Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

EFFECTIVE RESISTANCE MATRICES OF WEIGHTED THRESHOLD GRAPHS*

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Abstract. A threshold graph is generated from a single node by repeatedly adding either a node i connected to all existing nodes with a common link weight $w_i > 0$ or a node i connected to none. Let G_w be a weighted threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \geq 0$. A closed-form expression for the pseudoinverse of its Laplacian matrix Q_w is derived via spectral decomposition, which yields an explicit formula for the effective resistance matrix Ω_w. We present a detailed structural characterization of the matrix Ω_w and determine a subset of the spectrum of the matrix Ω_w in terms of the weights w_i. As an application, we show that when the missing links of a threshold graph are sequentially added in nondecreasing order of effective resistance, the threshold property of the graph is preserved at each step until the complete graph of the same size is obtained.

Key words. Threshold graph, Moore–Penrose inverse, Effective resistance matrix.

AMS subject classifications. 15A09, 05C50, 05C75.

1. Introduction. We consider an undirected graph G with N nodes and L links. The $N \times N$ adjacency matrix A of graph G possesses entries a_{ij}, where $a_{ij} = 1$ if two nodes i and j are adjacent through a link $i \sim j$, otherwise $a_{ij} = 0$. Each link $i \sim j$ is assigned a link weight $w_{ij} \geq 0$. The $N \times N$ weighted matrix W represents the weight between all pairs (i, j) of adjacent nodes. The weighted adjacency matrix is $\tilde{A} = W \circ A$, where the Hadamard product $\circ$ defines the element-wise product $\tilde{a}_{ij} = w_{ij} a_{ij}$. Following the notations in the book [1], the weighted Laplacian matrix $\tilde{Q}$ of a graph is defined as $\tilde{Q} = \mathrm{diag}(\tilde{A}u) - \tilde{A}$, where u represents the all-one vector of dimension N and $\tilde{A}u$ represents the weighted degree vector with entries $\tilde{d}_i = (\tilde{A}u)_i = \sum_{j=1}^{N} w_{ij} a_{ij}$. An unweighted graph can be viewed as a special case in which all link weights are equal to one, i.e., $W = J$, where J is the all-one matrix of dimension $N \times N$. The Laplacian matrix Q of a connected graph has the spectral decomposition $Q = Z\mathrm{diag}(\mu_1, \cdots, \mu_N)Z^T$, where $\mu_1 \geq \cdots > \mu_N = 0$ represent the eigenvalues and the normalized matrix Z contains the Laplacian eigenvectors $z_1, z_2, \cdots, z_N$ in its columns, obeying $ZZ^T = Z^T Z = I$.

For a real symmetric matrix M with eigenvalues $\lambda_1, \ldots, \lambda_N$ and corresponding orthonormal eigenvectors $x_1, \cdots, x_N$, its Moore–Penrose inverse $M^\dagger$ is given by

$$M^\dagger = X\mathrm{diag}(\frac{1}{\lambda})X^T = \sum_{i=1}^{N} \frac{1}{\lambda_i} x_i x_i^T,$$

where the orthogonal matrix X has columns $x_1, \cdots, x_N$ and the matrix $\mathrm{diag}(\frac{1}{\lambda}) = \mathrm{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_N})$ contains the entries

$$\frac{1}{\lambda_i} = \begin{cases} \frac{1}{\lambda_i} & \lambda_i \neq 0 \\ 0 & \lambda_i = 0 \end{cases}.$$

*Received by the editors on December 1, 2025. Accepted for publication on June 15, 2026. Handling Editor: Michael Tait. Corresponding Author: Yingyue Ke.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

The Laplacian matrix Q of a connected graph has a single eigenvalue $\mu_N = 0$ as its row sum is zero, which allows the pseudoinverse $Q^\dagger$ to be expressed as $Q^\dagger = \sum_{k=1}^{N-1} \frac{1}{\mu_k} z_k z_k^T$. For notational convenience in the following discussion, we adopt the convention $\frac{1}{\mu_k} := 0$ whenever $\mu_k = 0$.

The effective resistance [2, 3] between two nodes i and j in a graph is defined as

$$\omega_{ij} = Q^\dagger_{ii} + Q^\dagger_{jj} - 2Q^\dagger_{ij},$$

which quantifies the dissipated power when the current of 1 Ampere is applied between the nodes i and j. The effective resistance matrix $\Omega \in \mathbb{R}^{N\times N}$ is defined as

$$\Omega = \zeta u^T + u\zeta^T - 2Q^\dagger,$$

where the $N \times 1$ vector $\zeta = (Q^\dagger_{11}, Q^\dagger_{22}, \ldots, Q^\dagger_{NN})^T$ contains the diagonal elements of the pseudoinverse $Q^\dagger$. The sum of all entries of the effective resistance matrix Ω defines the effective graph resistance

$$R_G = \frac{u^T \Omega u}{2}.$$

Since each eigenvector z_i of the Laplacian Q, for $1 \le i \le N-1$, is orthogonal to the all-one vector u, we have $Q^\dagger u = \sum_{k=1}^{N-1} \frac{1}{\mu_k} z_k z_k^T u = 0$, and hence $R_G = N u^T \zeta = N \mathrm{trace}(Q^\dagger) = N \sum_{k=1}^{N-1} \frac{1}{\mu_k}$.

Threshold graphs [4] form a special class of graphs constructed by starting from a single node and subsequently adding either an isolated node (not connected to any existing nodes) or a dominant node (connected to all existing nodes), until all N nodes are added to the graph. A weighted threshold graph defined over a node set $\mathcal{N} = \{1, \ldots, N\}$ is constructed as follows. The graph is initialized with a single node labeled 1. For each integer $i \ge 2$, a new node i is added and connected to each existing node $j \in \{1, 2, \ldots, i-1\}$ by a link of weight $w_i \ge 0$, where $w_i \ne 0$ implies that the links $i \sim j$ have the same link weight w_i for all $1 \le j < i$. The unweighted case can be regarded as a special case where $w_i \in \{0, 1\}$. The resulting graph is determined by the $N \times 1$ weight vector $w = (w_1, w_2, \ldots, w_N)$. Since node 1 has no neighbors that appear earlier than itself, the value of w_1 can be chosen arbitrarily; we assign $w_1 = 1$ to facilitate later computations. The weight w_i of all the links $i \sim j$, for $1 \le j < i$, is determined solely by the addition of node i. We refer to w_i as the weight of node i and denote by G_w a weighted threshold graph encoded by the weight vector w.

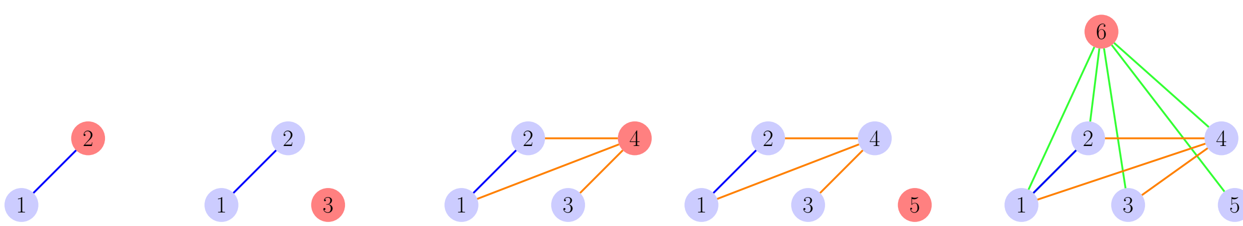

FIGURE 1. *Construction of the graph G_w encoded by weight vector $w = (1, 0, \sqrt{2}, 0, 2)$.*

Figure 1 illustrates the construction of the threshold graph G_w encoded by the weight vector $w = (1, 1, 0, \sqrt{2}, 0, 2)$. At each step, a node i (highlighted in red) is added and connected to all previous nodes j with the same weight w_i for $1 \le j < i$. The link colors represent the link weights. Blue, orange, and green links have weights 1, $\sqrt{2}$, and 2, respectively.

