rho_2-}_{\mathrm{loc}}(I \times \mathbb{T}^d)$ as the intersection of the spaces $C^{\theta_1,\theta_2}(J \times \mathbb{T}^d)$ for all $\theta_i \in (0,\rho_i) \setminus \mathbb{N}$ and intervals J which are compactly contained in I.

To ease the notation of these spaces we introduce for a given quadruple (p,κ,s,q) the notation $X_0 = H^{s-2,q}(\mathbb{T}^d)$ and $X_1 = H^{s+2,q}(\mathbb{T}^d)$. Following [5, 6], we write

$$X_\theta := [X_0, X_1]_\theta, \qquad X^{\mathrm{Tr}}_{\kappa,p} := (X_0, X_1)_{1-\frac{1+\kappa}{p},p}, \qquad X^{\mathrm{Tr}}_p := X^{\mathrm{Tr}}_{0,p}$$

for $\theta \in (0,1)$ and p,κ as in (5.1.7) for the complex and real interpolation spaces of X_0 and X_1, see [103, Chapter 1 and Chapter 2] for a definition. In particular, it holds

$$X_\theta = H^{s+2-4\theta,q}(\mathbb{T}^d), \qquad X^{\mathrm{Tr}}_{\kappa,p} = B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d) \tag{5.1.33}$$

by [119, Section 3.6.1] meaning that the Banach spaces coincide as sets and carry equivalent norms.

We also fix throughout the chapter a filtered probability space $(\Omega,\mathfrak{A},\mathscr{F},\mathbb{P})$ carrying a family of independent $\mathscr{F}$-Brownian motions $(\beta^{(k)})_{k\in\mathbb{N}}$. We write $\mathbb{E}$ for the expectation on $(\Omega,\mathfrak{A},\mathbb{P})$ and $\mathscr{P}$ for the progressive σ-field. For a Banach space $\mathscr{X}$ we denote the subspaces of $\mathscr{F}_t$ and progressively measurable random variables in $L^p(\Omega;\mathscr{X})$ by $L^p_{\mathscr{F}_t}(\Omega;\mathscr{X})$ and $L^p_{\mathscr{P}}(\Omega;\mathscr{X})$, respectively. If additionally $\mathscr{Y}$ is another Banach space and H is a Hilbert space, we write $\mathscr{L}(\mathscr{X},\mathscr{Y})$ for the space of bounded linear operators and $\gamma(H,\mathscr{X})$ for the space of γ-radonifying operators, see [85, Chapter 9]. The latter plays a central role in the definition of the $\mathscr{X}$-valued stochastic integral [109]. For a stopping time $\tau\colon \Omega \to [s,T]$, we define

$$[\![s,\tau)\!) := \{(\omega,t) \in \Omega \times [s,T] \,\big|\, t < \tau(\omega)\}.$$

Let $(\mathscr{X}_t)_{t\in[s,T]}$ be a family of Banach spaces such that $\mathscr{X}_t$ consists of functions from $[s,T]$ to another Banach space $\mathscr{X}$ such that $f|_{[s,t]} \in \mathscr{X}_t$ for each $f \in \mathscr{X}_T$ and $\|f|_{[s,t]}\|_{\mathscr{X}_t}$ is increasing in t. Then we define $L^p_{\mathscr{P}}(\Omega;\mathscr{X}_\tau)$ as the restrictions of processes from $L^p_{\mathscr{P}}(\Omega;\mathscr{X}_T)$ to $[\![s,\tau)\!)$ and equip it with the norm

$$\|u|_{[\![s,\tau)\!)}\|_{L^p(\Omega;\mathscr{X}_\tau)} := \mathbb{E}\big[\|u|_{[s,\tau]}\|^p_{\mathscr{X}_\tau}\big]^{\frac{1}{p}},$$

which is well-defined by [5, Lemma 2.15]. In the situation that $\mathscr{X}_t = L^p((s,t),w^s_\kappa;\mathscr{X})$, we write $L^p_{\mathscr{P}}((s,\tau)\times\Omega,w^s_\kappa;\mathscr{X})$ for $L^p_{\mathscr{P}}(\Omega;\mathscr{X}_\tau)$.

For two quantities x and y, we write $x \lesssim y$, if there exists a universal constant C such that $x \le Cy$. If a constant depends on parameters $(p_1,\dots)$ we either mention it explicitly or indicate this by writing $C_{(p_1,\dots)}$ and correspondingly $x \lesssim_{(p_1,\dots)} y$ whenever $x \le C_{(p_1,\dots)} y$. Finally, we write $x \eqsim_{(p_1,\dots)} y$, whenever $x \lesssim_{(p_1,\dots)} y$ and $y \lesssim_{(p_1,\dots)} x$.

5.2. LOCAL WELL-POSEDNESS OF THIN-FILM TYPE EQUATIONS IN ANY DIMENSION

The purpose of this section is to show local well-posedness, blow-up criteria and instantaneous regularization for (5.1.1) as stated in Subsection 5.1.1. To this end, we fix in this subsection the smooth coefficients $m\colon (0,\infty) \to (0,\infty)$ and $g,\Phi\colon (0,\infty) \to \mathbb{R}$ and do not indicate if an implicit constant depends on them. Our strategy is to use the general theory for quasilinear parabolic stochastic evolution equations developed in [5, 6] and apply the results [5, Theorem 4.7] and [6, Theorem 4.9, Theorem 6.3] to (5.1.1). However, since we allow for a degeneracy of the operator $A[u]$ when u approaches 0, we need to consider regularized versions of (5.1.1) instead. Specifically, we fix a smooth and increasing function $\eta\colon \mathbb{R} \to \mathbb{R}$ with $\eta(r) = 1$ for $r \geq 2$ and $\eta(r) = 0$ for $r \leq 1$ and define $\eta_j(r) = \eta(jr)$ for $j \in \mathbb{N}$. Setting

$$m_j(r) = \eta_j(r)m(r) + (1-\eta_j(r)), \quad \Phi_j(r) = \eta_j(r)\Phi(r), \quad g_j(r) = \eta_j(r)g(r), \tag{5.2.1}$$

gives rise to smooth functions $m_j, \Phi_j, g_j : \mathbb{R} \to \mathbb{R}$. The new coefficient m_j is positive and bounded away from 0 and consequently the leading order operator of

$$\begin{cases} \mathrm{d}u + \operatorname{div}(m_j(u)\nabla\Delta u)\,\mathrm{d}t = \operatorname{div}(\Phi_j(u)\nabla u)\,\mathrm{d}t + \sum_{k\in\mathbb{N}} \operatorname{div}(g_j(u)\psi_k)\,\mathrm{d}\beta^{(k)}, \\ u(0) = u_0, \end{cases} \tag{5.2.2}$$

is non-degenerate, so that the theory for parabolic stochastic evolution equations developed in [5, 6] becomes applicable. Analogously to (5.1.5), we define

$$A^{(j)}[u](f) = \operatorname{div}(m_j(u)\nabla\Delta f), \qquad F^{(j)}(u) = \operatorname{div}(\Phi_j(u)\nabla u), \qquad G_k^{(j)}(u) = \operatorname{div}(g_j(u)\psi_k)$$

and $G^{(j)}[u](e_k) = G_k^{(j)}(u)$, so that (5.2.2) takes the form

$$\mathrm{d}u + A^{(j)}[u](u)\,\mathrm{d}t = F^{(j)}(u)\,\mathrm{d}t + G^{(j)}[u]\,\mathrm{d}W, \qquad u(0) = u_0, \tag{5.2.3}$$

of [5, Eq. (1.1)].

The rest of this section is organized as follows. In Subsection 5.2.1 we check the local Lipschitz conditions on the coefficients $A^{(j)}$, $F^{(j)}$ and $G^{(j)}$ from [5, Hypothesis (H')]. Subsection 5.2.2 is devoted to proving the stochastic maximal regularity of the linear problem

$$\mathrm{d}u + \operatorname{div}(a\nabla\Delta u)\,\mathrm{d}t = f\,\mathrm{d}t + g\,\mathrm{d}W, \quad u(0) = u_0,$$

with a positive and bounded coefficient $a : [0,\infty) \times \Omega \to B_{q,p}^{s+2-4\frac{1+\kappa}{p}}(\mathbb{T}^d)$, which allows us to prove local well-posedness and the blow-up criteria for (5.2.2) in Subsection 5.2.3. In Subsection 5.2.4 we show instantaneous regularization of solutions to (5.2.2) for sufficiently smooth noise. Finally, in Subsection 5.2.5 we transfer these statements to the original equation (5.1.1) and prove Theorem 5.1.6 and Propositions 5.1.7–5.1.8.

Before proceeding, we point out that Assumption 5.1.5(s_ψ, q_ψ) is equivalent to demanding that the operator $\Psi : \ell^2(\mathbb{N}) \to H^{1+s_\psi,q_\psi}(\mathbb{T}^d;\mathbb{R}^d)$ defined by $\Psi(e_k) = \psi_k$ is γ-*radonifying*, see [85, Chapter 9]. More generally, by the γ-*Fubini theorem* [85, Theorem

9.4.8] and the fact that $(1-\Delta)^{r/2} : H^{s+r,q}(\mathbb{T}^d) \to H^{s,q}(\mathbb{T}^d)$ is an isomorphism, it follows that

$$\gamma(\ell^2(\mathbb{N}), H^{s,q}(\mathbb{T}^d)) = H^{s,q}(\mathbb{T}^d; \ell^2(\mathbb{N})) \text{ with equivalent norms.} \tag{5.2.4}$$

In the above $\gamma(\ell^2(\mathbb{N}), H^{s,q}(\mathbb{T}^d))$ denotes the set of γ-radonifying operators from $\ell^2(\mathbb{N})$ to $H^{s,q}(\mathbb{T}^d)$, cf., [85, Definition 9.1.4]. The identification (5.2.4) in particular implies

$$\begin{aligned}\|\Psi\|_{\gamma(\ell^2(\mathbb{N}), H^{1+s_\psi,q_\psi}(\mathbb{T}^d;\mathbb{R}^d))} &\eqsim \|\psi\|_{H^{1+s_\psi,q_\psi}(\mathbb{T}^d;\ell^2(\mathbb{N};\mathbb{R}^d))} \\ &\eqsim \left\| \left(\sum_{k\in\mathbb{N}} \left|(1-\Delta)^{(1+s_\psi)/2}\psi_k\right|^2 \right)^{1/2} \right\|_{L^{q_\psi}(\mathbb{T}^d)}.\end{aligned}$$

Below, we will frequently use the equivalence (5.2.4) without further mentioning it. For more information on the role of γ-radonifying operators in the context of the L^p-theory of stochastic evolution equations, the reader is referred to [4–6, 28, 102, 108, 109, 113] and the references therein.

Additionally, we recall the shorthand notation

$$X_\theta = H^{s+2-4\theta,q}(\mathbb{T}^d), \qquad X^{\mathrm{Tr}}_{\kappa,p} = B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d)$$

for function spaces, cf., (5.1.33).

5.2.1. LOCAL LIPSCHITZIANITY OF THE REGULARIZED COEFFICIENTS

As laid out earlier, this subsection is devoted to showing local Lipschitz and growth estimates of the coefficients of (5.2.3). The main ingredients are composition and product estimates in Bessel-potential estimates which result from a frequency decomposition of the involved functions, see [124, Chapter 2] or [115, Chapter 4 and 5] for an introduction. To this end, we fix a quadruple (p,κ,s,q) and a sequence $(\psi_k)_{k\in\mathbb{N}}$ subject to Assumptions 5.1.1 and 5.1.5(s_ψ, q_ψ) with $(s_\psi, q_\psi) = (s,q)$ for the remainder of this subsection and allow for all constants to depend on this particular choice.

Lemma 5.2.1. *For all $j,n \in \mathbb{N}$ there exist constants $C_{j,n}, L_{j,n} \in (0,\infty)$ such that*

$$\|A^{(j)}[u]\|_{\mathscr{L}(X_1,X_0)} \le C_{j,n}\big(1+\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}}\big), \tag{5.2.5}$$

$$\|A^{(j)}[u] - A^{(j)}[v]\|_{\mathscr{L}(X_1,X_0)} \le L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}} \tag{5.2.6}$$

for all $u,v \in X^{\mathrm{Tr}}_{\kappa,p}$ with $\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}}, \|v\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le n$.

Proof. We first verify the local Lipschitz condition (5.2.6) and observe that

$$\|(A^{(j)}[u] - A^{(j)}[v])f\|_{X_0} \lesssim \|(m_j(u)-m_j(v))\nabla\Delta f\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}. \tag{5.2.7}$$

We distinguish several cases and start with $s-1=0$. Due to Hölder's inequality we can bound (5.2.7) further by

$$\|m_j(u)-m_j(v)\|_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta f\|_{L^q(\mathbb{T}^d;\mathbb{R}^d)} \lesssim \|m_j(u)-m_j(v)\|_{L^\infty(\mathbb{T}^d)}\|f\|_{H^{s+2,q}(\mathbb{T}^d)}$$

and it remains to use that $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow L^\infty(\mathbb{T}^d)$ by (5.1.8) together with local Lipschitz continuity of m_j to deduce (5.2.6).

Next, we assume that $s-1>0$, in which case we employ the paraproduct estimate [8, Proposition 4.1 (1)] to bound (5.2.7) by

$$\begin{aligned} &\|m_j(u)-m_j(v)\|_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta f\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad + \|m_j(u)-m_j(v)\|_{H^{s-1,l}(\mathbb{T}^d)}\|\nabla\Delta f\|_{L^r(\mathbb{T}^d;\mathbb{R}^d)} \end{aligned} \tag{5.2.8}$$

for $l\in(1,\infty)$, $r\in(1,\infty]$ subject to

$$\tfrac{1}{q} = \tfrac{1}{l} + \tfrac{1}{r}. \tag{5.2.9}$$

We claim that we can choose l,r such that additionally $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow H^{s-1,l}(\mathbb{T}^d)$ and the embedding $H^{s-1,q}(\mathbb{T}^d) \hookrightarrow L^r(\mathbb{T}^d)$ holds. Indeed, if $s-1-\frac{d}{q}>0$, the choice $l=q$, $r=\infty$ is feasible. Otherwise, if $s-1-\frac{d}{q}\le 0$, we can choose r such that

$$s-1-\tfrac{d}{q} > \tfrac{-d}{r} > s-1-\tfrac{d}{q}-\varepsilon$$

for $\varepsilon>0$. Then, by (5.2.9)

$$\tfrac{d}{l} > s-1-\varepsilon \quad\Longleftrightarrow\quad s-1-\tfrac{d}{l}<\varepsilon$$

and by (5.1.8) we can choose ε smaller than

$$s+2-4\tfrac{1+\kappa}{p}-\tfrac{d}{q} \tag{5.2.10}$$

resulting in the embedding $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow H^{s-1,l}(\mathbb{T}^d)$. Thus, we can estimate (5.2.8) by

$$\begin{aligned} &\big(\|m_j(u)-m_j(v)\|_{L^\infty(\mathbb{T}^d)} + \|m_j(u)-m_j(v)\|_{H^{s-1,l}(\mathbb{T}^d)}\big)\|\nabla\Delta f\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad \lesssim \|m_j(u)-m_j(v)\|_{L^\infty(\mathbb{T}^d)\cap H^{s-1,l}(\mathbb{T}^d)}\|f\|_{H^{s+2,q}(\mathbb{T}^d)}. \end{aligned}$$

The desired estimate (5.2.6) follows by local Lipschitz continuity of $u\mapsto m_j(u)$ in the space $L^\infty(\mathbb{T}^d)\cap H^{s-1,l}(\mathbb{T}^d)$, see [115, Theorem 1, p.373], together with the embedding $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow L^\infty(\mathbb{T}^d)\cap H^{s-1,l}(\mathbb{T}^d)$.

Lastly, we consider the case $s-1<0$, in which also the condition (5.1.9) becomes relevant. Then, an application of [8, Proposition 4.1 (3)] yields the bound

$$\|m_j(u)-m_j(v)\|_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta f\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|m_j(u)-m_j(v)\|_{H^{\tau,\zeta}(\mathbb{T}^d)}\|\nabla\Delta f\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \tag{5.2.11}$$

on the right-hand side of (5.2.7) for any $\tau>\max\{\frac{d}{\zeta},1-s\}$ and $\zeta\in[q',\infty)$. If we can choose τ,ζ such that $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow H^{\tau,\zeta}(\mathbb{T}^d)$ the claimed estimate (5.2.6) follows as in the previous case. But we can choose simply $\zeta=q$ and τ slightly smaller than

$$s+2-4\frac{1+\kappa}{p}$$

by (5.1.8) and (5.1.9). The growth condition (5.2.5) follows in all cases by choosing $v=0$ in (5.2.6) and noticing that $A^{(j)}[0]=\Delta^2$, completing the proof. □

Lemma 5.2.2. *For all $j,n \in \mathbb{N}$ there exist constants $C_{j,n}, L_{j,n} \in (0,\infty)$ such that*

$$\|F^{(j)}(u)\|_{X_0} \le C_{j,n}\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}}, \tag{5.2.12}$$

$$\|F^{(j)}(u) - F^{(j)}(v)\|_{X_0} \le L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}} \tag{5.2.13}$$

for all $u,v \in X^{\mathrm{Tr}}_{\kappa,p}$ with $\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}}, \|v\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le n$.

Proof. We start again with (5.2.13) and calculate

$$\begin{aligned}\|F^{(j)}(u) - F^{(j)}(v)\|_{X_0} &\lesssim \|\Phi_j(u)\nabla u - \Phi_j(v)\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \\ &\le \|\Phi_j(u)(\nabla u - \nabla v)\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|(\Phi_j(u)-\Phi_j(v))\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}.\end{aligned}$$

Since these terms compare structurally to (5.2.7), we continue as in the proof of Lemma 5.2.1. If $s-1=0$, we proceed by estimating

$$\begin{aligned}\|\Phi_j(u)(\nabla u - \nabla v)\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} &\lesssim \|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)}\|\nabla u - \nabla v\|_{L^q(\mathbb{T}^d;\mathbb{R}^d)} \\ &\lesssim \|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)}\|u-v\|_{H^{s,q}(\mathbb{T}^d)} \lesssim \|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}},\end{aligned}$$

where we used in the last step that $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow X_{1/2}$. Moreover, we have

$$\begin{aligned}\|(\Phi_j(u)-\Phi_j(v))\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} &\lesssim \|\Phi_j(u)-\Phi_j(v)\|_{L^\infty(\mathbb{T}^d)}\|\nabla v\|_{L^q(\mathbb{T}^d;\mathbb{R}^d)} \\ &\lesssim \|\Phi_j(u)-\Phi_j(v)\|_{L^\infty(\mathbb{T}^d)}\|v\|_{X^{\mathrm{Tr}}_{\kappa,p}}\end{aligned}$$

by the same argument. The desired estimate (5.2.13) follows by local Lipschitz continuity of Φ_j and the embedding $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow L^\infty(\mathbb{T}^d)$ by (5.1.8).

If instead $s-1>0$, we choose l,r again such that (5.2.9) holds and additionally $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow H^{s-1,l}(\mathbb{T}^d)$ and $H^{s-1,q}(\mathbb{T}^d) \hookrightarrow L^r(\mathbb{T}^d)$. Then [8, Proposition 4.1 (1)] yields that

$$\begin{aligned}&\|\Phi_j(u)(\nabla u - \nabla v)\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad\lesssim \|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)}\|\nabla u - \nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|\Phi_j(u)\|_{H^{s-1,l}(\mathbb{T}^d)}\|\nabla u - \nabla v\|_{L^r(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad\lesssim \big(\|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)} + \|\Phi_j(u)\|_{H^{s-1,l}(\mathbb{T}^d)}\big)\|\nabla u - \nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad\le L_{j,n}\|u-v\|_{H^{s,q}(\mathbb{T}^d)},\end{aligned}$$

where in the last step we used $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow L^\infty(\mathbb{T}^d) \cap H^{s-1,l}(\mathbb{T}^d)$ together with local Lipschitz continuity of $u \mapsto \Phi_j(u)$ in the latter space, see again [115, Theorem 1, p.373]. Because of $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow X_{1/2}$, we obtain

$$\|\Phi_j(u)(\nabla u - \nabla v)\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \lesssim L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}}. \tag{5.2.14}$$

Analogously, we derive that

$$\begin{aligned}&\|(\Phi_j(u)-\Phi_j(v))\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad\lesssim \|\Phi_j(u)-\Phi_j(v)\|_{L^\infty(\mathbb{T}^d)}\|\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|\Phi_j(u)-\Phi_j(v)\|_{H^{s-1,l}(\mathbb{T}^d)}\|\nabla v\|_{L^r(\mathbb{T}^d;\mathbb{R}^d)} \\ &\quad\lesssim \big(\|\Phi_j(u)-\Phi_j(v)\|_{L^\infty(\mathbb{T}^d)} + \|\Phi_j(u)-\Phi_j(v)\|_{H^{s-1,l}(\mathbb{T}^d)}\big)\|v\|_{H^{s,q}(\mathbb{T}^d)}\end{aligned}$$

5

$$\lesssim L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}}.$$

Together with (5.2.14), we conclude that (5.2.13) holds.

