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Finding infinitely many even or odd continued fractions by introducing a new family of maps

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Finding infinitely many even or odd continued fractions by introducing a new family of maps

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Chapter 1

Preface

Before I started my thesis I went to Cor Kraaikamps office to see if he had any graduation projects that I could do. Instead of giving me a project he sent me home with a bunch of articles to pick a subject from. He gave me a lot of freedom of what I could do. I came with the suggestion of introducing a new continued fraction map and he was enthousiastic right away. Of course, this motivated me to spend a lot of time on the project. Therefore I want to thank Cor for his enthousiasm and also for the good ideas and input on the project. A room was assigned to me, at the Statistics and Probability department on the 6^{th} floor, to be able to work at the university and to be among the professors. I really enjoyed working there. Along with two other students (Harold and Saïd) I had a good time. The people of the department really tried to involve us in everything. Especially I would like to thank the techinician, Carl and the secretary, Leonie for this very reason.

The project itself was very interesting to me. Some things went relatively easy and others were far harder then expected. I hope the reader will enjoy reading the thesis as much as I enjoyed making it.

Contents

Chapter 2

Introduction

In recent years, much research has been done on continued fraction expansions. New variations of the classical regular continued fraction expansion have been introduced and their dynamical properties explored.
In this thesis, a combination of two different variations is introduced combining 2-expansions and flipped expansions. The 2-expansions are specific cases of N-expansions introduced in 2008 by E.B. Burger, J. Gell-Redman, R. Kravitz, D. Walton and N. Yates in [2] and studied in [1, 5, 18] amongst others. The flipped expansions have been studied in [3, 11]. The question is: why would you combine the two expansions? There are already so many different expansions! So why keep adding more animals to the zoo? One of the reasons is that various continued fraction expansions exhibit very interesting dynamical behavior. In this sense they are 'toy models' in ergodic theory and dynamical systems. Another reason is to find infinitely many expansions with odd or even digits. This is not the case when you make even and odd expansions in the classical way like in Schweiger's paper [16] from 1988, or as in [6].

The 2-expansions we will investigate, are determined by a set L and the flipped expansions will be determined by a set F. Logically, the ensuing expansion will be determined by L and F. Stating that there are infinitely many ways to make an even or odd expansion means there are infinitely many choices for L and F to make the expansion even or odd. Since these expansions have not been studied before a whole range of examples are given for L and F. If possible, an explicit density of the underlying dynamical system is given. In the past, however, it turned out to be very difficult to find such densities even though we can show that whenever our system is ergodic such invariant measures always exist! In this new family of expansions we will see that there are infinitely many mappings which are not ergodic but we can formulate a lemma and with the help of that lemma almost all the examples in this thesis can be proved to be ergodic. In cases where we do not have an explicit density a numerical estimate is given. We restricted ourselves to the 2-expansion because this already gives enough challenges and we expect the N-expansions to behave in a similar way or in a richer way. In some parts of the thesis we will look at cases where $N > 2$ but not too often since we do not want to deviate too much from the main topic.

First, in Chapter 2, we will review the 2-expansions followed by a chapter on flipped expansions. In the chapter on flipped expansions we will already see some new expansions with only odd or only even digits. Then, the two expansions are combined and a new map is introduced. Once we have the new map, we prove the convergence of the new expansions as well as the fact that rational numbers have a finite expansion. Before we look into specific examples of flipped 2-expansions we will study for which L and F the system is ergodic. Even though we did not get a description of the sets for which the system is not ergodic (which looked extremely hard) we will prove that for all maps exhibited in this thesis the system will be ergodic. Next, some specific choices for L and F are given. As they are not the main subject of this thesis it will be a quick overview to show what can be done with this new map (with the exception of the 3-divisible expansion which will be more elaborate) and, if possible, the invariant measure is given. After that we can finally look into ways of finding even expansions using the new map. In Chapter 9 it is shown why there are infinitely many systems with only even digits and some examples are given. In Chapter 10 the same is done but for expansions with only odd digits.

In the chapter on ergodicity some non-ergodic maps are given. We will look into two of those maps on which, if we restrict the mapping to a specific domain, is ergodic. This is done in Chapter 11 and will give interesting results. For simulations, a new method is introduced by using the idea behind the Gauss-Kuzmin Theorem. At last, before going to the conclusion and future work, some examples of mappings with finite digits for the case $N = 4$ are given. In the appendix we discuss the regular continued fraction map for those readers not familiar with some of the terminology of the field or simply forgot it for a moment. It might be also nice to read if the reader is unfamiliar with continued fraction expansions. An explanation on how the simulations are done can be found in the appendix as well together with all the Matlab code used for this thesis.

Chapter 3

2-expansions

In this thesis we will make 2-expansions by fixing a set L. For some choices of L it turns out to be easy to obtain their underlying dynamical system along with the invariant measure and natural extension.
Some of these choices are shown in [5] for general N. We will see some examples as well as a method of finding the invariant measures of these examples. But first we will give a short introduction to N-expansions.

3.0.1 N-expansions

Let us first recall some basic facts on N-expansions. An N-expansion of a number $x \in (0,1)$ is a continued fraction expansion of the form

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{d_3 + \ddots}}} .$$

where $N \in \mathbb{Z}\backslash\{0\}$ and $d_i \in \mathbb{N}$ for all i. It turns out that, whenever $N \geq 2$, there are infinitely many sequences $(d_1, d_2, d_3 \ldots)$ for which the equation holds. This is shown by Maxwell Anselm and Steven Weintraub in [1], see also [5] for a different approach. We will use the approach from [5]. Let $T : [0, N] \to [0, N]$ be defined by

$$T(x) = \frac{N}{x} - d(x) \quad \text{for } x \neq 0, \quad \text{and } T(0) = 0$$

where $d(x)$ is a natural number (greater than zero) such that $T(x) \in [0, N]$. As we will see for $N = 2$ we can use such maps to find the continued fraction expansion of any $x \in (0, N)$. Also for $N = 2$ we will see that for every $x \in (0,1)$ there are 2 options for $d(x)$. This gives us infinitely many different expansions. Note that $N = 1$ gives us the regular continued fraction expansion. For the regular continued fraction the digits are unique. Except for rational numbers which have a finite expansion and can be written in two ways (ending with a digit 1 and one ending with a digit other then 1).

3.0.2 Various 2-expansions

We will now start by introducing the 2-expansions. We will specify for each x which value for $d(x)$ to pick by giving a set L. Let $L \subset [0,1)$ be a Borel measurable set. We define $T_L : [0,2) \to [0,2)$ by

$$T_L(x) = \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor + 1_L(x) \ .$$

The set L is the set on which we use the top map which is also called the *lazy map*.

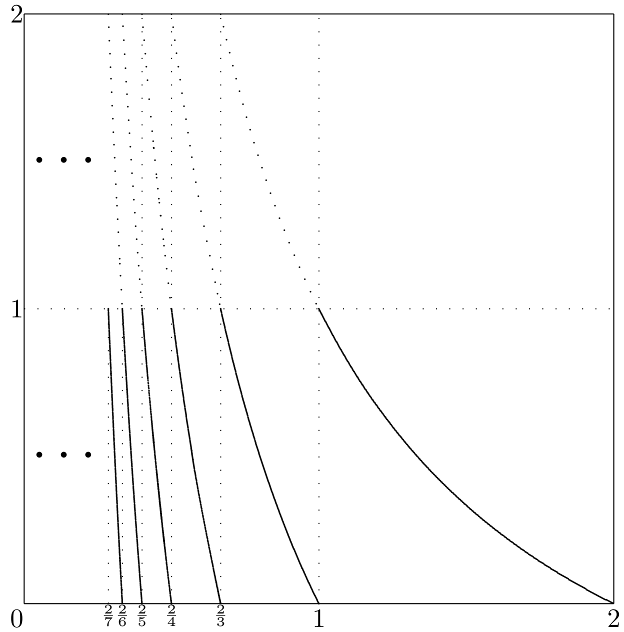

Figure 3.1: 2-expansions with the lazy map dotted.

Furthermore let $d_1 = d(x) = \left\lfloor \frac{2}{x} \right\rfloor - 1_L(x)$ and $d_n = d_n(x) = d(T_L^{n-1}(x))$ whenever $T_L^{n-1}(x) \neq 0$. We will see that we can make a continued fraction of any number $x \in (0,2)$ by applying $T_L(x)$ iteratively. We write

$$T_L(x) = \frac{2}{x} - d(x)$$

so

$$x = \frac{2}{d(x) + T_L(x)} \ . \tag{3.1}$$

Now, if we set $T_L^2(x) = T_L\left(T_L(x)\right)$, we find

$$T_L^2(x) = \frac{2}{T_L(x)} - d(T_L(x))$$

which gives us

$$T_L(x) = \frac{2}{d_2 + T_L^2(x)} \ .$$

Substituting this in 3.1 yields

$$x = \cfrac{2}{d_1 + \cfrac{2}{d_2 + T_L^2(x)}} \ .$$

In general, if $T_L^k(x) \neq 0$ for $0 \leq k < n$, we can write

$$x = \cfrac{2}{d_1 + \cfrac{2}{d_2 + \cfrac{\ddots}{d_n + T_L^n(x)}}} \ .$$

The numbers d_n are called the *digits* (or: partial quotients) of x.
We call

$$c_n = \frac{p_n}{q_n} = \cfrac{2}{d_1 + \cfrac{2}{d_2 + \cfrac{\ddots}{d_n}}}$$

the nth convergent of x. Occasionally we write $c_n = [2/d_1, 2/d_2, \ldots, 2/d_n]$ as a short hand notation. These convergents have the pleasant property that $\lim_{n\to\infty} c_n = x$. For the p_n's and q_n's we have the recurrence relations

$$\begin{array}{llll} p_{-1} := 1; & p_0 := 0; & p_n = d_n p_{n-1} + 2p_{n-2}, & n \geq 1, \\ q_{-1} := 0; & q_0 := 1; & q_n = d_n q_{n-1} + 2q_{n-2}, & n \geq 1. \end{array}$$

A proof of convergence can be found in Chapter 6 and, while proving the convergence, the recurrence formulas are obtained together with a bound on the convergents. In Chapter 6 such a proof is given for a larger class of maps of which these 2-expansions are a subfamily. Now a bound for the convergents of 2-expansions is already known and is given by

$$|x - c_n| < \frac{2^n}{q_n^2} \ .$$

Note that $\lim_{n\to\infty} \frac{2^n}{q_n^2} = 0$ (see also [18]). This can be derived from the above recurrence relation for q_n. We see that if q_n grows fast as $n \to \infty$ we will find better convergents. Now whenever $L = \emptyset$ we will always choose the largest digits possible resulting in the best convergence and whenever $L = [0,1]$ we will always choose the lowest digits possible resulting in the worst convergence. Because convergence is the fastest for $L = \emptyset$ we call the corresponding map the *greedy* 2-expansion map. The map with $L = [0,1]$ is called the *lazy* 2-expansion map. Note that we can only choose L for which $L \cap [1,2] = \emptyset$ since the digit will be 0 otherwise. This will result in dividing by 0 in the convergents. In Figure 3.1 the dotted lines give the lazy map and the solid lines indicate the greedy map.

Both for the greedy map and the lazy map the invariant measure is known (see [5]). Also their natural extension is known in both cases. For the definition of an invariant measure see Appendix A.1.2 and for the definition of the natural extension see Appendix A.1.4.

For the greedy 2-expansion it follows from [5] that the invariant measure is given by:

$$\mu(A) = \frac{1}{\ln(\frac{3}{2})} \int_{A\cap[0,1]} \frac{1}{2+x}\,\mathrm{d}x \ ,$$

where $A \subset [0,1]$ is a Borel set (for an explanation on Borel sets see Appendix A.1.2). The natural extension map $T_\emptyset : [0,2) \times [0,2) \to [0,2) \times [0,2)$ defined as

$$T_\emptyset(x,y) = \left(T_\emptyset(x), \frac{2}{d(x)+y}\right)$$

has as invariant measure μ, given by

$$\mu(A) = \frac{1}{\ln(\frac{3}{2})} \iint_A \frac{2}{(2+xy)^2}\,\mathrm{d}x\,\mathrm{d}y \ ,$$

where $A \subset [0,1] \times [0,1]$ is a Borel set. For the lazy 2-expansion we have the invariant measure:

$$\mu(A) = \frac{1}{2\ln(\frac{4}{3})} \left(\int_{A\cap[0,1]} \frac{1}{x^2+3x+2}\,\mathrm{d}x + \int_{A\cap[1,2]} \frac{1}{2+x}\,\mathrm{d}x \right) ,$$

again where $A \subset [0,1] \times [0,1]$ is a Borel set. In general, it is hard to find an invariant measure. Still these two invariant measures are obtained. In the next section we will give a strategy to find the invariant measure in some cases. This strategy is based on a strategy explained in [5]. The invariant measure of the lazy and the greedy 2-expansion could have been obtained using this method. The odd 2-expansion is used to illustrate this strategy.

3.1 Finding the invariant measure for the odd 2-expansion

For the odd 2-expansion we will lower the digit whenever its even so we take

$$L = \bigcup_{n=1}^{\infty} \left(\frac{2}{2n+1}, \frac{2}{2n} \right] .$$

This yields that for all $x \in (0,1)$ the digits $d_n(x)$ will be odd (hence the name). Note that this map is called a regular 2-expansion in [5].

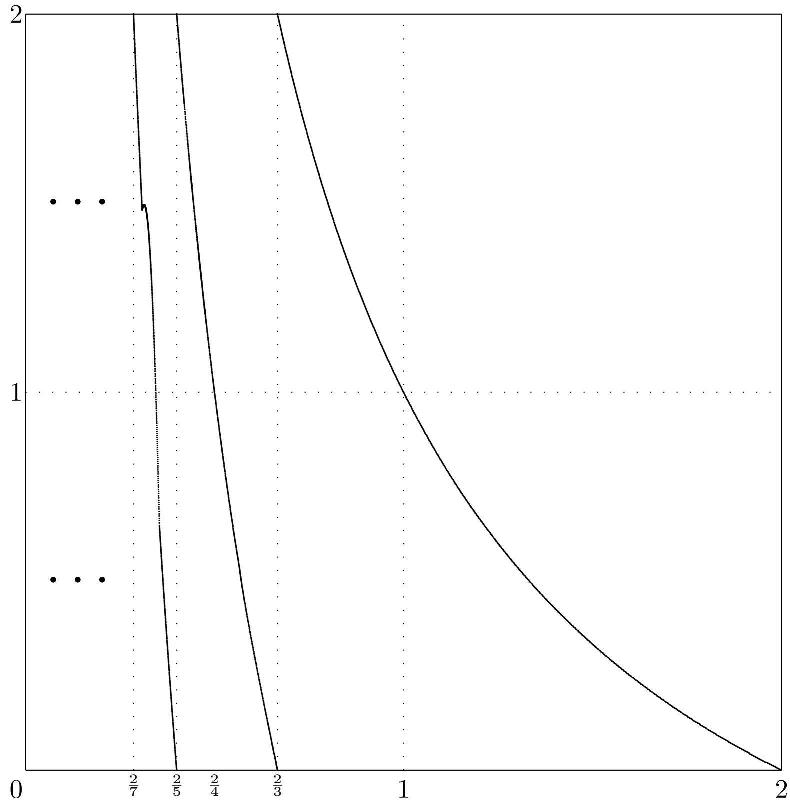

Figure 3.2: The odd 2-expansion.

Now the natural extension is known (see [5]) and the map $\mathcal{T}_L : [0,2) \times [0,2) \to [0,2) \times [0,2)$ is given by

$$\mathcal{T}_L(x,y) = \left(T_L(x), \frac{2}{d(x)+y} \right) .$$

We have that $T_L^n(x,0) = (t_n, v_n)$, where $t_n = T_L^n(x) = [2/d_{n+1}, 2/d_{n+2}, \ldots]$, and $v_n = [2/d_n, 2/d_{n-1}, \ldots, 2/d_1]$. In some sense the t_n gives all the information about the future at 'time' n while v_n captures the past.

We define the fundamental intervals $\Delta_i = \{(x,y) \in [0,2) \times [0,2) : d(x) = i\}$ for $i \in \mathbb{N}$. Of course, the odd 2-expansion only have the fundamental intervals with i odd. In Figure 3.3 we can see how $\mathcal{T}$ maps these fundamental intervals.

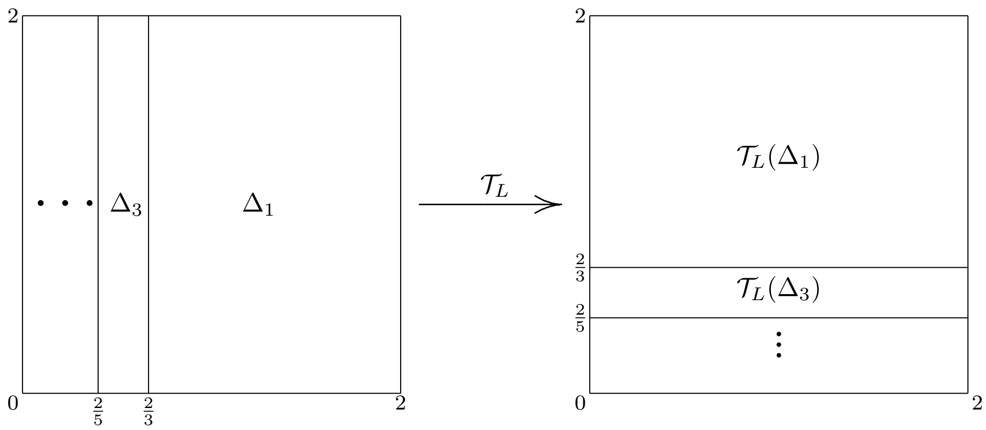

Figure 3.3: Natural extension map of the odd 2-expansion.

The invariant measure of the natural extension is also known (see [5]) and is given by

$$\mu(A) = C \iint_A \frac{2}{(2+xy)^2} \,\mathrm{d}x\,\mathrm{d}y \ ,$$

where

$$C^{-1} = \int_0^2 \int_0^2 \frac{2}{(2+xy)^2} \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\ln(3)}$$

is the normalizing constant, i.e. $\mu([0,2]\times[0,2]) = 1$. Now to find the invariant measure for $T_L(x)$ we simply integrate over y.
We find

$$\mu_x(A) = C \int_A \int_0^2 \frac{2}{(2+xy)^2} \,\mathrm{d}y\,\mathrm{d}x = C \int_A \frac{1}{x+1} \,\mathrm{d}x$$

for any $A \subset [0,2]$ measurable.

3.2 Simulating the density

In every section where a continued fraction map with $N = 2$ is discussed, a simulation of the density of its invariant measure is given. Whenever we have a analytic formula for the density of the invariant measure, the graph of this density will also be plotted. Not only will this show the strength of our simulations, but also confirms we did not make any calculation mistakes when obtaining the invariant measure. In Figure 3.4 we see a simulation of the odd 2-expansion together with the analytic formula $\frac{1}{\ln(3)} \frac{1}{1+x}$. Indeed the simulation is very close to the theoretic density. In Appendix B we will elaborate more on how we perform the simulations.

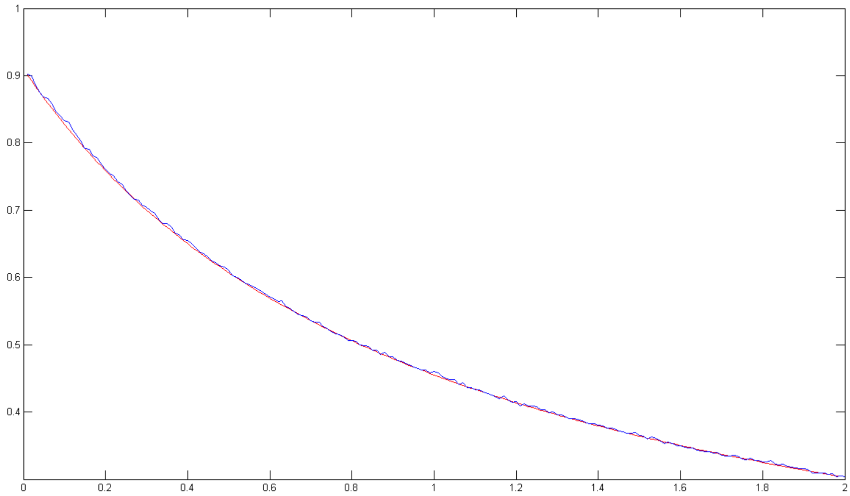

Figure 3.4: The invariant density of the (odd) regular 2-expansion with a simulation.

We have seen that we can make an expansion with odd digits only by using the 2-expansion and the right choice for L. Note that we can not make an even expansion because we have $L \cap [1, 2] = \emptyset$ so almost all $x \in (0, 2)$ will have digits equal to 1. Also note that there was only one option for L to make the expansions have only odd digits (up to a set with Lebesgue measure zero). In the next chapter we will introduce flipped 2-expansions. With a flipped 2-expansion you can make expansions for which for all $x \in (0, 2)$ all digits are odd or are even. Their invariant measure will be obtained.

Chapter 4

Flipped expansions

The flipped continued fraction expansion is extensively studied in [3, 11]. It is an expansion derived from the regular continued fraction map. By giving a set $F \subset [0,1)$ the flipped expansion can be obtained by flipping the regular map on the set F around the line $y = \frac{1}{2}$. In [11] it is shown that many of well known continued fraction expansions are flipped expansions with the right choice of F. Examples are: the odd or even continued fraction expansions, the backward continued fraction expansion, the folded α-continued fraction expansion and an expansion omitting one or more particular digits. Furthermore the metric and ergodic properties have been studied in [11].

In this thesis we adapt the idea of 'flipped expansions' to the case of 2-expansions. Instead of the regular continued fraction expansion we shall take the greedy 2-expansion as our 'starting point'. Though, since you will only be in $[1,2]$ for at most one step we restrict the domain to $[0,1)$. Let $F \subset [0,1]$ be a Borel measurable set. The map $T_F : [0,1) \to [0,1)$ is given by

$$T_F(x) = \begin{cases} \dfrac{2}{x} - \left\lfloor \dfrac{2}{x} \right\rfloor & \text{for } x \notin F \\[2ex] \left(1 + \left\lfloor \dfrac{2}{x} \right\rfloor\right) - \dfrac{2}{x} & \text{for } x \in F \ . \end{cases}$$

Setting

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x \notin F \\ -1 & \text{for } x \in F \end{cases} \quad \text{and} \quad d(x) = \begin{cases} \lfloor \frac{2}{x} \rfloor & \text{for } x \notin F \\ (1 + \lfloor \frac{2}{x} \rfloor) & \text{for } x \in F \ , \end{cases}$$

we have that

$$T_F(x) = \frac{2\varepsilon(x)}{x} - \varepsilon(x) d(x) \ .$$

Now let $d_1 = d(x)$ and $d_n = d_n(x) = d(T_F^{n-1}(x))$ as well as $\varepsilon_1 = \varepsilon(x)$ and $\varepsilon_n(x) = \varepsilon(T_F^{n-1}(x))$, whenever $T_F^{n-1}(x) \neq 0$. We find a continued fraction expansion

$$x = \cfrac{2}{d_1 + \cfrac{2\varepsilon_1}{d_2 + \cfrac{\ddots}{\ddots + \cfrac{2\varepsilon_{n-1}}{d_n + \varepsilon_n T_F^n(x)}}}}.$$

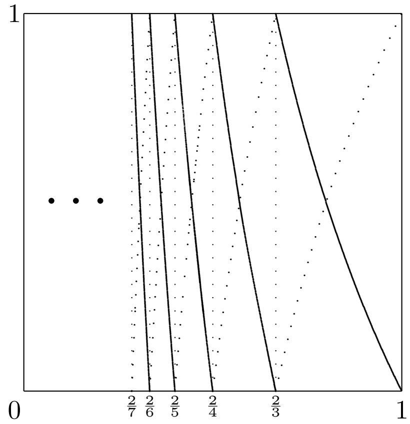

Figure 4.1: The greedy (solid lines) and the flipped expansions (dotted lines).

As usual, taking finite truncations yield the convergents

$$c_n = \cfrac{2}{d_1 + \cfrac{2\varepsilon_1}{d_2 + \ddots \cfrac{2\varepsilon_{n-1}}{d_n}}}.$$

A proof of convergence of c_n to x is given in Chapter 6. In fact in Chapter 6 the convergence of a family of expansions is given of which the flipped expansions are examples of. In the next sections we will study the odd flipped expansion and the even flipped expansion. We will slightly change the method of Section 3.1 to find the invariant measures of both. Also we will obtain the invariant measure of odd flipped expansions with $N > 2$ and also find the invariant measure of flipped expansions with $N > 2$ with only even digits.

4.1 Odd flipped expansion

To make an odd flipped expansion we have to flip every interval where the digit would be even. This gives $F = \bigcup_{n=1}^{\infty} \left(\frac{2}{2n+1}, \frac{2}{2n}\right]$.

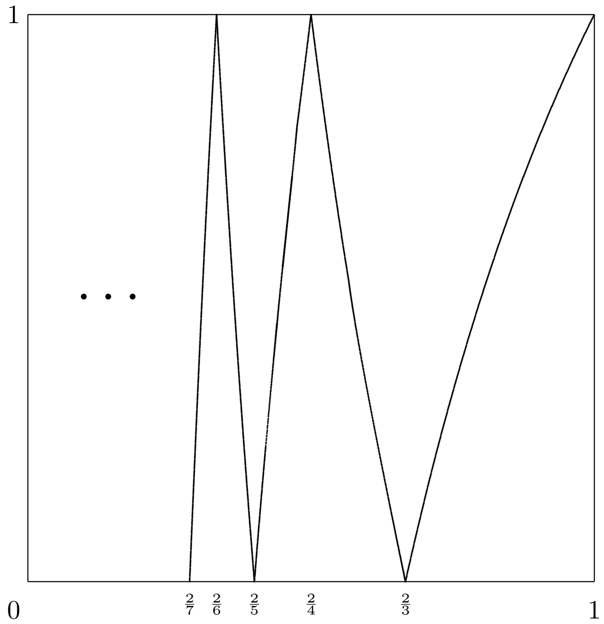

Figure 4.2: The odd flipped expansion.

Now we want to obtain the natural extension to be able to find the invariant measure of the odd flipped expansion. In order to find the natural extension let $\Omega = [0,1] \times [A,B]$. We will use the following natural extension map $\mathcal{T}_F : \Omega \to \Omega$, given by

$$\mathcal{T}_F(x,y) = \left(T_F(x), \frac{2\varepsilon(x)}{d(x)+y} \right) ,$$

where we choose $[A,B]$ in such a way that the map $\mathcal{T}_F$ is bijective on Ω (up to a set of Lebesgue measure 0). We use this map because we know that, up to a normalising constant, $\frac{2}{(2+xy)^2}$ is the density of the invariant measure.
Now we define the fundamental intervals

$$\Delta_{(a,b)} := \{(x,y) \in \Omega : d_1(x) = a, \varepsilon_1(x) = b\}$$

with short hand notation $\Delta_{-a} := \Delta_{(a,-1)}$ and, like in Section 3.1, $\Delta_a := \Delta_{(a,1)}$. Note that on $\Delta_{(a,b)}$ the map $\mathcal{T}_F$ is injective. Thus we need to pick A and B in such a way the fundamental intervals do not overlap and fill Ω (up to a set of Lebesgue measure 0).

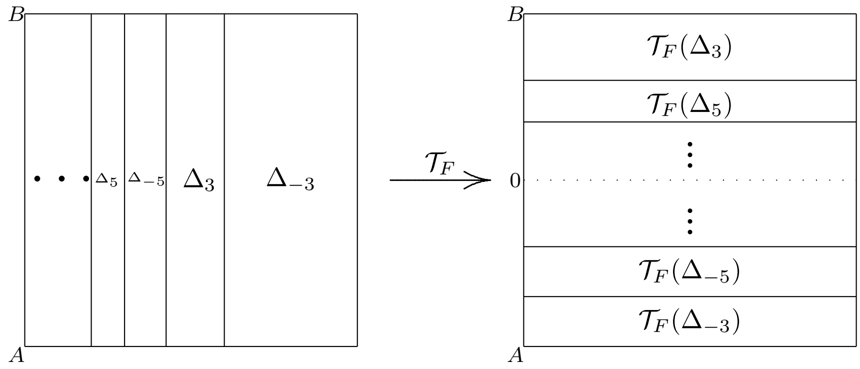

Figure 4.3: Ω and $\mathcal{T}_F(\Omega)$.