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A comprehensive overview of structural and theoretical properties of unweighted threshold graphs is provided in [5]. Closed-form expressions for the Laplacian spectrum and the enumeration of spanning trees in an unweighted threshold graph are presented in [6, 7]. Applications of unweighted threshold graphs in modeling real-world networks are explored in [8, 9, 10], highlighting their relevance in diverse domains such as ecology, social networks, and control theory. Recent studies have extended the theory of threshold graphs in various directions. Any two signed threshold graphs with $w_i \in \{0, -1, 1\}$ with cospectral Laplacian matrices are necessarily isomorphic [11]. Both combinatorial and algebraic approaches are employed to derive formulas for the pseudoinverse of the Laplacian of unweighted threshold graphs [12]. The equitable partition of the Laplacians and the relation between the weighted Ferrers diagram and the Laplacian eigenvalues of weighted threshold graphs are analyzed in [13]. Furthermore, the algebraic structure of the space of all Laplacians corresponding to weighted threshold graphs is studied in [14].

In this work, we investigate the effective resistance matrix of weighted threshold graphs. The paper is organized as follows. In Section 2, we introduce the partition of a weighted threshold graph G_w, the spectral decomposition of its weighted Laplacian matrix Q_w, and derive a closed-form expression of its pseudoinverse $Q_w^\dagger$. Section 3 establishes an explicit formula for the effective resistance matrix Ω_w, together with a comprehensive analysis of its structural properties. In Section 4, we investigate the eigenvalues and eigenvectors of the matrix Ω_w.

2. Preliminary.

2.1. Partition of a weighted threshold graph. Let G_w be a weighted threshold graph encoded by weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_N > 0$, which implies that the node N is adjacent to all other nodes and hence the graph is connected; conversely, if $w_N = 0$, then node N is isolated and the graph is disconnected. The weight vector w can be decomposed into alternating segments of nonzero and zero entries. Since G_w is connected, the weight vector w alternates between nonzero and zero segments, beginning and terminating with a nonzero segment. Consequently, the node set $\mathcal{N} = \{1, \ldots, N\}$ in a connected threshold graph G_w can be partitioned into disjoint subsets that alternate between nodes with nonzero weights (V subsets) and nodes with zero weights (U subsets). More precisely, if there are K sets of consecutive nodes of zero weights (U subsets), the nodal set $\mathcal{N} = \{1, \ldots, N\}$ can be expressed as the disjoint union:

$$\mathcal{N} = V_1 \cup U_1 \cup V_2 \cup U_2 \cup \cdots \cup V_K \cup U_K \cup V_{K+1} = \bigcup_{i=1}^{2K+1} S_i, \tag{2.1}$$

where V_i denotes the i-th nodal subset of nonzero weights and U_i denote the i-th nodal subset of zero weights. The subsets S alternate between V and U such that $S_{2j-1} = V_j$ and $S_{2j} = U_j$ and each node only belongs to exactly one subset S_i. Figure 2 illustrates the partition of a weighted threshold graph.

We define the union of the first k subsets $\mathcal{S}_k = \bigcup_{i=1}^{k} S_i$. The cumulative size of the first k subsets is denoted by

$$C_k = |\mathcal{S}_k| = \left| \bigcup_{i=1}^{k} S_i \right| = \sum_{i=1}^{k} |S_i|,$$

which follows that $C_{2K+1} = N$.

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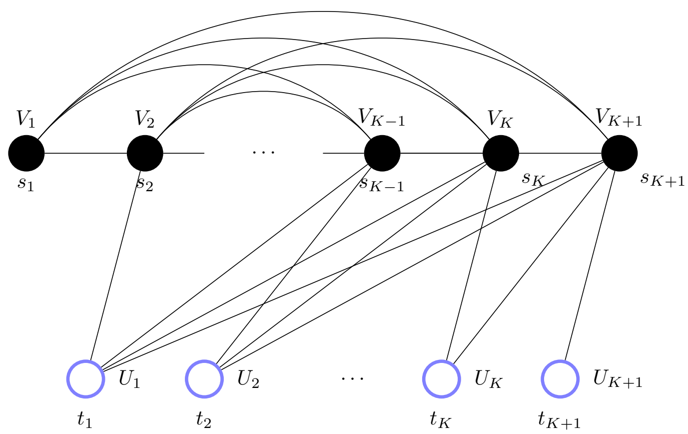

FIGURE 2. *The partition of node set* $\mathcal{N} = V_1 \cup U_1 \cup V_2 \cup U_2 \cup \cdots \cup V_K \cup U_K \cup V_{K+1}$ *in a weighted threshold graph. Each nodal set* V_i *forms a clique of size* s_i*. Each nodal set* U_i *forms an empty graph of size* t_i*. The link between two sets indicates a complete bipartite connection, where every node in one set is connected to every node in the other set.*

2.2. Laplacian matrices of weighted threshold graphs. We consider the weighted Laplacian matrices of weighted threshold graphs. We denote by A_w and Q_w the weighted adjacency matrix and the weighted Laplacian matrix of a weighted threshold graph G_w, respectively. We also use the following notations. 0_m denotes the all-zero vector of dimension m, and u_m denotes the all-one vector of dimension m. For matrices, we write $O_{m\times n}$ for the all-zero matrix of dimension $m \times n$ and $J_{m\times n}$ for the all-one matrix of dimension $m \times n$.

Based on the construction of a weighted threshold graph G_w, its weighted adjacency matrix A_w has the following structure

$$A_w = \begin{bmatrix} 0 & w_2 & w_3 & \cdots & \cdots & w_N \\ w_2 & 0 & w_3 & \cdots & \cdots & w_N \\ w_3 & w_3 & 0 & \cdots & \cdots & w_N \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & w_N \\ w_N & w_N & w_N & \cdots & w_N & 0 \end{bmatrix}.$$

Accordingly, the weighted Laplacian $Q_w = \operatorname{diag}(A_w u) - A_w$ is written as

$$Q_w = \begin{bmatrix} d_1 & -w_2 & -w_3 & \cdots & \cdots & -w_N \\ -w_2 & d_2 & -w_3 & \cdots & \cdots & -w_N \\ -w_3 & -w_3 & d_3 & \cdots & \cdots & -w_N \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & -w_N \\ -w_N & -w_N & -w_N & \cdots & -w_N & d_N \end{bmatrix},$$

where $d_i = (i-1)w_i + \sum_{j=i+1}^{N} w_j$ denotes the weighted degree of node i.

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With these definitions, we now present the following theorem on the eigenvalues and eigenvectors of the matrix Q_w.

Theorem 2.1 ([14]). *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \geq 0$. The eigenvalues of its Laplacian Q_w are $\mu = \{\mu_1, \mu_2, \cdots, \mu_N\}$, where $\mu_1 = 0$ and for $2 \leq k \leq N$*

$$\mu_k = kw_k + \sum_{j=k+1}^{N} w_j, \tag{2.2}$$

The corresponding normalized eigenvectors of the eigenvalue equation

$$Q_w z_k = \mu_k z_k,$$

are $z_1, z_2, \cdots, z_N$, where $z_1 = \frac{1}{\sqrt{N}} u$ and for $2 \leq k \leq N$, the j-th entry of the vector z_k is given by

$$(z_k)_j = \begin{cases} \frac{1}{\sqrt{k(k-1)}} & j < k \\ \frac{1-k}{\sqrt{k(k-1)}} & j = k \\ 0 & j > k \end{cases}.$$

We note that, unlike the convention in many studies [1], where the eigenvalues are ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N = 0$, the eigenvalues μ_k in (2.2) are not assumed to follow any specific magnitude ordering. Instead, we set $\mu_1 = 0$ and each μ_k given by (2.2) is associated with the addition of node k and without enforcing any ordering.

Lemma 2.2. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \geq 0$. If two nodes i and $i+1$ have identical node weights for $2 \leq i \leq N-1$, i.e., $w_i = w_{i+1}$, then the corresponding two Laplacian eigenvalues satisfy $\mu_i = \mu_{i+1}$.*

Proof. Substituting $w_i = w_{i+1}$ into (2.2) directly yields ∎

$$\mu_i = iw_i + w_{i+1} + \sum_{k=i+2}^{N} w_k = (i+1)w_{i+1} + \sum_{i=i+2}^{k} w_i = \mu_{i+1}.$$

Lemma 2.2 implies that if all nodes within a given nodal set share the same node weight, then these nodes correspond to the same Laplacian eigenvalue. Accordingly, we adopt the notation

$$\mu_U := \mu_i,\ \forall\, i \in U; \qquad \mu_V := \mu_i,\ \forall\, i \in V; \qquad \mu_S := \mu_i,\ \forall\, i \in S,$$

to indicate that all nodes belonging to the same nodal subset U, V, or S possess identical node weights and thus share a common Laplacian eigenvalue μ_U, μ_V, or μ_S, respectively.

