Conserved Currents of Spinor Fields

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Abstract

This thesis investigates the conserved currents of spinor fields. These can be computed through Noether's theorem, which links symmetries to conserved quantities. This requires the transformation behaviour of spinors, which is given using the Clifford algebra. Furthermore, other relevant concepts for describing spinors, such as vector bundles and connections, are introduced. Finally, the concept of a universal spinor bundle is introduced, which can be used to describe spin-$\frac{1}{2}$ particles under the effect of and as a cause of gravity.

Contents

Introduction

The main purpose of this thesis is to show how to derive the stress-energy tensor and other currents for spinor fields using Noether's theorem. This is done in a way that should be comprehensible to bachelor's students of physics and mathematics. The main prerequisite is basic differential geometry. This is covered by the reader *Introduction to differential geometry* by B. Janssens[1], from which most notation is also taken. The same information can be found in the books [2] and [3]. Some topology is also used, which can be found in [4].
Noether's theorem was first published by Emmy Noether in 1918 and provides a way to link symmetries to conservation laws. In particular given a Lagrangian and a symmetry of the Lagrangian it gives a way to compute conservation laws. This can be used to compute the stress-energy tensor, which corresponds to translational symmetries. This tensor serves as a source term in Einstein's field equations, which govern the curvature of the universe as a consequence of matter. In general relativity, this curvature of spacetime is the cause of gravity. Hereby, Noether's theorem provides a way to link types of matter to gravity.

The type of matter we want to describe in this thesis is an electron, or more generally a spin-$\frac{1}{2}$ particle. These are mathematically modelled by objects called spinors. Spinor fields are spinors defined on all of spacetime, which are used in quantum field theory. This theory describes the interactions of particles through the electromagnetic, weak nuclear, and strong nuclear forces.

We start in section 1 with Noether's theorem, which gives a way to calculate the stress-energy tensor from the Lagrangian of a theory. This process does have complications, as the stress-energy tensor obtained through this process is usually not symmetric, though general relativity uses symmetric stress-energy tensors. These symmetric stress-energy tensors are usually derived by varying the matter Lagrangian with respect to the metric tensor. This yields the formula

$$T^{\mu\nu} = -2\frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}} \tag{1}$$

in which the tensor $T^{\mu\nu}$ is clearly symmetric, as the metric tensor $g_{\mu\nu}$ is symmetric. This means the stress-energy tensor obtained through Noether's theorem should also be symmetric. We also immediately show how to apply the theorem by considering an electromagnetic field, which will be familiar to physics students.

Section 2 then gives a description of spinors as elements of a vector space $\mathbb{C}^4$, which transform under Lorentz transformation through the so-called spin group.

Section 3 introduces vector bundles, which are ways to put a copy of a vector space at each point of spacetime, and connections, which generalize directional derivatives of a function along a vector field. These are used to define spinor fields, along with their Lagrangian, which is shown to be Lorentz invariant, meaning they look the same in any inertial system. This then allows us to calculate the stress-energy tensor for spinor fields.

Finally, section 4 considers a description of spinors coupled to gravity. Using variational principles, like Noether's theorem, on the bundles used in this description requires some care, as a variation on spacetime can change the metric g, which describes gravity. This change in the metric then changes the bundle on which the spinor fields exist. To counteract this issue, the universal spinor bundle is introduced, the fields on this bundle are a combination of a gravitational field and a spinor field, which allows for variational techniques.

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1 Noether's Theorem

Noether's theorem links symmetries to conserved quantities. This theorem gives a way to calculate so-called Noether currents and in particular the stress-energy momentum tensor. We will introduce the theorem through Lagrangian formalism in subsection 1 and prove the theorem in subsection 2.

In subsection 3 we use Noether's theorem to derive some well-known results from classical mechanics.

Subsection 4 discusses the stress-energy tensor, which is important in general relativity. Using Noether's theorem to derive this tensor does have some complications, one of which can be solved using a method outlined in subsection 5, which discusses the moment tensor.

Finally, we compute these two tensors for the theory of electromagnetism in subsection 6.

1.1 Noether's Theorem

To formulate Noether's theorem we look at the description given by Forger and Römer [5], for which we recall Lagrangian formalism and introduce some conventions.

1.1.1 Lagrangian Formalism

A field is a function on spacetime taking values in some vector space V denoted as $\varphi : \mathbb{R}^4 \to V$. For a given coordinate system $x^\mu = (x^0, x^1, x^2, x^3)$ we can take the partial derivatives of the field with respect to the coordinates, we will denote these as $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The μ labels the different coordinates of spacetime, where we often give the time coordinate label 0. In standard coordinates for spacetime we have $x^\mu = (ct, x, y, z)$, with c the speed of light, used to give all coordinates the same units.

A Lagrangian is a real-valued function of the field values and the values of the partial derivates up to some finite order at some point in spacetime, and may furthermore depend explicitly on spacetime. The Lagrangian can be written as $\mathcal{L}(\varphi(x), \partial\varphi(x), x)$. If we leave the spacetime dependence of all the functions implicit, we instead have $\mathcal{L}(\varphi, \partial\varphi)$. This Lagrangian can be used to define the action functional

$$S[\varphi] = \int_\Omega d^n x \, \mathcal{L}(\varphi, \partial\varphi), \tag{2}$$

of which a physical field is a stationary point by the principle of least action. Such a stationary point is a field that satisfies the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi^i}, \tag{3}$$

where the i labels the coordinates for the vector space V, when given a basis.

1.1.2 Conventions

Before introducing Noether's theorem, we introduce Einstein summation convention and the Minkovski inner product. Hereafter spacetime coordinates will always be labelled by Greek letters such as μ, ν or κ and the vector space will often be labelled by Latin letters like i, j or k, unless they are directly related to the spacetime coordinates. Furthermore, any time these labels are repeated they are implicitly summed over. This means we can write a vector $v \in V$

as $v^i b_i = \sum_i v^i b_i$, where the b_i are vectors in a basis of V with the v^i the components of v with respect to this basis.

The Minkovski inner product is a bilinear form on $\mathbb{R}^4$. When spacetime is given the standard coordinates x^μ=(ct,x,y,z), the inner product is given by $\eta(v, w) = \eta_{\mu\nu} v^\mu w^\nu$, were the v^μ, w^ν are the coordinates of v, w with respect to the standard basis and

$$\eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then, to introduce Noether's theorem, we introduce the meaning of a symmetry. This is done using Lie groups, which are smooth manifolds G with a smooth group multiplication $\circ : G \times G \to G$ that makes the manifold a group. Suppose we have some group action of a Lie group G on the fields, which we denote $g \circ \varphi$ for $g \in G$ and fields φ. If $S[\varphi] = S[g \circ \varphi]$ for all $g \in G$ and fields φ, then the action functional S is invariant under a group action of the Lie group G and G is called a symmetry group of S. To calculate the conserved current we take a one-parameter subgroup of G. A one-parameter subgroup is given by a parametrisation

$$\varphi : \mathbb{R} \to G$$

that is also a group homomorphism. That is

$$\varphi(s) \circ \varphi(t) = \varphi(s + t).$$

We denote the parameter as ε, and subsequently denote the elements of the subgroup by Λ_ε with Λ_0 the identity element of G. The action of this subgroup gives a change in the spacetime coordinates of $x \to x' = \Lambda_\varepsilon(x)$. To determine the new field $\varphi'(x')$, we consider the value at $\Lambda_\varepsilon^{-1}(x')$, as this gets moved to x' by the change in coordinates. This field then gets another change by the action of Λ_ε that makes the transformation a symmetry. The precise nature of this transformation is determined by the transformation Λ_ε and the vector space V, though for the proof we assume that this action is known. For instance for scalar fields this action is trivial, i.e. $\Lambda_\varepsilon \circ \varphi = \varphi$. We will also see in subsection 6 that 1-forms, such as the electromagnetic potential, rotate the direction opposite to the rotation of the base space. The action of this subgroup gives a transformation

$$x \to \Lambda_\varepsilon(x) =: \Lambda_\varepsilon x \quad , \quad \varphi(x) \to \Lambda_\varepsilon \circ \varphi(\Lambda_\varepsilon^{-1} x) =: \varphi_\varepsilon(\Lambda_\varepsilon^{-1} x). \tag{4}$$

G is assumed to be a symmetry group of the action functional S, so the action of any element $g \in G$ leave S invariant. In particular the action of elements $\Lambda_\varepsilon \in G$ leave S invariant. Thus, $S[\varphi] = S[\varphi_\varepsilon]$, and in particular $\frac{d}{d\varepsilon} S[\varphi_\varepsilon] = 0$. We can use this identity to derive a so-called Noether current J^μ from this symmetry. This will be a conserved quantity as its divergence vanishes

$$\partial_\mu J^\mu = 0.$$

Before performing any of the computational steps of the proof we clarify some notation. The derivative ∂_μ is the total derivative with respect to a coordinate, where we assume the coordinates to be independent. The partial derivative of $\varphi(x)$ with respect to a coordinate is denoted $\frac{\partial \varphi}{\partial x^\mu}$. For functions that only depend explicitly on the coordinates, these two are the same. However, the Lagrangian depends implicitly on these coordinates through the field variables $\varphi(x)$ and derivatives $\partial\varphi$. This means that the total derivative is

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi^i}\partial_\mu \varphi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi^i)}\partial_\mu(\partial_\nu \varphi^i) + \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{5}$$

Moreover, we define the direction of the infinitesimal variation in the base space by $X_B^\mu := \frac{d}{d\varepsilon}(\Lambda_\varepsilon(x))^\mu|_{\varepsilon=0}$. Then, by the product rule

$$0 = \frac{d}{d\varepsilon}x\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Lambda_\varepsilon^{-1}\Lambda_\varepsilon(x)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\Lambda_\varepsilon^{-1}(x)\bigg|_{\varepsilon=0} + \frac{d}{d\varepsilon}\Lambda_\varepsilon(x)\bigg|_{\varepsilon=0},$$

so $\frac{d}{d\varepsilon}(\Lambda_\varepsilon^{-1}(x))^\mu\big|_{\varepsilon=0} = -X_B^\mu$. Similarly, we define $X_F^i = \frac{d}{d\varepsilon}\varphi_\varepsilon^i\big|_{\varepsilon=0}$ for the field space. Here, φ_ε^i are the components of the transformed field with respect to the basis of V. Lastly, we note that the total derivative ∂_μ is $\partial_\mu(\varphi_\varepsilon(\Lambda_\varepsilon^{-1}x)) = \partial_\nu(\varphi_\varepsilon)(\Lambda_\varepsilon^{-1}x)\partial_\mu(\Lambda_\varepsilon^{-1}x)^\nu$ by the chain rule.

1.2 Proof Noether's Theorem

Using conventions introduced in the previous subsection, we can now prove Noether's theorem. The main idea of this proof will be to rewrite the symmetry condition

$$\frac{d}{d\varepsilon}S[\varphi_\varepsilon] = 0$$

to an integral that, when evaluated at zero, has the form

$$\int_\Omega d^n x \; \partial_\mu J^\mu = 0.$$

This then gives an expression of the current J^μ, which is conserved as the integral is zero for any Ω. Then, to start the proof, we suppose that the transformation (4) leaves the action functional S invariant. Then

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\int_{\Omega'} d^n x' \, \mathcal{L}(\varphi_\varepsilon(\Lambda_\varepsilon^{-1}x'), \partial\varphi_\varepsilon(\Lambda_\varepsilon^{-1}x'), x') \\
&= \frac{d}{d\varepsilon}\int_{\Omega} d^n x \, \mathcal{L}(\varphi_\varepsilon(x), \partial\varphi_\varepsilon(x)\partial\Lambda_\varepsilon^{-1}(x), \Lambda_\varepsilon x) \det\left(\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\right) \\
&= \int_{\Omega} d^n x \, \frac{d}{d\varepsilon}\mathcal{L}(\varphi_\varepsilon(x), \partial\varphi_\varepsilon(x)\partial\Lambda_\varepsilon^{-1}(x), \Lambda_\varepsilon x) \det\left(\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\right).
\end{aligned}$$

Here, due to the change in the base space the region Ω changes to $\Omega' = \{\Lambda_\varepsilon(x) | x \in \Omega\}$. We first use the substitution $x' = \Lambda_\varepsilon x$ to make the boundary independent of ε, so that we can take $\frac{d}{d\varepsilon}$ inside of the integral. Then by the product rule the integrand splits into two parts

$$\frac{d}{d\varepsilon}(\mathcal{L}(\varphi_\varepsilon(x), \partial\varphi_\varepsilon(x)\partial\Lambda_\varepsilon^{-1}(x), \Lambda_\varepsilon x)) \det\left(\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\right) + \mathcal{L}(\varphi_\varepsilon(x), \partial\varphi_\varepsilon(x)\partial\Lambda_\varepsilon^{-1}(x), \Lambda_\varepsilon x)\frac{d}{d\varepsilon}\det\left(\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\right)$$

Then by Jacobi's identity second part is

$$\mathcal{L}_\varepsilon \det\left(\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\right) \operatorname{tr}\left(\frac{\partial x^\alpha}{\partial(\Lambda_\varepsilon x)^\nu}\frac{d}{d\varepsilon}\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\right).$$