We see that, if we want the image of the rectangles to fit nicely and not leave holes or overlap, we need the equality $A = -\frac{2}{3+A}$ and $B = \frac{2}{3+A}$ which implies:

$$A^2 + 3A + 2 = 0 \tag{4.1}$$

So $A = -1$ or $A = -2$. Inspection shows that we need $A = -1$ which yields $B = 1$ to make our map injective. Indeed we see that the rectangles nicely fit and do not overlap so that gives us injectivity. If we would make A larger and/or B smaller we would get 'holes'. When applying $\mathcal{T}_F$ iteratively more and more 'holes' will appear. Making the interval larger will give you overlap (which is forbidden by the definition of the natural extension). The map is (almost surely) surjective because

$$\lim_{n\to\infty} \frac{2\varepsilon(x)}{n+y} = 0$$

for both $\varepsilon(x) = 1$ and $\varepsilon(x) = -1$.
Note that the line $(x, 0)$ is not in the image of our map (but has Lebesgue measure 0). Now to find the invariant measure of $T_F(x)$ we need to integrate

$$\int_A^B \frac{2}{(2+xy)^2}\,\mathrm{d}y = \int_{-1}^1 \frac{2}{(2+xy)^2}\,\mathrm{d}y = \frac{1}{2+x} + \frac{1}{2-x}$$

and determine the constant C such that we get a density on $[0, 1]$ again

$$C = \int_0^1 \frac{1}{2+x} + \frac{1}{2-x}\,\mathrm{d}x = \ln 3\ ,$$

so the density is now given by

$$f(x) = \frac{1}{\ln 3}\left(\frac{1}{2+x} + \frac{1}{2-x}\right)\ . \tag{4.2}$$

This integration is a projection on the x coordinate and therefor measure preservant. Simulation shows that the method works (and the method shows the simulation is rather accurate see Figure 4.4).

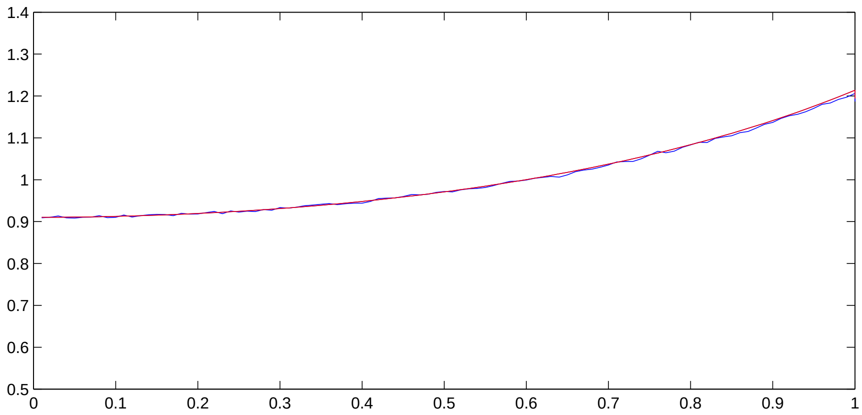

Figure 4.4: The density for the odd flipped expansion together with a simulation of the density.

4.1.1 The general odd case for $N \geq 2$

Note that we could find the invariant measure of any odd flipped N-expansion (starting from the greedy N-expansion) for an *even* N in a completely similar way. For A you would get the equation $A = -\frac{N}{N+1+A}$ and instead of equation (4.1) we would get

$$A^2 + (N+1)A + N = 0 \tag{4.3}$$

giving $A = -1$ or $A = -N$.
To make the domain in such a way the natural extension is injective we always need $A = -1$ and so $B = 1$. Resulting in the following densities:

$$f_N(x) = \frac{1}{\ln(\frac{N+1}{N-1})} \left(\frac{1}{N+x} + \frac{1}{N-x} \right) \tag{4.4}$$

Which is almost the same as in (4.2) but then there is an N instead of a 2.
For an odd flipped N-expansion with $N \in \mathbb{N}$ *odd* we find a slightly different equation for A, namely

$$A^2 + (N+2)A + N = 0 \tag{4.5}$$

giving $A = \frac{1}{2}(-(N+2) + \sqrt{N^2+4})$ or $A = \frac{1}{2}(-(N+2) - \sqrt{N^2+4})$. This time, in order to make our extension injective,
we need

$$A = \frac{1}{2}(-(N+2) + \sqrt{N^2+4})$$

giving

$$B = \frac{2N}{\sqrt{N^2+4} + N - 2}\ .$$

This gives us the following densities:

$$f_N(x) = C_N \left(\frac{2}{\sqrt{N^2+4}+N-2+2x} - \frac{\sqrt{N^2+4}-(N+2)}{2N+(\sqrt{N^2+4}-(N+2))x} \right) \tag{4.6}$$

where

$$\begin{aligned} C_N^{-1} &= 2\ln\left(\frac{\sqrt{N^2+4}+N}{\sqrt{N^2+4}+N-2}\right) \\ &\quad - \left(\sqrt{N^2+4}-(N+2)\right)\ln\left(\frac{N-2+\sqrt{N^2+4}}{2N}\right) . \end{aligned}$$

We could calculate all kind of things with these invariant measures (such as the percentage of a certain digit occurring in a continued fraction for almost all $x \in [0,1]$ wrt. the Lebesgue measure or constants of Khinchine). But this is beyond the scope of this thesis where we do not want to deviate too much from the 2-expansions.

4.2 Even flipped expansions

For the even flipped expansion we do the opposite as we did for the odd flipped expansion. Starting from the greedy expansion, by 'flipping', we will make the digit even on each interval it would be odd. This is achieved by setting $F = \bigcup_{n=1}^{\infty} \left(\frac{2}{2n+2}, \frac{2}{2n+1} \right]$. The graph of T_F is given in Figure 4.5.

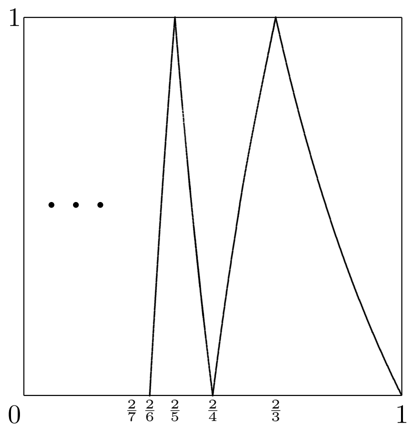

Figure 4.5: The even flipped expansion.

Again, using the method from in Section 3.1 to find the domain of the natural extension we can find the invariant measure. This time A is a solution of the polynomial

$$A^2 + 4A + 2 = 0$$

which has the solutions $A = \sqrt{2} - 2$ and $A = -\sqrt{2} - 2$. It turns out $A = \sqrt{2} - 2$ is the right one to pick giving $B = \sqrt{2}$. We find the following density:

$$f(x) = \frac{1}{\ln(\sqrt{2}+1)} \left(\frac{1}{\sqrt{2}+x} - \frac{\sqrt{2}-2}{2+(\sqrt{2}-2)x} \right)$$

A simulation of the density of the invariant measure is shown in Figure 4.6 as well as the density of the theoretic measure.

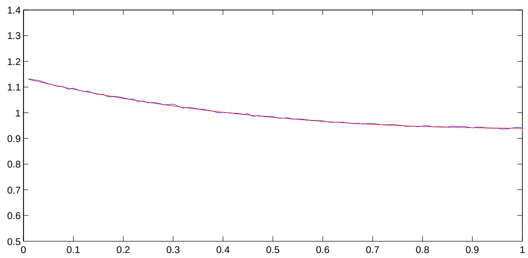

Figure 4.6: A simulation of the invariant measure for the even flipped expansion together with the exact density of this expansion.

4.2.1 The general even case for $N \geq 2$

Just as in previous chapter, we can find the invariant measure of any flipped N-expansion with only even digits. For N even we get the same equation for A as in the case of a flipped N-expansion with N odd with only odd digits (namely equation (4.5)). Leading to the densities (4.6) but now with N even. Likewise we find that if we want an N-expansion with only even digits and N is odd we find that the equation for A is given in (4.3) giving the invariant measure as in (4.4).

The map that is used in [16] to make an even expansion is not a flipped 2-expansion but a flipped expansion. In contrast to our map, the map in [16] leads to a σ-finite infinite measure for only even digits.

Note that (4.4) is an indication of this where the constant is undefined for $N = 1$ because we would have $\frac{1}{\ln(\frac{2}{0})}$ and therefore a division by 0. We also have that $\frac{1}{1-x} \to \infty$ as $x \to 1$. This formula is also given in [16] but without the constant.

4.3 Other possibilities

In Section 4.1 we started with the greedy 2-expansion map and flipped it on a set F in the line $y = \frac{1}{2}$. Of course, if we would have started with the lazy 2-expansion map and we would have flipped on a suitable set F in $y = 1\frac{1}{2}$ we would have obtained other odd or even expansions. In the next chapter we show that these and other expansions can be obtained by combining the flipped expansions and 2-expansions which we discussed.

Chapter 5

Combining the flipped and 2-expansions

To make things more interesting we will combine the two expansions given in previous chapters. Starting from the greedy 2-expansion, for the flipped expansions there is only one choice for F to make an expansion with only odd or even digits. The same holds for the 2-expansions. When combining the two, we will find infinitely many choices for L and F given some conditions. This will be discussed in Chapter 9 and 10. Besides these expansions, a whole range of different choices for L and F are studied. Before we can do anything we define our map $T_{L,F}$ with $L \subset [0,1]$ and $F \subset [0,2]$ both measurable. Let $T_{L,F} : [0,2] \to [0,2]$ be defined by

$$T_{L,F}(x) = \frac{2\varepsilon(x)}{x} - \varepsilon(x)d(x) \quad \text{if } x \neq 0 \text{ and } T_{L,F}(0) = 0 \ ,$$

where

$$\varepsilon(x) = \begin{cases} 1 & \text{for } x \notin F \\ -1 & \text{for } x \in F \end{cases}$$

and

$$d(x) = \begin{cases} \lfloor \frac{2}{x} \rfloor - 1_L(x) & \text{for } x \notin F \\ 1 + 1_L(x) + \lfloor \frac{2}{x} \rfloor & \text{for } x \in F \ . \end{cases}$$

Furthermore, whenever $T_{L,F}^{n-1}(x) \neq 0$, let $d_n(x) = d(T_{L,F}^{n-1}(x))$ and $\varepsilon_n(x) = \varepsilon(T_{L,F}^{n-1}(x))$. Note that ε and d are different functions for different choices of L and F. When we take $L = \emptyset$ we find the flipped continued fraction maps from Chapter 4 and whenever $F = \emptyset$ we find the continued fraction maps for 2-expansions from Chapter 3. When both L and F are empty we find the greedy map.

If we write $T_{L,F}(x)$ as

$$T_{L,F}(x) = \frac{2\varepsilon_1(x)}{x} - \varepsilon_1(x)d_1(x)$$

then

$$x = \frac{2\varepsilon_1}{\varepsilon_1 d_1 + T_{L,F}(x)} = \frac{2}{d_1 + \varepsilon_1 T_{L,F}(x)}$$

and, as long as $T^i_{L,F}(x) \neq 0$ for $i = 1, 2, \ldots, n$, we find

$$x = \cfrac{2}{d_1 + \cfrac{2\varepsilon_1}{d_2 + \ddots + \cfrac{2\varepsilon_{n-1}}{d_n + \varepsilon_n T^n_{L,F}(x)}}} \; .$$

Now we define the nth convergent c_n as

$$c_n = \cfrac{2}{d_1 + \cfrac{2\varepsilon_1}{d_2 + \ddots \cfrac{2\varepsilon_{n-1}}{d_n}}} \; . \tag{5.1}$$

In Chapter 6 we will show that $\lim_{n\to\infty} c_n = x$.

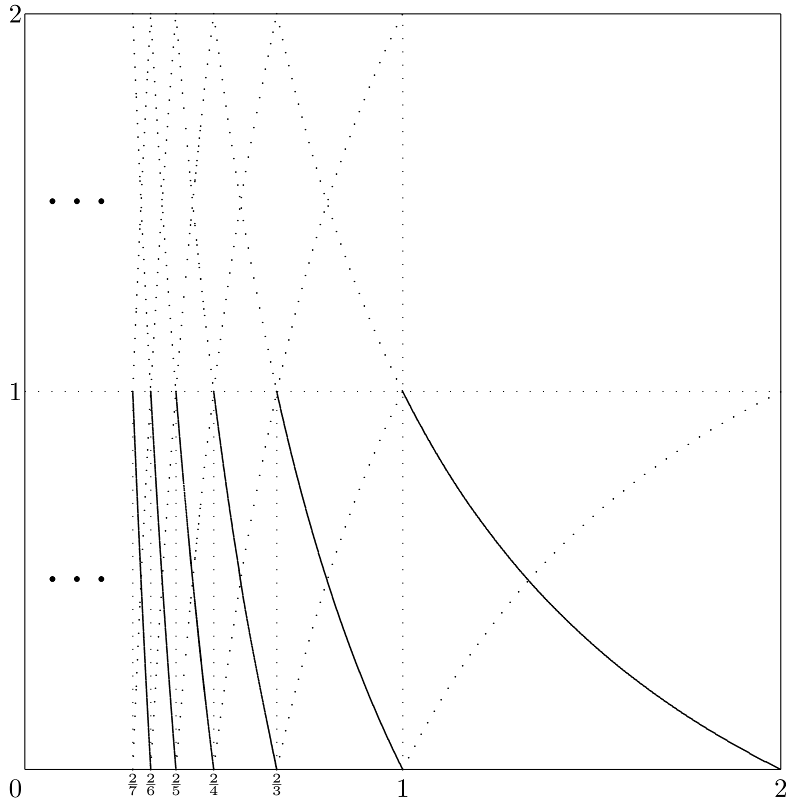

Figure 5.1: The flipped 2-expansions.

This allows us to write

$$x = \cfrac{2}{d_1 + \cfrac{2\varepsilon_1}{d_2 + \ddots}} \; . \tag{5.2}$$

However it only converges if L and F comply with the following condition:

$$T_{L,F}(F \cap [\tfrac{3}{2}, 2]) \cap (L \cap [\tfrac{2}{3}, 1]) = \emptyset \tag{5.3}$$

This condition assures that whenever $d_{n-1}(x) = 2$ and $\varepsilon_{n-1}(x) = -1$ we do not have $d_n(x) = 1$ and we would not divide by 0 in the n^{th} convergent. This can only happen if $T_{L,F}^{n-1}(x) \in [\frac{3}{2}, 2]$ and $x \in F$ and $x \in L \cap [\frac{2}{3}, 1]$ which is translated into this condition.
This seems a rather complicated restriction but luckily it does not restrict us in making expansions with only odd or even digits. In the next Subsection we will see a map that is not included whenever $L \subset [0, 1]$.

5.0.1 Another map

Remember that we took $L \subset [0, 1]$ because otherwise we would have digit 0. However, now that we combined it with flipped expansions, we can look at maps for which $L \cap [0, 2] \neq \emptyset$ but with $L \cap [0, 2] = F \cap [0, 2]$ which would be the map $f(x) = 3 - \frac{2}{x}$ and will result in a digit of 3. We will see that this map gives some obsticals.

Note that if we pick this map flipped on $[1, 2]$ for every $x \in [1, 2]$ then $[1, 2]$ is invariant under $T_{L,F}$ resulting in non-uniqueness of the continued fractions and in non-ergodic maps. A remedy for this is to demand that there is a δ such that

$$L \cap [2 - \delta, 2] = \emptyset \tag{5.4}$$

This ensures that you can not stay in $[1, 2]$ forever (see Figure 5.2).

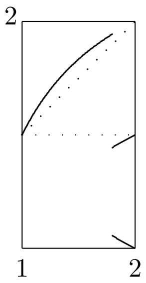

Figure 5.2: A hole to escape $[1, 2]$.

This solves the problem of non-uniqueness but since map makes a proof of convergence hard (or impossible) this map will never be chosen and we will always use $L \subset [0, 1]$ in this thesis.
In the next chapter we will discuss whether rational numbers have a finite expansion. This is a property that holds for the regular continued fraction map (see Appendix A.1, page 96). It turns out that this is dependent on the choice of L and F.

5.1 Do rational numbers have a finite expansion?

This question is not that trivial anymore for our new family of continued fraction maps as it is for the regular continued fraction. In fact, we already seen in Chapter 3 that for every $x \in (0,2)$ we have infinitely many 2-expansions. In [1] it is stated that every rational number has finite and infinite 2-expansions. This gives us reason to think depends on the choice of L and F whether a rational number will have a finite expansion or not. We give some sets for which we can prove that either all elements of the sets have a finite expansion or all the elements of that set have an infinite expansion. Though which rational numbers are in the particular set will depend on the choice of L and F. Let

$$\begin{aligned}
A &= \{q \in \mathbb{Q} \cap [0,2] : T^n_{L,F}(q) \in [0,1) \text{ for all } n \geq 1\} , \\
B &= \{q \in \mathbb{Q} \cap [0,2] : \text{there is an } m \text{ such that } T^n_{L,F}(q) \in [0,1) \text{ for all } n \geq m\} , \\
C &= \{q \in \mathbb{Q} \cap [0,2] : T^n_{L,F}(q) \notin F \cap [1,2] \text{ and } T^n_{L,F}(q) \neq 1 \text{ for all } n \geq 1\} , \\
D &= \{q \in \mathbb{Q} \cap [0,2] : \text{there is an } n \text{ such that } T^n_{L,F}(q) = 1\} .
\end{aligned}$$

Now let $q = \frac{r_0}{s_0}$ with r_0 and s_0 relatively prime and $T^n_{L,F}(q) = \frac{r_n}{s_n}$ with r_n and s_n relatively prime. Then we have the following equation

$$T_{L,F}\left(\frac{r_n}{s_n}\right) = \frac{2\varepsilon(\frac{r_n}{s_n})s_n - \varepsilon(\frac{r_n}{s_n})d(\frac{r_n}{s_n})r_n}{r_n} = \frac{r_{n+1}}{s_{n+1}} .$$

Note that a continued fraction expansion of q is finite if and only if there is an n such that $T^n_{L,F}(q) = 0$. We will first prove that for all measurable sets L and F all elements in A have a finite continued fraction expansion. Let $q \in A$ and suppose q has an infinite expansion. We have that

$$r_{n+1} \leq 2\varepsilon(\tfrac{r_n}{s_n})s_n - \varepsilon(\tfrac{r_n}{s_n})d(\tfrac{r_n}{s_n})r_n < r_n .$$

If $T_{L,F}(q) \neq 0$ for all $n \geq 1$, this gives us an infinite decreasing sequence $\ldots < r_{n+1} < r_n < \ldots < r_1 < r_0$ which is impossible so q has a finite expansion. The equation also gives us an upper bound on the number of iterates since $r_{n+1} < r_n$ we can have at most r_0 digits. Now let $q \in B$ then we have $r_{n+1} < r_n$ and we find the decreasing sequence $\ldots < r_{m+1} < r_m$ which gives us that q has at most $r_m + m$ digits. For $q \in C$ it will be a little trickier. In case $T_{L,F}(\frac{r_n}{s_n}) \in [0,1)$ then we have $r_{n+1} < r_n$ and in case $T_{L,F}(\frac{r_n}{s_n}) \in (1,2]$ we have

$$T_{L,F}(\frac{r_{n+1}}{s_{n+1}}) = \frac{2s_{n+1} - r_{n+1}}{r_{n+1}} = \frac{r_{n+2}}{s_{n+2}} .$$

Now $r_{n+1} > s_{n+1}$ which gives us $r_{n+2} \leq 2s_{n+1} - r_{n+1} < s_{n+1} \leq r_n$ so we find that either $r_{n+1} < r_n$ or $r_{n+2} < r_n$. Suppose q has an infinite expansion. We can now give a strictly decreasing subsequence which is impossible for positive integers and thus by contradiction we find that q has a finite expansion with at most $2r_0$ digits. Whenever $q \in D$ it depends on whether $1, 2$ are in L and or F or not. Suppose $1 \notin$ L and $1 \notin F$, then $T^{n+1}_{L,F}(q) = 0$ and q has a finite expansion. Suppose $1 \notin L$ and $1 \in F$, then $T^{n+1}_{L,F}(q) = 1$ and we find an infinite expansion. In case $1 \in L$ and $1 \notin F$, then $T_{L,F}^{n+1}(q) = 1$ and we find an infinite expansion again. Now for $1 \in L$ and $1 \in F$ then $T_{L,F}^{n+1}(q) = 2$ and we need to look at the value of $T_{L,F}(2)$. Suppose $2 \in F$ then we find an infinite expansion and if $2 \notin F$ we find a finite expansion for q.

We will see that the sets A, B, C, D are very different for different choices of L and F. For example in Section 8.1 we have that $C = [0, 2] \cap \mathbb{Q}$ and therefore all rationals have a finite expansion and in Section 11.4 we have that $D = [\frac{1}{2}, 1] \cap \mathbb{Q}$ and $T_{L,F}(1) = 1$ so we find that every $q \in [\frac{1}{2}, 1] \cap \mathbb{Q}$ has an infinite expansion.

In the following chapter we will prove the convergence of (5.2) (see page 28) for general choices of L and F using Möbius transformations. Recurrence relations are also obtained which will be of interest for the odd and even expansions. The proof is slightly more complicated then the proof of convergence for the regular continued fraction. For the standard continued fraction such proof can be found in [4] and in Appendix A.1.1. A proof of the statement that a number has a finite standard continued fraction expansion if and only if its rational can also be found in the book.

Chapter 6

A proof of convergence

In this chapter we will prove the convergence of the sequence of convergents c_n of any number $x \in (0, 2]$ using a map $T_{L,F}$ for those measurable sets L and F which are allowed and not restricted by (5.3).
Let $c_n = \frac{p_n}{q_n}$ be the truncated expansion of x having n digits (as in equation (5.1) page 28).In this chapter we prove $\lim_{n\to\infty} c_n = x$.

Essential in the proof are 2×2 matrices from $\mathcal{N}$, where

$$\mathcal{N} = \{A : \det(A) = \pm 2 \text{ and all entries are integers}\}$$

and their related Möbius transformation. Recurrence formulas are found in the process. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{N} \ .$$

The Möbius transformation, induced by A, is the map $A : \mathbb{C}^* \to \mathbb{C}^*$, given by

$$A \cdot x := \frac{ax + b}{cx + d} \ .$$

Furthermore we define for $n \geq 1$

$$A_n = \begin{bmatrix} 0 & 2\varepsilon_{n-1} \\ 1 & d_n \end{bmatrix}$$

where $\varepsilon_0 = 1$ and we set

$$M_n = A_1 A_2 \cdots A_n \text{ with } \quad M_0 = I_{2\times 2} \ .$$

Evaluating $M_n(x)$ in $x=0$ yields that

$$\begin{aligned}
M_n(0) &= (M_{n-1}A_n)(0) = M_{n-1}\left(\frac{2\varepsilon_{n-1}}{d_n}\right) \\
&= M_{n-2}A_{n-1}(\frac{2\varepsilon_{n-1}}{d_n}) = M_{n-2}\left(\frac{2\varepsilon_{n-2}}{d_{n-1}+\dfrac{2\varepsilon_{n-1}}{d_n}}\right) \\
&\vdots \\
&= c_n
\end{aligned}$$

Writing M_n as

$$M_n = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}, \text{ with } r_n, s_n, p_n, q_n \in \mathbb{Z}\,,$$

we find

$$\begin{aligned}
M_n &= M_{n-1}A_n = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix}\begin{bmatrix} 0 & 2\varepsilon_{n-1} \\ 1 & d_n \end{bmatrix} \\
&= \begin{bmatrix} p_{n-1} & d_n p_{n-1} + 2\varepsilon_{n-1} r_{n-1} \\ q_{n-1} & d_n q_{n-1} + 2\varepsilon_{n-1} s_{n-1} \end{bmatrix}
\end{aligned}$$

giving $r_n = p_{n-1}$ and $s_n = q_{n-1}$.
The recurrence relations are now found:

$$\begin{array}{llll}
p_{-1} := 1; & p_0 := 0; & p_n = d_n p_{n-1} + 2\varepsilon_{n-1} p_{n-2}, & n \geq 1, \\
q_{-1} := 0; & q_0 := 1; & q_n = d_n q_{n-1} + 2\varepsilon_{n-1} q_{n-2}, & n \geq 1.
\end{array}$$

Let $v_n = \frac{2q_{n-1}}{q_n}$. By using the recurrence relations this gives

$$v_n = [2/d_n, 2\varepsilon_{n-1}/d_{n-1}, \ldots, \varepsilon_1/d_1]\ .$$

Furthermore, let $t_n = T^n_{L,F}(x)$ which gives $t_n = [\varepsilon_n/d_{n+1}, \varepsilon_{n+1}/d_{n+2}, \ldots]$.
We will now introduce the last matrix.
Let

$$A_n^* = \begin{bmatrix} 0 & 2\varepsilon_{n-1} \\ 1 & d_n + \varepsilon_n t_n \end{bmatrix}$$

which gives

$$\begin{aligned}
M_{n-1}A_n^* &= \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix}\begin{bmatrix} 0 & 2\varepsilon_{n-1} \\ 1 & d_n + \varepsilon_n t_n \end{bmatrix} \\
&= \begin{bmatrix} p_{n-1} & 2\varepsilon_{n-1}p_{n-2} + d_n p_{n-1} + p_{n-1}\varepsilon_n t_n \\ q_{n-1} & 2\varepsilon_{n-1}q_{n-2} + d_n q_{n-1} + q_{n-1}\varepsilon_n t_n \end{bmatrix} \\
&= \begin{bmatrix} p_{n-1} & p_n + p_{n-1}\varepsilon_n t_n \\ q_{n-1} & q_n + q_{n-1}\varepsilon_n t_n \end{bmatrix}.
\end{aligned}$$

We can write x as $x = M_{n-1}A_n^*(0)$ yielding

$$x = \frac{p_n + p_{n-1}t_n\varepsilon_n}{q_n + q_{n-1}t_n\varepsilon_n}\ .$$

Note that on the one hand

$$\det(M_n) = q_n p_{n-1} - p_n q_{n-1}$$

while on the other hand

$$\det(M_n) = \det(A_1 A_2 \ldots A_n) = (-1)^n \prod_{k=0}^{n-1} 2\varepsilon_k = \pm 2^n \ .$$

Finally we can look at the convergence.
Now

$$\begin{aligned} \left| x - \frac{p_n}{q_n} \right| &= \left| \frac{p_n + p_{n-1} t_n \varepsilon_n}{q_n + q_{n-1} t_n \varepsilon_n} - \frac{p_n}{q_n} \right| \\ &= \left| \frac{(q_n p_{n-1} - p_n q_{n-1}) t_n \varepsilon_n}{q_n (q_n + q_{n-1} t_n \varepsilon_n)} \right| \\ &= \left| \frac{\pm 2^n t_n \varepsilon_n}{q_n (q_n + q_{n-1} t_n \varepsilon_n)} \right| \\ &= \frac{2^n}{q_n^2} \frac{2t_n}{2 + \varepsilon_n t_n v_n} \ . \end{aligned}$$

We will use the continued fraction of $x \in [0, 2]$ for which

$$\frac{2^n}{q_n^2} \tag{6.1}$$

converges the slowest so that for all $y \in [0, 2]$ we have that (6.1) converges to 0. This gives a lower bound for q_n. After that we will use another continued fraction to give a worst upper bound for

$$\frac{2t_n}{2 + \varepsilon_n t_n v_n} \ . \tag{6.2}$$

Then we will combine the two results and see that $\frac{2^n}{q_n^2} \frac{2t_n}{2+\varepsilon_n t_n v_n} \to 0$ as $n \to \infty$

The continued fraction with the slowest increase in q_n is

$$x = \cfrac{2}{1 + \cfrac{2}{4 - \cfrac{2}{1 + \ddots}}} = 4 - 2\sqrt{2} \ .$$

Clearly the number with the slowest increase in q_n would have $T^n_{L,F}(x) \in [\frac{2}{3}, 2]$ since here the lowest digits are obtained. Furthermore it has to be periodic with period at most 2. This gives us few candidates and this one is found by inspection. We can write the following recurrence relations for x:

$$\begin{aligned} q_n &= 4q_{n-1} + 2q_{n-2}, \quad n \text{ even}, \\ q_n &= q_{n-1} - 2q_{n-2}, \quad n \text{ odd} \end{aligned}$$

with $q_{-1} = 0$ and $q_0 = 1$. First we will prove that for all $n \in \mathbb{N}$ we have $q_n \geq 1$. This is done by induction. Clearly $q_1 = 1 \geq 1$ and $q_2 = 6 \geq 1$. Suppose $q_k \geq 1$ for all $k \leq n$. Suppose that $n + 1$ is even. We find

$$q_{n+1} = 4q_n + 2q_{n-1} \geq 1 \ .$$

Now suppose $n+1 \geq$ is odd. Then we have

$$q_{n+1} = q_n - 2q_{n-1} = 4q_{n-1} + 2q_{n-2} - 2q_{n-1} = 2q_{n-1} + 2q_{n-2} \geq 1$$

We find that for all $n \in \mathbb{N}$ we have $q_n \geq 1$. This gives us the following inequalities:

$$q_n \geq 4q_{n-1} \text{ for } n \text{ even}$$

and

$$q_n \geq 2q_{n-3} \text{ for } n \text{ odd.}$$

By combining these 2 we find $q_n \geq 8q_{n-4}$ for n even which will give us convergence. Let $n \geq 4$ be even. Then we can write $n = 4j + k$ with $k \in \{0, 2\}$. We find

$$\frac{2^n}{(q_n)^2} \leq \frac{2^{4j+k}}{((2^3)^j(q_k))^2} \leq \frac{2^{4j+k}}{2^{6j}}$$

This converges to 0 as $j \to \infty$ so all even numbers give a convergent sequence. Now we also have that $q_n \geq 2q_{n-3}$ for all odd n so all the odd numbers give a convergent sequence as well. We find that

$$\frac{2^n}{(q_n)^2} \to 0 \text{ as } n \to \infty$$

Since this is the slowest converging sequence for all numbers we find that for all $x \in (0, 2]$ we have convergence at least this fast.

