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WORKING COMMISSION W18 - TIMBER STRUCTURES

DISCUSSION OF THE FAILURE CRITERION FOR COMBINED BENDING AND COMPRESSION

by

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Summary

One of the conclusions of the stability group of CIB-W18A was that the Code must allow for analytical solutions for stability design based on quasi linear behaviour and must rever to these methods (p.e. to the Larsen-Theilgaard method for columns).

Although a new parabolic failure criterion for bending with compression is proposed for the Eurocode, based on glulam simulation, the interaction equation for in plane buckling of the code, being applicable for short columns as well, suggests a much less curved failure criterion. It therefore could be seen as a task of the stability group to reconsider the failure criterion.

For that purpose a derivation is given of a consistent simple improved failure criterion for bending with compression, that may account for the influence of quality and moisture content, leads to simple interaction equations for beam-columns and may meet the requirements of the Eurocode.

Together with the proposal of CIB-paper 23-15-2 a possible consistent design method for braced and free beam-columns is proposed for the Eurocode and is used in the new Dutch Code.

In the appendix an explanation is given of the bearing- and shear strength.

Introduction

At the CIB-W18-meeting in Lisbon 1990, a proposal for stability design was given, (paper 23-15-2), based on the quasi linear approach, as is used in timber engineering, and on a lower bound failure criterion for bending and compression as, for instance, can be deduced from the publication in the proceedings of the IUFRO 1982 paper 24. This lower bound criterion was felt to be too conservative for all cases and a differentiation ought to be possible for the different cases as for different moisture contents and grades.

Because pure experimental design methods should not be used and the proposed methods for stability design are based on the second order stress theory, there must be agreement on the failure criterion to be used. Therefore a discussion is given here of this criterion for bending with compression that is suitable for analytical solutions and for the Eurocode and is verified by the experimental research of the above mentioned IUFRO-paper 24.

The criterion should account for the elastic-plastic behaviour of wood, showing a plastic failure in compression with a high deformation capacity (like steel) and brittle-like failure in tension (thus showing a volume effect for tension). Thus after the flow strain at the end of the linear range, it need not be assumed (as for steel) that there is a limiting ultimate strain in compression because there is no indication of such a limit (Pure plastic flow is possible in a compression test if it is managed to keep the system stable at flow). For tension it can be assumed that the flow strain at the end of the elastic stage is also the ultimate strain.

Discussion of the basic equations for beam-columns

The stability equations for beam-columns, proposed in CIB-W18A-paper 23-15-2, are based on the quasi linear approach. In principle if a beam-column is elastic plastic, the sectional properties along the length of the member are not constant. The analysis then can be based on a deflection method assuming the deflected shape of the beam-column by a simple function such as an orthogonal function (depending upon the applied loading and boundary conditions) and establishing equilibrium at least at the end and at the most yielded cross section. For this purpose the equilibrium equations of Chen and Atsuta where chosen and extended for lateral loading. The derivation of these equations is based on the assumption that the warping torsional rigidity and the St. Venant torsional rigidity do not alter during yielding of the cross section. (This applies for lateral buckling by bending for low quality wood where the yielding is small or neglectable). When the influence of the smaller terms are neglected the dominating linear terms remain, leading to the proposed equations for beam-columns of paper 23-15-2 (It is assumed that the initial excentricity for compressional loading alone is chosen to be high enough to be able to neglect these non-linear terms).

The assumption of a quasi constant warping rigidity is in accordance with the method for wood to express the bending strength in an equivalent linear behaviour up to failure with an equivalent bending modulus. So the two opposite bending parts determining the warping rigidity are quasi linear and with that also the warping rigidity. This warping rigidity dominates for profiles (when "buckling" of the compressed flange is determining) and for short columns and the assumption of constant torsional rigidity has a minor influence.

For wood it is also a use to estimate also for torsion and shear the quasi elastic values. Because these shear stresses act only in the elastic part of the section a compensation for the reduced area is given by low ultimate shear strengths and a low torsional shear modulus. This thus has to be applied for high grade timber and for bending with compression.

The elastic-plastic approach doesn't account for the Engesser effect for low grade timber when by bending and compression just both the flow stress in compression and and ultimate tensile stress is reached. For this case a correction is possible by assuming earlier flow by increasing the value "s" of the failure criterion (see further).

Derivation of the failure criterion for bending with compression

The derivation can be based on the elastic-plastic behaviour of wood with the usual starting points:

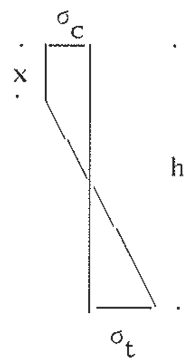
The modulus of elasticity is the same for compression and tension;

Plane sections remain plane (symmetry condition for bending);

In tension the behaviour is elastic untill failure at a critical stress or strain. How-

ever this stress may increase according to the volume effect. This effect can be accounted for in the parameter "s" of the failure criterion,
In compression the behaviour is elastic-plastic, being linear to the flow strain and then showing constant stress at increasing plastic strain.

Equilibrium of a section loaded in bending and compression.



For a beam of width b loaded by a moment M and normal force N is according to the figure:

$$\frac{M}{b} = \frac{\sigma_c + \sigma_t}{2} \cdot (h - x) \cdot \left(\frac{h}{2} - \frac{h - x}{3}\right) \tag{1}$$

$$\frac{N}{b} = \sigma_c \cdot h - \frac{\sigma_t + \sigma_c}{2} \cdot \left(1 - \frac{x}{h}\right) \cdot h \tag{2}$$

or after elimination of x/h:

$$\frac{6 \cdot M}{\sigma_t \cdot b \cdot h^2} = \frac{\sigma_c}{\sigma_t} \cdot \left(1 - \frac{N}{b \cdot h \cdot \sigma_c}\right) \cdot \left(\frac{-1 + 3 \cdot \sigma_t/\sigma_c + 4 \cdot N/(b \cdot h \cdot \sigma_c)}{1 + \sigma_t/\sigma_c}\right) \tag{3}$$

For N = 0 is: $\sigma_c = \frac{\sigma_t + \sigma_c}{2} \cdot \left(1 - \frac{x}{h}\right)$

or:

$$\frac{6 \cdot M}{b \cdot h^2} = \sigma_c \cdot \frac{3 \cdot \sigma_t - \sigma_c}{\sigma_t + \sigma_c} = \sigma_m \tag{4}$$

Thus:

$$\frac{6 \cdot M}{\sigma_m \cdot b \cdot h^2} = \left(1 - \frac{N}{b \cdot h \cdot \sigma_c}\right) \cdot \left(\frac{-1 + 3 \cdot \sigma_t/\sigma_c + 4 \cdot N/(b \cdot h \cdot \sigma_c)}{3 \cdot \sigma_t/\sigma_c - 1}\right) \tag{5}$$

So with:

$$Y = \frac{6 \cdot M}{\sigma_m \cdot b \cdot h^2}; \quad X = \frac{N}{b \cdot h \cdot \sigma_c} \quad \text{and} \quad s = \sigma_t/\sigma_c \ \left(= \varepsilon_t/\varepsilon_v\right),$$

or for the ultimate state: $\sigma_t = f_t$, $\sigma_c = f_c$, then $\sigma_m = f_m$, eq.(5) becomes:

$$Y = (1 - X) \cdot \left(1 + \frac{4 \cdot X}{3 \cdot s - 1}\right) = 1 - X + \frac{4 \cdot X \cdot (1 - X)}{3 \cdot s - 1} \tag{5'}$$

It is seen that: $Y \approx 1 - X$ as lower bound for high values of s and for s = 1.67:
$Y \approx 1 - X^2$ as is proposed now in the Code based on glulam.

Eq.(5') provides a simple design criterion that can be further simplified as shown later.