Lastly, if $s-1<0$, we choose τ and ζ such that $\tau > \max\{\frac{d}{\zeta}, 1-s\}$, $\zeta \in [q',\infty)$ and $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow H^{\tau,\zeta}(\mathbb{T}^d)$ and apply [8, Proposition 4.1 (3)] to estimate

$$\begin{aligned}
&\|\Phi_j(u)(\nabla u-\nabla v)\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\quad\lesssim \|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)}\|\nabla u-\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|\Phi_j(u)\|_{H^{\tau,\zeta}(\mathbb{T}^d)}\|\nabla u-\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\quad\lesssim \big(\|\Phi_j(u)\|_{L^\infty(\mathbb{T}^d)} + \|\Phi_j(u)\|_{H^{\tau,\zeta}(\mathbb{T}^d)}\big)\|\nabla u-\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\quad\le L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}}
\end{aligned}$$

and

$$\begin{aligned}
&\|(\Phi_j(u)-\Phi_j(v))\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\lesssim \|\Phi_j(u)-\Phi_j(v)\|_{L^\infty(\mathbb{T}^d)}\|\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|\Phi_j(u)-\Phi_j(v)\|_{H^{\tau,\zeta}(\mathbb{T}^d)}\|\nabla v\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\
&\lesssim \big(\|\Phi_j(u)-\Phi_j(v)\|_{L^\infty(\mathbb{T}^d)} + \|\Phi_j(u)-\Phi_j(v)\|_{H^{\tau,\zeta}(\mathbb{T}^d)}\big)\|v\|_{H^{s,q}(\mathbb{T}^d)}\\
&\lesssim L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}}
\end{aligned}$$

using once more [115, Theorem 1, p.373]. Also in this case we deduce (5.2.13) and the growth estimate (5.2.12) can be obtained by inserting $v=0$ in (5.2.13) and using that $F^{(j)}(0)=0$. □

Lemma 5.2.3. *For all $j,n\in\mathbb{N}$ there exist constants $C_{j,n}, L_{j,n}\in(0,\infty)$ such that*

(i) if $s+1-\frac{d}{q}>0$, then

$$\|G^{(j)}[u]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le C_{j,n}\|u\|_{X_{3/4}}, \tag{5.2.15}$$

$$\|G^{(j)}[u]-G^{(j)}[v]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le L_{j,n}\big(\|u-v\|_{X_{3/4}} + \|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}}\big(\|u\|_{X_{3/4}} + \|v\|_{X_{3/4}}\big)\big), \tag{5.2.16}$$

(ii) and if $s+1-\frac{d}{q}\le 0$, then

$$\|G^{(j)}[u]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le C_{j,n}\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}}, \tag{5.2.17}$$

$$\|G^{(j)}[u]-G^{(j)}[v]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le L_{j,n}\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}} \tag{5.2.18}$$

for all $u,v\in X^{\mathrm{Tr}}_{\kappa,p}$ with $\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}}, \|v\|_{X^{\mathrm{Tr}}_{\kappa,p}}\le n$.

Proof. To prove the assertions concerning the local Lipschitz estimates, we introduce the operator

$$H^{(j)}[u]\colon H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d)\to H^{s,q}(\mathbb{T}^d),\ f\mapsto \operatorname{div}(g_j(u)f)$$

so that $G^{(j)}[u] = H^{(j)}[u]\circ\Psi$ for the operator Ψ introduced below (5.1.15). Therefore, by the ideal property [85, Theorem 9.1.10] of γ-radonifying operators we deduce that

$$\begin{aligned}&\|G^{(j)}[u]-G^{(j)}[v]\|_{\gamma(\ell^2(\mathbb{N}),H^{s,q}(\mathbb{T}^d))}\\&\quad\le \|H^{(j)}[u]-H^{(j)}[v]\|_{\mathscr{L}(H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d),H^{s,q}(\mathbb{T}^d))}\|\Psi\|_{\gamma(\ell^2(\mathbb{N}),H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d))}\end{aligned}$$

and recall that the latter term is finite by Assumption 5.1.5(s_ψ, q_ψ). First, we treat the case from (i), namely that $s+1-\frac{d}{q}>0$. Then we can apply [8, Proposition 4.1 (1)] to obtain

$$\begin{aligned}&\|(H^{(j)}[u]-H^{(j)}[v])f\|_{H^{s,q}(\mathbb{T}^d)}\\&\quad\lesssim \|(g_j(u)-g_j(v))f\|_{H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\&\quad\lesssim \|g_j(u)-g_j(v)\|_{H^{s+1,q}(\mathbb{T}^d)}\|f\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)}+\|g_j(u)-g_j(v)\|_{L^\infty(\mathbb{T}^d)}\|f\|_{H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d)}\\&\quad\lesssim \|g_j(u)-g_j(v)\|_{H^{s+1,q}(\mathbb{T}^d)}\|f\|_{H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d)}.\end{aligned}$$

By [115, Theorem 1, p.373], we can estimate

$$\begin{aligned}&\|g_j(u)-g_j(v)\|_{H^{s+1,q}(\mathbb{T}^d)}\\&\quad\le L_{j,n}\big(\|u-v\|_{H^{s+1,q}(\mathbb{T}^d)}+\|u-v\|_{X^{\mathrm{Tr}}_{\kappa,p}}\big(\|u\|_{H^{s+1,q}(\mathbb{T}^d)}+\|v\|_{H^{s+1,q}(\mathbb{T}^d)}\big)\big)\end{aligned}$$

using the embedding $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow L^\infty(\mathbb{T}^d)$ implied by (5.1.8). Hence, (5.2.16) follows.

Next, we assume that $s+1-\frac{d}{q}\le 0$ as in (ii) and recall that $s+1>0$ by Assumption 5.1.1. Thus, we can use again [8, Proposition 4.1 (1)] to estimate

$$\begin{aligned}&\|(H^{(j)}[u]-H^{(j)}[v])f\|_{H^{s,q}(\mathbb{T}^d)}\\&\quad\lesssim \|g_j(u)-g_j(v)\|_{L^\infty(\mathbb{T}^d)}\|f\|_{H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d)}+\|g_j(u)-g_j(v)\|_{H^{s+1,l}(\mathbb{T}^d)}\|f\|_{L^r(\mathbb{T}^d;\mathbb{R}^d)}\end{aligned}\tag{5.2.19}$$

for $l\in(1,\infty)$, $r\in(1,\infty]$ subject to (5.2.9). As in the proof of Lemma 5.2.1 we find that l,r can be chosen such that $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow H^{s+1,l}(\mathbb{T}^d)$ and $H^{s+1,q}(\mathbb{T}^d)\hookrightarrow L^r(\mathbb{T}^d)$. Indeed for each $\varepsilon>0$, we can choose r such that

$$s+1-\tfrac{d}{q} > \tfrac{-d}{r} > s+1-\tfrac{d}{q}-\varepsilon.$$

Then we have

$$\tfrac{d}{l} > s+1-\varepsilon \quad\Longleftrightarrow\quad s+1-\tfrac{d}{l}<\varepsilon$$

due to (5.2.9). Invoking additionally (5.1.8) and choosing ε smaller than (5.2.10) yields that also $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow H^{s+1,l}(\mathbb{T}^d)$. We estimate (5.2.19) further by

$$\|g_j(u)-g_j(v)\|_{L^\infty(\mathbb{T}^d)\cap H^{s+1,l}(\mathbb{T}^d)}\|f\|_{H^{s+1,q}(\mathbb{T}^d;\mathbb{R}^d)}$$

and since $u\mapsto g_j(u)$ is locally Lipschitz continuous in $L^\infty(\mathbb{T}^d)\cap H^{s+1,l}(\mathbb{T}^d)$ by [115, Theorem 1, p.373] the embedding $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow L^\infty(\mathbb{T}^d)\cap H^{s+1,l}(\mathbb{T}^d)$ yields (5.2.18). The corresponding growth estimates (5.2.15) and (5.2.17) follow by inserting $v=0$ in (5.2.16) and (5.2.18) and using that $G^{(j)}[0]=0$. □

Remark 5.2.4. We convince ourselves that the assertions from Lemmas 5.2.1–5.2.3 imply [5, Hyptohesis (H')] for fixed $j \in \mathbb{N}$. Indeed, Lemma 5.2.1 yields [5, Hypothesis (HA)] and Lemma 5.2.2 yields the estimate in terms of the trace space from [5, Hypothesis (HF')]. Similarly, if $s+1-\frac{d}{q} \le 0$, Lemma 5.2.3 (ii) implies the estimate on the trace part from [5, Hypothesis (HG')]. If instead $s+1-\frac{d}{q} > 0$, Lemma 5.2.3 (i) yields

$$\|G^{(j)}[u]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le C_{j,n} \sum_{l=1}^{2} (1+\|u\|_{X_{\varphi_l}}^{\rho_l})\|u\|_{X_{\beta_l}},$$

$$\|G^{(j)}[u]-G^{(j)}[v]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le L_{j,n} \sum_{l=1}^{2} (1+\|u\|_{X_{\varphi_l}}^{\rho_l}+\|v\|_{X_{\varphi_l}}^{\rho_l})\|u-v\|_{X_{\beta_l}}$$

for $u,v \in X_{\kappa,p}^{\mathrm{Tr}}$ with $\|u\|_{X_{\kappa,p}^{\mathrm{Tr}}}, \|v\|_{X_{\kappa,p}^{\mathrm{Tr}}} \le n$, if we choose

$$\varphi_l \in \big(\max\{1-\tfrac{1+\kappa}{p},\tfrac{3}{4}\},1\big), \quad l \in \{1,2\},$$

$\beta_1 = \varphi_1$, $\rho_1 = 0$, $\beta_2 \in (1-\frac{1+\kappa}{p},\varphi_2]$ close to $1-\frac{1+\kappa}{p}$ and $\rho_2 = 1$. This implies the estimate on the (possibly) critical part from [5, Hypothesis (HG')]. In particular, we have

$$\rho_l\big(\varphi_l - 1 + \tfrac{1+\kappa}{p}\big) + \beta_l < 1, \quad l \in \{1,2\} \tag{5.2.20}$$

for sufficiently small β_2 and are therefore in the subcritical regime of the theory developed in [5, 6].

5.2.2. STOCHASTIC MAXIMAL REGULARITY OF THIN-FILM TYPE OPERATORS

Next to the local Lipschitz estimates established in the previous subsection, the local well-posedness theory from [5, 6] also requires optimal regularity estimates for linear problems of the form

$$\begin{cases} \mathrm{d}u + \mathscr{A}(u)\,\mathrm{d}t = f\,\mathrm{d}t + g\,\mathrm{d}W, & t \in [t_0,T], \\ u(t_0) = u_{t_0}, \end{cases} \tag{5.2.21}$$

called stochastic maximal regularity. Here $\mathscr{A} : [t_0,T] \times \Omega \to \mathscr{L}(X_1,X_0)$ is strongly progressively measurable, X_0, X_1 are UMD Banach spaces of type 2 (see, e.g., [85, Chapter 7]) and $T < \infty$, $t_0 \in [0,T]$ are fixed.

The purpose of this subsection is to verify that if (p,κ,s,q) are admissible,

$$\mathscr{A}(u) = \operatorname{div}(a\nabla\Delta u) \tag{5.2.22}$$

for a positive and $\mathscr{P}$-measurable coefficient $a : [0,\infty) \times \Omega \to X_{\kappa,p}^{\mathrm{Tr}}$ and (X_0,X_1) is as in Subsection 5.1.3, then the problem (5.2.21) indeed admits stochastic maximal regularity estimates, which can be seen as a generalization of [8, Theorem 5.2] to fourth-order operators. Let us note that, the time dependence of the coefficient a in (5.2.22) appears naturally when considering the nonlinear problem (5.2.2) by choosing $a = m(u)$. One of the key points here is that we will allow coefficients which are only measurable in time. This will be crucial in the proof of the blow-up criteria of Proposition 5.1.8 (cf., (5.2.41) below) used in the proof of the global well-posedness result in Subsection 5.1.2.

Before going into the details, we recall some definitions. For a stopping time $\tau\colon\Omega\to[t_0,T]$, an initial value $u_{t_0}\in L^0_{\mathscr{F}_{t_0}}(\Omega;X_0)$ and inhomogeneities $f\in L^0_{\mathscr{P}}(\Omega;L^1(t_0,\tau;X_0))$ and $g\in L^0_{\mathscr{P}}(\Omega;L^2(t_0,\tau;\gamma(l^2(\mathbb{N}),X_{1/2})))$ of (5.2.21), we call a strongly progressive measurable map $u\colon[\![t_0,\tau]\!]\to X_1$ a *strong solution* to (5.2.21) on $[\![t_0,\tau]\!]$, if $u\in L^0(\Omega;L^2(t_0,\tau;X_1))$ and

$$u(t)-u_{t_0}+\int_{t_0}^t\mathscr{A}(u)\,\mathrm{d}r=\int_{t_0}^t f\,\mathrm{d}r+\int_{t_0}^t g\,\mathrm{d}W_r$$

a.s. for all $t\in[t_0,\tau]$.

Definition 5.2.5 (Stochastic maximal regularity)**.** *Let $\widehat{\kappa}\in[0,\frac{p}{2}-1)$ for $p>2$ and $\widehat{\kappa}=0$ for $p=2$ be possibly different from κ and $\|\mathscr{A}\|_{\mathscr{L}(X_1,X_0)}\le C_{\mathscr{A}}$ for some constant $C_{\mathscr{A}}<\infty$. We write $\mathscr{A}\in\mathrm{SMR}_{p,\widehat{\kappa}}(t_0,T)$ if for every*

$$f\in L^p_{\mathscr{P}}((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_0),\quad \textit{and}\quad g\in L^p_{\mathscr{P}}((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};\gamma(H,X_{1/2}))$$

there exists a strong solution u to (5.2.21) with $u_{t_0}=0$ such that $u\in L^p_{\mathscr{P}}((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_1)$, and moreover for all stopping time $\tau\colon\Omega\to[t_0,T]$ and strong solutions $u\in L^p_{\mathscr{P}}((t_0,\tau)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_1)$ to (5.2.21) with $u_{t_0}=0$ the estimate

$$\|u\|_{L^p((t_0,\tau)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_1)}\lesssim\|f\|_{L^p((t_0,\tau)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_0)}+\|g\|_{L^p((t_0,\tau)\times\Omega,w^{t_0}_{\widehat{\kappa}};\gamma(H,X_{1/2}))}\tag{5.2.23}$$

holds, where the implicit constant is independent of f, g and τ. In addition:

(1) If $p\in(2,\infty)$, we write $\mathscr{A}\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ if $\mathscr{A}\in\mathrm{SMR}_{p,\widehat{\kappa}}(t_0,T)$ and

$$\begin{aligned}&\|u\|_{L^p(\Omega;H^{\theta,p}((t_0,T),w^{t_0}_{\widehat{\kappa}};X_{1-\theta}))}\\&\quad\lesssim\|f\|_{L^p((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_0)}+\|g\|_{L^p((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};\gamma(H,X_{1/2}))}\end{aligned}\tag{5.2.24}$$

for each $\theta\in[0,\frac{1}{2})$, where the implicit constant is independent of f and g.

(2) If $p=2$, we write $\mathscr{A}\in\mathrm{SMR}^\bullet_{2,0}(t_0,T)$ if $\mathscr{A}\in\mathrm{SMR}_{2,0}(t_0,T)$ and it holds

$$\|u\|_{L^2(\Omega;C([t_0,T];X_{1/2}))}\lesssim\|f\|_{L^2((t_0,T)\times\Omega;X_0)}+\|g\|_{L^2((t_0,T)\times\Omega;\gamma(H,X_{1/2}))},\tag{5.2.25}$$

where the implicit constant is independent of f and g.

The choice $u_{t_0}=0$ in the above definition is not essential. Indeed, stochastic maximal L^p-regularity estimates with zero initial data directly imply corresponding ones with non-trivial data [5, Proposition 3.10].

If $\mathscr{A}\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$, we define $C^{\det,\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)$ as the smallest implicit constant such that (5.2.23) for $\theta=0$ and (5.2.24)–(5.2.25) for $\theta>0$ holds for all $f\in L^p_{\mathscr{P}}((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};X_0)$ and $g=0$. Analogously, we define $C^{\mathrm{sto},\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)$ as the smallest implicit constant such that (5.2.23)–(5.2.25), respectively, holds for all $g\in L^p_{\mathscr{P}}((t_0,T)\times\Omega,w^{t_0}_{\widehat{\kappa}};\gamma(H,X_{1/2}))$ and $f=0$. For $\ell\in\{\det,\mathrm{sto}\}$, we set finally

$$K^{\ell,\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T):=C^{\ell,\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)+C^{\ell,0,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T).$$

We are ready to state the main result of this subsection.

Theorem 5.2.6 (Stochastic maximal regularity—Thin-film type operators)**.** *Let the parameters (p,κ,s,q) be admissible (i.e., Assumption 5.1.1 holds). Assume that the mapping $a:[t_0,T]\times\Omega\to X^{\mathrm{Tr}}_{\kappa,p}$ is $\mathscr{P}$-measurable and there exist positive constants $\lambda,\mu>0$ such that, for a.a. $(t,\omega)\in[t_0,T]\times\Omega$,*

$$\lambda\le\inf_{\mathbb{T}^d}a(t,\omega,\cdot)\quad\text{and}\quad\|a(t,\omega,\cdot)\|_{X^{\mathrm{Tr}}_{\kappa,p}}\le\mu.$$

Then $\mathscr{A}$ defined in (5.2.22) satisfies $\mathscr{A}\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ and

$$\max\{K^{\det,\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T),K^{\mathrm{sto},\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)\}\lesssim_{\lambda,\mu,\theta,T}1$$

for each $\theta\in[0,\frac{1}{2})$ and $\widehat{\kappa}\in\{0,\kappa\}$.

Note that no regularity of the coefficient a is assumed w.r.t. the time variable. The proof of the above result is given at the end of this subsection and it requires some preparation. We begin by considering the situation of a spatially constant coefficient a.

5

Lemma 5.2.7. *Assume that $a(t,x)=\bar{a}(t)$ for a progressive measurable random variable $\bar{a}:[t_0,T]\times\Omega\to\mathbb{R}$ satisfying $\lambda\le\bar{a}\le\lambda^{-1}$ for some $\lambda\in(0,1)$. Then $\mathscr{A}\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ and*

$$\max\{K^{\det,\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T),K^{\mathrm{sto},\theta,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)\}\lesssim_{\lambda,T}1\tag{5.2.26}$$

for $\widehat{\kappa}\in\{0,\kappa\}$.

Proof. Analogously to [85, Theorem 10.2.25], one can show that the Bi-Laplace operator $\Delta^2\colon H^{s+2,q}(\mathbb{T}^d)\to H^{s-2,q}(\mathbb{T}^d)$ admits a bounded H^∞-calculus of angle 0 and therefore $\Delta^2\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ by [8, Theorem 2.4]. Moreover, by the periodic version of [55, Theorem 5.1] (cf., Step 1 in the proof of [55, Theorem 5.4]), for a.e. (t,ω), $\mathscr{A}(t,\omega)=\bar{a}(t,\omega)\Delta^2$ has *deterministic* maximal L^p_κ-regularity on X_1 with constants depending only on λ and is therefore independent of (t,ω). Note that, to apply the results of [55], one needs to recall that $(1-\Delta)^{r/2}:H^{r+t,\zeta}(\mathbb{T}^d)\to H^{t,\zeta}(\mathbb{T}^d)$ is an isomorphism for all $r,t\in\mathbb{R}$ and $\zeta\in(1,\infty)$. Using that $\Delta^2\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ and the transference result [113, Theorem 3.9] yields that $\mathscr{A}\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ and that (5.2.26) holds. □

Using estimates similar to the ones in the proof of Lemma 5.2.1, we can include also small perturbations of a spatially constant coefficient.

Lemma 5.2.8. *Let $\bar{a}$ and λ as in Lemma 5.2.7 and let a be as in the statement of Theorem 5.2.6. Then there exists some $\varepsilon>0$ depending on λ,T such that if*

$$\operatorname*{ess\,sup}_{[t_0,T]\times\Omega}\|a-\bar{a}\|_{L^\infty(\mathbb{T}^d)}<\varepsilon,\tag{5.2.27}$$

then we have $\mathscr{A}\in\mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ with

$$\max\{C^{\det,0,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T),C^{\mathrm{sto},0,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)\}\lesssim_{\lambda,\mu,T}1$$

for $\widehat{\kappa}\in\{0,\kappa\}$.