2.3. Pseudoinverse of the Laplacian of weighted threshold graphs. By exploiting the spectral properties in Theorem 2.1, we obtain the closed-form formula of the pseudoinverse $Q_w^\dagger$ for any weighted threshold graph. The following theorem formalizes the result and provides a foundation for subsequent analysis.

Lemma 2.3. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \geq 0$. The pseudoinverse $Q_w^\dagger$ of its Laplacian Q_w is given by*

$$Q_w^\dagger = \sum_{k=1}^{N} \frac{1}{\mu_k} z_k z_k^T,$$

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with the $N \times N$ *matrix* $z_k z_k^T$ *defined as*

$$z_k z_k^T = \begin{bmatrix} \frac{1}{k(k-1)} J_{(k-1)\times(k-1)} & -\frac{1}{k-1} u_{k-1} & O_{(k-1)\times(N-k)} \\ -\frac{1}{k-1} u_{k-1}^T & \frac{k-1}{k} & 0_{N-k}^T \\ O_{(N-k)\times(k-1)} & 0_{N-k} & O_{(N-k)\times(N-k)} \end{bmatrix}, \tag{2.3}$$

where $\mu_1 = 0$ *and* $\mu_j = jw_j + \sum_{k=j+1}^N w_k$ *for* $2 \le j \le N$.

Proof. With the explicit formula for the eigenvector z_k from Theorem 2.1, the entries of the matrix $z_k z_k^T$ are given by □

$$\left(z_k z_k^T\right)_{ij} = (z_k)_i (z_k)_j = \begin{cases} \frac{1}{k(k-1)} & i < k, j < k \\ \frac{-1}{k} & i < k, j = k \text{ or } i = k, j < k \\ \frac{k-1}{k} & i = j = k \\ 0 & \text{otherwise} \end{cases},$$

from which the matrix $z_k z_k^T$ in (2.3) follows. Substituting $z_k z_k^T$ into $Q_w^\dagger = \sum_{k=1}^N \frac{1}{\mu_k} z_k z_k^T$ completes the proof.

3. Effective resistance matrix Ω_w. In this section, we provide the closed formula of the effective resistance matrix Ω_w of a weighted threshold graph G_w and present a comprehensive view of the structure of the matrix Ω_w. For notational convenience, when referring to an entry $(\Omega_w)_{ij}$, we assume $i < j$, due to the symmetry of the matrix Ω_w.

THEOREM 3.1. *Let* G_w *be a threshold graph encoded by the weight vector* $w = (w_1, w_2, \ldots, w_N)$ *with* $w_i \ge 0$, *its effective resistance matrix* Ω_w *is given by*

$$\Omega_w = \sum_{k=1}^N \frac{1}{\mu_k} P_k,$$

with the $N \times N$ *symmetric matrix* P_k *defined as*

$$P_k = \begin{bmatrix} O_{(k-1)\times(k-1)} & \frac{k}{k-1} u_{k-1} & \frac{1}{k(k-1)} J_{(k-1)\times(N-k)} \\ \frac{k}{k-1} u_{k-1}^T & 0 & \frac{k-1}{k} u_{N-k}^T \\ \frac{1}{k(k-1)} J_{(N-k)\times(k-1)} & \frac{k-1}{k} u_{N-k} & O_{(N-k)\times(N-k)} \end{bmatrix}, \tag{3.4}$$

where $\mu_1 = 0$ *and* $\mu_j = jw_j + \sum_{k=j+1}^N w_k$ *for* $2 \le j \le N$.

Proof. Substituting $Q_w^\dagger$, computed in Lemma 2.3, into $\Omega_{ij} = Q_{ii}^\dagger + Q_{jj}^\dagger - 2Q_{ij}^\dagger$ yields

$$\begin{aligned} (\Omega_w)_{ij} &= \sum_{k=i+1}^N \frac{1}{k(k-1)\mu_k} + \frac{i-1}{i\mu_i} + \sum_{k=j+1}^N \frac{1}{k(k-1)\mu_k} + \frac{j-1}{j\mu_j} - 2\sum_{k=j+1}^N \frac{1}{k(k-1)\mu_k} + \frac{2}{j\mu_j} \\ &= \frac{i-1}{i\mu_i} + \frac{j+1}{j\mu_j} + \sum_{k=i+1}^j \frac{1}{k(k-1)\mu_k} \\ &= \frac{i-1}{i\mu_i} + \sum_{k=i+1}^{j-1} \frac{1}{k(k-1)\mu_k} + \frac{j}{(j-1)\mu_j}. \end{aligned} \tag{3.5}$$

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Now fix $i < j$, the entries of $\sum_{k=1}^{N} \frac{1}{\mu_k} P_k$ are given by

$$\begin{aligned}\sum_{k=1}^{N} \frac{1}{\mu_k} (P_k)_{ij} &= \sum_{k=1}^{i-1} \frac{1}{\mu_k} (P_k)_{ij} + \frac{1}{\mu_i} (P_i)_{ij} + \sum_{k=i+1}^{j-1} \frac{1}{\mu_k} (P_k)_{ij} + \sum_{k=j+1}^{N} \frac{1}{\mu_k} (P_k)_{ij} \\ &= \sum_{k=1}^{i-1} 0 \cdot \frac{1}{\mu_k} + \frac{i-1}{i\mu_i} + \sum_{k=i+1}^{j-1} \frac{1}{k(k-1)\mu_k} + \frac{j}{(j-1)\mu_j} + \sum_{k=j+1}^{N} 0 \cdot \frac{1}{\mu_k},\end{aligned}$$

which is exactly $(\Omega_w)_{ij}$. ∎

For weighted threshold graphs, the effective graph resistance $R = N \sum_{k=1}^{N} \frac{1}{\mu_k}$ can be directly computed with the definition $R_G = \frac{u^T \Omega u}{2}$ without using $Q_w^{\dagger} u = 0$ because the entries $(\Omega_w)_{ij}$ are explicit given in Theorem 3.1. Due to the symmetry of the matrix Ω_w, we find that

$$\begin{aligned}R_{G_w} &= \frac{u^T \Omega_w u}{2} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (\Omega_w)_{ij} = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} (\Omega_w)_{ij} \\ &= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left(\frac{i-1}{i\mu_i} + \sum_{k=i+1}^{j-1} \frac{1}{k(k-1)\mu_k} + \frac{j}{(j-1)\mu_j} \right) \\ &= \sum_{i=1}^{N} (N-i) \frac{i-1}{i\mu_i} + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sum_{k=i+1}^{j-1} \frac{1}{k(k-1)\mu_k} + \sum_{j=1}^{N} (j-1) \frac{j}{(j-1)\mu_j}.\end{aligned}$$

Since the term $\frac{1}{k(k-1)\mu_k}$ appears when $1 \leq i \leq k-1$ and $k+1 \leq j \leq N$, we reindex the variants and obtain

$$\begin{aligned}R_{G_w} &= \sum_{k=1}^{N} (N-k) \frac{k-1}{k\mu_k} + \sum_{k=1}^{N} (k-1)(N-k) \frac{1}{k(k-1)\mu_k} + \sum_{k=1}^{N} \frac{k}{\mu_k} \\ &= \sum_{k=1}^{N} \left(\frac{(N-k)(k-1)}{k} + \frac{N-k}{k} + k \right) \frac{1}{\mu_k} = N \sum_{k=1}^{N} \frac{1}{\mu_k}.\end{aligned}$$

3.1. Structural properties of Ω_w. The explicit form of the effective resistance Ω_w serves as a powerful tool for investigating the structural properties of the matrix Ω_w. In the following, we provide a detailed discussion of the relations between the entries of the matrix Ω_w.