At $\varepsilon = 0$ this becomes

$$\mathcal{L}\mathrm{tr}\left(\delta^\alpha_\nu \frac{d}{d\varepsilon}\frac{\partial(\Lambda_\varepsilon x)^\nu}{\partial x^\mu}\bigg|_{\varepsilon=0}\right) = \mathcal{L}\frac{d}{d\varepsilon}\frac{\partial(\Lambda_\varepsilon x)^\mu}{\partial x^\mu}\bigg|_{\varepsilon=0} = \mathcal{L}\partial_\mu X^\mu_B(x), \tag{6}$$

where $X^\mu_B(x) := \frac{d}{d\varepsilon}(\Lambda_\varepsilon x)^\mu|_{\varepsilon=0}$ By use of the chain and product rules, the first part is

$$\frac{\partial\mathcal{L}}{\partial\varphi^i}\frac{d\varphi^i_\varepsilon(x)}{d\varepsilon} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}\left(\frac{d(\partial_\nu\varphi^i_\varepsilon(x))}{d\varepsilon}\frac{\partial(\Lambda^{-1}_\varepsilon x)^\nu}{\partial x^\mu} + \partial_\nu\varphi^i_\varepsilon(x)\frac{d}{d\varepsilon}\frac{\partial(\Lambda^{-1}_\varepsilon x)^\nu}{\partial x^\mu}\right) + \frac{\partial\mathcal{L}}{\partial x^\nu}\frac{d(\Lambda_\varepsilon x)^\nu}{d\varepsilon}.$$

Again evaluating at $\varepsilon = 0$ and using the Euler-Lagrange equation (3) this becomes

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}\right)X^i_F(\varphi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}(\partial_\mu X^i_F(\varphi) - \partial_\nu\varphi^i\partial_\mu X^\nu_B) + \frac{\partial\mathcal{L}}{\partial x^\nu}X^\nu_B. \tag{7}$$

Where we used the definition $X^i_F(x) := \frac{d}{d\varepsilon}\varphi^i_\varepsilon|_{\varepsilon=0}$. We now put the two terms together and use (5) to get the expression

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}X^i_F\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}\partial_\nu\varphi^i X^\nu_B\right) + \partial_\nu(\mathcal{L}X^\nu_B).$$

Here the first term comes from the first two terms in (7). The last two terms come from (6) and the last two terms of (7) after correctly applying (5). Then as $0 = \frac{d}{d\varepsilon}\int_\Omega \mathcal{L}d^n x$ for any region Ω, the integrand is identically 0. Thus we obtain the conserved current

$$\partial_\mu J^\mu = 0,$$

where we implicitly sum over the μ and the current is given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}X^i_F(\varphi) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^i)}\partial_\nu\varphi^i - \delta^\mu_\nu\mathcal{L}\right)X^\nu_B. \tag{8}$$

Here we use the Kronecker delta δ^μ_ν, which is 1 precisely when $\mu = \nu$, so that the last two terms can be grouped together, as we rewrote the last term to $\partial_\mu\delta^\mu_\nu X^\nu_B$. Though the current was derived from invariance under a group action, the same current is obtained from the corresponding infinitesimal variation

$$x^\mu \to x^\mu + \varepsilon X^\mu_B(x) \quad , \quad \varphi^i(x) \to \varphi^i(x) + \varepsilon(X^i_F(\varphi(x)) - X^\mu_B(x)\partial_\mu\varphi^i(x)). \tag{9}$$

Here, the group actions are only considered up to first order with $X^\mu_B(x) := \frac{d}{d\varepsilon}(\Lambda_\varepsilon x)^\mu|_{\varepsilon=0}$ and $X^i_F = \frac{d}{d\varepsilon}\varphi^i_\varepsilon|_{\varepsilon=0}$ giving changes in the base space and field space respectively. Such infinitesimal variations can also be used without deriving them from a group action. In this case the functional S must be constant up to first order in ε, i.e. $S[\varphi] = S[\varphi_\varepsilon] + \mathcal{O}(\varepsilon^2)$. The proof that this also yields a current (8) is similar to the proof already given.

1.3 Using Noether's Theorem

Now that we have proven Noether's theorem we can use it to derive some well known results from classical mechanics, which will help with some intuition on the theorem. For this we do some simplifications. Instead of four dimensional spacetime we consider only time, this turns the base space into $\mathbb{R}$ and the fields become generalised coordinates $q^i(t)$. Furthermore the conserved current becomes

$$0 = \partial_\mu J^\mu = \frac{d}{dt} J^0 = \frac{d}{dt} Q.$$

If the Lagrangian does not explicitly depend on the generalised coordinate q^j, then we have an infinitesimal symmetry given by the transformation

$$t \to t \quad , \quad q^i \to q^i + \varepsilon \delta^i_j \delta q.$$

Then $X^i_F = \delta^i_j \delta q$ and $X^\mu_B = 0$, which gives a conserved current $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^j} \delta q^j = 0$. This means the generalised momentum $p_j := \frac{\partial \mathcal{L}}{\partial \dot{q}^j}$ is constant. Similarly if the Lagrangian does not depend explicitly on the time we have the symmetry transformation

$$t \to t + \varepsilon \delta t \quad , \quad q \to q.$$

This time $X^\mu_B = \delta t$ and $X^i_F = 0$. Then Noether's theorem yields a conserved current $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial (\dot{q}^i)} \dot{q}^i - \mathcal{L} \right) \delta t = 0$. This means the Hamiltonian $H = \frac{\partial \mathcal{L}}{\partial (\dot{q}^i)} \dot{q}^i - \mathcal{L}$ is constant and as the Hamiltonian gives the energy of the system the energy is constant.

For a more concrete example consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2$$

of a free particle. Then the generalised momentum is $p = m\dot{x}$ and the hamiltonian is $H = p\dot{q} - \mathcal{L} = \frac{1}{2} m \dot{x}^2$. Thus the linear momentum and the kinetic energy of the free particle are constant.

1.4 Stress-Energy Tensor

We now derive some common currents for fields. First is the canonical stress-energy tensor of the field. This is given by the quantity in (8)

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^i)} \partial^\nu \varphi^i - \eta^{\mu\nu} \mathcal{L} \tag{10}$$

This tensor is obtained from the currents associated to the translational invariance of a Lagrangian, under the assumption that there is no change in the field values associated to a translation. We will see that holds for both electromagnetism and spinor fields in flat spacetime. This translational symmetry gives the conservation law

$$\partial_\mu \Theta^{\mu\nu} = 0. \tag{11}$$

This tensor is not generally symmetric, nor are its components gauge invariant.

The concept of gauge invariance requires the introduction of a gauge first. When a mathematical system has more degrees of freedom than the physical system it describes, the extra degrees of freedom are part of the gauge freedom. A gauge invariant quantity is then a quantity that is the same regardless of the values of these additional degrees of freedom. An example of a gauge freedom is the choice of reference point for the potential energy of a force in classical mechanics. Physically all reference points give the same information, so this degree of freedom exists only in the mathematical description. gauge independent quantities here are the force and the difference in potential energy between two points.

Physically measurable quantities must therefore be gauge invariant, as they are the same regardless of the mathematical description. This means that the different components of the stress-energy tensor must be gauge invariant as they correspond to energy density, energy flux, and stress, which are measurable quantities, which must be gauge invariant[5].

It is harder to see why the stress energy tensor should be symmetric, therefore consider the Einstein Hilbert action, the Lagrangian of this is of the form $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{matter}$, with the first term depending on the symmetric metric tensor $g_{\mu\nu}$ and its first and second order derivatives, and the second being the matter term. The Euler-Lagrange equations for this action are the Einstein field equations, where the stress-energy tensor is

$$T^{\mu\nu} = \frac{\delta \mathcal{L}_{matter}}{\delta g_{\mu\nu}},$$

which is symmetric as the metric tensor is symmetric. The derivation should yield the same stress-energy tensor regardless of whether it is through the Einstein field equations or Noether's theorem. This means that any stress-energy tensor obtained using Noether's theorem should be symmetric.

1.5 Moment Tensor

There is a way to make the stress-energy tensor symmetric if the Lagrangian is invariant under Lorentz transformations. Physically this means that the Lagrangian must have the same form in any inertial system with the same origin, as a Lorentz transformation is the coordinate transformation done between two frames S and S', where S' moves with a fixed velocity v compared to S and with their origins O coinciding at t=0. Mathematically a Lorentz transformation is an element $\Lambda \in GL_4(\mathbb{R})$ that leaves the Minkovski inner product invariant, which means that $\eta(\Lambda v, \Lambda w) = \eta(v, w)$ for all vectors $v, w \in \mathbb{R}^4$. These matrices form a group called the Lorentz group, which we will denote $O(1, 3)$. This group is often further restricted by only considering the transformations that maintain orientation, $\det(\Lambda) > 0$, and time orientation, which means that if a vector v has $v^0 > 0$ then the transformed vector $w = \Lambda v$ also has $w^0 > 0$. This group is called the proper orthochronous Lorentz group and denoted $SO^+(1, 3)$. Mathematically the Lagrangian must then be invariant under an action of the Lorentz group. This action is matrix multiplication for the spacetime $\mathbb{R}^4$, though the appropriate action on V depends on the Lagrangian. This action on V corresponding to a Lorentz transform of spacetime is often called the transformation behaviour of elements of V.

The symmetric tensor is obtained by the following procedure: first calculate the current belonging to an infinitesimal action of the Lorentz group. To this end we first show that the Lie algebra of the Lorentz group consists of matrices $\omega^{\mu}{}_{\nu}$ such that $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$. For a path through the Lorentz group $\Lambda(t)$ with $\Lambda(0) = I$ we have that

$$\eta_{\mu\nu} v^{\mu} w^{\nu} = \eta(v, w) = \eta(\Lambda(t)v, \Lambda(t)w) = \eta_{\mu\nu} \Lambda(t)^{\mu}{}_{\alpha} v^{\alpha} \Lambda(t)^{\nu}{}_{\beta} w^{\beta}.$$

In particular using the definition $\omega^\mu{}_\nu = \frac{d}{dt}\Lambda^\mu{}_\nu|_{t=0}$ for the elements of the Lie algebra

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\eta_{\mu\nu}\Lambda(t)^\mu{}_\alpha v^\alpha \Lambda(t)^\nu{}_\beta w^\beta\right)\Big|_{t=0} \\
&= \left(\frac{d}{dt}\Lambda(t)^\mu{}_\alpha \Lambda(t)^\nu{}_\beta + \frac{d}{dt}\Lambda(t)^\nu{}_\beta \Lambda(t)^\mu{}_\alpha\right)\Big|_{t=0} \eta_{\mu\nu} v^\alpha w^\beta \\
&= (\omega^\mu{}_\alpha \delta^\nu{}_\beta + \omega^\nu{}_\beta \delta^\mu{}_\alpha)\eta_{\mu\nu} v^\alpha w^\beta \\
&= \omega^\mu{}_\alpha \eta_{\mu\nu} v^\alpha w^\nu + \omega^\nu{}_\beta \eta_{\mu\nu} v^\mu w^\beta \\
&= \omega_{\nu\alpha} v^\alpha w^\nu + \omega_{\mu\beta} v^\mu w^\beta \\
&= (\omega_{\nu\mu} + \omega_{\mu\nu})(v^\mu w^\nu).
\end{aligned}$$

Then, as v and w are arbitrary, $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$, which means the Lie algebra consists of all such matrices as claimed.