Proof. By repeating the proof of the estimate (5.2.5) for the operator $\mathscr{A}$, one obtains that, a.e. on $[t_0,T]\times\Omega$,

$$\|\mathscr{A}\|_{\mathscr{L}(X_1,X_0)} \lesssim \|a\|_{X^{\mathrm{Tr}}_{\kappa,p}} \leq \mu. \tag{5.2.28}$$

By the previous Lemma 5.2.7, we have $\bar{a}\Delta^2 \in \mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ and $\bar{a}\Delta^2 \in \mathrm{SMR}^\bullet_{p,0}(t_0,T)$. We define $C^*_{\lambda,T}$ as the maximum of the corresponding upper bounds on $K^{\det,0,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)$ and $K^{\det,0,p,0}_{\mathscr{A}}(t_0,T)$ from (5.2.26). Then, since we can write

$$\mathscr{A}(u) = \mathrm{div}(\bar{a}\nabla\Delta u) + \mathrm{div}((a-\bar{a})\nabla\Delta u)$$

the claim follows by the perturbation result [8, Theorem 3.2] as soon as we can verify that

$$\|\mathrm{div}((a-\bar{a})\nabla\Delta u)\| \leq (2C^*_{\lambda,T})^{-1}\|u\|_{X_1} + L\|u\|_{X_0} \tag{5.2.29}$$

for a suitable constant $L\in(0,\infty)$. To this end, we assume (5.2.27) and obtain

$$\|\mathrm{div}((a-\bar{a})\nabla\Delta u)\|_{X_0} \leq \|a-\bar{a}\|_{L^\infty(\mathbb{T}^d)}\|u\|_{X_1} < \varepsilon\|u\|_{X_1}$$

for $s-1=0$ implying (5.2.29) for sufficiently small $\varepsilon>0$. For $s-1>0$, we derive

$$\begin{aligned}&\|\mathrm{div}((a-\bar{a})\nabla\Delta u)\|_{X_0}\\ &\quad\lesssim \|a-\bar{a}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta u\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|a-\bar{a}\|_{H^{s-1,l}(\mathbb{T}^d)}\|\nabla\Delta u\|_{L^r(\mathbb{T}^d;\mathbb{R}^d)}\end{aligned}$$

for l,r as in the proof of Lemma 5.2.1, analogously to (5.2.8). We observe moreover that the embedding $H^{s-1,q}(\mathbb{T}^d)\hookrightarrow L^r(\mathbb{T}^d)$ is not sharp and hence it also holds $H^{s-1-\delta,q}(\mathbb{T}^d)\hookrightarrow L^r(\mathbb{T}^d)$ for a sufficiently small $\delta>0$. Then, using (5.2.27), that $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow H^{s-1,l}(\mathbb{T}^d)$, the interpolation inequality

$$\|u\|_{X_{1-\delta/4}} \leq \|u\|^{\delta/4}_{X_0}\|u\|^{1-\delta/4}_{X_1} \tag{5.2.30}$$

and Young's inequality we can estimate

$$\begin{aligned}\|\mathrm{div}((a-\bar{a})\nabla\Delta u)\|_{X_0} &\lesssim \varepsilon\|u\|_{X_1} + \|a-\bar{a}\|_{X^{\mathrm{Tr}}_{\kappa,p}}\|u\|_{X_{1-\delta/4}}\\ &\leq \varepsilon\|u\|_{X_1} + C_{\lambda,\mu}\|u\|^{\delta/4}_{X_0}\|u\|^{1-\delta/4}_{X_1} \leq 2\varepsilon\|u\|_{X_1} + C_{\delta,\varepsilon,\lambda,\nu}\|u\|_{X_0}.\end{aligned}$$

Therefore, (5.2.29) follows also in this case as long as $\varepsilon>0$ is small enough. For $s-1<0$, we proceed again as in Lemma 5.2.1 and apply [8, Proposition 4.1 (3)] to obtain

$$\begin{aligned}&\|\mathrm{div}((a-\bar{a})\nabla\Delta u)\|_{X_0}\\ &\quad\lesssim \|a-\bar{a}\|_{L^\infty(\mathbb{T}^d)}\|\nabla\Delta u\|_{H^{s-1,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|a-\bar{a}\|_{H^{\tau,\zeta}(\mathbb{T}^d)}\|\nabla\Delta u\|_{H^{s-1-\delta,q}(\mathbb{T}^d;\mathbb{R}^d)}\end{aligned}$$

for τ,ζ as in Lemma 5.2.1 and some $\delta>0$. We remark that we used the slightly worse estimate (5.2.11) previously because it was sufficient to verify Lemma 5.2.1. Employing again (5.2.27), the inequality (5.2.30), Young's inequality and the embedding $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow H^{\tau,\zeta}(\mathbb{T}^d)$ we conclude

$$\|\mathrm{div}((a-\bar{a})\nabla\Delta u)\|_{X_0} \lesssim 2\varepsilon\|u\|_{X_1} + C_{\delta,\varepsilon,\lambda,\mu}\|u\|_{X_0}.$$

The desired estimate (5.2.29) follows again for sufficiently small $\varepsilon>0$. □

Before giving a proof of stochastic maximal regularity in the general situation we provide first the a-priori estimate on sufficiently small time intervals. The proof relies on a spatial localization procedure known as freezing the coefficients, which allows us to apply Lemma 5.2.8 locally in space.

Lemma 5.2.9. *Let a be as in the statement of Theorem 5.2.6. Then there exists a time $T^* > 0$ depending on λ, μ, T for which the following holds. For any $t \in [t_0, T]$, $\widehat{\kappa} \in \{0, \kappa\}$, stopping time $\tau\colon \Omega \to [t, T \wedge (t+T^*)]$, $f \in L^p_{\mathscr{P}}((t,\tau)\times\Omega, w^t_{\widehat{\kappa}}; X_0)$, $g \in L^p_{\mathscr{P}}((t,\tau)\times\Omega, w^t_{\widehat{\kappa}}; \gamma(H, X_{1/2}))$ and strong solution $u \in L^p_{\mathscr{P}}((t,\tau)\times\Omega, w^t_{\widehat{\kappa}}; X_1)$ to*

$$\begin{cases} \mathrm{d}u + \mathscr{A}(u)\,\mathrm{d}r = f\,\mathrm{d}r + g\,\mathrm{d}W, & r \in [t,\tau], \\ u(t) = 0, \end{cases} \tag{5.2.31}$$

it holds

$$\|u\|_{L^p((t,\tau)\times\Omega, w^t_{\widehat{\kappa}}; X_1)} \lesssim_{\lambda,\mu,T} \|f\|_{L^p((t,\tau)\times\Omega, w^t_{\widehat{\kappa}}; X_0)} + \|g\|_{L^p((t,\tau)\times\Omega, w^t_{\widehat{\kappa}}; \gamma(H, X_{1/2}))}. \tag{5.2.32}$$

5

Proof. We follow the strategy employed to prove [8, Lemma 5.4] and split the proof into three steps to improve its readability. We do not display the time dependence of a as it does not play any role here.

Construction of suitable extension operators. We first derive the existence of extension operators with a uniform operator norm which can be seen as a variant of [8, Lemma 5.8] for Besov spaces. The main difference compared to [8, Lemma 5.8] is that constant functions are contained in Hölder spaces on $\mathbb{R}^d$, but not necessarily in $B^s_{q,p}(\mathbb{R}^d)$. Let E be Stein's total extension operator from the unit ball $B_{\mathbb{R}^d}(0,1)$ to $\mathbb{R}^d$ as introduced in [122, Theorem 5, p.181], then

$$E\colon W^{l,r}(B_{\mathbb{R}^d}(0,1)) \to W^{l,r}(\mathbb{R}^d)$$

continuously for each $l \in \mathbb{N}_0$ and $r \in [1,\infty]$. We let $\varphi \in C^\infty_c(\mathbb{R}^d)$ such that $\varphi = 1$ on $B_{\mathbb{R}^d}(0,1)$ and $\varphi = 0$ outside of $B_{\mathbb{R}^d}(0,2)$ and define

$$E^{\mathbb{R}^d}_{y,r}[f](x) = (\varphi \cdot E[f(y + r\cdot)])\big(\tfrac{x-y}{r}\big)$$

for $y \in \mathbb{R}^d$ and $r \in (0, \frac{1}{8})$ and obtain extension operators from $B_{\mathbb{R}^d}(y,r)$ to $\mathbb{R}^d$ with

$$\|E^{\mathbb{R}^d}_{y,r}\|_{\mathscr{L}(L^\infty(B_{\mathbb{R}^d}(y,r)), L^\infty(\mathbb{R}^d))}$$

independent of y and r. Analogously to [8, Lemma 5.8] they induce extension operators $E^{\mathbb{T}^d}_{y,r}$ from $B_{\mathbb{T}^d}(y,r)$ to $\mathbb{T}^d$ with the same property. To ensure also that constant functions are mapped to constant functions, we define new extension operators by setting

$$\widetilde{E}^{\mathbb{T}^d}_{y,r} f = E^{\mathbb{T}^d}_{y,r}\Big[f - \fint_{B_{\mathbb{T}^d}(y,r)} f\,\mathrm{d}x\Big] + \fint_{B_{\mathbb{T}^d}(y,r)} f\,\mathrm{d}x,$$

where $\fint_{B_{\mathbb{T}^d}(y,r)} := |B_{\mathbb{T}^d}(y,r)|^{-1} \int$, and observe that again

$$\|\widetilde{E}^{\mathbb{T}^d}_{y,r}\|_{\mathscr{L}(L^\infty(B_{\mathbb{T}^d}(y,r)), L^\infty(\mathbb{T}^d))}$$

is independent of y and r. By real interpolation, we conclude that also

$$\widetilde{E}^{\mathbb{T}^d}_{y,r}\colon B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(B_{\mathbb{T}^d}(y,r)) \to X^{\mathrm{Tr}}_{\kappa,p}$$

continuously.

Spatial localization. Next, analogously to the proof of [8, Lemma 5.4], we introduce the operators

$$\mathscr{A}_y(u) = \operatorname{div}(a(y)\nabla\Delta u), \qquad \mathscr{A}^E_{y,r}(u) = \operatorname{div}\big(\widetilde{E}^{\mathbb{T}^d}_{y,r}[a|_{B_{\mathbb{T}^d}(y,r)}]\nabla\Delta u\big).$$

Let $(y_\iota)_{\iota\in\mathscr{I}} \subset \mathbb{T}^d$ and $r\in(0,\frac{1}{8})$ such that $(B_{\mathbb{T}^d}(y_\iota,r))_{\iota\in\mathscr{I}}$ is a finite cover of $\mathbb{T}^d$. Moreover, let $(\varphi_\iota)_{\iota\in\mathscr{I}}$ be a partition of unity subordinate to $(B_{\mathbb{T}^d}(y_\iota,r))_{\iota\in\mathscr{I}}$. We observe that for sufficiently small $\gamma>0$ we have $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow C^\gamma(\mathbb{T}^d)$ by (5.1.8) and consequently

$$\|a(y_\iota)-a|_{B_{\mathbb{T}^d}(y_\iota,r)}\|_{L^\infty(B_{\mathbb{T}^d}(y_\iota,r))} \lesssim_\mu r^\gamma.$$

After applying the extension operators $\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}$ we constructed, we conclude that

$$\big\|a(y_\iota)-\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}[a|_{B_{\mathbb{T}^d}(y_\iota,r)}]\big\|_{L^\infty(\mathbb{T}^d)} = \big\|\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}[a(y_\iota)-a|_{B_{\mathbb{T}^d}(y_\iota,r)}]\big\|_{L^\infty(\mathbb{T}^d)} \lesssim_\mu r^\gamma.$$

Thus, if we choose ε as in Lemma 5.2.8 according to the given constant λ, we can ensure that

$$\big\|a(y)-\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}[a|_{B_{\mathbb{T}^d}(y,r)}]\big\|_{L^\infty(\mathbb{T}^d)} < \varepsilon,$$

if we choose $r<r^*_{\lambda,\mu,T}\in(0,\frac{1}{8})$ small enough. Setting then

$$C^*_{\lambda,\mu,T} = \max_{\iota\in\mathscr{I}} \|\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}\|_{\mathscr{L}\left(B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(B_{\mathbb{T}^d}(y_\iota,r)),X^{\mathrm{Tr}}_{\kappa,p}\right)},$$

we deduce that

$$\big\|\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}[a|_{B_{\mathbb{T}^d}(y_\iota,r)}]\big\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le C^*_{\lambda,\mu,T}\|a|_{B_{\mathbb{T}^d}(y_\iota,r)}\|_{B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(B_{\mathbb{T}^d}(y_\iota,r))} \le C^*_{\lambda,\mu,T}\|a\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le C^*_{\lambda,\mu,T}\mu.$$

Therefore, the functions $\widetilde{E}^{\mathbb{T}^d}_{y_\iota,r}[a|_{B_{\mathbb{T}^d}(y_\iota,r)}]$ satisfy the assumptions of Lemma 5.2.8 with uniform constants and thus $\mathscr{A}^E_{y_\iota,r}\in \mathrm{SMR}^\bullet_{p,\widehat{\kappa}}(t_0,T)$ with

$$C^{\ell,0,p,\widehat{\kappa}}_{\mathscr{A}^E_{y_\iota,r}}(s,T) \lesssim_{\lambda,\mu,T} 1, \qquad \ell\in\{\det,\mathrm{sto}\}. \tag{5.2.33}$$

With this at hand, we finally consider the strong solution u to (5.2.31) and using the partition of unity $(\varphi_\iota)_{\iota\in\mathscr{I}}$, we can write $u=\sum_{\iota\in\mathscr{I}}u_\iota$, if we set $u_\iota = u\varphi\iota$. Introducing analogously $f_\iota = f\varphi_\iota$ and $g_\iota = g\varphi_\iota$, we find that

$$\begin{aligned} \mathrm{d}u_\iota + \mathscr{A}^E_{y_\iota,r}(u_\iota)\,\mathrm{d}t &= \mathrm{d}u_\iota + \mathscr{A}(u_\iota)\,\mathrm{d}t \\ &= \varphi_\iota\big(\mathrm{d}u+\mathscr{A}(u)\,\mathrm{d}t\big) + [\mathscr{A},\varphi_\iota]u\,\mathrm{d}t = \big(f_\iota + [\mathscr{A},\varphi_\iota]u\big)\mathrm{d}t + g_\iota\,\mathrm{d}W, \end{aligned}$$

where we use the commutator notation $[\mathscr{A},\varphi_\iota]u = \mathscr{A}(\varphi_\iota u) - \varphi_\iota \mathscr{A} u$. Hence, applying (5.2.33), we deduce that

$$\begin{aligned}
&\|u\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;X_1)} \\
&\leq \sum_{\iota\in\mathscr{I}} \|u_\iota\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;X_1)} \\
&\lesssim_{\lambda,\mu,T} \sum_{\iota\in\mathscr{I}} \Big(\|[\mathscr{A},\varphi_\iota]u\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;X_0)} + \|f_\iota\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;X_0)} + \|g_\iota\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;\gamma(H,X_{1/2}))}\Big) \\
&\lesssim_{\lambda,\mu,T} \sum_{\iota\in\mathscr{I}} \Big(\|[\mathscr{A},\varphi_\iota]u\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;X_0)}\Big) + \|f\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;X_0)} + \|g\|_{L^p((t,\tau)\times\Omega,w_{\widehat{\kappa}}^t;\gamma(H,X_{1/2}))},
\end{aligned} \tag{5.2.34}$$

where in the last step we used that multiplication φ_ι is a bounded linear mapping on $H^{s+2,q}(\mathbb{T}^d)$ and $H^{s,q}(\mathbb{T}^d)$ for each $\iota\in\mathscr{I}$.

Absorbing the commutator terms. To conclude the proof, we show that the commutator terms $[\mathscr{A},\varphi_\iota]u$ are of lower order and expand

$$\begin{aligned}
\mathscr{A}(\varphi_\iota u) &= \operatorname{div}(a\nabla\Delta(\varphi_\iota u)) = \operatorname{div}(a\nabla(\varphi_\iota\Delta u + 2\nabla\varphi_\iota\cdot\nabla u + u\Delta\varphi_\iota)) \\
&= \operatorname{div}(a(\varphi_\iota\nabla\Delta u + \nabla\varphi_\iota\Delta u)) + \operatorname{div}(a\nabla(2\nabla\varphi_\iota\cdot\nabla u + u\Delta\varphi_\iota)) \\
&= \varphi_\iota\mathscr{A}u + a\nabla\varphi_\iota\cdot\nabla\Delta u + \operatorname{div}(a\nabla\varphi_\iota\Delta u) + \operatorname{div}(a\nabla(2\nabla\varphi_\iota\cdot\nabla u + u\Delta\varphi_\iota))
\end{aligned}$$

to conclude

$$[\mathscr{A},\varphi_\iota]u = a\nabla\varphi_\iota\cdot\nabla\Delta u + \operatorname{div}(a\nabla\varphi_\iota\Delta u) + \operatorname{div}(a\nabla(2\nabla\varphi_\iota\cdot\nabla u + u\Delta\varphi_\iota)). \tag{5.2.35}$$

We claim that

$$\|[\mathscr{A},\varphi_\iota]u\|_{X_0} \lesssim_{\lambda,\mu,T} \|u\|_{X_{1-\delta/4}} \tag{5.2.36}$$

for sufficiently small $\delta>0$ and estimate only the first term on the right-hand side of (5.2.35), since the others are easier to treat. Firstly, we observe that

$$\|a\nabla\varphi_\iota\|_{B^{s+2-4\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d;\mathbb{R}^d)} \lesssim_{\lambda,\mu,T} \|a\|_{X^{\mathrm{Tr}}_{\kappa,p}} \lesssim_\mu 1 \tag{5.2.37}$$

since the φ_ι are smooth. If $s-1\leq 0$, we choose $\delta\in(0,1)$ such that

$$s+2-4\tfrac{1+\kappa}{p} > 1+\delta-s \tag{5.2.38}$$

by (5.1.9). Since $s-1-\delta<0$, we can use [8, Proposition 4.1 (3)] to estimate

$$\begin{aligned}
\|a\nabla\varphi_\iota\cdot\nabla\Delta u\|_{X_0} &\lesssim \|a\nabla\varphi_\iota\cdot\nabla\Delta u\|_{H^{s-1-\delta,q}(\mathbb{T}^d)} \\
&\lesssim \|a\nabla\varphi_\iota\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)}\|\nabla\Delta u\|_{H^{s-1-\delta,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|a\nabla\varphi_\iota\|_{H^{\sigma,\zeta}(\mathbb{T}^d;\mathbb{R}^d)}\|\nabla\Delta u\|_{H^{s-1-\delta,q}(\mathbb{T}^d;\mathbb{R}^d)}
\end{aligned}$$

as soon as $\sigma>\max\{\frac{d}{\zeta},1+\delta-s\}$ and $\zeta\in[q',\infty)$. Arguing as in Lemma 5.2.1, we see that we can choose σ and ζ such that $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow H^{\sigma,\zeta}(\mathbb{T}^d)$ due to (5.1.8) and (5.2.38). Because also $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow L^\infty(\mathbb{T}^d)$ by (5.1.8), we obtain

$$\|a\nabla\varphi_\iota\cdot\nabla\Delta u\|_{X_0} \lesssim_{\lambda,\mu,T} \|u\|_{H^{s+2-\delta,q}(\mathbb{T}^d)}$$

by (5.2.37). If instead $s-1>0$, we choose $\delta \in (0,\min\{1,s-1\})$. Then an application of [8, Proposition 4.1 (1)] as in the proof of Lemma 5.2.1 yields

$$\begin{aligned}\|a\nabla\varphi_\iota\cdot\nabla\Delta u\|_{X_0} &\le \|a\nabla\varphi_\iota\cdot\nabla\Delta u\|_{H^{s-1-\delta,q}(\mathbb{T}^d)}\\ &\lesssim \|a\nabla\varphi_\iota\|_{L^\infty(\mathbb{T}^d;\mathbb{R}^d)}\|\nabla\Delta u\|_{H^{s-1-\delta,q}(\mathbb{T}^d;\mathbb{R}^d)} + \|a\nabla\varphi_\iota\|_{H^{s-1-\delta,l}(\mathbb{T}^d;\mathbb{R}^d)}\|\nabla\Delta u\|_{L^r(\mathbb{T}^d;\mathbb{R}^d)}\end{aligned}$$

for $l\in(1,\infty)$, $r\in(1,\infty]$ whenever (5.2.9) holds. If we can show that l,r can be chosen in a way such that $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow H^{s-1-\delta,l}(\mathbb{T}^d)$ and $H^{s-1-\delta,q}(\mathbb{T}^d)\hookrightarrow L^r(\mathbb{T}^d)$, we can conclude as in Lemma 5.2.1 that

$$\|a\nabla\varphi_\iota\cdot\nabla\Delta u\|_{X_0}\lesssim_{\lambda,\mu,T}\|u\|_{H^{s+2-\delta,q}(\mathbb{T}^d)},$$

using additionally $X^{\mathrm{Tr}}_{\kappa,p}\hookrightarrow L^\infty(\mathbb{T}^d)$ and (5.2.37). If $s-1-\delta-\frac{d}{q}>0$, the choice $l=q$ and $r=\infty$ suffices. If $s-1-\delta-\frac{d}{q}\le 0$, we take r in accordance with

$$s-1-\delta-\tfrac{d}{q} > \tfrac{-d}{r} > s-1-\delta-\tfrac{d}{q}-\eta$$

for $\eta>0$. Using (5.2.9) we obtain that

$$\tfrac{d}{l} > s-1-\delta-\eta \quad\Longleftrightarrow\quad s-1-\delta-\tfrac{d}{l}<\eta.$$

Choosing $\eta>0$ smaller than the left-hand side of (5.1.8) ensures that the desired embeddings hold. We conclude that (5.2.36) is valid in any case.

With this at hand, we can proceed as in the proof of [8, Lemma 5.4], namely we use that

$$\begin{aligned}\|[\mathscr{A},\varphi_\iota]u\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_0)} &\lesssim_{\lambda,\mu,T}\|u\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_{1-\delta/4})}\\ &\le \|u\|^{\delta/4}_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_0)}\|u\|^{1-\delta/4}_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_1)}\\ &\le C_{\delta,\eta}\|u\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_0)}+\eta\|u\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_1)}\end{aligned}$$

for any $\eta>0$ by the interpolation inequality for Bochner spaces [84, Theorem 2.2.6] and Young's inequality. Choosing η sufficiently small and inserting this in (5.2.34) we obtain that

$$\begin{aligned}&\|u\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_1)}\\ &\quad\lesssim_{\lambda,\mu,T}\|u\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_0)}+\|f\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};X_0)}+\|g\|_{L^p((t,\tau)\times\Omega,w^t_{\hat\kappa};\gamma(H,X_{1/2}))}.\end{aligned}$$

If $\tau\le t+T^*$ for sufficiently small T^* depending on λ,μ,T, the term containing u on the right-hand side can once more be absorbed by the first statement of [5, Lemma 3.13] together with (5.2.28) leading to the desired a-priori estimate (5.2.32). □

Having established the necessary a-priori estimate, Theorem 5.2.6 now follows by the method of continuity together with a partition of the time interval $[t_0,T]$ into pieces $t_0<t_1<\cdots<t_N=T$ of the length T^* from Lemma 5.2.9.