For (6.2) we will not find a fixed number as upper bound but we will see that it does not grow fast enough to beat the convergence speed of (6.1). The number which has the worst upper bound is the number with all epsilons negative with the lowest digits so for $x = 2$ with expansion $[-2, -4, -2, -4, \ldots]$. Note that for this x we have $t_n = 1$ for n is odd and $t_n = 2$ for n is even. Furthermore note that $v_n = c_n(1)$ for n is even and $v_n = c_n(2)$ for n is odd due to symmetry of the sequence of digits. We will first look at n is even. This gives us that we need to find an upper bound for

$$\frac{4}{2 - 2v_n} \ . \tag{6.3}$$

Now the interesting thing is that the v_n's are convergents of $x = 1$ so if it has a fast convergence we will see this fraction will grow fast and it will result in a bad convergence for $x = 2$. By induction we will see that we can give the values of v_n in terms of n explicitly. We claim $v_n = \frac{n}{n+1}$ for all n even.
For $n = 2$ we find that $v_n = \frac{2}{3}$. Fix n even. Suppose it is true for $n - 2$. To prove is that it holds also for n.
We have that

$$v_n = \frac{2}{4 - \frac{2}{2 - v_{n-2}}} = \frac{2 - v_{n-2}}{3 - 2v_{n-2}} \ .$$

By using the induction hypothesis we find

$$\frac{2 - v_{n-2}}{3 - 2v_{n-2}} = \frac{2 - \frac{n-2}{n-1}}{3 - 2\frac{n-2}{n-1}} = \frac{n}{n+1}$$

which is what we wanted to prove. Now subtituting v_n for $\frac{n}{n+1}$ in (6.3) gives us

$$\frac{4}{2-2v_n} = \frac{4}{2-\frac{2n}{n+1}} = 2n+2\ .$$

For n is odd we have

$$v_n = \frac{2}{2-v_{n-1}} = \frac{2}{2-\frac{n-1}{n}} = \frac{2n}{n+1}$$

and $t_n = 1$. Substituting both in (6.2) gives

$$\frac{2t_n}{2-t_nv_n} = \frac{2}{2-\frac{2n}{n+1}} = n+1\ .$$

Since this is the worst upper bound the bound also holds for all other $x \in (0,2]$.

Combining the two results yields

$$\frac{2^n}{q_n^2}\frac{2t_n}{2+\varepsilon_n t_n v_n} \to 0 \quad \text{as } n \to \infty$$

Now that we have convergence we would want ergodicity and the existence of an invariant measure for all L and F. However not for every choice we have ergodicity. In the next chapter we show ergodicity of practically all maps studied in this thesis. For all maps studied in this thesis it will be clear whether they are ergodic or not. Whenever they are ergodic we know that there exists an invariant measure. Some examples of non-ergodic maps in our family of continued fraction expansions will be given as well.

Chapter 7

Existence and ergodicity

When introducing a new continued fraction one should always check if the new map is ergodic and has an invariant measure. For several flipped expansions this is proved in [11]. In general you can prove the existence of an invariant measure together with ergodicity by using checking Rényi's condition (see [14]) which can be written as the following theorem:

Theorem 7.1. *If there is an M such that for all $n \in \mathbb{N}$ we have that for all cylinder sets Δ_i the following holds:*
if $x, y \in \Delta_i$ then

$$\left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| < M \,.$$

Then T is ergodic and has an invariant measure which is finite.

A cylinder set should be understood as a one-dimensional fundamental interval (so x and y are two points from the same fundamental interval). This theorem, however, is not that helpfull for checking the existence of an invariant measure and ergodicity. Instead of checking Rényi's condition, we will use Adlers (Folklore) theorem which is Rényi's condition but reformulated in a way we do not need to iterate n times T. But before we can give Adler's theorem we need to define when a measurable space is Markov.

Theorem 7.2. *A map $T : X \to X$ on the measurable space $([a, b], \mathcal{F})$ is Markov if there is an at most countable collection $\{(B(k)_{k\in\mathbb{N}}\}$ for which the following holds:*

- *T is defined on $\cup_k B(k)$ and $X \setminus \cup_k B(k)$ has measure zero*
- *$T|_{B(k)}$ is strictly monotonic and is twice differentiable on $\overline{B(k)}$ for all k*
- *For all k, j if $T(B(k)) \cap B(j) \neq \emptyset$ then $B(j) \subset T(B(k))$*
- *For all k, j there exists an R such that $B(j) \subset \cup_{n=1}^{R} T^n(B(k))$*

Note that $B(k)$ does not need to be the partition of cylinder sets even though the cylinder sets will often be used as partition. If we have a measurable space that is Markov it is easy to check whether we have an ergodic map with an invariant measure by using Adler's Theorem.

Theorem 7.3. *Let $T : X \to X$ be Markov on the measurable space $(X = [a, b], \mathcal{F})$. If*

$$M = \sup_{B(k)} \sup_{x,y \in B(k)} \left| \frac{T''(x)}{T'(y)^2} \right| < \infty$$

and

$$\inf_x |(T^n)'(x)| > 1 \quad \textit{for some } n \in \mathbb{N},$$

then T is ergodic and has an invariant measure which is finite.

It turns out that for some choices of L and F the map $T_{L,F}$ is **not** ergodic. This is due to the fact the map is not Markov and will be explained later. For those which are Markov we can prove that there is a finite invariant measure and the corresponding map is ergodic. We can formulate this in the following lemma.

Lemma 7.4. *For all measurable sets $L \subset [0, 1]$ and $F \subset [0, 2]$ for which $T_{L,F}$ is Markov on $([0, 2], \mathcal{F})$ the map $T_{L,F}$ is ergodic and has a corresponding invariant measure.*

Proof. Let $L, F \subset [0, 2]$. For the first and second derivative of $T_{L,F}(x)$ we find

$$T'_{L,F}(x) = -\frac{2\varepsilon(x)}{x^2}$$

and

$$T''_{L,F}(x) = \frac{4\varepsilon(x)}{x^3} \ .$$

Now for $\{(B(k)_{k\in\mathbb{N}}\}$ we take $B(k) = [\frac{2}{2k+1}, \frac{2}{2k-1}]$ which gives us

$$\begin{aligned} M &= \sup_{B(k)} \sup_{x,y \in B(k)} \left| \frac{T''(x)}{T'(y)^2} \right| \\ &= \sup_{B(k)} \sup_{x,y \in B(k)} \frac{y^4}{x^3} \\ &= \sup_{k\in\mathbb{N}} \frac{(\frac{2}{2k-1})^4}{(\frac{2}{2k+1})^3} \\ &= \sup_{k\in\mathbb{N}} \frac{2(2k+1)^3}{(2k-1)^4} = 54 < \infty \ . \end{aligned}$$

Left is to find an n for which $\inf_x |(T^n)'(x)| > 1$. It turns out $n = 2$ works. We find

$$\begin{aligned} (T^2)'(x) &= \frac{-2\varepsilon(x)}{T(x)^2} \frac{-2\varepsilon(x)}{x^2} \\ &= \frac{4}{\frac{2\varepsilon(x)}{x} - \varepsilon(x)d(x)} \frac{1}{x^2} \\ &= \frac{4\varepsilon(x)}{2x - d(x)x^2} \ . \end{aligned}$$

Now

$$\inf_x \left| \frac{4\varepsilon(x)}{2x - d(x)x^2} \right| = \sup_x x(2 - d(x)x)$$

note that if $d(x) = m$ then $x \in [\frac{2}{m+2}, \frac{2}{m}]$ and

$$\sup_x x(2 - mx) = \frac{1}{m}$$

we find

$$\inf_x \left| \frac{4\varepsilon(x)}{2x - mx^2} \right| = 4m > 1 \ .$$

Now, by using Theorem 7.3, we find Lemma 7.4. □

We will investigate for what choices of L and F the map $T_{L,F}$ is Markov and therefore is ergodic and has a finite invariant measure by using Lemma 7.4. First we will see some choices that do give ergodicity and after that we will see some choices which do not give ergodicity. All the different choices for L and F in this thesis (outside this chapter) will give a map that is Markov. Up to Chapter 11 it follows from the following lemma.

Lemma 7.5. *Let $B(k) = [\frac{2}{k+1}, \frac{2}{k}]$ If $L = \cup_{k \in I} B(k)$ and $F = \cup_{k \in I} B(k)$ for some $I, J \subset \mathbb{N}$ with $1 \notin I$ and $I \neq \emptyset$ then $T_{L,F}$ is Markov.*

Proof. Take as an at most countable selection of sets $\{(B(k)_{k \in \mathbb{N}}\}$. The first and second condition of Theorem 7.2 are easily verified.
For the third condition note that for all k we have that if $B(k) \in L$ then $T_{L,F}(B(k)) = [1, 2] = B(1)$ and if $B(k) \notin L$ then $T_{L,F} = [0, 1]$ and for all $j \neq 1$ we have $B(j) \subset T_{L,F}(B(k))$ thus the third condition also holds.
For the fourth condition we have that for all k if $B(k) \in L$ then $T_{L,F}(B(k)) = [1, 2]$ and $T^2_{L,F}(B(k)) = [0, 1]$ so if we take $R = 2$ we find $\cup_{n=1}^{R} T^n(B(k)) = [0, 2]$ and the fourth condition is satisfied. If $B(k) \notin L$ then $T^2_{L,F}(B(k)) = [0, 2]$ and with $R = 2$ the fourth condition is satisfied. □

7.0.1 Non-ergodic examples

We will now see some examples of choices for L and F such that the map $T_{L,F}$ is not ergodic. First, we will find an example where L is empty. As a second example, we will find a 2-expansion which is not ergodic. After that we will find infinitely many examples for our map $T_{L,F}$ with both L and F non empty.

The case $L = \emptyset$

If we take $L = \emptyset$ we can find the following map $T_{\emptyset,F}$ which is not ergodic. Choose $F = \{[0, \frac{1}{2}] \cap \{x : T_{\emptyset,\emptyset}(x) > \frac{1}{2}\}\} \cup \{[\frac{1}{2}, 1] \cap \{x : T_{\emptyset,\emptyset}(x) < \frac{1}{2}\}\}$ (see Figure 7.1). Observe that for this F we have that

$$T^{-1}_{\emptyset,F}([0, \frac{1}{2}]) = [0, \frac{1}{2}]$$

and

$$T^{-1}_{\emptyset,F}([\frac{1}{2}, 1]) = [\frac{1}{2}, 1] \ .$$

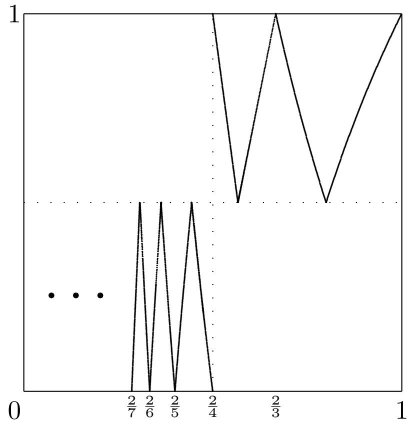

Figure 7.1: A non ergodic map.

Therefore the system is not ergodic on $[0,1]$. However, if we restrict the map to the interval $[0, \frac{1}{2}]$ it is easy to see the map is ergodic. This map is studied in Section 11.3. If we restrict the map to $[\frac{1}{2}, 1]$ we also get an ergodic map which is studied in Section 11.4.

We could ask ourselves whether this is the only map with $L = \emptyset$ which is not ergodic. Note that there is no way of making two intervals $[0, a]$ and $[a, 1]$ with $a \in (0,1)\backslash\{\frac{1}{2}\}$ such that

$$T_{\emptyset,F}^{-1}([0,a]) = [0,a]$$

and

$$T_{\emptyset,F}^{-1}([a,1]) = [a,1] \ .$$

This is due to the fact that when $a < \frac{1}{2}$ then there is an interval D_i such that $D_i \subset [0,a]$ and $\lambda(T_{\emptyset,F}(D_i)) \geq \frac{1}{2}$ where λ is the Lebesgue measure so that we can not have $T_{\emptyset,F}(D_i) \subset [0,a]$. When $a > \frac{1}{2}$ we can see that $\lambda(T_{\emptyset,F}([\frac{4}{5},1])) = \frac{1}{2}$ so we can not have $T_{\emptyset,F}([\frac{4}{5},1]) \subset [a,1]$. Note that this does not exclude the existence of any set F, different from the example above, for which the map $T_{\emptyset,F}$ is not ergodic.

The case $F = \emptyset$

Inspection of Figure 3.1 shows that there is **no** $a \in [0,2]$ such that

$$T_{L,\emptyset}^{-1}([0,a]) = [0,a]$$

and

$$T_{L,\emptyset}^{-1}([a,2]) = [a,2] \ .$$

Therefore we will try to find an interval $[a,b]$ such that

$$T_{L,\emptyset}([a,b]) = [a,b]$$

and

$$T_{L,\emptyset}([a,b]^c) = [a,b]^c$$

and then we found a non-ergodic map. Again by inspection you can see that there is not an interval $[a, b]$ without a discontinuity for which the map is not ergodic. Now note that whenever we have a discontinuity the 'jump' is 1 so we are going to try to make an interval $[a, b]$ with $b - a = 1$ furthermore we have that $T_{L,\emptyset}([b, 2]) = [0, \frac{2}{b} - 1]$ so by setting $a = \frac{2}{b} - 1$ we find the following equation

$$b - \left(\frac{2}{b} - 1\right) = 1$$

so $b - \frac{2}{b} = 0$ which gives us $b = \sqrt{2}$ and $a = \sqrt{2} - 1$ This yields the following set for L

$$\begin{aligned} L &= \left([0, \sqrt{2} - 1] \cap \{x : T_{\emptyset,\emptyset}(x) \in [\sqrt{2} - 1, \sqrt{2}]\right) \\ &\cup \left([\sqrt{2} - 1, 1] \cap \{x : T_{\emptyset,\emptyset}(x) \in [\sqrt{2} - 1, \sqrt{2}]^c\}\right). \end{aligned}$$

This indeed gives a non-ergodic map. Again, when we restrict $T_{L,F}$ to $[a, b]$ or $[a, b]^c$ then $T_{L,F}$ will be ergodic. These two ergodic maps are are studied in Section 11.1 and Section 11.2.

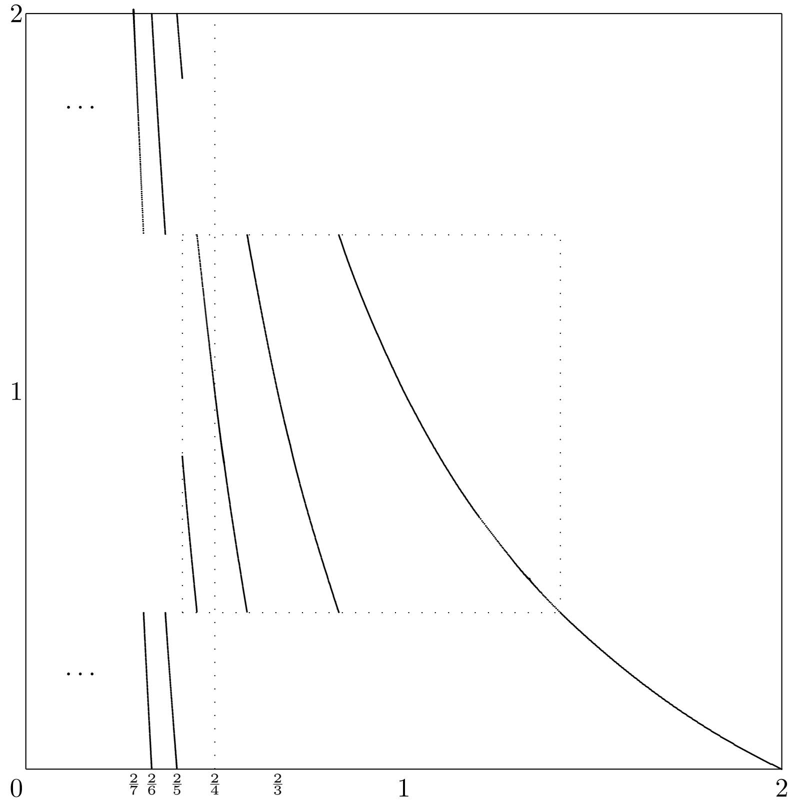

Figure 7.2: A non ergodic 2-expansion.

Note that this is the only way of making a non-ergodic map with an interval $[a,b]$ and $b-a=1$. Suppose there is an interval $[a',b']$ with $a' < \sqrt{2}-1$ then $b' < \sqrt{2}$ and $T_{L,\emptyset}((b',\sqrt{2})) \subset [a',b']$. Now suppose we have $[a',b']$ with $a' > \sqrt{2}-1$ then $b' > \sqrt{2}$ and $T_{L,\emptyset}((\sqrt{2},b')) \subset [0,a']$.

Other examples

If we drop the restriction of $L=\emptyset$ or $F=\emptyset$ we have infinitely many choices for L and F for which the map $T_{L,F}$ is not ergodic. Based on the non-ergodic map with $L=\emptyset$ we can find non-ergodic maps in two ways. First, we will choose L and F in such a way we will get

$$T_{L,F}^{-1}([0,\tfrac{1}{2}]) = [0,\tfrac{1}{2}]$$

and

$$T_{L,F}^{-1}([\tfrac{1}{2},2]) = [\tfrac{1}{2},2] \ .$$

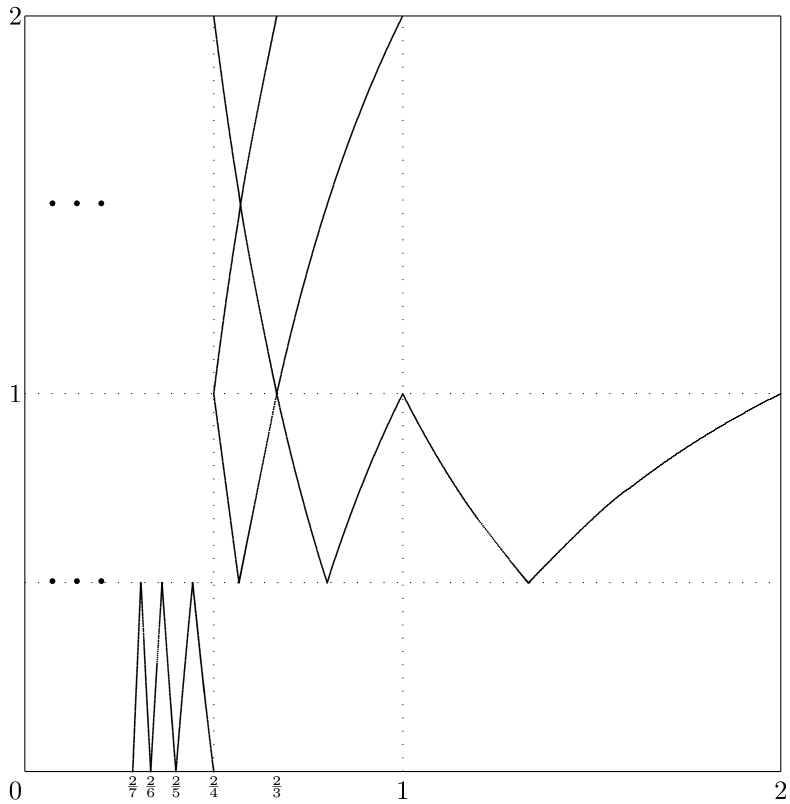

Figure 7.3: Choices for non-ergodic mappings.

In Figure 7.3 the maps from which we can choose are shown. We observe that for each $x \in [\frac{1}{2},\frac{2}{3}]$ we can choose between 3 maps and for each $x \in [\frac{2}{3},1]$ we can choose between 2 maps. This gives us infinitely many choices for L and F. Note that we would not make 2 separate parts $[0,a]$ and $[a,2]$ with $a \neq \frac{1}{2}$ since if $a < \frac{1}{2}$ the same argument as for $L = \emptyset$ holds. If $a > \frac{1}{2}$ then we have $T_{L,F}([1,2]) \cap [0,a] \neq \emptyset$ for any choice of L and F.

Another way of making non-ergodic maps, based on the non-ergodic map with $L = \emptyset$, is to expand the domain to $[0,2]$ and flip the map on $[1,\frac{4}{3}] \cup [\frac{3}{2},2]$. So

$$F = \left\{[0,\tfrac{1}{2}] \cap \{x : T_{\emptyset,\emptyset}(x) > \tfrac{1}{2}\}\right\} \cup \left\{[\tfrac{1}{2},1] \cap \{x : T_{\emptyset,\emptyset}(x) < \tfrac{1}{2}\}\right\} \cup [1,\tfrac{4}{3}] \cup [\tfrac{3}{2},2] \ .$$

Now for any measurable set $L \subset [0,1]$ we find

$$T_{L,F}\left((0,\tfrac{1}{2}) \cup (1,\tfrac{3}{2})\right) \cap \left((\tfrac{1}{2},1) \cup (\tfrac{3}{2},2)\right) = \emptyset$$

and

$$T_{L,F}\left((\tfrac{1}{2},1) \cup (\tfrac{3}{2},2)\right) \cap \left((0,\tfrac{1}{2}) \cup (1,\tfrac{3}{2})\right) = \emptyset \ .$$

This gives that the map is not ergodic (see also Figure 7.4).

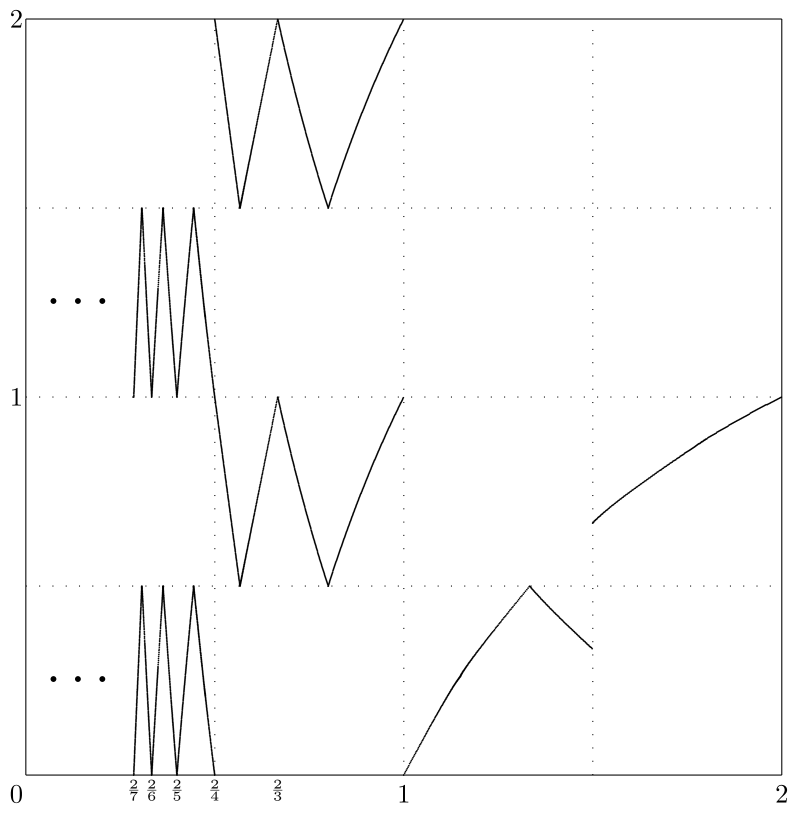

Figure 7.4: More choices for non-ergodic mappings.

We can also find infinitely many choices for L and F based on the non-ergodic map with $F = \emptyset$. This can be done by keeping the interval $[\sqrt{2}-1, \sqrt{2}]$ but we will see that there are more intervals for which we have

$$T_{L,\emptyset}([a,b]) = [a,b]$$

and

$$T_{L,\emptyset}([a,b]^c) = [a,b]^c \ .$$

When keeping the interval $[\sqrt{2}-1, \sqrt{2}]$ we can choose any measurable set $F \subset [0,1]$ and find a set L such that the map is not ergodic by setting

$$\begin{aligned} L &= \Big([0, \sqrt{2}-1] \cap \{x : T_{\emptyset,F}(x) \in [\sqrt{2}-1, \sqrt{2}]\Big) \\ &\quad \cup \Big([\sqrt{2}-1, 1] \cap \{x : T_{\emptyset,F}(x) \in [\sqrt{2}-1, \sqrt{2}]^c\}\Big) \ . \end{aligned}$$

Now we can also choose an interval $[\sqrt{2}-1+\delta, \sqrt{2}+\delta]$ with $\delta \in [0, \frac{3}{2} - \sqrt{2}]$ and choose a set $F' \subset [0,1]$ and let $F = F' \cup [\sqrt{2}, \frac{3}{2}]$. By setting

$$\begin{aligned} L &= \Big([0, \sqrt{2}-1+\delta] \cap \{x : T_{\emptyset,F}(x) \in [\sqrt{2}-1+\delta, \sqrt{2}+\delta]\Big) \\ &\quad \cup \Big([\sqrt{2}-1+\delta, 1] \cap \{x : T_{\emptyset,F}(x) \in [\sqrt{2}-1+\delta, \sqrt{2}+\delta]^c\}\Big) \end{aligned}$$

we find a map $T_{L,F}(x)$ which is not ergodic.