PROPERTY 3.2. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \geq 0$. If two nodes i and $i+1$ have identical node weights for $2 \leq i \leq N-1$, i.e., $w_i = w_{i+1}$, then the effective resistance satisfies $(\Omega_w)_{i,i+1} = \frac{2}{\mu_i}$, where the Laplacian eigenvalue is $\mu_i = iw_i + \sum_{j=i+1}^{N} w_j$.*

Proof. From Lemma 2.2, it follows that $\mu_i = \mu_{i+1}$. The effective resistance $(\Omega_w)_{ij}$ in (3.5) can then be simplified as

$$(\Omega_w)_{i,i+1} = \frac{i-1}{i\mu_i} + \frac{i+1}{i\mu_j} = \frac{2}{\mu_i}.$$

PROPERTY 3.3. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \geq 0$, whose node set is partitioned as in (2.1). Assume that all nodes belonging to the same set share an identical node weight. Then, the effective resistance entries satisfy $(\Omega_w)_{ab} = (\Omega_w)_{pq}$ for any pair of nodes a, b and p, q such that nodes a, p belong to the same set and nodes b, q belong to another set.*

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Proof. Let nodes $a, p \in S_j$ and nodes $b, q \in S_{j+m}$ with $1 \leq m \leq N - j$. The entry $(\Omega_w)_{ij}$ in (3.5) can be rewritten as

$$\begin{aligned}(\Omega_w)_{ab} &= \frac{a-1}{a\mu_a} + \sum_{k=a+1}^{b-1} \frac{1}{k(k-1)\mu_k} + \frac{b}{(b-1)\mu_b} \\ &= \frac{a-1}{a\mu_a} + \sum_{k=a+1}^{C_j} \frac{1}{k(k-1)\mu_k} + \sum_{k=1+C_j}^{C_{j+m-1}} \frac{1}{k(k-1)\mu_k} + \sum_{k=1+C_{j+m-1}}^{b-1} \frac{1}{k(k-1)\mu_k} + \frac{b}{(b-1)\mu_b},\end{aligned}$$

where $C_k = \left|\bigcup_{i=1}^{k} S_i\right|$ denotes the cumulative size of the first k sets S_k. Noting that $\mu_i = \mu_{S_j}$ for $i \in S_j$, we further simplify

$$\begin{aligned}(\Omega_w)_{ab} &= \frac{1}{\mu_a}\left(1 - \frac{1}{a} + \sum_{k=a+1}^{C_j}\left(\frac{1}{k-1} - \frac{1}{k}\right)\right) + \sum_{k=1+C_j}^{C_{j+m-1}} \frac{1}{k(k-1)\mu_k} + \frac{1}{\mu_b}\left(\sum_{k=1+C_{j+m-1}}^{b-1}\left(\frac{1}{k-1} - \frac{1}{k}\right) + \frac{1}{b-1} + 1\right) \\ &= \frac{1}{\mu_{S_j}}\left(1 - \frac{1}{C_j}\right) + \sum_{p=1}^{m-1} \frac{1}{\mu_{S_{j+p}}} \sum_{k=1+C_{j+p-1}}^{C_{j+p}} \frac{1}{k(k-1)} + \frac{1}{\mu_{S_{j+m}}}\left(1 + \frac{1}{C_{j+m-1}}\right) \\ &= \frac{1}{\mu_{S_j}}\left(1 - \frac{1}{C_j}\right) + \sum_{p=1}^{m-1} \frac{1}{\mu_{S_{j+p}}}\left(\frac{1}{C_{j+p-1}} - \frac{1}{C_{j+p}}\right) + \frac{1}{\mu_{S_{j+m}}}\left(1 + \frac{1}{C_{j+m-1}}\right).\end{aligned} \tag{3.6}$$

This final expression depends only on the set $S_j, \cdots, S_{j+m}$ but not on the specific node indices within these sets. Hence, the equality $(\Omega_w)_{ab} = (\Omega_w)_{pq}$ holds for any four nodes $a, p \in S_j$ and $b, q \in S_{j+m}$. ◻

3.2. Equitable partition of Ω_w. Property 3.3 establishes that the effective resistance between any two nodes depends only on the subsets to which they belong, provided that all nodes within each subset S_i share the same link weight. In particular, given the partition (2.1), for any nodes $a, p \in S_i$ and $b, q \in S_j$, the effective resistances $(\Omega_w)_{ab}$ and $(\Omega_w)_{pq}$ coincide, which allows us to define the effective resistance between the subsets S_i and S_j by

$$\Omega_{S_i,S_j} := (\Omega_w)_{ab}, \quad \text{for any } \forall\ a \in S_i,\ \forall\ b \in S_j.$$

Consequently, the matrix Ω_w admits the block structure

$$\Omega_w = \begin{bmatrix} \alpha_1 I_1 & \beta_{1,2} J_{1,2} & \beta_{1,3} J_{1,3} & \cdots & \beta_{1,2K+1} J_{1,2K+1} \\ \beta_{1,2} J_{1,2}^\top & \alpha_2 I_2 & \beta_{2,3} J_{2,3} & \cdots & \beta_{2,2K+1} J_{2,2K+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{1,2K+1} J_{1,2K+1}^\top & \beta_{2,2K+1} J_{2,2K+1}^\top & \beta_{3,2K+1} J_{3,2K+1}^\top & \cdots & \alpha_{2K+1} I_{2K+1} \end{bmatrix},$$

where $\alpha_i = \frac{2}{\mu_i}$ for each subset S_i, and $\beta_{i,j} = \frac{1}{\mu_{S_i}}\left(1 - \frac{1}{C_i}\right) + \sum_{k=i+1}^{j-1} \frac{1}{\mu_{S_k}}\left(\frac{1}{C_{k-1}} - \frac{1}{C_k}\right) + \frac{1}{\mu_{S_j}}\left(1 + \frac{1}{C_{j-1}}\right)$ for any $a \in S_i$, $b \in S_j$, as derived in (3.6). The matrix I_i denotes the $|S_i| \times |S_i|$ identity matrix, and $J_{i,j}$ denotes the $|S_i| \times |S_j|$ all-ones matrix.
Moreover, the partition defined in Eq.(2.1) induces an equitable partition $\pi = \{S_1, S_2, \cdots, S_{2K+1}\}$ of the matrix Ω_w. The corresponding $(2K+1) \times (2K+1)$ quotient matrix Ω_w^π is given by

$$\Omega_w^\pi = \begin{bmatrix} \alpha_1 & \beta_{1,2}|S_2| & \beta_{1,3}|S_3| & \cdots & \beta_{1,2K+1}|S_{2K+1}| \\ \beta_{1,2}|S_1| & \alpha_2 & \beta_{2,3}|S_3| & \cdots & \beta_{2,2K+1}|S_{2K+1}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{1,2K+1}|S_1| & \beta_{2,2K+1}|S_2| & \beta_{3,2K+1}|S_3| & \cdots & \alpha_{2K+1} \end{bmatrix}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

For a more general setting, one may consider an equitable partition in which each subset S_{2i-1} (V_i) consists of nodes with distinct weights. In this case, only the blocks corresponding to sets S_{2i} (U_i) can be aggregated into single nodes in the quotient matrix, while the remaining nodes must be treated individually. For more details about equitable partitions, readers are referred to the book[1, p.45]. Furthermore, Alazemi [12] derived an explicit formula for $Q_w^\dagger(\pi)$ in terms of $Q_w(\pi)$ and established a direct relationship between $Q_w^\dagger(\pi)$ and $Q_w^\dagger$ for unweighted threshold graphs. Given the close connection between effective resistance and the Moore–Penrose inverse of the Laplacian matrix, it may be possible to adapt similar techniques to derive analogous reconstruction results for effective resistance matrices. Such an approach may provide further insight into the relationship between effective resistance and equitable partitions in threshold graphs.

3.3. Ω_w of unweighted threshold graphs. In the following, we present that additional properties emerge when the node weights are restricted to $w_i \in \{0,1\}$, corresponding to unweighted threshold graphs.

Property 3.4. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \in \{0,1\}$. Let the node set of G_w be partitioned as in (2.1) and denote $\mathcal{S}_j = \bigcup_{i=1}^{j} S_i$ for $2 \le j \le N$. For nodes a, b, and x, the following holds:*

1. *If $a, b \in S_j$, then*

$$\Omega_{ab} = \begin{cases} \max\limits_{x\in\mathcal{S}_j}\{\Omega_{xb}\} & \text{if } S_j = U \\ \min\limits_{x\in\mathcal{S}_j}\{\Omega_{xb}\} & \text{if } S_j = V \end{cases}.$$

2. *If $a \in S_{j-1}$ and $b \in S_j$, then*

$$\Omega_{ab} = \begin{cases} \min\limits_{x\in\mathcal{S}_j}\{\Omega_{xb}\} & \text{if } S_j = U \\ \max\limits_{x\in\mathcal{S}_j}\{\Omega_{xb}\} & \text{if } S_j = V \end{cases}.$$

Proof. Let node $x \in S_{j-m}$ with $0 \le m \le j-1$ and denote $\mu_x = \mu_{S_{j-m}}$, $\mu_a := \mu_{S_{j-1}}$, $\mu_b := \mu_{S_j}$. We distinguish two configurations.