To derive the infinitesimal action we first consider the action of the Lorentz group. This changes the spacetime coordinates to $x' = \Lambda x$ and the field variables to $\varphi' = f(\Lambda)\varphi$, where f is the representation of the Lorentz group on the space of field variables, i.e. a homomorphism $f : SO^+(1,3) \to GL(V)$. We again take a path $\Lambda(t)$ in the Lorentz group with $\Lambda(0) = I$. Then the infinitesimal change in the variables of the base space is $X_B^\mu(x) = \frac{d}{dt}(\Lambda(t)^\mu{}_\nu x^\nu)|_{t=0} = \omega^\mu{}_\nu x^\nu$. The infinitesimal change in the field variables is

$$X_F^i(\varphi) = \frac{d}{dt} f(\Lambda(t))^i_j \varphi^j \Big|_{t=0} = \frac{\partial f}{\partial \omega^\mu{}_\nu}\Big|_{\Lambda=I} \frac{d}{dt}(\Lambda(t))\Big|_{t=0} = (F_\mu{}^\nu \omega^\mu{}_\nu)^i_j \varphi^j = \omega_{\mu\nu}(F^{\mu\nu})^i_j \varphi^j$$

This means we can write the infinitesimal action in the form

$$x^\mu \to x^\mu + \varepsilon\, \omega^\mu{}_\nu x^\nu \quad , \quad \varphi^i(x) \to \varphi^i + \varepsilon\left(\frac{1}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^i_j \varphi^j - \omega^\mu{}_\nu x^\nu \partial_\mu \varphi^i(x)\right). \tag{12}$$

Here we can obtain the representation Σ by calculating the pushforward of f. This yields a conservation law

$$\partial_\mu \Theta^{\mu\kappa\lambda} = 0 \tag{13}$$

for the canonical moment tensor where

$$\Theta^{\mu\kappa\lambda} = x^\kappa \Theta^{\mu\lambda} - x^\lambda \Theta^{\mu\kappa} + \Sigma^{\mu\kappa\lambda}, \tag{14}$$

where the last term is called the internal or spin part, and can be calculated by

$$\Sigma^{\mu\kappa\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)}(\Sigma^{\kappa\lambda})^i_j \varphi^j.$$

To derive these expressions we note that the infinitesimal action in (12) corresponds to a change $X_B^\mu(x) = \omega^\mu{}_\nu x^\nu$ of the base space with a change $X_F^i(\varphi(x)) = \frac{1}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^i_j \varphi^j$ of the field space. plugging this into (8) gives the conserved current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \frac{1}{2}\omega_{\kappa\lambda}(\Sigma^{\kappa\lambda})^i_j \varphi^j - \Theta^\mu{}_\kappa \omega^\kappa{}_\lambda x^\lambda,$$

now

$$\Theta^\mu{}_\kappa \omega^\kappa{}_\lambda x^\lambda = \Theta^{\mu\kappa}\omega_{\kappa\lambda} x^\lambda = \frac{1}{2}(\omega_{\kappa\lambda} - \omega_{\lambda\kappa}) x^\lambda \Theta^{\mu\kappa} = \frac{1}{2}\omega_{\kappa\lambda}(x^\lambda \Theta^{\mu\kappa} - x^\kappa \Theta^{\mu\lambda}),$$

as ω is antisymmetric, so $\frac{1}{2}(\omega_{\kappa\lambda} - \omega_{\lambda\kappa}) = \frac{1}{2}(\omega_{\kappa\lambda} + \omega_{\kappa\lambda}) = \omega_{\kappa\lambda}$ and the last step is done by relabelling $\lambda \leftrightarrow \kappa$ in the second sum. This means the current is given by

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} \frac{1}{2}\omega_{\kappa\lambda}(\Sigma^{\kappa\lambda})^i_j \varphi^j - \frac{1}{2}\omega_{\kappa\lambda}(x^\lambda \Theta^{\mu\kappa} - x^\kappa \Theta^{\mu\lambda}).$$

ω was arbitrary, so we get the canonical moment tensor as in (14), after multiplying by 2.

Using these expressions we can obtain a symmetric stress-energy tensor is given by

$$T^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{2}\partial_\kappa(\Sigma^{\kappa\mu\nu} + \Sigma^{\mu\nu\kappa} - \Sigma^{\nu\kappa\mu}) \tag{15}$$

1.6 Electromagnetic Field

For a concrete example of using Noether's theorem to derive the stress-energy tensor consider an electromagnetic field. This is defined by the electromagnetic four potential, which is described by a 1-form valued field written as $A_\mu = (\frac{\phi}{c}, -\vec{A})$, which are an electric and magnetic potential respectively. Using the partial derivatives $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$. The electromagnetic tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then using the fact that $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$ we get that

$$F_{\mu\nu} = \begin{bmatrix} 0 & -\frac{1}{c}\frac{\partial A_x}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial x} & -\frac{1}{c}\frac{\partial A_y}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial y} & -\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial z} \\ \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial \phi}{\partial x} & 0 & -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} & -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z} \\ \frac{1}{c}\frac{\partial A_y}{\partial t} + \frac{1}{c}\frac{\partial \phi}{\partial y} & -\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} & 0 & -\frac{\partial A_z}{\partial y} + \frac{\partial A_y}{\partial z} \\ \frac{1}{c}\frac{\partial A_z}{\partial t} + \frac{1}{c}\frac{\partial \phi}{\partial z} & -\frac{\partial A_x}{\partial z} + \frac{\partial A_z}{\partial x} & -\frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{bmatrix}. \tag{16}$$

The Lagrangian density of free electrodynamics is $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}(B^2 - \frac{E^2}{c^2})$, which has the well known Maxwell equations as its Euler-Lagrange equations. To calculate $\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^i)}$ we rewrite the Lagrangian to

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu)$$

Then as

$$\frac{\partial}{\partial(\partial_\kappa A^\gamma)}(\partial_\mu A_\nu \partial^\mu A^\nu) = \frac{\partial}{\partial(\partial_\kappa A^\gamma)}(\partial_\mu \eta_{\nu\nu'} A^{\nu'} \eta^{\mu\mu'} \partial_{\mu'} A^\nu) = \eta_{\nu\nu'}\eta^{\mu\mu'} \frac{\partial}{\partial(\partial_\kappa A^\gamma)}(\partial_\mu A^{\nu'} \partial_{\mu'} A^\nu) =$$
$$\eta_{\nu\gamma}\eta^{\kappa\mu'}\partial_{\mu'}A^\nu + \eta_{\gamma\nu'}\eta^{\mu\kappa}\partial_\mu A^{\nu'} = 2\partial^\kappa A_\gamma$$

and similarly

$$\frac{\partial}{\partial(\partial_\kappa A^\gamma)}(\partial_\mu A_\nu \partial^\mu A^\nu) = 2\partial_\gamma A^\kappa.$$

This yields the generalized momentum density

$$\pi_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^i)} = -(\partial^\mu A_i - \partial_i A^\mu) = F_i{}^\mu$$

Plugging this into (8) for the infinitesimal translation

$$X_B^\mu(x) = a^\mu, \; X_F^i = 0$$

we obtain currents

$$J^\mu = -\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^i)}\partial_\nu A^i - \delta^\mu_\nu \mathcal{L}\right) a^\nu .$$

Bundling these for separate ν and raising ν, which is done by multiplying on both sides by $\eta^{\nu'\nu}$, summing over ν and then relabelling $\nu' \to \nu$, gives the canonical stress-energy tensor

$$\Theta^{\mu\nu} = F_i{}^{\mu}\partial^\nu A^i - \eta^{\mu\nu}\mathcal{L},$$

which is certainly not symmetric, and its energy density component

$$\Theta^{00} = \frac{1}{2}\left(B^2 - \frac{E^2}{c^2}\right) - \mathbf{E}\cdot\frac{\partial \mathbf{A}}{\partial t}$$

is different from the known electromagnetic energy density $u_{EM} = \left(\frac{1}{\mu_0}B^2 + \varepsilon_0 E^2\right)$.

The canonical moment tensor can be made symmetric through the procedure outlined in the previous section. To show that this can be done we first have to show the Lorentz invariance of the Lagrangian. Then the action of $\Lambda \in SO^+(1,3)$ on $x \in \mathbb{R}^4$ sends x^μ to $\Lambda^\mu{}_\nu x^\nu = y^\mu$ with the inverse transformation written as $\Lambda^{-1} = \Lambda_\mu{}^\nu$ so that $x^\nu = \Lambda_\mu{}^\nu y^\mu$. The electromagnetic potential is a 1-form, so it transforms as

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu = y^\mu \quad , \quad A_\mu(x^\nu) \to \hat{A}_\mu(y) = \Lambda_\mu{}^\alpha A_\alpha(\Lambda_\beta{}^\nu y^\beta).$$

Then as $\hat{\partial}_\mu = \frac{\partial}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu$ we get that

$$\hat{F}_{\mu\nu} = \hat{\partial}_\mu \hat{A}_\nu - \hat{\partial}_\nu \hat{A}_\mu = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}.$$

This means the Lagrangian becomes

$$\hat{\mathcal{L}} = -\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} = \eta^{\mu\mu'}\eta^{\nu\nu'}\hat{F}_{\mu'\nu'}\hat{F}_{\mu\nu} = \eta^{\mu\mu'}\eta^{\nu\nu'}\Lambda_{\mu'}{}^{\alpha'}\Lambda_{\nu'}{}^{\beta'}F_{\alpha'\beta'}\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta F_{\alpha\beta}.$$

Then $\Lambda \in SO^+(1,3)$ means that $\eta^{\mu\mu'}\Lambda_\mu{}^\alpha\Lambda_{\mu'}{}^{\alpha'} = \eta^{\alpha\alpha'}$. Thus the Lagrangian becomes

$$\eta^{\beta\beta'}\eta^{\alpha\alpha'}F_{\alpha'\beta'}F_{\alpha\beta} = \mathcal{L},$$

which means the Lagrangian is Lorentz invariant. Now we want to calculate the quantity $\Sigma^{\mu\nu}$ in (12) for Electromagnetism. First we raise the index of A_μ to get A^i. Then the infinitesimal Lorentz transformation is

$$x^\mu \to x^\mu + \varepsilon\ \omega^\mu{}_\nu x^\nu \quad , \quad A^i \to A^i + \varepsilon\left(\omega^i{}_j A^j - \omega^\mu{}_\nu x^\nu \partial_\mu A^i\right),$$

which means $(\Sigma^{\mu\nu})^i_j = \eta^{\mu i}\delta^\nu_j - \eta^{\nu i}\delta^\mu_j$, as then $\Sigma^{\mu\nu} = -\Sigma^{\mu\nu}$ and

$$\frac{1}{2}\omega_{\mu\nu}(\Sigma^{\mu\nu})^i_j = \frac{1}{2}\omega_{\mu\nu}(\eta^{\mu i}\delta^\nu_j - \eta^{\nu i}\delta^\mu_j) = \frac{1}{2}(\omega_{\mu\nu}\eta^{\mu i}\delta^\nu_j + \omega_{\nu\mu}\eta^{\nu i}\delta^\mu_j) = \omega^i{}_j.$$

Thus the spin part of the moment tensor is

$$\Sigma^{\mu\kappa\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^i)}(\Sigma^{\kappa\lambda})^i_j A^j = F_i^\mu(\eta^{\kappa i}\delta^\lambda_j - \eta^{\lambda i}\delta^\kappa_j)A^j = F^{\kappa\mu}A^\lambda - F^{\lambda\mu}A^\kappa.$$

The extra correction term for the stress-energy tensor is then

$$\begin{aligned}&\frac{1}{2}\partial_\kappa(\Sigma^{\kappa\mu\nu}+\Sigma^{\mu\nu\kappa}-\Sigma^{\nu\kappa\mu}) =\\ &\frac{1}{2}\partial_\kappa(F^{\mu\kappa}A^\nu - F^{\nu\kappa}A^\mu + F^{\nu\mu}A^\kappa - F^{\kappa\mu}A^\nu - F^{\kappa\nu}A^\mu + F^{\mu\nu}A^\kappa) =\\ &\partial_\kappa(F^{\mu\kappa}A^\nu) = -\partial_\kappa(F^{\kappa\mu}A^\nu) = -F^{\kappa\mu}A^\nu,\end{aligned}$$

where the last line is due to the antisymmetry of $F^{\mu\nu}$ and the fact that $\partial_\kappa F^{\kappa\mu} = 0$ are the Euler-Lagrange equations. Thus the symmetric stress-energy tensor is

$$T^{\mu\nu} = \Theta^{\mu\nu} - \partial_\kappa(F^{\kappa\mu}A^\nu) = F_i{}^\mu\partial^\nu A^i - \eta^{\mu\nu}\mathcal{L} - F_i{}^\mu\partial^i A^\nu = -F_i{}^\mu F^{i\nu} - \eta^{\mu\nu}\mathcal{L}.$$

There is another way to obtain this tensor. This is done by using another symmetry of electrodynamics, the gauge symmetry of the 4-potential. Gauge symmetry means that the Lagrangian is invariant under the addition of a gradient $\partial_\mu f$ for some function f of the spacetime coordinates. Indeed then

$$F_{\mu\nu} \to \partial_\mu(A_\nu + \partial_\nu f) - \partial_\nu(A_\mu + \partial_\mu f) = F_{\mu\nu}$$

as partial derivatives commute, so $\partial_\mu\partial_\nu f - \partial_\nu\partial_\mu f = 0$, which means that the Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, is gauge invariant. Therefore we instead try the following symmetry

$$X_B^\mu(x) = a^\mu,\ X_F^i = \partial^i(A_\mu a^\mu),$$

which can be found in [6]. This is a symmetry as $X_B^\mu = a^\mu$ corresponds to the translational symmetry and $X_F^i = \partial^i(A_\mu a^\mu)$ is a gauge symmetry with $f = A_\mu a^\mu$ and the combination of two symmetries is still a symmetry. We then obtain the currents

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^i)}\partial^i(A_\kappa a^\kappa) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^i)}\partial_\nu A^i - \delta^\mu_\nu\mathcal{L}\right)a^\nu$$

Then the conserved current relations become

$$0 = -\partial_\mu J^\mu = -\partial_\mu\left(F_i{}^\mu\partial^i A_\nu - F_i{}^\mu\partial_\nu A^i + \delta^\mu_\nu\mathcal{L}\right)a^\nu.$$

Again combining these relations for different ν and raising ν we obtain the stress-energy tensor

$$T^{\mu\nu} = -F_i{}^\mu F^{i\nu} - \eta^{\mu\nu}\mathcal{L} = F^\mu{}_i F^{i\nu} - \eta^{\mu\nu}\mathcal{L}.$$

This one is symmetric and has the correct energy term. This tensor is also identical to the one obtained by adding the correction term obtained from the spin tensor.