Curvature

The relation for the radius of bending is with $s = \sigma_t/f_c$:

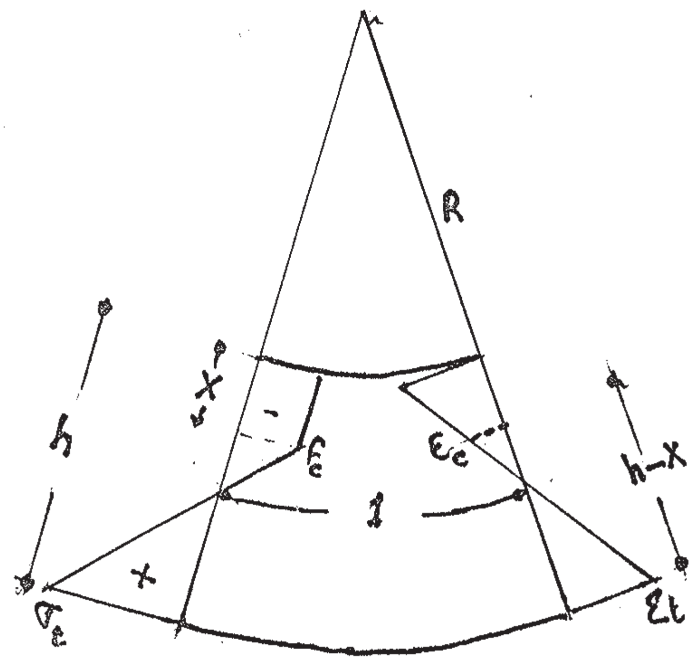
$$\varkappa = \frac{1}{R} = \frac{\varepsilon_v + \varepsilon_t}{h - x} = \frac{\varepsilon_v(1+s)}{h - x} = \frac{f_c(1+s)}{E(h-x)} = \frac{f_c(1+s)^2}{2Eh} = \frac{2\sigma_m}{Eh}\cdot\frac{(s+1)^3}{4(3s-1)} \qquad (6)$$

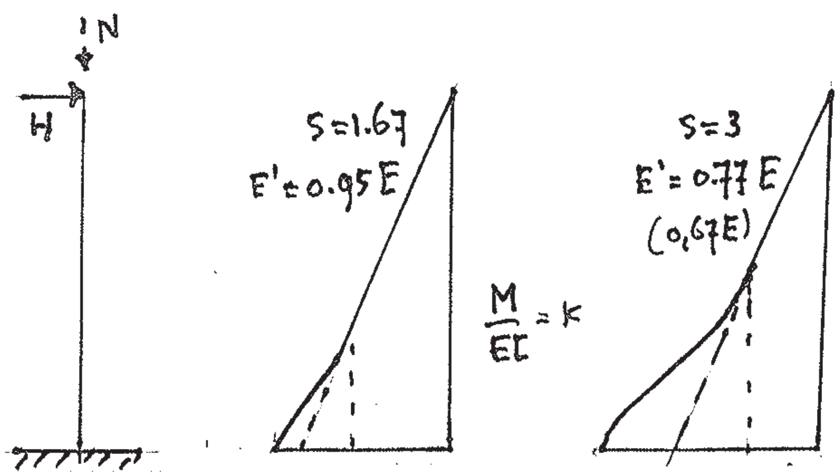
To eliminate the varying values of s along the beam this can be approximated to:

$$\varkappa = \frac{2\sigma_m}{Eh}\cdot\frac{(s+1)^3}{4(3s-1)} = \frac{2\sigma_m}{Eh}\cdot\frac{(s+1)^4}{4(3s-1)^2}\cdot\frac{\sigma_m}{f_c} \approx \frac{2\sigma_m}{Eh}\cdot\frac{\sigma_m}{f_c}\cdot 0.8 \qquad \left(1.25 \le \frac{\sigma_m}{f_c} \le \frac{f_m}{f_c}\right) \qquad (7)$$

s	$\frac{(s+1)^4}{4(3s-1)^2}$
1	1
1.25	0.85
1.5	0.8
1.75	0.8
2	0.8
2.25	0.8
2.5	0.9
2.75	0.9
3	1

giving the curvature in the elastic-plastic range. In the elastic part of the column: $\varkappa = 2\sigma_m/(Eh)$ $(\le 2f_c/(Eh))$
It is seen that a reduced modulus has to be applied as quasi linear modulus for extreme cases (s = 3).





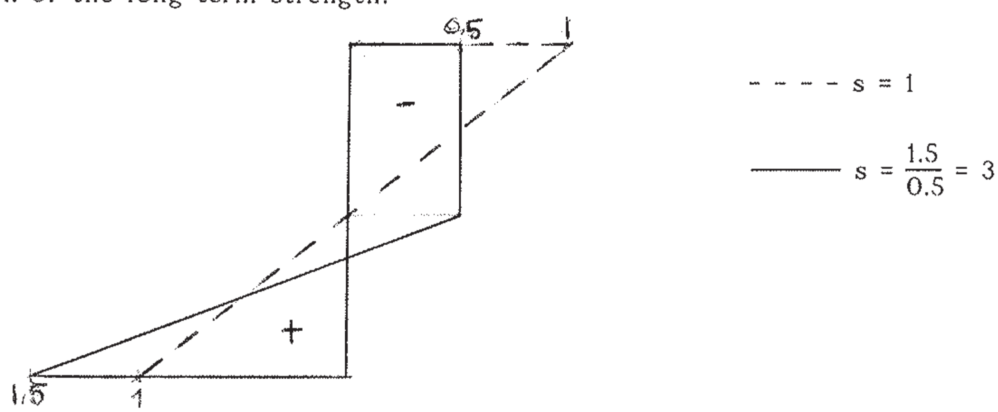
Elimination of (h - x) from (2) and (6) gives:

$$(s + 1)^2 = 2\varkappa h\frac{E}{f_c}\left(1 - \frac{N}{f_c bh}\right) = 2\varkappa h\frac{E}{f_c}\left(1 - X\right) \tag{8}$$

or for a combination with a normal force N: $\varkappa = \dfrac{\varkappa_{bending}}{1 - X}$ (9)

Extreme values

For high quality wood is $f_t/f_c \approx 1.3$ according to the measured values of the tension and compression strength tests of paper 24 of IUFRO 1982. Because there is a strong volume effect (k_{dis}) for tension only, increasing "s", and moisture effects reducing mainly compression and not tension, also increasing "s", the value of "s" may reach values of about 3 to 4 for high quality green wood (especially for small dimensions when the volume effect is the highest and changing moisture content effects are quick). Mechano-sorptive slip is much higher for compression than for tension and may differ nearly one order at working stress level. If it therefore is assumed, as first approximation, that tension is not affected, the slope of th elastic line must be two times steeper for a two times increase of deformation by this slip. This causes the stress distribution according to the figure below, differing an internal equilibrium system with the initial stresses to carry the same moment. Thus "s" may change from 1 to 3 and eq.(11) should be used for these cases. The increase of deformation by a factor 2 may occur within m.c. class 2 by the 6% m.c. change at working stess level. Thus it can be predicted that a long lasting climate change may strongly flatten the failure criterion for high loaded structures and thus flattens the failure criterion of the long term strength.