Chapter 8

Several examples of different L and F

We have seen in Chapter 6 that for all L and F, that are allowed, the expansion converges . Of course, there are infinitely many choices for L and F for which the map $T_{L,F}$ is ergodic. We will look into some of which we find interesting. In particular, those which make our continued fraction have only odd or only even digits. We will first introduce some notation to make it easier to describe L and F.
First we define for $i \in \mathbb{N}$,,

$$D_i = \{x \in (0,2] : \left\lfloor \frac{2}{x} \right\rfloor = i\} \ .$$

Now we define

$$D_{even} = \bigcup_{i:i \text{ even}} D_i$$

and

$$D_{odd} = \bigcup_{i:i \text{ odd}} D_i \ .$$

Note that $\Delta_{(a,b)}$ from Section 4.1 depends on the choice of F and L while D_i is independent of this choice.

For each example in this chapter, the sets L and F are specified. A simulation of the density of the invariant measure is given, and whenever possible the theoretical density is as well. If the theoretical density is found this is done by using the natural extension method of Section 3.1. Furthermore typical characteristics are explained.

8.1 The lazy backward expansion

Let $L = F = [0,1)$ we find that

$$T_{L,F}(x) = \begin{cases} 2 + \lfloor \frac{2}{x} \rfloor - \frac{2}{x} & \text{for } x \in [0,1] \\ \frac{2}{x} - 1 & \text{for } x \in (1,2] \ . \end{cases}$$

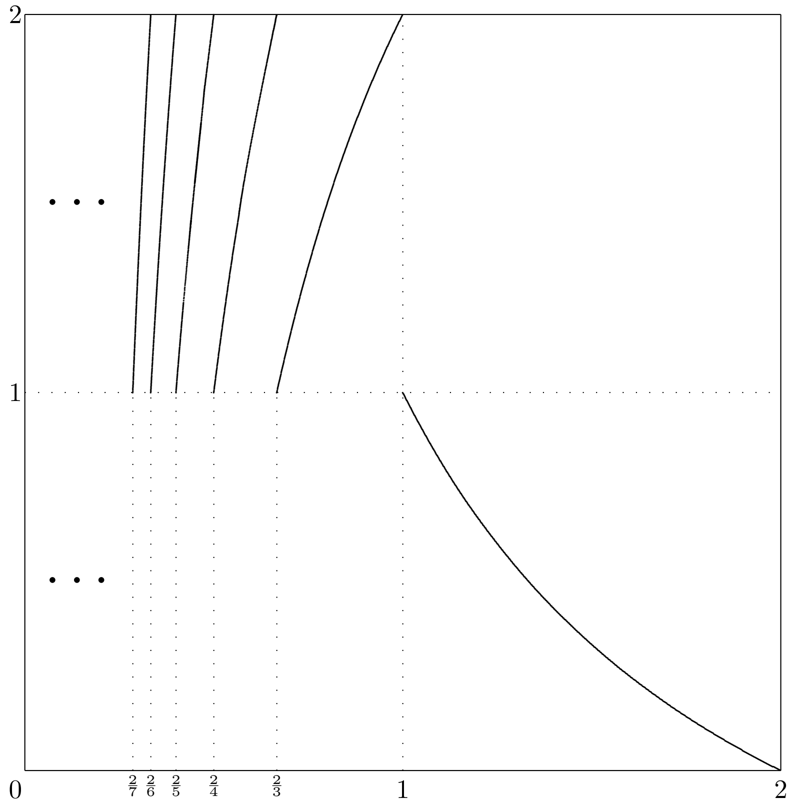

Figure 8.1: The lazy backward expansion.

Note that whenever $x \in [0,1]$ then $T_{L,F}(x) \in [1,2]$ and whenever $x \in [1,2]$ then $T_{L,F}(x) \in [0,1]$. So the map $T^2_{L,F}$ is not ergodic. Of course $T_{L,F}$ is ergodic due to Lemma 7.5. Also digits 2 and 3 are omitted. Note that for all $x \in [0,2]$ it has digits 1 and $\varepsilon = 1$ in 50% of the time since it is alternating digit 1 with other digits and digit 1 is the only digit with positive ε. We will find the density by using the method of Section 3.1. In figure 8.2 the domain is shown with the variables A, B and C in the second coordinate. We have the following equations for A, B and C: $A = \frac{-2}{4+B}$, $B = \frac{2}{1}$ and $C = \frac{2}{1+A}$ Solving the equation for B results in $B = 2$ which gives $A = -\frac{1}{3}$ and we find $C = 3$.

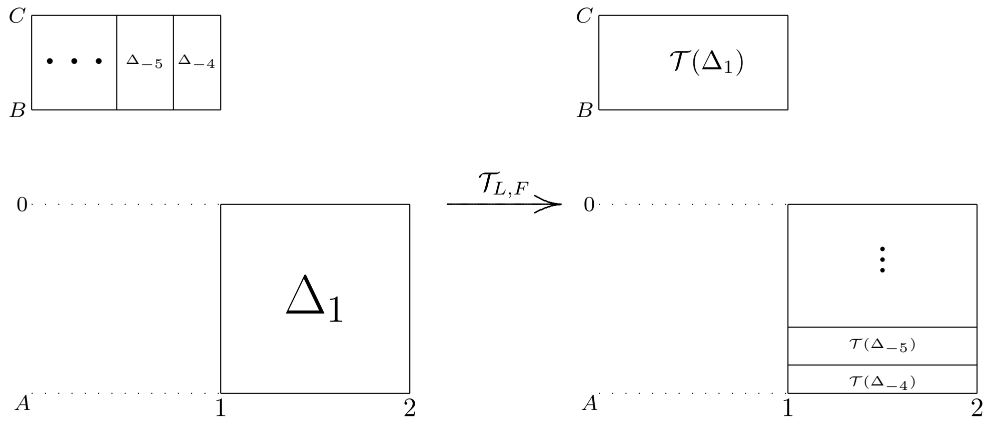

Figure 8.2: Ω and $\mathcal{T}_{L,F}(\Omega)$.

After projecting onto the first coordinate, this gives the following density

$$f(x) = \begin{cases} C_1\left(\frac{3}{2+3x} - \frac{1}{1+x}\right) & \text{for } x \in [0,1]\ , \\ C_1\left(\frac{3}{6-x}\right) & \text{for } x \in [1,2]\ , \end{cases}$$

where $C_1^{-1} = 2\ln(\frac{5}{4})$.

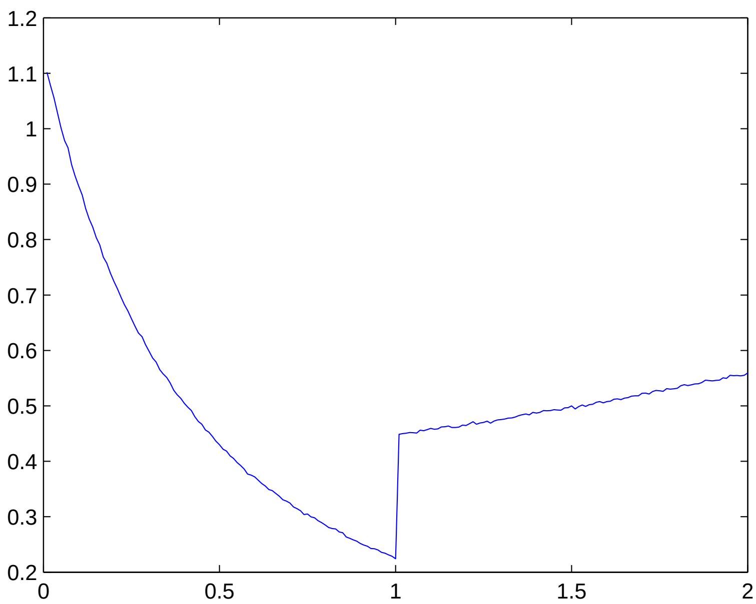

Figure 8.3: A simulation of the density of the lazy backward map.

8.2 The lazy backward expansion (v2)

With $L = [0,1]$ and $F = [0,2]$ we find the same expansion as in Section 8.1 but now the interval $[1,2]$ is also flipped. This results in an expansion without digits 1 and 3. Having the interval $[1,2]$ flipped drastically changes the shape of the invariant measure (see Figure 8.5). Whenever we use the method to find the natural extension as in 3.1 and change the domain we will find the following domain:

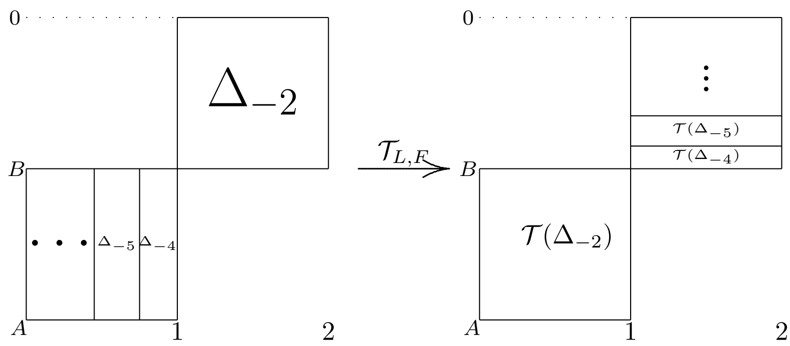

Figure 8.4: Ω and $\mathcal{T}_{L,F}(\Omega)$.

Now for A and B we find the equations $A = \frac{-2}{2+B}$ and $B = \frac{-2}{2}$ so $A = -2$. This gives us the following formula

$$f(x) = \begin{cases} \frac{1}{1-x} - \frac{1}{2-x} & \text{for } x \in [0,1), \\ \frac{1}{2-x} & \text{for } x \in [1,2)\ . \end{cases}$$

We find that we have a σ-finite infinite measure (as shown in Figure 8.5). This is due to the fact that points close to 1 are mapped to points close to 2 and visa versa.

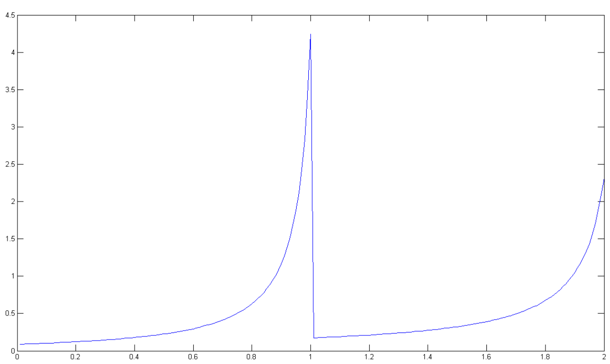

Figure 8.5: A simulation of the invariant measure of the lazy backward v2.

8.3 Digits only divisible by 3 or digit 1

We can also make a continued fraction expansion for which the $d_n(x)$ is always divisible by 3 except on the interval $[1,2]$ where we will pick digit 1. Even though this map generates digits of 1 we will refer to it as the 3 divisible map. Let

$$I = \{n \in \mathbb{N} : n = 1 \mod 3\} \backslash \{1\}$$

and

$$J = \{n \in \mathbb{N} : n = 2 \mod 3\} \ .$$

The lazy set and flipped set are easily made, namely

$$L = \bigcup_{i \in I} D_i \quad \text{and} \quad F = \bigcup_{i \in J} D_i \ .$$

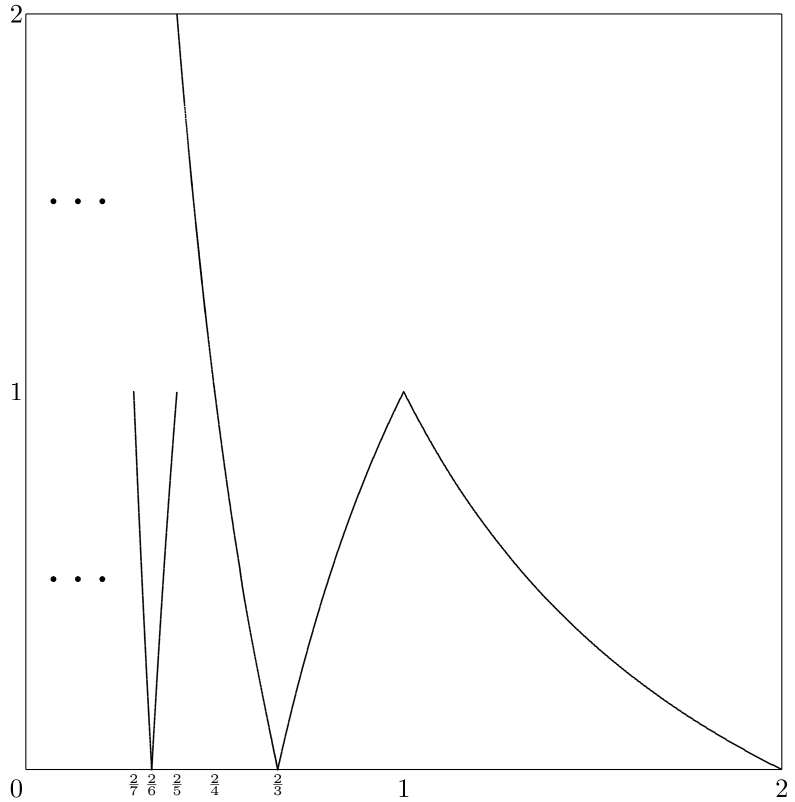

Figure 8.6: An expansion with digits only divisible by 3 or digit 1.

We will find the invariant measure by using method 3.1.

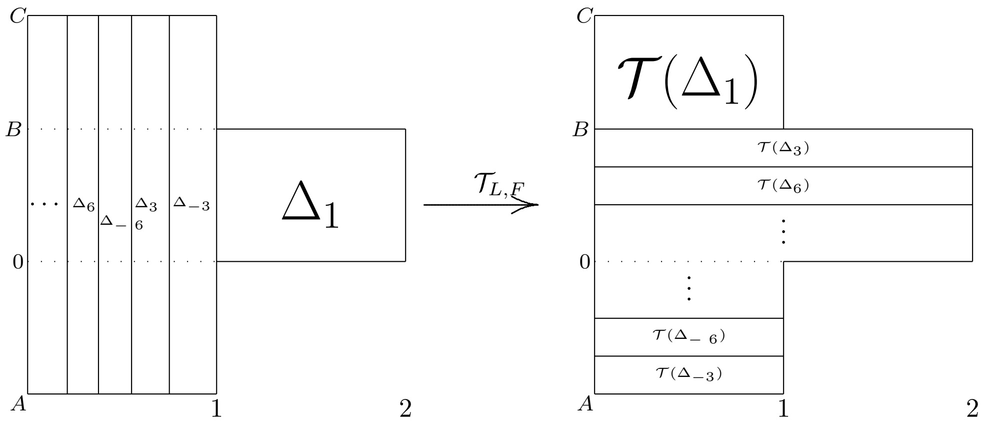

Figure 8.7: Ω and $\mathcal{T}_{L,F}(\Omega)$.

We get the following equations for A, B and C:

$$A = \frac{-2}{3+A}, \quad B = \frac{2}{1+B} \quad \text{and} \quad C = 2 \ .$$

Now we have that $B > 0$ so we find $B = 1$ and $A = -1$. The density is now given by

$$f(x) = c\left(\frac{1}{1+x} + \frac{1}{2-x}\right) \text{ for } x \in [0,1] \text{ and}$$

$$f(x) = c\left(\frac{1}{2+x}\right) \text{ for } x \in [1,2]$$

where

$$c^{-1} = \ln\left(\frac{16}{3}\right) \ .$$

A simulation of this density is given in Figure 8.8.

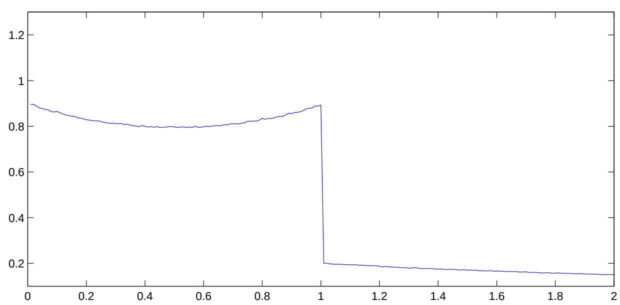

Figure 8.8: A simulation of the density for the 3 divisible map.

In this section we will also give the percentages a certain digit is occurring for almost all $x \in [0,2]$. For the percentages of a specific digit with a specific ε we will use Birkhoffs ergodic theorem with the indicator function $1_{\Delta_{i,\varepsilon}}$ for digit i with ε. Birkhoffs ergodic theorem, which can be viewed as a generalisation of the law of large numbers, can be found in Appendix A.1.3. If we use the function 1_F we will find the percentage of occurrence of $\varepsilon = -1$. Birkhoffs ergodic theorem provides us that this is given by

$$\frac{1}{\ln(\frac{16}{3})} \sum_{n=0}^{\infty} \int_{\frac{2}{3n+3}}^{\frac{2}{3n+2}} \frac{1}{1+x} + \frac{1}{2-x} \, \mathrm{d}x = 0.4141$$

So the percentage of occurrence of $\varepsilon = -1$ is 41.41%. In Table 8.1 the percentages of the digits are given.

	1	3	6	9	12	15	other
$\varepsilon = 1$	17.19	21.31	6.77	3.35	2.00	1.33	6.64
$\varepsilon = -1$	0	28.07	5.35	2.23	1.27	0.81	3.68

Table 8.1: Percentages of the occurrence of certain digits with their sign ε.

In the next subsection we will see a continued fraction map with the same density as the map of the 3 divisible map.

8.3.1 3 divisibles spouse

Let

$$L = \bigcup_{i \in I} D_i$$

and

$$F = L^c \backslash \bigcup_{i \in J} D_i$$

where I, J are defined as in previous section. We can find the invariant density of $T_{L,F}$ without any calculations. The following lemma will show that the density is the same as the density corresponding to the map of the 3 divisible expansion.

Lemma 8.1. *Let $T_1(x)$ be defined by the measurable sets L_1 and F_1 and let $T_2(x)$ be defined by $L_2 = L_1$ and $F_2 = F_1 \cap L_1 \cup (F_1^c \cap L_1^c)$ then the following holds.*
The density of the invariant measure corresponding to $T_1(x)$ is symmetric on $[0,1]$
if and only if
the density of the invariant measure corresponding to $T_2(x)$ is the same density as the invariant measure corresponding to $T_1(x)$.

Proof. " $\Rightarrow$ "
Let μ be the invariant measure corresponding to $T_1(x)$. It suffices to prove that $\mu([0,z]) = \mu\left(T_2^{-1}([0,z])\right)$ for all $z \in [0,2]$. Take $z \in [0,2]$ at arbitrary.
Suppose $z \in [0,1]$ then we have $T_2^{-1}([0,z]) = T_1^{-1}([1-z,1])$.
This gives

$$\mu\left(T_2^{-1}([0,z])\right) = \mu\left(T_1^{-1}([1-z,1])\right) = \mu([1-z,1]) = \mu([0,z])$$

Now suppose $z \in (1,2]$. We know that $T_1(x) = T_2(x)$ for all $x \in T_1^{-1}([1,2])$, yielding that

$$\begin{aligned}
\mu\left(T_2^{-1}([0,z])\right) &= \mu\left(T_2^{-1}([0,1])\right) + \mu\left(T_2^{-1}([1,z])\right) \\
&= \mu([0,1]) + \mu\left(T_1^{-1}([1,z])\right) \\
&= \mu([0,1]) + \mu([1,z]) \\
&= \mu([0,z])
\end{aligned}$$

"$\Leftarrow$"

Left to prove is that if the invariant measures of $T_1(x)$ and $T_2(x)$ are the same then it is symmetric on $[0,1]$. Now let $z \in [0,1]$, we find that

$$\mu([0,z]) = \mu\left(T_2^{-1}([0,z])\right) = \mu\left(T_1^{-1}([1-z,1])\right) = \mu([1-z,1])$$

and this gives that the density is symmetric on $[0,1]$. □

8.4 A 'switching digits' expansion

In this section we will have

$$L = D_{odd} \setminus D_1 \quad \text{and} \quad F = D_{even} \ .$$

In Figure 8.9 we can see why this section is called switching digits.

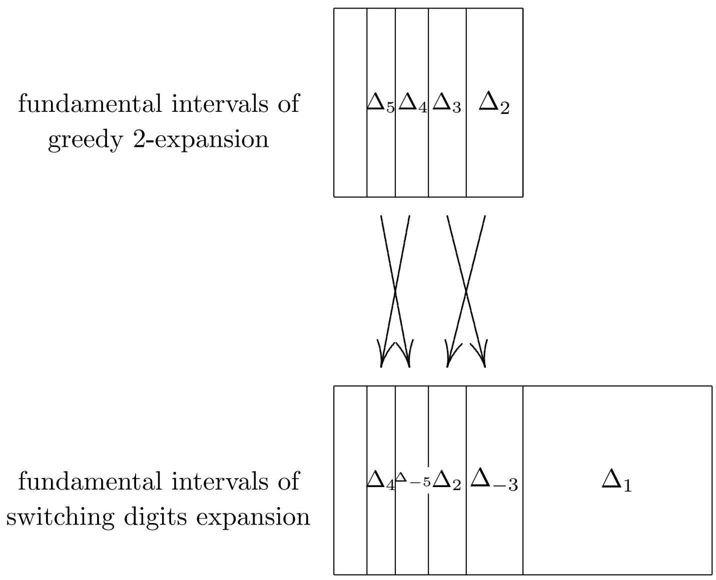

Figure 8.9: The reasoning behind the name 'switching digits'.

For this choice of L and F we can not find the invariant measure by using the natural extension and making the domain bijective.

Note that, even though in general it is hard to find an invariant measure, we succeeded in most instances in finding the invariant measure. A simulation shows that the density looks symmetric on $[0, 1]$ (see Figure 8.10).

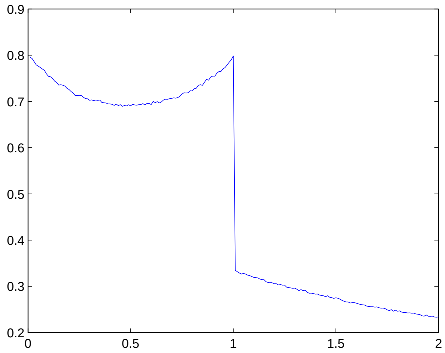

Figure 8.10: A simulation of the density for the switching digits map.

By using Lemma 8.1 this would mean it has the same density as the density for the even regular expansion with $F = [1, 2]$ (which will be discussed in Chapter 9) but we could not find the density of that map either. What we can do is simulate random points from the line $[0, 2] \times \{0\}$ and iterate them over $\mathcal{T}_{L,F}(x, y)$. Maybe this gives an indication of how the natural extension looks like. Note that we have no guarantee that the line is in the domain at all and neither all of the iterates. Though, whenever points 'enter' the domain they cannot 'leave'. The result is shown in Figure 8.11. A description of the programming can be found in Appendix B and the program itself in Appendix B.0.6.

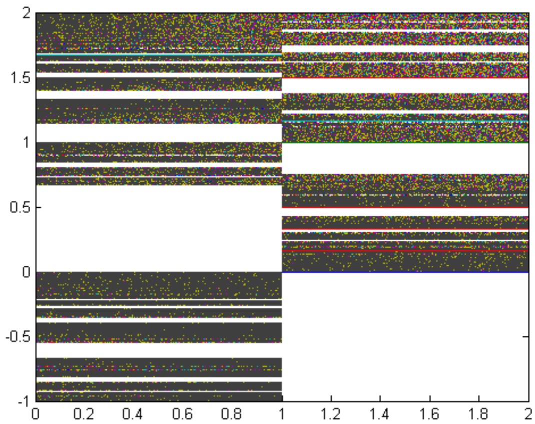

Figure 8.11: A simulation of the domain of the natural extension of the switchingdigit map.

Chapter 9

Continued fraction expansions with only even digits

In this chapter we will finally look into continued fractions for which all digits are even. We have seen an even expansion in Section 4.2 and not in Chapter 3. This is because with 2-expansions we could not make a map which would lead to only even digits. Now with the new family of maps $T_{L,F}(x)$ we will have infinitely many maps that would lead to only even digits. For the sets L and F of such map we have the following restrictions:

$$D_{odd} \subset F \cup L, \quad D_{odd} \cap F \cap L = \emptyset, \quad D_{even} \cap F = D_{even} \cap L\,.$$

Under these restrictions we can rewrite $d(x)$ and it becomes clear why this would give an expansion with only even digits

$$d(x) = \begin{cases} \lfloor \frac{2}{x} \rfloor - 1 & \text{for } x \in L \cap D_{odd} \\ \lfloor \frac{2}{x} \rfloor + 1 & \text{for } x \in F \cap D_{odd} \\ \lfloor \frac{2}{x} \rfloor & \text{for } x \in (F \cup L)^c \cap D_{even} \\ \lfloor \frac{2}{x} \rfloor + 2 & \text{for } x \in (F \cup L) \cap D_{even}\ . \end{cases}$$

Now we have two choices for the image of x for all $x \in [0, 1]$ (see Figure 9.1). Furthermore we can pick any measurable set $L \subset (0, 1]$ and find a unique set F such that the corresponding continued fraction expansion has only even digits and the other way around. Note that for each $x \in [0, 2]$ both $p_n(x)$ and $q_n(x)$ are even for all $n > 0$ (see page 34 for recurrence relations).

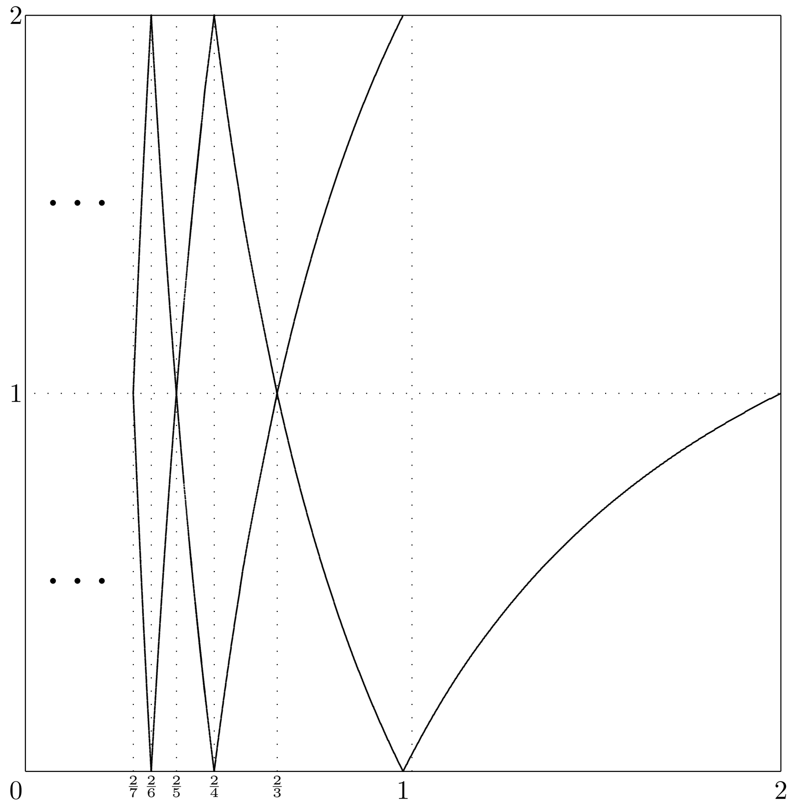

Figure 9.1: The options for an expansion with only even digits.

9.1 Examples of continued fraction expansions with only even digits

In Section 3.1 it is mentioned that we could not make an even 'regular' expansion because we could not make the digit even on the interval $[1, 2]$. However, now that we can also flip our continued fraction map $\frac{2}{x} - 1$ we have this option. This example is given first. Afterwards we will see the even expansion with $F = [0, 2]$ and the even expansion with $L = [0, 1]$.

9.2 'Regular' even continued fraction map with $[1, 2]$ flipped

If we take $L = D_{odd}$ and $F = [1, 2]$ we will find an even continued fraction map. In the spirit of [5] we will call the corresponding expansion a regular even 2-expansion and $T_{L,F}$ the regular even map.