2 Spinors

To model electrons and to calculate the stress-energy tensor of an electron we will need to introduce spinors. Spinors are used to describe particles with any spin, though we focus on spin-$\frac{1}{2}$ particles, such as the well-known electrons, protons and neutrons but also quarks and neutrinos. Spinors are elements of a vector space $\mathbb{C}^4$ and electrons can be described as functions from spacetime to the space of spinors. These spinor fields transform somewhat peculiarly when spacetime is rotated. If one rotates spacetime by 360° around a given axis then the points of space will be back where they started, but the spinor field will have obtained a minus sign. This happens because these fields do not transform by the Lorentz group, but instead by the spin group. This group contains two elements for every rotation and a rotation of 360° takes one of the elements to the other, thereby gaining a minus sign. This transformation behaviour allows us to show the relativistic invariance of the Dirac equation, and the Lagrangian belonging to it in the next section.

To properly describe the transformation behaviour of spinors we introduce the spin group. This spin group can be described in two ways. One is as a double cover of the proper orthochronous Lorentz group $SO^+(1,3) \subset O(1,3)$. This gives the exact sequence

$$1 \to \mathbb{Z}_2 \to \text{Spin(1,3)} \to SO^+(1,3) \to 1. \tag{17}$$

The other way is to consider the Clifford algebra, Cl(1,3), generated by $(\mathbb{R}^4, \eta)$, where η is the Minkowski inner product defined last section. This means Cl(1,3) is the associative algebra generated by $\mathbb{R}^4$ with relations

$$e_i e_j + e_j e_i = 2\eta_{ij} \tag{18}$$

and then identify the spin group with a subgroup generated by products of pairs of unit vectors, so

$$\text{Spin}(1,3) = \{e_{i_1} \cdots e_{i_n} | e_{i_k} \in \text{Cl}(1,3) \text{ unit vectors}, n \text{ even}\}. \tag{19}$$

Subsection 1 briefly discusses the topological construction of the spin group using the universal cover.

The Clifford algebra is constructed in subsection 2, which also shows that the algebra gives a splitting of $\mathbb{C}^4$ into two subspaces.

This last fact is used in subsection 3 to show that the spin group and SL(2,$\mathbb{C}$) are isomorphic, which together with the the double cover of SO$^+$(1,3) by SL(2,$\mathbb{C}$) given in subsection 4 gives the double cover of SO$^+$(1,3) by the spin group.

2.1 Topological Spin Group

For the characterisation as a double cover we define the universal cover, which is a way to "cover" a topological space with a simply connected space that for small neighbourhoods looks like multiple copies of those neighbourhoods.

Definition. *Let E be a simply connected topological space, B a topological space, and $\pi : E \to B$ continuous and surjective. If for every point $p \in B$ there is a neighbourhood U for which the pre-image $\pi^{-1}(U)$ can be written as a union of disjoint open sets $V_\alpha \subset E$ such the restriction $\pi|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism of V_α to U, then π is called a universal covering map and E is the universal covering space of B*

Sufficiently nice topological spaces have a universal cover, and any universal cover is unique up to homeomorphism[4]. $SO^+(1,3)$ is sufficiently nice, so it has a universal cover, which we will call Spin(1,3). Spin(1,3) becomes a group by equipping it with a multiplication that is the "lift" of the multiplication on $SO^+(1,3)$. Thus for any two elements $\widetilde{\Lambda}, \widetilde{\Lambda}' \in Spin(1,3)$ we have that $\pi(\widetilde{\Lambda} \cdot \widetilde{\Lambda}') = \pi(\widetilde{\Lambda}) \cdot \pi(\widetilde{\Lambda}')$. Spin(1,3) is called a double cover because for every point $\Lambda \in SO^+(1,3)$ the pre-image $\pi^{-1}(\Lambda)$ consists of two points. This will be shown through the explicit construction of Spin(1,3) using the Clifford algebra, but also follows from the fact that the fundamental group of $SO^+(1,3)$ has only two elements[7], as any path through $SO^+(1,3)$ can be continuously deformed to one of two paths.

2.2 Clifford Algebra

The second way to characterise the spin group was by considering the Clifford algebra, Cl(1,3), generated by $(\mathbb{R}^4, \eta)$. This algebra is generated by defining a formal multiplication for the vectors in $\mathbb{R}^4$ that satisfies the relations

$$\{e_i, e_j\} = e_i e_j + e_j e_i = 2\eta_{ij}, \tag{20}$$

where the e_i are the canonical basis vectors. The spin group is then obtained as the subgroup generated by products of pairs of unit vectors, so

$$\mathrm{Spin}(1,3) = \{e_{i_1} \cdots e_{i_n} | e_{i_k} \in \mathrm{Cl}(1,3) \text{ unit vectors}, n \text{ even}\}. \tag{21}$$

This means we can restrict a representation of Cl(1,3) to obtain one of Spin(1,3). We construct this representation as shown by Baez and Huerta [8] and then show that it is indeed a double cover as required. To this end we consider the action of Cl(4) on the exterior algebra of $\bigwedge \mathbb{C}^2$. The exterior algebra of a vector space V is obtained by defining a product, usually denoted by a wedge $\wedge$, such that $v \wedge w = -w \wedge v$ for all $v, w \in V$. In $\mathbb{C}^2$ this becomes a 4 dimensional vector space, which can be given the basis $\{\hat{x}, \hat{y}, 1, \hat{x} \wedge \hat{y}\}$. We choose this specific order of the basis, as then we first have 2 elements that are the product of an odd number of vectors, and then 2 which are the product of an even number of vectors. Using this basis we can consider the linear map obtained by left multiplication with a basis vector

$$\begin{gathered} a_j : \bigwedge \mathbb{C}^2 \to \bigwedge \mathbb{C}^2 \\ v \mapsto e_j \wedge v \end{gathered} \tag{22}$$

Then we can obtain matrices

$$a_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } a_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

in this basis along with their Hermitian adjoints

$$a_1^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } a_2^\dagger = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

These matrices have the following anticommutation relations

$$\begin{aligned}\{a_i, a_j\} &= 0\\ \{a_i^\dagger, a_j^\dagger\} &= 0\\ \{a_i^\dagger, a_j\} &= \delta_{ij} I\end{aligned} \tag{25}$$

Thus we can take

$$\begin{aligned}\gamma_1 &= a_2 - a_2^\dagger\\ \gamma_2 &= i(a_2 + a_2^\dagger)\\ \gamma_3 &= a_1 - a_1^\dagger\\ \gamma_4 &= i(a_1 + a_1^\dagger)\end{aligned} \tag{26}$$

such that $\{\gamma_i, \gamma_j\} = -2\delta_{ij}$

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix} \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & i & 0 & 0\\ -i & 0 & 0 & 0\end{bmatrix} \gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \text{ and } \gamma_4 = \begin{bmatrix} 0 & 0 & i & 0\\ 0 & 0 & 0 & i\\ i & 0 & 0 & 0\\ 0 & i & 0 & 0\end{bmatrix}. \tag{27}$$

Here we have chosen to order the matrices in this specific way to simplify calculations later on. Then to obtain a representation for Cliff(1,3) we define $\gamma_0 = i\gamma_4$ such that $\{\gamma_i, \gamma_j\} = 2\eta_{ij}$. This satisfies the relations 20, so we identify γ_i with e_i, then the unit vectors are simply given by the vectors $v \in \mathbb{R}^4$ with $\eta(v, v) = \pm 1$.

2.2.1 Splitting of the Vector Space

Before showing that the spin group is a double cover of SO$^+$(1,3), we will show that Cl(1,3) naturally provides a way to split $\mathbb{C}^4$ into two linear subspaces. This splitting will help construct a homomorphism φ : Spin(1,3) $\to$SO$^+$(1,3). This homomorphism proves that Spin(1,3) is a double cover of SO$^+$(1,3) and gives the transformation behaviour spinors that we require in the next section. This splitting occurs regardless of the basis chosen for $\bigwedge \mathbb{C}^2$. To see this note that a change of basis changes all the gamma matrices to $\gamma^\nu \to B^{-1}\gamma^\mu B$, where B is the change of basis matrix from the old basis to the new one. Then for the new matrices

$$\begin{aligned}\{B^{-1}\gamma^\mu B, B^{-1}\gamma^\nu B\} &= B^{-1}\gamma^\mu B B^{-1}\gamma^\nu B + B^{-1}\gamma^\nu B B^{-1}\gamma^\mu B\\ &= B^{-1}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)B\\ &= B^{-1}\eta^{\mu\nu} I B\\ &= \eta^{\mu\nu} I,\end{aligned}$$

which means they still satisfy (18). To define the splitting we first define the complex volume form $\omega = i\gamma^0\gamma^1\gamma^2\gamma^3$. This has two useful properties namely

$$\omega^2 = I \text{ and } \gamma^\mu\omega = -\omega\gamma^\mu \tag{28}$$

for the first note that

$$\begin{aligned}\omega^2 &= (i\gamma^0\gamma^1\gamma^2\gamma^3)^2\\ &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\\ &= -(-1)^6\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3\\ &= -(-1)^3I = I,\end{aligned}$$

where we first swap the matrices 6 times to get 6 minus signs and then get 3 minus as $\gamma^0\gamma^0 = I$ and $\gamma^\mu\gamma^\mu = -I$ otherwise, all by the commutation relations. Similarly the commutation relations give

$$\gamma^\mu\omega = \gamma^\mu(i\gamma^0\gamma^1\gamma^2\gamma^3) = -(i\gamma^0\gamma^1\gamma^2\gamma^3)\gamma^\mu = -\omega\gamma^\mu,$$

by swapping places 3 times, as one of the matrices is to γ^μ. Using the complex volume form we can define the left and right projections $P_L = \frac{I-\omega}{2}$ and $P_R = \frac{I+\omega}{2}$. Then

$$P_L^2 = \frac{I-\omega}{2}^2 = \frac{I-2\omega+\omega^2}{4} = \frac{2I-2\omega}{4} = P_L.$$

Similarly $P_R^2 = P_R$, so both are indeed projections. Moreover

$$P_LP_R = \frac{I-\omega}{2}\frac{1+\omega}{2} = \frac{I-\omega^2}{4} = 0$$

and similarly $P_RP_L = 0$. These projections define two subspaces

$$V_L := \{P_L(x)|x \in \bigwedge\mathbb{C}^2\} \quad , \quad V_R := \{P_R(x)|x \in \bigwedge\mathbb{C}^2\},$$

which collectively span the space. Indeed if $x \in \bigwedge\mathbb{C}^2$, then $P_L(x) + P_R(x) = \frac{I-\omega}{2}x + \frac{I+\omega}{2}x = x$. To see that the two spaces are linearly independent take two arbitrary vectors $P_L(x) \in V_L$ and $P_R(y) \in V_R$ and assume $P_L(x) + P_R(y) = 0$. Applying P_L to both sides gives

$$0 = P_L(P_L(x) + P_R(y)) = P_L^2(x) + P_LP_R(y) = P_L(x)$$

and by applying P_R to both sides we get that $P_R(y)$=0 as well. This means the subspaces have no vectors linearly dependent on the other subspace, so the subspaces are linearly independent. Consequently, we have a decomposition

$$\bigwedge\mathbb{C}^2 = V_L \oplus V_R.$$

Furthermore, if $y \in V_L$, then $y = \frac{I-\omega}{2}x$ for some $x \in \bigwedge\mathbb{C}^2$. This implies that $\gamma^\mu y = \gamma^\mu\frac{I-\omega}{2}x = \frac{I+\omega}{2}\gamma^\mu x \in V_R$. Similarly, multiplication with γ^μ takes elements of V_R to V_L. Thus, multiplication with any vector $a = a_\mu\gamma^\mu$ switches elements between V_R and V_L, and since these mappings are invertible and linear V_L and V_R must have the same dimension. Then V_R and V_L must have dimension two as their dimensions must add up to dim($\bigwedge\mathbb{C}^2$)=4. Multiplying with an even number of vectors then defines linear mappings $V_R \to V_R$ and $V_L \to V_L$. In particular, this means that elements of the spin group can be identified with 2 by 2 complex matrices through their action on V_L or V_R.

2.3 Spin Group

We will use this to find a spin homomorphism in the following manner: First we find an isomorphism $f : \mathrm{Spin} \xrightarrow{\sim} \mathrm{SL}(2, \mathbb{C})$, and then use the well known double cover of $\mathrm{SL}(2, \mathbb{C})$ on $\mathrm{SO}^+(1, 3)$. We will find the isomorphism f by considering the action of Spin on V_L. The projection on the left subspace, given our gamma matrices, is

$$P_L = \frac{1}{2}\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} + \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\right) = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Thus $V_L = \mathrm{span}(\{\hat{x}, \hat{y}\})$. Then the action on V_L is simply given by the upper left block. This leads us to try a homomorphism that takes out the upper left block. Thus

$$f : \begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{bmatrix} \mapsto \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$$

becomes our homomorphism. The spin group acts separately on V_L and V_R, so any matrix can be written as

$$\begin{bmatrix}A&0\\0&B\end{bmatrix}.$$

The product of two such matrices is then

$$\begin{bmatrix}A&0\\0&B\end{bmatrix}\begin{bmatrix}C&0\\0&D\end{bmatrix} = \begin{bmatrix}AC&0\\0&BD\end{bmatrix},$$

which means f is a homomorphism.