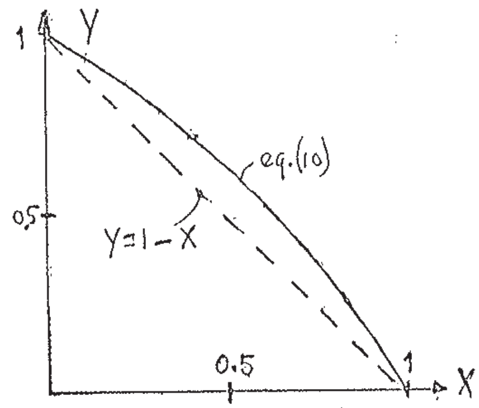


Prediction of stress re-distribution by mechano-sorptive slip

For s = 3, as for instance applies for wet clear wood or for every grade when a climate change occurs, eq.(5') becomes:

$$Y = 1 - X + 0.5 \cdot X \cdot (1 - X) \tag{10}$$

and the maximum deviation from the line: $Y = 1 - X$ is obtained for $X = 0.5$. Then the third term of the right side of eq.(10) is: 0.13 or:
$Y = 1.13 - X$. The slopes are about: $Y'|_{X=0} = -0.5$ and $Y'|_{X=1} = -1.5$ (see fig. below)



M - N - diagram for high quality wood

This curve will be flattened by the higher volume effect for bending with compression than for bending alone, especially for higher values of X and by the Engesser effect. So the failure condition for this case will be not far from:

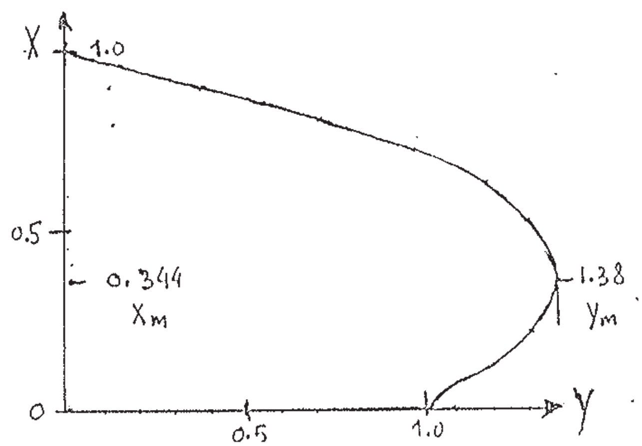
$$Y + X = 1 \tag{11}$$

as earlier was proposed in the Eurocode.

For very low quality wood is according to the mentioned paper 24: $f_t/f_c \approx 3/4$ and for bending failure f_t is determining and the compression stress will be equal to f_t. This means that $s = 1$ at bending and "s" decreases at application of compression. The boundary value, where "s" becomes constant, is reached when in eq.(2), $x = 0$ or when: $X = (1 - s)/2$. Thus for $s = 3/4$ is $X = 1/8$. For $X \geq 1/8$ is eq.(5'):

$$Y = 1 - X + 3.2 \cdot X \cdot (1 - X) \tag{12}$$

and the slope is: $Y' = 2.2 - 6.4 \cdot X$. thus the curve shows a maximum for $X = 0.34$, giving $Y_{max} = 1.38$. At $X = 1/8$ is $Y = 1.22$ and the slope is $Y' = 1.4$. For $X = 0$ is: $Y' = 1$ and $f_m = f_c$ ($s = 1$).
The curve shows that there is quite a reserve with respect to line, eq.(11), and eq.(12) of this curve will be applicable if major defects determ the tensile strength following fracture mechanics thus showing no volume effect. For common cases the curve will be flattened by the volume effect of the tensile strength as will be shown in the following section. Around Y_{max}, the point of first flow of the section in this case, a correction for the Engesser effect is necessary. Because this effect acts as an earlier flow of the section this can be done by increasing "s", flattening the curve there.



M - N - diagram for low quality wood

Curves like eq.(12), or based on the 5 % lower percentiles of the combined compression with bending strengths of the lowest grade are not general applicable because accounting for these lower percentiles is advantageous for the failure criterion.
To use a curved failure criterion at constant moisture conditions, there must be a guarantee that the compression strength is high enough in all circumstances. According to the strength class system of the Code the value of s is close to $s \approx 1$ for lowest grade (as will be shown later) but the different influence of the moisture content on tension and compression is not introduced in the code. Thus there is a problem that a separate moisture effect on s has to be introduced in the Code.
When for the lowest strength class a value of $s \approx 1$ is introduced eq.(5') becomes:

$$Y = 1 - X + 2X(1 - X) \qquad (13)$$

and: $Y' = 1 - 4X$, showing a maximum at: $X = 0.25$, being: $Y = 9/8$. Y is again 1 at $X = 0.5$. Thus an approximation of eq.(13) for these calculations can be in the form:

$$Y = 2\cdot(1 - X) \leq 1 \qquad (14)$$

(also in agreement with the Code to use nowhere values above $Y = 1$).
For wet conditions "s" will be a factor 1.6 higher with respect to the value at 10 % m.c. and the lowest value for s will be about 1.3.

Bi-axial bending

For the combination of "double" bending in the two main directions the interaction line will be straight (see CIB paper 18-2-1):

$$Y_y + Y_z = 1 \qquad (15)$$

for low grades if there is no volume effect (tensile failure by great defects) and this straight line will be a real lower bound. Due to the volume effect of the maximal tensile stress (acting at one point) the interaction line for combined bending will be curved and can be represented according to paper 18-2-1 by:

$$Y_y^{1.3} + Y_z^{1.3} = 1 \tag{16}$$

or approximately in a linear form:

$$Y_1 + 0.7 \cdot Y_2 = 1 \quad \text{when } Y_1 > Y_2 \tag{17}$$

$$0.7 \cdot Y_1 + Y_2 = 1 \quad \text{when } Y_1 \leq Y_2 \tag{18}$$

The form of this curve, eq.(16), doesn't change much when bending failure becomes elastic-plastic for high grades and high m.c. (being a reason to maintain a quasi linear approach for bending failure in wood).
Because the interaction line is not far from the straight line eq.(15) there is no coercive reason to replace the usual applied eq.(15) by eq.(17) and (18) as strength criterion.

Experimental verification

The measurements of IUFRO paper 24 are based on a sample from one mill at one time of all grades of 2x6" lumber, kiln dried to 14 % m.c. There were done 21 different tests on each 100 boards. It was tried to make a quality distinction by visual grading. However this didn't lead to consistent differences of the different quality classes. Thus the measured curve is for all grades and the 5 % lower percentile, showing an advantageous failure criterion for bending with compression, is not representative for structural timber.
It can be derived from the failure criterion that for the top-values, for lower grades, where $Y' = 0$ and Y is maximal equal to Y_m at $X = X_m$, the following applies:

$$Y_m = \frac{\left(1 - X_m\right)^2}{1 - 2 \cdot X_m} \qquad \text{where:} \quad X_m = \frac{5 - 3 \cdot s}{8} \tag{19}$$

Thus Y_m may also be expressed in s, being:

$$Y_m = \frac{9 \cdot (1 + s)^2}{16 \cdot (3 \cdot s - 1)} \tag{20}$$

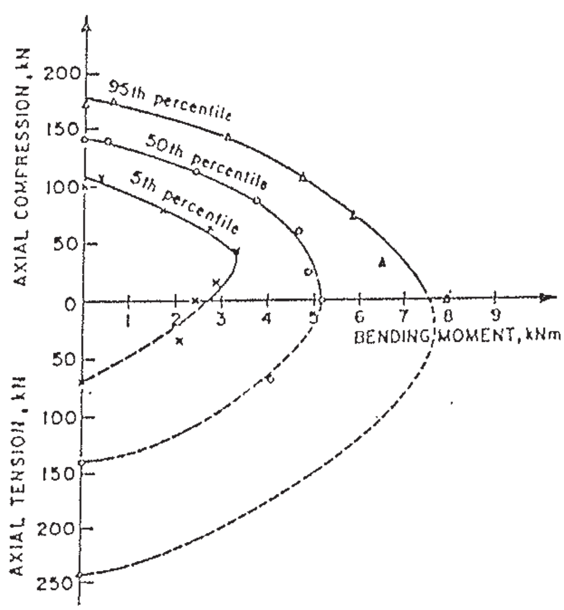
These relations can be used by constructing the interachtion curve for bending failure with compression.
In the fiure below the measured values of the interaction curve are given with the 5 % percentiles for the 0.9 m long members. This may be regarded as the failure criterion because of the only small buckling effects (lateral buckling in the weak direction was prevented).
It can be seen that a good fit of the derived curve is possible by:

s = 2.33 for the 95 percentile,

s = 1.67 for the 50 percentile, showing the top-value at $X_m = 0$ giving $Y_m = 1$,

s = 0.95 for the 5 percentile, showing the top-value at $X_m = 0.27$ giving $Y_m = 1.16$,