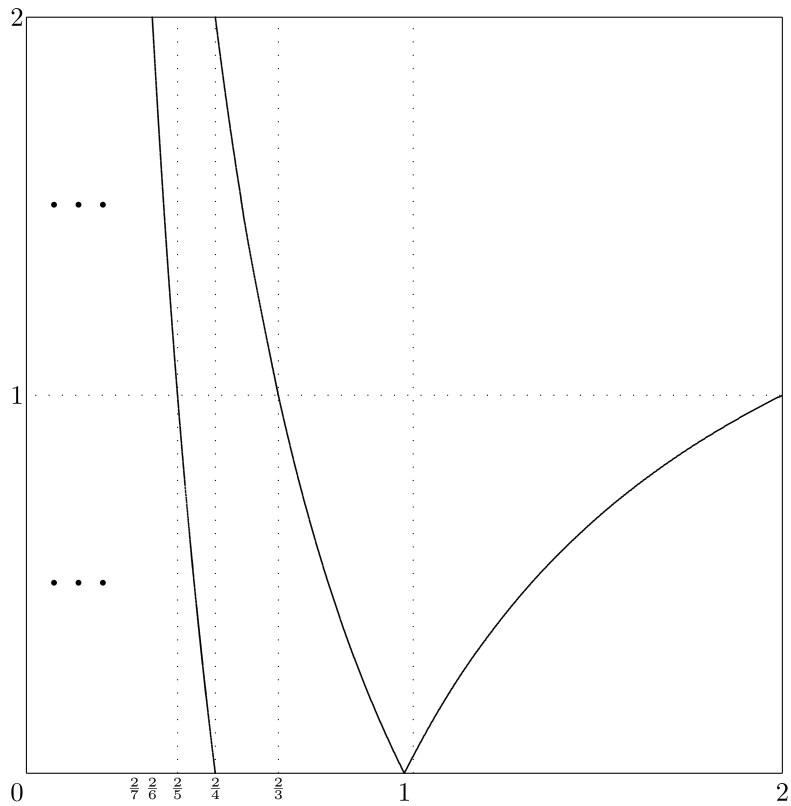

Figure 9.2: The regular even map.

Unfortunately, we could not find the invariant measure. Since this map $T_{L,F}(x)$ belongs to the main topic of this thesis, we will be ellaborate on the process of searching the domain of the natural extension such that $\mathcal{T}_{L,F}(x, y)$ is bijective. We will see that all tricks we try do not work. Also the same way of simulating as in Section 8.4 is used to give an idea of how the domain could look like. But first we will try the method as in Section 3.1 adjusting the domain. We get the following equations for A, B and C:

$$A = \frac{-2}{2+0}, \quad B = \frac{-2}{2+C} \quad \text{and } C = \frac{2}{2+A}$$

which gives us $A = -1$, $B = -\frac{1}{2}$ and $C = 2$.

Even though this works for the boundaries of the domain it will not give a bijection because all the fundamental intervals in the image of $\mathcal{T}_{L,F}$ should fit exactly and they do not (see Figure 9.3). In order to fit we need that $\frac{2}{n+B} = \frac{2}{n+2+A}$ for $n = 2, 4, \ldots$ such that the images of the edges fundamental intervals $\Delta_2, \Delta_4, \ldots$ connect. This gives $B - A = 2$ and we have that $B - A = \frac{1}{2}$. We end up with holes between the images of the fundamental intervals. Another problem is that $\mathcal{T}(\Delta_{n+2}) \subset \mathcal{T}(\Delta'_n)$ which gives overlap.

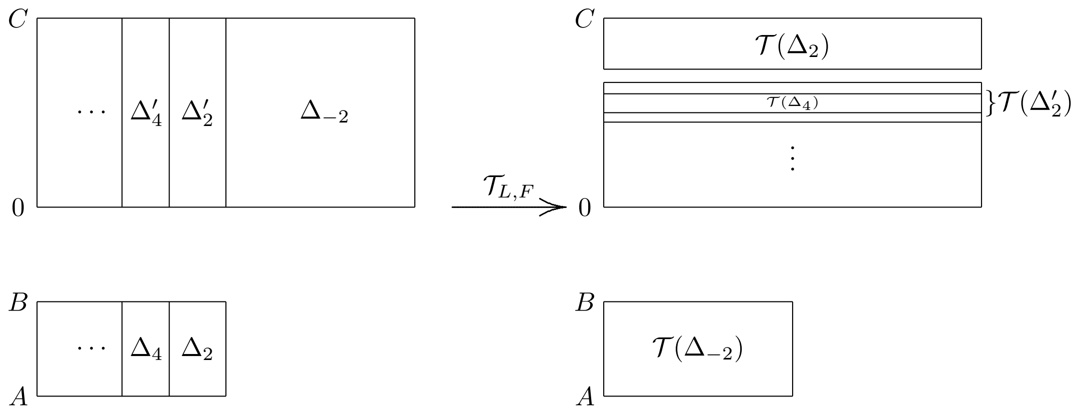

Figure 9.3: Ω and $\mathcal{T}_{L,F}(\Omega)$.

In fact, a simulation shows that we do not have such a domain as we hoped for but a domain with infinitely many holes. Figure 9.4 shows an indication of how the natural extension could look like. Note that, even though in the simulation the number of holes seems to be finite, we think that there are infinitely many holes. The reason for the simulation not to show that many holes is that simulated points have a thickness in Figure 9.4.

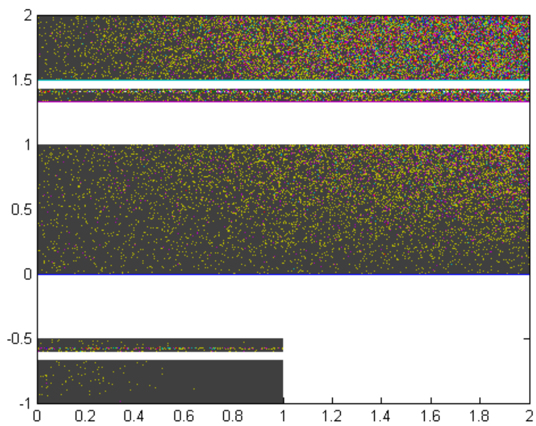

Figure 9.4: A simulation of the domain of the natural extension of the regular even map.

For our next attempt we will try to use a different formula for the map of the natural extension which does allow a domain on which it is bijective.
Let $\mathcal{T}_{L,F} : ([0,2]\times[0,2])\backslash([1,2]\times[1,2]) \to ([0,2]\times[0,2])\backslash([1,2]\times[1,2])$ be defined as

$$\mathcal{T}_{L,F}(x,y) = \left(T_{L,F}(x), \frac{2}{d(x)+\varepsilon(x)y} \right) .$$

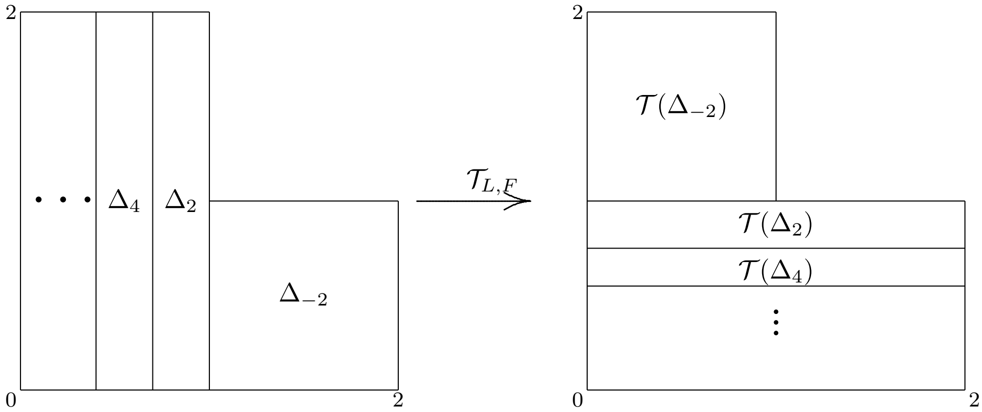

Figure 9.5: Ω and $\mathcal{T}_{L,F}(\Omega)$.

However, if we now use

$$\mu(A) = C \iint_A \frac{2}{(2+xy)^2} \, \mathrm{d}x \, \mathrm{d}y ,$$

where C is a normalising constant, as invariant measure for the natural extension and integrate out y we will find $C(\frac{2}{2+2x})$ as density on $[0,1]$ and $C(\frac{1}{2+x})$ on $[1,2]$. Simulation shows this is totally incorrect.

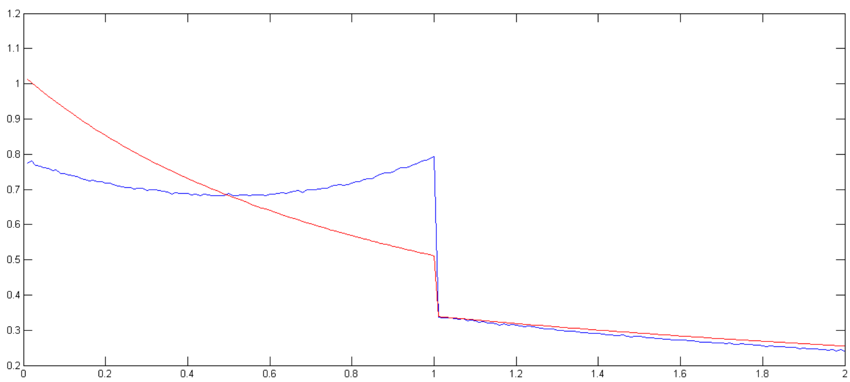

Figure 9.6: A simulation of the invariant measure as well as the theoretic result.

In fact, whenever we would have used this natural extension on the same map but without flipping on $[1,2]$ the fundamental intervals would have been on the exact same spots. It turns out the densities found are the ones of this map! The reason for this is that the invariant measure is not invariant anymore for the extension when we use the flipped map on $[1,2]$. Now the density looks very symmetric. If this is indeed the case then, by using the symmetry lemma, it is the same density as the density of the Switching digit expansion on page 54.

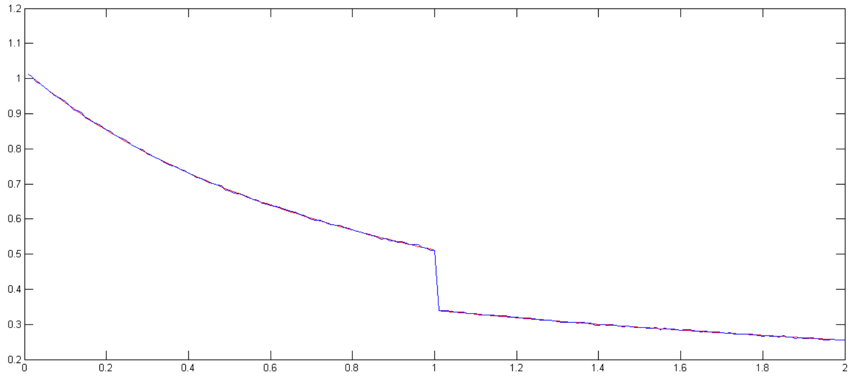

Figure 9.7: A simulation of the invariant measure with no flip on $[1,2]$ as well as the theoretic result.

We also tried to find the density by trying functions

$$c_1\left(\frac{\alpha}{2+\alpha x}-\frac{\beta}{2+\beta x}\right) \text{ for } x \in [0,1]$$

and

$$c_2\left(\frac{\gamma}{2+\gamma x}\right) \text{ for } x \in [1,2]$$

and search for a triple α, β, γ which fits the simulation the best (by using the least squared error method). This did not lead to the desired result.

Even though we could not find the density theoretically we do have the density numerically. This gives us the possibility to calculate the occurrence of a certain digit numerically by using Birkhoffs Theorem. The digits are given in Table 9.1.

2, $\varepsilon=-1$	2, $\varepsilon=1$	4	6	8	10	other
27.59	36.15	12.29	5.90	3.60	3.18	4.65

Table 9.1: Percentages of the occurrence of certain digits.

9.3 Even expansion with $F = [0, 2]$

In this section we choose F to be $F = [0, 2]$. In order to make the map even we have $L = D_{even}$. In Figure 9.8 the map is shown for this choice of L and F.

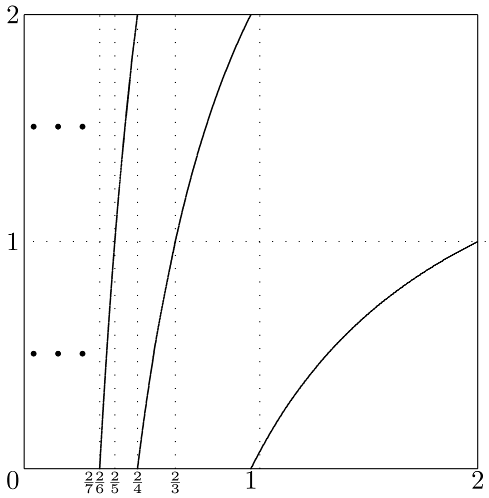

Figure 9.8: An even expansion with $F = [0, 2]$.

This map can be obtained by flipping the even regular expansion map in the line $y = 1$ on $[0, 1]$. Whenever we use the method as in Section 3.1 adjusting the domain we will find the following equations for A and B:

$$A = \frac{-2}{2 + B} \quad \text{and } B = \frac{-2}{4 + A} .$$

we find that $A = -2$ and $B = -1$ see Figure 9.9.

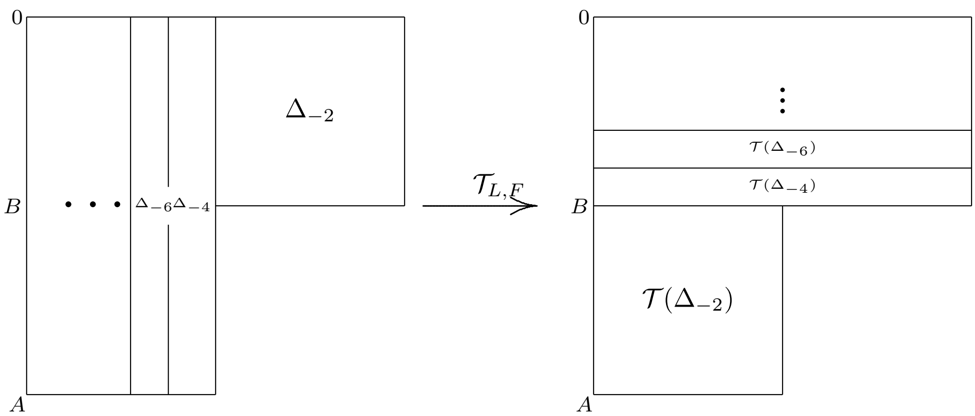

Figure 9.9: Ω and $\mathcal{T}_{L,F}(\Omega)$.

Whenever we integrate over the second coordinate we find the following formulas: $f(x) = \frac{1}{1-x}$ for $x \in [0,1)$ and $f(x) = \frac{1}{2-x}$ for $x \in [1,2)$.
Again we find a σ-finite infinite measure just as in Section 8.2. A simulation is shown in Figure 9.10.

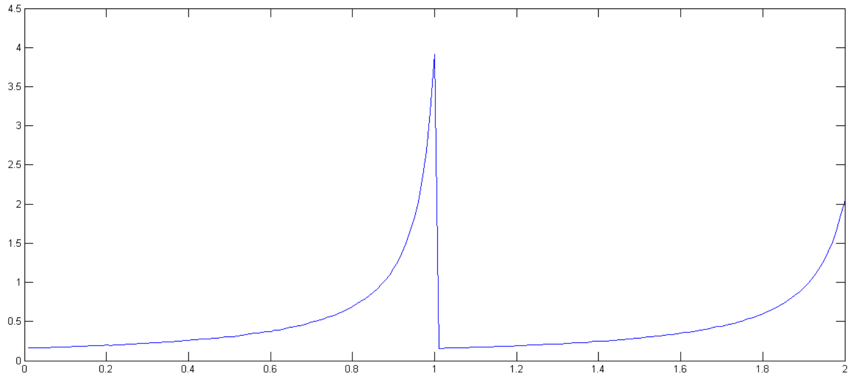

Figure 9.10: A simulation of the density of the even expansion with $F = [0, 2]$.

9.4 Even expansion with $L = [0, 1]$

Now as a last even expansion we will pick $L = [0, 1]$ which gives us that $F = D_{even}$. In Figure 9.11 the corresponding graph is shown.

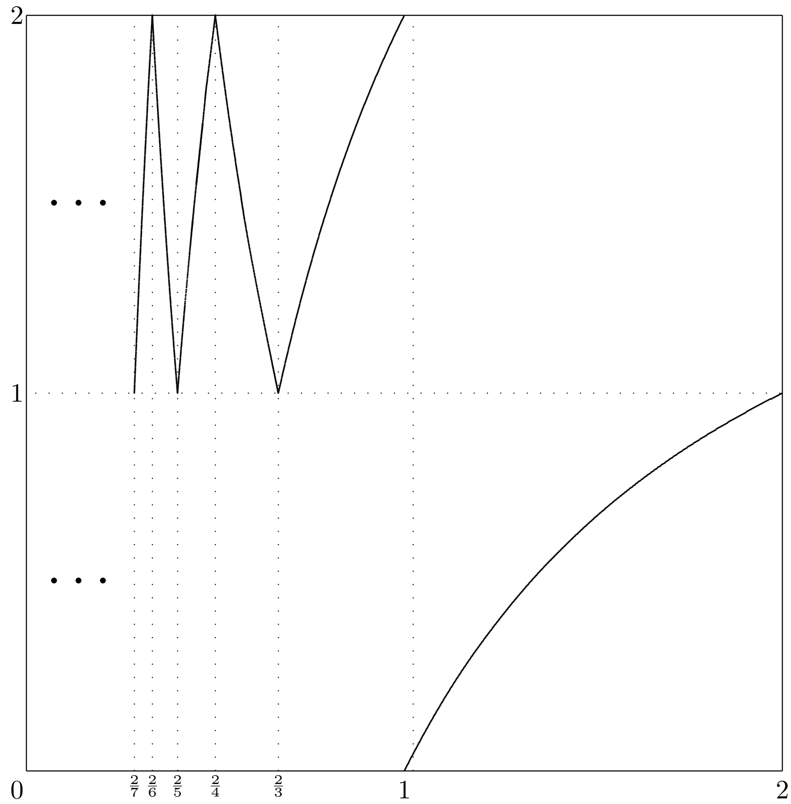

Figure 9.11: The even expansion with $L = [0, 1]$.

This maps looks like the odd flipped 2-expansion but then with $L = [0, 1]$ instead of $L = \emptyset$. Now we will find that the invariant measure is again σ-finite infinite. We can show this by using the method as in Section 3.1 adjusting the domain. We find the following equations for A, B, C and D (see Figure 9.12):

$$A = \frac{-2}{2 + C} \quad B = \frac{-2}{2 + D} \quad C = \frac{-2}{4 + A} \quad \text{and } D = \frac{2}{2 + A} \ .$$

We find that $A = -2$ and $C = -1$. Now in the equation for D we get a division by zero. It turns out that whenever we put $D = \infty$ we find a bijective map. This implies $B = 0$.

From this we find the following formula for $f(x)$:

$$f(x) = \begin{cases} \frac{1}{1-x} & \text{for} \quad x \in [0, 1) \\ \frac{1}{x} + \frac{1}{2-x} & \text{for} \quad x \in [1, 2) \ . \end{cases}$$

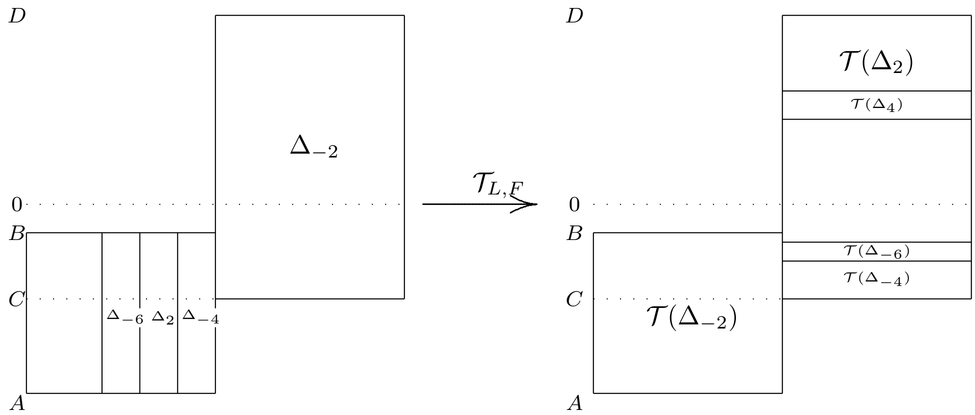

Figure 9.12: Ω and $\mathcal{T}_{L,F}(\Omega)$.

In Figure 9.13 a simulation is shown. In the next Chapter we will discus continued fraction expansions with only odd digits. There will be no dynamical system with a σ-finite infinite measure.

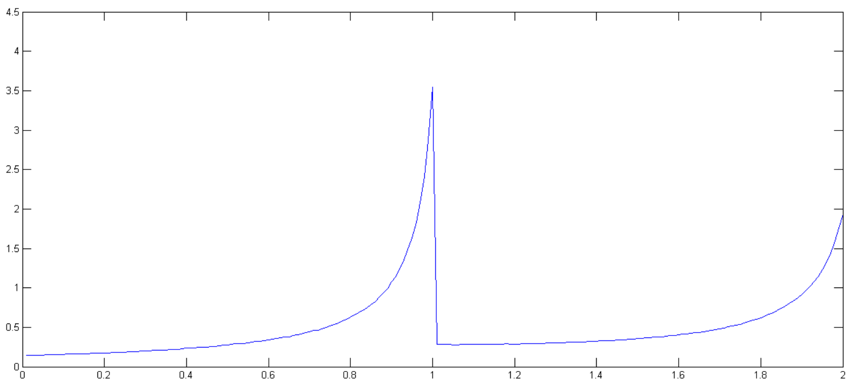

Figure 9.13: A simulations of the density of the invariant measure of the even expansion with $L = [0, 1]$.

Chapter 10

Continued fraction expansions with only odd digits

We already have seen two ways of making a continued fraction expansion with only odd digits in Section 3.1 and Section 4.1. In this chapter we will use both sets L and F to find more examples of such continued fraction expansions We will have restrictions on L and F in a way similar to the even expansions. These restrictions are given by

$$D_{even} \subset F \cup L, \quad D_{even} \cap F \cap L = \emptyset, \quad D_{odd} \cap F = D_{odd} \cap L \ .$$

These restrictions give exactly those maps prohibited in the previous chapter. In Figure 10.1 we see the different maps we can choose from. Note that for all $x \in [0,1]$ we have two options which gives us infinitely many ways of making odd expansions. Also note that given any measurable set $L \subset [0,1]$ there is a unique F such that the expansion is giving only odd digits (and the other way around). If we look at the recurrence relations in Chapter 6 on page 34 we see that by picking the digits only odd for all $x \in [0,2]$ the q_n's will be odd for all $n \in \mathbb{N}$. The p_n's will always be even no matter what the digits are. In the next sections we will see several examples of continued fraction expansions with only odd digits.

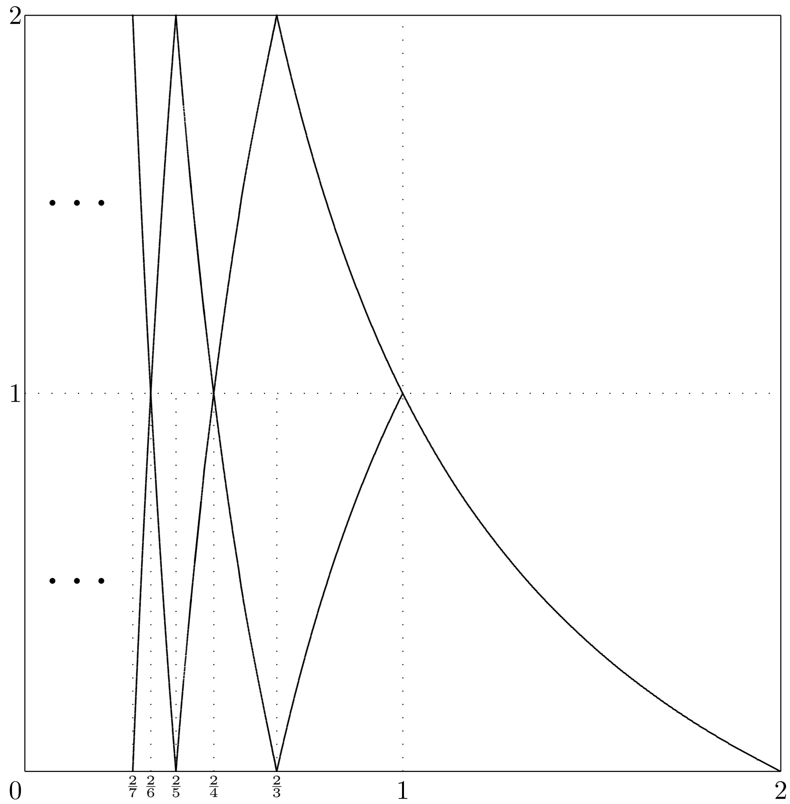

Figure 10.1: The options for an expansion with only odd digits.

10.1 Examples of continued fraction expansions with only odd digits

We examine two examples which are somewhat similar to the ones we already encountered in Section 3.1 and Section 4.1. The example we consider first is the odd expansion with $F = [0, 1]$. This map looks like the backward continued fraction map (see Figure 10.2). Afterwards we study the odd expansion with $L = [0, 1]$ which looks like the flipped even continued fraction map but then lifted to $[1, 2]$. As a last example we will see an expansion with $d_i = 1 \mod 4$ for all $i \geq 1$.

10.2 Odd expansion with $F = [0, 1]$

As mentioned, we can first pick a set F and then find a set L such that $T_{L,F}$ gives only odd digits. In this section we choose F to be $F = [0, 1]$. Now we observe that we need to pick $L = D_{odd}$ in order to make an odd expansion. In Figure 10.2 the graph is shown.

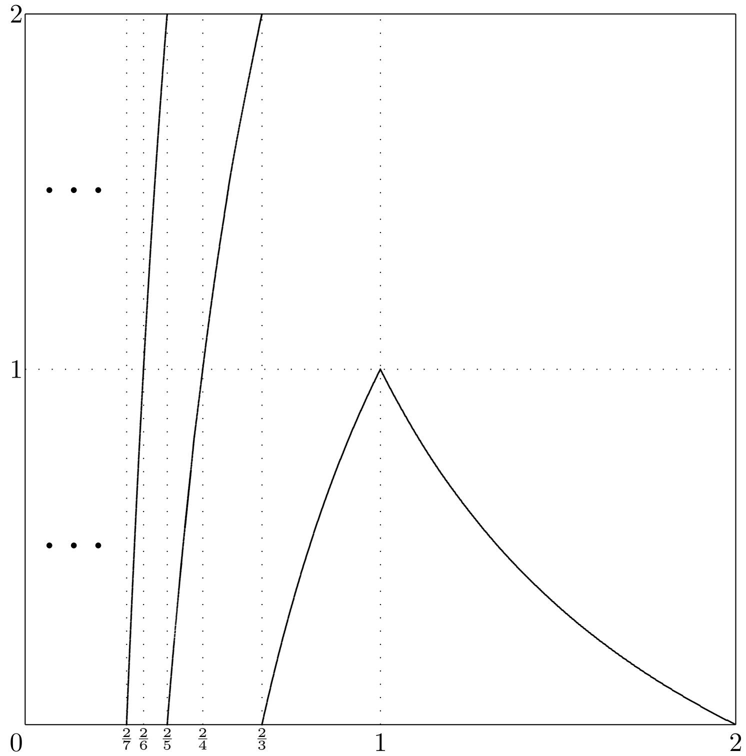

Figure 10.2: The odd expansion for $F = [0, 1]$.

