STOCHASTIC EVOLUTION EQUATIONS IN UMD BANACH SPACES

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Abstract. We discuss existence, uniqueness, and space-time Hölder regularity for solutions of the parabolic stochastic evolution equation

$$\begin{cases} dU(t) = (AU(t) + F(t,U(t)))\,dt + B(t,U(t))\,dW_H(t), & t \in [0,T_0], \\ U(0) = u_0, \end{cases}$$

where A generates an analytic C_0-semigroup on a UMD Banach space E and W_H is a cylindrical Brownian motion with values in a Hilbert space H. We prove that if the mappings $F : [0,T] \times E \to E$ and $B : [0,T] \times E \to \mathscr{L}(H,E)$ satisfy suitable Lipschitz conditions and u_0 is $\mathcal{F}_0$-measurable and bounded, then this problem has a unique mild solution, which has trajectories in $C^\lambda([0,T];\mathcal{D}((-A)^\theta)$ provided $\lambda \geq 0$ and $\theta \geq 0$ satisfy $\lambda + \theta < \frac{1}{2}$. Various extensions of this result are given and the results are applied to parabolic stochastic partial differential equations.

1. Introduction and statement of the results

In this paper we prove existence, uniqueness, and space-time regularity results for the abstract semilinear stochastic Cauchy problem

$$\text{(SCP)} \qquad \begin{cases} dU(t) = (AU(t) + F(t,U(t)))\,dt + B(t,U(t))\,dW_H(t), & t \in [0,T_0], \\ U(0) = u_0. \end{cases}$$

Here A is the generator of an analytic C_0-semigroup $(S(t))_{t\geq 0}$ on a UMD Banach space E, H is a separable Hilbert space, and for suitable $\eta \geq 0$ the functions $F : [0,T] \times \mathcal{D}((-A)^\eta) \to E$ and $B : [0,T] \times \mathcal{D}((-A)^\eta) \to \mathcal{L}(H,E)$ enjoy suitable Lipschitz continuity properties. The driving process W_H is an H-cylindrical Brownian motion adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$. In fact we shall allow considerably less restrictive assumptions on F and B; both functions may be unbounded and may depend on the underlying probability space.

A Hilbert space theory for stochastic evolution equations of the above type has been developed since the 1980s by the schools of Da Prato and Zabczyk [10]. Much of this theory has been extended to martingale type 2-spaces [2, 3]; see also the earlier work [35]. This class of Banach spaces covers the L^p-spaces in the range

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$2 \le p < \infty$, which is enough for many practical applications to stochastic partial differential equations. Let us also mention an alternative approach to the L^p-theory of stochastic partial differential equations has been developed by Krylov [23].

Extending earlier work of McConnell [27], the present authors have developed a theory of stochastic integration in UMD spaces [31, 32] based on decoupling inequalities for UMD-valued martingale difference sequences due to Garling [14, 15]. This work is devoted to the application of this theory to stochastic evolution equations in UMD spaces. In this introduction we will sketch in an informal way the main ideas of our approach. For the simplicity of presentation we shall consider the special case $H = \mathbb{R}$ and make the identifications $\mathcal{L}(\mathbb{R}, E) = E$ and $W_{\mathbb{R}} = W$, where W is a standard Brownian motion. For precise definitions and statements of the results we refer to the main body of the paper.

A solution of equation (SCP) is defined as an E-valued adapted process U which satisfies the variation of constants formula

$$U(t) = S(t)u_0 + \int_0^t S(t-s)F(s,U(s))\,ds + \int_0^t S(t-s)B(s,U(s))\,dW(s).$$

The relation of this solution concept with other type of solutions is considered in [44]. The principal difficulty to be overcome for the construction of a solution, is to find an appropriate space of processes which is suitable for applying the Banach fixed point theorem. Any such space V should have the property that $U \in V$ implies that the deterministic convolution

$$t \mapsto \int_0^t S(t-s)F(s,U(s))\,ds$$

and the stochastic convolution

$$t \mapsto \int_0^t S(t-s)B(s,U(s))\,dW(s)$$

belong to V again. To indicate why this such a space is difficult to construct we recall a result from [30] which states, loosely speaking, that if E is a Banach space which has the property that $f(u)$ is stochastically integrable for every E-valued stochastically integrable function u and every Lipschitz function $f : E \to E$, then E is isomorphic to a Hilbert space. Our way out of this apparent difficulty is by strengthening the definition of Lipschitz continuity to L^2_γ-Lipschitz continuity, which can be thought of as a Gaussian version of Lipschitz continuity. From the point of view of stochastic PDEs, this strengthening does not restrict the range of applications of our abstract theory. Indeed, we shall prove that under standard measurability and growth assumptions, Nemytskii operators are L^2_γ-Lipschitz continuous in L^p. Furthermore, in type 2 spaces the notion of L^2_γ-Lipschitz continuity coincides with the usual notion of Lipschitz continuity.

Under the assumption that F is Lipschitz continuous in the second variable and B is L^2_γ-Lipschitz continuous in the second variable, uniformly with respect to bounded time intervals in their first variables, the difficulty described above is essentially reduced to finding a space of processes V having the property that $\phi \in V$ implies that the pathwise deterministic convolutions

$$t \mapsto \int_0^t S(t-s)\phi(s)\,ds$$

and the stochastic convolution integral

$$t \mapsto \int_0^t S(t-s)\phi(s)\,dW(s) \tag{1.1}$$

define processes which again belong to V. The main tool for obtaining estimates for this stochastic integral is γ-boundedness. This is the Gaussian version of the notion of R-boundedness which in the past years has established itself as a natural generalization to Banach spaces of the notion of uniform boundedness in the Hilbert space context and which played an essential role in much recent progress in the area of parabolic evolution equations. The power of both notions derives from the fact that they connect probability in Banach spaces with harmonic analysis.

From the point of view of stochastic integration, the importance of γ-bounded families of operators is explained by the fact that they act as pointwise multipliers in spaces of stochastically integrable processes. This would still not be very useful if it were not the case that one can associate γ-bounded families of operators with an analytic C_0-semigroup $(S(t))_{t\geq 0}$ with generator A. In fact, for all $\eta > 0$ and $\varepsilon > 0$, families such as

$$\big\{t^{\eta+\varepsilon}(-A)^{\eta}S(t):\ t\in(0,T_0)\big\}$$

are γ-bounded. Here, for simplicity, we are assuming that the fractional powers of A exist; in general one has to consider translates of A. This suggests to rewrite the stochastic convolution (1.1) as

$$t \mapsto \int_0^t \big[(t-s)^{\eta+\varepsilon}(-A)^{\eta}S(t-s)\big](t-s)^{-\eta-\varepsilon}(-A)^{-\eta}\phi(s)\,dW(s). \tag{1.2}$$

By γ-boundedness we can estimate the L^p-moments of this integral by the L^p-moments of the simpler integral

$$t \mapsto \int_0^t (t-s)^{-\eta-\varepsilon}(-A)^{-\eta}\phi(s)\,dW(s). \tag{1.3}$$

Thus we are led to define $V^p_{\alpha,\infty}([0,T_0]\times\Omega;\mathcal{D}((-A)^{\eta}))$ as the space of all continuous adapted processes $\phi:(0,T_0)\times\Omega\to\mathcal{D}((-A)^{\eta})$ for which the norm

$$\begin{aligned}&\|\phi\|_{V^p_{\alpha,\infty}([0,T_0]\times\Omega;\mathcal{D}((-A)^{\eta}))}\\ &:= \Big(\mathbb{E}\|\phi\|^p_{C([0,T_0];\mathcal{D}((-A)^{\eta}))}\Big)^{\frac1p} + \sup_{t\in[0,T_0]}\Big(\mathbb{E}\|(t-\cdot)^{-\alpha}\phi(\cdot)\|^p_{\gamma(L^2(0,t),\mathcal{D}((-A)^{\eta}))}\Big)^{\frac1p}\end{aligned}$$

is finite. Here, $\gamma(L^2(0,t),F)$ denotes the Banach space of γ-radonifying operators from $L^2(0,t)$ into the Banach space F; by the results of [33], a function $f:(0,t)\to F$ is stochastically integrable on $(0,t)$ with respect to W if and only if it is the kernel of an integral operator belonging to $\gamma(L^2(0,t),F)$.

Now we are ready to formulate a special case of one of the main results (see Theorems 6.2, 6.3, 7.3).

Theorem 1.1. *Let E be a UMD space and let $\eta\geq 0$ and $p>2$ satisfy $\eta+\frac1p<\frac12$. Assume that:*

(i) *A generates an analytic C_0-semigroup on E;*
(ii) *$F:[0,T_0]\times\mathcal{D}((-A)^{\eta})\to E$ is Lipschitz continuous and of linear growth in the second variable, uniformly on $[0,T_0]$;*
(iii) *$B:[0,T_0]\times\mathcal{D}((-A)^{\eta})\to\mathcal{L}(H,E)$ is L^2_{γ}-Lipschitz continuous and of linear growth in the second variable, uniformly on $[0,T_0]$;*

(iv) $u_0 \in L^p(\Omega, \mathcal{F}_0; \mathcal{D}((-A)^\eta))$.

Then:

(1) (Existence and uniqueness) *For all* $\alpha > 0$ *such that* $\eta + \frac{1}{p} < \alpha < \frac{1}{2}$ *the problem* (SCP) *admits a unique solution* U *in* $V^p_{\alpha,\infty}([0,T_0]\times\Omega; \mathcal{D}((-A)^\eta))$.

(2) (Hölder regularity) *For all* $\lambda \geq 0$ *and* $\delta \geq \eta$ *such that* $\lambda + \delta < \frac{1}{2}$ *the process* $U - S(\cdot)u_0$ *has a version with paths in* $C^\lambda([0,T_0]; \mathcal{D}((-A)^\delta))$.

For martingale type 2 spaces E, Theorem 1.1 was proved by Brzeźniak [3]; in this setting the L^2_γ-Lipschitz assumption in (iii) reduces to a standard Lipschitz assumption. As has already been pointed out, the class of martingale type 2 spaces includes the spaces L^p for $2 \leq p < \infty$, whereas the UMD spaces include L^p for $1 < p < \infty$. The UMD assumption in Theorem 1.1 can actually be weakened so as to include L^1-spaces as well; see Section 9. The assumptions on F and B as well as the integrability assumption on u_0 can be substantially weakened; we shall prove versions of Theorem 1.1 assuming that F and B are merely locally Lipschitz continuous and locally L^2_γ-Lipschitz continuous, respectively, and u_0 is $\mathcal{F}_0$-measurable.

Let us now briefly discuss the organization of the paper. Preliminary material on γ-radonifying operators, stochastic integration in UMD spaces, and γ-boundedness of families of operators, is collected in Section 2. In Sections 3 and 4 we prove estimates for deterministic and stochastic convolutions. After introducing the notion of L^2_γ-Lipschitz continuity in Section 5 we take up the study of problem (SCP) in Section 6, where we prove Theorem 1.1. The next two sections are concerned with refinements of this theorem. In Section 7 we consider arbitrary $\mathcal{F}_0$-measurable initial values, still assuming that the functions F and B are globally Lipschitz continuous and L^2_γ-Lipschitz continuous respectively. In Section 8 we consider the locally Lipschitz case and prove existence and uniqueness of solutions up to an explosion time. In Section 9 we discuss how the results of this paper can be extended to a larger class of Banach spaces including the UMD spaces as well as the spaces L^1.

The final Section 10 is concerned with applications to stochastic partial differential equations. On bounded smooth domains $S \subseteq \mathbb{R}^d$ we consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t,s) &= A(s,D)u(t,s) + f(t,s,u(t,s)) \\ &\qquad\qquad + g(t,s,u(t,s))\,\frac{\partial w}{\partial t}(t,s), \quad s \in S,\ t \in (0,T], \\ B_j(s,D)u(t,s) &= 0, \quad s \in \partial S,\ t \in (0,T], \\ u(0,s) &= u_0(s), \quad s \in S. \end{aligned}$$

Here A is of the form

$$A(s,D) = \sum_{|\alpha| \leq 2m} a_\alpha(s) D^\alpha$$

with $D = -i(\partial_1, \dots, \partial_d)$ and for $j = 1, \dots, m$,

$$B_j(s,D) = \sum_{|\beta| \leq m_j} b_{j\beta}(s) D^\beta$$

where $1 \leq m_j < 2m$ is an integer. As a sample existence result, we prove that if f and g satisfy standard measurability assumptions and are locally Lipschitz and of linear growth in the third variable, uniformly with respect to the first and second variables, and if $u \in H^{2m\eta,p}_{\{B_j\}}(S)$, then the above problem admits a solution with paths in $C^\lambda([0,T]; H^{2m\delta,p}_{\{B_j\}}(S))$ for all $\delta > \frac{d}{2mp}$ and $\lambda > 0$ that satisfy $\delta + \lambda < \frac{1}{2} - \frac{d}{4m}$ and $2m\delta - \frac{1}{p} \neq m_j$, for all $j = 1, \dots, m$. Uniqueness results are obtained as well.

All vector spaces in this paper are real. Throughout the paper, H and E denote a separable Hilbert space and a Banach space, respectively. We study the problem (SCP) on a time interval $[0, T_0]$ which is always considered to be fixed. In many estimates below we are interested on bounds on sub-intervals $[0, T]$ of $[0, T_0]$ and it will be important to keep track of the dependence upon T of the constants appearing in these bounds. For this purpose we shall use the convention that the letter C is used for generic constants which are independent of T but which may depend on T_0 and all other relevant data in the estimates. The numerical value of C may vary from line to line.

We write $Q_1 \lesssim_A Q_2$ to express that there exists a constant c, only depending on A, such that $Q_1 \leq cQ_2$. We write $Q_1 \eqsim_A Q_2$ to express that $Q_1 \lesssim_A Q_2$ and $Q_2 \lesssim_A Q_1$.

2. Preliminaries

The purpose of this section is to collect the basic stochastic tools used in this paper. For proofs and further details we refer the reader to our previous papers [32, 33], where also references to the literature can be found.

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{P})$ always denotes a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$. For a Banach space F and a finite measure space (S, Σ, μ), $L^0(S; F)$ denotes the vector space of strongly measurable functions $\phi : S \to F$, identifying functions which are equal almost everywhere. Endowed with the topology induced by convergence in measure, $L^0(S; F)$ is a complete metric space.

γ-Radonifying operators. A linear operator $R : H \to E$ from a separable Hilbert space H into a Banach space E is called *γ-radonifying* if for some (and then for every) orthonormal basis $(h_n)_{n\geq 1}$ of H the Gaussian sum $\sum_{n\geq 1} \gamma_n R h_n$ converges in $L^2(\Omega; E)$. Here, and in the rest of the paper, $(\gamma_n)_{n\geq 1}$ is a *Gaussian sequence*, i.e., a sequence of independent standard real-valued Gaussian random variables. The space $\gamma(H, E)$ of all γ-radonifying operators from H to E is a Banach space with respect to the norm

$$\|R\|_{\gamma(H,E)} := \Big(\mathbb{E}\Big\| \sum_{n\geq 1} \gamma_n R h_n \Big\|^2\Big)^{\frac{1}{2}}.$$

This norm is independent of the orthonormal basis $(h_n)_{n\geq 1}$. Moreover, $\gamma(H, E)$ is an operator ideal in the sense that if $S_1 : H' \to H$ and $S_2 : E \to E'$ are bounded operators, then $R \in \gamma(H, E)$ implies $S_2 R S_1 \in \gamma(H', E')$ and

$$\|S_2 R S_1\|_{\gamma(H',E')} \leq \|S_2\| \|R\|_{\gamma(H,E)} \|S_1\|. \tag{2.1}$$

We will be mainly interested in the case where $H = L^2(0, T; \mathcal{H})$, where $\mathcal{H}$ is another separable Hilbert space.

The following lemma gives necessary and sufficient conditions for an operator from H to an L^p-space to be γ-radonifying. It unifies various special cases in the literature, cf. [4, 43] and the references given therein. In passing we note that by using the techniques of [25] the lemma can be generalized to arbitrary Banach function spaces with finite cotype.

Lemma 2.1. *Let (S,Σ,μ) be a σ-finite measure space and let $1 \le p < \infty$. For an operator $T \in \mathcal{L}(H, L^p(S))$ the following assertions are equivalent:*

(1) *$T \in \gamma(H, L^p(S))$;*
(2) *For some orthonormal basis $(h_n)_{n=1}^\infty$ of H the function $\big(\sum_{n\ge1} |Th_n|^2\big)^{\frac12}$ belongs to $L^p(S)$;*
(3) *For all orthonormal bases $(h_n)_{n=1}^\infty$ of H the function $\big(\sum_{n=1}^\infty |Th_n|^2\big)^{\frac12}$ belongs to $L^p(S)$;*
(4) *There exists a function $g \in L^p(S)$ such that for all $h \in H$ we have $|Th| \le \|h\|_H \cdot g$ μ-almost everywhere;*
(5) *There exists a function $k \in L^p(S;H)$ such that $Th = [k(\cdot), h]_H$ μ-almost everywhere.*

Moreover, in this situation we may take $k = \big(\sum_{n=1}^\infty |Th_n|^2\big)^{\frac12}$ and have

$$\|T\|_{\gamma(H,L^p(S))} \eqsim_p \Big\| \Big(\sum_{n=1}^\infty |Th_n|^2 \Big)^{\frac12} \Big\| \le \|g\|_{L^p(S)}. \tag{2.2}$$

Proof. By the Kahane-Khintchine inequalities and Fubini's theorem we have, for all $f_1, \dots, f_N \in L^p(S)$,

$$\begin{aligned} &\Big\| \Big(\sum_{n=1}^N |f_n|^2 \Big)^{\frac12} \Big\|_{L^p(S)} \\ &\quad = \Big\| \Big(\mathbb{E} \Big| \sum_{n=1}^N \gamma_n f_n \Big|^2 \Big)^{\frac12} \Big\|_{L^p(S)} \eqsim_p \Big\| \Big(\mathbb{E} \Big| \sum_{n=1}^N \gamma_n f_n \Big|^p \Big)^{\frac1p} \Big\|_{L^p(S)} \\ &\quad = \Big(\mathbb{E} \Big\| \sum_{n=1}^N \gamma_n f_n \Big\|_{L^p(S)}^p \Big)^{\frac1p} \eqsim_p \Big(\mathbb{E} \Big\| \sum_{n=1}^N \gamma_n f_n \Big\|_{L^p(S)}^2 \Big)^{\frac12}. \end{aligned}$$

The equivalences (1)⇔(2)⇔(3) follow by taking $f_n := Th_n$, $n = 1, \dots, N$. This also gives the first part of (2.2).

(2)⇒(4): Let $g \in L^p(S)$ be defined as $g = \big(\sum_{n=1}^\infty |Th_n|^2\big)^{\frac12}$. For $h = \sum_{n=1}^N a_n h_n$ we have, for μ-almost all $s \in S$,

$$|Th(s)| = \Big| \sum_{n=1}^N a_n Th_n(s) \Big| \le \Big(\sum_{n=1}^N |a_n|^2 \Big)^{\frac12} \Big(\sum_{n=1}^N |Th_n(s)|^2 \Big)^{\frac12} \le g(s) \|h\|_H.$$

The case of a general $h \in H$ follows by an approximation argument.

(4)⇒(5): Let H_0 be a countable dense set in H which is closed under taking $\mathbb{Q}$-linear combinations. Let $N \in \Sigma$ be a μ-null set such that for all $s \in \complement N$ and for all $h \in H_0$, $|Th(s)| \le g(s)\|f\|_H$ and $h \mapsto Th(s)$ is $\mathbb{Q}$-linear on H_0. By the Riesz representation theorem, applied for each fixed $s \in \complement N$, the mapping $h \to Th(s)$ has a unique extension to an element $k(s) \in H$ with $Th(s) = [h, k(s)]_H$ for all $h \in H_0$. By an approximation argument we obtain that for all $h \in H$ we have $Th(s) = [h, k(s)]_H$ for μ-almost all $s \in S$. For all $s \in \complement N$,

$$\|k(s)\|_H = \sup_{\|h\|_H \le 1, h \in H_0} |[h, k(s)]| = \sup_{\|h\|_H \le 1, h \in H_0} |Th(s)| \le g(s).$$

Putting $k(s) = 0$ for $s \in N$, we obtain (5) and the last inequality in (2.2).