This shows that "s" is about 1.7 times higher for the 95 percentile as is to be expected from the tensile test. This can be explained by the volume effect as mentioned in paper 24. At bending already 0.4 times the height of the beam flows in compression for this high percentile and thus the volume factor is higher than for linear bending when compared with pure tension failure of the member. The value of 1.7 is in accordance with the usual applied k = 5 value of the Weibull model. For the 50 percentile this factor is 1.67, where the Weibull value is 1.66 and for the 5 percentile this factor is about 1.5, where the Weibull value for bending without flow of the compression zone predicts 1.64. This difference of 10 % is not astonishing because not the pure tension strengths of the members were measured but combined values with bending and the percentiles were found by transposing the results into polar coordinates, using the found radius for each percentile. The values above show that it is reasonable to account for the volume effect by the stress distribution that is incorporated in the bending strength, replacing $s = f_t/f_c$ (ratio of tensile/compressional strengths) by: $s = f_m/f_c$ (ratio of bending/compression strenghts) or to use safely $s \approx 1.7 \cdot f_t/f_c$.



Measurements of proceedings IUFRO Boras 1982, paper 24.

Higher values of "s" are to be expected from the measurements of paper 24 at higher moisture contents. The mentioned measured values of "s" were for a m.c. of 15 %. At 25 % m.c. "s" will be about a factor 1.3 higher than at 15 % m.c.. Thus:

for the 95 percentile at 25 % m.c., $s \approx 3.1$;
for the 50 percentile at 25 % m.c., $s \approx 2.2$, and
for the 5 percentile at 25 % m.c., $s \approx 1.2$.
However the Code makes no distinction between compression and tension strengths depending on moisture content and other moisture effects and if that is retained these lower bound values have to be used or a separate moisture effect on "s" has to be introduced.
The results of paper 19-12-2 show that also for gluelam a volume effect has to be assumed.

Strenth classes

To use a curved failure criterion for constant moisture conditions, there must be a guarantee that the compression strength is high enough with respect to the tensile strength in all circumstances. the strength classes may contain timber with a compression strength value of a lower class giving values of $s \approx 1.7 \cdot f_t/f_c$:

s - values	15% m.c.	25% m.c.
class C_7 to C_{10}	1.7	~ 2.2
C_3 to C_6	1.3	~ 1.7
C_1 and C_2	~ 1.2	

It can be seen that the now in the Eurocode proposed parabolic criterion wherefore $s = 1.67$, applies or is safe at 15% m.c. but is unsafe for the 4 highest classes above 15% and a m.c. correction is necessary and can be:

Class C_7 to C_{10}: $s = 1.7/k_{moist}$ (21)

C_1 to C_6: $s = 1.3/k_{moist}$ (22)

with: $k_{moist} = 1 - 2.5(\omega - 0.15)$ (ω is relative moisture content)

Shear strength

Another reduction of the curved form of the failure criterion for bending with compression can be due to shear. This is derived in the appendix. Because this criterion is a straigth line, there can always be a cut off of a curved failure criterion and it is necessary to introduce this additional criterion in the Code when a curved criterion for bending with compression is used thus:

$$\tau_d/f_{v,d} \leq 1 - \sigma_{c,d}/f_{c,0,d} \qquad (23)$$

Conclusion

- It is shown that a simple design failure criterion for bending and compression is possible that can be explained by the elastic-full-plastic approach, with linear elasticity for tension up to failure, determined by the volume effect, and by unlimited flow in compression. This criterion:

$$Y = 1 - X + \frac{4 \cdot X \cdot (1 - X)}{3 \cdot s - 1}$$

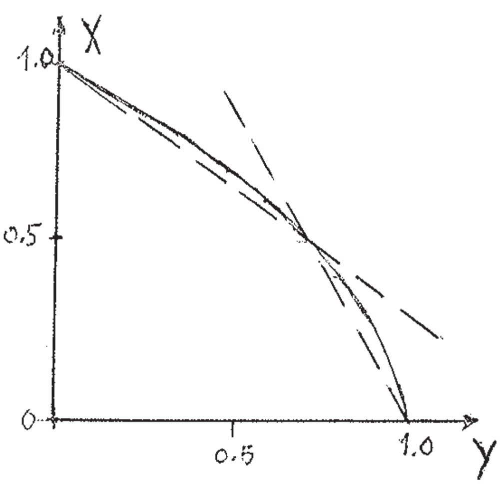
accounts for quality and moisture effects by the value of "s".

- For s = 1.67 this is:

$$Y = 1 - X^2$$

the new criterion of the Eurocode, applicable at constant m.c. except for the highest four grades above 15% m.c. Here s = 2.3 is safe giving: $Y = 1 - X/3 - 2X^2/3$.

- Because there will always be a linear cut off of the failure criterion, a simple linear approach of the parabolic failure criterion is appropriate, and lines can be drawn through the Y values of point X = 0 and point X = 0.5 and the Y values of the points X = 1 and X = 0.5 (avoiding too high estimates at Y_m).



bi-linear failure criterium for bending with compression and shear

Thus the failure criterion then becomes in general:

$Y + c_1 X = 1$ when $X \leq 0.5$ with $c_1 = (s - 1)/(s - 0.33)$

$c_2 Y + X = 1$ when $X > 0.5$ with $c_2 = (s - 0.33)/(s + 0.33)$
or when smaller: $c_2 = f_m \tau_d / f_{v,d} \sigma_{m,d}$ (or eq.(23))

where for rectangular cross sections:

$$Y = \frac{6 \cdot M}{f_m \cdot b \cdot h^2} \; ; \quad X = \frac{N}{b \cdot h \cdot f_{c,0}}$$ and "s" is given by eq.(21) and (22).

Or as proposal for the Code:

$c_2 Y + c_1 X \leq 1$

with: $c_2 = 1$ and $c_1 = (s - 1)/s + 0.33)$ when $X \leq 0.5$,

or: $c_2 = (s - 0.33)/(s + 0.33) \leq f_m \tau_d/(f_{v,d} \sigma_{m,d})$ and $c_1 = 1$, when $X \geq 0.5$.

The advantage of this criterion is that simple consistent interaction equations are possible for twist-bend buckling. The proposed stability equations of CIB paper 23-15-2 can be retained replacing the compression strength f_c by a higher value:

$f_c \cdot \frac{s - 0.33}{s - 1}$ below the 50% normal load level or:

replacing the bending strength f_m by a higher value:

$f_m \frac{s + 0.33}{s - 0.33}$ above the 50% normal load level.

- As lower bound:
  - when a climate change can be expected at high loading;
  - at high shear loading;
  - for reinforced timber, improved gluelam or veneerwood, wet clear wood, etc.,

$c_1 = c_2 = 1$ in the criterion above and the criterion becomes:

$$Y = 1 - X$$

and this lower bound criterion can better be retained in the Code (as is done for the Dutch Code) because of the possible restrictions on the use of a parabolic criterion.