2.3.1 Lie Algebra

To show f is also an isomorphism, we will use Lie's second theorem. By this theorem f is an isomorphism if both groups are simply connected and its pushforward at the identity f_* is an isomorphism of Lie algebras. Therefore, we want to determine the Lie algebra of the spin group, which is the tangent space to the identity element. To this end we consider paths $x(t)$, with $x(0) = 1$, where $x'(0)$ are then elements of the Lie algebra. First we consider a path parametrised by $x(t) = (\cos(t)\gamma_1 + \sin(t)\gamma_2)(-\gamma_1)$ then $x(0) = -\gamma_1^2 = 1$ and $x'(0) = \gamma_1\gamma_2$, so $\gamma_1\gamma_2$ is in the Lie algebra of Spin(1,3). Through similar parametrisations, noting that $\cosh(t)\gamma_0 + \sinh(t)\gamma_1$ is also a unit vector, it can be shown that the other products of 2 different matrices are also in the Lie algebra. These are independent, and the Lie algebra is 6 dimensional, so we have a basis of $\{\gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_1\gamma_3, \gamma_0\gamma_3, \gamma_0\gamma_1, \gamma_0\gamma_2\}$ for the Lie algebra. We then note that we have the following commutation relations for elements in the basis

$$\begin{aligned}&[\gamma^i\gamma^j, \gamma^k\gamma^l] = 0 \text{ if i,j,k,l all different}\\&[\gamma^0\gamma^k, \gamma^k\gamma^j] = -2\gamma^0\gamma^j\\&[\gamma^0\gamma^k, \gamma^0\gamma^j] = -2\gamma^k\gamma^j\end{aligned} \tag{29}$$

Furthermore we can calculate the pushforward, f_*, of this mapping at the identity matrix. Note that f is linear, so its total derivative is equal to itself. Therefore, we find that f_* sends our basis elements $\{\gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3, \gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3\}$ to

$$\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{30}$$

respectively. Then as pushforwards are linear maps and f_* sends a basis to a basis it is a linear isomorphism. Furthermore, it trivially respects the Lie bracket [A,B]=AB-BA by again considering that the spin matrices act separately on the two parts of the exterior algebra. Thus this is a Lie algebra isomorphism. Then finally as Spin is simply connected and $\mathrm{SL}(2,\mathbb{C})$ is the simply connected Lie group whose Lie algebra has the basis given in 30, f is a Lie group isomorphism by Lie's second theorem.

2.4 Spin map

Now that we have an isomorphism $f : \mathrm{Spin} \xrightarrow{\sim} \mathrm{SL}(2,\mathbb{C})$, we find the double cover $\varphi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$. We can define a homomorphism $\varphi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ as follows. First, we identify $\mathbb{R}^{1,3}$ with the Hermitian matrices by

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix}, \tag{31}$$

then

$$\varphi : X \mapsto (M \mapsto XMX^\dagger). \tag{32}$$

To show that the image of φ is indeed $^+(1,3)$ we first note that $(M \mapsto XMX^\dagger) \in O(1,3)$ as $\det(XMX^\dagger) = \det(X)\det(M)\det(X^\dagger) = \det(M) = t^2 - x^2 - y^2 - z^2$, so the mapping preserves the Minkowski metric. Furthermore, since φ is continuous and SL(2,$\mathbb{C}$) is connected, the image of φ is connected. Thus as φ is a homomorphism its image contains the identity, and thus must be at most $SO^+(1,3) \subset O(1,3)$, the connected component of the identity. $X \in \mathrm{Ker}(\varphi)$ must satisfy $XMX^\dagger = M \ \forall M$, which holds precisely when $X = \pm I$. Then if φ is surjective it is a double covering map, which makes SL(2,$\mathbb{C}$) the double cover of $\mathrm{SO}^+(1,3)$. To show that φ is surjective we note that SO(1,3) is generated by boosts and rotations around the axes. To obtain a rotation we take the element

$$f\left(\cos\left(\frac{\varphi}{2}\right) + \sin\left(\frac{\varphi}{2}\right)\gamma^1\gamma^2\right) = \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix}, \tag{33}$$

then this maps to

$$\begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} \mapsto \begin{bmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{bmatrix} \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} \begin{bmatrix} e^{i\frac{\varphi}{2}} & 0 \\ 0 & e^{-i\frac{\varphi}{2}} \end{bmatrix} = \begin{bmatrix} t+z & e^{-i\varphi}(x-iy) \\ e^{i\varphi}(x+iy) & t-z \end{bmatrix} \tag{34}$$

and we obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SO^+(1,3).$$

Similarly

$$\varphi\left(f\left(\cosh\left(\frac{r}{2}\right)+\sinh\left(\frac{r}{2}\right)\gamma^0\gamma^3\right)\right)=\varphi\left(\begin{bmatrix} e^{\frac{r}{2}} & 0\\ 0 & e^{-\frac{r}{2}}\end{bmatrix}\right)=\begin{bmatrix}\cosh(r)&0&0&\sinh(r)\\0&1&0&0\\0&0&1&0\\\sinh(r)&0&0&\cosh(r)\end{bmatrix},$$

$$\varphi\left(f\left(\cosh\left(\frac{r}{2}\right)+\sinh\left(\frac{r}{2}\right)\gamma^0\gamma^1\right)\right)=\varphi\left(\begin{bmatrix} \cosh\left(\frac{r}{2}\right) & \sinh\left(\frac{r}{2}\right)\\ \sinh\left(\frac{r}{2}\right) & \cosh\left(\frac{r}{2}\right)\end{bmatrix}\right)=\begin{bmatrix}\cosh(r)&\sinh(r)&0&0\\\sinh(r)&\cosh(r)&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$\varphi\left(f\left(\cosh\left(\frac{r}{2}\right)+\sinh\left(\frac{r}{2}\right)\gamma^0\gamma^2\right)\right)=\varphi\left(\begin{bmatrix} \cosh\left(\frac{r}{2}\right) & -i\sinh\left(\frac{r}{2}\right)\\ i\sinh\left(\frac{r}{2}\right) & \cosh\left(\frac{r}{2}\right)\end{bmatrix}\right)=\begin{bmatrix}\cosh(r)&0&\sinh(r)&0\\0&1&0&0\\\sinh(r)&0&\cosh(r)&0\\0&0&0&1\end{bmatrix},$$

$$\varphi\left(f\left(\cos\left(\frac{\varphi}{2}\right)+\sin\left(\frac{\varphi}{2}\right)\gamma^2\gamma^3\right)\right)=\varphi\left(\begin{bmatrix} \cos\left(\frac{\varphi}{2}\right) & -i\sin\left(\frac{\varphi}{2}\right)\\ -i\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)\end{bmatrix}\right)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos(\varphi)&-\sin(\varphi)\\0&0&\sin(\varphi)&\cos(\varphi)\end{bmatrix}, \text{and}$$

$$\varphi\left(f\left(\cos\left(\frac{\varphi}{2}\right)+\sin\left(\frac{\varphi}{2}\right)\gamma^1\gamma^3\right)\right)=\varphi\left(\begin{bmatrix} \cos\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right)\\ -\sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right)\end{bmatrix}\right)=\begin{bmatrix}1&0&0&0\\0&\cos(\varphi)&0&-\sin(\varphi)\\0&0&1&0\\0&\sin(\varphi)&0&\cos(\varphi)\end{bmatrix}.$$

Thus, we can obtain any boost or rotation around an axis by an element of Spin(1,3). Then. as any boost or rotation can be written as the product of such boosts and rotations, φ is a surjective homomorphism. Thus $\mathrm{SL}(2,\mathbb{C}) \cong$ Spin(1,3) is a double cover of $\mathrm{SO}^+(1,3)$, with the homomorphism given by f and $\varphi \circ f$ respectively. The previous calculations hold without proof that f is an isomorphism, therefore we could instead have used these calculations to conclude that Spin(1,3) is simply connected as it is a double cover of $\mathrm{SO}^+(1,3)$, making it the universal cover, and therefore homeomorphic to $\mathrm{SL}(2,\mathbb{C})$ by something called the universal covering property. This also lets us conclude that Spin(1,3) and $\mathrm{SL}(2,\mathbb{C})$ have isomorphic Lie algebras, as the isomorphism f can be differentiated to gain the pushforward f_*, which is a Lie algebra isomorphism. In particular their dimensions must be the same, so Spin(1,3) has a 6 dimensional Lie algebra.

3 Spinor Fields

A spinor field is the assignment of a spinor to every point in spacetime. For flat Minkowski spacetime, by which we mean $\mathbb{R}^4$ with the Minkowski metric, this assignment is simply a function $\varphi : \mathbb{R}^4 \to \mathbb{C}^4$. However, if the chosen spacetime has curvature, as is the case when gravitational effects are included, the vector spaces in which the spinors live are different at each point in spacetime. In this case, we need the concept of a vector bundle, which we will introduce here. However, we will consider the Dirac equation and Lagrangian for spinor fields in flat spacetime as the inclusion of gravity in derivation of Noether currents has some more intricacies that we will touch on in the next section.

Subsection 1 gives some required tools for describing spinors in a more general framework and also includes an aside into the different ways a connection may be chosen to transform.

In subsection 2 we provide the description of Spinor fields in flat spacetime. This description is given by the Dirac equation and a Lagrangian, both of which are shown to be Lorentz invariant.

This allows us in subsection 3 to calculate some important Noether currents like the canonical stress-energy and moment tensors.

3.1 Connections on Vector Bundles

In order to formulate spinor fields on general manifolds, or even $\mathbb{R}^4$ with a metric other than the Minkovski metric, we will need to define vector bundles. Essentially, a vector bundle on a manifold M is a copy of a given vector space V at each element $x \in M$, called a fibre, such that the fibres vary smoothly with the change of $x \in M$ or more formally[2]:

Definition. *Let M be a smooth manifold. A smooth vector bundle over M is a smooth manifold E, with a smooth mapping $\pi : E \to M$. Such that for all $x \in M$, the fibre $E_x := \pi^{-1}(\{x\})$ is a vector space, and that there exists an open set U containing x with a diffeomorphism $\phi : U \times \mathbb{R}^n \xrightarrow{\sim} \pi^{-1}(U)$, satisfying*
1. $\pi \circ \phi(x, v) = x \; \forall u \in U, v \in R^n$
2. $v \mapsto \phi(x, v)$ is a linear isomorphism between $\mathbb{R}^n$ and $\pi^{-1}(\{x\})$

Here E is the vector bundle, π is the bundle projection, and the open set U with diffeomorphism ϕ is a requirement that the bundle locally looks like $U \times \mathbb{R}^n$, which is also called a local trivialisation. A section of a vector bundle is then a smooth function $s : M \to E$, such that $\pi \circ s = id$, or, in other words, a function that chooses one element of the fibre E_p for each $p \in M$. A vector bundle is called trivial if it admits a global trivialisation, that is there is a diffeomorphism ϕ for $U = M$. Since such a bundle is diffeomorphic to $M \times \mathbb{R}^n$, it is often denoted as such.

We will also define connections, which gives a way to differentiate sections of vector bundles along vector fields of a manifold. Connections are used in quantum field theory to model the different fields [7] and in general relativity to model gravity[2], which makes them very important for the study of spinors affected by gravity or Electromagnetism.

Definition. *A connection on a vector bundle $\pi : E \to M$ is a is a bilinear map*

$$\nabla : \mathit{Vec}(M) \times \Gamma(E) \to \Gamma(E)$$

denoted $(v, s) \mapsto \nabla_v s$, such that

$$\begin{aligned} \nabla_{fv} s &= f \nabla_v s \\ \nabla_v (fs) &= f \nabla_v s + v(f) s. \end{aligned}$$

To see what this looks like in coordinates, we equip an open neighbourhood U_p around a point $p \in M$ with local coordinates and obtain a basis ∂_μ for the vector fields on M,Vec(M) , and basis e_i for $\Gamma(E)$. Then we can determine the connection in terms of this basis, so

$$\begin{aligned}
\nabla_v s &= \nabla_{v^\mu \partial_\mu}(s^i e_i) \\
&= v^\mu \nabla_{\partial_\mu}(s^i e_i) \\
&= v^\mu (s^i \nabla_{\partial_\mu}(e_i) + \partial_\mu(s^i) e_i) \\
&= v^\mu (s^i A^j_{\mu i} e_j + \partial_\mu(s^i) e_i) \\
&= v^\mu (s^j A^i_{\mu j} + \partial_\mu(s^i)) e_i,
\end{aligned}$$

where we first use linearity and the two product rules, then write $\nabla_{\partial_\mu} e_i \in \Gamma(E)$ in the basis e_j and call the coefficients $A^j_{\mu i}$, and lastly in the first term rename the dummy variables i, j. We now note that the first term seems $C^\infty(M)$ linear in both v and s, and the second term appears independent of the connection. Therefore, we consider the difference between two different connections ∇ and ∇', $A = \nabla - \nabla'$. For a vector field $v \in \text{Vec}(M)$, $A(v) = \nabla_v - \nabla'_v : \Gamma(E) \to \Gamma(E)$. This means that

$$A(v)(fs) = \nabla_v(fs) - \nabla'_v(fs) = f\nabla_v s + v(f)s - f\nabla'_v s - v(f)s = f\nabla_v s - f\nabla'_v s = fA(v)(s)$$

for all sections $s \in \Gamma(E)$ and functions $f \in C^\infty(M)$, so A is linear over both v and s. In particular, evaluating at a point $p \in M$, $A(v)_p$ is a linear map from E_p to itself, thus $A_p : T_pM \to \text{End}(E_p)$. This means that given a choice of base connection ∇^0 any connection can be written as $\nabla_v = \nabla^0_v + A(v)$ for some 1-form A(v). In case of a manifold with globally defined coordinates and global basis e_i for E, ∇^0_v can be taken to be the Lie derivative $\mathcal{L}_v$, which is evaluated by

$$\mathcal{L}_v s = \mathcal{L}_{v^\mu \partial_\mu}(s^i e_i) = v^\mu \partial_\mu(s^i) e_i.$$

For trivial bundles over $\mathbb{R}^4$ there are the obvious global identity coordinates, turning the derivatives into the standard euclidean derivatives, and global basis e_i for E given by sections $e_i(x) = (x, b_i)$, where the b_i are some basis for the vector space.