(5)⇒(3): Let $(h_n)_{n=1}^{\infty}$ be an orthonormal basis for H. Let $N \in \Sigma$ be a μ-null set such that for all $s \in \complement N$ and all $n \geq 1$ we have $Th_n(s) = [h_n, k(s)]$. Then for $s \in \complement N$,

$$\Big(\sum_{n=1}^{\infty} |Th_n(s)|^2\Big)^{\frac{1}{2}} = \Big(\sum_{n=1}^{\infty} |[h_n, k(s)]|^2\Big)^{\frac{1}{2}} = \|k(s)\|_H.$$

This gives (3) and the middle equality of (2.2). □

Recall that for domains $S \subseteq \mathbb{R}^d$ and $\lambda > \frac{d}{2}$ one has $H^{\lambda,2}(S) \hookrightarrow C_{\rm b}(\overline{S})$ (cf. [42, Theorem 4.6.1]). Applying Lemma 2.1 with $g \equiv C \cdot 1_S$ we obtain the following result.

Corollary 2.2. *Assume $S \subseteq \mathbb{R}^d$ is a bounded domain. If $\lambda > \frac{d}{2}$, then for all $p \in [1, \infty)$, the embedding $I : H^{\lambda,2}(S) \to L^p(S)$ is γ-radonifying.*

From the lemma we obtain an isomorphism of Banach spaces

$$L^p(S; H) \simeq \gamma(H, L^p(S)),$$

which is given by $f \mapsto (h \mapsto [f(\cdot), h]_H)$. The next result generalizes this observation:

Lemma 2.3 ([32]). *Let (S, Σ, μ) be a σ-finite measure space and let $p \in [1, \infty)$ be fixed. Then $f \mapsto (h \mapsto f(\cdot)h)$ defines an isomorphism of Banach spaces*

$$L^p(S; \gamma(H, E)) \simeq \gamma(H, L^p(S; E)).$$

Stochastic integration. In this section we recall some aspects of stochastic integration in UMD Banach spaces. For proofs and more details we refer to our paper [32], whose terminology we follow.

A Banach space E is called a *UMD* space if for some (equivalently, for all) $p \in (1, \infty)$ there exists a constant $\beta_{p,E} \geq 1$ such that for all L^p-integrable E-valued martingale difference sequences $(d_j)_{j=1}^n$ and all $\{-1, 1\}$-valued sequence $(\varepsilon_j)_{j=1}^n$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n \varepsilon_j d_j\Big\|^p\Big)^{\frac{1}{p}} \leq \beta_{p,E} \Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}}. \tag{2.3}$$

The class of UMD spaces was introduced in the 1970s by Maurey and Burkholder and has been studied by many authors. For more information and references to the literature we refer the reader to the review articles [5, 38]. Examples of UMD spaces are all Hilbert spaces and the spaces $L^p(S)$ for $1 < p < \infty$ and σ-finite measure spaces (S, Σ, μ). If E is a UMD space, then $L^p(S; E)$ is a UMD space for $1 < p < \infty$.

Let H be a separable Hilbert space. An *H-cylindrical Brownian motion* is family $W_H = (W_H(t))_{t \in [0,T]}$ of bounded linear operators from H to $L^2(\Omega)$ with the following two properties:

(1) $W_H h = (W_H(t)h)_{t \in [0,T]}$ is real-valued Brownian motion for each $h \in H$,
(2) $\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s \wedge t)\, [g, h]_H$ for all $s, t \in [0, T]$, $g, h \in H$.

The stochastic integral of the indicator process $1_{(a,b] \times A} \otimes (h \otimes x)$, where $0 \leq a < b < T$ and the subset A of Ω is $\mathcal{F}_a$-measurable, is defined as

$$\int_0^T 1_{(a,b] \times A} \otimes (h \otimes x)\, dW_H := 1_A (W_H(b)h - W_H(a)h)x.$$

By linearity, this definition extends to adapted step processes $\Phi : (0,T) \times \Omega \to \mathcal{L}(H,E)$ whose values are finite rank operators.

In order to extend this definition to a more general class of processes we introduce the following terminology. A process $\Phi : (0,T) \times \Omega \to \mathcal{L}(H,E)$ is called *H-strongly measurable* if Φh is strongly measurable for all $h \in H$. Here, $(\Phi h)(t,\omega) := \Phi(t,\omega)h$. Such a process is called *stochastically integrable* with respect to W_H if it is adapted and there exists a sequence of adapted step processes $\Phi_n : (0,T) \times \Omega \to \mathcal{L}(H,E)$ with values in the finite rank operators from H to E and a pathwise continuous process $\zeta : [0,T] \times \Omega \to E$, such that the following two conditions are satisfied:

(1) $\lim_{n\to\infty} \Phi_n h = \Phi h$ in $L^0((0,T) \times \Omega; E)$ for all $h \in H$;
(2) $\displaystyle\lim_{n\to\infty} \int_0^{\cdot} \Phi_n \, dW_H = \zeta$ in $L^0(\Omega; C([0,T]; E))$.

In this situation, ζ is determined uniquely as an element of $L^0(\Omega; C([0,T]; E))$ and is called the *stochastic integral* of Φ with respect to W_H, notation:

$$\zeta = \int_0^{\cdot} \Phi \, dW_H.$$

The process ζ is a continuous local martingale starting at zero. The following result from [31, 32] states necessary and sufficient conditions for stochastic integrability.

Proposition 2.4. *Let E be a UMD space. For an adapted H-strongly measurable process $\Phi : (0,T) \times \Omega \to \mathcal{L}(H,E)$ the following assertions are equivalent:*

(1) *the process Φ is stochastically integrable with respect to W_H;*
(2) *for all $x^* \in E^*$ the process $\Phi^* x^*$ belongs to $L^0(\Omega; L^2(0,T;H))$, and there exists a pathwise continuous process $\zeta : [0,T] \times \Omega \to E$ such that for all $x^* \in E^*$ we have*
$$\langle \zeta, x^* \rangle = \int_0^{\cdot} \Phi^* x^* \, dW_H \quad \text{in } L^0(\Omega; C([0,T]);$$
(3) *for all $x^* \in E^*$ the process $\Phi^* x^*$ belongs to $L^0(\Omega; L^2(0,T;H))$, and there exists an operator-valued random variable $R : \Omega \to \gamma(L^2(0,T;H),E))$ such that for all $f \in L^2(0,T;H)$ and $x^* \in E^*$ we have*
$$\langle Rf, x^* \rangle = \int_0^T [f(t), \Phi^*(t)x^*]_H \, dt \quad \text{in } L^0(\Omega).$$

In this situation we have $\zeta = \int_0^{\cdot} \Phi \, dW_H$ in $L^0(\Omega; C([0,T]; E))$. Furthermore, for all $p \in (1,\infty)$,

$$\mathbb{E} \sup_{t\in[0,T]} \Big\| \int_0^t \Phi \, dW_H \Big\|^p \eqsim_{p,E} \mathbb{E} \|R\|^p_{\gamma(L^2(0,T;H),E)}.$$

In the situation of (3) we shall say that R is *represented* by Φ. Since Φ is uniquely determined almost everywhere on $(0,T) \times \Omega$ by R and vise versa (this readily follows from [32, Lemma 2.7 and Remark 2.8]), in what follows we shall frequently identify R and Φ.

The next lemma will be useful in Section 7.

Lemma 2.5. *Let $\Phi : (0,T) \times \Omega \to \mathcal{L}(H,E)$ be stochastically integrable with respect to W_H. Suppose $A \in \mathcal{F}$ is a measurable set such that for all $x^* \in E^*$ we have*

$$\Phi^*(t,\omega)x^* = 0 \ \text{ for almost all } (t,\omega) \in (0,T) \times A.$$

Then almost surely in A, for all $t \in [0,T]$ we have $\int_0^t \Phi\, dW_H = 0$.

Proof. Let $x^* \in E^*$ be arbitrary. By strong measurability it suffices to show that, almost surely in A, for all $t \in [0,T]$ we have

$$M_t := \int_0^t \Phi^* x^* \, dW_H = 0.$$

For the quadratic variation of the continuous local martingale M we have

$$[M]_T = \int_0^T \|\Phi^*(s)x^*\|^2 \, ds = 0 \quad \text{a.s. on } A.$$

Therefore, $M = 0$ a.s. on A. Indeed, let

$$\tau := \inf\{t \in [0,T] : [M]_t > 0\},$$

where we take $\tau = T$ if the infimum is taken over the empty set. Then M^τ is a continuous local martingale with quadratic variation $[M^\tau] = [M]^\tau = 0$. Hence $M^\tau = 0$ a.s. This implies the result. □

R-Boundedness and γ-boundedness. Let E_1 and E_2 be Banach spaces and let $(r_n)_{n\geq 1}$ be a *Rademacher sequence*, i.e., a sequence of independent random variables satisfying $\mathbb{P}\{r_n = -1\} = \mathbb{P}\{r_n = 1\} = \frac{1}{2}$. A family $\mathscr{T}$ of bounded linear operators from E_1 to E_2 is called *R-bounded* if there exists a constant $C \geq 0$ such that for all finite sequences $(x_n)_{n=1}^N$ in E_1 and $(T_n)_{n=1}^N$ in $\mathscr{T}$ we have

$$\mathbb{E}\Big\| \sum_{n=1}^N r_n T_n x_n \Big\|^2 \leq C^2 \mathbb{E}\Big\| \sum_{n=1}^N r_n x_n \Big\|^2.$$

The least admissible constant C is called the *R-bound* of $\mathscr{T}$, notation $R(\mathscr{T})$. By the Kahane-Khintchine inequalities the exponent 2 may be replaced by any $p \in [1,\infty)$. This only affects the value of the R-bound; we shall use the notation $R_p(\mathscr{T})$ for the R-bound of $\mathscr{T}$ relative to exponent p.

Upon replacing the Rademacher sequence by a Gaussian sequence we arrive at the notion of a *γ-bounded* family of operators, whose *γ-bound* will be denoted by $\gamma(\mathscr{T})$. A standard randomization argument shows that every R-bounded family is γ-bounded, and both notions are equivalent if the range space has finite cotype (the definitions of type and cotype are recalled in the next section).

The notion of R-boundedness has played an important role in recent progress in the regularity theory of parabolic evolution equations. Detailed accounts of these developments are presented in [12, 24], where more about the history of this concept and further references to the literature can be found.

Here we shall need various examples of R-bounded families, which are stated in the form of lemmas.

Lemma 2.6 ([46])**.** *If $\Phi : (0,T) \to \mathcal{L}(E_1,E_2)$ is differentiable with integrable derivative, the family*

$$\mathscr{T}_\Phi = \big\{\Phi(t) : \ t \in (0,T)\big\}$$

is R-bounded in $\mathcal{L}(E_1,E_2)$, with

$$R(\mathscr{T}_\Phi) \leq \|\Phi(0+)\| + \int_0^T \|\Phi'(t)\| \, dt.$$

We continue with a lemma which connects the notions of R-boundedness and γ-radonification. Let H be a Hilbert space and E a Banach space. For each $h \in H$ we obtain a linear operator $T_h : E \to \gamma(H,E)$ by putting

$$T_h x := h \otimes x, \qquad x \in E.$$

Lemma 2.7 ([17])**.** *If E has finite cotype, the family*

$$\mathscr{T} = \big\{T_h : \ \|h\|_H \le 1\big\}$$

is R-bounded in $\mathcal{L}(E, \gamma(H,E))$.

Following [21], a Banach space E is said to have *property* (Δ) if there exists a constant C_Δ such that if $(r_n')_{n=1}^N$ and $(r_n'')_{n=1}^N$ are Rademacher sequences on probability spaces $(\Omega', \mathbb{P}')$ and $(\Omega'', \mathbb{P}'')$ respectively, and $(x_{mn})_{m,n=1}^N$ is a doubly indexed sequence of elements of E, then

$$\mathbb{E}'\mathbb{E}''\Big\|\sum_{n=1}^N\sum_{m=1}^n r_m' r_n'' x_{mn}\Big\|^2 \le C_\Delta^2 \mathbb{E}'\mathbb{E}''\Big\|\sum_{n=1}^N\sum_{m=1}^N r_m' r_n'' x_{mn}\Big\|^2.$$

Every UMD space has property (Δ) [6] and every Banach space with property (Δ) has finite cotype. Furthermore the spaces $L^1(S)$ with (S,Σ,μ) σ-finite have property (Δ). The space of trace class operators does not have property (Δ) (see [21]).

The next lemma is a variation of Bourgain's vector-valued Stein inequality for UMD spaces [1, 6] and was kindly communicated to us by Tuomas Hytönen.

Lemma 2.8. *Let W_H be an H-cylindrical Brownian motion, adapted to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, on a probability space (Ω, P). If E is a Banach space enjoying property (Δ), then for all $1 \le p < \infty$ the family of conditional expectation operators*

$$\mathscr{E}_p = \big\{\mathbb{E}(\cdot|\mathcal{F}_t) : \ t \in [0,T]\big\}$$

is R-bounded, with R-bound C_Δ, on the closed linear subspace $G^p(\Omega;E)$ of $L^p(\Omega;E)$ spanned by all random variables of the form $\int_0^T \Phi\, dW_H$ with $\Phi \in \gamma(L^2(0,T;H),E)$.

Proof. Let $1 \le p < \infty$ be fixed and choose $\mathbb{E}_1, \dots, \mathbb{E}_N \in \mathscr{E}_p$, say $\mathbb{E}_n = \mathbb{E}(\cdot|\mathcal{F}_{t_n})$ with $0 \le t_n \le T$. By relabeling the indices we may assume that $t_1 \le \dots \le t_N$. We must show that for all $F_1, \dots, F_N \in L^p(\Omega;E)$ of the form $F_n = \int_0^T \Phi_n\, dW_H$ we have

$$\mathbb{E}'\Big\|\sum_{n=1}^N r_n' \mathbb{E}_n F_n\Big\|^2 \le C_\Delta^2 \mathbb{E}'\Big\|\sum_{n=1}^N r_n' F_n\Big\|^2.$$

We write $\mathbb{E}_n = \sum_{j=1}^n D_j$, where $D_j := \mathbb{E}_j - \mathbb{E}_{j-1}$ with the convention that $\mathbb{E}_0 = 0$. The important point to observe is that if $\Psi_j \in \gamma(L^2(0,T;H),E)$ and $G_j := \int_0^T \Psi_j\, dW_H$, the random variables $D_j G_j$ are symmetric and independent. Hence, by a standard randomization argument,

$$
\begin{aligned}
\mathbb{E}'\Big\| \sum_{n=1}^{N} r_n' \mathbb{E}_n F_n \Big\|^2_{G^p(\Omega;E)} &= \mathbb{E}'\Big\| \sum_{n=1}^{N} \sum_{j=1}^{n} r_n' D_j F_n \Big\|^2_{G^p(\Omega;E)} \\
&= \mathbb{E}'\Big\| \sum_{j=1}^{N} D_j \sum_{n=j}^{N} r_n' F_n \Big\|^2_{G^p(\Omega;E)} = \mathbb{E}'\mathbb{E}''\Big\| \sum_{j=1}^{N} r_j'' D_j \sum_{n=j}^{N} r_n' F_n \Big\|^2_{G^p(\Omega;E)} \\
&\le C_\Delta^2 \mathbb{E}'\mathbb{E}''\Big\| \sum_{j=1}^{N} r_j'' D_j \sum_{n=1}^{N} r_n' F_n \Big\|^2_{G^p(\Omega;E)} = C_\Delta^2 \mathbb{E}'\Big\| \sum_{j=1}^{N} D_j \sum_{n=1}^{N} r_n' F_n \Big\|^2_{G^p(\Omega;E)} \\
&= C_\Delta^2 \mathbb{E}'\Big\| \mathbb{E}_N \sum_{n=1}^{N} r_n' F_n \Big\|^2_{G^p(\Omega;E)} \le C_\Delta^2 \mathbb{E}'\Big\| \sum_{n=1}^{N} r_n' F_n \Big\|^2_{G^p(\Omega;E)}.
\end{aligned}
$$

□

The next lemma, obtained in [20] for the case $H = \mathbb{R}$, states that γ-bounded families act boundedly as pointwise multipliers on spaces of γ-radonifying operators. The proof of the general case is entirely similar.

Lemma 2.9. *Let E_1, E_2 be Banach spaces and let H be a separable Hilbert space. Let $T > 0$. Let $M : (0,T) \to \mathcal{L}(E_1, E_2)$ be function with the following properties:*

(1) *for all $x \in E_1$ the function $M(\cdot)x$ is strongly measurable in E_2;*
(2) *the range $\mathscr{M} = \{M(t) : \ t \in (0,T)\}$ is γ-bounded in $\mathcal{L}(E_1, E_2)$.*

Then for all step functions $\Phi : (0,T) \to \mathcal{L}(H, E_1)$ with values in the finite rank operators from H to E_1 we have

$$\|M\Phi\|_{\gamma(L^2(0,T;H),E_2)} \le \gamma(\mathscr{M}) \|\Phi\|_{\gamma(L^2(0,T;H),E_1)}. \tag{2.4}$$

Here, $(M\Phi)(t) := M(t)\Phi(t)$. As a consequence, the mapping $\Phi \mapsto M\Phi$ has a unique extension to a bounded operator from $\gamma(L^2(0,T;H),E_1)$ to $\gamma(L^2(0,T;H),E_2)$ of norm at most $\gamma(\mathscr{M})$.

In [20] it is shown that under slight regularity assumptions on M, the γ-boundedness is also a necessary condition.

3. Deterministic convolutions

After these preliminaries we take up our main line of study and begin with some estimates for deterministic convolutions. The main tool will be a multiplier lemma for vector-valued Besov spaces, Lemma 3.1, to which we turn first.

Let E be a Banach space, let $I = (a,b]$ with $-\infty \le a < b \le \infty$ be a (possibly unbounded) interval, and let $s \in (0,1)$ and $1 \le p, q \le \infty$ be fixed. Following [22, Section 3.b], the Besov space $B^s_{p,q}(I;E)$ is defined as follows. For $h \in \mathbb{R}$ and a function $f : I \to E$, we define $T(h)f : I \to E$ as the translate of f by h, i.e.,

$$(T(h)f)(t) := \begin{cases} f(t+h) & \text{if } t+h \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Put

$$I[h] := \{t \in I : \ t+h \in I\}$$

and, for $f \in L^p(I;E)$ and $t > 0$,

$$\varrho_p(f,t) := \sup_{|h|\le t} \|T(h)f - f\|_{L^p(I[h];E)}.$$

Now define

$$B^s_{p,q}(I;E) := \{f \in L^p(I;E) : \|f\|_{B^s_{p,q}(I;E)} < \infty\},$$

where

$$\|f\|_{B^s_{p,q}(I;E)} := \|f\|_{L^p(I;E)} + \Big(\int_0^1 \big(t^{-s}\varrho_p(f,t)\big)^q \,\frac{dt}{t}\Big)^{\frac1q} \tag{3.1}$$

with the obvious modification for $q = \infty$. Endowed with the norm $\|\cdot\|_{B^s_{p,q}(I;E)}$, $B^s_{p,q}(I;E)$ is a Banach space.

The following continuous inclusions hold for all $s, s_1, s_2 \in (0,1)$, $p, q, q_1, q_2 \in [1,\infty]$ with $q_1 \le q_2$, $s_2 \le s_1$:

$$B^s_{p,q_1}(I;E) \hookrightarrow B^s_{p,q_2}(I;E),\ B^{s_1}_{p,q}(I;E) \hookrightarrow B^{s_2}_{p,q}(I;E).$$

If I is bounded, then also

$$B^s_{p_1,q}(I;E) \hookrightarrow B^s_{p_2,q}(I;E)$$

for $1 \le p_2 \le p_1 \le \infty$.

The next lemma will play an important role in setting up our basic framework. We remind the reader of the convention, made at the end of Section 1, that constants appearing in estimates may depend upon the number T_0 which is kept fixed throughout the paper.