Proof of Theorem 5.2.6. We firstly remind ourselves of the uniform estimate (5.2.28) on the operator norm of $\mathscr{A}$. By Lemma 5.2.7 the set $\mathrm{SMR}^{\bullet}_{p,\widehat{\kappa}}(t_0,T)$ is non-empty and therefore it suffices by [8, Proposition 2.6] to show that $\mathscr{A}\in\mathrm{SMR}_{p,\widehat{\kappa}}(t_0,T)$ with

$$\max\{C^{\mathrm{det},0,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T), C^{\mathrm{sto},0,p,\widehat{\kappa}}_{\mathscr{A}}(t_0,T)\}\lesssim_{\lambda,\mu,T} 1.$$

This again can be deduced by [8, Proposition 3.1], if we can provide a partition $t_0<t_1<\dots<t_N=T$ such that $\mathscr{A}\in\mathrm{SMR}_{p,\widehat{\kappa}}(t_0,t_1)$ and $\mathscr{A}\in\mathrm{SMR}_{p,0}(t_i,t_{i+1})$ for $i\geq 1$ with

$$\max\{C^{\mathrm{det},0,p,\kappa_i}_{\mathscr{A}}(t_i,t_{i+1}), C^{\mathrm{sto},0,p,\kappa_i}_{\mathscr{A}}(t_i,t_{i+1})\}\lesssim_{\lambda,\mu,T} 1,\qquad \kappa_i=\widehat{\kappa}\delta_{i0}$$

for $i\in\{0,\dots,N-1\}$. The required a-priori estimate follows from Lemma 5.2.9 if we set $t_i=(t_0+iT^*)\wedge T$ for $i\geq 1$. Moreover, the existence of strong solutions on the respective intervals $[t_i,t_{i+1}]$ follows from the method of continuity [6, Proposition 3.13] applied to the family of operators

$$\mathscr{A}_r(u)=r\mathscr{A}(u)+(1-r)\lambda\Delta^2u=\operatorname{div}\big((ra+(1-r)\lambda)\nabla\Delta u\big),\qquad r\in[0,1],$$

because the coefficients $a_r=ra+(1-r)\lambda$ suffice the assumptions of Lemma 5.2.9 with the same constants λ,μ as the original coefficient a. □

5.2.3. LOCAL WELL-POSEDNESS AND BLOW-UP CRITERIA I

In this subsection, we use the preceding findings to apply the framework for quasilinear stochastic evolution equations from [5, 6] to the regularized equation (5.2.2). To this end, we fix again compatible parameters (p,κ,s,q) and a sequence $(\psi_k)_{k\in\mathbb{N}}$ subject to Assumptions 5.1.1 and 5.1.5(s_ψ,q_ψ) with $(s_\psi,q_\psi)=(s,q)$, and start with the definition of maximal local solutions to (5.2.2), similar to the one of (5.1.1). We remark that this definition deviates from the corresponding Definition 5.1.3 and Definition 5.1.4 for (5.1.1) in which the solution is required to be positive.

Definition 5.2.10. *Let (p,κ,s,q) be as in Assumption 5.1.1. Let $\sigma\colon\Omega\to[0,\infty]$ be a stopping time and $u\colon[\![0,\sigma)\!)\to X_1$ progressively measurable. Then the tuple (u,σ) is a* local (p,κ,s,q)-solution *to (5.2.2), if there exists a localizing sequence $0\leq\sigma_l\nearrow\sigma$ of stopping times such that for all $l\in\mathbb{N}$, we have a.s.*

$$u\in L^p(0,\sigma_l,w_\kappa;X_1)\cap C([0,\sigma_l];X^{\mathrm{Tr}}_{\kappa,p}),$$
$$F^{(j)}(u)\in L^p(0,\sigma_l,w_\kappa;X_0),\quad G^{(j)}[u]\in L^p(0,\sigma_l,w_\kappa;\gamma(\ell^2(\mathbb{N}),X_{1/2})),$$

and a.s. for all $t\in[0,\sigma_l]$

$$u(t)-u(0)+\int_0^t A^{(j)}[u(r)](u(r))\mathrm{d}r=\int_0^t F^{(j)}(u(r))\mathrm{d}r+\int_0^t G^{(j)}[u(r)]\mathrm{d}W_r. \tag{5.2.39}$$

As for Definition 5.1.3, all the integrals in (5.2.39) are well-defined due to the required integrability conditions.

Definition 5.2.11. *We call a local (p,κ,s,q)-solution (u,σ) to (5.2.2)* maximal unique (p,κ,s,q)-solution, *if for every local (p,κ,s,q)-solution (v,τ) to (5.2.2), one has $\tau\leq\sigma$ a.s. and $u=v$ a.e. on $[\![0,\sigma\wedge\tau)\!)$.*

The existence of a maximal unique (p,κ,s,q)-solution to (5.2.2) follows from our preceding findings.

Proposition 5.2.12. *Let the Assumptions 5.1.1 and 5.1.5(s_ψ, q_ψ) with $(s_\psi, q_\psi) = (s,q)$ be satisfied. Let $u_0 \in L^0_{\mathscr{F}_0}(\Omega; X^{\mathrm{Tr}}_{\kappa,p})$. Then there exists a maximal unique (p,κ,s,q)-solution (u,σ) to (5.2.2) in the sense of Definition 5.2.11 such that a.s. $\sigma > 0$ and*

$$u \in H^{\theta,p}_{\mathrm{loc}}([0,\sigma), w_\kappa; X_{1-\theta}) \cap C((0,\sigma); X^{\mathrm{Tr}}_p) \tag{5.2.40}$$

for all $\theta \in [0,\frac{1}{2})$, if $p > 2$. Additionally, for all $T < \infty$, the following blow-up criterion holds

$$\mathbb{P}\Big(\sigma < T,\ \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathrm{Tr}}_{\kappa,p}} < \infty\Big) = 0. \tag{5.2.41}$$

Proof. Let $T < \infty$ be arbitrary. We check the assumptions of [5, Theorem 4.7] and define the localization

$$u_{0,n} = \mathbf{1}_{\{\|u_0\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le n\}} u_0, \qquad n \in \mathbb{N},$$

of the initial value. Since $m_j : \mathbb{R} \to \mathbb{R}$ is smooth, we deduce that $m_j(u_{0,n}) \in L^\infty_{\mathscr{F}_0}(\Omega; X^{\mathrm{Tr}}_{\kappa,p})$ by [115, Theorem 1, p.373] and moreover uniformly bounded away from 0. Thus, Theorem 5.2.6 applies to the operator $A^{(j)}[u_{0,n}](f) = \operatorname{div}(m_j(u_{0,n})\nabla\Delta f)$ and yields that

$$A^{(j)}[u_{0,n}] \in \mathrm{SMR}^{\bullet}_{p,\kappa}(0,T)$$

for each $n \in \mathbb{N}$. Therefore, the assumption regarding stochastic maximal regularity holds and we convinced ourselves already in Remark 5.2.4 that [5, Hypothesis (H')] is satisfied. Furthermore, due to Theorem 5.2.6 and [6, Remark 5.6], our definition of maximal unique (p,κ,s,q)-solution to (5.2.2) is equivalent to [5, Definition 4.4] and therefore an application of [5, Theorem 4.7] yields the existence of a maximal unique (p,κ,s,q)-solution (u,σ) to (5.2.2) and the regularity assertions. Next, we prove the blow-up criterion (5.2.41). Firstly, as pointed out in the aforementioned Remark 5.2.4, the equation (5.2.2) is subcritical in our choice of spaces so that we obtain

$$\mathbb{P}\Big(\sigma < T,\ \lim_{t\nearrow\sigma} u(t) \text{ exists in } X^{\mathrm{Tr}}_{\kappa,p}\Big) = 0 \tag{5.2.42}$$

from [6, Theorem 4.9 (2)]. Before going further, let us comment on the technical requirements needed to apply [6, Theorem 4.9 (2)]. The condition from [6, Assumption 4.5] regarding stochastic maximal regularity follows as before from Theorem 5.2.6. Following again Remark 5.2.4 we can choose $\beta_1 = \varphi_1$ and $\rho_2 = 1$ to estimate the nonlinearity $G^{(j)}$ and therefore [6, Assumption 4.7] is satisfied by [6, Remark 4.8].

To conclude the proof, we show that (5.2.42) implies the seemingly weaker statement of (5.2.41). To this end, we use that the stochastic maximal regularity estimates of Theorem 5.2.6 hold for coefficients with measurable dependence on time. To prove (5.2.41), it is enough to show that, for all $M < \infty$,

$$\mathbb{P}(\Omega_M) = 0 \quad \text{where} \quad \Omega_M := \Big\{\sigma < T,\ \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le M\Big\}. \tag{5.2.43}$$

For the sake of clarity, we split the proof of (5.2.43) into three steps. Moreover, we define $\widetilde{u}_0 = \mathbf{1}_{\{\|u_0\|_{X^{\mathrm{Tr}}_{\kappa,p}}\le M\}} u_0$ which coincides with u_0 on Ω_M and satisfies $\widetilde{u}_0 \in L^p_{\mathscr{F}_0}(\Omega; X^{\mathrm{Tr}}_{\kappa,p})$.

For all $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that, for* $v \in X_1$,

$$\|F^{(j)}(v)\|_{X_0} + \|G^{(j)}[v]\|_{\gamma(\ell^2(\mathbb{N}),X_{1/2})} \le \varepsilon\|v\|_{X_1} + C_\varepsilon(1+\|v\|^\delta_{X^{\mathrm{Tr}}_{\kappa,p}}) \tag{5.2.44}$$

where $\delta > 0$ *is independent of* $\varepsilon > 0$. The estimate (5.2.44) is a straightforward consequence of the *subcriticality* of the nonlinearity for the regularized thin-film equation (5.2.2), see Remark 5.2.4. Indeed, by the reiteration theorem for real-interpolation (see, e.g., [12, Theorem 3.5.3]), for all $\beta \in (1-\frac{1+\kappa}{p}, 1)$ one has

$$\|v\|_{X_\beta} \lesssim \|v\|^{1-\theta}_{X^{\mathrm{Tr}}_{\kappa,p}} \|v\|^\theta_{X_1} \quad \text{for } v \in X_1, \quad \text{with} \quad \theta = \frac{\beta - 1 + (1+\kappa)/p}{(1+\kappa)/p}.$$

One can readily check that (5.2.44) follows from the above and the subcriticality condition (5.2.20).

For all $M < \infty$*, it holds that*

$$\mathbb{E}\|u\|^p_{L^p(0,\tau,w_\kappa;X_1)} < \infty, \quad \textit{where} \quad \tau := \inf\{t \in [0,\sigma) : \|u\|_{X^{\mathrm{Tr}}_{\kappa,p}} \ge M\} \wedge T \tag{5.2.45}$$

with $\inf\varnothing := \sigma$. Let σ_n be the stopping time given by

$$\sigma_n := \inf\{t \in [0,\sigma) : \|u\|_{L^p(0,t,w_\kappa;X_1)} \ge n\} \wedge T,$$

where, as above, $\inf\varnothing := \sigma$. Now the idea is to apply stochastic maximal regularity estimates of Theorem 5.2.6 to the problem (5.2.2). To this end, note that $u \in L^p([\![0,\sigma_n \wedge \tau]\!], w_\kappa; X_1)$ solves, on $[\![0,\tau)\!)$,

$$\begin{cases} du + \operatorname{div}(a\nabla\Delta u)\,dt = \mathbf{1}_{[\![0,\tau)\!)} F^{(j)}(u)\,dt + \mathbf{1}_{[\![0,\tau)\!)} G^{(j)}[u]\,dW, \\ u(0) = \widetilde{u}_0, \end{cases} \tag{5.2.46}$$

where $a := \mathbf{1}_{[\![0,\tau)\!)} m_j(u) + \mathbf{1}_{[\![\tau,T]\!]}$. Recall that $X^{\mathrm{Tr}}_{\kappa,p} \hookrightarrow L^\infty(\mathbb{T}^d)$ by (5.1.8). Hence, by [115, Theorem 1, p.373],

$$\operatorname*{ess\,inf}_{[0,T]\times\Omega} a \gtrsim_j 1 \qquad \text{and} \qquad \operatorname*{ess\,sup}_{[0,T]\times\Omega} \|a\|_{X^{\mathrm{Tr}}_{\kappa,p}} < \infty.$$

Combining (5.2.46), Theorem 5.2.6 and [5, Proposition 3.12(b)], we obtain the existence of a constant $C^{(0)}_M > 0$ such that, for all $n \ge 1$,

$$\begin{aligned} &\mathbb{E}\|u\|^p_{L^p(0,\tau\wedge\sigma_n,w_\kappa;X_1)} \\ &\le C^{(0)}_M \mathbb{E}\|\widetilde{u}_0\|^p_{X^{\mathrm{Tr}}_{\kappa,p}} + C^{(0)}_M \mathbb{E}\int_0^{\tau\wedge\sigma_n} \big(\|F^{(j)}(u)\|^p_{X_0} + \|G^{(j)}[u]\|^p_{\gamma(\ell^2(\mathbb{N}),X_{1/2})}\big)\, w_\kappa\,dt \\ &\le C^{(0)}_M \mathbb{E}\|\widetilde{u}_0\|^p_{X^{\mathrm{Tr}}_{\kappa,p}} + \tfrac{1}{2}\mathbb{E}\|u\|^p_{L^p(0,\tau\wedge\sigma_n,w_\kappa;X_1)} + C^{(1)}_{\delta,M} \end{aligned}$$

where we applied (5.2.44) with $\varepsilon = (2C^{(0)}_M)^{-1}$ and used that $\sup_{t\in[0,\tau)}\|u\|_{X^{\mathrm{Tr}}_{\kappa,p}} \le M$. The above implies

$$\mathbb{E}\|u\|^p_{L^p(0,\tau\wedge\sigma_n,w_\kappa;X_1)} \le C^{(0)}_M \mathbb{E}\|\widetilde{u}_0\|^p_{X^{\mathrm{Tr}}_{\kappa,p}} + C^{(1)}_{\delta,M}.$$

The claim of Step 2 follows by letting $n \to \infty$ and using Fatou's lemma.

Proof of (5.2.43). Let τ be defined as in (5.2.45). By the first two steps we have

$$F^{(j)}(u) \in L^p((0,\tau)\times\Omega, w_\kappa; X_0) \quad \text{and} \quad G^{(j)}[u] \in L^p((0,\tau)\times\Omega, w_\kappa; \gamma(\ell^2(\mathbb{N}), X_{1/2})).$$

Again, by (5.2.46), Theorem 5.2.6 with $a := \mathbf{1}_{[\![0,\tau)\!)} m_j(u) + \mathbf{1}_{[\![\tau,T]\!]}$ and [5, Proposition 3.12(b)],

$$u \in L^p(\Omega; C([0,\tau]; X^{\mathrm{Tr}}_{\kappa,p})).$$

In particular, $\lim_{t\nearrow\sigma} u(t)$ exists in $X^{\mathrm{Tr}}_{\kappa,p}$ a.s. on $\{\tau=\sigma\}$. Since $\{\tau=\sigma\}\supseteq\Omega_M$ by definition of τ, we have

$$\begin{aligned}\mathbb{P}(\Omega_M) &= \mathbb{P}\Big(\Omega_M \cap \Big\{\sigma < T, \lim_{t\nearrow\sigma} u(t) \text{ exists in } X^{\mathrm{Tr}}_{\kappa,p}\Big\}\Big)\\ &\le \mathbb{P}\Big(\sigma < T, \lim_{t\nearrow\sigma} u(t) \text{ exists in } X^{\mathrm{Tr}}_{\kappa,p}\Big) \overset{(5.2.42)}{=} 0.\end{aligned}$$

Thus (5.2.43) is proved. The claimed blow-up criterion (5.2.41) follows from the arbitrariness of M. □

5

5.2.4. INSTANTANEOUS REGULARIZATION AND BLOW-UP CRITERIA II

This subsection is dedicated to understanding how the regularity of the noise affects the regularity of solutions to the regularized problem (5.2.2) and its consequences in terms of blow-up criteria.

The following result is the key ingredient in the proof of Proposition 5.1.7.

Proposition 5.2.13 (Instantaneous regularization—Regularized problem)**.** *Let* (p,κ,s,q) *be as in Assumption 5.1.1. Let* $u_0 \in L^0_{\mathscr{F}_0}(\Omega; X^{\mathrm{Tr}}_{\kappa,p})$ *and* (u,σ) *the maximal unique* (p,κ,s,q)*-solution to* (5.2.2) *provided by Proposition 5.2.12. Suppose that Assumption 5.1.5*(s_ψ,q_ψ) *holds for some* $s_\psi \ge s$ *and all* $q_\psi \in [2,\infty)$*. Then, a.s.,*

$$u \in H^{\theta,r}_{\mathrm{loc}}(0,\sigma; H^{2+s_\psi-4\theta,\zeta}(\mathbb{T}^d)) \ \textit{for all}\ \theta\in[0,\tfrac12) \ \textit{and}\ r,\zeta\in[2,\infty). \tag{5.2.47}$$

In particular $u \in C^{\theta_1,\theta_2}_{\mathrm{loc}}((0,\sigma)\times\mathbb{T}^d)$ *a.s. for all* $\theta_1\in[0,\frac12)$ *and* $\theta_2\in(0,2+s_\psi)$*.*

Proof. The last assertion follows from (5.2.47) and Sobolev embeddings. Thus, below we only prove (5.2.47). Our strategy is to apply the instantaneous regularization theory for stochastic parabolic evolution equations from [6, Section 6] to (5.2.2) which, as we have commented in the proof of Proposition 5.2.12, we see as the stochastic evolution equation (5.2.3). The proof of improved regularity is structured as follows:

- Reduction to the case of a positive weight $\kappa > 0$.
- Bootstrap time integrability.
- Bootstrap spatial integrability.
- Bootstrap spatial smoothness.

The reduction in the first step is convenient in light of instantaneous regularization results for weighted function spaces, see, e.g., [2, Theorem 1.2].

We can assume that $p > 2$ and $\kappa > 0$. In this step, we assume that (5.2.47) is valid in the case $p > 2$ and $\kappa > 0$ and show that it carries over to the general case $p \geq 2$, $\kappa \geq 0$. To this end, it suffices to show that if either $p = 2$ or $\kappa = 0$, there exists a new set of parameters $(\widehat{p}, \widehat{\kappa}, \widehat{s}, q)$ with $\widehat{p} > 2$ and $\widehat{\kappa} > 0$, such that (u, σ) coincides with the maximal unique $(\widehat{p}, \widehat{\kappa}, \widehat{s}, q)$-solution to (5.2.2) in the sense of Definitions 5.2.10 and 5.2.11. Let us start with the case $p = 2$. By (5.1.7) in Assumption 5.1.1, this forces $\kappa = 0$ and $q = 2$. By standard interpolation inequalities, a.s.,

$$
\begin{aligned}
u \in\ & L^2_{\mathrm{loc}}([0,\sigma); H^{s+2}(\mathbb{T}^d)) \cap C([0,\sigma); H^s(\mathbb{T}^d)) \\
&\hookrightarrow L^{2/\theta}_{\mathrm{loc}}([0,\sigma); H^{s+2\theta}(\mathbb{T}^d)) \hookrightarrow L^{2/\theta}_{\mathrm{loc}}([0,\sigma), w_{\widehat{\kappa}}; H^{s+2\theta}(\mathbb{T}^d)),
\end{aligned} \tag{5.2.48}
$$

where $\theta \in (0,1)$ and in the last embedding we used that the weight w_κ is bounded. Now, by continuity, there exist $\widehat{\theta} \in (0,1)$ and $\widehat{\kappa} > 0$ for which Assumption 5.1.1 with (p, κ, s) replaced by $(\widehat{p}, \widehat{\kappa}, \widehat{s})$ holds, where $\widehat{p} := 2/\widehat{\theta}$, and that $\widehat{s} + 2 - 4/\widehat{p} < s$. The previous condition ensures that $H^s(\mathbb{T}^d) \hookrightarrow B^{s+2-4\frac{1+\widehat{\kappa}}{\widehat{p}}}_{2,\widehat{p}}(\mathbb{T}^d)$. Thus, by (5.2.48), a.s.,

$$
u \in L^{\widehat{p}}_{\mathrm{loc}}([0,\sigma); H^{\widehat{s}+2}(\mathbb{T}^d)) \cap C([0,\sigma); B^{s+2-4\frac{1+\widehat{\kappa}}{\widehat{p}}}_{2,\widehat{p}}(\mathbb{T}^d)).
$$

In particular (u, σ) is a local $(\widehat{p}, \widehat{\kappa}, \widehat{s}, 2)$-solution to (5.2.2). Let $(\widehat{u}, \widehat{\sigma})$ be the maximal unique $(\widehat{p}, \widehat{\kappa}, \widehat{s}, 2)$-solution to (5.2.2) provided by Proposition 5.2.12. By maximality (cf., Definition 5.2.11), we obtain

$$
\sigma \leq \widehat{\sigma} \text{ a.s.} \qquad \text{and} \qquad u = \widehat{u} \text{ a.e. on } [\![0,\sigma)\!).
$$

Hence, the claim of this step follows in this case, since the regularity of the $(\widehat{p}, \widehat{\kappa}, \widehat{s}, 2)$-solution $(\widehat{u}, \widehat{\sigma})$ transfers to (u, σ). For completeness, let us show that such improved regularity also yields $\sigma = \widehat{\sigma}$ a.s. Indeed, if (5.2.47) holds with (u, σ) replaced by $(\widehat{u}, \widehat{\sigma})$, it follows that

$$
u = \widehat{u} \in C((0,\sigma]; H^s(\mathbb{T}^d)) \text{ a.s. on } \{\sigma < \widehat{\sigma}\}.
$$

Thus, for all $T < \infty$,

$$
\mathbb{P}(\sigma < \widehat{\sigma}, \sigma < T) \leq \mathbb{P}\Big(\sigma < T, \sup_{t \in [0,\sigma)} \|u(t)\|_{H^s(\mathbb{T}^d)} < \infty\Big) = 0,
$$

where the last equality follows from (5.2.41) with $p = 2$ and $\kappa = 0$. The arbitrariness of $T < \infty$ yields $\sigma = \widehat{\sigma}$ a.s. on $\{\sigma < \infty\}$. Since $\sigma \leq \widehat{\sigma}$ a.s., it follows that $\sigma = \widehat{\sigma}$ a.s., as desired.