3.1.1 Transforming a Connection

The rest of this subsection is of most interest when considering more complicated spacetime manifolds or including gravitational fields. Since this thesis mostly considers flat Minkowski spacetime the following may be treated as an aside and can be skipped without issue.

In general relativity any diffeomorphism $a : M \to M$ is supposed to be a symmetry of space and connections are used to formulate Lagrangians. Therefore, it is useful to consider how connections can transform under a. Generally, there is no one way an object on a manifold should transform, unless further structure is specified. For instance for a vector field $v : M \to TM$ its transformation is dictated by the structure of the tangent bundle TM, here we obtain the transformed v^a at a point $p \in M$ by first considering the value $v_{a^{-1}(p)} \in T_{a^{-1}(p)}M$, to then transport this value to T_pM, the tangent space at p, we simply use the pushforward of a, $a_* : T_qM \to T_{a(q)}M$. Then $v^a = a_* \circ v \circ a^{-1}$, which is again a vector field $v^a : M \to TM$.

For the transformation of a section of a more general vector bundle, in this case a section $s : \mathbb{R}^4 \to \mathbb{R}^4 \times \mathbb{C} = E$ of the trivial line bundle over flat spacetime, we try to adopt a similar procedure to obtain s^a. First we consider the value $s_{a^{-1}(p)} \in \{a^{-1}(p)\} \times \mathbb{C} = E_{a^{-1}(p)}$, but now we have no natural way to transport a value in the fibre over q to a value in the fibre over $a(q)$. Luckily, as the bundle is trivial we can simply copy its value to the other fibre, for lack of a better name we also name this $a_* : E_q \to E_{a(q)}$, which is given by $a_*((q,z)) = (a(q), z)$. Then a section transforms as $s^a = id_a \circ s \circ a^{-1}$.

Now we can try to define the transformation of the connection ∇, To do this we transform the inputs $v \to v^a$ and $s \to s^a$, the section we get out of the connection we transform in the opposite way, so $\nabla_v(s) = t \to t^{a^{-1}}$. We put these together to obtain that ∇^a is given by

$$\nabla^a_v(s) = id_{a^{-1}} \circ \nabla_{v^a}(id_a \circ s \circ a^{-1}) \circ a.$$

We then claim that if we consider the connection locally, so that it splits as $\nabla_v(s) = \mathcal{L}_v(s) + A_v(s)$, then the transformed connection becomes $\nabla^a_v(s) = \mathcal{L}_v(s) + (a^*A)(s)$, so the connection changes as the 1-form A is pulled back through a. The connection is simply a sum, so its components can be considered separately, so we first see how A_v transforms. To make the process clearer we rewrite $A_v(\phi)$ at a point $p \in M$ to $A_p(v)(\phi)$. Then

$$\begin{aligned}
&(a^{-1})_*(A_{v^a}(a_* \circ \phi \circ a^{-1}) \circ a)(p)\\
&= (a^{-1})_*(A(a_* \circ v \circ a^{-1}) \cdot (a_* \circ \phi \circ a^{-1})(a(p))\\
&= (a^{-1})_*(A_{a(p)}(a_* v(p))(a_*\phi(p))) = ((a^*A)_p(v)(\phi)).
\end{aligned}$$

To determine the transformation of $\mathcal{L}_v$ we consider local coordinates ∂_μ. Then

$$\mathcal{L}_v(\phi) = v^\mu \partial_\mu \phi.$$

Then we consider its transformation

$$(\mathcal{L}_{v^a}(\phi \circ a^{-1})) \circ a = ((v^a)^\mu \partial_\mu(\phi \circ a^{-1})) \circ a = \frac{\partial a^\mu}{\partial x^\nu} v^\nu \partial_\kappa \phi \frac{\partial (a^{-1})^\kappa}{\partial x^\mu} = \delta^\kappa_\nu v^\nu \partial_\kappa \phi = v^\nu \partial_\nu \phi,$$

as

$$\delta^\kappa_\nu = \frac{\partial x^\kappa}{\partial x^\nu} = \frac{\partial (a^{-1} \circ a)^\kappa}{\partial x^\nu} = \frac{\partial (a^{-1})^\kappa}{\partial x^\mu} \frac{\partial a^\mu}{\partial x^\nu}$$

Thus $(\mathcal{L}_{v^a}(\phi \circ a^{-1})) \circ a = \mathcal{L}_v(\phi)$, combining these results the connection transforms as

$$\begin{aligned}
\nabla^a_v(\phi) &= ((\mathcal{L}_{v^a} + A_{v^a})(\phi \circ a^{-1})) \circ a\\
&= \mathcal{L}_v(\phi) + (a^*A)(\phi)
\end{aligned}$$

3.1.2 Determinant Transformation

Alternatively, we can use a more complex transformation rule for the section ϕ, to do this we consider the determinant line bundle instead of the standard line bundle, at each point the fibre consists of the volume forms on $\mathbb{C}^n$, where n=4 is the dimension of $M = \mathbb{R}^4$. As a vector space this is isomorphic to $\mathbb{C}$, so we still have the same trivial bundle $E = \mathbb{R}^4 \times \mathbb{C}$. However now to transport an element of E_q to $E_{a(q)}$, it instead makes more sense to multiply it with the Jacobian determinant of a. Thus we replace our trivial identification, a_*, by $a_* : E_q \to E_{a(q)}$ given by $a_* : (p,z) \mapsto (a(p), \det(Da)z)$. Now

$$\begin{aligned}
&(a^{-1})_*(A_{v^a}(a_* \circ \phi \circ a^{-1}) \circ a)(p)\\
&= (a^{-1})_*(A(a_* \circ v \circ a^{-1}) \cdot (a_* \circ \phi \circ a^{-1})(a(p))\\
&= \det(Da^{-1})(A_{a(p)}(a_* v(p))(\det(Da)\phi(p))) = ((a^*A)_p(v)(\phi \circ a^{-1}))
\end{aligned}$$

as $A_p(v)$ is linear in ϕ, so we may simply take out $\det(Da)$. The Lie derivative transforms differently now though, as

$$\begin{aligned}&\det(Da^{-1})\mathcal{L}_v(\det(Da)\phi) =\\&\det(Da^{-1})(\det(Da)\mathcal{L}_v(\phi) + \mathcal{L}_v(\det(Da))\phi) =\\&\mathcal{L}_v(\phi) + \det(Da)^{-1}\mathcal{L}_v(\det(Da))\phi\end{aligned}$$

Thus the 1-form A_v gains an extra term $\det(Da)^{-1}\mathcal{L}_v(\det(Da))$, which is 0 if a is a Poincaré transformation as then

$$\det(Da) = 1.$$

The only other allowed transformations are of the form $a_* : (p, z) \mapsto (a(p), \det(Da)^\lambda z)$ for $\lambda \in \mathbb{Z}$, where we by similar calculations obtain that A_v gains an extra term

$$\det(Da)^{-\lambda}\mathcal{L}_v(\det(Da)^\lambda) = \det(Da)^{-\lambda}\lambda\det(Da)^{\lambda-1}\mathcal{L}_v(\det(Da)) = \lambda\det(Da)^{-1}\mathcal{L}_v(\det(Da)),$$

which is again zero for Poincaré transforms. Due to the chain rule this additional term is the same as

$$\mathcal{L}_v(\lambda\ln(\det(Da))),$$

which means A_μ changes by the gradient of a function $\partial_\mu f$, which is similar to the gauge for electromagnetic fields. To again conclude that such a transformation is a symmetry we will need a Lagrangian.

We choose the Yang-Mills Lagrangian, which is a function of curvature form of the connection[5]. The curvature form is defined by

$$F(v, w)s = \nabla_v\nabla_w s - \nabla_w\nabla_v s - \nabla_{[v,w]}s,$$

where v and w are vector fields and s is a section of the vector bundle. This can be calculated using local coordinates and a local trivialisation of the vector bundle. Since the fibres are 1 dimensional, the basis is given by e_1. Then as $\nabla_\mu e_1 =: A_\mu e_1$ and $[\partial_\mu, \partial_\nu] = 0$

$$\begin{aligned}F_{\mu\nu}(e_1) &= \nabla_{\partial_\mu}\nabla_{\partial_\nu}e_1 - \nabla_{\partial_\nu}\nabla_{\partial_\mu}e_1\\&= \nabla_{\partial_\mu}A_\nu e_1 - \nabla_{\partial_\nu}A_\mu e_1\\&= \partial_\mu A_\nu e_1 + A_\mu A_\nu e_1 - \partial_\nu A_\mu e_1 - A_\nu A_\mu e_1\\&= (\partial_\mu A_\nu - \partial_\nu A_\mu)e_1.\end{aligned}$$

Thus $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, as was the case for the electromagnetic tensor, so by the same arguments the Lagrangian $L \propto F^{\mu\nu}F_{\mu\nu}$ is unchanged.

For more general bundles we note that for any $p \in M$ with open neighbourhood U_p that gives a local trivialization $\varphi : U \times V \xrightarrow{\sim} \pi^{-1}(U)$, and that if $a(p) \notin U_p$, $\mathcal{L}_v$ and $A_{a(p)}$ are not well defined, nor can fibres E_p and $E_{a(p)}$ be trivially identified with each other, so this construction requires more care to do globally, and in some cases may not even be possible globally. In these cases the construction could be done locally instead, using something called infinitesimally natural bundles[9].

3.2 Spinor Fields

Now all the required components for describing spinor fields have been introduced. This will allow us to formulate a Lagrangian for the spinors and show that it has Poincaré invariance, which will finally allow us to use Noether's theorem to derive the canonical stress-energy and moment tensors. Thus we will now describe them based on section 2.4 Spin One-Half Electrodynamics of [7]. An electron is a spin half particle and thus can be described using a wavefunction ϕ taking values in $\mathbb{C}^4$. We then use the concept of a vector bundle introduced in the previous subsection to describe a spinor field as a vector bundle over spacetime with fibre isomorphic to $\mathbb{C}^4$.