Lemma 3.1. *Let $1 \le q < p < \infty$, $s > 0$ and $\alpha \ge 0$ satisfy $s < \frac1q - \frac1p$ and $\alpha < \frac1q - \frac1p - s$, and let $1 \le r < \infty$. For all $T \in [0,T_0]$ and $\phi \in B^s_{p,r}(0,T;E)$ the function $t \mapsto t^{-\alpha}\phi(t)\mathbf{1}_{(0,T)}(t)$ belongs to $B^s_{q,r}(0,T_0;E)$ and there exists a constant $C \ge 0$, independent of $T \in [0,T_0]$, such that*

$$\|t \mapsto t^{-\alpha}\phi(t)\mathbf{1}_{(0,T)}(t)\|_{B^s_{q,r}(0,T_0;E)} \le CT^{\frac1q-\frac1p-s-\alpha}\|\phi\|_{B^s_{p,r}(0,T;E)}.$$

Proof. We prove the lemma under the additional assumption that $\alpha > 0$; the proof simplifies for case $\alpha = 0$. We shall actually prove the following stronger result

$$\|t \mapsto t^{-\alpha}\phi(t)\mathbf{1}_{(0,T)}(t)\|_{B^s_{q,r}(\mathbb{R};E)} \le CT^{\frac1q-\frac1p-s-\alpha}\|\phi\|_{B^s_{p,r}(0,T;E)}$$

with a constant C independent of $T \in [0,T_0]$.

Fix $u \in [0,T]$ and $|h| \le u$. First assume that $h \ge 0$. Then $I[h] = [0,T-h]$ and, by Hölder's inequality,

$$\begin{aligned}
&\Big(\int_{\mathbb{R}}\Big\|\frac{\phi(t+h)\mathbf{1}_{(0,T)}(t+h) - \phi(t)\mathbf{1}_{(0,T)}(t)}{(t+h)^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \\
&\quad\le \Big(\int_{-h}^0 \Big\|\frac{\phi(t+h)}{(t+h)^\alpha}\Big\|^q\,dt\Big)^{\frac1q} + \Big(\int_0^{T-h}\Big\|\frac{\phi(t+h)-\phi(t)}{(t+h)^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \\
&\qquad + \Big(\int_{T-h}^T \Big\|\frac{\phi(t)}{(t+h)^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \\
&\quad\le Cu^{\frac1q-\frac1p-\alpha}\|\phi\|_{L^p(0,T;E)} + CT^{\frac1q-\frac1p-\alpha}\Big(\int_{I[h]}\|\phi(t+h)-\phi(t)\|^p\,dt\Big)^{\frac1p}.
\end{aligned}$$

Again by Hölder's inequality,

$$\Big(\int_{\mathbb{R}}\Big\|\frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{(t+h)^\alpha}-\frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{t^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \le \Big(\int_0^T \big|(t+h)^{-\alpha}-t^{-\alpha}\big|^{\frac{pq}{p-q}}\,dt\Big)^{\frac{p-q}{pq}}\|\phi\|_{L^p(0,T;E)}$$

with

$$\int_0^T \big|(t+h)^{-\alpha}-t^{-\alpha}\big|^{\frac{pq}{p-q}}\,dt \le \int_0^T t^{-\frac{\alpha pq}{p-q}}-(t+h)^{-\frac{\alpha pq}{p-q}}\,dt \le Ch^{1-\frac{\alpha pq}{p-q}} \le Cu^{1-\frac{\alpha pq}{p-q}}.$$

Combining these estimates with the triangle inequality we obtain

$$\Big(\int_{\mathbb{R}}\Big\|\frac{\phi(t+h)\mathbf{1}_{(0,T)}(t+h)}{(t+h)^\alpha}-\frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{t^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \le Cu^{\frac1q-\frac1p-\alpha}\|\phi\|_{L^p(0,T;E)}+CT^{\frac1q-\frac1p-\alpha}\Big(\int_{I[h]}\|\phi(t+h)-\phi(t)\|^p\,dt\Big)^{\frac1p}.$$

A similar estimate holds for $h\le 0$.

Next we split $[0,1]=[0,T\wedge 1]\cup[T\wedge 1,1]$ and estimate the integral in (3.1). For the first we have

$$\begin{aligned}
&\Big(\int_0^{T\wedge 1} u^{-sr}\sup_{|h|\le u}\Big\|t\mapsto \frac{\phi(t+h)\mathbf{1}_{(0,T)}(t+h)}{(t+h)^\alpha}-\frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{t^\alpha}\Big\|^r_{L^q(\mathbb{R};E)}\frac{du}{u}\Big)^{\frac1r}\\
&\le C\Big(\int_0^{T\wedge 1} u^{-sr}\Big[T^{\frac1q-\frac1p-\alpha}\sup_{|h|\le u}\|\phi(\cdot+h)-\phi(\cdot)\|_{L^p(I[h];E)}\\
&\qquad\qquad + u^{\frac{p-q}{pq}-\alpha}\|\phi\|_{L^p(0,T;E)}\Big]^r\frac{du}{u}\Big)^{\frac1r}\\
&\overset{\text{(i)}}{\le} CT^{\frac1q-\frac1p-\alpha}\Big(\int_0^1 u^{-sr}\Big[\sup_{|h|\le u}\|\phi(\cdot+h)-\phi(\cdot)\|_{L^p(I[h];E)}\Big]^r\frac{du}{u}\Big)^{\frac1r}\\
&\qquad\qquad + C\Big(\int_0^T u^{-sr}u^{\frac{(p-q)r}{pq}-\alpha r}\frac{du}{u}\Big)^{\frac1r}\|\phi\|_{L^p(0,T;E)}\\
&\overset{\text{(ii)}}{\le} CT^{\frac1q-\frac1p-\alpha}\|\phi\|_{B^s_{p,r}(0,T;E)}+CT^{\frac1q-\frac1p-s-\alpha}\|\phi\|_{L^p(0,T;E)}.
\end{aligned}$$

In (i) we used the triangle inequality in $L^r(0,T\wedge 1,\frac{du}{u})$ and in (ii) we noted that $\alpha<\frac1q-\frac1p-s$.

Next,

$$\Big(\int_{\mathbb{R}}\Big\|\frac{\phi(t+h)\mathbf{1}_{(0,T)}(t+h)}{(t+h)^\alpha}-\frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{t^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \le 2\Big(\int_0^T\Big\|\frac{\phi(t)}{t^\alpha}\Big\|^q\,dt\Big)^{\frac1q} \le CT^{\frac1q-\frac1p-\alpha}\|\phi\|_{L^p(0,T;E)}.$$

Using this we estimate the second part:

$$\begin{aligned}
&\Big(\int_{T\wedge 1}^1 u^{-sr}\sup_{|h|\le u}\Big\|\frac{\phi(t+h)\mathbf{1}_{(0,T)}(t+h)}{(t+h)^\alpha}-\frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{t^\alpha}\Big\|^r_{L^q(I[h];E)}\frac{du}{u}\Big)^{\frac1r}\\
&\le CT^{\frac1q-\frac1p-\alpha}\|\phi\|_{L^p(0,T;E)}\Big(\int_{T\wedge 1}^1 u^{-sr}\frac{du}{u}\Big)^{\frac1r}\\
&\le CT^{\frac1q-\frac1p-s-\alpha}\|\phi\|_{L^p(0,T;E)}.
\end{aligned}$$

Putting everything together and using Hölder's inequality to estimate the L^q-norm of $t^{-\alpha}\phi(t)$ we obtain

$$\begin{aligned}
&\|t \mapsto t^{-\alpha}\phi(t)\|_{B^s_{q,r}(0,T;E)} \\
&= \|t \mapsto t^{-\alpha}\phi(t)\|_{L^q(0,T;E)} \\
&\quad + \Big(\int_0^1 u^{-sr} \sup_{|h|\le u} \Big\| \frac{\phi(t+h)\mathbf{1}_{(0,T)}(t+h)}{(t+h)^\alpha} - \frac{\phi(t)\mathbf{1}_{(0,T)}(t)}{t^\alpha}\Big\|^r_{L^q(\mathbb{R};E)} \frac{du}{u}\Big)^{\frac1r} \\
&\le CT^{\frac1q-\frac1p-\alpha}\|\phi\|_{L^p(0,T;E)} + CT^{\frac1q-\frac1p-\alpha}\|\phi\|_{B^s_{p,r}(0,T;E)} + CT^{\frac1q-\frac1p-s-\alpha}\|\phi\|_{L^p(0,T;E)}.
\end{aligned}$$

□

A Banach space E has *type* p, where $p \in [1,2]$, if there exists a constant $C \ge 0$ such that for all $x_1, \dots, x_n \in E$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j\, x_j\Big\|^2\Big)^{\frac12} \le C\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{\frac1p}.$$

Here $(r_j)_{j\ge1}$ is a Rademacher sequence. Similarly E has *cotype* q, where $q \in [2,\infty]$, if there exists a constant $C \ge 0$ such that for all $x_1, \dots, x_n \in E$ we have

$$\Big(\sum_{j=1}^n \|x_j\|^q\Big)^{\frac1q} \le C\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j\, x_j\Big\|^2\Big)^{\frac12}.$$

In these definitions the Rademacher variables may be replaced by Gaussian variables without changing the definitions; for a proof and more information see [13]. Every Banach space has type 1 and cotype ∞, the spaces $L^p(S)$, $1 \le p < \infty$, have type $\min\{p,2\}$ and cotype $\max\{p,2\}$, and Hilbert spaces have type 2 and cotype 2. Every UMD space has nontrivial type, i.e., type p for some $p \in (1,2]$.

In view of the basic role of the space $\gamma(L^2(0,T;H),E)$ in the theory of vector-valued stochastic integration, it is natural to look for conditions on a function $\Phi : (0,T) \to \mathcal{L}(H,E)$ ensuring that the associated integral operator $I_\Phi : L^2(0,T;H) \to E$,

$$I_\Phi f := \int_0^T \Phi(t)f(t)\,dt, \qquad f \in L^2(0,T;H),$$

is well-defined and belongs to $\gamma(L^2(0,T;H),E)$. The next proposition, taken from [31], states such a condition for functions Φ belonging to suitable Besov spaces of $\gamma(H,E)$-valued functions.

Lemma 3.2. *If E has type $\tau \in [1,2)$, then $\Phi \mapsto I_\Phi$ defines a continuous embedding*

$$B^{\frac1\tau-\frac12}_{\tau,\tau}(0,T_0;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T_0;H),E),$$

where the constant of the embedding depends on T_0 and the type τ constant of E.

Conversely, if $\Phi \mapsto I_\Phi$ defines a continuous embedding $B^{\frac1\tau-\frac12}_{\tau,\tau}(0,T_0;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T_0;H),E)$, then E has type τ (see [19]); we will not need this result.

Lemma 3.3. *Let E be a Banach space with type $\tau \in [1,2)$. Let $\alpha \ge 0$ and $q > 2$ be such that $\alpha < \frac12 - \frac1q$. There exists a constant $C \ge 0$ such that for all $T \in [0,T_0]$ and $\Phi \in B^{\frac1\tau-\frac12}_{q,\tau}(0,T;\gamma(H,E))$ we have*

$$\sup_{t\in(0,T)} \|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{\gamma(L^2(0,t;H),E)} \le CT^{\frac12-\frac1q-\alpha}\|\Phi\|_{B^{\frac1\tau-\frac12}_{q,\tau}(0,T;\gamma(H,E))}.$$

Proof. Fix $T \in [0, T_0]$ and $t \in [0, T]$. Then,

$$\begin{aligned}
\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{\gamma(L^2(0,t;H),E)} &= \|s \mapsto s^{-\alpha}\Phi(t-s)\|_{\gamma(L^2(0,t;H),E)} \\
&= \|s \mapsto s^{-\alpha}\Phi(t-s)\mathbf{1}_{(0,t)}(s)\|_{\gamma(L^2(0,T_0;H),E)} \\
&\overset{\text{(i)}}{\le} C\|s \mapsto s^{-\alpha}\Phi(t-s)\mathbf{1}_{(0,t)}(s)\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{\tau,\tau}(0,T_0;\gamma(H,E))} \\
&\overset{\text{(ii)}}{\le} Ct^{\frac{1}{2}-\frac{1}{q}-\alpha}\|s \mapsto \Phi(t-s)\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,t;\gamma(H,E))} \\
&\le CT^{\frac{1}{2}-\frac{1}{q}-\alpha}\|\Phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;\gamma(H,E))}.
\end{aligned}$$

In (i) we used Lemma 3.2 and (ii) follows from Lemma 3.1. □

In the remainder of this section we assume that A is the infinitesimal generator of an analytic C_0-semigroup $S = (S(t))_{t\ge 0}$ on E. We fix an arbitrary number $w \in \mathbb{R}$ such that the semigroup generated by $A - w$ is uniformly exponentially stable. The fractional powers $(w - A)^\eta$ are then well-defined, and for $\eta > 0$ we put

$$E_\eta := \mathcal{D}((w - A)^\eta).$$

This is a Banach space with respect to the norm

$$\|x\|_{E_\eta} := \|x\| + \|(w - A)^\eta x\|.$$

As is well known, up to an equivalent norm this definition is independent of the choice of w. The basic estimate

$$\|S(t)\|_{\mathcal{L}(E,E_\eta)} \le Ct^{-\eta}, \qquad t \in [0, T_0], \tag{3.2}$$

valid for $\eta > 0$ with C depending on η, will be used frequently.

The extrapolation spaces $E_{-\eta}$ are defined, for $\eta > 0$, as the completion of E with respect to the norm

$$\|x\|_{E_{-\eta}} := \|(w - A)^{-\eta}x\|.$$

Up to an equivalent norm, this space is independent of the choice of w.

We observe at this point that the spaces E_η and $E_{-\eta}$ inherit all isomorphic Banach space properties of E, such as (co)type, the UMD property, and property (Δ), via the isomorphisms $(w - A)^\eta : E_\eta \simeq E$ and $(w - A)^{-\eta} : E_{-\eta} \simeq E$.

The following lemma is well-known; a sketch of a proof is included for the convenience of the reader.

Lemma 3.4. *Let $q \in [1, \infty)$ and $\tau \in [1, 2)$ be given, and let $\eta \ge 0$ and $\theta \ge 0$ satisfy $\eta + \theta < \frac{3}{2} - \frac{1}{\tau}$. There exists a constant $C \ge 0$ such that for all $T \in [0, T_0]$ and $\phi \in L^\infty(0, T; E_{-\theta})$ we have $S * \phi \in B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0, T; E_\eta)$ and*

$$\|S * \phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;E_\eta)} \le CT^{\frac{1}{q}}\|\phi\|_{L^\infty(0,T;E_{-\theta})}.$$

Proof. Without loss of generality we may assume that $\eta, \theta > 0$. Let $\varepsilon > 0$ be such that $\eta + \theta < \frac{3}{2} - \frac{1}{\tau} - \varepsilon$. Then

$$\|S * \phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;E_\eta)} \le CT^{\frac{1}{q}}\|S * \phi\|_{C^{\frac{1}{\tau}-\frac{1}{2}-\varepsilon}([0,T];E_\eta)} \le CT^{\frac{1}{q}}\|\phi\|_{L^\infty(0,T;E_{-\theta})}.$$

The first estimate is a direct consequence of the definition of the Besov norm, and the second follows from [26, Proposition 4.2.1]. □

From the previous two lemmas we deduce the next convolution estimate.

Proposition 3.5. *Let E be a Banach space with type $\tau \in [1,2]$ and let $0 \le \alpha < \frac{1}{2}$. Let $\eta \ge 0$ and $\theta \ge 0$ satisfy $\eta + \theta < \frac{3}{2} - \frac{1}{\tau}$. Then there is a constant $C \ge 0$ such that for all $0 \le t \le T \le T_0$ and $\phi \in L^\infty(0,T;E)$,*

$$\|s \mapsto (t-s)^{-\alpha}(S * \phi)(s)\|_{\gamma(L^2(0,t),E_\eta)} \le CT^{\frac{1}{2}-\alpha}\|\phi\|_{L^\infty(0,T;E_{-\theta})}.$$

Proof. First assume that $1 \le \tau < 2$. It follows from Lemmas 3.3 and 3.4 that for any $q > 2$ such that $\alpha < \frac{1}{2} - \frac{1}{q}$,

$$\begin{aligned}\|s \mapsto (t-s)^{-\alpha} S * \phi(s)\|_{\gamma(L^2(0,t),E_\eta)} &\le CT^{\frac{1}{2}-\frac{1}{q}-\alpha}\|S * \phi\|_{B^{\frac{1}{\tau}-\frac{1}{2}}_{q,\tau}(0,T;E_\eta)} \\ &\le CT^{\frac{1}{2}-\alpha}\|\phi\|_{L^\infty(0,T;E_{-\theta})}.\end{aligned}$$

For $\tau = 2$ we argue as follows. Since E_η has type 2, we have a continuous embedding $L^2(0,t;E_\eta) \hookrightarrow \gamma(L^2(0,t),E_\eta)$; see [37]. Therefore, using (3.2),

$$\begin{aligned}\|s \mapsto (t-s)^{-\alpha} S * \phi(s)\|_{\gamma(L^2(0,t),E_\eta)} &\le C\|s \mapsto (t-s)^{-\alpha} S * \phi(s)\|_{L^2(0,t;E_\eta)} \\ &\le C\|s \mapsto (t-s)^{-\alpha}\|_{L^2(0,t)}\|S * \phi\|_{L^\infty(0,T;E_\eta)} \\ &\le CT^{\frac{1}{2}-\alpha}T^{1-\eta-\theta}\|\phi\|_{L^\infty(0,T;E_{-\theta})}.\end{aligned}$$

□

The following lemma, due to Da Prato, Kwapień and Zabczyk [9, Lemma 2] in the Hilbert space case, gives a Hölder estimate for the convolution

$$R_\alpha\phi(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}S(t-s)\phi(s)\,ds.$$

The proof carries over to Banach spaces without change.

Lemma 3.6 ([9])**.** *Let $0 < \alpha \le 1$, $1 < p < \infty$, $\lambda \ge 0$, $\eta \ge 0$, and $\theta \ge 0$ satisfy $\lambda + \eta + \theta < \alpha - \frac{1}{p}$. Then there exist a constant $C \ge 0$ and an $\varepsilon > 0$ such that for all $\phi \in L^p(0,T;E)$ and $T \in [0,T_0]$,*

$$\|R_\alpha\phi\|_{C^\lambda([0,T];E_\eta)} \le CT^\varepsilon\|\phi\|_{L^p(0,T;E_{-\theta})}.$$

4. Stochastic convolutions

We now turn to the problem of estimating stochastic convolution integrals. We start with a lemma which, in combination with Lemma 2.9, can be used to estimate stochastic convolutions involving analytic semigroups.

Lemma 4.1. *Let S be an analytic C_0-semigroup on a Banach space E. For all $0 \le a < 1$ and $\varepsilon > 0$ the family*

$$\big\{t^{a+\varepsilon}S(t) \in \mathcal{L}(E,E_a) : \ t \in [0,T]\big\}$$

is R-bounded in $\mathcal{L}(E,E_a)$, with R-bound of order $O(T^\varepsilon)$ as $T \downarrow 0$.

Proof. Let $N : [0,T] \to \mathcal{L}(E,E_a)$ be defined as $N(t) = t^{a+\varepsilon}S(t)$. Then N is continuously differentiable on $(0,T)$ and $N'(t) = (a+\varepsilon)t^{a+\varepsilon-1}S(t) + t^{a+\varepsilon}AS(t)$, where A is the generator of S. Hence, by (3.2),

$$\|N'(t)\|_{\mathcal{L}(E,E_a)} \le Ct^{\varepsilon-1} \ \text{ for } t \in (0,T).$$

By Lemma 2.6 the R-bound on $[0,T]$ can now be bounded from above by

$$\int_0^T \|N'(t)\|_{\mathcal{L}(E,E_a)}\,dt \le CT^{\varepsilon}.$$

□

We continue with an extension of the Da Prato-Kwapień-Zabczyk factorization method [9] for Hilbert spaces to UMD spaces. For deterministic Φ, the assumption that E is UMD can be dropped. A related regularity result for arbitrary C_0-semigroups is due to Millet and Smoleński [28].