For the case that $p > 2$ and $\kappa = 0$ the same procedure works by passing to the new parameters $(p, \widehat{\kappa}, s, q)$ with $\widehat{\kappa}$ slightly larger than 0.

Temporal regularity. We continue with the initial set of parameters (p, κ, s, q) and can assume by the previous step that $p > 2$ and $\kappa > 0$. To deduce additional temporal regularity of (u, σ), we verify the assumptions of [6, Corollary 6.5] with $Y_0 = X_0$, $Y_1 = X_1$, $r = p$ and $\alpha = \kappa$. The technical conditions [6, Assumption 4.5, Assumption 4.7] follow as in the proof of Proposition 5.2.12. Assumption [6, Corollary 6.5 (1)] holds by $\kappa > 0$ and the regularity assertion (5.2.40) of Proposition 5.2.12 for $p>2$. To check [6, Corollary 6.5 (2)], we let $\widehat{r}\in[p,\infty)$ and $\widehat{\alpha}\in[0,\frac{\widehat{r}}{2}-1)$ with $\frac{1+\widehat{\alpha}}{\widehat{r}}<\frac{1+\kappa}{p}$ and observe that Assumption 5.1.1 holds for the parameters $(\widehat{r},\widehat{\alpha},s,q)$ too. Thus, again by the same reasoning as in the proof of Proposition 5.2.12 we obtain that [6, Assumption 4.5, Assumption 4.7] holds for the choice of parameters $(\widehat{r},\widehat{\kappa},s,q)$ as well. Consequently [6, Corollary 6.5] indeed applies and yields that

$$u\in\bigcap_{\theta\in[0,\frac{1}{2})}H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma;H^{s+2-4\theta,q}(\mathbb{T}^d))\ \text{ a.s. for all } \widehat{r}\in[p,\infty). \tag{5.2.49}$$

Integrability in space. We again consider the maximal unique (p,κ,s,q)-solution (u,σ) with $p>2$ and $\kappa>0$. As a next step, we show that, for all $\widehat{r}\geq p$, there exists a sequence $q_l\nearrow\infty$ with $q_1=q$ such that

$$u\in\bigcap_{\theta\in[0,\frac{1}{2})}H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma;H^{s+2-4\theta,q_l}(\mathbb{T}^d))\ \text{ a.s.} \tag{5.2.50}$$

implies

$$u\in\bigcap_{\theta\in[0,\frac{1}{2})}H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma;H^{s+2-4\theta,q_{l+1}}(\mathbb{T}^d))\ \text{ a.s.}$$

Then, by a recursive application starting from (5.2.49), one concludes that

$$u\in\bigcap_{\theta\in[0,\frac{1}{2})}H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma;H^{s+2-4\theta,\widehat{q}}(\mathbb{T}^d)) \tag{5.2.51}$$

a.s. for all $\widehat{r}\in[2,\infty)$ and $\widehat{q}\in[2,\infty)$. To this end, we apply [6, Theorem 6.3] with

$$Y_0=H^{s-2,q_l}(\mathbb{T}^d),\qquad Y_1=H^{s+2,q_l}(\mathbb{T}^d)\qquad \widehat{Y}_0=H^{s-2,q_{l+1}}(\mathbb{T}^d),\qquad \widehat{Y}_1=H^{s+2,q_{l+1}}(\mathbb{T}^d)$$

and $r=\widehat{r}$, $\alpha=\widehat{\alpha}=\kappa>0$ and $q_{l+1}\geq q_l\geq q_1=q$ for $l\in\mathbb{N}$. As in the previous step, the technical conditions of [6, Theorem 6.3] follow from the fact that Assumption 5.1.1 remains valid if q and p increase. Therefore, (5.2.50) implies that [6, Theorem 6.3 (1)] holds and [6, Theorem 6.3 (2)] follows because we are in the subcritical regime as observed in Remark 5.2.4. Since [6, Eq. (6.1)] trivializes by [6, Lemma 6.2 (1)] it remains to ensure that $Y^{\mathrm{Tr}}_r\hookrightarrow\widehat{Y}^{\mathrm{Tr}}_{r,\kappa}$ for [6, Theorem 6.3 (3)] to hold. The desired inclusion

$$B^{s+2-\frac{4}{r}}_{q_l,p}(\mathbb{T}^d)\hookrightarrow B^{s+2-4\frac{1+\kappa}{r}}_{q_{l+1},p}(\mathbb{T}^d)$$

holds if the condition

$$s+2-\frac{4}{r}-\frac{d}{q_l}>s+2-4\frac{1+\kappa}{r}-\frac{d}{q_{l+1}}$$

is satisfied. The previous inequality is equivalent to

$$\frac{d}{q_l}-\frac{d}{q_{l+1}}<\frac{4\kappa}{r}$$

for which it suffices to choose $q_{l+1}<q_l+\frac{4\kappa q^2}{rd}$. Then [6, Theorem 6.3] is indeed applicable and yields the improved regularity (5.2.51), finishing this part of the proof.

Smoothness in space. For the last step, we apply once more [6, Theorem 6.3] to the maximal unique (p,κ,s,q)-solution (u,σ) with $p>2$ and $\kappa>0$. Similar to the previous step, we show the existence of a finite set of values $(s_l)_{l=1}^{L}$ with $s_1=s$ and $s_L=s_\psi$ (here s_ψ is as in Assumption 5.1.5(s_ψ,q_ψ)) such that

$$u\in\bigcap_{\theta\in[0,\frac{1}{2})}H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma;H^{s_l+2-4\theta,\widehat{q}}(\mathbb{T}^d)) \tag{5.2.52}$$

a.s. for all $\widehat{r},\widehat{q}\in[2,\infty)$, implies

$$u\in\bigcap_{\theta\in[0,\frac{1}{2})}H^{\theta,\widehat{r}}_{\mathrm{loc}}(0,\sigma;H^{s_{l+1}+2-4\theta,\widehat{q}}(\mathbb{T}^d)) \tag{5.2.53}$$

a.s. for all $\widehat{r},\widehat{q}\in[2,\infty)$.

We prove the above implication with $s_{l+1}:=(s_l+1)\wedge s_\psi$. By the previous steps, we can assume that $\widehat{r}>p\vee 4$ and $\widehat{q}\geq q$. To begin, note that (5.2.52) with $s_1=s$ holds due to the previous step. Now we prove that (5.2.52) implies (5.2.53) by using [6, Theorem 6.3] with

$$Y_0=H^{s_l-2,\widehat{q}}(\mathbb{T}^d),\qquad Y_1=H^{s_l+2,\widehat{q}}(\mathbb{T}^d),\qquad \widehat{Y}_0=H^{s_{l+1}-2,\widehat{q}}(\mathbb{T}^d),\qquad \widehat{Y}_1=H^{s_{l+1}+2,\widehat{q}}(\mathbb{T}^d)$$

for $\widehat{q}\in[q,\infty)$ and we set $r=\widehat{r}$, $\alpha=0$, and

$$\widehat{\alpha}:=\frac{r(s_{l+1}-s_l)}{4}>0. \tag{5.2.54}$$

Note that, if $r>4$, then $\widehat{\alpha}<\frac{r}{2}-1$ since $s_{l+1}\leq s_l+1$. The motivation behind the choice of $\widehat{\alpha}$ is that the trace spaces corresponding to the settings (Y_0,Y_1,α,r) and $(\widehat{Y}_0,\widehat{Y}_1,\widehat{\alpha},\widehat{r})$ coincide:

$$Y^{\mathrm{Tr}}_r=\widehat{Y}^{\mathrm{Tr}}_{\widehat{\alpha},\widehat{r}}=B^{2+s_l-\frac{4}{r}}_{\widehat{q},r}(\mathbb{T}^d). \tag{5.2.55}$$

It remains to check the assumptions of [6, Theorem 6.3]. Firstly, note that Assumption 5.1.1 is valid for $(r,\alpha,s_l,\widehat{q})$ as it is valid for (p,κ,s,q) and $s_l\geq s$, $r\geq p$ as well as $\alpha=0$. Since $s_{l+1}-\frac{1+\widehat{\alpha}}{r}=s_l-\frac{1}{r}$ and $s_{l+1}>s_l$ by construction, Assumption 5.1.1 is valid for $(r,\widehat{\alpha},s_{l+1},\widehat{q})$ as well. Hence, by Remark 5.2.4, we are again in the subcritical regime and consequently (5.2.52), which holds by the inductive assumption, yields that the conditions [6, Theorem 6.3 (1)] and [6, Theorem 6.3 (2)] are fulfilled. Due to (5.2.55), to check the conditions in [6, Theorem 6.3 (3)] it remains to verify [6, Eq. (6.1)]. Then, choosing $\varepsilon=\frac{s_{l+1}-s_l}{4}$ it holds that

$$\frac{1+\widehat{\alpha}}{\widehat{r}}=\frac{1}{r}+\frac{s_{l+1}-s_l}{4}=\frac{1+\alpha}{r}+\varepsilon$$

by (5.2.54) and $\alpha=0$. Moreover $\varepsilon\in(0,\frac{1}{2}-\frac{1}{r})$ as $s_{l+1}-s_l\leq 1$ and $r>4$. Finally, it is straightforward to check that $\widehat{Y}_{1-\varepsilon}=Y_1$ and $\widehat{Y}_0=Y_\varepsilon$ and therefore [6, Lemma 6.2 (4)] implies [6, Eq. (6.1)]. In conclusion [6, Theorem 6.3] is applicable and yields the desired regularity (5.2.53). □

As commented below the statement of Proposition 5.1.8, the independence of the regularity of solutions at positive times allows us to deduce the independence of the blow-up criteria from the original choice of the admissible parameters. This observation and the blow-up criterion in Proposition 5.2.12 readily imply the following result, which is the key ingredient in the proof of Proposition 5.1.8.

Corollary 5.2.14 (Blow-up criteria—Regularized problem)**.** *Let (p,κ,s,q) be as in Assumption 5.1.1. Moreover, suppose that Assumption 5.1.5(s_ψ,q_ψ) holds for some $s_\psi \geq s$ and all $q_\psi \in [2,\infty)$. Let $u_0 \in L^0_{\mathscr{F}_0}(\Omega;X^{\mathrm{Tr}}_{\kappa,p})$ and (u,σ) the maximal unique (p,κ,s,q)-solution to (5.2.2) provided by Proposition 5.2.12. Then, for any quadruple of admissible parameters (p_0,κ_0,s_0,q_0) with $s_0 \leq s_\psi$ and $0<\varepsilon<T<\infty$,*

$$\mathbb{P}\Big(\varepsilon<\sigma<T,\ \sup_{t\in[\varepsilon,\sigma)}\|u(t)\|_{B^{\gamma_0}_{q_0,p_0}(\mathbb{T}^d)}<\infty\Big)=0 \tag{5.2.56}$$

where $\gamma_0 := s_0+2-4\frac{1+\kappa_0}{p_0}$.

Note that the norm in (5.2.56) can be evaluated even if $(p_0,\kappa_0,s_0,q_0)\neq(p,\kappa,s,q)$ due to Proposition 5.2.13.

Proof. By Proposition 5.2.13 and the blow-up criterion in Proposition 5.2.12, (5.2.56) follows as in the proof of [7, Theorem 2.10]. We include some details for the reader's convenience. We begin by collecting some useful facts. Let (u,σ) be the maximal unique (p,κ,s,q)-solution to (5.2.2) provided by Proposition 5.2.12. Proposition 5.2.13 ensures that, for all $\varepsilon>0$,

$$\mathbf{1}_{\{\sigma>\varepsilon\}}u(\varepsilon)\in L^0_{\mathscr{F}_\varepsilon}(\Omega;B^{\gamma_0}_{q_0,p_0}(\mathbb{T}^d)).$$

Since Assumption 5.1.5(s_ψ,q_ψ) holds for some $s_\psi\geq s_0\vee s$ and all $q_\psi\in[2,\infty)$, Proposition 5.2.12 yields the existence of maximal unique (p_0,κ_0,s_0,q_0)-solution (v,τ) to

$$\begin{cases} \mathrm{d}v = -\mathrm{div}(m_j(v)\nabla\Delta v)\,\mathrm{d}t+\mathrm{div}(\Phi_j(v)\nabla v)\,\mathrm{d}t+\sum_{k\in\mathbb{N}}\mathrm{div}(g_j(v)\psi_k)\,\mathrm{d}\beta^{(k)},\\ v(0) = \mathbf{1}_{\{\sigma>\varepsilon\}}u(\varepsilon). \end{cases} \tag{5.2.57}$$

with

$$\mathbb{P}\Big(\varepsilon<\tau<T,\ \sup_{t\in[\varepsilon,\tau)}\|v(t)\|_{B^{\gamma_0}_{q_0,p_0}(\mathbb{T}^d)}<\infty\Big)=0. \tag{5.2.58}$$

Finally, Proposition 5.2.13 and a translation argument ensure that

$$v\in C((\varepsilon,\tau);C^{2+s_\psi-}(\mathbb{T}^d))\ \text{ a.s.} \tag{5.2.59}$$

We now turn to the proof of Corollary 5.2.14. By (5.2.58), it is enough to prove that

$$\sigma=\tau \text{ a.s. on } \{\varepsilon<\sigma\leq T\} \quad\text{and}\quad u=v \text{ a.e. on } [\varepsilon,\sigma)\times\{\varepsilon<\sigma\leq T\}. \tag{5.2.60}$$

To this end, we note that, due to Proposition 5.2.13, $(u|_{[\![\varepsilon,\sigma)\!)},\sigma\mathbf{1}_{\{\sigma>\varepsilon\}}+\varepsilon\mathbf{1}_{\{\sigma\leq\varepsilon\}})$ is a local (p_0,κ_0,s_0,q_0)-solution to (5.2.57). The maximality of (v,τ) yields

$$\sigma\leq\tau \text{ a.s. on } \{\varepsilon<\sigma\leq T\} \quad\text{and}\quad u=v \text{ a.e. on } [\varepsilon,\sigma)\times\{\varepsilon<\sigma\leq T\}. \tag{5.2.61}$$

5

Therefore, since $C^{2+s_\psi-}(\mathbb{T}^d)\hookrightarrow B^{2+s-4\frac{1+\kappa}{p}}_{q,p}(\mathbb{T}^d)=X^{\mathrm{Tr}}_{\kappa,p}$ as $s\le s_\psi$, (5.2.59) and the previous yield

$$\mathbb{P}(\varepsilon<\sigma<\tau\le T)\le\mathbb{P}\Big(\{\varepsilon<\sigma<\tau\}\cap\Big\{\sigma<T,\ \sup_{t\in[0,\sigma)}\|u(t)\|_{X^{\mathrm{Tr}}_{\kappa,p}}<\infty\Big\}\Big)\overset{(5.2.41)}{=}0.$$

Hence $\sigma=\tau$ a.s. on $\{\varepsilon<\sigma\le T\}$, and therefore (5.2.60) follows from (5.2.61). □

5.2.5. Transference to the original equation

Finally, we are concerned with transferring the previously obtained results on (5.2.2) to positive solutions to the original equation (5.1.1) which leads to the results stated in Subsection 5.1.1. The main idea is that as long as a solution to (5.2.2) remains above the threshold $2j^{-1}$ it is also a solution to (5.2.3) and vice versa due to the choice of the regularization (5.2.1).

In the proof of Theorem 5.1.6 below, as an intermediate step to obtain maximal solutions, we also employ the notion of positive local *unique* solutions to (5.1.1). Recall that positive local solutions of (5.1.1) are defined in Definition 5.1.3.

5

Definition 5.2.15 (Local unique positive solution for (5.1.1))**.** *Let (p,κ,s,q) be as in Assumption 5.1.1. A positive local (p,κ,s,q)-solution (u,σ) to (5.1.1) is called* positive local unique (p,κ,s,q)-solution, *if for every positive local (p,κ,s,q)-solution (v,τ) to (5.1.1) one has $u=v$ a.s. on $[0,\sigma\wedge\tau)$.*

In contrast to *maximal solutions* as defined in Definition 5.1.4, the lifetime σ of the unique solution u does not necessarily extend the one of the other solution v, i.e., τ.

Proof of Theorem 5.1.6. Recall that (p,κ,s,q) are as in Assumption 5.1.1, $(\psi_k)_{k\in\mathbb{N}}$ satisfy Assumption 5.1.5(s_ψ,q_ψ) with $(s_\psi,q_\psi)=(s,q)$ and $\inf_{\mathbb{T}^d}u_0>0$ a.s., respectively.

Existence of a positive local unique solution. Set

$$\Omega_1:=\Big\{\inf_{\mathbb{T}^d}u_0\ge1\Big\},\quad\text{and}\quad\Omega_j:=\Big\{\tfrac{1}{j+1}\le\inf_{\mathbb{T}^d}u_0<\tfrac{1}{j}\Big\}\ \text{ for } j\ge1.$$

From the positivity assumption on the initial data, it follows that $\mathbb{P}(\cup_{j\in\mathbb{N}}\Omega_j)=1$. Hence, to construct a positive local (p,κ,s,q)-solution to (5.1.1) it is enough to construct a solution on Ω_j for $j\in\mathbb{N}$. To this end, recall that, Proposition 5.2.12 ensures the existence of maximal unique (p,κ,s,q)-solution $(u^{(j)},\sigma^{(j)})$ to

$$\begin{cases}\mathrm{d}u^{(j)}=\big[-\mathrm{div}(m_{2j+3}(u^{(j)})\nabla\Delta u^{(j)})+\mathrm{div}(\Phi_{2j+3}(u^{(j)})\nabla u^{(j)})\big]\,\mathrm{d}t\\ \qquad\qquad+\sum_{k\in\mathbb{N}}\mathrm{div}(g_{2j+3}(u^{(j)})\psi_k)\,\mathrm{d}\beta^{(k)},\\ u^{(j)}(0)=\mathbf{1}_{\Omega_j}u_0.\end{cases}\tag{5.2.62}$$

Now, let us define

$$\widetilde{\sigma}^{(j)}:=\inf\Big\{t\in[0,\sigma^{(j)}):\inf_{\mathbb{T}^d}u(t,\cdot)\le\tfrac{2}{2j+3}\Big\}$$

where $\inf\varnothing:=\sigma^{(j)}$. Note that $\widetilde{\sigma}^{(j)}>0$ a.s. on Ω_j by construction, and a.e. on $[0,\widetilde{\sigma}^{(j)})$,

$$m_{2j+3}(u^{(j)})=m(u^{(j)}),\quad\Phi_{2j+3}(u^{(j)})=\Phi(u^{(j)}),\qquad g_{2j+3}(u^{(j)})=g(u^{(j)}).$$

In particular, for $j \in \mathbb{N}$, letting

$$\widetilde{\sigma} := \widetilde{\sigma}^{(j)} \text{ on } \Omega_j \quad \text{and} \quad u := u^{(j)} \text{ on } [0, \widetilde{\sigma}_j) \times \Omega_j,$$

one can check that $(u, \widetilde{\sigma})$ is a positive local (p, κ, s, q)-solution to the original equation (5.1.1). To conclude this step, it remains to discuss its uniqueness, see Definition 5.2.15. To this end, let (v, τ) be a positive local (p, κ, s, q)-solution to (5.1.1). We define

$$\tau^{(j)} := \inf\Big\{t \in [0, \tau) : \inf_{\mathbb{T}^d} v(t, \cdot) \le \tfrac{2}{2j+3}\Big\}$$

where, as usual, $\inf \varnothing := \tau$. Now, arguing as above, it readily follows that $(v|_{\Omega_j}, \mathbf{1}_{\Omega_j}\tau^{(j)})$ is a local (p, κ, s, q)-solution to (5.2.62). By maximality of the solution $(u^{(j)}, \sigma^{(j)})$ we have $\tau^{(j)} \le \sigma^{(j)}$ a.s. on Ω_j and $v = u^{(j)}$ a.e. on $[0, \tau^{(j)}) \times \Omega_j$. Hence, for all $j \in \mathbb{N}$, we have $v = u^{(j)}$ a.e. on $[0, \widetilde{\sigma}^{(j)} \wedge \tau^{(j)}) \times \Omega_j$. The latter implies $\widetilde{\sigma}^{(j)} \wedge \tau^{(j)} = \widetilde{\sigma}^{(j)} \wedge \tau$ a.s. and therefore $v = u$ a.e. on $[\![0, \widetilde{\sigma} \wedge \tau)\!)$.

Maximality within the class of local unique solutions. In this step, we build a positive unique local (p, κ, s, q)-solution to (5.1.1) which is maximal within the set of positive unique local (p, κ, s, q)-solution. We set

$$\mathscr{T} := \Big\{\tau : \tau \text{ is a stopping time for which there exists a positive local unique } (p, \kappa, s, q)\text{-solution } (v, \tau) \text{ to (5.1.1)}\Big\}. \tag{5.2.63}$$

Note that, the above set is non-empty as $\widetilde{\sigma} \in \mathscr{T}$, where $\widetilde{\sigma}$ is as in the previous step. Proceeding as in [5, Step 5b, proof of Theorem 4.5], the uniqueness requirement for local solutions in $\mathscr{T}$ yields $\tau_0, \tau_1 \in \mathscr{T}$ implies $\tau_0 \vee \tau_1 \in \mathscr{T}$. In particular, by [91, Theorem A.3], we conclude that $\sigma := \operatorname{ess\,sup}_{\tau \in \mathscr{T}} \tau$ is a stopping time and there exists a positive local (p, κ, s, q)-solution (u, σ) to (5.1.1) with a localizing sequence $(\sigma_l)_{l \in \mathbb{N}}$, cf., Definition 5.1.3. Let us conclude by noticing that $\sigma \ge \widetilde{\sigma} > 0$ a.s.