Case 1. $a, b \in S_j$. If $b \in U$, then $w_b = 0$ and $\mu_b = \sum_{k=b+1}^{N} w_k$. Hence,

$$\mu_x = xw_x + \sum_{k=x+1}^{N} w_k = xw_x + \sum_{k=x+1}^{b} w_k + \sum_{k=b+1}^{N} w_k = xw_x + \sum_{k=x+1}^{b} w_k + \mu_b \ge \mu_b.$$

Substituting $\mu_{s_{j-m}} \ge \mu_{s_j}$ into (3.6) gives

$$\begin{aligned}\Omega_{xb} &= \frac{1}{\mu_{S_{j-m}}}\left(1 - \frac{1}{C_{j-m}}\right) + \sum_{p=j-m+1}^{j-1} \frac{1}{\mu_{S_p}}\left(\frac{1}{C_{p-1}} - \frac{1}{C_p}\right) + \frac{1}{\mu_{S_j}}\left(1 + \frac{1}{C_{j-1}}\right) \\ &\le \frac{1}{\mu_{S_j}}\left(1 - \frac{1}{C_{j-m}} + \sum_{p=j-m+1}^{j-1}\left(\frac{1}{C_{p-1}} - \frac{1}{C_p}\right) + 1 + \frac{1}{C_{j-1}}\right) = \frac{2}{\mu_{S_j}} = \Omega_{ab},\end{aligned}$$

where the last equality follows from Property 3.2. Thus, $\Omega_{ab} = \max\limits_{x\in\mathcal{S}_j}\{\Omega_{xb}\}$ when $S_j = U$.

For $b \in V$, we have $w_b = 1$ and then $\mu_x = xw_x + \sum_{k=x+1}^{N} w_k = xw_x + \sum_{k=x+1}^{b} w_k + \sum_{k=b+1}^{N} w_k \le bw_b + \sum_{k=b+1}^{N} w_k = \mu_b$. Therefore, the inequality reverses, giving $\Omega_{ab} = \min\limits_{x\in\mathcal{S}_j}\{\Omega_{xb}\}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

Case 2. $a \in S_{j-1}$, $b \in S_j$. If $b \in U$, then $a \in V$ $(w_a = 1)$ and

$$\mu_x = xw_x + \sum_{k=x+1}^{N} w_k = xw_x + \sum_{k=x+1}^{a} w_k + \sum_{k=a+1}^{N} w_k \leq aw_a + \sum_{k=a+1}^{N} w_k = \mu_a.$$

Substituting $\mu_{s_{j-m}} \leq \mu_{s_{j-1}}$ into (3.6) yields

$$\begin{aligned}\Omega_{xb} &= \frac{1}{\mu_{S_{j-m}}}\left(1 - \frac{1}{C_{j-m}}\right) + \sum_{p=j-m+1}^{j-1} \frac{1}{\mu_{S_p}}\left(\frac{1}{C_{p-1}} - \frac{1}{C_p}\right) + \frac{1}{\mu_{S_j}}\left(1 + \frac{1}{C_{j-1}}\right)\\ &\geq \frac{1}{\mu_{S_{j-1}}}\left(1 - \frac{1}{C_{j-m}} + \sum_{p=j-m+1}^{j-1}\left(\frac{1}{C_{p-1}} - \frac{1}{C_p}\right)\right) + \frac{1}{\mu_{S_j}}\left(1 + \frac{1}{C_{j-1}}\right)\\ &= \frac{1}{\mu_{S_{j-1}}}\left(1 - \frac{1}{C_{j-1}}\right) + \frac{1}{\mu_{S_j}}\left(1 + \frac{1}{C_{j-1}}\right) = \Omega_{ab}.\end{aligned}$$

Therefore, $\Omega_{ab} = \min_{x \in \mathcal{S}_j}\{\Omega_{xb}\}$ for $S_j = U$. Conversely, if $b \in V$, then we have $a \in U$ and $\mu_x \geq \mu_a$, leading to the reverse inequality and $\Omega_{ab} = \max_{x \in \mathcal{S}_j}\{\Omega_{xb}\}$. □

Property 3.4 further establishes the following bounds on the effective resistance for nodes from different sets given the partition (2.1).

PROPERTY 3.5. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$ with $w_i \in \{0, 1\}$. Let the node set of G_w be partitioned as in (2.1). For set S_j and all sets S_i for $1 \leq i \leq j$, the following bounds on the effective resistance Ω_{S_i,S_j} holds:*

1. $\Omega_{S_{j-1},S_j} \leq \Omega_{S_i,S_j} \leq \Omega_{S_j,S_j}$, *if* $S_j = U$
2. $\Omega_{S_j,S_j} \leq \Omega_{S_i,S_j} \leq \Omega_{S_{j-1},S_j}$, *if* $S_j = V$

Example. Given a threshold graph G_w with $N = 11$ nodes and encoded by the weight vector $w = (1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1)$. All the nodes in G_w can be partitioned as $\mathcal{N} = V_1 \cup U_1 \cup V_2 \cup U_2 \cup V_3$, where $V_1 = \{1, 2\}$, $U_1 = \{3, 4\}$, $V_2 = \{5, 6\}$, $V_2 = \{7, 8, 9\}$ and $V_1 = \{10, 11\}$. Theorem 2.1 indicates that its Laplacian eigenvalues are $\mu = \{0, 6, 4, 4, 8, 8, 2, 2, 2, 11, 11\}$. Table 1 presents the effective resistance matrix Ω_w of the graph G_w.

Due to the symmetry of the matrix Ω_w, it suffices to examine only its upper triangular part, which is enclosed by the red dashed line. Within each block, all entries of the matrix Ω_w are identical, which confirms Property 3.3. More precisely, if two nodes belong to the same set S_i, then their mutual effective resistance is given by $\frac{2}{\mu_{S_i}}$, which is formalized in Property 3.2. For two nodes belonging to different sets, the effective resistance between them lies within analytically derived bounds in Property 3.5.

4. Eigenvalues and eigenvectors of Ω_w.

4.1. Eigenvalues and eigenvectors of the matrix P_k.

We recall that the effective resistance matrix Ω_w of a weighted threshold graph G_w is given by $\Omega_w = \sum_{k=1}^{N} \frac{1}{\mu_k} P_k$ with the matrix P_k defined in (3.4). The matrix $J_{(k-1)\times(N-k)}$ has eigenvalue 0 with multiplicity $N - k - 1$ and an eigenvalue of $N - k$ with multiplicity 1. The vector u_{N-k} is the eigenvector associated with its nonzero eigenvalue $N - k$. Therefore,

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

TABLE 1
The effective resistance matrix Ω_w corresponding to the threshold graph G_w encoded by the weight vector $w = (1,1,0,0,1,1,0,0,0,1,1)$. The matrix is divided into block structures (highlighted by red dashed lines). For each column of blocks, the blue and pink backgrounds indicate the lower and upper bounds of the effective resistance, respectively.

$V_1=\{1,2\}$		$U_1=\{3,4\}$		$V_3=\{5,6\}$		$U_2=\{7,8,9\}$			$V_3=\{10,11\}$		
0	0.333	0.458	0.458	0.302	0.302	0.74	0.74	0.74	0.285	0.285	$V_1=\{1,2\}$
0.333	0	0.458	0.458	0.302	0.302	0.74	0.74	0.74	0.285	0.285	
0.458	0.458	0	0.5	0.344	0.344	0.781	0.781	0.781	0.327	0.327	$U_1=\{3,4\}$
0.458	0.458	0.5	0	0.344	0.344	0.781	0.781	0.781	0.327	0.327	
0.302	0.302	0.344	0.344	0	0.25	0.688	0.688	0.688	0.233	0.233	$V_3=\{5,6\}$
0.302	0.302	0.344	0.344	0.25	0	0.688	0.688	0.688	0.233	0.233	
0.74	0.74	0.781	0.781	0.688	0.688	0	1	1	0.545	0.545	$U_2=\{7,8,9\}$
0.74	0.74	0.781	0.781	0.688	0.688	1	0	1	0.545	0.545	
0.74	0.74	0.781	0.781	0.688	0.688	1	1	0	0.545	0.545	
0.285	0.285	0.327	0.327	0.233	0.233	0.545	0.545	0.545	0	0.182	$V_3=\{10,11\}$
0.285	0.285	0.327	0.327	0.233	0.233	0.545	0.545	0.545	0.182	0	

the block structure of P_k enables a systematic characterization of its eigenvalues and eigenvectors, which we establish in Theorem 4.1.