It will be convenient to introduce the notation

$$S \diamond \Phi(t) := \int_0^t S(t-s)\Phi(s)\,dW_H(s)$$

for the stochastic convolution with respect to W_H of S and Φ, where W_H is an H-cylindrical Brownian motion.

Proposition 4.2. *Let $0 < \alpha < \frac{1}{2}$, $\lambda \ge 0$, $\eta \ge 0$, $\theta \ge 0$, and $p > 2$ satisfy $\lambda + \eta + \theta < \alpha - \frac{1}{p}$. Let A be the generator of an analytic C_0-semigroup S on a UMD space E and let $\Phi : (0,T) \times \Omega \to \mathcal{L}(H, E_{-\theta})$ be H-strongly measurable and adapted. Then there exist $\varepsilon > 0$ and $C \ge 0$ such that*

$$\mathbb{E}\|S \diamond \Phi\|^p_{C^\lambda([0,T];E_\eta)} \le C^p T^{\varepsilon p} \int_0^T \mathbb{E}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}\,dt.$$

Here, and in similar formulations below, it is part of the assumptions that the right-hand side is well-defined and finite. In particular it follows from the proposition there exist $\varepsilon > 0$ and $C \ge 0$ such that

$$\mathbb{E}\|S \diamond \Phi\|^p_{C^\lambda([0,T];E_\eta)} \le C^p T^{\varepsilon p} \sup_{t\in[0,T]} \mathbb{E}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}$$

provided the right-hand side is finite.

Proof. The idea of the proof is the same as in [9], but there are some technical subtleties which justify us to outline the main steps.

Let $\beta \in (0, \frac{1}{2})$ be such that $\lambda + \eta < \beta - \frac{1}{p} < \alpha - \theta - \frac{1}{p}$. It follows from Lemmas 2.9 and 4.1 that, for almost all $t \in [0,T]$, almost surely we have

$$\begin{aligned} &\|s \mapsto (t-s)^{-\beta} S(t-s)\Phi(s)\|_{\gamma(L^2(0,t;H),E)} \\ &\qquad \le Ct^{\alpha-\beta-\theta}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|_{\gamma(L^2(0,t;H),E_{-\theta})}. \end{aligned} \tag{4.1}$$

By Proposition 2.4, the process $\zeta_\beta : [0,T] \times \Omega \to E$,

$$\zeta_\beta(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} S(t-s)\Phi(s)\,dW_H(s),$$

is well-defined for almost all $t \in [0,T]$ and satisfies

$$\big(\mathbb{E}\|\zeta_\beta(t)\|^p\big)^{\frac{1}{p}} \le Ct^{\alpha-\beta-\theta}\big(\mathbb{E}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}\big)^{\frac{1}{p}}.$$

By Proposition A.1 the process ζ_β is strongly measurable. Therefore, by Fubini's theorem,

$$\|\zeta_\beta\|_{L^p(\Omega;L^p(0,T;E))} \le CT^{\alpha-\beta-\theta} \int_0^T \mathbb{E}\|s \mapsto (t-s)^{-\alpha}\Phi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}\,dt.$$

By Lemma 3.6, the paths of $R_\beta\zeta_\beta$ belong to $C^\lambda([0,T];E_\eta)$ almost surely, and for some $\varepsilon'>0$ independent of $T\in[0,T_0]$ we have

$$\begin{aligned}\|R_\beta\zeta_\beta\|&_{L^p(\Omega;C^\lambda([0,T];E_\eta))}\\ &\le CT^{\varepsilon'}\|\zeta_\beta\|_{L^p(\Omega;L^p(0,T;E))}\\ &\le CT^{\alpha-\beta-\theta+\varepsilon'}\Big(\int_0^T \mathbb{E}\|s\mapsto (t-s)^{-\alpha}\Phi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}\,dt\Big)^{\frac1p}.\end{aligned}\tag{4.2}$$

The right ideal property (2.1), (4.1), and Proposition 2.4 imply the stochastic integrability of $s\mapsto S(t-s)\Phi(s)$ for almost all $t\in[0,T]$. The proof will be finished (with $\varepsilon=\alpha-\beta-\theta+\varepsilon'$) by showing that almost surely on $(0,T)\times\Omega$,

$$S\diamond\Phi = R_\beta\zeta_\beta.$$

It suffices to check that for almost all $t\in[0,T]$ and $x^*\in E^*$ we have, almost surely,

$$\langle S\diamond\Phi(t),x^*\rangle = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\langle S(t-s)\zeta_\beta(s),x^*\rangle\,ds.\tag{4.3}$$

This follows from a standard argument via the stochastic Fubini theorem, cf. [9], which can be applied here since almost surely we have, writing $\langle\Phi(r),x^*\rangle := \Phi^*(r)x^*$,

$$\begin{aligned}\int_0^t &\big\|\big\langle (t-s)^{\beta-1}S(t-s)(s-\cdot)^{-\beta}S(s-\cdot)\Phi(\cdot)\mathbf{1}_{[0,s]}(\cdot),x^*\big\rangle\big\|_{L^2(0,t;H)}\,ds\\ &=\int_0^t \big\|\big\langle (s-\cdot)^{-\beta}S(s-\cdot)\Phi(\cdot),(t-s)^{\beta-1}S^*(t-s)x^*\big\rangle\big\|_{L^2(0,t;H)}\,ds\\ &\le\int_0^t \|(s-\cdot)^{-\beta}S(s-\cdot)\Phi(\cdot)\|_{\gamma(L^2(0,t;H),E)}\|(t-s)^{\beta-1}S^*(t-s)x^*\|\,ds,\end{aligned}$$

which is finite for almost all $t\in[0,T]$ by Hölder's inequality. □

Remark 4.3. The stochastic integral $S\diamond\Phi$ in Proposition 4.2 may be defined only for almost all $t\in[0,T]$. If in addition one assumes that $\Phi\in L^p((0,T)\times\Omega;\gamma(H,E_{-\theta}))$, then $S\diamond\Phi(t)$ is well-defined in E_η for all $t\in[0,T]$. This follows readily from (4.3), [32, Theorem 3.6(2)] and the density of E^* in $(E_\eta)^*$. Since we will not need this in the sequel, we leave this to the interested reader.

As a consequence we have the following regularity result of stochastic convolutions in spaces with type $\tau\in[1,2)$. We will not need this result below, but we find it interesting enough to state it separately.

Corollary 4.4. *Let E be a UMD space with type $\tau\in[1,2)$. Let $p>2$, $q>2$, $\lambda\ge0$, $\eta\ge0$, $\theta\ge0$ be such that $\lambda+\eta+\theta<\frac12-\frac1p-\frac1q$. Then there is an $\delta>0$ such that for all H-strongly strongly measurable and adapted $\Phi:(0,T)\times\Omega\to\mathcal{L}(H,E_{-\theta})$,*

$$\mathbb{E}\|S\diamond\Phi\|^p_{C^\lambda([0,T];E_\eta)}\le C^pT^{\delta p}\mathbb{E}\|\Phi\|^p_{B^{\frac1\tau-\frac12}_{q,\tau}(0,T;\gamma(H,E_{-\theta}))}.\tag{4.4}$$

Proof. By assumption we may choose $\alpha\in(0,\frac12)$ such that $\lambda+\eta+\theta+\frac1p<\alpha<\frac12-\frac1q$. The result now follows from Proposition 4.2 and Lemma 3.3 (noting that $E_{-\theta}$ has type τ):

$$\begin{aligned}\mathbb{E}\|S\diamond\Phi\|^p_{C^\lambda([0,T];E_\eta)} &\le C^pT^{\varepsilon p}\sup_{t\in[0,T]}\mathbb{E}\|s\mapsto(t-s)^{-\alpha}\Phi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}\\ &\le C^pT^{(\frac12-\frac1q-\alpha+\varepsilon)p}\mathbb{E}\|\Phi\|^p_{B^{\frac1\tau-\frac12}_{q,\tau}(0,T;\gamma(H,E_{-\theta}))}.\end{aligned}$$

□

The main estimate of this section is contained in the next result.

Proposition 4.5. *Let E be a UMD Banach space. Let $\eta\ge 0$, $\theta\ge 0$, $\alpha>0$ satisfy $0\le\eta+\theta<\alpha<\frac12$. Let $\Phi:(0,T)\times\Omega\to\mathcal{L}(H,E_{-\theta})$ be adapted and H-strongly measurable. Then for all $1<p<\infty$ and all $0\le t\le T\le T_0$,*

$$\mathbb{E}\|(t-\cdot)^{-\alpha}S\diamond\Phi(\cdot)\|^p_{\gamma(L^2(0,t;H),E_\eta)}\le C^pT^{(\frac12-\eta-\theta)p}\mathbb{E}\|(t-\cdot)^{-\alpha}\Phi(\cdot)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}.$$

Proof. Fix $0\le t\le T\le T_0$. As in Proposition 4.2 one shows that the finiteness of the right-hand side implies that $s\mapsto S(t-s)\Phi(s)$ is stochastically integrable on $[0,t]$. We claim that $s\mapsto S(t-s)\Phi(s)$ takes values in E_η almost surely and is stochastically integrable on $[0,t]$ as an E_η-valued process. Indeed, let $\varepsilon>0$ be such that $\beta:=\eta+\theta+\varepsilon<\alpha$ and put

$$N_\beta(t):=t^\beta(\mu-A)^{\eta+\theta}S(t).$$

It follows from Lemmas 2.9 and 4.1 that

$$\begin{aligned}\mathbb{E}\|S(t-\cdot)\Phi(\cdot)\|^p_{\gamma(L^2(0,t;H),E_\eta)} &\le C\mathbb{E}\|N_\beta(t-\cdot)(t-\cdot)^{-\beta}\Phi(\cdot)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})}\\ &\le CT^{\varepsilon p}\mathbb{E}\|(t-\cdot)^{-\beta}\Phi(\cdot)\|^p_{\gamma(L^2(0,t;H),E_{-\theta})},\end{aligned}$$

and the expression on the right-hand side is finite by the assumption. The stochastic integrability now follows from Proposition 2.4. This proves the claim. Moreover, by Proposition A.1, the stochastic convolution process $S\diamond\Phi$ is adapted and strongly measurable as an E_η-valued process.

Let $G^p(\Omega;E_\eta)$ and $G^p(\Omega\times\widetilde{\Omega};E_\eta)$ denote the closed subspaces in $L^p(\Omega;E_\eta)$ and $L^p(\Omega\times\widetilde{\Omega};E_\eta)$ spanned by all elements of the form $\int_0^T\Psi\,dW_H$ and $\int_0^T\Psi\,d\widetilde{W}_H$, respectively, where $\widetilde{W}_H$ is an independent copy of W_H and Ψ ranges over all adapted elements in $L^p(\Omega;\gamma(L^2(0,T;H),E))$. Since E_η is a UMD space, by Proposition 2.4 the operator

$$D_p\int_0^T\Psi\,d\widetilde{W}_H:=\int_0^T\Psi\,dW_H,$$

is well defined and bounded from $G^p(\Omega\times\widetilde{\Omega};E_\eta)$ to $G^p(\Omega;E_\eta)$. Using the Fubini isomorphism of Lemma 2.3 twice, we estimate

$$\begin{aligned}&\big\|s\mapsto(t-s)^{-\alpha}S\diamond\Phi(s)\big\|_{L^p(\Omega;\gamma(L^2(0,t),E_\eta))}\\ &\eqsim\Big\|s\mapsto\int_0^s(t-s)^{-\alpha}S(s-r)\Phi(r)\,dW_H(r)\Big\|_{\gamma(L^2(0,t),G^p(\Omega;E_\eta))}\\ &=\Big\|s\mapsto D_p\int_0^t 1_{(0,s)}(r)(t-s)^{-\alpha}S(s-r)\Phi(r)\,d\widetilde{W}_H(r)\Big\|_{\gamma(L^2(0,t),G^p(\Omega;E_\eta))}\\ &\lesssim\Big\|s\mapsto\int_0^t 1_{(0,s)}(r)(t-s)^{-\alpha}S(s-r)\Phi(r)\,d\widetilde{W}_H(r)\Big\|_{\gamma(L^2(0,t),G^p(\Omega\times\widetilde{\Omega};E_\eta))}\\ &\eqsim\Big\|s\mapsto\int_0^s(t-s)^{-\alpha}S(s-r)\Phi(r)\,d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_\eta)))}.\end{aligned}$$

Rewriting the right-hand side in terms of the function $N_\beta(t) = t^\beta(\mu - A)^{\eta+\theta}S(t)$ introduced above and using the stochastic Fubini theorem to interchange the Lebesgue integral and the stochastic integral, the right-hand side can be estimated as

$$\begin{aligned}
&\Big\| s \mapsto \int_0^s (t-s)^{-\alpha} S(s-r)\Phi(r)\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_\eta)))} \\
&\eqsim \Big\| s \mapsto \int_0^s (t-s)^{-\alpha}(\mu - A)^{\eta+\theta} S(s-r)\Phi(r)\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} \\
&= \Big\| s \mapsto \int_0^s (t-s)^{-\alpha}(s-r)^{-\beta} N(s-r)\Phi(r)\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} \\
&= \Big\| s \mapsto \int_0^s (t-s)^{-\alpha} \\
&\qquad \times (s-r)^{-\beta} \int_0^{s-r} N_\beta'(w)\Phi(r)\, dw\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} \\
&= \Big\| s \mapsto \int_0^s N_\beta'(w) \\
&\qquad \times \int_0^{s-w} (t-s)^{-\alpha}(s-r)^{-\beta}\Phi(r)\, d\widetilde{W}_H(r)\, dw\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} \\
&= \Big\| s \mapsto \int_0^t N_\beta'(w) 1_{(0,s)}(w) \\
&\qquad \times \mathbb{E}_{\widetilde{\mathscr{F}}_{s-w}} \int_0^s (t-s)^{-\alpha}(s-r)^{-\beta}\Phi(r)\, d\widetilde{W}_H(r)\, dw\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))},
\end{aligned}$$

where $\mathbb{E}_{\widetilde{\mathscr{F}}_t}(\xi) := \mathbb{E}(\xi|\widetilde{\mathscr{F}}_t)$ is the conditional expectation with respect to $\widetilde{\mathscr{F}}_t = \sigma(\widetilde{W}_H(s)h:\ 0 \le s \le t,\ h \in H\}$. Next we note that

$$\int_0^t \|N_\beta'(w)\|\, dw \lesssim T^\varepsilon.$$

Applying Lemmas 2.8 and 2.9 pointwise with respect to $\omega \in \Omega$, we may estimate the right-hand side above by

$$\begin{aligned}
&\int_0^t \|N_\beta'(w)\| \Big\| s \mapsto 1_{(w,t)}(s) \\
&\qquad \times \mathbb{E}_{\widetilde{\mathscr{F}}_{s-w}} \int_0^s (t-s)^{-\alpha}(s-r)^{-\beta}\Phi(r)\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} dw \\
&\lesssim T^\varepsilon \Big\| s \mapsto \mathbb{E}_{\widetilde{\mathscr{F}}_{s-w}} \int_0^s (t-s)^{-\alpha}(s-r)^{-\beta}\Phi(r)\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} \\
&\lesssim T^\varepsilon \Big\| s \mapsto \int_0^s (t-s)^{-\alpha}(s-r)^{-\beta}\Phi(r)\, d\widetilde{W}_H(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t),L^p(\widetilde{\Omega};E_{-\theta})))} \\
&\lesssim T^\varepsilon \big\| s \mapsto \big[r \mapsto (t-s)^{-\alpha}(s-r)^{-\beta} 1_{(0,s)}(r)\Phi(r)\big]\big\|_{L^p(\Omega;\gamma(L^2(0,t),\gamma(L^2(0,t;H),E_{-\theta})))}.
\end{aligned}$$

Using the isometry

$$\gamma(H_1, \gamma(H_2, F)) \simeq \gamma(H_2, \gamma(H_1; F)),$$

and the Fubini isomorphism, the right hand side is equivalent to

$$\eqsim T^{\varepsilon}\big\|s\mapsto \big[r\mapsto (t-s)^{-\alpha}(s-r)^{-\beta}1_{(0,s)}(r)\Phi(r)\big]\big\|_{L^p(\Omega;\gamma(L^2(0,t),\gamma(L^2(0,t;H),E_{-\theta})))}$$
$$\eqsim T^{\varepsilon}\big\|r\mapsto \big[s\mapsto (t-s)^{-\alpha}(s-r)^{-\beta}1_{(0,s)}(r)\Phi(r)\big]\big\|_{L^p(\Omega;\gamma(L^2(0,t;H),\gamma(L^2(0,t),E_{-\theta})))}.$$

To proceed further we want to apply, pointwise with respect to Ω, Lemma 2.9 to the multiplier

$$M:(0,t)\to\mathcal{L}(E_{-\theta},\gamma(L^2(0,t),E_{-\theta}))$$

defined by

$$M(r)x:=f_{r,t}\otimes x,\quad s\in(0,t),\quad x\in E_{-\theta},$$

where $f_{r,t}\in L^2(0,t)$ is the function

$$f_{r,t}(s):=(t-r)^{\alpha}(t-s)^{-\alpha}(s-r)^{-\beta}\mathbf{1}_{(r,t)}(s).$$

We need to check that the range of M is γ-bounded in $\mathcal{L}(E_{-\theta},\gamma(L^2(0,t),E_{-\theta}))$. For this we invoke Lemma 2.7, keeping in mind that R-bounded families are always γ-bounded and that UMD spaces have finite cotype. To apply the lemma we check that functions $f_{s,t}$ are uniformly bounded in $L^2(0,t)$:

$$\begin{aligned}\int_0^t |f_{r,t}(s)|^2\,ds &= (t-r)^{2\alpha}\int_r^t (t-s)^{-2\alpha}(s-r)^{-2\beta}\,ds\\ &= (t-r)^{1-2\beta}\int_0^1 (1-u)^{-2\alpha}u^{-2\beta}\,du\\ &\le T^{1-2\beta}\int_0^1 (1-u)^{-2\alpha}u^{-2\beta}\,du.\end{aligned}$$

It follows from Lemma 2.9 that

$$\begin{aligned}&\Big\|s\mapsto (t-s)^{-\alpha}(s-\cdot)^{-\beta}1_{(0,s)}(\cdot)\Phi(\cdot)\Big\|_{L^p(\Omega;\gamma(L^2(0,t;H),\gamma(L^2(0,t),E_{-\theta})))}\\ &\quad\le CT^{\frac12-\beta}\Big\|r\mapsto (t-r)^{-\alpha}\Phi(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t;H),E_{-\theta}))}\\ &\quad= CT^{\frac12-\eta-\theta-\varepsilon}\Big\|r\mapsto (t-r)^{-\alpha}\Phi(r)\Big\|_{L^p(\Omega;\gamma(L^2(0,t;H),E_{-\theta}))}.\end{aligned}$$

Combining all estimates we obtain the result. □

5. L^2_γ-Lipschitz functions

Let (S,Σ) be a countably generated measurable space and let μ be a finite measure on (S,μ). Then $L^2(S,\mu)$ is separable and we may define

$$L^2_\gamma(S,\mu;E):=\gamma(L^2(S,\mu);E)\cap L^2(S,\mu;E).$$

Here, $\gamma(L^2(S,\mu);E)\cap L^2(S,\mu;E)$ denotes the Banach space of all strongly μ-measurable functions $\phi:S\to E$ for which

$$\|\phi\|_{L^2_\gamma(S,\mu;E)}:=\|\phi\|_{\gamma(L^2(S,\mu);E)}+\|\phi\|_{L^2(S,\mu;E)}$$

is finite. One easily checks that the simple functions are dense in $L^2_\gamma(S,\mu;E)$.