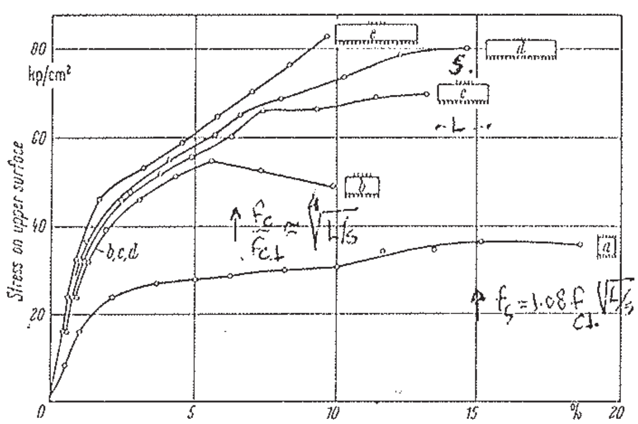
Appendix

a. Bearing strength perpendicular to the grain of locally loaded blocks

The compression strength perpendicular to the grain may increase due to confined dilatation perpendicular to the loading direction.
From the figure below it can be seen that the strength increases with the increasing possibility op spreading of the load. Further it is seen that there is a maximal spreading of about 4 x h. An increase of the strength is then only possible by increasing h. At plastic flow the increase of strength is about proportional with $\sqrt{L/s}$ (see figure).

$$f_s = c \cdot f_{c,90} \cdot \sqrt{L/s} = 1.08 \cdot f_{c,90} \cdot \sqrt{L/s} \qquad (1)$$

At lower strain this is about proportional to $\sqrt[4]{L/s}$ when the cube strength is not regarded and it is seen that this empirical relation, proposed for the Eurocode, is not very well to represent the ultimate state.



Bearing strength ⊥, specimen 15x15 cm, lengths: 15, 30, 45, 60, 75 cm, of a to e, [1]

The dependence of the strength upon spreading can be explained by the equilibrium method of the theory of plasticity.
In the plastic region a stress field can be constructed in the specimen that satisfies the equilibrium conditions:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \qquad (2)$$

and the boundary conditions and nowhere surmounts the failure criterion.
This failure criterion is of the Mohr-type in the radial plane: $\tau/f_v = \sqrt{1 - \sigma_{c,0}/f_{c,0}}$, [2], or can be approached by a Coulomb criterion. In the radial plane an inscibed Tresca criterion can be used being a maximum stress criterion:

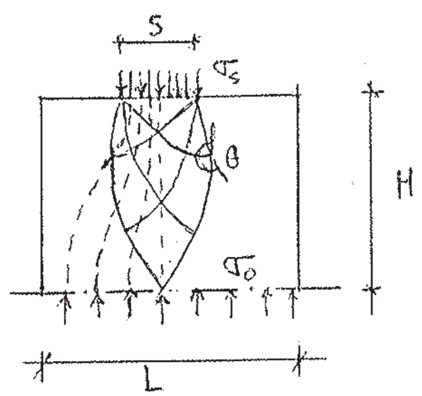
$$(\sigma_1 - \sigma_2)/2 = k = f'_v \tag{3}$$

However the result of the derivation does not depend much on the failure criterion used. The Coulomb or the Mohr criterion gives a higher value of c of eq.(1) than the Tresca criterion, the strength $f_{c,90}$ however is related to a prism strength and is lower than the cube strength associated with the Tresca criterion. Both criteria therefore will give comparable values of f_s.

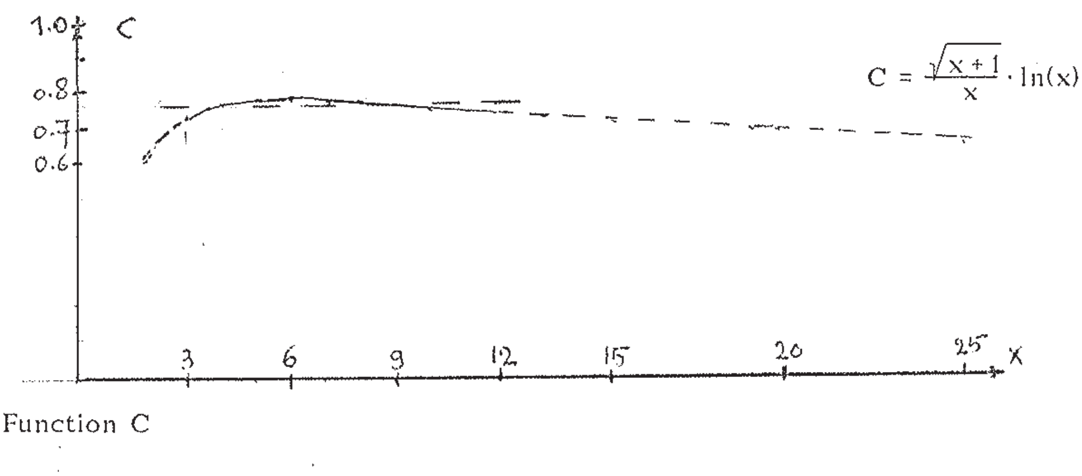
Although "shear-flow" is in the weak direction as in the compression test perpendicular to the grain, the behaviour is simular to a reinforced material (in the strong direction) having the shear strength of the weak direction and confined pressure may build up in all directions when there is friction between bearing plate and specimen in the width direction where the width of the bearing plate is equal to the width of the block. Because failure, according to a shear plane in the weak direction, not affecting the reinforcement, is not determining in this case, the upper stress is determined by the spreading possibility in the strong direction.

Equations (1) to (3) can be written as equations along discontinuity lines (characteristics as for instance Prandtl slip lines) and from the construction of these lines it follows that $\sigma_s = 4k\vartheta + \sigma_0$ and $\vartheta \approx 0.62 \cdot \ln(2H/s)$, see [3], giving:

$$\sigma_s - \sigma_0 = 2.48 \cdot k \cdot \ln(2H/s) \tag{4}$$



"Slip-lines" determining the direction of the main stresses



Function C

and because $\sigma_s \cdot s = \sigma_0 \cdot L$ (see figure below) is: $\sigma_s(1 - s/L) = 2.48 \cdot k \cdot \ln(2H/s)$. Further the construction for a finite block gives the indication that the spreading of the stress is below: $L \approx 2H + s$ or: $H \approx (L - s)/2$ when $H > s$, thus: $L/s > 3$ (Below this value the spreading is less strong and finally failure is simular to the cube test). Substitution of the values for σ_0 and H in eq.(4) gives:

$$\sigma_s = 2.48 \cdot k \cdot \ln\left(\frac{L}{s} - 1\right) \cdot \frac{\sqrt{L/s}}{L/s - 1} \cdot \sqrt{L/s} = 2.48 \cdot k \cdot C \cdot \sqrt{L/s} \tag{5}$$

where C is a function of L/s only and can be regarded constant of about 0.78.
Thus:

$$\sigma_s = 0.97 \cdot 2k \cdot \sqrt{L/s} \tag{6}$$

The value of k follows from the compression test (cube test) with $\sigma_1 = f_{c,90}$ and $\sigma_2 = 0$, or: $k = f_{c,90}/2$. Thus eq.(6) becomes:

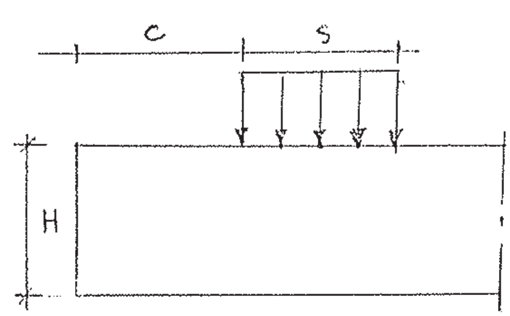
$$f_s = c \cdot f_{c,90} \cdot \sqrt{L/s} \qquad \text{with } c \approx 1 \tag{7}$$

The higher experimental value of c given in eq.(1) shows the lower bound approach of the chosen method (the real slip-lines must give a higher value). At lower flow strains c also will be lower in experiments. Thus c gives the possibility to adapt the model to test results.