At this stage, we do not know whether (u, σ) constructed in this step is a positive maximal unique (p, κ, s, q)-solution as we are not excluding the existence of a positive local (but not unique) (p, κ, s, q)-solution (v, τ) satisfying $\mathbb{P}(\tau > \sigma) > 0$. To prove that (u, σ) is actually a positive maximal unique (p, κ, s, q)-solution to (5.1.1) we employ a blow-up criterion for (u, σ) as constructed above, cf., [6, Remark 5.6].

A blow-up criterion. Let (u, σ) be the positive unique local (p, κ, s, q)-solution constructed in the previous step. Then, for all $T < \infty$,

$$\mathbb{P}\Big(\sigma < T,\ \sup_{t \in [0, \sigma)} \|u(t)\|_{X^{\mathrm{Tr}}_{\kappa, p}} < \infty,\ \inf_{[0, \sigma) \times \mathbb{T}^d} u > 0\Big) = 0. \tag{5.2.64}$$

We prove the claim by contradiction using the maximality among unique solutions of (u, σ) and the blow-up criterion (5.2.41). To begin, let us assume that (5.2.64) is false, and therefore, for some $T_\star > 0$ and $j_\star \in \mathbb{N}$,

$$\mathbb{P}(\mathscr{O}_\star) > 0 \quad \text{where} \quad \mathscr{O}_\star := \Big\{\sigma < T_\star,\ \sup_{t \in [0, \sigma)} \|u(t)\|_{X^{\mathrm{Tr}}_{\kappa, p}} < \infty,\ \inf_{[0, \sigma) \times \mathbb{T}^d} u \ge \tfrac{2}{j_\star}\Big\}.$$

Let $\tau_\star$ be the following stopping time

$$\tau_\star := \inf\Big\{t \in [0,\sigma) : \inf_{\mathbb{T}^d} u(t,\cdot) \le \tfrac{2}{j_\star}\Big\}$$

with $\inf\varnothing := \sigma$. By construction $\tau_\star = \sigma$ on $\mathscr{O}_\star$. By the choice of the cut-off in the regularized equation (5.2.2), $(u,\tau_\star)$ is a local (p,κ,s,q)-solution to (5.2.2) for all $j \ge j_\star$.

Now, by Proposition 5.2.12, there exists a maximal unique (p,κ,s,q)-solution $(u_\star,\sigma_\star)$ to (5.2.2) with $j = j_\star + 1$. By maximality of $(u_\star,\sigma_\star)$, it follows that $\tau_\star \le \sigma_\star$ a.s. and $u = u_\star$ a.e. on $[\![0,\tau_\star)\!)$. The latter fact and the definition of $\mathscr{O}_\star$ yield

$$\mathbb{P}(\{\tau_\star = \sigma_\star\}\cap\mathscr{O}_\star) \le \mathbb{P}\Big(\{\tau_\star = \sigma_\star\}\cap\Big\{\sigma_\star < T_\star,\ \sup_{t\in[0,\sigma_\star)} \|u_\star(t)\|_{X^{\mathrm{Tr}}_{\kappa,p}} < \infty\Big\}\Big) \overset{(5.2.41)}{=} 0.$$

Thus, $\sigma_\star > \tau_\star$ a.s. on $\Omega_\star$. Consider the following stopping time

$$\tau_{\star\star} := \inf\Big\{t \in [0,\sigma_\star) : \inf_{\mathbb{T}^d} u_\star(t,\cdot) \le \tfrac{2}{j_\star+1}\Big\}$$

with $\inf\varnothing := \sigma_\star$. Since $\tau_\star < \sigma_\star$ and $\tau_\star = \sigma$ a.s. on $\mathscr{O}_\star$, it follows that $\tau_{\star\star} > \sigma$ a.s. on $\mathscr{O}_\star$. Arguing as in the first step of the current proof, by maximality of $(u_\star,\sigma_\star)$, one can check that $(u_\star,\tau_{\star\star})$ is a positive local unique (p,κ,s,q)-solution to the original problem (5.1.1) which extends (u,σ) as $\tau_{\star\star} > \sigma$ a.s. on a set of positive probability $\mathscr{O}_\star$. This contradicts the maximality of σ in the set of positive local unique solutions $\mathscr{T}$, see (5.2.63). Therefore (5.2.64) holds.

Maximality in the class of positive local solution. Now we show that the positive local unique (p,κ,s,q)-solution (u,σ) to (5.1.1) constructed above is actually maximal. Indeed, let (v,τ) be another positive local (p,κ,s,q)-solution to (5.1.1). By uniqueness of (u,σ) we have $u = v$ a.s. on $[\![0,\tau\wedge\sigma)\!)$. Hence, it remains to prove that $\tau \le \sigma$ a.s. By the regularity of local (p,κ,s,q)-solutions, a.s. on $\{\sigma < \tau\}$, we have

$$u = v \in C([0,\sigma]; X^{\mathrm{Tr}}_{\kappa,p}) \quad \text{and} \quad \inf_{(0,\sigma)\times\mathbb{T}^d} u = \inf_{(0,\sigma)\times\mathbb{T}^d} v > 0.$$

Therefore, for all $T < \infty$,

$$\mathbb{P}(\sigma < \tau, \sigma < T) = \mathbb{P}\Big(\{\sigma<\tau\}\cap\Big\{\sigma < T,\ \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathrm{Tr}}_{\kappa,p}} < \infty,\ \inf_{[0,\sigma)\times\mathbb{T}^d} u > 0\Big\}\Big) \overset{(5.2.64)}{=} 0.$$

The arbitrariness of $T < \infty$ implies that $\tau \le \sigma$ a.s. on $\{\sigma = \infty\}$. Hence $\tau \le \sigma$ a.s. as desired.

Additional regularity. Next, we assume that $p > 2$ and prove the additional assertions regarding the regularity of the positive maximal unique (p,κ,s,q)-solution (u,σ). To this end, for all $j \ge 1$, let

$$\tau_j := \inf\Big\{t \in [0,\sigma) : \inf_{\mathbb{T}^d} u(t,\cdot) \le \tfrac{2}{j}\Big\} \quad \text{with} \quad \inf\varnothing := \sigma.$$

Arguing as in the previous step, $(u|_{[\![0,\tau_j)\!)}, \tau_j)$ is a local (p,κ,s,q)-solution to (5.2.2). Hence, it is extended by the maximal unique (p,κ,s,q)-solution to (5.2.2) provided by Proposition 5.2.12 with j replaced by $j+1$, and admits consequently the regularity stated in Proposition 5.2.12. □

Proof of Proposition 5.1.7. Analogously to the proof of the regularity assertion of Theorem 5.1.6, by a stopping time argument and the maximality of solutions to (5.2.2), we conclude that u inherits the regularity from the solutions to the regularized problems stated in Proposition 5.2.13. □

Proof of Proposition 5.1.8. The proof is analogous to the one of (5.2.64), where instead of the blow-up criterion (5.2.41) in Proposition 5.2.12, one uses the blow-up criteria of Corollary 5.2.14. □

5.3. GLOBAL WELL-POSEDNESS IN ONE DIMENSION

The aim of this section is to show the global well-posedness of the stochastic thin-film equation (5.1.2). As laid out in the introduction of this chapter, (5.1.2) can be cast into the form (5.1.1) if one sets

$$\Phi(u) = m(u)\phi''(u), \qquad g(u) = m^{1/2}(u), \tag{5.3.1}$$

so that Theorem 5.1.6 yields that the equation is well-posed locally in time. It remains to use the blow-up criterion from Proposition 5.1.8 to deduce that the unique solution exists even globally in time. We achieve this by first establishing an a-priori estimate on the α-entropy (5.1.17) and subsequently estimating the energy (5.1.16) along the trajectory of a solution. To this end, we restrict ourselves to $d = 1$ and $s \in (1/2, 1]$ and impose Assumption 5.1.11 on the noise and Assumption 5.1.9 and 5.1.10 on the mobility function m and the effective interface potential ϕ, respectively. Accordingly, we denote the exponent of degeneracy and growth exponent of m by n and ν, and let ϑ and c_0 be as in (5.1.21) and allow for all implicit constants in this section to depend on $(\psi_k)_{k\in\mathbb{N}}$, m and ϕ and in particular on n, ν, ϑ, and c_0. Moreover, we fix an initial value $u_0 \in L^0_{\mathscr{F}_0}(\Omega; H^s(\mathbb{T}))$ with $\inf_{\mathbb{T}} u_0 > 0$ a.s. and let (u,σ) be the maximal unique positive local $(2,0,s,2)$-solution to (5.1.1) with coefficients (5.3.1) given by Theorem 5.1.6. We focus our analysis on the Itô formulation, i.e., Theorem 5.1.12, which is more delicate since no cancellations occur. The proof of Theorem 5.1.13 on the Stratonovich formulation can be obtained analogously by estimating the terms due to the Stratonovich correction in the same way as the Itô correction terms. We recall both the energy and the α-entropy functional

$$\mathscr{E}(u) = \int_{\mathbb{T}} \left[\tfrac{1}{2}|u_x|^2 + \phi(u)\right] \mathrm{d}x,$$

$$\mathscr{H}_\beta(u) = \int_{\mathbb{T}} h_\beta(u)\,\mathrm{d}x, \qquad h_\beta(r) = \int_1^r \int_1^{r'} \frac{(r'')^\beta}{m(r'')}\,\mathrm{d}r''\,\mathrm{d}r', \qquad \beta \in (-1/2, 1),$$

which appear in the following a-priori estimate on the energy at the heart of Theorem 5.1.12.

Lemma 5.3.1. *For any* $0 < t_0 < T < \infty$, $q \in [1,\infty)$ *and* $\max\{0, \nu - 5\} < \beta < 1$ *holds the energy estimate*

$$\mathbb{E}\left[\mathbf{1}_\Gamma \sup_{t_0\le t<\sigma\wedge T} \mathscr{E}^q(u(t))\right] + \mathbb{E}\left[\left(\int_{t_0}^{\sigma\wedge T}\int_{\mathbb{T}} \mathbf{1}_\Gamma m(u)(u_{xx} - \phi'(u))_x^2\,\mathrm{d}x\,\mathrm{d}t\right)^q\right] \tag{5.3.2}$$

$$\lesssim_{\beta,q,T} \mathbb{E}\Big[\mathbf{1}_\Gamma \mathscr{E}^q(u(t_0)) + \mathbf{1}_\Gamma \mathscr{H}_\beta^{6q/(5+\beta-\tilde{\nu})}(u(t_0)) + \mathbf{1}_\Gamma \mathscr{H}_0^{(\vartheta-2)q/(\vartheta+2\tilde{n}-6)}(u(t_0))\Big]$$
$$+ \mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}} u_0\,\mathrm{d}x\Big)^{\min\left\{\frac{6(\beta-\vartheta)q}{5+\beta-\tilde{\nu}},\frac{-\vartheta(\vartheta-2)q}{\vartheta+2\tilde{n}-6}\right\}} + \mathbf{1}_\Gamma\Big(\int_{\mathbb{T}} u_0\,\mathrm{d}x\Big)^{\max\left\{\frac{9(\beta+3)q}{5+\beta-\tilde{\nu}},\frac{9(\vartheta-2)q}{2\vartheta+4\tilde{n}-12}\right\}}\Big],$$

where $\tilde{n} = \min\{2,n\}$, $\tilde{\nu} = \max\{3,\nu\}$ *and*

$$\Gamma = \Big\{\sigma > t_0,\ \mathscr{E}(u(t_0)) + \mathscr{H}_\beta(u(t_0)) + \mathscr{H}_0(u(t_0)) \le l,\ \tfrac{1}{l} \le \int_{\mathbb{T}} u_0\,\mathrm{d}x \le l\Big\}$$

for some $l \in \mathbb{N}$.

We proceed to show an α-entropy estimate in Subsection 5.3.1 which is used to give the proof of Lemma 5.3.1 in Subsection 5.3.2. Before all this, however, we demonstrate how Lemma 5.3.1 leads to the global well-posedness result Theorem 5.1.12.

Proof of Theorem 5.1.12. Any process satisfying (5.1.24)–(5.1.26) constitutes a positive $(2,0,s,2)$-solution to (5.1.1) with infinite lifetime. Therefore, the claim follows if we can verify that the positive maximal unique $(2,0,s,2)$-solution (u,σ) to (5.1.1) provided by Theorem 5.1.6 satisfies a.s. $\sigma = \infty$. Indeed, this immediately yields the existence of a process satisfying (5.1.24)–(5.1.26) and uniqueness follows by the uniqueness part of Definition 5.1.4. The additional regularity assertions (5.1.27) and (5.1.28) will then follow from Proposition 5.1.7 and the fact that Assumption 5.1.11 implies Assumption 5.1.5(s_ψ, q_ψ) for $s_\psi = 1$ and any $q_\psi < \infty$.

To this end, we recall that u preserves mass, as follows by integrating (5.1.1), and therefore

$$\sup_{t\in(0,\sigma)} \int_{\mathbb{T}} u(t,x)\,\mathrm{d}x = \int_{\mathbb{T}} u_0(x)\,\mathrm{d}x. \tag{5.3.3}$$

Moreover, Lemma 5.3.1 and Proposition 5.1.7 show that, for all $0 < \varepsilon < T < \infty$ and a.s. on $\{\sigma > \varepsilon\}$,

$$\sup_{t\in[\varepsilon,\sigma\wedge T)} \mathscr{E}(u(t)) < \infty. \tag{5.3.4}$$

Combining the previous with (5.1.22), which is rigorously stated in Lemma 5.3.4, and $\vartheta > 2$ as imposed in Assumption 5.1.10 we deduce that

$$\begin{aligned}\Big(\inf_{[\varepsilon,\sigma\wedge T)\times\mathbb{T}} u\Big)^{-1} &= \sup_{t\in[\varepsilon,\sigma\wedge T)}\ \sup_{x\in\mathbb{T}} u^{-1} \\ &\lesssim \sup_{t\in[\varepsilon,\sigma\wedge T)} \mathscr{E}^{2/(\vartheta-2)}(u) + \Big(\int_{\mathbb{T}} u_0(x)\,\mathrm{d}x\Big)^{-1} < \infty,\end{aligned} \tag{5.3.5}$$

a.s. on $\{\sigma > \varepsilon\}$. In particular, u is bounded from below on $(\varepsilon, \sigma\wedge T)\times\mathbb{T}$. Since Assumption 5.1.11 implies Assumption 5.1.5(s_ψ, q_ψ) for $s_\psi = 1$ and all $q_\psi < \infty$, Proposition 5.1.8 with $s_0 = 1$, $p_0 = q_0 = 2$ and $\kappa_0 = 0$ is applicable, and combined with (5.3.3)–(5.3.5) and the Poincaré-Wirtinger inequality, it yields

$$\mathbb{P}(\varepsilon < \sigma < T) = 0 \ \text{ for all } 0 < \varepsilon < T < \infty.$$

Now the fact that $\sigma = \infty$ a.s. follows by letting $\varepsilon \searrow 0$ and $T \nearrow \infty$ as well as $\sigma > 0$ a.s. by Theorem 5.1.6. □

5.3.1. ALPHA-ENTROPY ESTIMATES

As a tool to close an α-entropy estimate we show how to control the minimum and maximum of a function in terms of the α-entropy dissipation, see [51, Lemma 4.1] and Lemma 4.2.3 from Chapter 4 for similar estimates.

Lemma 5.3.2. *Let* $\beta \in (-1/2, 1)$, *then it holds*

$$\sup_{x\in\mathbb{T}} f^{\beta-\vartheta}(x) \le \tfrac{(\beta-\vartheta)^2}{2} \int_{\mathbb{T}} f^{\beta-\vartheta-2} f_x^2 \,\mathrm{d}x + 2\Big(\int_{\mathbb{T}} f \,\mathrm{d}x\Big)^{\beta-\vartheta}, \tag{5.3.6}$$

$$\sup_{x\in\mathbb{T}} f^{\beta+5}(x) \lesssim_\beta \Big(\int_{\mathbb{T}} f^{\beta-2} f_x^4 \,\mathrm{d}x\Big)\Big(\int_{\mathbb{T}} f \,\mathrm{d}x\Big)^3 + \Big(\int_{\mathbb{T}} f \,\mathrm{d}x\Big)^{\beta+5} \tag{5.3.7}$$

for every positive function $f \in C^1(\mathbb{T})$ *bounded away from* 0.

Proof. By the fundamental theorem of calculus

$$\sup_{x\in\mathbb{T}} f^{(\beta-\vartheta)/2}(x) - \inf_{x\in\mathbb{T}} f^{(\beta-\vartheta)/2}(x) \le \Big(\int_{\mathbb{T}} |(f^{(\beta-\vartheta)/2})_x|^2 \,\mathrm{d}x\Big)^{1/2}$$
$$= \tfrac{\vartheta-\beta}{2}\Big(\int_{\mathbb{T}} f^{\beta-\vartheta-2} f_x^2 \,\mathrm{d}x\Big)^{1/2},$$

and therefore

$$\sup_{x\in\mathbb{T}} f^{(\beta-\vartheta)/2}(x) \le \tfrac{\vartheta-\beta}{2}\Big(\int_{\mathbb{T}} f^{\beta-\vartheta-2} f_x^2 \,\mathrm{d}x\Big)^{1/2} + \Big(\int_{\mathbb{T}} f \,\mathrm{d}x\Big)^{(\beta-\vartheta)/2}.$$

By squaring both sides, we conclude (5.3.6).

For (5.3.7), we introduce the function

$$g = f^{(\beta+2)/4} - \int_{\mathbb{T}} f^{(\beta+2)/4} \,\mathrm{d}x$$

so that

$$\int_{\mathbb{T}} |g|^{4/(\beta+2)} \,\mathrm{d}x \lesssim_\beta \int_{\mathbb{T}} f \,\mathrm{d}x. \tag{5.3.8}$$

We can estimate again by the fundamental theorem of calculus

$$\sup_{x\in\mathbb{T}} |g|^{(\beta+5)/(\beta+2)}(x) \lesssim_\beta \int_{\mathbb{T}} |g_x||g|^{3/(\beta+2)} \,\mathrm{d}x \le \Big(\int_{\mathbb{T}} |g_x|^4 \,\mathrm{d}x\Big)^{1/4}\Big(\int_{\mathbb{T}} |g|^{4/(\beta+2)} \,\mathrm{d}x\Big)^{3/4},$$

because g is mean free. Thus, we deduce that

$$\sup_{x\in\mathbb{T}} f^{(\beta+5)/4}(x) = \Big(\sup_{x\in\mathbb{T}} f^{(\beta+2)/4}(x)\Big)^{(\beta+5)/(\beta+2)} = \Big(\sup_{x\in\mathbb{T}} g(x) + \int_{\mathbb{T}} f^{(\beta+2)/4} \,\mathrm{d}x\Big)^{(\beta+5)/(\beta+2)}$$
$$\lesssim_\beta \sup_{x\in\mathbb{T}} |g|^{(\beta+5)/(\beta+2)}(x) + \Big(\int_{\mathbb{T}} f \,\mathrm{d}x\Big)^{(\beta+5)/4}$$
$$\lesssim_\beta \Big(\int_{\mathbb{T}} |g_x|^4 \,\mathrm{d}x\Big)^{1/4}\Big(\int_{\mathbb{T}} |g|^{4/(\beta+2)} \,\mathrm{d}x\Big)^{3/4} + \Big(\int_{\mathbb{T}} f \,\mathrm{d}x\Big)^{(\beta+5)/4}$$

and it remains to insert $g_x = \frac{\beta+2}{4} f^{\frac{\beta-2}{4}} f_x$, (5.3.8) and raise both sides to the power 4. □

To give the following proof we recall the additional regularity properties of u stated in Proposition 5.1.7 justifying all the performed integrations by parts.

Lemma 5.3.3. *Let* $0 < t_0 < T < \infty$, $q \in [1,\infty)$, $\beta \in (-1/2, 1)$ *and*

$$\gamma \in \left[\tfrac{\beta+2-\sqrt{(1-\beta)(1+2\beta)}}{3}, \tfrac{\beta+2+\sqrt{(1-\beta)(1+2\beta)}}{3}\right],$$

then holds the α*-entropy estimate*

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_\Gamma \sup_{t_0\le t<\sigma\wedge T} \mathscr{H}_\beta^q(u(t))\Big] + \mathbb{E}\Big[\Big(\int_{t_0}^{\sigma\wedge T}\int_{\mathbb{T}} \mathbf{1}_\Gamma u^{\beta-\vartheta-2} u_x^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big] \\
&+ \mathbb{E}\Big[\Big(\int_{t_0}^{\sigma\wedge T}\int_{\mathbb{T}} \mathbf{1}_\Gamma u^{\beta-2\gamma+2} (u^\gamma)_{xx}^2 \,\mathrm{d}x\,\mathrm{d}t\Big)^q + \Big(\int_{t_0}^{\sigma\wedge T}\int_{\mathbb{T}} \mathbf{1}_\Gamma u^{\beta-2} u_x^4 \,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big] \qquad (5.3.9)\\
&\lesssim_{\beta,\gamma,q,T} \mathbb{E}[\mathbf{1}_\Gamma \mathscr{H}_\beta^q(u(t_0))] + \mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}} u_0\,\mathrm{d}x\Big)^{(\beta-\vartheta)q} + \mathbf{1}_\Gamma\Big(\int_{\mathbb{T}} u_0\,\mathrm{d}x\Big)^{3(\beta+3)q/2}\Big],
\end{aligned}$$

for any $\mathscr{F}_{t_0}$*-measurable subset* Γ *of*

$$\Big\{\sigma > t_0,\ \mathscr{H}_\beta(u(t_0)) \le l,\ \tfrac{1}{l} \le \int_{\mathbb{T}} u_0\,\mathrm{d}x \le l\Big\}$$

for some $l \in \mathbb{N}$.