Next let H be a nonzero separable Hilbert space, let E_1 and E_2 be Banach spaces, and let $f:S\times E_1\to\mathcal{L}(H,E_2)$ be a function such that for all $x\in E_1$ we have $f(\cdot,x)\in\gamma(L^2(S,\mu;H),E_2)$. For simple functions $\phi:S\to E_1$ one easily checks that $s\mapsto f(s,\phi(s))\in\gamma(L^2(S,\mu;H),E_2)$. We call f *L^2_γ-Lipschitz function* *with respect to* μ if f is strongly continuous in the second variable and for all simple functions $\phi_1, \phi_2 : S \to E_1$,

$$\|f(\cdot,\phi_1) - f(\cdot,\phi_2)\|_{\gamma(L^2(S,\mu;H),E_2)} \le C\|\phi_1 - \phi_2\|_{L^2_\gamma(S,\mu;E_1)}. \tag{5.1}$$

In this case the mapping $\phi \mapsto S_{\mu,f}\phi := f(\cdot,\phi(\cdot))$ extends uniquely to a Lipschitz mapping from $L^2_\gamma(S,\mu;E_1)$ into $\gamma(L^2(S,\mu;H),E_2)$. Its Lipschitz constant will be denoted by $L^\gamma_{\mu,f}$.

It is evident from the definitions that for simple functions $\phi : S \to E_1$, the operator $S_f(\phi) \in \gamma(L^2(S,\mu;H),E_2)$ is represented by the function $f(\cdot,\phi(\cdot))$ The next lemma extends this to arbitrary functions $\phi \in L^2_\gamma(S,\mu;E_1)$.

Lemma 5.1. *If* $f : S \times E_1 \to \mathcal{L}(H,E_2)$ *is an* L^2_γ*-Lipschitz function, then for all* $\phi \in L^2_\gamma(S,\mu;E_1)$ *the operator* $S_{\mu,f}\phi \in \gamma(L^2(S,\mu;H),E_2)$ *is represented by the function* $f(\cdot,\phi(\cdot))$.

Proof. Let $(\phi_n)_{n\ge1}$ be a sequence of simple functions such that $\phi = \lim_{n\to\infty}\phi_n$ in $L^2_\gamma(S,\mu;E_1)$. We may assume that $\phi = \lim_{n\to\infty}\phi_n$ μ-almost everywhere. It follows from (5.1) that $(f(\cdot,\phi_n(\cdot)))_{n\ge1}$ is a Cauchy sequence in $\gamma(L^2(S,\mu;H),E_2)$. Let $R \in \gamma(L^2(S,\mu;H),E_2)$ be its limit. We must show that R is represented by $f(\cdot,\phi(\cdot))$. Let $x^* \in E_2^*$ be arbitrary. Since $R^*x^* = \lim_{n\to\infty} f^*(\cdot,\phi_n(\cdot))x^*$ in $L^2(S,\mu;H)$ we may choose a subsequence $(n_k)_{k\ge1}$ such that $R^*x^* = \lim_{k\to\infty} f^*(\cdot,\phi_{n_k}(\cdot))x^*$ μ-almost everywhere. On the other hand since f is strongly continuous in the second variable we have

$$\lim_{k\to\infty} f^*(s,\phi_{n_k}(s))x^* = f^*(s,\phi(s))x^* \quad \text{for } \mu\text{-almost all } s \in S.$$

This proves that for all $h \in H$ we have $R^*x^* = f^*(\cdot,\phi(\cdot))x^*$ μ-almost everywhere and the result follows. □

Justified by this lemma, in what follows we shall always identify $S_{\mu,f}\phi$ with $f(\cdot,\phi(\cdot))$.

If f is L^2_γ-Lipschitz with respect to all finite measures μ on (S,Σ) and

$$L^\gamma_f := \sup\{L^\gamma_{\mu,f} : \mu \text{ is a finite measure on } (S,\Sigma)\}$$

is finite, we say that f is a L^2_γ*-Lipschitz function.* In type 2 spaces there is the following easy criterium to check whether a function is L^2_γ-Lipschitz.

Lemma 5.2. *Let* E_2 *have type* 2*. Let* $f : S \times E_1 \to \gamma(H,E_2)$ *be such that for all* $x \in E_1$, $f(\cdot,x)$ *is strongly measurable. If there is a constant* C *such that*

$$\|f(s,x)\|_{\gamma(H,E_2)} \le C(1+\|x\|), \quad s \in S, \quad x \in E_1, \tag{5.2}$$

$$\|f(s,x) - f(s,y)\|_{\gamma(H,E_2)} \le C\|x-y\|, \quad s \in S, \quad x,y \in E_1, \tag{5.3}$$

then f *is a* L^2_γ*-Lipschitz function and* $L^\gamma_f \le C_2C$*, where* C_2 *is the Rademacher type* 2 *constant of* E_2*. Moreover, it satisfies the following linear growth condition*

$$\|f(\cdot,\phi)\|_{\gamma(L^2(S,\mu;H),E_2)} \le C_2C(1+\|\phi\|_{L^2(S,\mu;E_1)}).$$

If f does not depend on S, one can check that (5.1) implies (5.2) and (5.3).

Proof. Let $\phi_1,\phi_2 \in L^2(S,\mu;E_1)$. Via an approximation argument and (5.3) one easily checks that $f(\cdot,\phi_1)$ and $f(\cdot,\phi_2)$ are strongly measurable. It follows from (5.2) that $f(\cdot,\phi_1)$ and $f(\cdot,\phi_2)$ are in $L^2(S,\mu;\gamma(H,E_2))$ and from (5.3) we obtain

$$\|f(\cdot,\phi_1) - f(\cdot,\phi_2)\|_{L^2(S,\mu;\gamma(H,E_2))} \le C\|\phi_1 - \phi_2\|_{L^2(S,\mu;E_1)}. \tag{5.4}$$

Recall from [34] that $L^2(S,\mu;\gamma(H,E_1)) \hookrightarrow \gamma(L^2(S,\mu;H),E_1)$ where the norm of the embedding equals C_2. From this and (5.4) we conclude that

$$\|f(\cdot,\phi_1) - f(\cdot,\phi_2)\|_{\gamma(L^2(S,\mu;H),E_2)} \le C_2 C\|\phi_1 - \phi_2\|_{L^2(S,\mu;E_1)}.$$

This clearly implies the result. The second statement follows in the same way. □

A function $f : E_1 \to \mathcal{L}(H,E_2)$ is said to be *L^2_γ-Lipschitz* if the induced function $\tilde f : S\times E_1 \to \mathcal{L}(H,E_2)$, defined by $\tilde f(s,x) = f(x)$, is L^2_γ-Lipschitz for every finite measure space (S,Σ,μ).

Lemma 5.3. *For a function $f : E_1 \to \mathcal{L}(H,E_2)$, the following assertions are equivalent:*

1. *f is L^2_γ-Lipschitz;*
2. *There is a constant C such that for some (and then for every) orthonormal basis $(h_m)_{m\ge1}$ of H and all finite sequences $(x_n)_{n=1}^N, (y_n)_{n=1}^N$ in E_1 we have*

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^N\sum_{m\ge1}\gamma_{nm}(f(x_n)h_m - f(y_n)h_m)\Big\|^2 \\ \le C^2\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n(x_n-y_n)\Big\|^2 + C^2\sum_{n=1}^N\|x_n-y_n\|^2.\end{aligned}$$

Proof. (1) ⇒ (2): Let $(h_m)_{m\ge1}$ be an orthonormal basis and let $(x_n)_{n=1}^N$ and $(y_n)_{n=1}^N$ in E_1 be arbitrary. Take $S=(0,1)$ and μ the Lebesgue measure and choose disjoint sets $(S_n)_{n=1}^N$ in $(0,1)$ such that $\mu(S_n) = \frac{1}{N}$ for all $n = 1,\dots,N$. Now define $\phi_1 := \sum_{n=1}^N \mathbf{1}_{S_n}\otimes x_n$ and $\phi_2 := \sum_{n=1}^N \mathbf{1}_{S_n}\otimes y_n$. Then (2) follows from (5.1).

(2) ⇒ (1): Since the distribution of Gaussian vectors is invariant under orthogonal transformations, if (2) holds for one orthonormal basis $(h_m)_{m\ge1}$, then it holds for every orthonormal basis $(h_m)_{n\ge1}$. By a well-known argument (cf. [16, Proposition 1]), (2) implies that for all $(a_n)_{n=1}^N$ in $\mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^N\sum_{m\ge1}a_n\gamma_{nm}(f(x_n)h_m - f(y_n)h_m)\Big\|^2 \\ \le C^2\mathbb{E}\Big\|\sum_{n=1}^N a_n\gamma_n(x_n-y_n)\Big\|^2 + C^2\sum_{n=1}^N a_n^2\|x_n-y_n\|^2.\end{aligned}$$

Now (5.1) follows for simple functions ϕ, and the general case follows from this by an approximation argument. □

Clearly, every L^2_γ-Lipschitz function $f : E_1 \to \gamma(H,E_2)$ is a Lipschitz function. It is a natural question whether Lipschitz functions are automatically L^2_γ-Lipschitz. Unfortunately, this is not true. It follows from the proof of [30, Theorem 1] that if $\dim(H)\ge 1$, then every Lipschitz function $f : E_1 \to \gamma(H,E_2)$ is L^2_γ-Lipschitz if and only if E_2 has type 2.

A Banach space E has *property* (α) if for all $N\ge1$ and all sequences $(x_{mn})_{m,n=1}^N$ in E we have

$$\mathbb{E}\Big\|\sum_{m,n=1}^N r_{mn}x_{mn}\Big\|^2 \eqsim \mathbb{E}'\mathbb{E}''\Big\|\sum_{m,n=1}^N r'_m r''_n x_{mn}\Big\|^2.$$

Here, $(r_{mn})_{m,n\geq 1}$, $(r'_m)_{m\geq 1}$, and $(r''_n)_{n\geq 1}$ are Rademacher sequences, the latter two independent of each other. By a randomization argument one can show that the Rademacher random variables can be replaced by Gaussian random variables. It can be shown using the Kahane-Khintchine inequalities that the exponent 2 in the definition can be replaced by any number $1 \leq p < \infty$.

Property (α) has been introduced by Pisier [36]. Examples of spaces with this property are the Hilbert spaces and the spaces L^p for $1 \leq p < \infty$.

The next lemma follows directly from the definition of property (α) and Lemma 5.3.

Lemma 5.4. *Let E_2 be a space with property (α). Then $f : E_1 \to \gamma(H, E_2)$ is L^2_γ-Lipschitz if and only if there exists a constant C such that for all finite sequences $(x_n)_{n=1}^N$ and $(y_n)_{n=1}^N$ in E_1 we have*

$$\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n (f(x_n) - f(y_n))\Big\|^2_{\gamma(H,E_2)} \leq C^2 \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n (x_n - y_n)\Big\|^2 + C^2 \sum_{n=1}^N \|x_n - y_n\|^2.$$

In particular, every $f \in \mathcal{L}(E_1, \gamma(H, E_2))$ is L^2_γ-Lipschitz.

When H is finite dimensional, this result remains valid even if E_2 fails to have property (α).

The next example identifies an important class of L^2_γ-Lipschitz continuous functions.

Example 5.5 (Nemytskii maps). Fix $p \in [1, \infty)$ and let (S, Σ, μ) be a σ-finite measure space. Let $b : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function; in case $\mu(S) = \infty$ we also assume that $b(0) = 0$. Define the Nemytskii map $B : L^p(S) \to L^p(S)$ by $B(x)(s) := b(x(s))$. Then B is L^2_γ-Lipschitz with respect to μ. Indeed, it follows from the Kahane-Khintchine inequalities that

$$\begin{aligned}\Big(\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n (B(x_n) - B(y_n))\Big\|^2\Big)^{\frac12} &\eqsim_p \Big(\int_S \Big(\sum_{n=1}^N |b(x_n(s)) - b(y_n(s))|^2\Big)^{\frac{p}{2}} d\mu(s)\Big)^{\frac1p} \\ &\leq L_b \Big(\int_S \Big(\sum_{n=1}^N |x_n(s) - y_n(s)|^2\Big)^{\frac{p}{2}} d\mu(s)\Big)^{\frac1p} \\ &\eqsim_p L_b \Big(\mathbb{E}\Big\| \sum_{n=1}^N \gamma_n (x_n - y_n)\Big\|^2\Big)^{\frac12}.\end{aligned}$$

Now we apply Lemma 5.3.

6. Stochastic evolution equations I: integrable initial values

On the space E we consider the stochastic equation:

$$\text{(SCP)} \qquad \begin{cases} dU(t) = (AU(t) + F(t, U(t)))\, dt + B(t, U(t))\, dW_H(t), & t \in [0, T_0], \\ U(0) = u_0, \end{cases}$$

where W_H is an H-cylindrical Brownian motion. We make the following assumptions on A, F, B, u_0, the numbers $\eta, \theta_F, \theta_B \geq 0$:

- (A1) The operator A is the generator of an analytic C_0-semigroup S on a UMD Banach space E.

(A2) The function
$$F : [0, T_0] \times \Omega \times E_\eta \to E_{-\theta_F}$$
is Lipschitz of linear growth uniformly in $[0, T_0]\times\Omega$, i.e., there are constants L_F and C_F such that for all $t \in [0, T_0], \omega \in \Omega$ and $x, y \in E_\eta$,
$$\begin{aligned}\|(F(t,\omega,x) - F(t,\omega,y))\|_{E_{-\theta_F}} &\le L_F\|x-y\|_{E_\eta},\\ \|F(t,\omega,x)\|_{E_{-\theta_F}} &\le C_F(1+\|x\|_{E_\eta}).\end{aligned}$$
Moreover, for all $x \in E_\eta$, $(t,\omega) \mapsto F(t,\omega,x)$ is strongly measurable and adapted in $E_{-\theta_F}$.

(A3) The function
$$B : [0, T_0] \times \Omega \times E_\eta \to \mathcal{L}(H, E_{-\theta_B})$$
is L^2_γ-Lipschitz of linear growth uniformly in Ω, i.e., there are constants L^γ_B and C^γ_B such that for all finite measures μ on $([0, T_0], \mathcal{B}_{[0,T_0]})$, for all $\omega \in \Omega$, and all $\phi_1, \phi_2 \in L^2_\gamma((0, T_0), \mu; E_\eta)$,
$$\begin{aligned}\|(B(\cdot,\omega,\phi_1)-&\ B(\cdot,\omega,\phi_2))\|_{\gamma(L^2((0,T_0),\mu;H),E_{-\theta_B})}\\ &\le L^\gamma_B\|\phi_1-\phi_2\|_{L^2_\gamma((0,T_0),\mu;E_\eta)},\end{aligned}$$
and
$$B(\cdot,\omega,\phi)\|_{\gamma(L^2((0,T_0),\mu;H),E_{-\theta_B})} \le C^\gamma_B(1+\|\phi\|_{L^2_\gamma((0,T_0),\mu;E_\eta)}).$$
Moreover, for all $x \in E_\eta$, $(t,\omega) \mapsto B(t,\omega,x)$ is H-strongly measurable and adapted in $E_{-\theta_B}$.

(A4) The initial value $u_0 : \Omega \to E_\eta$ is strongly $\mathcal{F}_0$-measurable.

We call a process $(U(t))_{t\in[0,T_0]}$ a *mild E_η-solution* of (SCP) if

(i) $U : [0, T_0] \times \Omega \to E_\eta$ is strongly measurable and adapted,
(ii) for all $t \in [0, T_0]$, $s \mapsto S(t-s)F(s, U(s))$ is in $L^0(\Omega; L^1(0, t; E))$,
(iii) for all $t \in [0, T_0]$, $s \mapsto S(t-s)B(s, U(s))$ H-strongly measurable and adapted and in $\gamma(L^2(0, t; H), E)$ almost surely,
(iv) for all $t \in [0, T_0]$, almost surely
$$U(t) = S(t)u_0 + S * F(\cdot, U)(t) + S \diamond B(\cdot, U)(t).$$

By (ii) the deterministic convolution is defined pathwise as a Bochner integral, and since E is a UMD space, by (iii) and Proposition 2.4 the stochastic convolutions is well-defined.

We shall prove an existence and uniqueness result for (SCP) using a fixed point argument in a suitable scale of Banach spaces of E-valued processes introduced next. Fix $T \in (0, T_0]$, $p \in [1,\infty)$, $\alpha \in (0, \frac{1}{2})$. We define $V^p_{\alpha,\infty}([0, T] \times \Omega; E)$ as the space of all continuous adapted processes $\phi : [0, T] \times \Omega \to E$ for which
$$\begin{aligned}&\|\phi\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E)}\\ &\quad:= \big(\mathbb{E}\|\phi\|^p_{C([0,T];E)}\big)^{\frac{1}{p}} + \sup_{t\in[0,T]}\Big(\mathbb{E}\|s \mapsto (t-s)^{-\alpha}\phi(s)\|^p_{\gamma(L^2(0,t),E)}\Big)^{\frac{1}{p}}\end{aligned}$$

is finite. Similarly we define $V^p_{\alpha,p}([0,T]\times\Omega;E)$ as the space of pathwise continuous and adapted processes $\phi:[0,T]\times\Omega\to E$ for which

$$\begin{aligned}&\|\phi\|_{V^p_{\alpha,p}([0,T]\times\Omega;E)}\\&\quad:=\big(\mathbb{E}\|\phi\|^p_{C([0,T];E)}\big)^{\frac1p}+\Big(\int_0^T\mathbb{E}\|s\mapsto(t-s)^{-\alpha}\phi(s)\|^p_{\gamma(L^2(0,t),E)}\,dt\Big)^{\frac1p}\end{aligned}$$

is finite. Identifying processes which are indistinguishable, the above norm on $V^p_{\alpha,p}([0,T]\times\Omega;E)$ and $V^p_{\alpha,\infty}([0,T]\times\Omega;E)$ turn these spaces into Banach spaces.

The main result of this section, Theorem 6.2 below, establishes existence and uniqueness of a mild solution of (SCP) with initial value $u_0\in L^p(\Omega,\mathcal{F}_0;E_\eta)$ in each of the spaces $V^p_{\alpha,p}([0,T_0]\times\Omega;E)$ and $V^p_{\alpha,\infty}([0,T_0]\times\Omega;E)$. Since we have a continuous embedding $V^p_{\alpha,\infty}([0,T_0]\times\Omega;E)\hookrightarrow V^p_{\alpha,p}([0,T_0]\times\Omega;E)$, the existence result is stronger for $V^p_{\alpha,\infty}([0,T_0]\times\Omega;E)$ while the uniqueness result is stronger for $V^p_{\alpha,p}([0,T_0]\times\Omega;E)$.

For technical reasons, in the next section we will also need the space $\tilde V^p_{\alpha,p}([0,T]\times\Omega;E)$ which is obtained by 'pathwise continuous' replaced by 'pathwise bounded and $\mathcal{B}_{[0,T]}\otimes\mathcal{F}$-measurable' and $C([0,T];E)$ replaced by $B_{\rm b}([0,T];E)$ in the definition of $\tilde V^p_{\alpha,p}([0,T]\times\Omega;E)$. Here $B_{\rm b}([0,T];E)$ denotes the Banach space of bounded strongly Borel measurable functions on $[0,T]$ with values in E, endowed with the supremum norm.

Consider the fixed point operator

$$L_T(\phi)=\big[t\mapsto S(t)u_0+S*F(\cdot,\phi)(t)+S\diamond B(\cdot,\phi)(t)\big].$$

In the next proposition we show that L_T is well-defined on each of the three spaces introduced above and that it is a strict contraction for T small enough.

Proposition 6.1. *Let E be a UMD space with type $\tau\in[1,2]$. Suppose that* (A1)-(A4) *are satisfied and assume that $0\le\eta+\theta_F<\frac32-\frac1\tau$ and $0\le\eta+\theta_B<\frac12$. Let $p>2$ and $\alpha\in(0,\frac12)$ be such that $\eta+\theta_B<\alpha-\frac1p$. If $u_0\in L^p(\Omega;E_\eta)$, then the operator L_T is well-defined and bounded on each of the spaces*

$$V\in\big\{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta),\ V^p_{\alpha,p}([0,T]\times\Omega;E_\eta),\ \tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)\big\},$$

and there exist a constant C_T, with $\lim_{T\downarrow0}C_T=0$, such that for all $\phi_1,\phi_2\in V$,

$$\|L_T(\phi_1)-L_T(\phi_2)\|_V\le C_T\|\phi_1-\phi_2\|_V. \tag{6.1}$$

Moreover, there is a constant $C\ge0$, independent of u_0, such that for all $\phi\in V$,

$$\|L_T(\phi)\|_V\le C(1+(\mathbb{E}\|u_0\|^p_{E_\eta})^{\frac1p})+C_T\|\phi\|_V. \tag{6.2}$$

Proof. We give a detailed proof for the space $V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$. The proof for $V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)$ is entirely similar. For the proof for $\tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)$ one replaces $C([0,T];E)$ by $B_{\rm b}((0,T);E)$.