THEOREM 4.1. *Let P_k with $k \geq 4$ be the $N \times N$ matrix defined in (3.4). Its eigenvalue equation $P_k x = \lambda x$ admits the following eigenvalues and corresponding eigenspaces:*
(1) $\lambda = 0$ with multiplicity $N-3$. The corresponding eigenspace is

$$\left\{ x = \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} \in \mathbb{R}^N,\ x_1^T u_{k-1} = 0,\ x_3^T u_{N-k} = 0 \right\}. \tag{4.7}$$

(2) The nonzero eigenvalues are the zeros of the polynomial $-2 + \frac{2N}{k} + \frac{\left(3N - 3k(1+N) + k^2(3+N)\right)\lambda}{k(k-1)} - \lambda^3 = 0$. The corresponding eigenspace is

$$\left\{ x = \begin{bmatrix} \alpha u_{k-1} \\ \beta \\ \gamma u_{N-k} \end{bmatrix} \in \mathbb{R}^N,\ \alpha = k\lambda\gamma - (k-1)\beta,\ \beta = \frac{k^3\lambda + (k-1)(N-k)}{k^2(k-1) + k\lambda}\gamma,\ \gamma \in \mathbb{R} \right\}.$$

Proof. We discuss the two cases: $\lambda = 0$ and $\lambda \neq 0$.
Case 1: $\lambda = 0$.
Let $x = [x_1, x_2, x_3]^T$ be the eigenvector associated with $\lambda = 0$, where $x_1 \in \mathbb{R}^{k-1}$, $x_2 \in \mathbb{R}$, and $x_3 \in \mathbb{R}^{N-k}$. The eigenvalue equation $P_k x = \lambda x$ for $\lambda = 0$, by expressing $J_{(k-1)\times(N-k)}$ as $u_{k-1}u_{N-k}^T$, then becomes

$$P_k x = \begin{bmatrix} \frac{k}{k-1} u_{k-1} x_2 + \frac{1}{k(k-1)} u_{k-1} u_{N-k}^T x_3 \\ \frac{k}{k-1} u_{k-1}^T x_1 + \frac{k-1}{k} u_{N-k}^T x_3 \\ \frac{1}{k(k-1)} u_{N-k} u_{k-1}^T x_1 + \frac{k-1}{k} u_{N-k} x_2 \end{bmatrix} = 0_N.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

From the first and third rows, we obtain $x_2 = -\frac{1}{k^2} u_{N-k}^T x_3 = -\frac{1}{(k-1)^2} u_{k-1}^T x_1$. Substituting these expressions into the second row yields $2(k-1)kx_2 = 0$. Hence, $x_2 = 0$, and therefore, $u_{N-k}^T x_3 = 0$ and $u_{k-1}^T x_1 = 0$, which indicates that the vector x must obey the form in (4.7).

Therefore, the null space of the matrix P_k consists of vectors orthogonal to both u_{k-1} and u_{N-k}, giving that matrix P_k has the eigenvalue 0 with multiplicity $(k-2)+(N-k-1) = N-3$.

Case 2: $\lambda \neq 0$.
All the remaining eigenvectors of P_k should be orthogonal to those eigenvectors corresponding to $\lambda = 0$ and hence have the form

$$x = \begin{bmatrix} \alpha u_{k-1} \\ \beta \\ \gamma u_{N-k} \end{bmatrix}, \tag{4.8}$$

with scalars $\alpha, \beta, \gamma \in \mathbb{R}$.
Substituting (4.8) into the eigenvalue equation $P_k x = \lambda x$ gives

$$P_k x = \begin{bmatrix} \frac{k}{k-1}\beta u_{k-1} + \frac{N-k}{k(k-1)}\gamma u_{k-1} \\ k\alpha + \frac{(k-1)(N-k)}{k}\gamma \\ \frac{1}{k}\alpha u_{k-1} + \frac{k-1}{k}\beta u_{k-1} \end{bmatrix} = \lambda \begin{bmatrix} \alpha u_{k-1} \\ \beta \\ \gamma u_{N-k} \end{bmatrix},$$

which reduces to the 3×3 eigenvalue equation

$$\begin{bmatrix} 0 & \frac{k}{k-1} & \frac{N-k}{k(k-1)} \\ k & 0 & \frac{(k-1)(N-k)}{k} \\ \frac{1}{k} & \frac{k-1}{k} & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \lambda \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \tag{4.9}$$

Its characteristic polynomial is

$$-\lambda^3 + \frac{\left(3N - 3k(1+N) + k^2(3+N)\right)\lambda}{(-1+k)k} - 2 + \frac{2N}{k} = 0,$$

whose three roots yield the remaining nonzero eigenvalues of P_k.

Next, we determine the scalars α, β, and γ. We express α in terms of β, γ from the third row of (4.9) to obtain

$$\alpha = k\lambda\gamma - (k-1)\beta.$$

Substituting α into the second row of (4.9) leads to

$$\beta = \frac{k^3\lambda + (k-1)(N-k)}{k^2(k-1) + k\lambda}\gamma.$$

∎

Therefore, α, β are expressed in terms of λ and a free scaling factor γ.

While the reduction to the three-dimensional invariant subspace provides a direct route to the nonzero eigenvalues of P_k, the same result can also be obtained by treating $P_k - \lambda I_N$ as a 2×2 block matrix and applying the Schur complement formula [1]. Write $P_k - \lambda I_N$ in 2×2 blocks form

$$P_k - \lambda I_N = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

where

$$A = -\lambda I_{k-1} \qquad B = \begin{bmatrix} \frac{k}{k-1}u_{k-1} & \frac{1}{k(k-1)}J_{(k-1)\times(N-k)} \end{bmatrix}$$
$$C = \begin{bmatrix} \frac{k}{k-1}u_{k-1}^T \\ \frac{1}{k(k-1)}J_{(N-k)\times(k-1)} \end{bmatrix} \qquad D = \begin{bmatrix} -\lambda & \frac{k-1}{k}u_{N-k}^T \\ \frac{k-1}{k}u_{N-k} & -\lambda I_{N-k} \end{bmatrix}.$$

Since $A^{-1} = -\frac{1}{\lambda}I_{k-1}$ for $\lambda \neq 0$, the Schur complement formula gives

$$\det(P_k - \lambda I_N) = \det(A)\det(D - CA^{-1}B) = (-\lambda)^{k-1}\det\left(D + \frac{1}{\lambda}CB\right).$$

Applying $u_{k-1}^T u_{k-1} = k-1$ and $J_{(k-1)\times(N-k)} = u_{k-1}u_{N-k}^T$ leads to

$$CB = \begin{bmatrix} \frac{k^2}{k-1} & \frac{1}{k-1}u_{N-k}^T \\ \frac{1}{k-1}u_{N-k} & \frac{1}{k^2(k-1)}u_{N-k}u_{N-k}^T \end{bmatrix}.$$

Hence,

$$D + \frac{1}{\lambda}CB = \begin{bmatrix} -\lambda + \frac{k^2}{\lambda(k-1)} & \left(\frac{k-1}{k} + \frac{1}{\lambda(k-1)}\right)u_{N-k}^T \\ \left(\frac{k-1}{k} + \frac{1}{\lambda(k-1)}\right)u_{N-k} & -\lambda I_{N-k} + \frac{1}{\lambda k^2(k-1)}u_{N-k}u_{N-k}^T \end{bmatrix}.$$

Applying Schur complement again to $D + \frac{1}{\lambda}CB$ and denoting $E := -\lambda I + \frac{1}{\lambda k^2(k-1)}u_{N-k}u_{N-k}^T$, we obtain

$$\det\left(D + \frac{1}{\lambda}CB\right) = \det(E)\left(-\lambda + \frac{k^2}{\lambda(k-1)} - \left(\frac{k-1}{k} + \frac{1}{\lambda(k-1)}\right)^2 u_{N-k}^T E^{-1} u_{N-k}\right).$$

The matrix E has eigenvalue $-\lambda$ with multiplicity $N-k-1$ and eigenvalue $-\lambda + \frac{N-k}{\lambda k^2(k-1)}$ with the corresponding eigenvector u_{N-k}, which leads to

$$u_{N-k}^T E^{-1} u_{N-k} = \frac{1}{-\lambda + \frac{N-k}{\lambda k^2(k-1)}}u_{N-k}^T u_{N-k} = \frac{N-k}{-\lambda + \frac{N-k}{\lambda k^2(k-1)}}.$$