Proof. For a localizing sequence $0 \le \sigma_j \nearrow \sigma$ for (u,σ) as in Definition 5.1.3 we define

$$\tilde{\sigma}_j = \mathbf{1}_\Gamma \inf\Big\{t \in [t_0, \sigma_j \wedge T] : \inf_{x\in\mathbb{T}} u(t) \le \tfrac{1}{j} \text{ or } \|u(t)\|_{C^2(\mathbb{T})} + \|u\|_{L^2((t_0,t);H^3(\mathbb{T}))} \ge j\Big\} + \mathbf{1}_{\Gamma^c} t_0, \qquad (5.3.10)$$

so that $\tilde{\sigma}_j \nearrow \sigma \wedge T$ as $j \to \infty$ on Γ by Proposition 5.1.7. Correspondingly, we define the process

$$u^{(j)}(t) = \mathbf{1}_\Gamma u(t \wedge \tilde{\sigma}_j) + \mathbf{1}_{\Gamma^c} \mathbf{1}_{\mathbb{T}} \qquad (5.3.11)$$

for $t \in [t_0, T]$ and let $\tilde{h}_\beta : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with bounded second derivative such that $\tilde{h}_\beta = h_\beta$ on $[1/j, j]$. Then the assumptions of [39, Proposition A.1] are satisfied and an application of Itô's formula yields that

$$\begin{aligned}
\int_{\mathbb{T}} \tilde{h}_\beta(u^{(j)}(t))\,\mathrm{d}x &= \int_{\mathbb{T}} \tilde{h}_\beta(u^{(j)}(t_0))\,\mathrm{d}x \\
&+ \int_{t_0}^t \int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma} \tilde{h}_\beta''(u^{(j)}) u_x^{(j)} m(u^{(j)}) \big(u_{xxx}^{(j)} - \phi''(u^{(j)}) u_x^{(j)}\big)\,\mathrm{d}x\,\mathrm{d}r \\
&+ \tfrac{1}{2} \sum_{k\in\mathbb{N}} \int_{t_0}^t \int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma} \tilde{h}_\beta''(u^{(j)}) (g(u^{(j)})\psi_k)_x^2 \,\mathrm{d}x\,\mathrm{d}r \\
&+ \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma} \tilde{h}_\beta'(u^{(j)}) (g(u^{(j)})\psi_k)_x \,\mathrm{d}x\,\mathrm{d}\beta^{(k)}.
\end{aligned}$$

Moreover, we can replace again $\widetilde{h}_\beta$ by h_β since they coincide on the range of $u^{(j)}$ and $u^{(j)}$ by u to conclude that

$$\begin{aligned} \mathbf{1}_\Gamma \mathscr{H}_\beta(u(t\wedge\widetilde{\sigma}_j)) = \mathbf{1}_\Gamma \mathscr{H}_\beta(u(t_0)) &+ \int_{t_0}^t \int_{\mathbb{T}} \mathbf{1}_{[t_0,\widetilde{\sigma}_j]\times\Gamma} u^\beta u_x (u_{xxx} - \phi''(u)u_x)\,\mathrm{d}x\,\mathrm{d}r \\ &+ \tfrac{1}{2}\sum_{k\in\mathbb{N}} \int_{t_0}^t \int_{\mathbb{T}} \mathbf{1}_{[t_0,\widetilde{\sigma}_j]\times\Gamma} h_\beta''(u)(g(u)\psi_k)_x^2\,\mathrm{d}x\,\mathrm{d}r \\ &+ \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathbb{T}} \mathbf{1}_{[t_0,\widetilde{\sigma}_j]\times\Gamma} h_\beta'(u)(g(u)\psi_k)_x\,\mathrm{d}x\,\mathrm{d}\beta^{(k)} \end{aligned} \tag{5.3.12}$$

for $t\in[t_0,T]$. Using the classical calculation for the deterministic α-entropy [11, Proposition 2.1], see also Subsection 1.3.2, and the assumption (5.1.21) on ϕ'', we deduce

$$\begin{aligned} \int_{\mathbb{T}} u^\beta u_x(u_{xxx} - \phi''(u)u_x)\,\mathrm{d}x \le -\tfrac{1}{\gamma^2}\int_{\mathbb{T}} u^{\beta+2-2\gamma}(u^\gamma)_{xx}^2\,\mathrm{d}x - c(\beta,\gamma)\int_{\mathbb{T}} u^{\beta-2}u_x^4\,\mathrm{d}x \\ - c_1\int_{\mathbb{T}} u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x + c_2\int_{\mathbb{T}} u^\beta u_x^2\,\mathrm{d}x, \end{aligned}$$

on $[t_0,\widetilde{\sigma}_j]\times\Gamma$ for a constant $c(\beta,\gamma)>0$. To estimate the Itô correction we calculate moreover that

$$(g(u)\psi_k)_x^2 \lesssim (g'(u))^2u_x^2\psi_k^2 + m(u)(\psi_k')^2.$$

Since

$$(g'(u))^2 = \left(\frac{m'(u)}{2m^{1/2}(u)}\right)^2 \lesssim u^{-2}m(u) \tag{5.3.13}$$

by (5.1.20), we obtain

$$\sum_{k\in\mathbb{N}}\int_{\mathbb{T}} h_\beta''(u)(g(u)\psi_k)_x^2\,\mathrm{d}x \lesssim \int_{\mathbb{T}} u^{\beta-2}u_x^2 + u^\beta\,\mathrm{d}x.$$

Using Young's inequality twice we arrive at

$$\begin{aligned} &\int_{\mathbb{T}} u^\beta u_x(u_{xxx}-\phi''(u)u_x)\,\mathrm{d}x + \sum_{k\in\mathbb{N}}\int_{\mathbb{T}} h_\beta''(u)(g(u)\psi_k)_x^2\,\mathrm{d}x \\ &\le -\tfrac{1}{\gamma^2}\int_{\mathbb{T}} u^{\beta+2-2\gamma}(u^\gamma)_{xx}^2\,\mathrm{d}x - \tfrac{c(\beta,\gamma)}{2}\int_{\mathbb{T}} u^{\beta-2}u_x^4\,\mathrm{d}x \\ &\quad - \tfrac{c_1}{2}\int_{\mathbb{T}} u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x + C\int_{\mathbb{T}} u^{\beta+2} + u^\beta\,\mathrm{d}x \end{aligned}$$

on $[t_0,\widetilde{\sigma}_j]\times\Gamma$. Inserting this in (5.3.12), taking the q-th power on both sides, the supremum in time and using the Burkholder–Davis–Gundy inequality, we arrive at

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_\Gamma \sup_{t_0\le t\le\widetilde{\sigma}_j} \mathscr{H}_\beta^q(u(t))\Big] + \mathbb{E}\bigg[\bigg(\int_{t_0}^{\widetilde{\sigma}_j}\int_{\mathbb{T}} \mathbf{1}_\Gamma u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^q\bigg] \\
&+\mathbb{E}\bigg[\bigg(\int_{t_0}^{\widetilde{\sigma}_j}\int_{\mathbb{T}} \mathbf{1}_\Gamma u^{\beta-2\gamma+2}(u^\gamma)_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^q + \bigg(\int_{t_0}^{\widetilde{\sigma}_j}\int_{\mathbb{T}} \mathbf{1}_\Gamma u^{\beta-2}u_x^4\,\mathrm{d}x\,\mathrm{d}t\bigg)^q\bigg] \\
&\lesssim_{\beta,\gamma,q} \mathbb{E}\big[\mathbf{1}_\Gamma \mathscr{H}_\beta^q(u(t_0))\big] + \mathbb{E}\bigg[\bigg(\mathbf{1}_\Gamma\int_{t_0}^{\widetilde{\sigma}_j}\int_{\mathbb{T}} u^{\beta+2}+u^\beta\,\mathrm{d}x\,\mathrm{d}t\bigg)^q\bigg] \\
&+\mathbb{E}\bigg[\bigg(\mathbf{1}_\Gamma\sum_{k\in\mathbb{N}}\int_{t_0}^{\widetilde{\sigma}_j}\bigg(\int_{\mathbb{T}} h_\beta''(u)u_x g(u)\psi_k\,\mathrm{d}x\bigg)^2\mathrm{d}t\bigg)^{q/2}\bigg].
\end{aligned} \tag{5.3.14}$$

To estimate the latter term, we integrate by parts and obtain that

$$\begin{aligned}
\bigg(\int_{\mathbb{T}} h_\beta''(u)u_x g(u)\psi_k\,\mathrm{d}x\bigg)^2 &= \bigg(\int_{\mathbb{T}}\int_1^u h_\beta''(r)g(r)\,\mathrm{d}r\,\psi_k'\,\mathrm{d}x\bigg)^2 \\
&\le \int_{\mathbb{T}}\bigg(\int_1^u r^\beta m^{-1/2}(r)\,\mathrm{d}r\bigg)^2(\psi_k')^2\,\mathrm{d}x.
\end{aligned}$$

Using (5.1.18) and (5.1.19), we estimate separately

$$\bigg(\int_1^u r^\beta m^{-1/2}(r)\,\mathrm{d}r\bigg)^2 \lesssim \bigg(\int_1^u r^\beta\,\mathrm{d}r\bigg)^2 \lesssim_\beta u^{2\beta+2}, \quad \text{on } \{u>1\},$$

and

$$\bigg(\int_1^u r^\beta m^{-1/2}(r)\,\mathrm{d}r\bigg)^2 \lesssim \bigg(\int_1^u r^{\beta-n/2}\,\mathrm{d}r\bigg)^2 \lesssim_\beta \begin{cases} u^{2\beta-n+2}, & 2\beta-n+2<0, \\ \log^2(u), & 2\beta-n+2\ge 0, \end{cases} \quad \text{on } \{u\le 1\}.$$

Thus, we arrive at

$$\begin{aligned}
&\sum_{k\in\mathbb{N}}\int_{t_0}^{\widetilde{\sigma}_j}\bigg(\int_{\mathbb{T}} h_\beta''(u)u_x g(u)\psi_k\,\mathrm{d}x\bigg)^2\mathrm{d}t \\
&\lesssim_\beta \int_{t_0}^{\widetilde{\sigma}_j}\int_{\{u>1\}} u^{2\beta+2}\,\mathrm{d}x\,\mathrm{d}t + \int_{t_0}^{\widetilde{\sigma}_j}\int_{\{u\le 1\}} u^{2\beta-n+2}+\log^2(u)\,\mathrm{d}x,\mathrm{d}t
\end{aligned}$$

on $[t_0,\widetilde{\sigma}_j]\times\Gamma$. To estimate the power $u^{2\beta-n+2}$, we use that

$$h_\beta(r) \gtrsim \int_1^r\int_1^{r'} (r'')^{\beta-n}\,\mathrm{d}r''\,\mathrm{d}r' = \begin{cases} \frac{1}{\beta-n+1}\Big(\frac{r^{\beta-n+2}-1}{\beta-n+2}-r+1\Big), & \beta-n\notin\{-1,-2\}, \\ r-1-\log(r), & \beta-n=-2, \\ r\log(r)-r+1, & \beta-n=-1, \end{cases}$$

by (5.1.18) for $r\le 1$ so that

$$\int_{\{u\le 1\}} u^{\beta-n+2}\,\mathrm{d}x \lesssim_\beta \int_{\{u\le 1\}} h_\beta(u)+(u+1)\,\mathrm{d}x \le \mathscr{H}_\beta(u) + \int_{\mathbb{T}} u_0\,\mathrm{d}x + 1,$$

due to the positivity of u and conservation of mass. Therefore, we deduce that

$$\begin{aligned}\int_{t_0}^{\tilde\sigma_j}\int_{\{u\le 1\}} u^{2\beta-n+2}\,\mathrm{d}x\,\mathrm{d}t &\le \Big(\sup_{t_0\le t\le\tilde\sigma_j}\int_{\{u\le 1\}} u^{\beta-n+2}\,\mathrm{d}x\Big)\Big(\int_{t_0}^{\tilde\sigma_j}\sup_{x\in\mathbb{T}} u^\beta\,\mathrm{d}t\Big)\\ &\le \varepsilon\Big(\sup_{t_0\le t\le\tilde\sigma_j}\mathscr{H}_\beta(u(t))+\int_{\mathbb{T}} u_0\,\mathrm{d}x+1\Big)^2+C_{\beta,\varepsilon}\Big(\int_{t_0}^{\tilde\sigma_j}\sup_{x\in\mathbb{T}} u^\beta\,\mathrm{d}t\Big)^2\end{aligned}$$

for any $\varepsilon>0$. Inserting all this in (5.3.14) yields

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_\Gamma\sup_{t_0\le t\le\tilde\sigma_j}\mathscr{H}_\beta^q(u(t))\Big]+\mathbb{E}\Big[\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big]\\ &+\mathbb{E}\Big[\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma u^{\beta-2\gamma+2}(u^\gamma)_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\Big)^q+\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma u^{\beta-2}u_x^4\,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big]\\ &\lesssim_{\beta,\gamma,q}\mathbb{E}[\mathbf{1}_\Gamma\mathscr{H}_\beta^q(u(t_0))]+\mathbb{E}\Big[\Big(\mathbf{1}_\Gamma\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^q+\mathbf{1}_\Gamma\Big]\\ &+\mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_{t_0}^{\tilde\sigma_j}\sup_{x\in\mathbb{T}}\Big(u^{\beta+2}+u^\beta+u^{2\beta+2}+\log^2(u)\Big)\mathrm{d}t\Big)^q\Big].\end{aligned}\tag{5.3.15}$$

To estimate also the latter term, we observe that all the powers lie between $\beta-\vartheta$ and $\beta+3$ and hence an application of Young's inequality leads to

$$\sup_{x\in\mathbb{T}}\Big(u^{\beta+2}+u^\beta+u^{2\beta+2}+\log^2(u)\Big)\le\varepsilon\sup_{x\in\mathbb{T}}u^{\beta-\vartheta}+C_{\beta,\varepsilon}\sup_{x\in\mathbb{T}}u^{\beta+3},$$

for each $\varepsilon>0$. By applying Lemma 5.3.2, once more Young's inequality and conservation of mass, we can estimate this further by

$$\begin{aligned}&\tfrac{\varepsilon(\beta-\vartheta)^2}{2}\int_{\mathbb{T}}u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x+2\varepsilon\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\beta-\vartheta}\\ &+C_{\beta,\varepsilon}\Big(\int_{\mathbb{T}}u^{\beta-2}u_x^4\,\mathrm{d}x\Big)^{(\beta+3)/(\beta+5)}\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{3(\beta+3)/(\beta+5)}+C_{\beta,\varepsilon}\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\beta+3}\\ &\le\tfrac{\varepsilon(\beta-\vartheta)^2}{2}\int_{\mathbb{T}}u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x+2\varepsilon\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\beta-\vartheta}\\ &+\varepsilon\Big(\int_{\mathbb{T}}u^{\beta-2}u_x^4\,\mathrm{d}x\Big)+C_{\beta,\varepsilon}\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{3(\beta+3)/2}+C_{\beta,\varepsilon}\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\beta+3}.\end{aligned}$$

Choosing ε sufficiently small to absorb the resulting terms in the left-hand side of (5.3.15) and dropping the intermediate powers of the mass yields

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_\Gamma\sup_{t_0\le t\le\tilde\sigma_j}\mathscr{H}_\beta^q(u(t))\Big]+\mathbb{E}\Big[\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma u^{\beta-\vartheta-2}u_x^2\,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big]\\ &+\mathbb{E}\Big[\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma u^{\beta-2\gamma+2}(u^\gamma)_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\Big)^q+\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma u^{\beta-2}u_x^4\,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big]\\ &\lesssim_{\beta,\gamma,q,T}\mathbb{E}[\mathbf{1}_\Gamma\mathscr{H}_\beta^q(u(t_0))]+\mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{(\beta-\vartheta)q}+\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{3(\beta+3)q/2}\Big].\end{aligned}$$

We obtain the claimed estimate (5.3.9) by letting $j \to \infty$ and using Fatou's lemma. □

5.3.2. PROOF OF THE ENERGY ESTIMATE—LEMMA 5.3.1

To close the a-priori estimate from Lemma 5.3.1 on the energy of the solution u we use additionally an estimate on the minimum and maximum of a function in terms of the energy functional. Moreover, we recall once more the additional regularity properties of u from Proposition 5.1.7.

Lemma 5.3.4. *It holds*

$$\sup_{x\in\mathbb{T}} f^{(2-\vartheta)/2}(x) \lesssim \mathscr{E}(f) + \left(\int_{\mathbb{T}} f\,\mathrm{d}x\right)^{(2-\vartheta)/2}, \tag{5.3.16}$$

$$\sup_{x\in\mathbb{T}} f^{3}(x) \lesssim \mathscr{E}(f)\left(\int_{\mathbb{T}} f\,\mathrm{d}x\right) + \left(\int_{\mathbb{T}} f\,\mathrm{d}x\right)^{3} \tag{5.3.17}$$

for every positive function $f \in C^1(\mathbb{T})$ bounded away from 0.

5

Proof. The proof is completely analogous to the proof of Lemma 5.3.2 using additionally (5.1.21), for a discrete version of (5.3.16) see also [51, Lemma 4.1]. □

Proof of Lemma 5.3.1. We again make use of the stopping time $\tilde\sigma_j$ introduced in (5.3.10) and the process $u^{(j)}$ defined in (5.3.11). As in the proof of Lemma 5.3.3, we replace ϕ by a two times differentiable function $\tilde\phi\colon \mathbb{R}\to\mathbb{R}$ with bounded second derivative, which agrees with ϕ on $[1/j, j]$. Then the assumptions of [39, Proposition A.1] are satisfied and an application of Itô's formula yields

$$\begin{aligned}
\int_{\mathbb{T}} \tilde\phi(u^{(j)}(t))\,\mathrm{d}x &= \int_{\mathbb{T}} \tilde\phi(u^{(j)}(t_0))\,\mathrm{d}x \\
&\quad + \int_{t_0}^{t}\int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde\sigma_j]\times\Gamma}\tilde\phi''(u^{(j)})u_x^{(j)} m(u^{(j)})(u_{xxx}^{(j)} - \phi''(u^{(j)})u_x^{(j)})\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \tfrac{1}{2}\sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde\sigma_j]\times\Gamma}\tilde\phi''(u^{(j)})(g(u^{(j)})\psi_k)_x^2\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde\sigma_j]\times\Gamma}\tilde\phi'(u^{(j)})(g(u^{(j)})\psi_k)_x\,\mathrm{d}x\,\mathrm{d}\beta^{(k)}.
\end{aligned}$$

Itô's formula is also applicable to the functional $\|u_x^{(j)}\|^2_{L^2(\mathbb{T})}$. Indeed, as carried out in detail in Appendix 2.C from Chapter 2, one can for example identify $u^{(j)}$ with its equivalence class $\bar u^{(j)}$ of homogeneous distributions so that the functional $\|u_x\|^2_{L^2(\mathbb{T})}$ coincides with the squared $\dot H^1(\mathbb{T})$-norm of $\bar u^{(j)}$. Then [100, Theorem 4.2.5] becomes applicable on the Gelfand triple $\dot H^3(\mathbb{T}) \subset \dot H^1(\mathbb{T}) \subset \dot H^{-1}(\mathbb{T})$ leading to

$$\begin{aligned}
\tfrac{1}{2}\int_{\mathbb{T}} (u_x^{(j)})^2(t)\,\mathrm{d}x &= \tfrac{1}{2}\int_{\mathbb{T}} (u_x^{(j)})^2(t_0)\,\mathrm{d}x \\
&\quad - \int_{t_0}^{t}\int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde\sigma_j]\times\Gamma} u_{xxx}^{(j)} m(u^{(j)})\big(u_{xxx}^{(j)} - \phi''(u^{(j)})u_x^{(j)}\big)\,\mathrm{d}x\,\mathrm{d}r
\end{aligned}$$

$$+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}}\mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma}(g(u^{(j)})\psi_k)_{xx}^2\,\mathrm{d}x\,\mathrm{d}r$$
$$-\sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}}\mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma}u_{xx}^{(j)}(g(u^{(j)})\psi_k)_x\,\mathrm{d}x\,\mathrm{d}\beta^{(k)}.$$

By inserting the definition of $u^{(j)}$, using that $\tilde{\phi}$ and ϕ coincide on its range and adding the two Itô expansions together, we conclude that

$$\begin{aligned}\mathbf{1}_\Gamma\mathscr{E}(u(t\wedge\tilde{\sigma}_j)) &= \mathbf{1}_\Gamma\mathscr{E}(u(t_0))-\int_{t_0}^{t}\int_{\mathbb{T}}\mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma}m(u)(u_{xx}-\phi'(u))_x^2\,\mathrm{d}x\,\mathrm{d}r\\ &+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}}\mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma}\Big[\phi''(u)(g(u)\psi_k)_x^2+(g(u)\psi_k)_{xx}^2\Big]\,\mathrm{d}x\,\mathrm{d}r\\ &+\sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}}\mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma}(\phi'(u)-u_{xx})(g(u)\psi_k)_x\,\mathrm{d}x\,\mathrm{d}\beta^{(k)},\end{aligned}\tag{5.3.18}$$

for $t\in[t_0,T]$. To estimate the Itô correction, we use (5.1.18), (5.1.19), (5.1.21) and (5.3.13) to deduce that

$$\begin{aligned}\sum_{k\in\mathbb{N}}\int_{\mathbb{T}}\phi''(u)(g(u)\psi_k)_x^2\,\mathrm{d}x &\lesssim \sum_{k\in\mathbb{N}}\int_{\mathbb{T}}u^{-\vartheta-2}(g'(u))^2u_x^2\psi_k^2+u^{-\vartheta-2}m(u)(\psi_k')^2\,\mathrm{d}x\\ &\lesssim \int_{\mathbb{T}}u^{-\vartheta-4}m(u)u_x^2+u^{-\vartheta-2}m(u)\,\mathrm{d}x\\ &\lesssim \int_{\{u>1\}}u^{\nu-\vartheta-4}u_x^2+u^{\nu-\vartheta-2}\,\mathrm{d}x+\int_{\{u\le1\}}u^{n-\vartheta-4}u_x^2+u^{n-\vartheta-2}\,\mathrm{d}x.\end{aligned}$$