Step 1: Estimating the initial value part. Let $\varepsilon\in(0,\frac12)$. From Lemmas 2.9 and 4.1 we infer that

$$\begin{aligned}\|s\mapsto(t-s)^{-\alpha}S(s)u_0\|_{\gamma(L^2(0,t),E_\eta)}&\le C\|s\mapsto(t-s)^{-\alpha}s^{-\varepsilon}u_0\|_{\gamma(L^2(0,t),E_\eta)}\\&=C\|s\mapsto(t-s)^{-\alpha}s^{-\varepsilon}\|_{L^2(0,t)}\|u_0\|_{E_\eta}\\&\le C\|u_0\|_{E_\eta}.\end{aligned}$$

For the other part of the $V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$-norm we note that

$$\|Su_0\|_{C([0,T];E_\eta)} \le C\|u_0\|_{E_\eta}.$$

It follows that

$$\|Su_0\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} \le C\|u_0\|_{L^p(\Omega;E_\eta)}.$$

Step 2: Estimating the deterministic convolution. We proceed in two steps.

(a): For $\psi \in C([0,T];E_{-\theta_F})$ we estimate the $V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$-norm of $S*\psi$.

By Lemma 3.6 (applied with $\alpha = 1$ and $\lambda = 0$) $S*\psi$ is continuous in E_η. Using (3.2) we estimate:

$$\begin{aligned}\|S*\psi\|_{C([0,T];E_\eta)} &\le C\int_0^t (t-s)^{-\eta-\theta_F}\,ds\,\|\psi\|_{C([0,T];E_{-\theta_F})}\\ &\le CT^{1-\eta-\theta_F}\|\psi\|_{C([0,T];E_{-\theta_F})}.\end{aligned}\tag{6.3}$$

Also, since E has type τ, it follows from Proposition 3.5 that

$$\|s\mapsto (t-s)^{-\alpha}S*\psi(s)\|_{\gamma(L^2(0,t),E_\eta)} \le T^{\frac12-\alpha}\|\psi\|_{C([0,T];E_{-\theta_F})}.\tag{6.4}$$

Now let $\Psi \in L^p(\Omega;C([0,T];E_{-\theta_F}))$. By applying (6.3) and (6.4) to the paths $\Psi(\cdot,\omega)$ one obtains that $S*\Psi \in V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$ and

$$\|S*\Psi\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} \le CT^{\min\{\frac12-\alpha,1-\eta-\theta_F\}}\|\Psi\|_{L^p(\Omega;C([0,T];E_{-\theta_F}))}.\tag{6.5}$$

(b): Let $\phi_1,\phi_2 \in V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$. Since F is of linear growth, $F(\cdot,\phi_1)$ and $F(\cdot,\phi_2)$ belong to $L^p(\Omega;C([0,T];E_{-\theta_F}))$. From (6.5) and the fact that F is Lipschitz continuous in its E_η-variable we deduce that $S*(F(\cdot,\phi_1)), S*(F(\cdot,\phi_2)) \in V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$ and

$$\begin{aligned}&\|S*(F(\cdot,\phi_1)-F(\cdot,\phi_2))\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\\ &\quad\le CT^{\min\{\frac12-\alpha,1-\eta-\theta_F\}}\|(F(\cdot,\phi_1)-F(\cdot,\phi_2))\|_{L^p(\Omega;C([0,T];E_{-\theta_F}))}\\ &\quad\le CT^{\min\{\frac12-\alpha,1-\eta-\theta_F\}}L_F\|\phi_1-\phi_2\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}.\end{aligned}\tag{6.6}$$

Step 3: Estimating the stochastic convolution. Again we proceed in two steps.

(a): Let $\Psi : [0,T]\times\Omega \to \mathcal{L}(H,E_{-\theta_B})$ be H-strongly measurable and adapted and suppose that

$$\sup_{t\in[0,T]} \mathbb{E}\|s\mapsto (t-s)^{-\alpha}\Psi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta_B})} < \infty.\tag{6.7}$$

We estimate the $V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$-norm of $S\diamond\Psi$.

From Proposition 4.2 we obtain an $\varepsilon > 0$ such that

$$\Big(\mathbb{E}\|S\diamond\Psi\|^p_{C([0,T];E_\eta)}\Big)^{\frac1p} \le CT^\varepsilon \sup_{t\in[0,T]}\Big(\mathbb{E}\|s\mapsto (t-s)^{-\alpha}\Psi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta_B})}\Big)^{\frac1p}.$$

For the other part of the norm, by Proposition 4.5 we obtain that

$$\begin{aligned}&\Big(\mathbb{E}\|s\mapsto (t-s)^{-\alpha}S\diamond\Psi(s))\|^p_{\gamma(L^2(0,t;H),E_\eta)}\Big)^{\frac1p}\\ &\qquad\le CT^{\frac12-\eta-\theta_B}\Big(\mathbb{E}\|s\mapsto (t-s)^{-\alpha}\Psi\|^p_{\gamma(L^2(0,t;H),E_{-\theta_B})}\Big)^{\frac1p}.\end{aligned}$$

Combining things we conclude that

$$\begin{aligned}
&\|S\diamond\Psi\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\\
(6.8)\qquad &\le CT^{\min\{\frac12-\eta-\theta_B,\varepsilon\}}\Big(\sup_{t\in[0,T]}\Big(\mathbb{E}\|s\mapsto(t-s)^{-\alpha}\Psi(s)\|^p_{\gamma(L^2(0,t;H),E_{-\theta_B})}\Big)^{\frac1p}.
\end{aligned}$$

(b): For $t\in[0,T]$ let $\mu_{t,\alpha}$ be the finite measure on $((0,t),\mathcal{B}_{(0,t)})$ defined by

$$\mu_{t,\alpha}(B)=\int_0^t(t-s)^{-2\alpha}\mathbf{1}_B(s)\,ds.$$

Notice that for a function $\phi\in C([0,t];E)$ we have

$$\phi\in\gamma(L^2((0,t),\mu_{t,\alpha}),E)\Longleftrightarrow s\mapsto(t-s)^{-\alpha}\phi(s)\in\gamma(L^2(0,t),E).$$

Trivially,

$$\|\phi\|_{L^2((0,t),\mu_{t,\alpha};E)}=\|(t-\cdot)^{-\alpha}\phi(\cdot)\|_{L^2(0,t;E)}\le Ct^{\frac12-\alpha}\|\phi\|_{C([0,T];E)}.$$

Now let $\phi_1,\phi_2\in V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$. Since B is L^2_γ-Lipschitz and of linear growth and ϕ_1 and ϕ_2 belong to $L^2_\gamma((0,t),\mu_{t,\alpha};E_\eta)$ uniformly, $B(\cdot,\phi_1)$ and $B(\cdot,\phi_2)$ satisfy (6.7). Since $B(\cdot,\phi_1)$ and $B(\cdot,\phi_2)$ are H-strongly measurable and adapted, it follows from (6.8) that $B(\cdot,\phi_1)$, $B(\cdot,\phi_2)\in V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$ and

(6.9)

$$\begin{aligned}
&\|S\diamond(B(\cdot,\phi_1)-B(\cdot,\phi_2))\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\\
&\quad\lesssim T^{\min\{\frac12-\eta-\theta_B,\varepsilon\}}\\
&\quad\times\Big(\sup_{t\in[0,T]}\Big(\mathbb{E}\|s\mapsto(t-s)^{-\alpha}[B(s,\phi_1(s))-B(s,\phi_2(s))]\|^p_{\gamma(L^2(0,t;H),E_{-\theta_B})}\Big)^{\frac1p}\\
&\quad=T^{\min\{\frac12-\eta-\theta_B,\varepsilon\}}\sup_{t\in[0,T]}\Big(\mathbb{E}\|B(\cdot,\phi_1)-B(\cdot,\phi_2)\|^p_{\gamma(L^2((0,t),\mu_{t,\alpha};H),E_{-\theta_B})}\Big)^{\frac1p}\\
&\quad\lesssim L^\gamma_B T^{\min\{\frac12-\eta-\theta_B,\varepsilon\}}\sup_{t\in[0,T]}\Big(\mathbb{E}\|\phi_1-\phi_2\|^p_{L^2_\gamma((0,t),\mu_{t,\alpha};E_\eta)}\Big)^{\frac1p}\\
&\quad\lesssim L^\gamma_B T^{\min\{\frac12-\eta-\theta_B,\varepsilon\}}\Big[\sup_{t\in[0,T]}\Big(\mathbb{E}\|s\mapsto(t-s)^{-\alpha}[\phi_1-\phi_2]\|^p_{\gamma(L^2(0,t),E_\eta)}\Big)^{\frac1p}\\
&\qquad\qquad+T^{\frac p2-\alpha p}\mathbb{E}\|\phi_1-\phi_2\|^p_{C([0,T];E_\eta)}\Big)^{\frac1p}\Big]\\
&\quad\lesssim L^\gamma_B T^{\min\{\frac12-\eta-\theta_B,\varepsilon\}}\|\phi_1-\phi_2\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}.
\end{aligned}$$

Step 4: Collecting the estimates. It follows from the above considerations that L_T is well-defined on $V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$ and there exist constants $C\ge0$ and $\beta>0$ such that for all $\phi_1,\phi_2\in V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)$ we have

$$(6.10)\qquad \|L_T(\phi_1)-L_T(\phi_2)\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\le CT^\beta\|\phi_1-\phi_2\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}.$$

The estimate (6.2) follows from (6.10) and

$$\|L_T(0)\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\le C(1+(\mathbb{E}\|u_0\|^p_{E_\eta})^{\frac1p}.$$

□

Theorem 6.2 (Existence and uniqueness)**.** *Let E be a UMD space with type $\tau\in[1,2]$. Suppose that* (A1)-(A4) *are satisfied and assume that $0\le\eta+\theta_F<\frac32-\frac1\tau$ and $0\le\eta+\theta_B<\frac12$. Let $p>2$ and $\alpha\in(0,\frac12)$ be such that $\eta+\theta_B<\alpha-\frac1p$.*

If $u_0 \in L^p(\Omega, \mathcal{F}_0; E_\eta)$, then there exists a mild solution U in $V^p_{\alpha,\infty}([0,T_0] \times \Omega; E_\eta)$ of (SCP)*. As a mild solution in $V^p_{\alpha,p}([0,T] \times \Omega; E_\eta)$, this solution U is unique. Moreover, there exists a constant $C \geq 0$, independent of u_0, such that*

$$\|U\|_{V^p_{\alpha,\infty}([0,T_0]\times\Omega;E_\eta)} \leq C(1 + (\mathbb{E}\|u_0\|^p_{E_\eta})^{\frac{1}{p}}). \tag{6.11}$$

Proof. By Proposition 6.1 we can find $T \in (0, T_0]$, independent of u_0, such that $C_T < \frac{1}{2}$. It follows from (6.1) and the Banach fixed point theorem that L_T has a unique fixed point $U \in V^p_{\alpha,\infty}([0,T] \times \Omega; E_\eta)$. This gives a continuous adapted process $U : [0,T] \times \Omega \to E_\eta$ such that almost surely for all $t \in [0,T]$,

$$U(t) = S(t)u_0 + S * F(\cdot, U)(t) + S \diamond B(\cdot, U)(t). \tag{6.12}$$

Noting that $U = \lim_{n\to\infty} L^n_T(0)$ in $V^p_{\alpha,\infty}([0,T] \times \Omega; E_\eta)$, (6.2) implies the inequality

$$\|U\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} \leq C(1 + (\mathbb{E}\|u_0\|^p_{E_\eta})^{\frac{1}{p}}) + C_T\|U\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)},$$

and then $C_T < \frac{1}{2}$ implies

$$\|U\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} \leq C(1 + (\mathbb{E}\|u_0\|^p_{E_\eta})^{\frac{1}{p}}). \tag{6.13}$$

Via a standard induction argument one may construct a mild solution on each of the intervals $[T, 2T], \ldots, [(n-1)T, nT], [nT, T_0]$ for an appropriate integer n. The induced solution U on $[0, T_0]$ is the mild solution of (SCP). Moreover, by (6.13) and induction we deduce (6.11).

For small $T \in (0, T_0]$, uniqueness on $[0,T]$ follows from the uniqueness of the fixed point of L_T in $V^p_{\alpha,p}([0,T] \times \Omega; E_\eta)$. Uniqueness on $[0, T_0]$ follows by induction. □

In the next theorem we deduce regularity properties of the solution. They are formulated for $U - Su_0$; if u_0 is regular enough, regularity of U can be deduced.

Theorem 6.3 (Regularity)**.** *Let E be a UMD space with type $\tau \in [1,2]$ and suppose that* (A1)-(A4) *are satisfied. Assume that $0 \leq \eta+\theta_F < \frac{3}{2} - \frac{1}{\tau}$ and $0 \leq \eta+\theta_B < \frac{1}{2} - \frac{1}{p}$ with $p > 2$. Let $\lambda \geq 0$ and $\delta \geq \eta$ satisfy $\lambda + \delta < \min\{\frac{1}{2} - \frac{1}{p} - \theta_B, 1 - \theta_F\}$. Then there exists a constant $C \geq 0$ such that for all $u_0 \in L^p(\Omega; E_\eta)$,*

$$\Big(\mathbb{E}\|U - Su_0\|^p_{C^\lambda([0,T_0];E_\delta)}\Big)^{\frac{1}{p}} \leq C(1 + (\mathbb{E}\|u_0\|^p_{E_\eta})^{\frac{1}{p}}. \tag{6.14}$$

Proof. Choose $r \geq 1$ and $0 < \alpha < \frac{1}{2}$ such that $\lambda + \delta < 1 - \frac{1}{r} - \theta_F$, $\eta + \theta_B < \alpha - \frac{1}{p}$, and $\lambda + \delta + \theta_B < \alpha - \frac{1}{p}$. Let $\tilde{U} \in V^p_{\alpha,\infty}([0,T_0] \times \Omega; E_\eta)$ be the mild solution from Theorem 6.2. It follows from Lemma 3.6 (with $\alpha = 1$) that we may take a version of $S * F(\cdot, \tilde{U})$ with

$$\begin{aligned}\mathbb{E}\|S * F(\cdot, \tilde{U})\|^p_{C^\lambda([0,T_0];E_\delta)} &\leq C\mathbb{E}\|F(\cdot, \tilde{U})\|^p_{L^r(0,T_0;E_{-\theta_F})} \\ &\leq C\mathbb{E}\|F(\cdot, \tilde{U})\|^p_{C([0,T_0];E_{-\theta_F})}.\end{aligned}$$

Similarly, via Proposition 4.2 we may take a version of $S * B(\cdot, \tilde{U})$ with

$$\begin{aligned}&\mathbb{E}\|S \diamond B(\cdot, \tilde{U})\|^p_{C^\lambda([0,T_0];E_\delta)} \\ &\quad\leq C \sup_{t\in[0,T_0]} \mathbb{E}\|s \mapsto (t-s)^{-\alpha} B(\cdot, \tilde{U}(s))\|^p_{\gamma(L^2(0,t;H),E_{-\theta_B})}.\end{aligned}$$

Define $U : [0, T_0] \times \Omega \to E_\eta$ as

$$U(t) = S(t)u_0 + S * F(\cdot, \tilde{U})(t) + S \diamond B(\cdot, \tilde{U})(t),$$

where we take the versions of the convolutions as above. By uniqueness we have almost surely $U \equiv \tilde{U}$. Arguing as in (6.9) deduce that

$$\mathbb{E}\|U - Su_0\|^p_{C^\lambda([0,T_0];E_\delta)} \le C(1 + \|U\|^p_{V^p_{\alpha,\infty}([0,T_0]\times\Omega;E_\eta)}).$$

Now (6.14) follows from (6.11). □

7. Stochastic evolution equations II: measurable initial values

So far we have solved the problem (SCP) for initial values $u_0 \in L^p(\Omega, \mathcal{F}_0; E_\eta)$. In this section we discuss the case of initial values $u_0 \in L^0(\Omega, \mathcal{F}_0; E_\eta)$.

Fix $T \in (0, T_0]$. For $p \in [1, \infty)$ and $\alpha \in (0, \frac{1}{2})$ we define $V^0_{\alpha,p}([0, T] \times \Omega; E)$ as the linear space of continuous adapted processes $\phi : [0, T] \times \Omega \to E$ such that almost surely,

$$\|\phi\|_{C([0,T];E)} + \Big(\int_0^T \|s \mapsto (t-s)^{-\alpha}\phi(s)\|^p_{\gamma(L^2(0,t),E)}\,dt\Big)^{\frac{1}{p}} < \infty.$$

As usual we identify indistinguishable processes.

Theorem 7.1 (Existence and uniqueness)**.** *Let E be a UMD space of type $\tau \in [1, 2]$ and suppose that* (A1)-(A4) *are satisfied. Assume that $0 \le \eta + \theta_F < \frac{3}{2} - \frac{1}{\tau}$ and $\eta + \theta_B < \frac{1}{2}$. If $\alpha \in (0, \frac{1}{2})$ and $p > 2$ are such that $\eta + \theta_B < \alpha - \frac{1}{p}$, then there exists a unique mild solution $U \in V^0_{\alpha,p}([0, T_0] \times \Omega; E_\eta)$ of* (SCP)*.*

For the proof we need the following uniqueness result.

Lemma 7.2. *Under the conditions of Theorem 6.2 let U_1 and U_2 in $V^p_{\alpha,\infty}([0, T] \times \Omega; E_\eta)$ be the mild solutions of* (SCP) *with initial values u_1 and u_2 in $L^p(\Omega, \mathcal{F}_0; E_\eta)$. Then almost surely on the set $\{u_1 = u_2\}$ we have $U_1 \equiv U_2$.*

Proof. Let $\Gamma = \{u_1 = u_2\}$. First consider small $T \in (0, T_0]$ as in Step 1 in the proof of Theorem 6.2. Since Γ is $\mathcal{F}_0$-measurable we have

$$\begin{aligned}\|U_1\mathbf{1}_\Gamma - U_2\mathbf{1}_\Gamma\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} &= \|L_T(U_1)\mathbf{1}_\Gamma - L_T(U_2)\mathbf{1}_\Gamma\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\\ &= \|(L_T(U_1\mathbf{1}_\Gamma) - L_T(U_2\mathbf{1}_\Gamma))\mathbf{1}_\Gamma\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)}\\ &\le \tfrac{1}{2}\|U_1\mathbf{1}_\Gamma - U_2\mathbf{1}_\Gamma\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)},\end{aligned}$$

hence almost surely $U_1|_{[0,T]\times\Gamma} \equiv U_2|_{[0,T]\times\Gamma}$.

To obtain uniqueness on the interval $[0, T_0]$ one may proceed as in the proof of Theorem 6.2. □

Proof of Theorem 7.1. (Existence): Define $(u_n)_{n\ge1}$ in $L^p(\Omega, \mathcal{F}_0; E_\eta)$ as

$$u_n := \mathbf{1}_{\{\|u_0\|_{E_\eta}\le n\}}u_0.$$

By Theorem 6.2, for each $n \ge 1$ there is a unique solution $U_n \in V^p_{\alpha,\infty}([0, T] \times \Omega; E_\eta)$ of (SCP) with initial value u_n. By Lemma 7.2 we may define $U : (0, T_0) \times \Omega \to E_\eta$ as $U(t) = \lim_{n\to\infty} U_n(t)$ if this limit exists and 0 otherwise. Then, U is strongly measurable and adapted, and almost surely on $\{\|u_0\|_{E_\eta} \le n\}$, for all $t \in (0, T_0)$ we have $U(t) = U_n(t)$. Hence, $U \in V^0_{\alpha,p}([0, T] \times \Omega; E_\eta)$. It is routine to check that U is a solution of (SCP).