Note that this map looks like the regular odd expansion map but then flipped. In fact, if we flip the regular odd expansion map on the interval $[0, 1]$ in the line $y = 1$ we will get this map. Unfortunately we could not find the invariant measure. We will give a short description of why the method as in Section 3.1 did not work. We get the following equations for A, B, C and D:

$$A = \frac{-2}{3 + A} \quad B = \frac{-2}{5 + A} \quad C = \frac{2}{1} \quad \text{and } D = \frac{2}{1 + B} \ .$$

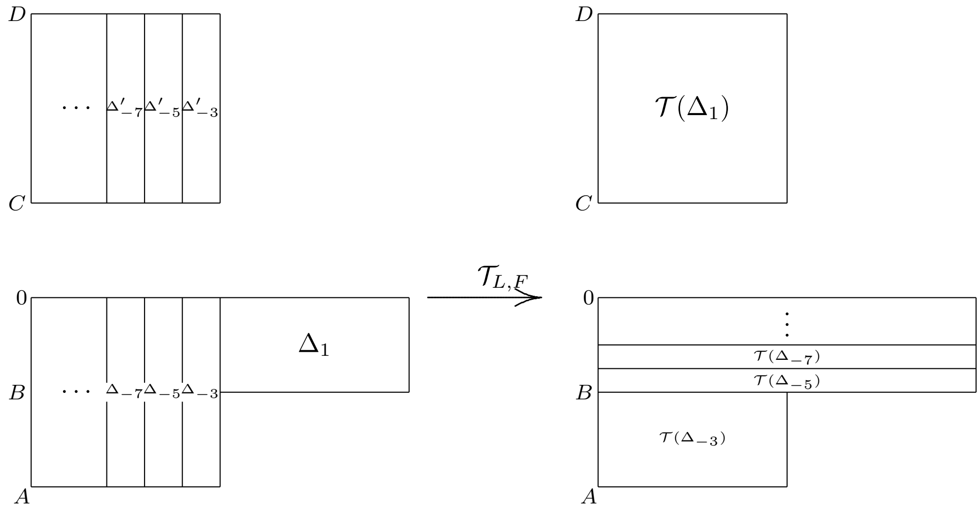

Figure 10.3: Ω and $\mathcal{T}(\Omega)$ but without the images of the rectangles with a prime.

This gives us $A = -1$ or $A = -2$. We will explain that if we pick $A = -1$ we will get overlap and that implies that you will get overlap when you take $A = -2$. Now $A = -1$ gives $B = -\frac{1}{2}$ and $D = 4$. We find that the image of Δ'_{-2i-1} overlaps with the image of Δ'_{-2i-3} with $i \geq 1$. The images of Δ'_{-2i-1} with $i \geq 1$ are not drawn in Figure 10.3. Not only do we have overlap we also have holes. Note that the image of Δ_{-3} does not fit to the rest anymore since the highest y value is $\frac{-2}{3}$. Figure 10.4 shows a simulation of the natural extension. We can see that it indeed has holes.

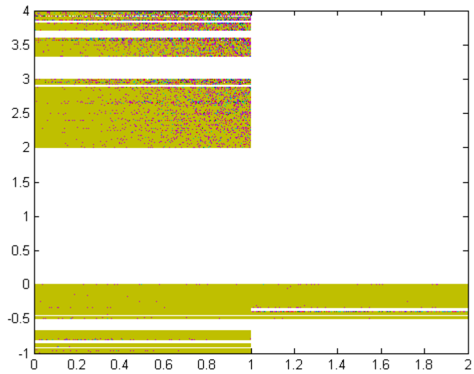

Figure 10.4: A simulation of the domain of the natural extension of the odd expansion with $F = [0, 1]$.

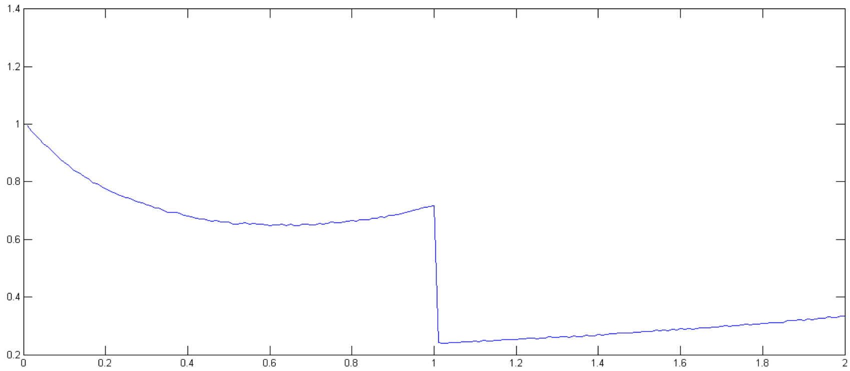

Figure 10.5: A simulation of the density for the odd expansion with $F = [0, 1]$.

Even though we do not have the theoretical density we can still simulate the density (see Figure 10.5). With this simulated density we can calculate the percentage of occurrence of certain digits using Birkhoffs ergodic theorem. The result is found in Table 10.1. For the percentage of $\varepsilon = 1$ we find it is the same as the percentage of occurrence of digit 1. Since on all other fundamental intervals $\varepsilon = -1$.

1	3	5	7	9	11	other
28.11	23.17	16.90	8.79	4.18	3.08	15.77

Table 10.1: Percentages of the occurrence of certain digits.

It might be disappointing that we did not find the invariant measure but in the next two examples we will find the invariant measure by using the method as in Section 3.1.

10.3 Odd expansion with $L = [0, 1]$

Now we will pick the set $L = [0, 1]$ and choose a set F such that our expansion will become odd. It turns out we have to choose $F = D_{odd}$. In Figure 10.6 the graph is shown.

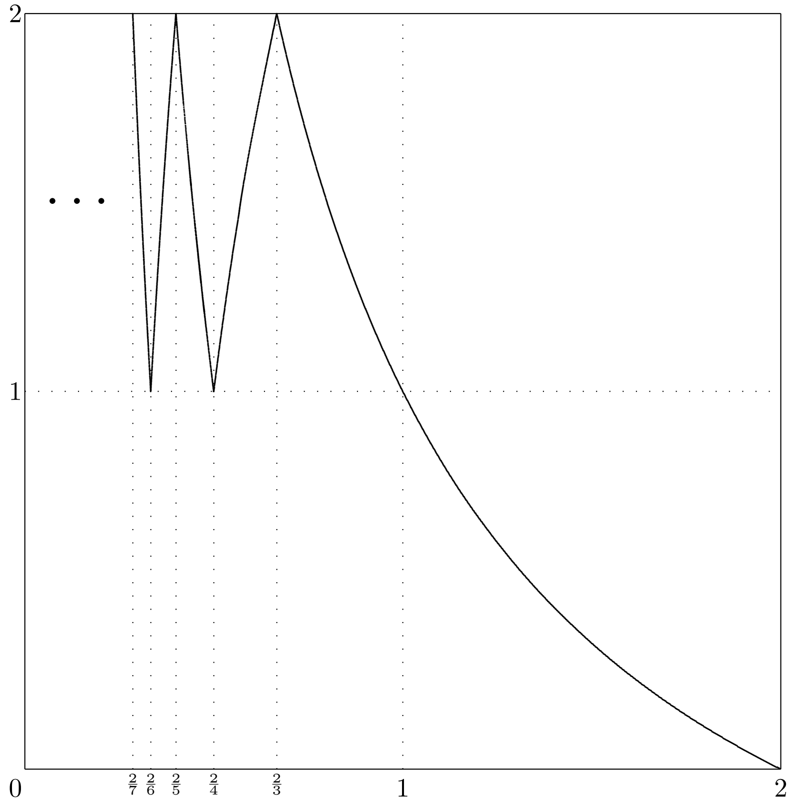

Figure 10.6: The odd expansion for $L = [0, 1]$.

This map could have been obtained by taking the flipped even expansion and then putting $L = [0, 1]$.
To find the invariant measure we will use the method of Section 3.1 again adjusting the domain. This time we have to determine A, B and D for which the map is bijective. We get the following equations:

$$A = -\frac{2}{5+D} \quad B = \frac{2}{1+A} \quad \text{and } D = \frac{2}{1+D} \ .$$

The natural extension is shown in Figure 10.7.

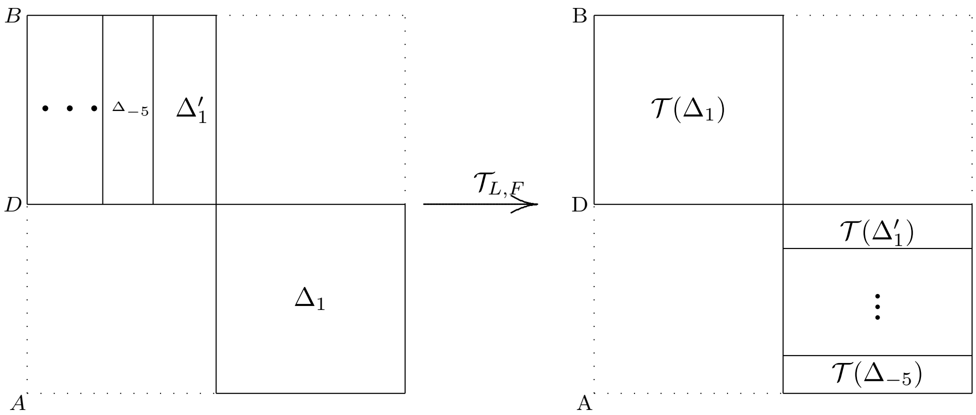

Figure 10.7: Ω and $\mathcal{T}_{L,F}(\Omega)$.

The third equation gives us $D = 1$ or $D = -2$. Now $D = -2$ gives us $A > D$ which is impossible so we find $D = 1$. This gives us $A = -\frac{1}{3}$ and $B = 3$. This gives us that the density on $[0, 1]$ is

$$f(x) = C\left(\frac{3}{2+3x} - \frac{1}{2+x}\right)$$

and on $[1, 2]$ we find

$$f(x) = C\left(\frac{1}{2+x} + \frac{1}{6-x}\right)$$

where $C^{-1} = \ln\left(\frac{25}{4}\right)$. A simulation of the density is shown in Figure 10.8.

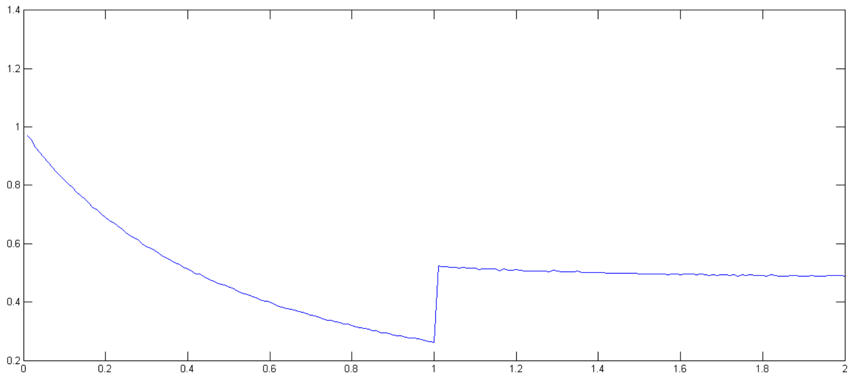

Figure 10.8: A simulation of the density for the odd expansion with $L = [0, 1]$.

10.4 The 'large' flip

For the following expansion we will take

$$L = \{x : \quad \left\lfloor \frac{2}{x} \right\rfloor \equiv 2 \mod 4 \quad \vee \quad \left\lfloor \frac{2}{x} \right\rfloor \equiv 3 \mod 4\}$$

and

$$F = \{x : \quad \left\lfloor \frac{2}{x} \right\rfloor \equiv 0 \mod 4 \quad \vee \quad \left\lfloor \frac{2}{x} \right\rfloor \equiv 3 \mod 4\}$$

The map we get is shown in Figure 10.9.

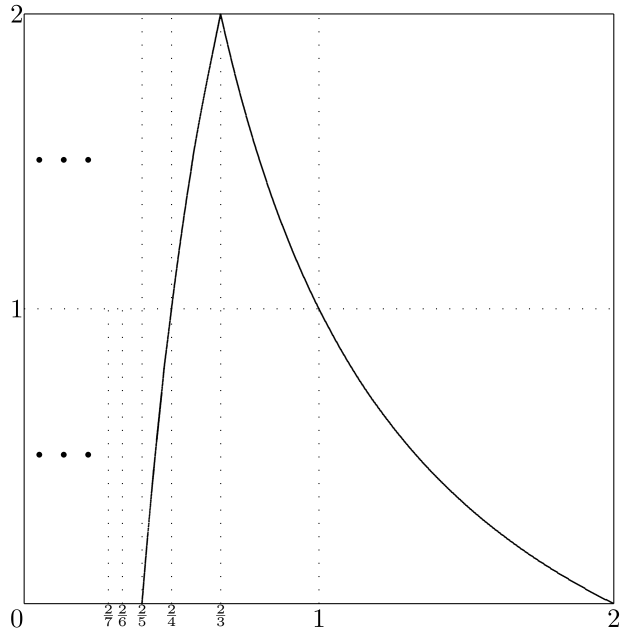

Figure 10.9: The large flip map.

Note that this map gives only digits d_i with $d_i \equiv 1 \mod 4$ for all $x \in [0, 2]$. Using the method of Section 3.1 adjusting the domain we get the following equations for A and B:

$$A = \frac{-2}{5 + A} \quad \text{and } B = \frac{2}{1 + A} \ .$$

This gives us $A = \frac{1}{2}(\sqrt{17} - 5)$ and $B = \frac{4}{\sqrt{17}-3} = \frac{1}{2}(\sqrt{17} + 3)$. The density is now given by

$$f(x) = C \left(\frac{\sqrt{17} + 3}{4 + (\sqrt{17} - 5)x} - \frac{\sqrt{17} - 5}{4 + (\sqrt{17} - 5)x} \right)$$

with $C^{-1} = \ln \left(\frac{5+\sqrt{17}}{\sqrt{17}-3} \right)$.

Chapter 11

The ergodic parts of the non-ergodic mappings

In Chapter 7 we saw that in the family of new continued fraction maps there are infinitely many maps which are not ergodic. In this chapter we will look into two specific cases. For $L = \emptyset$ we found an F for which the map is not ergodic and for $F = \emptyset$ we also found one map for which $T_{L,F}$ is not ergodic. In both cases the domain splits into two parts on which $T_{L,F}$ acts seperately. Whenever we restrict the map to one of those parts we get a map that is ergodic. In this way we get four different ergodic maps which are discussed in this chapter. We will see that we can find the invariant measure for two of them. Two of them will have finitely many digits and two of them will have infinitely many digits. For one of the maps with finitely many digits we could not find the invariant measure (even though we know it exists due to the results in Chapter 7). We will discus a classic theorem of Gauss and Kuzmin and give a new method of approximating the density of the invariant measure of this map. This new approximation method will only be usable for expansions with finitely many digits. But first we start with the two maps having $F = \emptyset$.

11.1 A 2-expansion, part 1

The first map we will discuss with $F = \emptyset$ has as domain $[\sqrt{2}-1, \sqrt{2}]$. Rather than giving the set L we will define it on each of its fundamental intervals. Note that these intervals are different from D_i.
Let $T(x) : [\sqrt{2}-1, \sqrt{2}] \to [\sqrt{2}-1, \sqrt{2}]$ be defined by

$$T(x) = \begin{cases} \dfrac{2}{x} - 1 & \text{for} \quad 2(\sqrt{2}-1) < x \leq \sqrt{2} \\ \dfrac{2}{x} - 2 & \text{for} \quad 2-\sqrt{2} < x \leq 2(\sqrt{2}-1) \\ \dfrac{2}{x} - 3 & \text{for} \quad \frac{1}{7}(6-2\sqrt{2}) < x \leq 2-\sqrt{2} \\ \dfrac{2}{x} - 4 & \text{for} \quad \sqrt{2}-1 \leq x \leq \frac{1}{7}(6-2\sqrt{2}) \, . \end{cases}$$

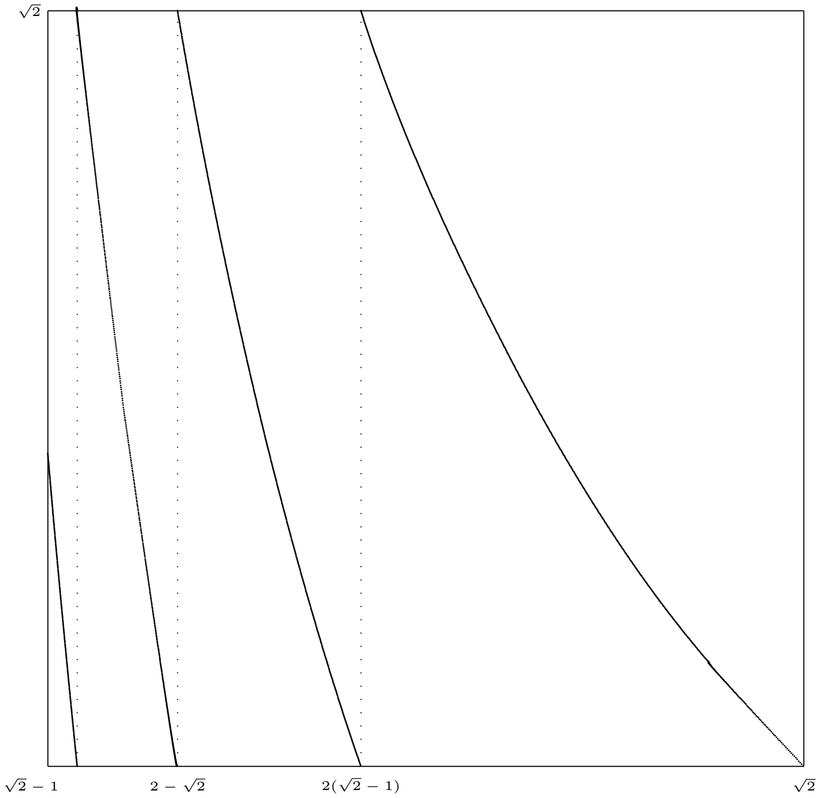

Figure 11.1: A 2-expansion on the interval $[\sqrt{2}-1, \sqrt{2}]$.

A graph of this map is shown in Figure 11.1. Note that on the fundamental interval $[\sqrt{2}-1), \frac{1}{7}(6-2\sqrt{2})]$ the map is not full (which means that $T\left([\sqrt{2}-1), \frac{1}{7}(6-2\sqrt{2})]\right)$ is not the entire domain). Still we can find the invariant measure for this map by using the method as in Section 3.1 adjusting the domain. Also note that we could not use Lemma 7.5 to prove ergodicity. Though it is easy to check that, whenever you take the fundamental intervals as partition, the system is Markov and therefore ergodic by using Lemma 7.4. When making the natural extension as in Section 3.1 we find the following equations for A, B and C:

$$A = \frac{2}{4+C} \quad B = \frac{2}{3+C} \quad \text{and } C = \frac{2}{1+B} \ .$$

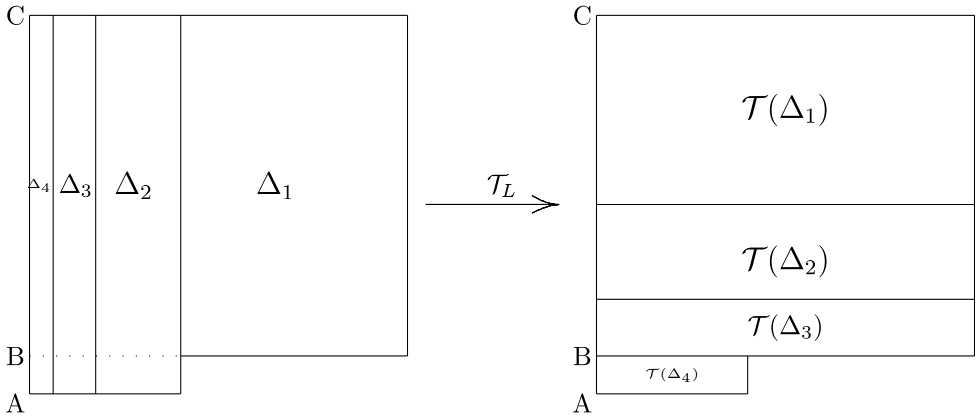

Figure 11.2: Ω and $\mathcal{T}_L(\Omega)$.

This results in $A = \frac{1}{2}(\sqrt{33}-5)$, $B = \frac{1}{6}(\sqrt{33}-3)$ and $C = \frac{1}{2}(\sqrt{33}-3)$. We find the following density up to an integration constant

$$f(x) = \begin{cases} \frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-5}{4+(\sqrt{33}-5)x} & \text{for } \sqrt{2}-1 < x \leq 2(\sqrt{2}-1) \\ \frac{\sqrt{33}-3}{4+(\sqrt{33}-3)x} - \frac{\sqrt{33}-3}{12+(\sqrt{33}-3)x} & \text{for } 2(\sqrt{2}-1) < x \leq \sqrt{2} \end{cases}.$$

The graph of the density is shown in Figure 11.3.

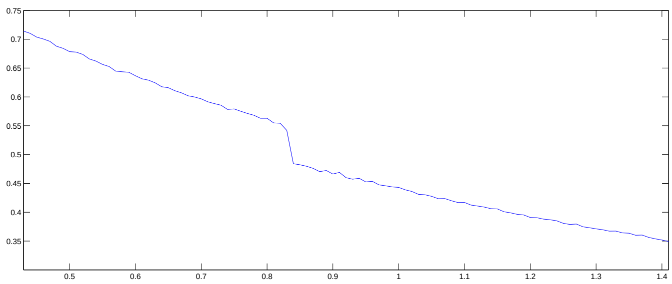

Figure 11.3: A simulation of the 2-expansion on $[\sqrt{2}-1, \sqrt{2}]$.

In the next section we will see the map with $F = \emptyset$ which has $[\sqrt{2}-1, \sqrt{2}]^c$ as domain.

11.2 A 2-expansion, part 2

For the map which has the domain $[\sqrt{2}-1, \sqrt{2}]^c$ and has $F = \emptyset$ we have that $L = [0, \sqrt{2}-1] \cap \{x : T_{\emptyset,\emptyset}(x) \in [\sqrt{2}-1, \sqrt{2}]\}$. This gives a 2-expansion without digits 2 and 3. A graph of the map is shown in Figure 11.4.

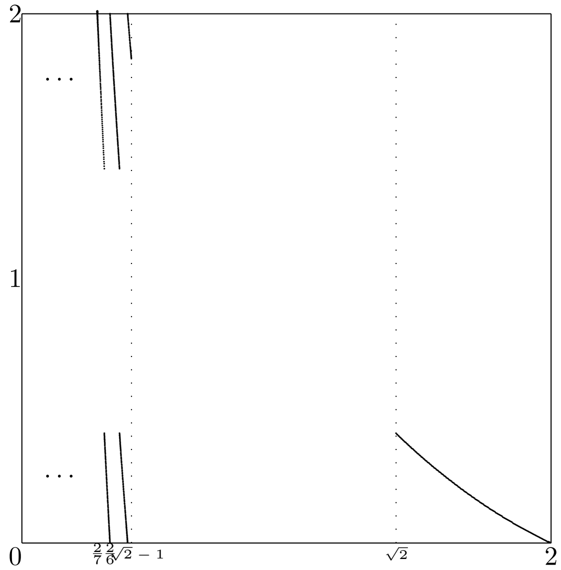

Figure 11.4: A non-ergodic 2-expansion on $[\sqrt{2}-1, \sqrt{2}]^c$.

Again we can use Lemma 7.4 to prove ergodicity since this map is Markov. Now we would like to find the natural extension. However the method at in Section 3.1 did not give any results. Whenever we simulate the domain of the natural extension we get the domain as in Figure 11.5.

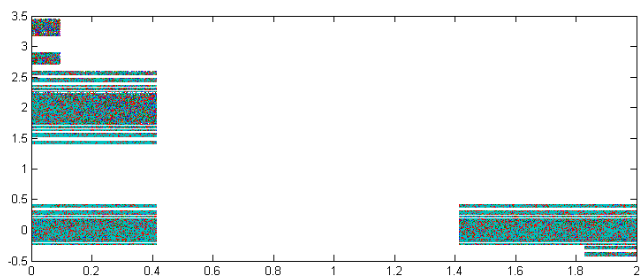

Figure 11.5: A simulation of the domain of the extension of the 2-expansion on $[\sqrt{2}-1, \sqrt{2}]^c$.

Observe that it has a lot of holes and therefore it is not surprising we could not find the natural extension. The program for simulating the domain can be found in Appendix B.0.6.

11.3 The 'tiny flip'

In this section we will choose L and F in such a way that $T_{L,F}(x) < \frac{1}{2}$. This yields $T_{L,F}(x) \in [0, \frac{1}{2}]$ for all $x \in [0, 2]$ so we will take $[0, \frac{1}{2}]$ as our domain. Note that the digits $1, 2$ and 3 are omitted. For digit 4 we have that $\varepsilon = 1$ and for the remaining digits ε can be both positive and negative.

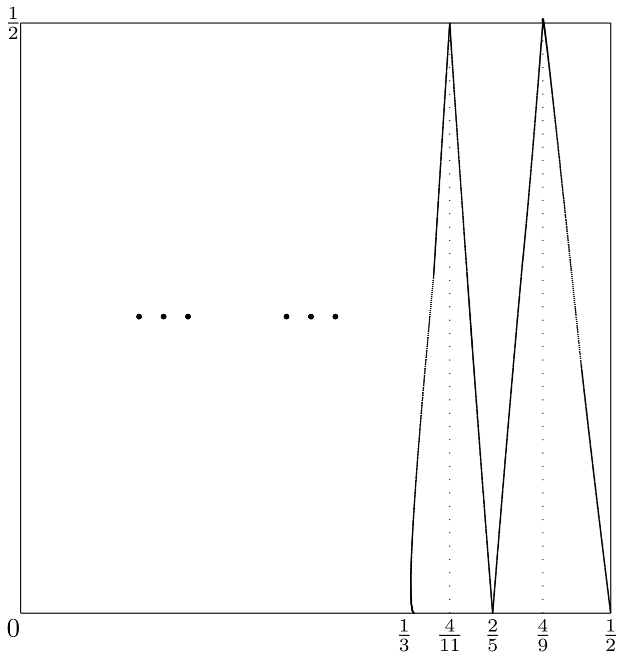

Figure 11.6: The tiny flip map.

To obtain the invariant measure we will use the method of Section 3.1 again. Adjusting the boundaries A and B, we see that for A we have the equation

$$A = A^2 + 5A + 2 = 0 \quad \text{and for } B, \quad B = \frac{2}{4 + A} \ .$$

Now for A it turns out $A = \frac{1}{2}\left(\sqrt{17} - 5\right)$ is the right choice to pick. This yields $B = \frac{1}{2}(\sqrt{17} - 3)$ and we find the following invariant measure

$$f(x) = C\left(\frac{\sqrt{17} - 3}{4 + (\sqrt{17} - 3)x} - \frac{\sqrt{17} - 5}{4 + (\sqrt{17} - 5)x}\right)$$

with

$$C^{-1} = \ln\left(\frac{5 + \sqrt{17}}{3 + \sqrt{17}}\right) \ .$$

In Figure 11.7 we can see that the density is almost uniform. We may expected this when looking at Figure 11.6.

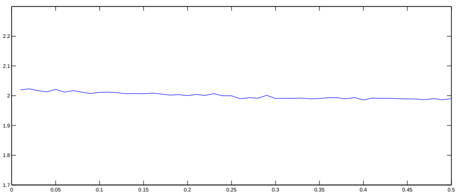

Figure 11.7: A simulation of the density for the tiny flip.

In the next section we will see the ergodic map with $L = \emptyset$ and has $[\frac{1}{2}, 1]$ as its domain. Since the natural extension is not found and the map has finitely many digits we can use an approximation method based on the idea behind the Gauss Kuzmin Theorem. This method is also explained in the next section.