Similarly, using that

$$(g(u)\psi_k)_{xx}=(g'(u)u_{xx}+g''(u)u_x^2)\psi_k+2g'(u)u_x\psi_k'+g(u)\psi_k'',$$

we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{N}}\int_{\mathbb{T}}(g(u)\psi_k)_{xx}^2\,\mathrm{d}x\\ &\quad\lesssim\int_{\mathbb{T}}(g'(u))^2u_{xx}^2+(g''(u))^2u_x^4+(g'(u))^2u_x^2+m(u)\,\mathrm{d}x.\end{aligned}\tag{5.3.19}$$

Because of (5.1.20) it holds

$$(g''(u))^2=\left(\frac{2m''(u)m(u)-(m'(u))^2}{4m^{3/2}(u)}\right)^2\lesssim u^{-4}m(u),$$

which together with (5.1.18), (5.1.19) and (5.3.13) yields that (5.3.19) is bounded by

$$\begin{aligned}&\int_{\mathbb{T}}u^{-2}m(u)u_{xx}^2+u^{-4}m(u)u_x^4+u^{-2}m(u)u_x^2+m(u)\,\mathrm{d}x\\ &\quad\lesssim\int_{\mathbb{T}}u^{-2}m(u)u_{xx}^2+u^{-4}m(u)u_x^4+m(u)\,\mathrm{d}x\end{aligned}$$

$$\lesssim \int_{\{u>1\}} u^{\nu-2}u_{xx}^2 + u^{\nu-4}u_x^4 + u^\nu \,\mathrm{d}x + \int_{\{u\le 1\}} u^{n-2}u_{xx}^2 + u^{n-4}u_x^4 + u^n \,\mathrm{d}x.$$

In summary, we have verified the estimate

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \int_{\mathbb{T}} \phi''(u)(g(u)\psi_k)_x^2 + (g(u)\psi_k)_{xx}^2 \,\mathrm{d}x \\
&\quad \lesssim \int_{\{u>1\}} u^{\nu-\vartheta-4}u_x^2 + u^{\nu-\vartheta-2} + u^{\nu-2}u_{xx}^2 + u^{\nu-4}u_x^4 + u^\nu \,\mathrm{d}x \\
&\qquad + \int_{\{u\le 1\}} u^{n-\vartheta-4}u_x^2 + u^{n-\vartheta-2} + u^{n-2}u_{xx}^2 + u^{n-4}u_x^4 + u^n \,\mathrm{d}x
\end{aligned}$$

on $[t_0, \tilde{\sigma}_j] \times \Gamma$. Inserting this in (5.3.18) and taking the supremum in time, we arrive at

$$\begin{aligned}
&\mathbf{1}_\Gamma \sup_{t_0\le t\le \tilde{\sigma}_j} \mathscr{E}(u(t)) + \int_{t_0}^{\tilde{\sigma}_j}\int_{\mathbb{T}} \mathbf{1}_\Gamma m(u)(u_{xx} - \phi'(u))_x^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \mathbf{1}_\Gamma \mathscr{E}(u(t_0)) + \mathbf{1}_\Gamma \int_{t_0}^{\tilde{\sigma}_j}\int_{\{u>1\}} u^{\nu-\vartheta-4}u_x^2 + u^{\nu-\vartheta-2} + u^{\nu-2}u_{xx}^2 + u^{\nu-4}u_x^4 + u^\nu \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \mathbf{1}_\Gamma \int_{t_0}^{\tilde{\sigma}_j}\int_{\{u\le 1\}} u^{n-\vartheta-4}u_x^2 + u^{n-\vartheta-2} + u^{n-2}u_{xx}^2 + u^{n-4}u_x^4 + u^n \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sup_{t_0\le t\le T} \left| \sum_{k\in\mathbb{N}} \int_{t_0}^{t}\int_{\mathbb{T}} \mathbf{1}_{[t_0,\tilde{\sigma}_j]\times\Gamma}(\phi'(u) - u_{xx})(g(u)\psi_k)_x \,\mathrm{d}x\,\mathrm{d}\beta^{(k)} \right|.
\end{aligned} \tag{5.3.20}$$

For the sake of clarity, we divide the rest of the proof into three steps.

Estimate on the integral over $\{u>1\}$. We define $\tilde{\nu} = \max\{3,\nu\}$ and estimate using (5.3.6), (5.3.7), (5.3.17) and Young's inequality

$$\begin{aligned}
&\int_{\{u>1\}} u^{\nu-\vartheta-4}u_x^2 + u^{\nu-\vartheta-2} + u^{\nu-2}u_{xx}^2 + u^{\nu-4}u_x^4 + u^\nu \,\mathrm{d}x \\
&\le \int_{\mathbb{T}} u^{\tilde{\nu}-\vartheta-4}u_x^2 + u^{\tilde{\nu}-\vartheta-2} + u^{\tilde{\nu}-2}u_{xx}^2 + u^{\tilde{\nu}-4}u_x^4 + u^{\tilde{\nu}} \,\mathrm{d}x \\
&\le \left(\sup_{x\in\mathbb{T}} u^{\tilde{\nu}-\beta-2}\right) \int_{\mathbb{T}} u^{\beta-\vartheta-2}u_x^2 + u^{\beta-\vartheta} + u^{\beta}u_{xx}^2 + u^{\beta-2}u_x^4 + u^{\beta+2} \,\mathrm{d}x \\
&\lesssim_\beta \left(\mathscr{E}^{(\tilde{\nu}-\beta-2)/3}(u)\left(\int_{\mathbb{T}} u_0 \,\mathrm{d}x\right)^{(\tilde{\nu}-\beta-2)/3} + \left(\int_{\mathbb{T}} u_0 \,\mathrm{d}x\right)^{\tilde{\nu}-\beta-2}\right) \\
&\quad \times \left(\int_{\mathbb{T}} u^{\beta-\vartheta-2}u_x^2 + u^{\beta}u_{xx}^2 + u^{\beta-2}u_x^4 \,\mathrm{d}x + \left(\int_{\mathbb{T}} u_0 \,\mathrm{d}x\right)^{\beta-\vartheta} + \left(\int_{\mathbb{T}} u_0 \,\mathrm{d}x\right)^{\beta+2}\right).
\end{aligned}$$

We denote the terms in the last row as $\mathscr{I}(u)$ and obtain due to Young's inequality the bound

$$\int_{t_0}^{\tilde{\sigma}_j}\int_{\{u>1\}} u^{\nu-\vartheta-4}u_x^2 + u^{\nu-\vartheta-2} + u^{\nu-2}u_{xx}^2 + u^{\nu-4}u_x^4 + u^\nu \,\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&\lesssim_\beta \left(\sup_{t_0\le t\le\tilde\sigma_j} \mathscr{E}^{(\tilde\nu-\beta-2)/3}(u) + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{2(\tilde\nu-\beta-2)/3}\right) \times \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{(\tilde\nu-\beta-2)/3} \times \int_{t_0}^{\tilde\sigma_j} \mathscr{I}(u)\,\mathrm{d}t \\
&\le \varepsilon\left(\sup_{t_0\le t\le\tilde\sigma_j} \mathscr{E}(u) + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^2\right) + C_{\beta,\varepsilon}\left(\left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{(\tilde\nu-\beta-2)/3} \times \int_{t_0}^{\tilde\sigma_j} \mathscr{I}(u)\,\mathrm{d}t\right)^{3/(5+\beta-\tilde\nu)} \\
&\le \varepsilon\left(\sup_{t_0\le t\le\tilde\sigma_j} \mathscr{E}(u) + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^2\right) \\
&\quad + C_{\beta,\varepsilon}\left(\left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{2(\tilde\nu-\beta-2)/(5+\beta-\tilde\nu)} + \left(\int_{t_0}^{\tilde\sigma_j} \mathscr{I}(u)\,\mathrm{d}t\right)^{6/(5+\beta-\tilde\nu)}\right)
\end{aligned} \tag{5.3.21}
$$

on Γ for any $\varepsilon > 0$. To estimate the $\mathscr{I}(u)$-term later on, we remark that

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{1}_\Gamma\left(\int_{t_0}^{\tilde\sigma_j} \mathscr{I}(u)\,\mathrm{d}t\right)^{6q/(5+\beta-\tilde\nu)}\right] &\lesssim_{\beta,q,T} \mathbb{E}\left[\mathbf{1}_\Gamma \mathscr{H}_\beta^{6q/(5+\beta-\tilde\nu)}(u(t_0))\right] \\
&+ \mathbb{E}\left[\mathbf{1}_\Gamma\left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{6(\beta-\vartheta)q/(5+\beta-\tilde\nu)} + \mathbf{1}_\Gamma\left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{9(\beta+3)q/(5+\beta-\tilde\nu)}\right]
\end{aligned} \tag{5.3.22}
$$

by an application of Lemma 5.3.3 with $\gamma = 1$ and additionally absorbing the intermediate power of the mass.

Estimate on the integral over $\{u \le 1\}$. We set $\tilde n = \min\{2, n\}$, so that

$$
\begin{aligned}
&\int_{\{u\le 1\}} u^{n-\vartheta-4}u_x^2 + u^{n-\vartheta-2} + u^{n-2}u_{xx}^2 + u^{n-4}u_x^4 + u^n\,\mathrm{d}x \\
&\le \int_{\mathbb{T}} u^{\tilde n-\vartheta-4}u_x^2 + u^{\tilde n-\vartheta-2} + u^{\tilde n-2}u_{xx}^2 + u^{\tilde n-4}u_x^4 + u^{\tilde n}\,\mathrm{d}x \\
&\le \left(\sup_{x\in\mathbb{T}} u^{\tilde n-2}\right) \int_{\mathbb{T}} u^{-\vartheta-2}u_x^2 + u^{-\vartheta} + u_{xx}^2 + u^{-2}u_x^4 + u^2\,\mathrm{d}x \\
&\lesssim \left(\mathscr{E}^{2(2-\tilde n)/(\vartheta-2)}(u) + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{\tilde n-2}\right) \\
&\quad \times \left(\int_{\mathbb{T}} u^{-\vartheta-2}u_x^2 + u_{xx}^2 + u^{-2}u_x^4\,\mathrm{d}x + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{-\vartheta} + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^2\right).
\end{aligned} \tag{5.3.23}
$$

because of (5.3.6), (5.3.7) and (5.3.16). We define $\mathscr{K}(u)$ as the last row of (5.3.23) and observe that for $\tilde n = 2$ its prefactor equals 1 so that the following estimate trivializes. Otherwise we have $0 < \frac{2(2-\tilde n)}{\vartheta-2} < 1$ by Assumption 5.1.10 and an application of Young's inequality yields the bound

$$
\begin{aligned}
&\int_{t_0}^{\tilde\sigma_j} \int_{\{u\le 1\}} u^{n-\vartheta-4}u_x^2 + u^{n-\vartheta-2} + u^{n-2}u_{xx}^2 + u^{n-4}u_x^4 + u^n\,\mathrm{d}x\,\mathrm{d}t \\
&\lesssim \left(\sup_{t_0\le t\le\tilde\sigma_j} \mathscr{E}(u)^{2(2-\tilde n)/(\vartheta-2)}(u) + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{\tilde n-2}\right) \times \int_{t_0}^{\tilde\sigma_j} \mathscr{K}(u)\,\mathrm{d}t \\
&\le \varepsilon\left(\sup_{t_0\le t\le\tilde\sigma_j} \mathscr{E}(u) + \left(\int_{\mathbb{T}} u_0\,\mathrm{d}x\right)^{(2-\vartheta)/2}\right) + C_\varepsilon\left(\int_{t_0}^{\tilde\sigma_j} \mathscr{K}(u)\,\mathrm{d}t\right)^{(\vartheta-2)/(\vartheta+2\tilde n-6)}
\end{aligned} \tag{5.3.24}
$$

on Γ. As in the previous step we remark that Lemma 5.3.3 with $\beta'=0$, $\gamma'=1$ yields

$$\begin{aligned}\mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_0^{\tilde\sigma_j}\mathcal{K}(u)\,\mathrm{d}t\Big)^{(\vartheta-2)q/(\vartheta+2\tilde n-6)}\Big] &\lesssim_{\beta,q} \mathbb{E}\big[\mathbf{1}_\Gamma\mathcal{H}_0^{(\vartheta-2)q/(\vartheta+2\tilde n-6)}(u(t_0))\big]\\ &+\mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{-\vartheta(\vartheta-2)q/(\vartheta+2\tilde n-6)}+\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{9(\vartheta-2)q/(2\vartheta+4\tilde n-12)}\Big]\end{aligned}\tag{5.3.25}$$

for later purposes.

Closing the energy estimate. Inserting (5.3.21) and (5.3.24) in (5.3.20) and choosing ε sufficiently small yields

$$\begin{aligned}&\mathbf{1}_\Gamma\sup_{t_0\le t\le\tilde\sigma_j}\mathcal{E}(u(t))+\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma m(u)(u_{xx}-\phi'(u))_x^2\,\mathrm{d}x\,\mathrm{d}t\\ &\lesssim_\beta \mathbf{1}_\Gamma\mathcal{E}(u(t_0))+\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{(2-\vartheta)/2}+\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\max\{\frac{2(\tilde\nu-\beta-2)}{5+\beta-\tilde\nu},2\}}\\ &\quad+\mathbf{1}_\Gamma\Big(\int_{t_0}^{\tilde\sigma_j}\mathcal{I}(u)\,\mathrm{d}t\Big)^{6/(5+\beta-\tilde\nu)}+\mathbf{1}_\Gamma\Big(\int_{t_0}^{\tilde\sigma_j}\mathcal{K}(u)\,\mathrm{d}t\Big)^{(\vartheta-2)/(\vartheta+2\tilde n-6)}\\ &\quad+\sup_{t_0\le t\le T}\Big|\sum_{k\in\mathbb{N}}\int_{t_0}^{t}\int_{\mathbb{T}}\mathbf{1}_{[t_0,\tilde\sigma_j]\times\Gamma}(\phi'(u)-u_{xx})(g(u)\psi_k)_x\,\mathrm{d}x\,\mathrm{d}\beta^{(k)}\Big|.\end{aligned}$$

We take the q-th power on both sides, use the estimates (5.3.22) and (5.3.25) and the Burkholder–Davis–Gundy inequality to estimate further

$$\begin{aligned}&\mathbb{E}\Big[\mathbf{1}_\Gamma\sup_{t_0\le t\le\tilde\sigma_j}\mathcal{E}^q(u(t))\Big]+\mathbb{E}\Big[\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}\mathbf{1}_\Gamma m(u)(u_{xx}-\phi'(u))_x^2\,\mathrm{d}x\,\mathrm{d}t\Big)^q\Big]\\ &\lesssim_{\beta,q,T}\mathbb{E}\big[\mathbf{1}_\Gamma\mathcal{E}^q(u(t_0))+\mathbf{1}_\Gamma\mathcal{H}_\beta^{6q/(5+\beta-\tilde\nu)}(u(t_0))+\mathbf{1}_\Gamma\mathcal{H}_0^{(\vartheta-2)q/(\vartheta+2\tilde n-6)}(u(t_0))\big]\\ &\quad+\mathbb{E}\Big[\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\min\{\frac{6(\beta-\vartheta)q}{5+\beta-\tilde\nu},\frac{-\vartheta(\vartheta-2)q}{\vartheta+2\tilde n-6}\}}+\mathbf{1}_\Gamma\Big(\int_{\mathbb{T}}u_0\,\mathrm{d}x\Big)^{\max\{\frac{9(\beta+3)q}{5+\beta-\tilde\nu},\frac{9(\vartheta-2)q}{2\vartheta+4\tilde n-12}\}}\Big]\\ &\quad+\mathbb{E}\Big[\Big(\mathbf{1}_\Gamma\sum_{k\in\mathbb{N}}\int_{t_0}^{\tilde\sigma_j}\Big(\int_{\mathbb{T}}(\phi'(u)-u_{xx})(g(u)\psi_k)_x\,\mathrm{d}x\Big)^2\mathrm{d}t\Big)^{q/2}\Big],\end{aligned}$$

where we absorb again the intermediate power of the mass. Since

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{N}}\int_{t_0}^{\tilde\sigma_j}\Big(\int_{\mathbb{T}}(\phi'(u)-u_{xx})(g(u)\psi_k)_x\,\mathrm{d}x\Big)^2\mathrm{d}t\Big)^{q/2}\lesssim\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}(\phi'(u)-u_{xx})_x^2g^2(u)\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/2}\\ &\le\varepsilon\Big(\int_{t_0}^{\tilde\sigma_j}\int_{\mathbb{T}}m(u)(\phi'(u)-u_{xx})_x^2\,\mathrm{d}x\,\mathrm{d}t\Big)^q+C_\varepsilon,\end{aligned}$$

by another application of Young's inequality, we deduce (5.3.2) by taking the limit $j\to\infty$ and using Fatou's lemma. □

5

6

Conclusion and outlook

We summarize the findings from this thesis regarding the solution theory of the stochastic thin-film equation with temporally white and spatially colored Gaussian noise as follows.

In Chapter 5 based on [3] we showed that while the film height remains strictly positive, unique solutions exist and become instantaneously smooth in space if the noise is sufficiently regular. We proved that such a positivity preserving behavior of the equation can be observed when the repulsive van der Waals forces between the molecules are incorporated in the model. This is the first result on uniqueness of solutions to the stochastic thin-film equation available in the literature and expresses additionally that, if the intermolecular forces are included in the model, the equation is very well-behaved. Moreover, since the results of Chapter 5 apply to a wide range of mobility functions, the positive and spatially smooth solutions can serve as approximate solutions for the equation without an interface potential in future works.

We proved moreover that if the film height touches down, martingale solutions to the stochastic thin-film equation exist. In particular, in Chapter 2 based on [116] we pioneered the two-dimensional setting together with the independently developed [107] in the linear noise case and considerably advanced the nonlinear noise case $n \in (2,3)$ in one spatial dimension in Chapters 3 and 4 based on [36, 117]. While the current literature, including the results of this thesis, do not cover all scenarios of interest—for example nonlinear noise in higher dimensions is untreated—the presented results indicate that some compactness argument is expected to work as long as one accepts low regularity solutions.

As key ingredients in the proofs of these results we additionally provided insight on the subtle interplay of higher-order degenerate parabolic operators, their non-negativity preserving mechanisms and non-Lipschitz continuous, conservative noise. Specifically, we demonstrated the usefulness of α-entropy estimates for the stochastic thin-film equation and investigated in Chapters 3 and 5 their relation to the energy estimate. This will certainly play a role in future research on the stochastic thin-film equation, but can also be inspiring for other nonlinear stochastic partial differential equations.

Knowing about the existence of solutions does surely not constitute a full mathematical understanding of the stochastic thin-film equation, however, it does stand at its beginning. In particular, the existence results from of this thesis together with the literature discussed in Section 1.5 invite for a rigorous investigation of the various constructed solutions concerning their qualitative properties. Since for the deterministic thin-film equation many interesting phenomena like quantified spreading rates [13, 14, 18, 48, 64, 71, 72, 83], asymptotics in the complete wetting [29, 30, 101, 125] and partial wetting regime [44, 105] as well as the occurrence of waiting times [21, 34, 38, 49, 50, 61, 74] have been verified, it is a natural question how the noise affects these properties. A step in this direction is taken in a series of three works starting with [76, 77], in which finite propagation speed of a regularized stochastic thin-film equation is shown.

We remark that weak solutions to the thin-film equation are, in general, not unique [11], but uniqueness holds in the smaller class of waiting time solutions [31]. Complementing the weak solution theory, the local in time existence and uniqueness of strong solutions was shown in [40, 62] for the case that a contact line is present. Also addressing the case of a non-fully supported profile, global in time well-posedness of the thin-film equation close to source type, stationary and traveling-wave solutions has been shown. Source type and self-similiar solutions are studied in [16, 47, 65, 104, 120], while global in time existence and uniqueness of solutions close to stationary profiles is obtained in [27, 63, 89, 92, 93] and traveling-wave solutions are treated in [60, 66–68]. It would be interesting to see if existence and uniqueness of solutions can be shown in the case that a contact line is present also for the stochastic thin-film equation.

Finally, it is an intriguing problem to eventually treat the stochastic thin-film equation with a spatio-temporal white noise $\mathscr{W}$. Not only is this the thermodynamically prescribed noise, but also the resulting stochastic gradient flow structure makes the equation interesting. By following up on the work [80] a pathwise existence and uniqueness result of a renormalized version seems to be in reach.

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Curriculum Vitæ

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List of publications

Preprints

[P1] A. Agresti, and M. Sauerbrey. "Well-posedness of the stochastic thin-film equation with an interface potential". In: *arXiv preprint arXiv:2403.12652* (2024).

[P2] K. Dareiotis, B. Gess, M. V. Gnann, and M. Sauerbrey. "Solutions to the stochastic thin-film equation for initial values with non-full support". In: *arXiv preprint arXiv:2305.06017* (2023).

[P3] M. Sauerbrey. "Solutions to the stochastic thin-film equation for the range of mobility exponents $n \in (2,3)$". In: *arXiv preprint arXiv:2310.02765* (2024).

Published articles

[A1] M. Grothaus, and M. Sauerbrey. "Dirichlet form analysis of the Jacobi process". *Stochastic Process. Appl.* 157 (2023), pp. 376–412.

[A2] M. Sauerbrey. "Martingale solutions to the stochastic thin-film equation in two dimensions". In: *Ann. Inst. Henri Poincaré Probab. Stat.* 60.1 (2024), pp. 373–412.