(Uniqueness): The argument is more or less standard, but there are some subleties due to the presence of the radonifying norms.

Let $U, V \in V^0_{\alpha,p}([0,T_0]\times\Omega; E_\eta)$ be mild solutions of (SCP). For each $n \geq 1$ let the stopping times μ_n^U and ν_n^U be defined as

$$\mu_n^U = \inf\Big\{r \in [0,T_0] : \int_0^{T_0} \|s \mapsto (t-s)^{-\alpha} U(s)\mathbf{1}_{[0,r]}(s)\|^p_{\gamma(L^2(0,t),E_\eta)}\,dt \geq n\Big\},$$
$$\nu_n^U = \inf\Big\{r \in [0,T] : \|U(r)\|_{E_\eta} \geq n\Big\}.$$

This is well-defined since

$$r \mapsto \int_0^{T_0} \|s \mapsto (t-s)^{-\alpha} U(s)\mathbf{1}_{[0,r]}(s)\|^p_{\gamma(L^2(0,t),E_\eta)}\,dt$$

is a continuous adapted process by [32, Proposition 2.4] and the dominated convergence theorem. The stopping times μ_n^V and ν_n^V are defined in a similarly. For each $n \geq 1$ let

$$\tau_n = \mu_n^U \wedge \nu_n^U \wedge \mu_n^V \wedge \nu_n^V,$$

and let $U_n = U\mathbf{1}_{[0,\tau_n]}$ and $V_n = V\mathbf{1}_{[0,\tau_n]}$. Then for all $n \geq 1$, U_n and V_n are in $\tilde{V}^p_{\alpha,p}([0,T_0]\times\Omega; E_\eta)$. One easily checks that

$$U_n = \mathbf{1}_{[0,\tau_n]}(L_T(U_n))^{\tau_n} \quad \text{and} \quad V_n = \mathbf{1}_{[0,\tau_n]}(L_T(V_n))^{\tau_n},$$

where L_T is the map introduced preceding Proposition 6.1 and $(L_T(U_n))^{\tau_n}(t) := (L_T(U_n))(t\wedge\tau_n)$. By Proposition 6.1 we can find $T \in (0,T_0]$ such that $C_T \leq \frac{1}{2}$. A routine computation then implies

$$\|U_n - V_n\|_{\tilde{V}^p_{\alpha,p}([0,T]\times\Omega;E_\eta)} \leq \tfrac{1}{2}\|U_n - V_n\|_{\tilde{V}^p_{\alpha,p}([0,T]\times\Omega;E_\eta)}.$$

We obtain that $U_n = V_n$ in $\tilde{V}^p_{\alpha,p}([0,T]\times\Omega; E_\eta)$, hence $\mathbb{P}$-almost surely, $U_n \equiv V_n$. Letting n tend to infinity, we may conclude that almost surely, $U \equiv V$ on $[0,T]$. This gives the uniqueness on the interval $[0,T]$. Uniqueness on $[0,T_0]$ can obtained by the usual induction argument. □

Note that in the last paragraph of the proof we needed to work in the space $\tilde{V}^p_{\alpha,p}([0,T]\times\Omega; E_\eta)$ rather than in $V^p_{\alpha,p}([0,T]\times\Omega; E_\eta)$ because the truncation with the stopping time destroys the pathwise continuity.

By applying Theorem 6.3 to the unique solution U_n with initial value $u_n := \mathbf{1}_{\{\|u_0\|_{E_\eta}\leq n\}}u_0$, the solution $U := \lim_{n\to\infty} U_n$ constructed in Theorem 7.1 enjoys the following regularity property.

Theorem 7.3 (Hölder regularity)**.** *Let E be a UMD space and type $\tau \in [1,2]$ and suppose that* (A1)-(A4) *are satisfied. Assume that $0 \leq \eta + \theta_F < \frac{3}{2} - \frac{1}{\tau}$ and $0 \leq \eta + \theta_B < \frac{1}{2}$. Let $\lambda \geq 0$ and $\delta \geq \eta$ satisfy $\lambda + \delta < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F\}$. Then the mild solution U of* (SCP) *has a version such that almost all paths satisfy $U - Su_0 \in C^\lambda([0,T_0]; E_\delta)$.*

Proof of Theorem 1.1. Part (1) is a the special case of Theorem 6.2 corresponding to $\tau = 1$ and $\theta_F = \theta_B = 0$. For part (2) we apply Theorem 7.3, again with $\tau = 1$ and $\theta_F = \theta_B = 0$. □

8. Stochastic evolution equations III: the locally Lipschitz case

Consider the following assumptions on F and B.

(A2)$'$ The function $F : [0,T_0]\times\Omega\times E_\eta \to E_{-\theta_F}$ is locally Lipschitz, uniformly in $[0,T_0]\times\Omega$, i.e., for all $R>0$ there exists a constant L_F^R such that for all $t\in[0,T_0]$, $\omega\in\Omega$ and $\|x\|_{E_\eta}, \|y\|_{E_\eta}\le R$,

$$\|F(t,\omega,x)-F(t,\omega,y)\|_{E_{-\theta_F}} \le L_F^R\|x-y\|_{E_\eta}.$$

Moreover, for all $x\in E_\eta$, $(t,\omega)\mapsto F(t,\omega,x)\in E_{-\theta_F}$ is strongly measurable and adapted, and there exists a constant $C_{F,0}$ such that for all $t\in[0,T_0]$ and $\omega\in\Omega$,

$$\|F(t,\omega,0)\|_{E_{-\theta_F}} \le C_{F,0}.$$

(A3)$'$ The function $B : [0,T_0]\times\Omega\times E_\eta\to\mathcal{L}(H,E_{-\theta_B})$ is locally L^2_γ-Lipschitz, uniformly in Ω, i.e., there exists a sequence of L^2_γ-Lipschitz functions $B_n : [0,T_0]\times\Omega\times E_\eta\to\mathcal{L}(H,E_{-\theta_B})$ such that $B(\cdot,x)=B_n(\cdot,x)$ for all $\|x\|_{E_\eta}<n$. Moreover, for all $x\in E_\eta$, $(t,\omega)\mapsto B(t,\omega,x)\in E_{-\theta_B}$ is H-strongly measurable and adapted, and there exists a constant $C_{B,0}$ such that for all finite measures μ on $([0,T_0],\mathcal{B}_{[0,T_0]})$ and all $\omega\in\Omega$,

$$\|t\mapsto B(t,\omega,0)\|_{\gamma(L^2((0,T_0),\mu;H),E_{-\theta_B})}\le C_{B,0}.$$

One may check that the locally Lipschitz version of Lemma 5.2 holds as well. This gives an easy way to check (A3)$'$ for type 2 spaces E.

Let ϱ be a stopping time with values in $[0,T_0]$. For $t\in[0,T_0]$ let

$$\begin{aligned}\Omega_t(\varrho) &= \{\omega\in\Omega : t<\varrho(\omega)\},\\ [0,\varrho)\times\Omega &= \{(t,\omega)\in[0,T_0]\times\Omega : 0\le t<\varrho(\omega)\},\\ [0,\varrho]\times\Omega &= \{(t,\omega)\in[0,T_0]\times\Omega : 0\le t\le\varrho(\omega)\}.\end{aligned}$$

A process $\zeta:[0,\varrho)\times\Omega\to E$ (or $(\zeta(t))_{t\in[0,\varrho)}$) is called *admissible* if for all $t\in[0,T_0]$, $\Omega_t(\varrho)\ni\omega\to\zeta(t,\omega)$ is $\mathcal{F}_t$-measurable and for almost all $\omega\in\Omega$, $[0,\varrho(\omega))\ni t\mapsto\zeta(t,\omega)$ is continuous.

Let E be a UMD space. An admissible E_η-valued process $(U(t))_{t\in[0,\varrho)}$ is called a *local solution* of (SCP) if $\varrho\in(0,T_0]$ almost surely and there exists an increasing sequence of stopping times $(\varrho_n)_{n\ge1}$ with $\varrho=\lim_{n\to\infty}\varrho_n$ such that

(i) for all $t\in[0,T_0]$, $s\mapsto S(t-s)F(\cdot,U(s))\mathbf{1}_{[0,\varrho_n]}(s)\in L^0(\Omega;L^1(0,t;E_\eta))$,
(ii) for all $t\in[0,T_0]$, $s\mapsto S(t-s)B(\cdot,U(s))\mathbf{1}_{[0,\varrho_n]}(s)\in L^0(\Omega;\gamma(L^2(0,t;H),E_\eta))$,
(iii) almost surely for all $t\in[0,\rho_n]$,

$$U(t)=S(t)u_0+S*F(\cdot,U)(t)+S\diamond B(\cdot,U)(t).$$

By (i) the deterministic convolution is defined pathwise as a Bochner integral. Since E is a UMD space, by (ii) and Proposition 2.4 we may define the stochastic convolution as

$$S\diamond B(\cdot,U)(t)=\int_0^t S(t-s)B(s,U(s))\mathbf{1}_{[0,\varrho_n]}(s)\,dW_H(s),\quad t\in[0,\rho_n].$$

A local solution $(U(t))_{t\in[0,\varrho)}$ is called *maximal* for a certain space V of E_η-valued admissible processes if for any other local solution $(\tilde U(t))_{t\in[0,\tilde\varrho)}$ in V, almost surely we have $\tilde\varrho\le\varrho$ and $\tilde U\equiv U|_{[0,\tilde\varrho)}$. Clearly, a maximal local solution for such a space V is always unique in V. We say that a local solution $(U(t))_{t\in[0,\varrho)}$ of (SCP) is a *global solution* of (SCP) if $\varrho=T_0$ almost surely and U has an extension to a solution $\hat{U} : [0, T_0] \times \Omega \to E_\eta$ of (SCP). In particular, almost surely "no blow" up occurs at $t = T_0$.

We say that ϱ is an *explosion time* if for almost all $\omega \in \Omega$ with $\varrho(\omega) < T_0$,

$$\limsup_{t \uparrow \varrho(\omega)} \|U(t, \omega)\|_{E_\eta} = \infty.$$

Notice that if $\varrho = T_0$ almost surely, then ϱ is always an explosion time in this definition. However, there need not be any "blow up" in this case.

Let ϱ be a stopping time with values in $[0, T_0]$. For $p \in [1, \infty)$, $\alpha \in [0, \frac{1}{2})$ and $\eta \in [0, 1]$ we define $V^{0,\mathrm{loc}}_{\alpha,p}([0, \varrho) \times \Omega; E)$ as all E-valued admissible processes $(\phi(t))_{t \in [0,\varrho)}$ such that there exists an increasing sequence of stopping times $(\varrho_n)_{n \geq 1}$ with $\varrho = \lim_{n\to\infty} \varrho_n$ and almost surely

$$\|\phi\|_{C([0,\varrho_n];E)} + \Big(\int_0^T \|s \mapsto (t - s)^{-\alpha} \phi(s) \mathbf{1}_{[0,\varrho_n]}(s)\|^p_{\gamma(0,t;E)}\, dt\Big)^{\frac{1}{p}} < \infty.$$

In the case that for almost all ω, $\varrho_n(\omega) = T$ for n large enough,

$$V^{0,\mathrm{loc}}_{\alpha,p}([0, \varrho) \times \Omega; E) = V^0_{\alpha,p}([0, T] \times \Omega; E).$$

Theorem 8.1. *Let E be a UMD space with type $\tau \in [1, 2]$ and suppose that* (A1), (A2)′, (A3)′, (A4) *are satisfied, and assume that $0 \leq \eta + \theta_F < \frac{3}{2} - \frac{1}{\tau}$.*

1. *For all $\alpha \in (0, \frac{1}{2})$ and $p > 2$ such that $\eta + \theta_B < \alpha - \frac{1}{p}$ there exists a unique maximal local solution $(U(t))_{[0,\varrho)}$ in $V^{0,\mathrm{loc}}_{\alpha,p}([0, \varrho) \times \Omega; E_\eta)$ of* (SCP).
2. *For all $\lambda > 0$ and $\delta \geq \eta$ such that $\lambda + \delta < \min\{\frac{1}{2} - \theta_B, 1 - \theta_F\}$, U has a version such that for almost all $\omega \in \Omega$,*
$$t \mapsto U(t, \omega) - S(t)u_0(\omega) \in C^\lambda_{\mathrm{loc}}([0, \varrho(\omega)); E_\delta),$$

If in addition the linear growth conditions of (A2) *and* (A3) *hold, then the above function U is the unique global solution of* (SCP) *in $V^0_{\alpha,p}([0, T_0] \times \Omega; E_\eta)$ and the following assertions hold:*

3. *The solution U satisfies the statements of Theorems 7.1 and 7.3.*
4. *If $\alpha \in (0, \frac{1}{2})$ and $p > 2$ are such that $\alpha > \eta + \theta_B + \frac{1}{p}$ and $u_0 \in L^p(\Omega, \mathcal{F}_0; E_\eta)$, then the solution U is in $V^p_{\alpha,\infty}([0, T_0] \times \Omega; E_\eta)$ and* (6.11) *and the statements of Theorem 6.3 hold.*

Before we proceed, we prove the following local uniqueness result.

Lemma 8.2. *Suppose that the conditions of Theorem 8.1 are satisfied and let $(U_1(t))_{t\in[0,\varrho_1)}$ in $V^{0,\mathrm{loc}}_{\alpha,p}([0, \varrho_1) \times \Omega; E_\eta)$ and $(U_2(t))_{t\in[0,\varrho_2)}$ in $V^{0,\mathrm{loc}}_{\alpha,p}([0, \varrho_2) \times \Omega; E_\eta)$ be local solutions of* (SCP) *with initial values u^1_0 and u^2_0. Let $\Gamma = \{u^1_0 = u^2_0\}$. Then almost surely on Γ, $U_1|_{[0,\varrho_1\wedge\varrho_2)} \equiv U_2|_{[0,\varrho_1\wedge\varrho_2)}$. Moreover, if ϱ_1 is an explosion time for U_1, then almost surely on Γ, $\varrho_1 \geq \varrho_2$. If ϱ_1 and ϱ_2 are explosion times for U_1 and U_2, then almost surely on Γ, $\varrho_1 = \varrho_2$ and $U_1 \equiv U_2$.*

Proof. Let $\varrho = \varrho_1 \wedge \varrho_2$. Let $(\mu_n)_{n\geq1}$ be an increasing sequences of bounded stopping times such that $\lim_{n\to\infty} \mu_n = \varrho$ and for all $n \geq 1$, $U_1\mathbf{1}_{[0,\mu_n]}$ and $U_2\mathbf{1}_{[0,\mu_n]}$ are in $\tilde{V}^p_{\alpha,p}([0, T_0] \times \Omega; E_\eta)$. Let

$$\nu^1_n = \inf\{t \in [0, T_0] : \|U_1(t)\|_{E_\eta} \geq n\} \text{ and } \nu^2_n = \inf\{t \in [0, T_0] : \|U_2(t)\|_{E_\eta} \geq n\}$$

and let $\sigma^i_n = \mu_n \wedge \nu^i_n$ and let $\sigma_n = \sigma^1_n \wedge \sigma^2_n$. On $[0, T_0] \times \Omega \times \{x \in E_\eta : \|x\|_{E_\eta} \leq n\}$ we may replace F and B by F_n (for a possible definition of F_n, see the proof of

Theorem 8.1) and B_n which satisfy (A2) and (A3). As in the proof of Theorem 7.1 it follows that for all $0 < T \le T_0$,

$$\begin{aligned}
\|U_1^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma} - U_2^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma}\|_{\tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)} \\
&= \|(L_T(U_1^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma}) - L_T(U_2^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma}))\mathbf{1}_{[0,\sigma_n]\times\Gamma}\|_{\tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)} \\
&\le \|L_T(U_1^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma}) - L_T(U_2^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma})\|_{\tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)} \\
&\le C_T\|U_1^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma} - U_2^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma}\|_{\tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)},
\end{aligned}$$

where C_T satisfies $\lim_{T\downarrow 0} C_T = 0$. Here $\mathbf{1}_{[0,\sigma_n]\times\Gamma}$ should be interpreted as the process $(t,\omega)\mapsto \mathbf{1}_{[0,\sigma_n(\omega)]\times\Gamma}(t,\omega)$. For T small enough it follows that $U_1^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma} = U_2^{\sigma_n}\mathbf{1}_{[0,\sigma_n]\times\Gamma}$ in $\tilde V^p_{\alpha,p}([0,T]\times\Omega;E_\eta)$. By an induction argument this holds on $[0,T_0]$ as well. By path continuity it follows that almost surely, $U_1\equiv U_2$ on $[0,\sigma_n]\times\Gamma$. Since $\varrho = \lim_{n\to\infty}\sigma_n$ we may conclude that almost surely, $U_1\equiv U_2$ on $[0,\varrho)\times\Gamma$.

If ϱ_1 is an explosion time, then as in [40, Lemma 5.3] this yields $\varrho_1\ge\varrho_2$ on Γ almost surely. Indeed, if for some $\omega\in\Gamma$, $\varrho_1(\omega)<\varrho_2(\omega)$, then we can find an n such that $\varrho_1(\omega)<\nu_n^2(\omega)$. We have $U_1(t,\omega)=U_2(t,\omega)$ for all $0\le t\le \nu^1_{n+1}(\omega)<\varrho_1(\omega)$. If we combine both assertions we obtain that

$$n+1 = \|U_1(\nu^1_{n+1}(\omega),\omega)\|_{E_a} = \|U_2(\nu^1_{n+1}(\omega),\omega)\|_{E_a} \le n.$$

This is a contradiction. The final assertion is now obvious. □

Proof of Theorem 8.1. We follow an argument of [3, 40].

For $n\ge 1$ let $\Gamma_n = \{\|u_0\|\le \frac{n}{2}\}$ and $u_n = u_0\mathbf{1}_{\Gamma_n}$. Let $(B_n)_{n\ge1}$ be the sequence of L^2_γ-Lipschitz functions from (A3)$'$. Fix an integer $n\ge 1$. Let $F_n:[0,T_0]\times\Omega\times E_\eta\to E_{-\theta_F}$ be defined by

$$F_n(\cdot,x) = F(\cdot,x) \ \text{ for } \|x\|_{E_\eta}\le n,$$

and $F_n(\cdot,x) = F\big(\cdot,\frac{nx}{\|x\|_{E_\eta}}\big)$ otherwise. Clearly, F_n and B_n satisfy (A2) and (A3). It follows from Theorem 6.2 that there exists a solution $U_n\in V^p_{\alpha,\infty}([0,T_0]\times\Omega;E_\eta)$ of (SCP) with u_0, F and B replaced by u_n, F_n and B_n. In particular, U_n has a version with continuous paths. Let ϱ_n be the stopping time defined by

$$\varrho_n(\omega) = \inf\{t\in[0,T_0] : \|U_n(t,\omega)\|_{E_\eta}\ge n\}.$$

It follows from Lemma 8.2 that for all $1\le m\le n$, almost surely, $U_m\equiv U_n$ on $[0,\varrho_m\wedge\varrho_n]\times\Gamma_m$. By path continuity this implies $\varrho_m\le\varrho_n$. Therefore, we can define $\varrho=\lim_{n\to\infty}\varrho_n$ and on Γ_n, $U(t)=U_n(t)$ for $t\le\varrho_n$. By approximation and Lemma 2.5 it is clear that $U\in V^{0,\mathrm{loc}}_{\alpha,p}([0,\varrho)\times\Omega;E_\eta)$ is a local solution of (SCP). Moreover, ϱ is an explosion time. This proves the existence part of (1). Maximality is a consequence of Lemma 8.2. Therefore, $(U(t))_{t\in[0,\varrho)}$ is a maximal local solution. This concludes the proof of (1).

We continue with (2). By Corollary 6.3, each U_n has the regularity as stated by (2). Therefore, the construction yields the required pathwise regularity properties of U.