3.2.1 Lorentz invariance of Dirac Equation

In the case of a trivial bundle over standard flat spacetime we have the bundle $\mathbb{R}^4 \times \mathbb{C}^4$. Such a spinor field must then satisfy the Dirac equation

$$i\hbar\gamma^\mu\partial_\mu\phi - m\phi = 0. \tag{35}$$

The Dirac equation is Lorentz invariant by construction. We use this to show that our chosen transformation behaviour is the correct transformation behaviour for spinors. If $\tilde{\Lambda} \in Spin(1,3)$ covers $\Lambda \in SO(1,3)$ a spinor transforms as

$$x \to \Lambda x = y \quad , \quad \phi(x) \to \hat{\phi}(y) = \tilde{\Lambda}\phi(\Lambda^{-1}y).$$

This transformation is a symmetry of the Dirac equation if the transformed fields are a solution to the Dirac equation whenever the untransformed fields are. To verify this we first note that $\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu} = \Lambda^\nu{}_\mu\hat{\partial}_\nu$. Thus we can rewrite 35 to

$$i\hbar\gamma^\mu\Lambda^\nu{}_\mu\hat{\partial}_\nu(\tilde{\Lambda}^{-1}\hat{\phi}) - m\tilde{\Lambda}^{-1}\hat{\phi} = 0.$$

We then multiply both sides by $\tilde{\Lambda}$ to obtain

$$i\hbar\tilde{\Lambda}\gamma^\mu\Lambda^\nu{}_\mu\tilde{\Lambda}^{-1}\hat{\partial}_\nu\hat{\phi} - m\hat{\phi} = 0,$$

which shows that $\hat{\phi}$ is a solution if

$$\tilde{\Lambda}\gamma^\mu\Lambda^\nu{}_\mu\tilde{\Lambda}^{-1} = \gamma^\nu$$

Then as $\Lambda^\nu{}_\mu$ is just a number we can rewrite this to

$$\Lambda^\nu{}_\mu\gamma^\mu = \tilde{\Lambda}^{-1}\gamma^\nu\tilde{\Lambda},$$

which can be shown to be true if $\tilde{\Lambda} = \cos(t) + \sin(t)\gamma^1\gamma^2$, as then the right hand side is

$$\left(\cos(t) - \sin(t)\gamma^1\gamma^2\right)\gamma^\nu\left(\cos(t) + \sin(t)\gamma^1\gamma^2\right),$$

so if $\nu = 0$ or $\nu = 3$ the right hand side is

$$\left(\cos(t) - \sin(t)\gamma^1\gamma^2\right)\left(\cos(t) + \sin(t)\gamma^1\gamma^2\right)\gamma^\nu = \gamma^\nu.$$

For $\nu = 1$ we get

$$\begin{aligned}\left(\cos(t) - \sin(t)\gamma^1\gamma^2\right)\gamma^1\left(\cos(t) + \sin(t)\gamma^1\gamma^2\right) &=\\ \left(\cos(t) - \sin(t)\gamma^1\gamma^2\right)\left(\cos(t) - \sin(t)\gamma^1\gamma^2\right)\gamma^1 &=\\ (\cos(t)^2 - \sin(t)^2 - 2\cos(t)\sin(t)\gamma^1\gamma^2)\gamma^1 &=\\ (\cos(2t) - \sin(2t)\gamma^1\gamma^2)\gamma^1 = \cos(2t)\gamma^1 - \sin(2t)\gamma^2&\end{aligned}$$

and for $\nu = 2$ we obtain

$$\left(\cos(t) - \sin(t)\gamma^1\gamma^2\right)\gamma^2\left(\cos(t) + \sin(t)\gamma^1\gamma^2\right) = \sin(2t)\gamma^1 + \cos(2t)\gamma^2.$$

Then as

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) & 0 \\ 0 & \sin(2t) & \cos(2t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we indeed have

$$\Lambda^{\nu}{}_{\mu}\gamma^{\mu} = \tilde{\Lambda}^{-1}\gamma^{\nu}\tilde{\Lambda}.$$

Furthermore using similar calculations we can show that this holds for any elements of such a form, then as Spin(1,3) is generated by such elements, the equation must hold for any arbitrary $\tilde{\Lambda} \in$ Spin(1,3). Therefore the transformation

$$x \to \Lambda x \quad , \quad \phi(x) \to \tilde{\Lambda}\phi(\Lambda^{-1}x) \tag{36}$$

is a symmetry of the Dirac equation and the Dirac equation is Lorentz invariant if the spinor transformation behaviour is chosen correctly. Note that if we had chosen our gamma matrices in a different order, the chosen cover would not have worked, though this can be fixed by changing the basis on $\mathbb{R}^4$, which gives an equivalent representation of the Lorentz group on $\mathbb{R}^4$. Thus once we have chosen our gamma matrices we need to select one of the equivalent spin covers so that we obtain the physically correct transformation behaviour for spinor fields.

3.2.2 Lorentz Invariant Lagrangian

To use Noether's theorem we will need a Lagrangian for spinor fields. This Lagrangian should have the Dirac equation as its Euler-Lagrange equation and also be Lorentz invariant using the same transformation behaviour for spinors. The Dirac Lagrangian is

$$\mathcal{L} = \bar{\phi}(i\gamma^{\mu}\partial_{\mu} - m)\phi, \tag{37}$$

where $\bar{\phi} = \phi^{\dagger}\gamma^0$ is the conjugate field[5]. The Euler-Lagrange equations of this are indeed the Dirac equation and we will now show its Lorentz invariance. The second term is Lorentz invariant if

$$\bar{\phi}\phi = \hat{\phi}^{\dagger}(\tilde{\Lambda}^{-1})^{\dagger}\gamma^0\tilde{\Lambda}^{-1}\hat{\phi} = \hat{\phi}^{\dagger}\gamma^0\hat{\phi} = \bar{\hat{\phi}}\hat{\phi},$$

which is easily seen for matrices of the form $\tilde{\Lambda} = \cos(t) + \sin(t)\gamma^1\gamma^2$, as then

$$\begin{aligned}\tilde{\Lambda}^{\dagger}\gamma^0\tilde{\Lambda} &= (\cos(t) + \sin(t)\gamma^1\gamma^2)^{\dagger}\gamma^0(\cos(t) + \sin(t)\gamma^1\gamma^2) \\ &= (\cos(t) + \sin(t)\gamma^{2\dagger}\gamma^{1\dagger})(\cos(t) + \sin(t)\gamma^1\gamma^2)\gamma^0 \\ &= (\cos(t) + \sin(t)\gamma^2\gamma^1)(\cos(t) + \sin(t)\gamma^1\gamma^2)\gamma^0 \\ &= (\cos(t) - \sin(t)\gamma^1\gamma^2)(\cos(t) + \sin(t)\gamma^1\gamma^2)\gamma^0 = \gamma^0,\end{aligned}$$

as $\gamma^1, \gamma^2, \gamma^3$ are skew Hermitian and γ^0 is Hermitian. and similarly for matrices of the form $\tilde{\Lambda} = \cosh(t) + \sinh(t)\gamma^0\gamma^1$.

$$\begin{aligned}\tilde{\Lambda}^{\dagger}\gamma^0\tilde{\Lambda} &= (\cosh(t) + \sinh(t)\gamma^0\gamma^1)^{\dagger}\gamma^0(\cosh(t) + \sinh(t)\gamma^0\gamma^1) \\ &= (\cosh(t) + \sinh(t)\gamma^{1\dagger}\gamma^{0\dagger})(\cosh(t) - \sinh(t)\gamma^0\gamma^1)\gamma^0 \\ &= (\cosh(t) - \sinh(t)\gamma^1\gamma^0)(\cosh(t) - \sinh(t)\gamma^0\gamma^1)\gamma^0 \\ &= (\cosh(t) + \sinh(t)\gamma^0\gamma^1)(\cosh(t) - \sinh(t)\gamma^0\gamma^1)\gamma^0 = \gamma^0.\end{aligned}$$

Thus $\tilde{\Lambda}^\dagger\gamma^0\tilde{\Lambda} = \gamma^0$, and also $\gamma^0 = (\tilde{\Lambda}^{-1})^\dagger\gamma^0\tilde{\Lambda}^{-1}$, which holds regardless of the choice of gamma matrices, so long as γ^0 is chosen to be Hermitian and $\gamma^1, \gamma^2, \gamma^3$ skew-Hermitian. Then all we still need to show is that

$$\bar{\phi} i\gamma^\mu\partial_\mu\phi = (\tilde{\Lambda}^{-1}\hat{\phi})^\dagger\gamma^0 i\gamma^\mu\Lambda^\nu{}_\mu\hat{\partial}_\nu\tilde{\Lambda}^{-1}\hat{\phi},$$

which is true if

$$(\tilde{\Lambda}^{-1})^\dagger\gamma^0\gamma^\mu\Lambda^\nu{}_\mu\tilde{\Lambda}^{-1} = \gamma^0\gamma^\nu.$$

As $\gamma^0 = (\tilde{\Lambda}^{-1})^\dagger\gamma^0\tilde{\Lambda}^{-1}$ we also have that $(\tilde{\Lambda}^{-1})^\dagger\gamma^0 = \gamma^0\tilde{\Lambda}$, so we may rewrite to

$$\gamma^0\tilde{\Lambda}\Lambda^\nu{}_\mu\gamma^\mu\tilde{\Lambda}^{-1} = \gamma^0\gamma^\nu.$$

Multiplying both sides by $\tilde{\Lambda}^{-1}\gamma^0$ on the left and $\tilde{\Lambda}$ on the right we then get

$$\Lambda^\nu{}_\mu\gamma^\mu = \tilde{\Lambda}^{-1}\gamma^\nu\tilde{\Lambda},$$

which holds as we have already shown. Thus the transformation given by (36) is a symmetry of the Dirac Lagrangian. Finally, we notice that a translation of the underlying space, given by

$$x \to x + a \quad , \quad \phi(x) \to \phi(x - a) \tag{38}$$

for $a \in \mathbb{R}^4$ is also a symmetry as then $\partial_\mu = \frac{\partial\hat{x}^\mu}{\partial x^\nu}\hat{\partial}_\mu = \hat{\partial}_\mu$, so trivially the Lagrangian keeps the same form, which is then Poincaré invariant.

3.3 Noether Currents for Spinor Fields

Now we can finally use Noether's theorem to calculate the stress-energy and moment tensors.

3.3.1 Stress-Energy Tensor

To calculate the stress-energy tensor we use the translational symmetry of the Lagrangian. Thus we plug the infinitesimal version of (38), $a = a^\mu\partial_\mu$

$$x^\mu \to x^\mu + \varepsilon\ a^\mu \quad , \quad \phi(x) \to \phi(x) - \varepsilon\ a^\mu\partial_\mu\phi(x)$$

into Noether's theorem. For which we extract the changes in base and field variables

$$X_B^\mu = a^\mu \quad , \quad X_F = 0$$

respectively. Then by equation (8) we get a current

$$\begin{aligned} J^\mu &= -\left(\frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu L\right)a^\nu \\ &= -\left(\bar{\phi} i\gamma^\mu\partial_\nu\phi - \delta^\mu_\nu\bar{\phi}(i\gamma^\alpha\partial_\alpha - m)\phi\right)a^\nu \end{aligned}$$

giving a canonical stress-energy tensor of

$$\Theta^{\mu\nu} = \bar{\phi} i\gamma^\mu\partial^\nu\phi - \eta^{\mu\nu}\bar{\phi} i\gamma^\alpha\partial_\alpha\phi - \eta^{\mu\nu}m\bar{\phi}\phi.$$

3.3.2 Moment Tensor

The other symmetry shown last section was Lorentz symmetry, which will yield the moment tensor. To compute the Noether current associated with Lorentz symmetry we turn it into an infinitesimal symmetry, which can be plugged into (14). To do this first consider $\widetilde{\Lambda}(t) = \cos\left(\frac{t}{2}\right) + \sin\left(\frac{t}{2}\right)\gamma^1\gamma^2$, which covers

$$\Lambda(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) & 0 \\ 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Linearizing this by dropping terms of order t^2 and higher gives that infinitesimally $\widetilde{\Lambda} \approx I + t\gamma^1\gamma^2$ and

$$\Lambda(t) \approx \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus the vector $\frac{1}{2}\gamma^1\gamma^2$ is pushed forward to the antisymmetric matrix $\omega = E^2{}_1 - E^1{}_2$, where $E^i{}_j$ is the matrix with a 1 in the ith row of the jth column. This means that if the base space changes by $\omega^\mu{}_\nu x^\nu$, the spinor space changes with

$$T = \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{4}\eta_{\mu\mu'}\omega^{\mu'}{}_\nu\gamma^\mu\gamma^\nu = \frac{1}{4}(\eta_{11}\omega^1{}_2\gamma^1\gamma^2 + \eta_{22}\omega^2{}_1\gamma^2\gamma^1) = \frac{1}{4}(--\gamma^1\gamma^2 - \gamma^2\gamma^1) = \frac{1}{2}\gamma^1\gamma^2.$$

Similarly $\frac{1}{2}\gamma^0\gamma^1$ is pushed forward to the symmetric matrix $\omega = E^0{}_1 + E^1{}_0$. Then for this boost too a change of the base space by $\omega^\mu{}_\nu x^\nu$ gives a change in the spinor space of

$$T = \frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{4}\eta_{\mu\mu'}\omega^{\mu'}{}_\nu\gamma^\mu\gamma^\nu = \frac{1}{4}(\eta_{00}\omega^0{}_1\gamma^0\gamma^1 + \eta_{11}\omega^1{}_0\gamma^1\gamma^0) = \frac{1}{4}(\gamma^0\gamma^1 - \gamma^1\gamma^0) = \frac{1}{2}\gamma^0\gamma^1.$$

The fact that this formula also holds for changes corresponding to the other rotations and boost holds by nearly identical calculations again, where only a few indices are replaced. Therefore, this holds for any general infinitesimal Lorentz transformation $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Thus the transformation in (12) is a symmetry by defining $\Sigma^{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu$. By the computations in section 1 this yields the conserved current

$$\partial_\mu \Theta^{\mu\kappa\lambda} = 0,$$

where

$$\Theta^{\mu\kappa\lambda} = x^\kappa\Theta^{\mu\lambda} - x^\lambda\Theta^{\mu\kappa} + \Sigma^{\mu\kappa\lambda}$$

is the canonical moment tensor for spinor fields and

$$\Sigma^{\mu\kappa\lambda} = \frac{\partial L}{\partial(\partial_\mu\phi)}(\Sigma^{\kappa\lambda})\phi = \frac{i}{2}\bar{\phi}\gamma^\mu\gamma^\kappa\gamma^\lambda\phi.$$

3.3.3 Gauge Symmetry

Since the gauge symmetry of electromagnetism was used in section 1, we also compute the current associated to the U(1) gauge symmetry of spinors. This is invariance under a U(1) gauge transformation

$$x^\mu \to x^\mu, \quad \phi(x) \to e^{ia}\phi(x),$$

with $a \in \mathbb{R}$ a constant. This changes the Lagrangian as

$$\begin{aligned}\mathcal{L} &= e^{-ia}\bar{\phi}(i\gamma^\mu\partial_\mu - m)e^{ia}\phi \\ &= \bar{\phi}(i\gamma^\mu\partial_\mu - m)\phi,\end{aligned}$$

so this is a symmetry of the Lagrangian. The corresponding infinitesimal phase change is

$$X_B^\mu = 0, X_F = ia\phi,$$

which has a corresponding Noether current

$$J^\mu = \frac{\partial L}{\partial(\partial_\mu)\phi}X_F = \bar{\phi}i\gamma^\mu ia\phi = -a\bar{\phi}\gamma^\mu\phi.$$

This means that $\partial_\mu\bar{\phi}\gamma^\mu\phi = 0$.