We rewrite $\det\left(D + \frac{1}{\lambda}CB\right)$ as

$$\begin{aligned}\det\left(D + \frac{1}{\lambda}CB\right) &= (-\lambda)^{N-k-1}\left(-\lambda + \frac{N-k}{\lambda k^2(k-1)}\right)\left(-\lambda + \frac{k^2}{\lambda(k-1)} - \left(\frac{\lambda(k-1)^2+k}{k\lambda(k-1)}\right)^2 \frac{(N-k)}{-\lambda + \frac{N-k}{\lambda k^2(k-1)}}\right) \\ &= (-\lambda)^{N-k-1}\left[\lambda^2 - \frac{k^2}{k-1} - (N-k)\left(\frac{1}{k^2(k-1)} + \frac{2}{\lambda k} + \frac{(k-1)^2}{k^2}\right)\right]\end{aligned}.$$

Finally, we reach

$$\begin{aligned}\det(P_k - \lambda I) &= (-\lambda)^{N-2}\left[\lambda^2 - \frac{k^2}{k-1} - (N-k)\left(\frac{1}{k^2(k-1)} + \frac{2}{\lambda k} + \frac{(k-1)^2}{k^2}\right)\right] \\ &= (-\lambda)^{N-3}\left[\lambda^3 - \frac{k^2\lambda}{k-1} - (N-k)\left(\frac{\lambda}{k^2(k-1)} + \frac{(k-1)^2\lambda}{k^2}\right) - \frac{2N}{k} + 2\right] \\ &= (-\lambda)^{N-3}\left[\lambda^3 - \frac{k^4\lambda + (N-k)[(k-1)^3\lambda + \lambda]}{k^2(k-1)} - \frac{2N}{k} + 2\right] \\ &= (-\lambda)^{N-3}\left[\lambda^3 - \frac{k^3\lambda + (N-k)(k^2\lambda - 3k\lambda + 3\lambda)}{k^2(k-1)} - \frac{2N}{k} + 2\right],\end{aligned}$$

which gives the same spectrum obtained in the earlier proof.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

4.2. Eigenvalues and eigenvectors of Ω_w. The eigenvalue equation of the effective resistance matrix Ω is defined as

$$\Omega v_k = \rho_k v_k,$$

where ρ_j is the j-th eigenvalue of Ω belonging to the normalized eigenvector v_j for $1 \leq j \leq N$. The relation between the eigenvalues of the Laplacian Q_w and the effective resistance matrix Ω_w is stated in the following interlacing Theorem.

Theorem 4.2 ([15, 16]). *In a connected graph G, the eigenvalues $\rho_1, \rho_2, \ldots, \rho_N$ of its effective resistance matrix Ω interlace with those of the Laplacian matrix Q as*

$$\rho_1 > 0 > -\frac{2}{\mu_1} \geq \rho_2 \geq -\frac{2}{\mu_2} \geq \cdots \geq -\frac{2}{\mu_{N-2}} \geq \rho_{N-1} \geq -\frac{2}{\mu_{N-1}} \geq \rho_N,$$

where $\mu_1 \geq \cdots > \mu_N = 0$ represents the eigenvalues of the Laplacian Q of the graph.

The Interlacing Theorem shows that, if $\mu_k = \mu_{k-1}$, then $\rho_k = -\frac{2}{\mu_k}$. In other words, when a graph has Laplacian eigenvalues of multiplicity larger than 1, then equality $\rho_k = -\frac{2}{\mu_k}$ occurs. In the case of threshold graphs, Property 3.2 shows that the multiplicity of an eigenvalue is determined by the weight vector w. Hence, we have the following result.

Theorem 4.3. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$, where the node weights satisfy $w_i \geq 0$, for $1 \leq i < N-1$ and $w_N > 0$. If two node weights $w_i = w_{i+1}$ for $2 \leq i \leq N-1$, then the effective resistance matrix Ω_w of graph G_w has eigenvalue $\rho_i = \frac{-2}{(i+1)w_i + \sum_{k=i+2}^{N} w_k}$.*

Proof. The assumption $w_i = w_{i+1}$ implies, by Lemma 3.2, that the two Laplacian eigenvalues satisfy $\mu_i = \mu_{i+1}$ Thus, we directly obtain

$$\rho = \frac{-2}{\mu_{i+1}} = \frac{-2}{(i+1)w_i + \sum_{k=i+2}^{N} w_k},$$

from Theorem 4.2, where $\mu_i = \mu_{i+1} = (i+1)w_i + \sum_{k=i+2}^{N} w_k$ is established in Theorem 2.1. ∎

Theorem 4.3 shows that whenever the weight vector w of graph G_w contains k pairs of consecutive equal weights, the effective resistance matrix Ω_w possesses at least k explicitly identifiable eigenvalues. In addition, for every weighted threshold graph G_w, one can always construct an eigenvector associated with the eigenvalue $-\frac{2}{\mu_2}$, guaranteeing at least one fully determined eigenpair of Ω_w independently of the particular choice of weights.

Property 4.4. *Let G_w be a threshold graph encoded by the weight vector $w = (w_1, w_2, \ldots, w_N)$, where the node weights satisfy $w_i \geq 0,\ 1 \leq i < N-1$, and $w_N > 0$, its effective resistance Ω_w has the eigenvalue $\rho = \frac{-2}{2w_2 + \sum_{k=3}^{N} w_k}$ and the corresponding normalized eigenvector $v = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0_{N-2}]^T$.*

Proof. By the definition of the matrices P_k in (3.4), each P_k has identical entries in its first two columns except in the first two rows. Since the effective resistance matrix is given by $\Omega_w = \sum_{k=1}^{N} \frac{1}{\mu_k} P_k$, the same structural property is inherited by Ω_w. Substituting v into the eigenvalue equation $\Omega_w v = \rho v$ yields ∎

$$\Omega_w v = \left[-\frac{\sqrt{2}}{\mu_2}, \frac{\sqrt{2}}{\mu_2}\ , 0,\ \cdots,\ 0 \right]^T = -\frac{2}{\mu_2} v.$$

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A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

5. An application. In this section, we provide an application of our results concerning the structural properties of the effective resistance matrices Ω_w, as discussed in Section 3. Consider an unweighted threshold graph G_w encoded by the weight vector $w = (w_1, \cdots, w_N)$ where $w_i \in \{0, 1\}$. The graph has N nodes and $M = \sum_{i=1}^{N}(i-1)w_i$ links. A natural problem is to determine how to add a link to G_w such that the resulting graph remains a threshold graph and to identify the corresponding updated weight vector.

Given graph G_w with N nodes and M links and its weight vector $w = (w_1, \cdots, w_N)$ with $w_i \in \{0, 1\}$, a new threshold graph with N nodes and $M + 1$ links can be constructed by performing the steps:
(I) Selection: Choose two nodes i and $i + 1$ satisfying $w_i = 1$ and $w_{i+1} = 0$.
(II) Link addition: Insert a link between nodes i and $i + 1$.
(III) Reordering: Interchange the nodes i and $i + 1$ to obtain a new graph G'_w with weight vector $w' = (w'_1, \cdots, w'_N)$ where $w'_i = 0,\ w'_{i+1} = 1$ and $w_{j'} = w_j$ for $j \notin \{i, i + 1\}$.

The structure of an unweighted threshold graph is determined by whether each node is dominant or isolated, together with the relative ordering of the nodes. In step (I), a dominant node i and an isolated node $i + 1$ are selected. In step (II), the addition of a link between these two nodes increases the total number of links from M to $M + 1$. In step (III), after interchanging their positions, node i becomes isolated, since it is no longer adjacent to any preceding node, while node $i + 1$ becomes dominant, as it is adjacent to all nodes that precede it. Because the modification affects only the local structure involving nodes i and $i + 1$, the adjacency relationships among the remaining $N - 2$ nodes remain unchanged. Consequently, the resulting graph G'_w preserves the threshold property and contains exactly one additional link.