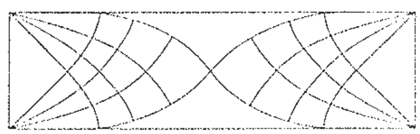
Eq.(7) provides a basis for design rules and is able to explain the different results. As mentioned before the rules of the Eurocode based on $\sqrt[4]{L/s}$ suggest to be based on small deformations (and not the ultimate state) and the dependence on H is omitted in the Code. However for very small values of H there is hardly any spreading and the given rules don't apply. Increase in bearing is then only possible after flow at hardening if the structure remains stable in that state. The given rules seem to apply for a special case: $H \geq (a + l_1)/3 = 250/3 = 84$ mm. This has to be mentioned in the Code.
As discussed in paper CIB-W18/5-10-1, the French rules show the dependence of H and the results are closer to the the ultimate state (see table):

$f_s/f_{c,90}$

s/H	c/H ≥ 1.5	c/H 1	c/H 0.5	c/H 0
1	2	1.5	1.25	1
2	1.5	1.25	1.12	1
≥ 3	1	1	1	1



When $c/H = (L - s)/2H \geq 1.5$, thus when $L \geq 3H + s$, the maximal spreading is reached according to the table and to the first figure above of [1]. (This indicates friction of the plates in the strong direction because without that: $L \approx 2H + s$ is expected). For $s/H \geq 3$, it is assumed that in the middle the same conditions appear as in the cube test. However this is even too low when friction is ignored here and test values will be higher than the table values. The same applies for $c = 0$ when $L > s$, only without friction (and $L = s$) the situation is comparable with the cube test and test values will be higher than 1. The confined pressure may be build up according to the following figure.

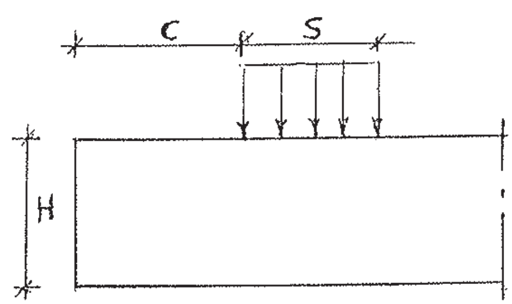


"Failure" between two plates

The influence of no friction along the bearing plate in the strong direction (and thus full friction in the width direction) can be assessed as lower bound by assuming that only symmetrical spreading is possible. Thus $L = 2c + s$. Further $s/H \geq 3$ must comprise $s/H >> 3$. $c/H \geq 1.5$ must be $c/H \geq 1$ and the first column has to be omitted when no friction is assumed. $c = 0$ must comprise $L = s$ giving a value 1.
According to eq.(7) is then:

$f_s/f_{c,90}$

s/H	c/H = (L - s)/2H			
	≥ 1.5	1	0.5	0
1	2	1.7	1.4	1
2	1.6	1.4	1.2	1
≥ 3	1	1	1	1



These values are close to the values of the French rules and are comparable when in eq.(7) $c \approx 0.9$ is used indicating that a limited flow is regarded as the ultimate state in the French tests or that safe lower bounds where given.

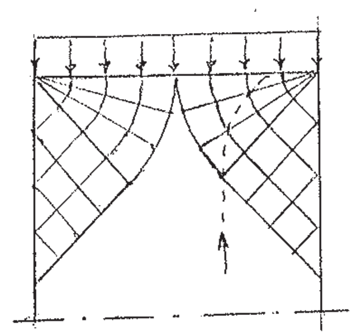
In paper CIB-W18A/23-6-1 test results are given of bearing in the range where H is not limiting for spreading because: $L < 2H + s$. The determination of $f_{c,90}$ is done on a specimen that is long in the strong direction and the results will be higher than those of the compression test. However the strain chosen as failure strain was lower than the ultimate giving compensating lower strength values. The comparison of this compression test with the ASTM-bearing test in the paper shows that the ASTM

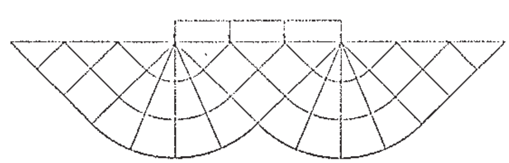
values are about $\sqrt{3}$ times higher according to the theory (L/s = 3 in the ASTM-specimens).
In the following table the test results are compaired with eq.(7) and it is seen that also non-symetrical spreading is possible of end loaded blocks because of the friction between plate and specimen (and the high value of H).
According to the Eurocode a limitting value occurs at $s/L \leq 0.125$ for central loading. The results here show that for end loaded blocks the limit of k_c is about $k_c \approx 2$ for $s/L \leq 0.25$. These limits are due to a local mechanism as for instance given in the figure below. It has to be remembered that the lines are not real slip lines here but means to construct a stress field that satisfies equilibrium, boundary conditions and the failure criterion. The theoretical value of the limit of k_c is higher for central loading and it seems to be possible to obtain higher values by tests.

$f_s/f_{c,90} = k_c$

s/L	measurements central loaded k_c	measurements end loaded k_c	$\sqrt{L/s}$	adaption c of eq.(7) central loaded $c = k_c/\sqrt{L/s}$	adaption c of eq.(7) end loaded $k_c/\sqrt{L/s}$
1	1	1	1	1	1
0.875	1.063	1.063	1.069	1	1
0.75	1.188	1.156	1.155	1.03	1
0.625	1.375	1.281	1.265	1.09	1
0.5	1.625	1.438	1.414	1.15	1
0.375	1.969	1.625	1.633	1.2	1
0.25	2.344	1.875	2.0	1.17	0.94 limit
0.125	2.781	2.156	2.83	1 limit	
mean of c:				1.08	1

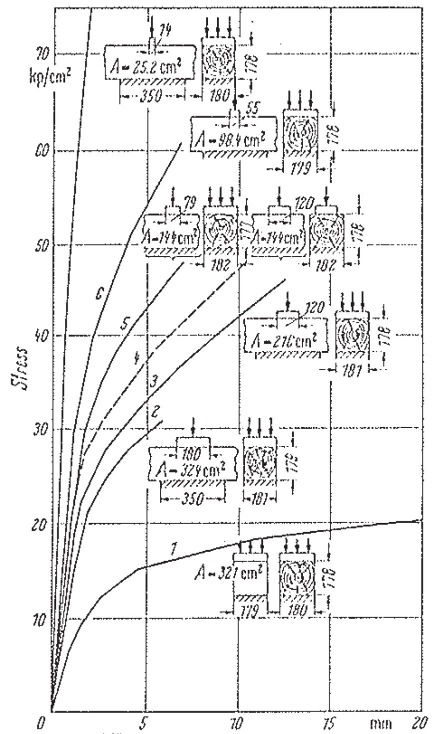
"slip-lines" when the plate transfers the entire shear stress

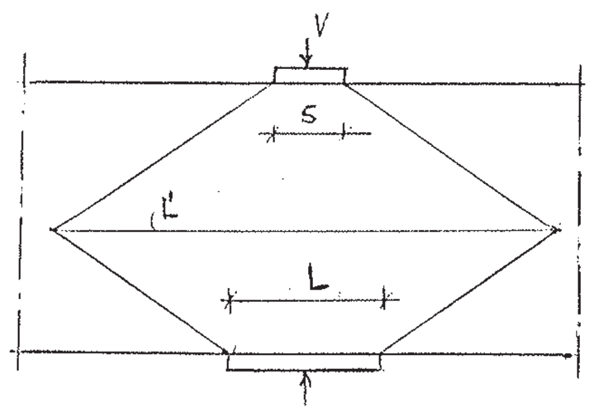




"Local failures"

For long blocks with respect to the bearing plates the maximal spreading will occur at both plates according to the figures below.