4 Spinor Bundles

In order to use Noether's theorem to obtain a stress moment tensor of spinor fields interacting with gravity, we need a Lagrangian for the system. The Lagrangian for spinor fields in flat spacetime as mentioned before is

$$\mathcal{L} = \bar{\phi}(i\gamma^\mu \partial_\mu - m)\phi \tag{39}$$

and the Lagrangian for gravity, when described using general relativity is the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} = -\frac{1}{2}(R + 2\Lambda),$$

where R is the scalar curvature associated to a metric g on spacetime and Λ is the cosmological constant [5]. This Lagrangian concerns empty space; to describe space with matter, one adds a matter term to the Lagrangian, which depends on the type of matter being considered. subsection 1 introduces principal bundles and associated bundles, which are used to describe gravity and spinors respectively. These are used to construct a spinor bundle, which describes a spinor field with negligible gravitational efffect. subsection 2 introduces the universal spinor bundle which is used to describe spinors whose gravitational effect is not neglected.

4.1 Principal Bundles

We now consider spinor fields in a gravitational background, i.e. the spinor has negligible gravitational effect and instead all gravity comes from some mass external to the system. This comes into effect as a fixed metric g on the spacetime M. For each tangent space T_xM at some point $x \in M$ the metric gives a way to call a basis orthonormal. That is, a basis b_i is orthonormal if $g_x(b_i, b_j) = \eta_{ij}$. Call the set of all such bases at x $OF(M)_x$, then $OF(M) = \dot{\bigcup}_{x \in M} OF(M)_x$ is the orthonormal frame bundle of M. A basis is equivalently defined by a frame $f : \mathbb{R}^4 \to T_xM, f : e_i \mapsto b_i$, for which $g_x(v, w) = \eta(f^{-1}v, f^{-1}w)$. An element of the orthogonal group $\Lambda \in O(1, 3)$ acts on these frames by $f \cdot \Lambda = f \circ \Lambda$, taking an orthogonal frame to another orthogonal frame, as for $f' = f \circ \Lambda$

$$g(v, w) = \eta(f^{-1}v, f{-}1w) = \eta(\Lambda f'^{-1}v, \Lambda f'^{-1}w) = \eta(f'^{-1}v, f'^{-1}w).$$

By introducing an orientation and time orientation on the bundle and restricting the frames to those that agree with the orientations, we obtain the oriented, time oriented frame bundle [7], which we will call $OF^+(M)$. The action of the proper orthochronous Lorentz group $SO^+(1, 3)$ on the fibres of this bundle is transitive and free. This is an example of a principal bundle, which similar to a vector bundle is a bundle for which each fibre is group isomorphic to a given group. Principal bundles have the following formal definition.

Definition. *Let M be a smooth manifold and G a Lie group. Then a smooth principal bundle over M with structure group G (or a G-bundle over M) is a smooth manifold P, with a smooth mapping $\pi : P \to M$ and a smooth right action $\sigma : P \times G \to P$, $\sigma(p, g) = p \cdot g$ of G on P such that the following hold:*
1. σ preserves the fibres of π, i.e. $\pi(p \cdot g) = \pi(p)$ $\forall p \in P, \forall g \in G$
2. for each $x \in M$ there exists an open set U containing x with a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$ of the form $\phi(p)(\pi(p), \varphi(p))$, where φ must satisfy $\varphi(p \cdot g) = \varphi(p)g$ for all $p \in \pi^{-1}(U)$ and $g \in G$

[7]

If we want to put vectors on this bundle, we will require the notion of an associated vector bundle. To define this, let V be a vector space, $\rho : G \to GL(V)$ a smooth representation on V, which gives a smooth left action of G on V, namely $g \cdot v = (\rho(g))(v)$. Then $\sigma : ((p, v), g) \mapsto (p \cdot g, g^{-1} \cdot v)$ is a smooth right action of G on $P \times V$. The orbit space of this action $P \times_\rho V$ is then the set of equivalence classes $[(p, v)] = \{(p \cdot g, g^{-1} \cdot v) | g \in G\}$. Then we define the projection mapping $\pi_\rho : P \times_\rho V \to M$, by $[(p, v)] \mapsto \pi(p)$, which is well defined as $\pi(p \cdot g) = \pi(p)$. Then the vector bundle associated with $G \hookrightarrow P \xrightarrow{\pi} M$ by this action is denoted is the fibre bundle $\pi_\rho : P \times_\rho V \to M$. To describe spinors, the action on which is determined by the double cover Spin(1,3) of $SO^+(1, 3)$, we thus need a principal bundle for which the fibres are isomorphic to Spin(1,3).A spin bundle Q^g is a Spin(1,3)-bundle that must double cover $OF^+(M)$. Q^g is equipped with an action of Spin(1,3) such that for any element $\tilde{p} \in Q^g$, and $\tilde{\Lambda} \in$ Spin(1,3) covering $p \in F$, $\Lambda \in SO^+(1, 3)$ respectively $\varphi(\tilde{p} \cdot \tilde{\Lambda}) = p \cdot \Lambda$. This is summarized by the following commutative diagram:

$$\begin{tikzcd}
Q^g \arrow[d] & Spin(1,3) \arrow[l, bend right] \arrow[d] \\
F^+(M) & SO^+(1,3) \arrow[l, bend right]
\end{tikzcd}$$

Spin bundles do not always exist and need not be unique. The existence of spin bundles depends on the topology of the spacetime M [7]. Then with the representation of Spin(1,3) already given before we obtain the spinor bundle $\pi_{spin} : Q^g \times_{spin} \mathbb{C}^4 \to M$. A problem with this construction is that a transformation of spacetime can change the metric. Then this change in the metric changes the principal Spin-bundle, which automatically changes the spinor field. In order to determine each of these changes we will consider a construction called the universal spinor bundle.

4.2 Universal Spinor Bundle

In order to construct the universal spinor bundle we follow an article by Müller and Nowaczyk[10]. We consider the oriented frame bundle of manifold M, in our case this will be $\mathbb{R}^4$ and we will denote this bundle as $F^+(M)$. The fiber at each point consists of all bases whose orientation is consistent with the orientation of the manifold. This can also be written as the set of orientation preserving linear maps from $\mathbb{R}^4$ to the tangent space T_xM.

$$F_x^+ = \{f : \mathbb{R}^4 \to T_xM | f \text{ linear and preserves orientation}\}$$

given an element $A \in GL_4^+$, the 4 by 4 matrices with positive determinant, and an element $f \in F_x^+$ the multiplication $f \circ A$, acts freely and transitively on the fibres. This makes $F^+(M)$ a principal GL_4^+ bundle over M. GL_4^+ is connected, so it has an universal cover, denoted $\widetilde{GL}_4^+$. To make the associated spinor bundle we first take a spin structure over M, which is a double cover Q of $F^+(M)$ with an action of $\widetilde{GL}_4^+$ on the fibres of Q, compatible with the action of GL_4^+ on $F^+(M)$. This means that for any frame $\tilde{E} \in \widetilde{F^+}(M)$ covering $E \in F^+(M)$ and element $\tilde{\Lambda} \in \widetilde{GL}_4^+$ covering $\Lambda \in GL_4^+$ $u(\tilde{E} \cdot \tilde{\Lambda}) = E \cdot \Lambda$, which can be summarized by the following commutative diagram:

$$\begin{tikzcd}
Q \arrow[d] \arrow[r, phantom, "\circlearrowright"] & \widetilde{GL_4^+} \arrow[d] \\
F^+(M) \arrow[r, phantom, "\circlearrowright"] & GL_4^+
\end{tikzcd}$$

We can restrict the actions to the proper orthochronous Lorentz group $SO^+(1,3) \subset GL_4^+$ and its cover $Spin \subset \widetilde{GL_4^+}$. This allows us to define the orbit spaces $Q/spin \cong F^+(M)/SO^+(1,3)$, which can be identified with the space of Lorentzian metrics on M, denoted $S_{1,3}M$, by identifying each section of the orbit space with the metric that makes the bases orthogonal. To do this, consider a section of the orbit space $s \in \Gamma(F^+(M)/SO^+(1,3))$. At each point $x \in M$ this is an equivalence class of oriented bases. Take one such oriented basis, $f \in s_x$, and define the metric as

$$g(v,w) = \eta(f^{-1}v, f^{-1}w) \forall v, w \in T_xM.$$

with η the Minkovski inner product with respect to the canonical basis of $\mathbb{R}^4$. This does not depend on the choice of f as for any $f' \in s_x$, we can write $f' = f \circ \Lambda^{-1}$ for some $\Lambda \in SO^+(1,3)$. Then

$$g(v,w) = \eta(f'^{-1}v, f'^{-1}w) = \eta(\Lambda f^{-1}v, \Lambda f^{-1}w) = \eta(f^{-1}v, f^{-1}w).$$

This construction gives mappings

$$\begin{aligned} \kappa^M : & F^+(M) \to S_{1,3}M \\ & f \mapsto \eta(f^{-1}\cdot, f^{-1}\cdot) \end{aligned}$$

and similarly $\widetilde{\kappa}^M : Q(M) \to S_{1,3}M$.

Given a choice of representation $\rho : Spin \to GL(\mathbb{C}^4)$ we can associate a $\mathbb{C}^4$ bundle to Q. This bundle is

$$\begin{aligned} \pi^{\Sigma}_{SM} : & \Sigma M = Q(M) \times_{Spin} \mathbb{C}^4 \to S_{1,3}M \\ & [\tilde{B}, v] \to \widetilde{\kappa}^M(\tilde{B}). \end{aligned}$$

This projection is well defined as all $\widetilde{B}$ in the equivalence class differ by an action of Spin, which does not change the metric. This can also be considered a bundle over M using the projection $\pi_M : S_{1,3}M \to M$, which gives the projection $\pi^{\Sigma}_M = \pi_M \circ \pi^{\Sigma}_{SM}$, which makes the diagram

$$\begin{tikzcd}
\Sigma M \arrow[r, "\pi^{\Sigma}_{SM}"] \arrow[dr, "\pi^{\Sigma}_M"'] & S_{1,3}M \arrow[d, "\pi_M"] \\
& M
\end{tikzcd}$$

commute. Since there are now multiple bundles over M, different bundle projections are denoted by using superscripts and subscripts associated with domain and codomain respectively, and bundles are referred to by their projections. The sections of π^{Σ}_M store the information of both the metric g and the spinor field. To see this we take a universal spinor field $\Phi \in \Gamma(\pi^{\Sigma}_M)$. To obtain a metric we note that $g = \pi^{\Sigma}_{SM} \circ \Phi$ is a metric on M by the construction of the bundle. Writing $\Phi = [\widetilde{B}, v]$ now that we have g, we can obtain the spinor field by considering the section $\phi \in \Gamma(\pi^{g}_{Spin})$ given by $[\tilde{B}, v]$, which is well defined, as $\tilde{B}$ is a section of the spinor bundle Q^g, by construction of g.

5 Conclusion

The transformation behaviour of spinors can be defined using the spin group, whose action on the spinors comes from the action of the Clifford algebra on $\bigwedge \mathbb{C}^2 \cong \mathbb{C}^4$. This transformation behaviour is a symmetry of the Dirac equation and the Lagrangian whose Euler-Lagrange equation is the Dirac equation. Using Noether's theorem this yields the canonical energy-momentum tensor

$$\Theta^{\mu\nu} = \bar{\phi} i\gamma^\mu \partial^\nu \phi - \eta^{\mu\nu} \bar{\phi} i\gamma^\alpha \partial_\alpha \phi - \eta^{\mu\nu} m\bar{\phi}\phi$$

and canonical moment tensor

$$\Theta^{\mu\kappa\lambda} = x^\kappa \Theta^{\mu\lambda} - x^\lambda \Theta^{\mu\kappa} + \Sigma^{\mu\kappa\lambda}.$$

For further study it may be interesting to construct the Lagrangian for the universal spinor field. This can be used to describe an electron, or other spin-$\frac{1}{2}$ particle, coupled to gravity, and as such its conserved currents may be of physical interest. This will likely also require the consideration of infinitesimally natural bundles to describe the infinitesimal variations used in Noether's theorem.

Similarly, a further study can show how to couple a spinor field to electromagnetism, or even the weak and strong nuclear forces and derive the conserved currents of such fields.

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