11.4 A 'tiny flip' on $[\frac{1}{2}, 1]$

In this section we will study the map mentioned in Chapter 7 which has as domain $[\frac{1}{2}, 1]$ with $L = \emptyset$ and $F = [\frac{4}{7}, \frac{2}{3}] \cup [\frac{4}{5}, 1]$. Note that for this map $d(x)$ only attains the values $2, 3, 4$.

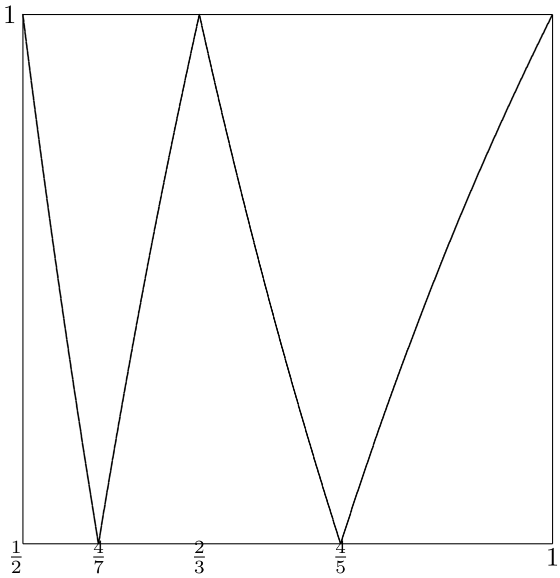

Figure 11.8: An expansion map on $[\frac{1}{2}, 1]$

Unfortunately, we were not able to obtain the invariant measure by finding the appropriate domain for the natural extension. Again we can simulate the natural extension which gives a not very promising result (see Figure 11.9).

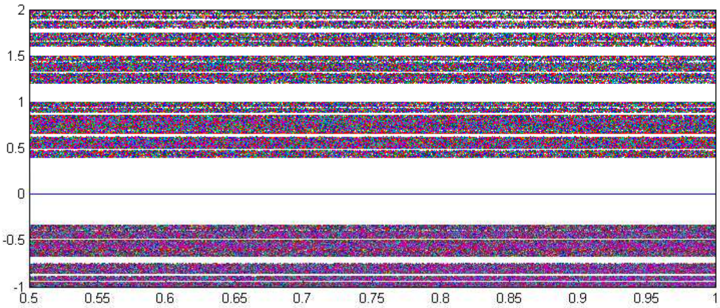

Figure 11.9: A simulation of the domain of the extension for the 'tiny flip' on $[\frac{1}{2}, 1]$.

Fortunately we can use Gauss Kuzmin Theorem to approximate the density.

Theorem 11.1 (Gauss Kuzmin)**.** *Let T be the regular continued fraction map, μ the Gauss measure and λ the Lebesgue measure.*
Then for any measurable set A we have

$$\lambda(T^{-n}(A)) \to \mu(A)$$

as $n \to \infty$.

Note that this is not the exact same theorem because in the origional theorem there is a speed of convergence given as well. This theorem was stated as an hypothesis by Gauss in his diary and was proven by Kuzmin in 1928 and (independently) by Paul Lévy in 1929. A proof can be found in [9]. Now this Theorem is stated and proved for the regular continued fraction expansion. We believe it is true for the new family of continued fraction expansions as well. Because the proof is long and fairly complicated we did not proof the theorem for this new family. Instead we will show a simulation of a case where it works perfectly. This will give us reason to believe it is also true for the new continued fraction maps. The idea of the theorem being applicable for other expansions is not new. In fact, a lot of work has been done to prove the theorem (or similar theorems) for other expansions then the regular one. For example in [17] the theorem is proved for θ-expansions and in [8] it is proved for a normal flipped expansion having only odd digits. The latter is closely related to the new continued fraction expansions.
In Figure 11.10 we can see that this methods approximation of the density is close to the simulation of random points. In the next chapter we will see an example of an expansion of which we know the invariant measure and we will see that this method gives such a good approximation that the density and the approximation coincide in a graph. Note that the estimate given by the "Gauss Kuzmin" approximation is much smoother then the one we got by simulation.

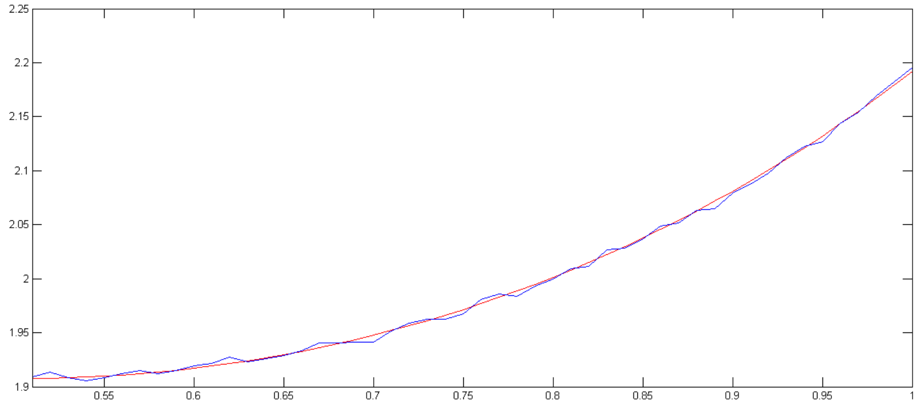

Figure 11.10: An approximation using the Gauss Kuzmin method on the attractor together with a simulation of the density of the 'tiny flip' on $[\frac{1}{2}, 1]$.

Chapter 12

Other mappings giving finitely many digits with $N = 4$

In the previous chapter we ended with an expansion with finitely many digits. This expansion both had even and only odd digits. In this chapter we will make an expansion with $N > 2$ allowing us to have expansions with finitely many digits with either only even or only odd digits. We will see three examples of expansions with finitely many digits with $N = 4$. First we will see an expansion with digits $\{1, 2\}$ then, by flipping, we will find an expansion with digits $\{4, 2\}$ and then the last one will have digits $\{5, 1\}$. In all three cases we will give a approximation of the density by using the Gauss-Kuzmin method as in 11.4.

12.1 A 4-expansion with digits 1 and 2

Let $T : [1, 2] \to [1, 2]$ be defined as

$$T(x) = \begin{cases} \dfrac{4}{x} - 1 & \text{for} \quad x \in (\frac{4}{3}, 2] \\ \dfrac{4}{x} - 2 & \text{for} \quad x \in [1, \frac{4}{3}] \end{cases}$$

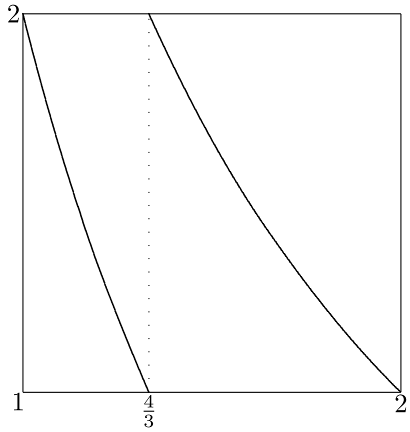

Figure 12.1: A 4-expansion with digits 1 and 2.

For this expansion we can find the natural extension my using the method as in Section 3.1. We find the following equations for A and B

$$A = \frac{4}{2 + B} \quad \text{and} \quad B = \frac{4}{1 + A}$$

which gives us $A = 1$ and $B = 2$. So the invariant density is

$$f(x) = C \left(\frac{1}{2 + x} - \frac{1}{4 + x} \right)$$

with $C^{-1} = \ln(\frac{10}{9})$.
Now if we apply the method of Gauss-Kuzmin we find a very good approximation. In Figure 12.2 both the theoretic and the approximation are shown. In the Gauss-Kuzmin method only 10 iterations are used. Note that with these axis the difference can hardly be seen by the eye! If we calculate difference in 2-norm we get

$$\left(\int_1^2 (f(x) - \hat{f}(x))^2 \, dx \right)^{\frac{1}{2}} = 1.1235 * 10^{-5}$$

where $f(x)$ is the true density and $\hat{f}(x)$ the approximation. In Figure 12.3 we zoomed in a lot to show there are indeed two functions plotted.

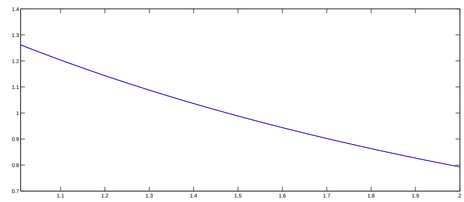

Figure 12.2: An approximation of the density for a 4-expansion with digits $\{1, 2\}$ and the density.

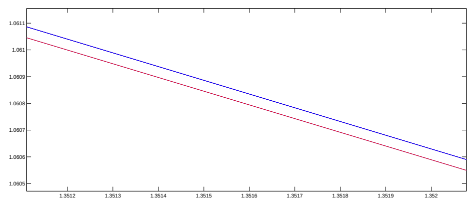

Figure 12.3: Zoomed in on a part of Figure 12.2

12.2 A 4-expansion with digits 2 and 4

In this section we will see an expansion with only even digits by using the map as in previous section but then flipping it on the first fundamental interval. Let $T : [1, 2] \to [1, 2]$ be defined as

$$T(x) = \begin{cases} 4 - \dfrac{4}{x} & \text{for} \quad x \in (\frac{4}{3}, 2] \\ \dfrac{4}{x} - 2 & \text{for} \quad x \in [1, \frac{4}{3}] \ . \end{cases}$$

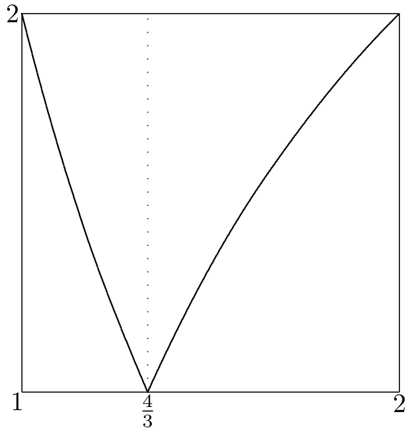

Figure 12.4: A 4 -expansion with digits 2 and 4.

When using method as in Section 3.1 we find the equations for A and B:

$$A = \frac{-4}{4+A} \quad \text{and} \quad B = \frac{4}{2+A} \ .$$

Now the first equation gives us $A = -2$ which results in a division by zero in the equation for B. If we set $B = \infty$ then we find that our natural extension is bijective. For the density $f(x)$ we find

$$f(x) = \frac{1}{x} + \frac{1}{2-x}$$

note that $f(x) \to \infty$ as $x \to 2$ and indeed we have a σ-finite infinite invariant measure. In Figure 12.5 an approximation is found by using the Gauss Kuzmin method. We can see it is tending to infinity in 2 but not that strong.

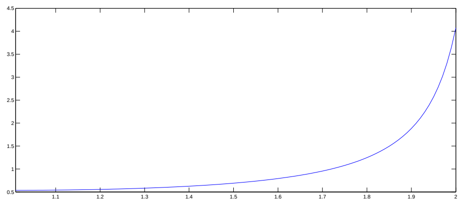

Figure 12.5: An approximation using the Gauss Kuzmin method for a 4-expansion with digits $\{2, 4\}$

12.3 A 4-expansion with digits 1 and 5

Let $T : [1, 2] \to [1, 2]$ be defined as

$$T(x) = \begin{cases} \frac{4}{x} - 1 & \text{for} \quad x \in (\frac{4}{3}, 2] \\ 5 - \frac{4}{x} & \text{for} \quad x \in [1, \frac{4}{3}] \ . \end{cases}$$

Again we will find a natural extension. This time the equations for A and B are

$$A = \frac{-4}{5+A} \quad \text{and} \quad B = \frac{4}{1+A}$$

which gives $A = -1$ and we set $B = \infty$ as before. Which leads to the density

$$f(x) = C\left(\frac{1}{x} + \frac{1}{4-x}\right)$$

with $C^{-1} = \ln(3)$.

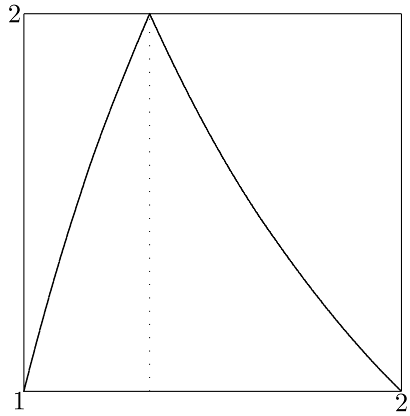

Figure 12.6: A 4 -expansion with digits 1 and 5.

Again we see the method of Gauss-Kuzmin is very good in approximating the density (see Figure 12.7)

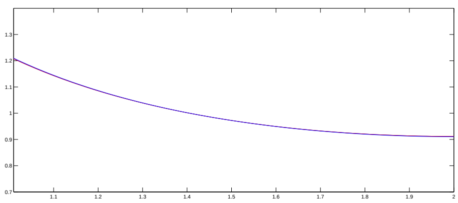

Figure 12.7: An approximation of the density for a 4-expansion with digits $\{1,5\}$ and the density.

This was the last expansion that is discussed in this thesis. There is still a lot of things that can be studied concerning the expansions studied in this project. But before we give suggestions on things to study in the future we will reflect on what we did in this thesis since a lot of results have been obtained during this project.

Chapter 13

Conclusion

Before this study on the new family of continued fraction maps started we did not expect to find so many invariant measures. The method of making a natural extension and then integrate over the second coordinate turned out to be extremely powerful. We have seen a large variety of shapes in the domains of different natural extensions. Initially the new family of continued fraction maps was introduced to find infinitely many continued fraction maps with only even or only odd digits. Since the new family of continued fractions almost always have 4 maps to choose from there is a lot of freedom. This resulted in having infinitely many maps with only odd or only even digits instead of finding two maps with only odd digits and two maps for only even digits. On the other hand, we could only find one way of choosing F for which the flipped map is not ergodic and one way of choosing L for which the 2-expansion was not ergodic and infinitely many ways of choosing L and F for which the map $T_{L,F}$ was not ergodic. At first glance, we did not expect this.

It was unfortunate that one of the mapping for which we could not find the invariant measure was the regular even expansion with $F = [1, 2]$. Still it was interesting to see that the corresponding density seemed to be symmetric and if it was symmetric then it was the same density of the mapping of switching digits by using Lemma 8.1. The lemma we used in the chapter on the 3 divisible expansion.

The map in Section 11.4 looked easy at first. Though, we could not find the invariant measure by using the natural extension. Simulations gave interesting insight in how the domain of the natural extension would look like. The simulations showed that the natural extension would look rather complicated. The Gauss-Kuzmin method looked promising. The simulation of random points and the approximation found by using the Gauss-Kuzmin method where very close. In the cases where we did know the density and had finitely many digits the Gauss-Kuzmin method gave a very good approximation.

In this project we encountered a wide variety of things. Some parts where harder then expected and others where far easier then expected. When looking back, I think it was really rewarding and interesting to look into. Still a lot of open questions and ideas are there yet to be studied.

Chapter 14

For future study

We have seen a lot of different expansions in this thesis. Though most of them are discussed briefly. More study on those can be done (for example calculating constants of Khinchine and almost sure percentages of digits occurring in a continued fraction). For the $T_{\emptyset,F}$ we found exactly one F for which we had a non-ergodic map. A natural question would be: Is this the only F which $T_{\emptyset,F}$ is not ergodic? If it is, a prove of that might be hard. On the other hand, if it is not the only F then a description of all choices for F for which the map is not ergodic might be a nice result. We can ask the same question for the mappings $T_{L,\emptyset}$ since we only found one non-ergodic map as well.

It was good to see we found so many invariant measures. Still one might wonder why we did not find the ones we could not find. What made these maps so different from the others? And can we give some criteria for $\{L, F\}$ in such a way we can say, in advance, whether the method of making a natural extension will work to find the invariant measure? Two maps of which we could not find the density seemed to be the same density and seemed to be symmetric (The even regular map with $F = [1, 2]$ and the switching digits map). Two other maps had a symmetric density which where the same (3-divisible and his spouse). Are these 2/4 mappings the only mappings which have a symmetric density? For flipped N-expansions with $N > 2$ we might find more dynamical systems with a symmetric density. These dynamical systems can be a topic of a study.

A study on other flipped N-expansions can be done. In particular n-divisible expansions for $n = 1, 2, \dots N + 1$. Note that for $n = 1, 2, \dots N$ we can always find infinitely many maps which give digits that are only n-divisible. and for $n = N + 1$ we will find exactly one map with digits only n-divisible.

A chapter that was initially in the thesis was a chapter on robustness of the density of a continued fraction map. Since for the even regular 2-expansion without flip it was easy to find the density and with flip we could not. We wondered why, by flipping one fundamental interval, everything changes. So the question arose of how the density changes if L or F is slightly changed. Eventually the chapter deviated too much from the thesis, but could be the base of a new study.

The method of Gauss-Kuzmin looked very promising. Though, the way it works uses the fact that you only have finitely many digits. An idea is to adjust the method in such a way that you can find a good approximation for expansions with infinitely many digits. Also a proof of the theorem for our new continued fraction maps is not yet been made.

In this thesis a study on approximation properties is left out. Study on approximation coefficients can still be done. A classical result for the regular continued fraction expansion is that whenever we have when

$$\theta_n(x) = q_n(x)^2 \left| x - \frac{p_n(x)}{q_n(x)} \right|$$

where $\frac{p_n(x)}{q_n(x)}$ is the n^{th} convergent then

$$\min\{\theta_{n-1}(x), \theta_n(x), \theta_{n+1}(x)\} < \frac{1}{\sqrt{5}} .$$

This is a result from Borel found in 1903. It would be interesting to see what kind of bounds on the θ_n's we could have if we do not use the regular continued fraction map but use maps which are studied in this thesis instead.

Another result for the regular continued fraction is that a continued fraction expansion of $x \in [0, 1]$ is periodic if and only if it is a quadratic irrational i.e. a solution to $ax^2 + bx + c = 0$ for $a, b, c \in \mathbb{N}$. To prove that for each periodic expansion the number it represents is a quadratic irrational is easy but the other way around is hard. In [7] a lot of information on quadratic irrationals is found. For other mappings the same theorem seems to be hard to prove. In our case questions concerning quadratic irrationals are:

1. Can we choose L, F in such a way every quadratic irrational is periodic and if so is there only one way or are there multiple ways?

2. Can we choose L, F in such a way no quadratic irrational is periodic and if so is there only one way or are there multiple ways?

3. For specific L, F how can you determine if every quadratic irrational is periodic?

These are hard questions but are interesting to say the least.

Appendices

Appendix A

The regular continued fraction expansion

In this first appendix the regular continued fraction expansion is explained together with concepts belonging to the field of ergodic theory and dynamical systems. Almost everything that is included in this first appendix can be found in [4] and also in [10] which is in Dutch.

A.1 Introducing the map for the regular continued fraction expansion

The regular continued fraction expansion is the most well-known fraction expansion there is. It is also the oldest continued fraction expansion and could be considered the mother of all continued fraction expansions. A regular continued fraction expansion is an expansion of a number $x \in \mathbb{R}$ of the form

$$x = d_0 + \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{\ddots}}}$$

Where $d_n \in \mathbb{N}$ for all n. The numbers d_n are called the digits of x. Sometimes we use the shorthand notation $x = [d_0; d_1, d_2, \ldots]$.

For all $x \in \mathbb{R}$ we can find the digits by using a map.
Let $T : [0,1) \to [0,1)$

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{if } x \neq 0 \quad \text{and} \quad T(0) = 0$$

We now define $d_0(x) = d(x) = \left\lfloor \frac{1}{x} \right\rfloor$ and $d_n(x) = d(T^{n-1}(x))$. Without loss of generality we can choose $d_0 = 0$ and look at $x \in [0,1]$ only. We can write x as follows:

$$x = \frac{1}{d(x) + T(x)} = \frac{1}{d_1 + T(x)} \; .$$

Now

$$T^2(x) = \frac{1}{T(x)} - \left\lfloor \frac{1}{T(x)} \right\rfloor = \frac{1}{\frac{1}{x} - d_1} - d_2$$

gives us

$$x = \cfrac{1}{d_1 + \cfrac{1}{d_2 + T^2(x)}}$$

In general we can write, if $T^k(x) \neq 0$ for $0 \leq k < n$,

$$x = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{\ddots}{d_n + T^n(x)}}}$$

We can now define the nth convergent of x as

$$c_n(x) = \frac{p_n(x)}{q_n(x)} = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{\ddots}{d_n}}} .$$

with short hand notation $[0; d_1, d_2, \ldots, d_n]$. We have that $c_n \to x$ as $n \to \infty$ in Appendix A.1.1 this is proved and recurrent relationships are found. But first we will prove that rational numbers have a finite expansion.

Suppose rational numbers have an infinite expansion. Let $\frac{p_0}{q_0}$ with $p_0, q_0 \in \mathbb{N}$ and $p_0 < q_0$. Note that $T(\mathbb{Q} \cap [0,1]) = \mathbb{Q} \cap [0,1]$ so we can define $T^n(\frac{p_0}{q_0}) = \frac{p_n}{q_n}$ with $p_n, q_n \in \mathbb{N}$, relatively prime and $p_n < q_n$. Now

$$T\left(\frac{p_n}{q_n}\right) = \frac{1}{\frac{p_n}{q_n}} - \left\lfloor \frac{1}{\frac{p_n}{q_n}} \right\rfloor = \frac{q_n}{p_n} - d(\frac{p_n}{q_n}) = \frac{q_n - p_n d(\frac{p_n}{q_n})}{p_n} = \frac{p_{n+1}}{q_{n+1}}$$

gives us $p_{n+1} \leq q_n - p_n d(\frac{p_n}{q_n}) < p_n$ because we divided by the greatest common divisor and because the rational is in $[0,1]$. This result is obtained for all $n \in \mathbb{N}$ which gives us an infinite strictly decreasing sequence

$$\ldots < p_{n+1} < p_n < \ldots < p_o$$

which is impossible for a sequence with values in $\mathbb{N}$ so we find that every rational number has a finite expansion.

A.1.1 Convergence and recurrence formulas

We will prove that the convergents of a given number $x \in [0,1]$ will indeed converge to x. This is done by using a Möbius transformation and matrices. It is basically the same proof as in [10] but in English instead of Dutch.
Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with } \det(A) \neq 0 .$$

We define

$$A(x) := \frac{ax+b}{cx+d} .$$

which is a Möbius transformation. Furthermore we define

$$A_n = \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} \quad \text{and} \quad M_n = A_1 A_2 \cdots A_n$$

where a_n are the digits of x. Now note that we have

$$\begin{aligned} M_n(0) &= (M_{n-1}A_n)(0) = M_{n-1}\left(\frac{1}{a_n}\right) \\ &= M_{n-2}A_{n-1}(\frac{1}{a_n}) = M_{n-2}\left(\frac{1}{a_{n-1} + \frac{1}{a_n}}\right) \\ &\vdots \\ &= c_n \end{aligned}$$

and $c_n = \frac{a0+p_n}{c0+q_n}$ so let us write $M_n = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}$. On the other hand

$$M_n = M_{n-1}A_n = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}$$

$$= \begin{bmatrix} p_{n-1} & a_n p_{n-1} + r_{n-1} \\ q_{n-1} & a_n q_{n-1} + s_{n-1} \end{bmatrix}$$

which gives us that

$$r_n = p_{n-1} \quad \text{(which leads to } r_{n-1} = p_{n-2})$$

$$s_n = q_{n-1} \quad \text{(which leads to } s_{n-1} = q_{n-2})$$

and

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n p_{n-1} + q_{n-2} \ .$$

We will now write a matrix that will give x as the möbius transform of 0. Let

$$M_n^* = M_{n-1} \begin{bmatrix} 0 & 1 \\ 1 & a_n + T_n \end{bmatrix} \quad \text{with } T_n = T^n(x) \ .$$

This will give us

$$M_n^*(0) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_n + T_n}}} = x \ .$$

We can now find an appropriate expression for x

$$\begin{aligned}
x &= M_n^*(0) = M_{n-1}\begin{bmatrix} 0 & 1 \\ 1 & a_n + T_n \end{bmatrix}(0) \\
&= \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix}\left(\frac{1}{a_n + T_n}\right) \\
&= \frac{p_{n-2}\,\dfrac{1}{a_n+T_n} + p_{n-1}}{q_{n-2}\,\dfrac{1}{a_n+T_n} + q_{n-1}} \\
&= \frac{p_{n-2} + p_{n-1}(a_n + T_n)}{q_{n-2} + q_{n-1}(a_n + T_n)} \\
&= \frac{p_n + p_{n-1}T_n}{q_n + q_{n-1}T_n}
\end{aligned}$$

This enables us to find an expression for the difference of x and the convergent c_n

$$\begin{aligned}
x - c_n &= \frac{p_n + p_{n-1}T_n}{q_n + q_{n-1}T_n} - \frac{p_n}{q_n} \\
&= \frac{q_n(p_n + p_{n-1}T_n) - (q_n + q_{n-1}T_n)p_n}{(q_n + T_n q_{n-1})q_n} \\
&= \frac{(p_{n-1}q_n - p_n q_{n-1})T_n}{q_n(q_n + T_n q_{n-1})} \ .
\end{aligned}$$

Now note that

$$p_{n-1}q_n - p_n q_{n-1} = \det(M_n) = \det(A_1)\det(A_2)\cdots\det(A_n) = (-1)^n$$

and by using this we find

$$x - c_n = \frac{(-1)^n T_n}{q_n(q_n + T_n q_{n-1})} \ .$$

This gives us

$$|x - c_n| = \frac{T_n}{q_n(q_n + T_n q_{n-1})} < \frac{1}{q_n^2}$$

so the faster q_n grows the better estimate we find. The number which converges the slowest is the one with $a_n = 1$ for all n. This is the number $\frac{1}{2}\sqrt{5} - \frac{1}{2}$. From the recurrent formulas we can see that for this number q_n will still grow exponentially fast which gives us that

$$c_n - \to x \text{ as } n \to \infty \qquad \text{for all } x \in [0, 1].$$

In the next appendix we will see what an invariant measure is.

A.1.2 Invariant measures

The existence of an invariant measure is very helpful in dynamical systems. A lot of machinery evolved around it so in this appendix we will give the definition. But before we can introduce the invariant measure the definition of a σ-algebra and the definition of a measure is given.

Definition 1. *Let $\mathcal{F}$ be a set of subsets of a set X. Then $\mathcal{F}$ is said to be a σ-algebra if the following conditions hold*

- $X \in \mathcal{F}$
- *if* $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$
- *if* $A_1, A_2, \ldots \in \mathcal{F}$ *then* $A = \cup_{n=1}^{\infty} A_n \in \mathcal{F}$.

Now the pair $(X, \mathcal{F})$ is called a measure space. We will now define a measure.

Definition 2. *Let $(X, \mathcal{F})$ be a measure space.*
A function $\mu : \mathcal{F} \to [0, \infty)$ satisfying $\mu(\emptyset) = 0$ and

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

for all $A_1, A_2, \ldots \in \mathcal{F}$ which are pairwise disjoint is called a measure on $(X, \mathcal{F})$.

Furthermore we call μ a probability measure if $\mu(X) = 1$. And $(X, \mathcal{F}, \mu)$ a probability space. The sigma algebra generated by intervals on $\mathbb{R}$ is called the Borel σ-algebra. How to make the σ-algebra from the intervals is explained in [4]. Now whenever we define $\lambda([a, b]) = b - a$ we can extend λ to the Borel σ-algebra which will give you the Lebesgue measure. This measure is well known and often used. Without going into any integration theory note that if we have a measure space $(X, \mathcal{B})$ where $\mathcal{B}$ is the Borel measure on X and we have a function $f : X \to \mathbb{R}_{>0}$ then we can define μ as

$$\mu(A) = \int_A f(x)dx \quad \text{for all } A \in \mathcal{B}.$$

Now if f is a density function we find that μ is a probability measure. We will now give the definition of an invariant measure.