Local loading perp. to the grain [1] Graf Possible spreading

From the figure it follows that: $s + 3\alpha H = L + 3\cdot(1 - \alpha)H$. Thus:

$$\alpha = 0.5 + \frac{L - s}{6H}$$

and thus the equivalent spreading factor (of the strength determining plate) is:

$$\frac{L'}{s} = \frac{s + 3\alpha H}{s} = 1 + \frac{3H}{s}\left(0.5 + \frac{L - s}{6H}\right) = 0.5 + \frac{3H + L}{2s}$$

With H =17.9; L = 35; b = 18.1 cm according to the measurements of O. Graf is:

$$k_{c,90} = c\cdot\sqrt{0.5 + 800/s\cdot b} = 1.1\cdot\sqrt{0.5 + 800/s\cdot b}$$

leading to the values of f_s at 5 mm deformation (see fig.) of the curves:
1: 16 - 2: 30 - 3: 36 - 5: 43 - 6: 52 kgf/cm^2, about the same as the measurements. The highest maximum is not shown (at f = s·b = 25.2, see fig.). Predicted according to the last formula is: f_s = 100 kgf/cm^2. However this will be cut off by a local mechanism. Because $f_s \geq 75$ is measured, the maximum value of $k_{c,90}$ is at least 75/16 = 4.7 (close to the theoretical value obtained from a local failure mechanism (giving an upper bound value) of about 5.5 to 6).

The measurements suggest a higher spreading possibility than to: $s + 3\alpha H$. However the model applies for high plastic deformation and after splitting softening may occur simular to the specimen with 30 cm length in the first figure of this appendix. Because complete curves are not given, rules have to be based on the limited deformation (~ 5 mm here).

It can be concluded that the theory is able to explain the, contradictory, test results and design proposals of the Eurocode, the French rules, the measurements of Suenson, Graf, CIB-W18A/23-6-1 and design rules should be adapted in this way.

As proposal for the Eurocode the following rules are possible for bearing blocks:

$\sigma_{d,90,d} \leq k_{c,90} \cdot f_{c,90,d}$

where:

$k_{c,90} = \sqrt{L/s}$

with: $L \leq a + s + l_1/2$; $L \leq 3H + s$ and:

$k_{c,90} = 2.8$ when $s/L \leq 0.125$ for central loads;

$k_{c,90} = 2$ when $s/L \leq 0.25$ for end loads.

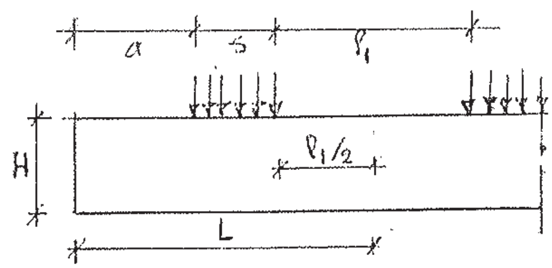
For safe rules (when friction is only in the width direction), the conditions are:

$L \leq 2a + s$; $L \leq s + l_1$; $L \leq 2H + s$,

$k_{c,90} = 2.8$ when $s/L \leq 0.125$

For the bearing strength of a middle section of a beam between two plates of lengths L and s is:

$$k_{c,90} = 1.1 \cdot \sqrt{0.5 + \frac{3H + L}{2s}} \leq 5$$



Literature

[1] F. Kollmann, Principles of wood science and technology, vol. I, 1984, Springer-Verlag, Berlin.

[2] T.A.C.M. van der Put, A general failure criterion for wood, Proc. IUFRO S5.02 paper 23, 1982, Boras, Sweden.

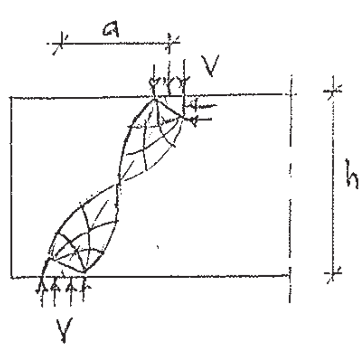
[3] H. Schwartz, PhD dissert. Stuttgart, 1969.

[4] P. Vermeijden, P.B.B.J. Kurstjens, Shear strength of close to support loaded beams on 3 supports, report 4-68-13 HE-2, Stevinlab. Delft, 1968 (in Dutch).

These results were discussed in a small groupe after the presentation of paper CIB-W18A/23-6-1. The general opinion was that failure perpendicular to the grain does not really exist and there would always occur enough hardening at the end to make any rule safe. It thus was decided not to present these results. However this hardening is only possible when the structure at failure remains stable p.e. by locking up the failed wood. This is not always achieved in a real structure.
Further understanding of bearing is also necessary to explain the mechanism of shear failure for loading close to the supports. Therefore the proposal for design rules based on the start of the first stable ultimate flow state is given here as annex of this paper and provide better and more consistent rules for the Eurocode.

b. Shear strength of beams

Shoring model:
For shear loading by a load on a beam close to the support, the bearing strength is determining for the strength as given in appendix a. The strength is: $V_u = f_s bs$ when $a/h \leq 1$ (see fig. below). Above about $a/h = 1$ the shear strength (along the grain) can be determining (when s is long enough) and a simular formula applies as for the bearing strength with f'_v in stead of $f_{c,90}$ that also can be given as: $V_u = 0.67 f'_v bh$, where f'_v depends on the volume effect (of the shearing plane along the grain; see test results of [4]). For the lowest grade, that may show early failure by bending-tension showing a linear stress distribution along the height of the beam, the shear strength V_u according to the beam theory is the same. For higher qualities, flow in compression may occur and the shear strength decreases while the bending strength increases according to the bending theory.
At about $a/h \geq 1.5$, depending on the grade, the bending theory may apply (showing the same value of V_u according to the shoring (or bearing) model for loads close to the supports).



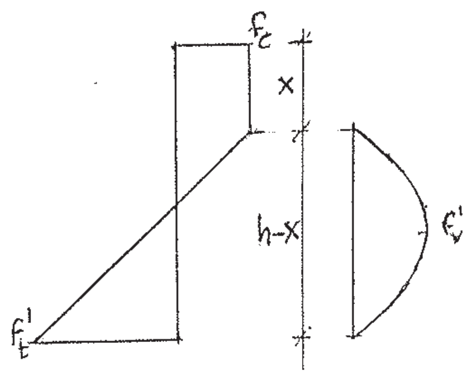
Bearing or shoring mechanism

Beam model:

When there is plastic flow in compression, shear can only be carried in the elastic region. According to the figure below is for bending:

$$V_u = \frac{2}{3}\cdot f'_v\cdot b\cdot h\cdot\left(1 - \frac{x}{h}\right) \qquad (1)$$

$$1 - \frac{x}{h} = \frac{2\cdot f_c}{f'_t + f_c} \qquad (2)$$



or from eq.(1) and (2):

$$V_u = \frac{4}{3}\cdot\frac{f'_t\cdot f_c}{f'_t + f_c}\cdot b.h = \frac{2}{3}\cdot f_v\cdot b\cdot h$$

or:

$$f_v = \frac{2\cdot f'_v\cdot f_c}{f'_t + f_c} \qquad (3)$$

where f_v is the quasi linear shear strength.

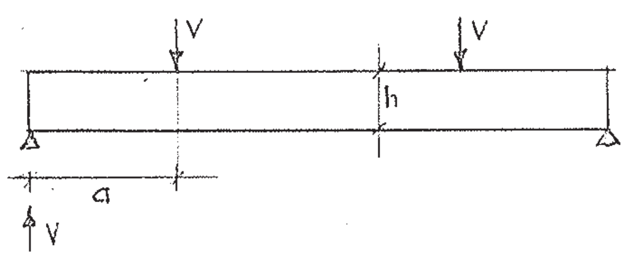
For bending with compression is:

$$V = \frac{2}{3}\cdot f'_v\cdot b\cdot h\cdot\left(1 - \frac{x}{h}\right) = \frac{2}{3}\cdot f'_v\cdot b\cdot h\cdot\left(1 - \frac{N}{f_c\cdot b\cdot h}\right)\cdot\frac{2\cdot f_c}{f_c + f'_t} = \frac{2}{3}\cdot f_v\cdot b\cdot h\cdot\left(1 - \frac{N}{f_c\cdot b\cdot h}\right)$$

or:

$$V = V_u\cdot\left(1 - \frac{N}{N_c}\right) = V_u\cdot\left(1 - X\right) \qquad (4)$$

For failure in bending and in shear there is a critical value of the shear slenderness M_u/V_uh where the ultimate bending strength is reached ($\sigma'_t = f'_t$) and at the same time the ultimate shear stress ($\tau = f_v$) is reached. In a four point bending test is:



$$M_u/V_uh = a_c/h = \frac{3f'_t - f_c}{8f'_v} = \frac{f_m}{4f_v} \approx 3$$

for most strength classes (that may contain timber with a bending strength value of one higher class).