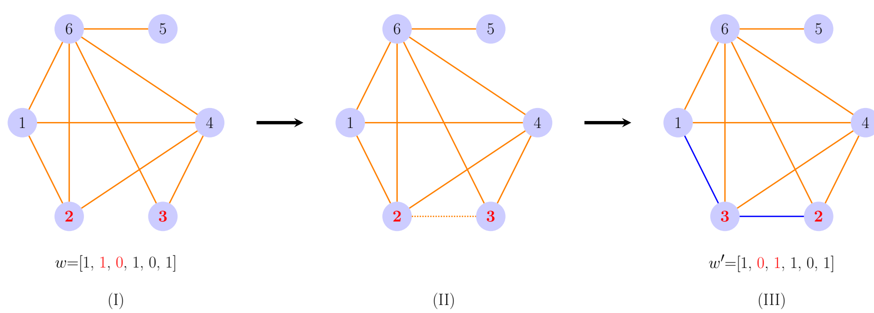

FIGURE 3. *From a threshold graph G_w with $M = 7$ links and weight vector $w = [1, 1, 0, 1, 0, 1]$ to a threshold graph G'_w with $M' = 8$ links and weight vector $w' = [1, 0, 1, 1, 0, 1]$. A link is added between the dominant node 2 and the isolated node 3 (dashed link). After relabeling, node 2 becomes isolated, and node 3 becomes dominant (blue links). All the remaining nodes are unchanged.*

The example in Fig. 3 illustrates the link-addition procedure described above. In step (I), the consecutive entries $[1, 0]$ corresponding to nodes 2 and 3 are chosen in the original threshold graph G_w. In step (II), a link is added between these two nodes, thereby increasing the total number of links by one. Finally, in step (III), after interchanging their positions, we obtain a new threshold graph $G_{w'}$, which preserves the threshold property and contains one additional link compared with the original graph G_w.

Since a weight vector w may contain multiple subsequences of the form $[1, 0]$, there may exist several possible choices in step (I) of the link-selection process. The entire process, from a threshold graph G_w to a complete graph K_N of the same order, can be viewed as a tree process with G_w serving as the root and K_N serving as the leaves. A threshold graph G_w contains $M = \sum_{i=1}^{N}(i - 1)w_i$ links, while the number of links

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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in a complete graph K_N is $L = \frac{N(N-1)}{2}$. Hence, the height of the tree from the root G_w is given by

$$L - M = \frac{N(N-1)}{2} - \sum_{i=1}^{N}(i-1)w_i.$$

We denote by $G(w,k)$ the threshold graph obtained by adding k links to an initial threshold graph G_w through iterative application of the above procedure. To construct $G(w,k)$, one starts from G_w and follows a branch of length k in the corresponding tree. Figure 4 illustrates how the evolution of the weight vector w from the threshold graph G_w encoded by weight vector $w = [1,1,0,1,0,1]$ to the complete graph K_6 encoded by weight vector $w = [1,1,1,1,1,1]$.

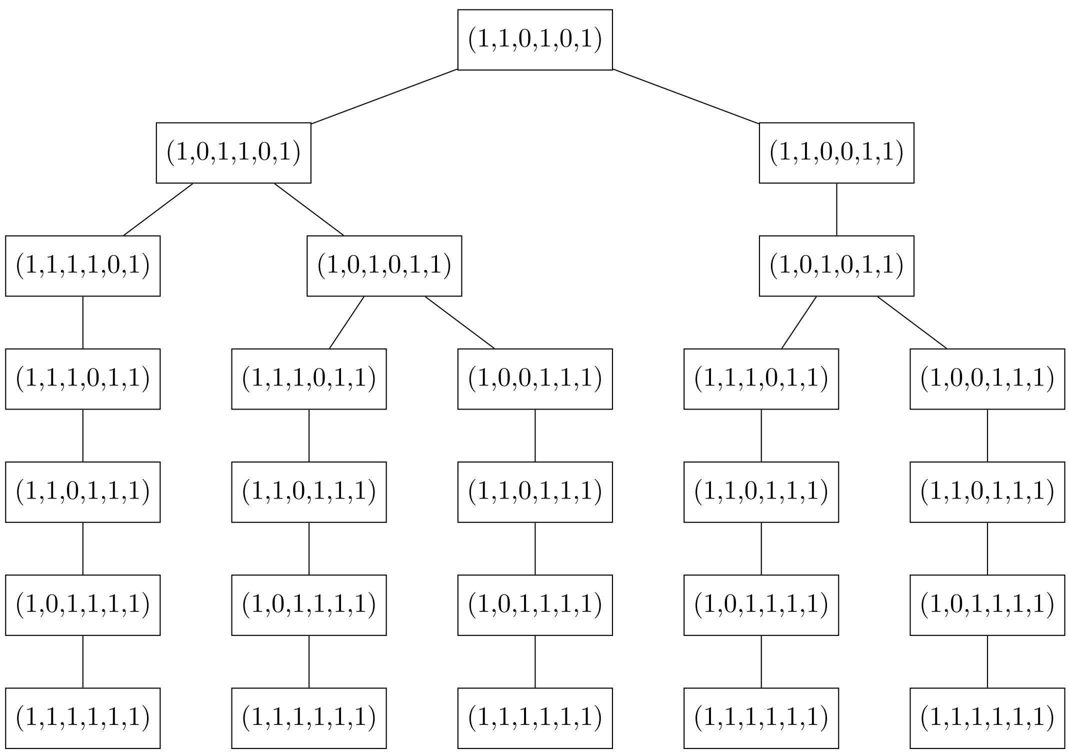

FIGURE 4. *Evolution of the weight vector w from an initial threshold graph G_w encoded by $w = (1,1,0,1,0,1)$, to the complete graph K_6 encoded by $w = (1,1,1,1,1,1)$, while preserving the threshold condition. After each step, the corresponding weight w_1 of node 1 is automatically updated to 1.*

The modification affects only the local structure of nodes i and $i+1$. Consequently, only the Laplacian eigenvalues μ_i and μ_{i+1} change to μ_i' and μ_{i+1}' , while the remaining $N-2$ Laplacian eigenvalues remain unchanged, i.e., $w_j' = w_j$ for $j \neq i$ and $j \neq i+1$. By Theorem 2.1, the Laplacian eigenvalues before and after the modification satisfy

$$\mu_i = (i-1) + \sum_{j=i+1}^{N} w_j = (i-1) + \sum_{j=i+2}^{N} w_j,\ \mu_{i+1} = \sum_{j=i+2}^{N} w_j,$$
$$\mu_i' = \sum_{j=i+1}^{N} w_j' = 1 + \sum_{j=i+2}^{N} w_j = 1 + \mu_{i+1},\ \mu_{i+1}' = i + \sum_{j=i+2}^{N} w_j' = 1 + \mu_i.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 551-568, June 2026.

The corresponding reduction of the effective graph resistance is therefore

$$R_{G_w} - R_{G_{w'}} = N\sum_{k=1}^{N}\left(\frac{1}{\mu_k} - \frac{1}{\mu_k'}\right).$$

Since only μ_i' and μ_{i+1}' are changed, this reduces to

$$\begin{aligned} R_{G_w} - R_{G_{w'}} &= N\left(\frac{1}{\mu_i} + \frac{1}{\mu_{i+1}} - \frac{1}{\mu_i'} - \frac{1}{\mu_{i+1}'}\right) = N\left(\frac{1}{\mu_i(\mu_i+1)} + \frac{1}{\mu_{i+1}(\mu_{i+1}+1)}\right) \\ &= N\left(\frac{1}{(\mu_{i+1}+i-1)(\mu_{i+1}+i)} + \frac{1}{\mu_{i+1}(\mu_{i+1}+1)}\right) \\ &= N\left(\frac{1}{(\sum_{j=i+2}^{N} w_j + i - 1)(\sum_{j=i+2}^{N} w_j + i)} + \frac{1}{\sum_{j=i+2}^{N} w_j(\sum_{j=i+2}^{N} w_j + 1)}\right) \end{aligned}.$$

Each isolated node $i \in U$ is associated with $i-1$ missing links in the threshold graph relative to the complete graph of the same order. The number of candidate node pairs k at step (II) equals the number of subsequences of the form $[1,0]$ in the weight vector w. Let nodes $a \in V_i$ and $b \in U_i$ form one of these $[1,0]$ subsequences. According to Property 3.5, the effective resistance $\Omega_{ab} = \Omega_{V_i,U_i}$ between nodes a and b is the smallest among all effective resistances Ω_{xb} for $x < b$. This observation implies that, among all node pairs with missing links, those forming $[1,0]$ subsequences exhibit the lowest effective resistances. Therefore, the link addition process prioritizes one of the k node pairs with the smallest effective resistances.

Declaration of competing interest. Nothing to declare.

Acknowledgments. We thank the referees for their constructive remarks on equitable partitions of the effective resistance matrices. Yingyue Ke has been funded by the China Scholarship Council (Grant No. 202206270016). Piet Van Mieghem has been funded by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (Grant Agreement No. 101019718).

Data availability. No data were used for the research described in the article.

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