Definition 3. *Let $(X, \mathcal{F}, \mu)$ be a probability space and let $T : X \to X$ be a map. Then μ is said to be invariant (or T-invariant) if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$.*

Sometimes we say that for a given probability space the map T is μ-invariant. For the continued fraction map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{if } x \neq 0 \quad \text{and} \quad T(0) = 0$$

Gauss found that whenever $\mathcal{B}$ is the borel measure on $[0, 1]$ then $\mu : \mathcal{B} \to \mathbb{R}$ given by

$$\mu(A) = \frac{1}{\ln(2)} \int_A \frac{1}{1 + x} dx$$

is a T-invariant measure (a proof of the invariantness of this measure is given in [10] in Dutch). This measure is called the Gauss measure. A historical sidenote is that nobody knows how Gauss came up with this measure.

A.1.3 Ergodicity and Birkhoffs Theorem

In ergodic theory it is not surprising that ergodicity plays the most important role. Ergon means work and odos means path in Greek. Heuristically a function $T : X \to X$ is ergodic when you have for almost all points $x \in X$ that if you let T work on x then the path $T^n(x)$ comes everywhere in X. Formally we have the following definition:

Definition 4. *Let $(X, \mathcal{F})$ be a measure space,μ a probability measure on that space and $T : X \to X$ a map. Then T is ergodic if for every $A \in \mathcal{F}$ satisfying $T^{-1}(A) = A$ we have that $\mu(A) \in \{0, 1\}$.*

For more characterisations see [4].

The following theorem plays an important role in ergodic theory and is on its strongest when applied to an ergodic function. It is named after George David Birkhoff.

Theorem A.1. *Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$. Then, for any f in $L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = f^*(x)$$

exists a.e. is T-invariant and $\int_X f\,d\mu = \int_X f^*\,d\mu$. If moreover T is ergodic, then f^* is a constant a.e. and $f^* = \int_X f\,d\mu$

It is well known that the map for the regular continued fraction is ergodic (see [4] for a proof). With Birkhoffs Theorem we can, for example, calculate how often a certain digit occurs in a continued fraction. Remember that we had $d(x) = \lfloor \frac{1}{x} \rfloor$ and $d_n(x) = d(T^{n-1}(x))$. So if we want to calculate the percentage of occurrence of 1 we have to see how often $T^n(x)$ is in the interval $[\frac{1}{2}, 1]$. Since the map is ergodic this gives the same number for almost all $x \in [0, 1]$. Now if we take the characteristic function $1_{[\frac{1}{2},1]}$ and use Birkhoffs theorem we will calculate exactly that and find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[\frac{1}{2},1]} \circ T^i(x) = \int_X 1_{[\frac{1}{2},1]}\,d\mu = \mu([\frac{1}{2}, 1]) = 0.415037\ldots$$

A.1.4 Natual extentions

A usefull tool in ergodic theory is the notion of the natural extension. First we will give the definition of a dynamical system. Then we will give the definition of a factor and after that we can give the definition of an extension which finally leads to the definition of the natural extension. Now the definition of a dynamical system is as followed

Definition 5. *A dynamical system is a quadruple* $(X, \mathcal{F}, \rho, T)$ *where* X *is a non empty set,* $\mathcal{F}$ *is a* σ*-algebra on* X*,* ρ *is a probability measure on* $(X, \mathcal{F})$ *and* $T : X \to X$ *is a surjective* ρ*-measure preserving transformation. Furthermore if* T *is injective the system is called an invertible dynamical system.*

An example of such a system is $([0, 1], \mathcal{B}, \mu, T)$ where $\mathcal{B}$ is the Borel σ-algebra, μ the gauss measure and T the regular continued fraction map. We will now give the definition of a factor.

Definition 6. *Let* $(X, \mathcal{F}, \mu, T)$ *and* $(Y, \mathcal{C}, \nu, S)$ *be two dynamical systems. Then* $(Y, \mathcal{C}, \nu, S)$ *is said to be a factor of* $(X, \mathcal{F}, \mu, T)$ *if there exists a measurable and surjective map* $\psi : X \to Y$ *such that*

- $\psi^{-1}\mathcal{C} \subset \mathcal{F}$ *(so* ψ *preserves the measure structure)*
- $\psi T = S\psi$ *(so* ψ *preserves the dynamics)*
- $\mu(\psi^{-1}E) = \nu(E)$ *for all* $E \in \mathcal{C}$ *(so* ψ *preserves the measure).*

The dynamical system $(X, \mathcal{F}, \mu, T)$ *is called an extension of* $(T, \mathcal{C}, \nu, S)$ *and* ψ *is called a factor map.*

We can finally give the definition of a natural extension.

Definition 7. *Let* $(Y, \mathcal{C}, \nu, S)$ *be a non invertible dynamical system then* $(X, \mathcal{F}, \mu, T)$ *is called a natural extension of* $(Y, \mathcal{C}, \nu, S)$ *if* Y *is a factor of* X *and the factor map* ψ *satisfies*

$$\bigvee_{m=0}^{\infty} T^m \psi^{-1}\mathcal{C} = \mathcal{F} \ .$$

Rohlin showed in [15] that natural extensions are unique up to isomorphisms. This allows us to talk about *the* natural extension of T. For the continued fraction map we have that whenever $([0, 1], \mathcal{B}, \mu, T)$ is the dynamical system as above. The natural extension $\mathcal{T} : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ is given by

$$\mathcal{T}(x, y) = \left(T(x), \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor + y} \right)$$

Note that $\mathcal{T}(x, 0) = (t_n, v_n)$ where $t_n = [d_{n+1}, d_{n+2}, \ldots]$ and $v_n = [d_n, d_{n-1}, \ldots, d_1]$ so in a sense t_n captures the future while v_n captures the past. In [12, 13] Nakada, Ito and Tanaka proved that the invariant measure for this natural extension is given by

$$\mu_{xy}(A) = \frac{1}{\ln(2)} \int_A \frac{1}{(1 + xy)^2}$$

with $A \in \mathcal{B} \otimes \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra on $[0, 1]$ and $\otimes$ is the tensor product. Furthermore they proved that this function is indeed ergodic with respect to this measure. Note that ψ is given by $\psi(x, y) = (x, 0)$ which is a projection onto the x coordinate.

Appendix B

The programming

The program used for my bachelor's project worked well and is written in a way it can be changed easily to be applicable for other map (see [10]). Therefor there I did so. The program can be used on any ergodic map T. It can iterate a lot of points from $[0, 2)$ over T. For all simulations of the density in this thesis the following procedure is done. In phase 1 we sample 2500 uniform points from $[0, 2)$ which we iterate 20 times over T. We repeat this process 400 times and find a density of 20 million points. Phase 2 is almost the same but instead of sampling uniformly from $[0, 2)$ we will sample from the density just found in phase 1. For an error analysis also see [10].

Not only the densities are simulated. A lot of other different scripts have been made. All the scripts for simulating densities will be given first and then the other programs used. For each script theres a description written as comment. Also they are ordered in a way that a script is only using other scripts that are previously explained.

B.0.5 The density machinery

```
function[y] = TDL(x)
%the function T(x) for different lazy and flipped sets. To apply
%one specific function just uncomment it. DO NOT FORGET TO COMMENT THE OLD
%ONE.
y=zeros(length(x),1);


%lazy expansion
%y=2./x-floor(2./x) + (x>=0 &x <=1);

%lazy backward
% for i=1:length(x)
%           if 0<= x(i)&& x(i)<=1
%               y(i)=2+floor(2/x(i))-2/x(i);
%           else
```

```
%               y(i)=2/x(i)-1;
%           end
%
% end

%lazy backward with F=[0,2]
% for i=1:length(x)
%           if  0<= x(i)&& x(i)<=1
%               y(i)=2+floor(2/x(i))-2/x(i);
%           else
%               y(i)=2-2/x(i);
%           end
%
% end

%regular odd
% for i=1:length(x)
%           if floor(floor(2/x(i))/2)==floor(2/x(i))/2 %
    gives if floor(2/x) is even
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           else
%               y(i)=2/x(i)-floor(2/x(i));
%           end
%
% end

%regular odd but with flip on non lazy [0,1]
% for i=1:length(x)
%           if floor(floor(2/x(i))/2)==floor(2/x(i))/2 %
    gives if floor(2/x) is even
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           else
%               y(i)=2/x(i)-floor(2/x(i));
%           end
%
% end

%odd flipped
% for i=1:length(x)
%           if floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5
    %gives if floor(2/x) is odd
%               y(i)=2/x(i)-floor(2/x(i));
%           else
%               y(i)=floor(2/x(i))-2/x(i)+1;
%           end
%
%end

%even flipped
% for i=1:length(x)
```

```
%           if  floor(floor(2/x(i))/2)==floor(2/x(i))/2 %gives if floor(2/x) is even
%               y(i)=2/x(i)-floor(2/x(i));
%           else
%               y(i)=floor(2/x(i))-2/x(i)+1;
%           end
%
% end

%regular even with F=[1,2]
% for i=1:length(x)
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2-2/x(i);
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is od
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           else
%               y(i)=2/x(i)-floor(2/x(i));
%           end
%
% end

%odd expansion with F=[0,1]
% for i=1:length(x)
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2/x(i)-1;
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is od
%               y(i)=floor(2/x(i))+2-2/x(i);
%           else
%               y(i)=floor(2/x(i))-2/x(i)+1;
%           end
%
% end

%odd expansion with L=[0,1]
% for i=1:length(x)
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2/x(i)-1;
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%               y(i)=floor(2/x(i))+2-2/x(i);
%           else
%               y(i)=2/x(i)-(floor(2/x(i))-1);
%           end
%
% end

%even expansion with F=[0,2]
% for i=1:length(x)
```

```
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2-2/x(i);
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%               y(i)=floor(2/x(i))+1-2/x(i);
%           else
%               y(i)=-2/x(i)+(floor(2/x(i))+2);
%           end
%
% end

%even expansion with L=[0,1]
% for i=1:length(x)
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2-2/x(i);
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           else
%               y(i)=-2/x(i)+(floor(2/x(i))+2);
%           end
%
% end

%even expansion with L=[0,1]but no flip on [1,2]
% for i=1:length(x)
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2/x(i)-1;
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           else
%               y(i)=-2/x(i)+(floor(2/x(i))+2);
%           end
%
% end

%large flip
% for i=1:length(x)
%           if floor((floor(2/x(i))-1)/4)==floor(2/x(i)-1)/4 %gives if floor(2/x) is 1mod4
%               y(i)=2/x(i)-floor(2/x(i));
%           elseif floor((floor(2/x(i))-2)/4)==floor(2/x(i)-2)/4 %gives 2mod 4
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           elseif floor((floor(2/x(i))-3)/4)==floor(2/x(i)-3)/4 %3mod4
%               y(i)=floor(2/x(i))+2-2/x(i);
%           else
%               y(i)=floor(2/x(i))-2/x(i)+1;
```

```
%           end
%
% end

%tiny flip
% for i=1:length(x)
%     if floor(floor(4/x(i))/2)==floor(4/x(i))/2-0.5 % gives if floor(4/x) is odd
%                y(i)=floor(2/x(i))-2/x(i)+1;
%     else
%           y(i)=2/x(i)-floor(2/x(i));
%           end
%
% end
%

%3 divisible
% for i=1:length(x)
%     if floor((floor(2/x(i))-1)/3)==floor(2/x(i)-1)/3 && x(i)<1 %gives if floor(2/x) is 1mod3
%           y(i)=2/x(i)-floor(2/x(i))+1;
%     elseif floor((floor(2/x(i))-2)/3)==floor(2/x(i)-2)/3 && x(i)<1 %gives if floor(2/x) is 2mod3
%           y(i)=floor(2/x(i))+1-2/x(i);
%     else
%           y(i)=2/x(i)-floor(2/x(i));
%     end
% end

%switchingdigits
% for i=1:length(x)
%     if x(i)>1
%           y(i)=2/x(i)-floor(2/x(i));
%     elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%          y(i)=2/x(i)-floor(2/x(i))+1;
%     else
%           y(i)=floor(2/x(i))-2/x(i)+1;
%     end
% end

%spouse odd expansion F=[0,1]
% for i=1:length(x)
%           if 1<= x(i)&& x(i)<=2
%                y(i)=2-2/x(i);
%           elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%               y(i)=floor(2/x(i))+2-2/x(i);
%           else
%               y(i)=2/x(i)-floor(2/x(i));
```

```
%           end
%
% end

%spouse  3 div
% for i=1:length(x)
%     if 1<= x(i)&& x(i)<=2
%         y(i)=2-2/x(i);
%     elseif floor((floor(2/x(i))-1)/3)==floor(2/x(i)-1)/3 && x(i)<1 %gives if floor(2/x) is 1mod3
%         y(i)=2/x(i)-floor(2/x(i))+1;
%     elseif floor((floor(2/x(i))-2)/3)==floor(2/x(i)-2)/3 && x(i)<1 %gives if floor(2/x) is 2mod3
%         y(i)=2/x(i)-floor(2/x(i));
%     else
%         y(i)=floor(2/x(i))+1-2/x(i);
%     end
% end

%L=[2/4,2/3] F= L^c on [0,1]
% for i=1:length(x)
%     if 1<= x(i)&& x(i)<=2
%         y(i)=2/x(i)-1;
%     elseif 1/2<= x(i)&& x(i)<=2/3
%          y(i)=2/x(i)-2;
%     else
%         y(i)=floor(2/x(i))+1-2/x(i);
%     end
% end
%

%experiment blz 3204 niels vd wekken artikel
% for i=1:length(x)
%     if x(i)<=0.25
%         y(i)=9/(x(i)+2)-4;
%     elseif x(i)>1
%         y(i)=x(i)-1;
%     else
%         y(i)=9/(x(i)+2)-3;
%     end
% end

%another with 2e part flipt
% for i=1:length(x)
%     if x(i)<=0.25
%         y(i)=9/(x(i)+2)-4;
%     elseif x(i)>1
%         y(i)=x(i)-1;
%     else
%         y(i)=4-9/(x(i)+2);
```

```
%       end
% end

%another one with with first part flipped
% for i=1:length(x)
%       if x(i)<=0.25
%           y(i)=5-9/(x(i)+2);
%       elseif x(i)>1
%           y(i)=x(i)-1;
%       else
%           y(i)=9/(x(i)+2)-3;
%       end
% end

%another one with second part flipped
% for i=1:length(x)
%       if x(i)<=1/3
%           y(i)=4/(x(i)+1)-3;
%       elseif x(i)>1
%           y(i)=x(i)-1;
%       else
%           y(i)=3-4/(x(i)+1);
%       end
% end

%one with first part flipped
% for i=1:length(x)
%       if x(i)<=1/3
%           y(i)=4-4/(x(i)+1);
%       elseif x(i)>1
%           y(i)=x(i)-1;
%       else
%           y(i)=4/(x(i)+1)-2;
%       end
% end

%atractor [0.5,1]
for i=1:length(x)
    if x(i)<=0.5
        y(i)=x(i)+0.5;
    elseif x(i)<= 4/7
        y(i)=2/x(i)-3;
    elseif x(i)<=2/3
        y(i)=4-2/x(i);
    elseif x(i)<=4/5
        y(i)=2/x(i)-2;
    elseif x(i)<=1
        y(i)=3-2/x(i);
    else
        y(i)=0.5*x(i);
```

```
        end
end

%non ergodic map 2-expansion
% for i=1:length(x)
%       if x(i)<=sqrt(2)-1
%           if 2/x(i)-floor(2/x(i))<(sqrt(2)-1)
%               y(i)=2/x(i)-floor(2/x(i));
%           else
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           end
%       elseif x(i)<=1
%           if 2/x(i)-floor(2/x(i))<(sqrt(2)-1)
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           else
%               y(i)=2/x(i)-floor(2/x(i));
%           end
%       else
%           y(i)=2/x(i)-floor(2/x(i));
%       end
% end


%non ergodic map 2-expansion only [a,b]^c
% for i=1:length(x)
%       if x(i)<=sqrt(2)-1
%           if 2/x(i)-floor(2/x(i))<(sqrt(2)-1)
%               y(i)=2/x(i)-floor(2/x(i));
%           else
%               y(i)=2/x(i)-floor(2/x(i))+1;
%           end
%       elseif x(i)<=sqrt(2)
%           y(i)=(sqrt(2)-1)*x(i)+2*sqrt(2)-3;
%       else
%           y(i)=2/x(i)-1;
%       end
% end
%
```

```
function [A] = Matmakers(n,x)
%makes a matrix with in the n^th column TDL^n(x).
m = length(x);
A = zeros(m,n);
A(:,1)=x;

for j= 2:n
    A(:,j)= TDL(A(:,j-1));
end
```

```
function [x] = randmakers(d,n)
%makes a collum vector of length n with random numbers
    between 0 and 2. You
%can also determine the number of digits it will generate
     for each number.
digits(d);
x=rand(n,1)*2;
```

```
function [A] = ketsims(d,n,m)
%performs m iterations of TDL(x) on n random points with
    d number of
%digits. Returns the matrix A with all the paths of the
     points in a row.
digits(d)
x =vpa(randmakers(d,n));

A =Matmakers(m,x);
vpa(A);
```

```matlab
function[s]= kollomtellers(x)
%counts how much values in x are between 0 and 0.01 between 0.01 and 0.02
%etc. Up to between 1.99 and 2.00
s = zeros(200,1);
l = length(x);
t = 0;

for n = 1:200
    for i = 1 : l
        if ((n-1)/100  < x(i,1) &&  x(i,1) < n/100)
            t = t+1;

        end
    end
    s(n,1) = t;
    t = 0;
end
```

```matlab
function[sc] =mattellers (A)
%applies kollomtellers to every column of A and adds all the results then
%scales it back.
l = size(A);
S = zeros(200,l(1));

%on every column kollomtellers is applied
for i = 1: l(2)
     S(:,i) =kollomtellers(A(:,i));
end


% now we need to add the rows of S and scale back
s= zeros(10,1);
for i = 1:200
    s(i,1)= sum(S(i,:));
end
sc = 1/sum(s)*s;
sc =100*sc;
```

```matlab
function[S] = ketsimpros(d,n,m,N)
%Applies ketsims(d,n,m) N times calculates the density of all points and
%plots the results.

B = zeros(200,N);
for i = 1:N
    A = ketsims(d,n,m);
    B(:,i) =mattellers(A);
end
```

```
S = zeros(200,1);
for i = 1:200
S(i) =(1/N)*sum(B(i,:));
end
xe= [0.01:0.01:2]';
 plot(xe,S)
```

```
function[v]= samplefromfs(n,f)
%takes n samples from the density f.

v= zeros(n,1);
l = length(f);
m=1;
for i=1:l
    k=floor(((2*f(i))/l)*n);%determines how many samples there should be from an interval
    for j= 1:k
        v(m)=(2/l)*rand(1)+(i-1)/(0.5*l);
        m=m+1;
    end
end

%here the empty spots are removed from v.
a=n;
while v(a) == 0
    a = a-1;
end

v = v(1:a,1);
```

```
function[A] = ketsimfromfs(d,n,m,f)
%performs m iterations of TDL(x) on n random points with d number of
%digits. Returns the matrix A with all the paths of the points in a row.
%But instead of sammpling uniform from [0,2] we now sample from the density
%f.
digits(d)
x =vpa(samplefromfs(n,f));

A =Matmakers(m,x);
vpa(A);
```

```
function [S] = ketsimprofromfs(d,n,m,N,f)
%Applies ketsimfromfs(d,n,m,f) N times calculates the
    density of all points and
%plots the results.

B = zeros(200,N);
for i = 1:N
    A = ketsimfromfs(d,n,m,f);
    B(:,i) =mattellers(A');
end
S = zeros(200,1);
for i = 1:200
S(i) =(1/N)*sum(B(i,:));
end
xe= [0.01:0.01:2]';
 plot(xe,S)
```

B.0.6 The natural extention program

The TDL2D will just specify the natural extention map. In itTDL2D random points will be itereated over this map gets and all the paths are stored. Then it will plot all the points ever visited. The program works quiet fast. For example to iterate 25000 points a 100 times will take only half a second.

```
function [z] = TDL2D(x,y)
%function TDL, 2epsilon/(d(x)+y) note that d(x) and
    epsilon are domain
%dependend! therefor specified here.

z=zeros(length(x),2);
z(:,1)=TDL(x);
%atractor [1/2,1]
for i=1:length(x)
     if x(i)<=4/7
          z(i,2)=2/(3+y(i));
     elseif x(i)<=2/3
          z(i,2)=-2/(4+y(i));
     elseif x(i)<=4/5
          z(i,2)=2/(2+y(i));
     else
          z(i,2)=-2/(3+y(i));
     end
end

%even regular flipped [1,2]
% for i=1:length(x)
%      if x(i)>=1
%           z(i,2)=-2/(2+y(i));
%      elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5
%           z(i,2)=2/(floor(2/x(i))-1+y(i));
```

```
%       else
%            z(i,2)=2/(floor(2/x(i))+y(i));
%       end
% end

%switching digits
% for i=1:length(x)
%       if x(i)>=1
%            z(i,2)=2/(1+y(i));
%       elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5
    %gives if floor(2/x) is odd
%            z(i,2)=2/(floor(2/x(i))-1+y(i));
%       else
%            z(i,2)=-2/(floor(2/x(i))+1+y(i));
%       end
% end


%odd F=[0,1]
% for i=1:length(x)
%       if 1<= x(i)&& x(i)<=2
%            z(i,2)=2/(1+y(i));
%       elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5
    %gives if floor(2/x) is od
%            z(i,2)=-2/(floor(2/x(i))+2+y(i));
%       else
%            z(i,2)=-2/(floor(2/x(i))+1+y(i));
%       end
% end

%mid atractor 2-expansion
% for i=1:length(x)
%       if 2*(sqrt(2)-1)<=x(i) && x(i)<=sqrt(2)
%            z(i,2)=2/(1+y(i));
%            %1
%       elseif (2-sqrt(2))<=x(i) && x(i)<=2*(sqrt(2)-1)
%            z(i,2)=2/(2+y(i));
%       elseif 2/(sqrt(2)+3)<=x(i) && x(i)<=(2-sqrt(2))
%            z(i,2)=2/(3+y(i));
%            %3
%       else
%            z(i,2)=2/(4+y(i));
%       end
% end

%2expansion atractor complement
% for i=1:length(x)
%       if 2/x(i)-floor(2/x(i))<(sqrt(2)-1)
%            z(i,2)=2/(floor(2/x(i))+y(i));
%       else
```

```
%           z(i,2)=2/(floor(2/x(i))+1+y(i));
%       end
% end


%3-div as a check
% for i=1:length(x)
%       if floor((floor(2/x(i))-1)/3)==floor(2/x(i)-1)/3 && x(i)<1 %gives if floor(2/x) is 1mod3
%           z(i,2)=2/(floor(2/x(i))-1+y(i));
%       elseif floor((floor(2/x(i))-2)/3)==floor(2/x(i)-2)/3 && x(i)<1 %gives if floor(2/x) is 2mod3
%           z(i,2)=-2/(floor(2/x(i))+1+y(i));
%       else
%           z(i,2)=2/(floor(2/x(i))+y(i));
%       end
% end

%lazy backward as a check in case y=0 is not in the domain
% for i=1:length(x)
%       if 0<= x(i)&& x(i)<=1
%           z(i,2)=-2/(2+floor(2/x(i))+y(i));
%       else
%           z(i,2)=2/(1+y(i));
%       end
% end

%even expansion L=[0,1]
% for i=1:length(x)
%       if x(i)>=1
%           z(i,2)=-2/(2+y(i));
%       elseif floor(floor(2/x(i))/2)==floor(2/x(i))/2-0.5 %gives if floor(2/x) is odd
%           z(i,2)=2/(floor(2/x(i))-1+y(i));
%       else
%           z(i,2)=-2/(floor(2/x(i))+2+y(i));
%       end
% end
%


%large flip
% for i=1:length(x)
%           if floor((floor(2/x(i))-1)/4)==floor(2/x(i)-1)/4 %gives if floor(2/x) is 1mod4
%               z(i,2)=2/(floor(2/x(i))+y(i));
%           elseif floor((floor(2/x(i))-2)/4)==floor(2/x(i)-2)/4 %gives 2mod 4
%               z(i,2)=2/(floor(2/x(i))-1+y(i));
```

```
%           elseif  floor((floor(2/x(i))-3)/4)==floor(2/x(i)-3)/4  %3mod4
%               z(i,2)=-2/(floor(2/x(i))+2+y(i));
%           else
%               z(i,2)=-2/(floor(2/x(i))+1+y(i));
%           end
%
% end
```

```
function[]=itTDL2D(m,n)
%iterates over TDL2D n times and puts the x values in A1 and y values in A2 starting
%points are from [1/2,1]! m is the number of random starting points

x0=rand(m,1)/2+0.5;
A1=zeros(m,n);
A2=zeros(m,n);
A1(:,1)=x0;
%A2(:,1)=rand(m,1)*0.2;

for i=2:n
    z=TDL2D(A1(:,i-1),A2(:,i-1));
    A1(:,i)=z(:,1);
    A2(:,i)=z(:,2);
end
plot(A1,A2,'.','MarkerSize',1)
```

B.0.7 The program for Gauss Kuzmin method

These programs work relatively fast for 10 iterations of a vector with length 200 (x in gausskuzminvcalc) it will be done in less then a tenth of a second and already will give a good approximation.

```
function[s]=gausskuzminv(z,n)
%this function calculates the lebesgue measure of the set T^{-n}([0,z])
%it will iterate a vector with in the first collum a_i and second b_i when
%the intervals are [a_i,b_i]
%where T is the atractor on [0,0.5]




itv=[2/(3+z),2/(4-z);2/(2+z),2/(3-z)];


A=zeros(2,8);
for i= 1:n-1
```

```
    A(:,1)=2./(3+itv(:,2));
    A(:,2)=2./(3+itv(:,1));
    A(:,3)=2./(4-itv(:,1));
    A(:,4)=2./(4-itv(:,2));
    A(:,5)=2./(2+itv(:,2));
    A(:,6)=2./(2+itv(:,1));
    A(:,7)=2./(3-itv(:,1));
    A(:,8)=2./(3-itv(:,2));
    l=length(A(:,1));
    itv(:,1:2)=A(:,1:2);
    itv(l+1:2*l,1:2)=A(:,3:4);
    itv(2*l+1:3*l,1:2)=A(:,5:6);
    itv(3*l+1:4*l,1:2)=A(:,7:8);
    A=zeros(4*l,8);
end
length(itv);
s=sum(itv(:,2))-sum(itv(:,1));
```

```
function [s]=gausskuzminv2(z,n)
%this function calculates the lebesgue measure of the set
    T^{-n}([0,z])
%it will iterate a vector with in the first collum a_i
    and second b_i when
%the intervals are [a_i,b_i]
%where T is the atractor on [1,2] with N=4 having digits
    {1,2}




itv=[1,4/(5-z);4/(1+z),2];


A=zeros(2,4);
for i= 1:n-1
    A(:,1)=4./(1+itv(:,2));
    A(:,2)=4./(1+itv(:,1));
    A(:,3)=4./(5-itv(:,1));
    A(:,4)=4./(5-itv(:,2));
    l=length(A(:,1));
    itv(:,1:2)=A(:,1:2);
    itv(l+1:2*l,1:2)=A(:,3:4);
    A=zeros(2*l,4);
end
length(itv);
s=sum(itv(:,2))-sum(itv(:,1));
```

```
function [y]=gausskuzminvcalc(x,n)
%calculates gausskuzminv for each value of x when you
    iterate n times.
y=zeros(length(x),1);
for i=1:length(x)
    %y(i,1)=gausskuzminv(x(i),n);
    y(i,1)=gausskuzminv2(x(i),n);
end

%if we want the density
yb=zeros(length(y),1);
yb(2:length(y))=y(1:length(y)-1);

y=(y-yb)/(x(2)-x(1));
%y(1,1)=y(2,1);
plot(x,y,'red')
```

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