Above this critical value shear is not determining and there is bending failure. Below this value rotation and bending strength is reduced by the high shear force reducing the length x until x = 0. Then the maximal possible shear strength is reached: $V_u = 0.67f'_vbh$ at a moment: $M = f_mbh^2/6 = f_cbh^2/6$ ($\leq f'_tbh^2/6$).

When both terms of eq.(4) are multiplied by a_c, then this equation becomes:

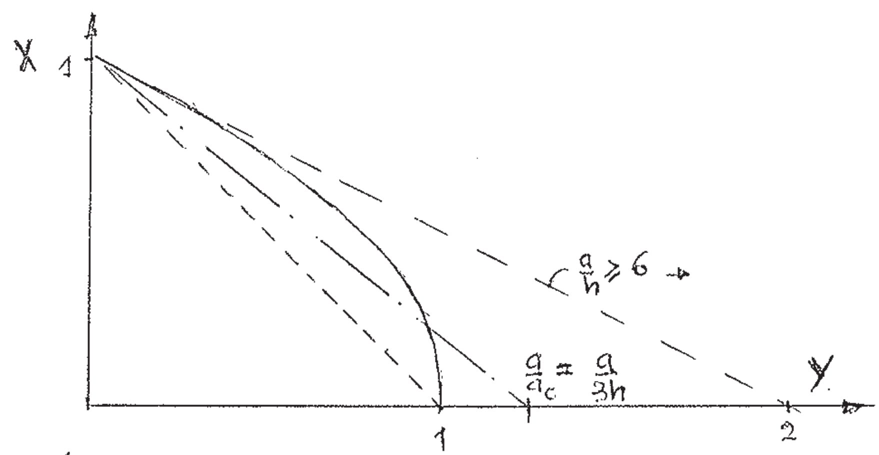
$$Va_c = V_ua_c(1 - X) = M_u(1 - X) = \frac{M}{a}\cdot a_c \quad \text{or:}$$

$$\frac{M}{M_u} \cdot \frac{a_c}{a} = 1 - X \qquad \text{or:} \qquad Y(a_c/a) = 1 - X$$

being a straight line and giving a cut off of a curved failure criterion. See for instance the figure below where the now proposed parabolic criterion is given. Only for $a/h \geq \sim 6$, the situation of the loads in the 4-points bending test, there is no cut off of the failure criterion. For loads closer to the support this reduction has to be accounted for bending with compression.

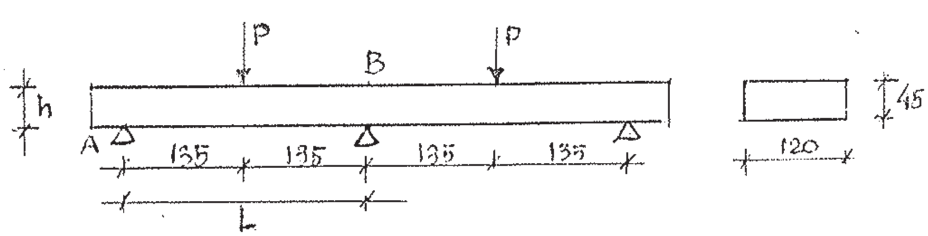
The experimental verification of this model is p.e. given by the test results of paper CIB-W18/24-10-1

Although for an explanation of test results a refined elastic plastic model has to be used, accounting precisely for the volume effects of shear and bending-tension and with adapted compression strengths at reduce rotation according to the work-curve of compressional failure, the method can be applied in its simple form as lower bound and has to be used when a non-linear failure criterion for bending with compression is introduced.



Cut off of the parabolic failure criterion for combined bending and compression by high shear loading

<u>Eplanation of the test results of paper CIB- W18/24-10-1</u>



For the beam on 3 supports a cut of the beam at the middle support at point B will give a rotation at B by the loading P of: $\varphi = PL/16EI$. The shear strain will give no rotation at B. Because the beam is fixed at point B the moment at the support will close the gap by $\varphi' = M_B/3EI$. However the shear by the reaction M_B/L of this moment will also close the gap by: $\gamma = \tau/G \approx M_B/(LAG)$. Thus:

$$\varphi - \gamma = \frac{PL^2}{16EI} - \frac{M_B}{LAG} = \frac{M_B}{3EI} \quad \text{or:} \quad M_B = \frac{3PL}{16} \cdot \frac{1}{1 + 4h^2/L^2}$$

With: h = 45; L =270 is: $M_B = 0.9 \cdot 3PL/16$. Thus $\sigma_B = 0.9 \cdot 50 = 45$ MPa (see paper). Now the field- and support moments are equal but bending failures are initiated from the field if there is a volume effect.

The shear slenderness: M/Vh of the field moment at the side of the free support is: M/Vh = L/2h = 3 is not determining. At the midsupport is $M_B/V_Bh \approx L/4h \approx 1.5$. In general is:

$$\frac{M}{Vh} = \frac{3s - 1}{s + 1} \cdot \frac{f_c}{4f_v} \quad \text{or at point B:} \quad 1{,}5 = \frac{3s - 1}{s + 1} \cdot \frac{45}{4 \cdot 7.6} \quad \text{giving:} \quad s \approx 1$$

showing that there is just no plastic flow and indicating that: $f_c \approx 45$ MPa and the maximal shear stress is: $f'_v \approx 7.6$ MPa.

The value of f_c may be used to explain the strength of the centre-point loading, single span test of the paper where: M/Vh = L/2h = 3 and $f_v = 5.4$ MPa:

$$3 = \frac{3s - 1}{s + 1} \cdot \frac{45}{4 \cdot 5.4} \quad \text{or} \quad s = 1.56 \quad \text{giving a bending strength of:}$$

$$\sigma_m = 45 \cdot (3 \cdot 1.56 - 1)/(1 + 1.56) = 64.9 \text{ MPa (measured is 64.8 MPa)}$$

The pure bending strength of the 4 point bending test is: $f_m = 77.8$ MPa. Thus:

$$77.8 = 45 \cdot \frac{3s - 1}{s + 1} \quad \text{or} \quad s = 2.15$$

The maximal shear stress of 7.6 occurs at the neutral line. For shear failure at plastic flow in compression the maximal shear stress is combined with a tension stress and will be about 0.9 times lower. Thus: $f''_v = 0.9 \cdot f'_v = 0.9 \cdot 7.6 = 6.8$ MPa.

This means that the shear strength at the maximal bending strength will be:

$$\tau_m = \frac{2f''_v}{s + 1} = 2 \cdot 6.8/(1 + 2.15) = 4.3 \text{ MPa}$$

and will occur at: $a/h = (3 \cdot 2.15 - 1) \cdot 45/(3.15 \cdot 4 \cdot 4.3) = 4.5$.

Thus the 4-point bending test can be repeated with loads at a distance of 203 mm from the support to obtain the shear strength at ultimate bending.

It is assumed that corrections for volume effects, as for clear wood, can be ignored here for LVL (this has to be checked first).

It then can be concluded that the quasi linear shear strength has to be 4.3 MPa in order to predict a correct reduction of the bending strength by ultimate shear control. This of course is a prediction because data are lacking.