Turning to (4), let $(U_n)_{n\ge1}$ be as before. As in the proof of Proposition 6.1 one can check that by the linear growth assumption,

$$\begin{aligned}
\|U_n\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} &= \|L_T(U_n)\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} \\
&\le C_T\|U_n\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} + C + C\|u_n\|_{L^p(\Omega;E_\eta)},
\end{aligned}$$

where the constants do not depend on n and u_0 and we have $\lim_{T\downarrow 0} C_T = 0$. Since $\|u_n\|_{L^p(\Omega;E_\eta)} \le \|u_0\|_{L^p(\Omega;E_\eta)}$, it follows that for T small we have

$$\|U_n\|_{V^p_{\alpha,\infty}([0,T]\times\Omega;E_\eta)} \le C(1 + \|u_0\|_{L^p(\Omega;E_\eta)}),$$

where C is a constant independent of n and u_0. Repeating this inductively, we obtain a constant C independent of n and u_0 such that $\|U_n\|_{V^p_{\alpha,\infty}([0,T_0]\times\Omega;E_\eta)} \le C(1 + \|u_0\|_{L^p(\Omega;E_\eta)})$. In particular,

$$\mathbb{E} \sup_{s\in[0,T_0]} \|U_n(s)\|^p_{E_\eta} \le C^p(1 + \|u_0\|_{L^p(\Omega;E_\eta)})^p.$$

It follows that

$$\mathbb{P}(\sup_{s\in[0,T_0]} \|U_n(s)\|_{E_\eta} \ge n) \le C^p n^{-p}.$$

Since $\sum_{n\ge 1} n^{-p} < \infty$, the Borel-Cantelli Lemma implies that

$$\mathbb{P}\Big(\bigcap_{k\ge 1} \bigcup_{n\ge k} \Big\{ \sup_{s\in[0,T_0]} \|U_n(s)\|_{E_\eta} \ge n\Big\}\Big) = 0.$$

This gives that almost surely, $\varrho_n = T_0$ for all n large enough, where ϱ_n is as before. In particular, $\varrho = T_0$ and by Fatou's lemma

$$\|U\|_{V^p_{\alpha,\infty}([0,T_0]\times\Omega;E_\eta)} \le \liminf_{n\to\infty} \|U_n\|_{V^p_{\alpha,\infty}([0,T_0]\times\Omega;E_\eta)} \le C(1 + \|u_0\|_{L^p(\Omega;E_\eta)}).$$

Via an approximation argument one can check that U is a global solution. The final statement in (4) can be obtained as in Theorem 6.3.

For the proof of (3) one may repeat the construction of Theorem 7.1, using Lemma 8.2 instead of Lemma 7.2. □

9. Generalizations to one-sided UMD spaces

In this section we explain how the theory of the preceding sections can be extended to a class of Banach spaces which contains, besides all UMD spaces, the spaces L^1.

A Banach space E is called a *UMD$^+$-space* if for some (equivalently, for all) $p \in (1,\infty)$ there exists a constant $\beta^+_{p,E} \ge 1$ such that for all E-valued L^p-martingale difference sequences $(d_j)_{j=1}^n$ we have

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j d_j\Big\|^p\Big)^{\frac{1}{p}} \le \beta^+_{p,E} \Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}}$$

where $(r_j)_{j=1}^n$ is a Rademacher sequence independent of $(d_j)_{j=1}^n$. The space E is called a *UMD$^-$ space* if the reverse inequality holds:

$$\Big(\mathbb{E}\Big\|\sum_{j=1}^n d_j\Big\|^p\Big)^{\frac{1}{p}} \le \beta^-_{p,E} \Big(\mathbb{E}\Big\|\sum_{j=1}^n r_j d_j\Big\|^p\Big)^{\frac{1}{p}}$$

Both classes of spaces were introduced and studied by Garling [15]. By a standard randomization argument, every UMD spaces is both UMD$^+$ and UMD$^-$, and conversely a Banach space which is is both UMD$^+$ and UMD$^-$ is UMD. At present, no examples are known of UMD$^+$-spaces which are not UMD. For the UMD$^-$property the situation is different: if E is UMD$^-$, then also $L^1(S;E)$ is UMD$^-$. In particular, every L^1-space is UMD$^-$ (cf. [29]).

Assume that $(\mathcal{F}_t)_{t\geq 0}$ is the complete filtration induced by W_H. If E is a UMD$^-$-space, condition (3) still gives a *sufficient* condition for stochastic integrability of Φ, and instead of a norm equivalence one obtains the one-sided estimate

$$\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^p \lesssim_{p,E} \mathbb{E}\|R\|^p_{\gamma(L^2(0,T;H),E)}$$

for all $p \in (1,\infty)$, where we use the notations of Proposition 2.4. The condition on the filtration is needed for the approximation argument used in [14]. By using Fubini's theorem it is obvious that the result also holds if the probability space has the following product structure $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F} \otimes \mathcal{G}$, $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$, and the filtration is of the form $(\mathcal{F}_t \otimes \mathcal{G})_{t\geq 0}$.

Mutatis mutandis, the theory presented in the previous sections extends to UMD$^-$ spaces E, with two exceptions: (i) Proposition 4.5 relies, via the use of Lemma 2.8, on the fact that UMD spaces have property (Δ); this property should now be included into the assumptions. (ii) One needs the above assumption on the filtration. We note that it follows from [7] that for $E = L^1$ the assumption on the filtration is not needed.

10. Applications to stochastic PDEs

Case of bounded A. We start with the case of a bounded operator A. By putting $\tilde{F} := A + F$ it suffices to consider the case $A = 0$.

Let E be a UMD$^-$ space with property (α) (see Section 5). Consider the equation

$$\text{(10.1)} \qquad \begin{aligned} dU(t) &= F(t, U(t))\, dt + B(U(t))\, dW_E(t), \quad t \in [0,T], \\ U(0) &= u_0, \end{aligned}$$

where W_E is an E-valued Brownian motion. With every E-valued Brownian motion W_E one can canonically associate an H-cylindrical Brownian motion W_H, where H is the so-called reproducing kernel Hilbert space associated with $W_E(1)$ (see the proof of Theorem 10.1 below). Using this H-cylindrical Brownian motion W_H, the problem (10.1) can be rewritten as a special instance of (SCP).

We make the following assumptions:

(1) $F : [0,T] \times \Omega \times E \to E$ satisfies (A2) with $a = \theta_F = 0$;
(2) $B \in \mathcal{L}(E, \mathcal{L}(E))$;
(3) $u_0 : \Omega \to E$ is $\mathcal{F}_0$-measurable.

Theorem 10.1. *Under these assumptions, for all $\alpha > 0$ and $p > 2$ such that $\alpha < \frac{1}{2} - \frac{1}{p}$ there exists a unique strong and mild solution $U : [0,T] \times \Omega \to E$ of (10.1) in $V^0_{\alpha,p}([0,T] \times \Omega; E)$. Moreover, for all $0 \leq \lambda < \frac{1}{2}$, U has a version with paths in $C^\lambda([0,T]; E)$.*

Proof. Let H be the reproducing kernel Hilbert space associated with $W_E(1)$. Then H is a separable Hilbert space which is continuously embedded into E by means of an inclusion operator $i : H \hookrightarrow E$ which belongs to $\gamma(H,E)$. Putting $W_H(t)i^*x^* := \langle W_E(t), x^*\rangle$ (cf. [33, Example 3.2]) we obtain an H-cylindrical Brownian motion.

Assumption (A1) is trivially fulfilled, and (A2) and (A4) hold by assumption. Let $\hat{B} \in \mathcal{L}(E, \gamma(H,E))$ be given by $\hat{B}(x)h = B(x)ih$. Using Lemma 5.4 one checks that $\hat{B}$ satisfies (A3) with $a = \theta_B = 0$. Therefore, the result follows from Theorems 7.1 and 7.3 (applied to $\hat{B}$ and the H-cylindrical Brownian motion W_H). Here we use the extension to UMD$^-$ space as explained in Section 9. ∎

Elliptic equations on bounded domains. Below we will consider an elliptic equation of order $2m$ on a domain $S \subseteq \mathbb{R}^d$. We will assume the noise is white in space and time. The regularizing effect of the elliptic operator will be used to be able to consider the white-noise in a suitable way. Space-time white noise equations seem to be studied in the literature in the case $m = 1$ (cf. [3, 11]).

Let $S \subseteq \mathbb{R}^d$ be a bounded domain with C^∞ boundary. We consider the problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t,s) &= A(s,D)u(t,s) + f(t,s,u(t,s)) \\ &\quad + g(t,s,u(t,s))\,\frac{\partial w}{\partial t}(t,s), && s \in S,\ t \in (0,T], \\ B_j(s,D)u(t,s) &= 0, && s \in \partial S,\ t \in (0,T], \\ u(0,s) &= u_0(s), && s \in S. \end{aligned} \tag{10.2}$$

Here A is of the form

$$A(s,D) = \sum_{|\alpha| \le 2m} a_\alpha(s) D^\alpha$$

where $D = -i(\partial_1, \ldots, \partial_d)$ and for $j = 1, \ldots, m$,

$$B_j(s,D) = \sum_{|\beta| \le m_j} b_{j\beta}(s) D^\beta$$

where $1 \le m_j < 2m$ is an integer. We assume that $a_\alpha \in C(\overline{S})$ for all $|\alpha| = 2m$. For $|\alpha| < 2m$ the coefficients a_α are in $L^\infty(S)$. For the principal part $\sum_{|\alpha|=2m} a_\alpha(s) D^\alpha$ of A we assume that there is a $\kappa > 0$ such that

$$(-1)^{m+1} \sum_{|\alpha|=2m} a_\alpha(s)\xi^\alpha \ge \kappa |\xi|^{2m}, \quad s \in S,\ \xi \in \mathbb{R}^d.$$

For the coefficients of the boundary value operator we assume that for $j = 1, \ldots, m$ and $|\beta| \le m_j$ we have $b_{j\beta} \in C^\infty(\overline{S})$. The boundary operators $(B_j)_{j=1}^m$ define a normal system of Dirichlet type, i.e. $0 \le m_j < m$ (cf. [41, Section 3.7]). The C^∞ assumption on the boundary of S and on the coefficients $b_{j\beta}$ is made for technical reasons. We will need complex interpolation spaces for Sobolev spaces with boundary conditions. It is well-known to experts that one can reduce the the assumption to S has a C^{2m}-boundary and $b_{j\beta} \in C^{2m-m_j}(\overline{S})$. However, this seems not to be explicitly contained in the literature.

The functions $f, g : [0,T] \times \Omega \times S \times \mathbb{R} \to \mathbb{R}$ are jointly measurable, and adapted in the sense that for each $t \in [0,T]$, $f(t,\cdot)$ and $g(t,\cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}_S \otimes \mathcal{B}_\mathbb{R}$-measurable. Finally, w is a space-time white noise (see, e.g., [45]) and $u_0 : S \times \Omega \to \mathbb{R}$ is an $\mathcal{B}_S \otimes \mathcal{F}_0$-measurable initial value condition. We say that $u : [0,T] \times \Omega \times S \to \mathbb{R}$ is a *solution* of (10.2) if the corresponding functional analytic model (SCP) has a mild solution U and $u(t,s,\omega) = U(t,\omega)(s)$.

Consider the following conditions:

(C1) The functions f and g are locally Lipschitz in the fourth variable, uniformly on $[0,T] \times \Omega \times S$, i.e., for all $R > 0$ the exist constants L_f^R and L_g^R such that

$$|f(t,\omega,s,x) - f(t,\omega,s,y)| \le L_f^R |x-y|,$$
$$|g(t,\omega,s,x) - g(t,\omega,s,y)| \le L_g^R |x-y|,$$

for all $t \in [0,T]$, $\omega \in \Omega$, $s \in S$, and $|x|, |y| < R$. Furthermore, f and g satisfy the boundedness conditions

$$\sup |f(t,\omega,s,0)| < \infty, \qquad \sup |g(t,\omega,s,0)| < \infty,$$

where the suprema are taken over $t \in [0,T]$, $\omega \in \Omega$, and $s \in S$.

(C2) The functions f and g are of linear growth in the fourth variable, uniformly in $[0,T] \times \Omega \times S$, i.e., there exist constants C_f and C_g such that

$$|f(t,\omega,s,x)| \le C_f(1+|x|), \qquad |g(t,\omega,s,x)| \le C_g(1+|x|),$$

for all $t \in [0,T]$, $\omega \in \Omega$, $s \in S$, and $x \in \mathbb{R}$.

Obviously, if f and g are Lipschitz and $f(\cdot,0)$ and $g(\cdot,0)$ are bounded, i.e., if (C1) holds with constants L_f and L_g not depending on R, then (C2) is automatically fulfilled.

The main theorem of this section will be formulated in the terms of the spaces $B^s_{p,1,\{B_j\}}(S)$. For their definition and further properties we refer to [42, Section 4.3.3] and references therein. For $p \in [1,\infty]$, $q \in [1,\infty]$ and $s > 0$, let

$$H^{s,p}_{\{B_j\}}(S) := \big\{f \in H^{s,p}(S) : \ B_j f = 0 \text{ for } m_j < s - \frac{1}{p}, \ j = 1,\dots,m\big\},$$

$$C^s_{\{B_j\}}(\overline{S}) := \big\{f \in C^s(S) : \ B_j f = 0 \text{ for } m_j \le s, \ j = 1,\dots,m\big\}.$$

For $p \in (1,\infty)$ let A_p be the realization of A on the space $L^p(S)$ with domain $H^{2m,p}_{\{B_j\}}(S)$. In this way $-A_p$ is the generator of an analytic C_0-semigroup $(S_p(t))_{t\ge 0}$. Since we may replace A and f in (10.2) by $A - w$ and $w + f$, we may assume that $(S_p(t))_{t\ge0}$ is uniformly exponentially stable. From [39, Theorem 4.1] and [42, Theorem 1.15.3] (also see [8]) we deduce that if $\theta \in (0,1)$ and $p \in (1,\infty)$ are such that

$$2m\theta - \frac{1}{p} \neq m_j, \text{ for all } j = 1,\dots,m, \tag{10.3}$$

then

$$[L^p(S), D(A_p)]_\theta = [L^p(S), H^{2m,p}_{\{B_j\}}(S)]_\theta = H^{2m\theta,p}_{\{B_j\}}(S)$$

isomorphically.

Theorem 10.2. *Assume that* (C1) *holds, let* $\frac{d}{m} < 2$, *and let* $p \in (1,\infty)$ *be such that* $\frac{d}{2mp} < \frac12 - \frac{d}{4m}$.

(1) *If* $\eta \in (\frac{d}{2mp}, \frac12 - \frac{d}{4m})$ *is such that* (10.3) *holds for the pair* (η,p) *and if* $u_0 \in H^{2m\eta,p}_{\{B_j\}}(S)$ *almost surely, then for all* $r > 2$ *and* $\alpha \in (\eta + \frac{d}{4m}, \frac12 - \frac1r)$ *there exists a unique maximal solution* $(u(t))_{t\in[0,\varrho)}$ *of* (10.2) *in* $V^{0,\mathrm{loc}}_{\alpha,r}([0,\varrho) \times \Omega; H^{2m\eta,p}_{\{B_j\}}(S))$.

(2) *Moreover, if* $\delta > \frac{d}{2mp}$ *and* $\lambda \ge 0$ *are such that* $\delta + \lambda < \frac12 - \frac{d}{4m}$ *and* (10.3) *holds for the pair* (δ,p), *and if* $u_0 \in H^{m-\frac{d}{2},p}_{\{B_j\}}(S)$ *almost surely, then* u *has paths in* $C^\lambda_{\mathrm{loc}}([0,\tau); H^{2m\delta,p}_{\{B_j\}}(S))$ *almost surely.*

Furthermore, if condition (C2) *holds as well, then:*

(3) *If* $\eta \in (\frac{d}{2mp}, \frac12 - \frac{d}{4m})$ *is such that* (10.3) *holds for the pair* (η,p) *and if* $u_0 \in H^{2m\eta,p}_{\{B_j\}}(S)$ *almost surely, then for all* $r > 2$ *and* $\alpha \in (\eta + \frac{d}{4m}, \frac12 - \frac1r)$ *there exists a unique global solution* u *of* (10.2) *in* $V^0_{\alpha,r}([0,T] \times \Omega; H^{2m\eta,p}_{\{B_j\}}(S))$.

(4) *Moreover, if* $\delta > \frac{d}{2mp}$ *and* $\lambda \geq 0$ *are such that* $\delta + \lambda < \frac{1}{2} - \frac{d}{4m}$ *and* (10.3) *holds for the pair* (δ, p), *and if* $u_0 \in H^{m-\frac{d}{2},p}_{\{B_j\}}(S)$ *almost surely, then* u *has paths in* $C^\lambda([0,T]; H^{2m\delta,p}_{\{B_j\}}(S))$ *almost surely.*

Remark 10.3.

(i) For $p \in [2,\infty)$ the uniqueness result in (1) and (3) can be simplified. In that case one obtains a unique solution in

$$L^0(\Omega; C([0,T]; H^{2m\eta,p}_{\{B_j\}}(S))) \subseteq V^0_{\alpha,r}([0,T]\times\Omega; H^{2m\eta,p}_{\{B_j\}}(S)).$$

For this case on could also apply martingale type 2 integration theory from [3] to obtain the result.

(ii) By the Sobolev embedding theorem one obtains Hölder continuous solutions in time and space. For instance, assume in (4) that $u_0 \in C^{m-\frac{d}{2}}_{\{B_j\}}(\overline{S})$ almost surely. It follows from

$$C^{m-\frac{d}{2}}_{\{B_j\}}(\overline{S}) \hookrightarrow H^{\eta,p}_{\{B_j\}}(S) = [E, D(-A_p)]_{\frac{\eta}{2m}} \hookrightarrow D((-A_p)^{\frac{\eta-\varepsilon}{2m}})$$

for all $p \in (1,\infty)$ and $\eta < m - \frac{d}{2}$ and $\varepsilon > 0$, that $t \mapsto S(t)u_0$ is in $C^\lambda([0,T]; D((-A_p)^\delta)$ for all $\delta, \lambda > 0$ that satisfy $\delta + \lambda < \frac{1}{2} - \frac{d}{4m}$. Since

$$D((-A_p)^\delta) \hookrightarrow [E, D(-A_p)]_{\delta-\varepsilon} = H^{2m(\delta-\varepsilon),p}_{\{B_j\}}(S)$$

for all $p \in (1,\infty)$ and $\varepsilon > 0$, by Sobolev embedding we obtain that the solution u has paths in $C^\lambda([0,T]; C^{2m\delta}_B(\overline{S}))$ for all $\delta, \lambda > 0$ that satisfy $\delta + \lambda < \frac{1}{2} - \frac{d}{4m}$.

Proof of Theorem 10.2. Let $p \in (1,\infty)$ be as in the theorem and take $E := L^p(S)$. For $b \in (0,1)$ let E_b denote the complex interpolation space $[E, \mathcal{D}(A_p)]_b$. Note that we use the notation E_b for complex interpolation spaces instead of fractional domain spaces as we did before. This will be more convenient, since we do not assume that A_p has bounded imaginary powers, and therefore we do not know the fractional domain spaces explicitly. Recall (cf. [26]) that $E_a \hookrightarrow \mathcal{D}((-A)^b)$ and that $\mathcal{D}((-A)^a) \hookrightarrow E_b$ for all $a \in (b,1)$ for all $b \in (0,1)$.

If $b > \frac{d}{2mp}$, then by [42, Theorem 4.6.1] we have

$$[E, \mathcal{D}(A_p)]_b \hookrightarrow C(\overline{S}).$$

Assume now that $\eta \in (\frac{d}{2mp}, \frac{1}{2} - \frac{d}{4m})$. Let $F, G : [0,T]\times\Omega\times E_\eta \to L^\infty(S)$ be defined as

$$(F(t,\omega,x))(s) = f(t,\omega,s,x(s)) \text{ and } G(t,\omega,x))(s) = g(t,\omega,s,x(s)).$$

We show that F and G are well-defined and locally Lipschitz. Fix $x, y \in E_\eta$ and let

$$R := \max\{\operatorname*{ess\,sup}_{s\in S} |x(s)|, \operatorname*{ess\,sup}_{s\in S} |y(s)|\} < \infty.$$

From the measurability of x, y and f it is clear that $s \mapsto (F(t,\omega,x))(s)$ and $s \mapsto (F(t,\omega,y))(s)$ are measurable. By (C1) it follows that for almost all $s \in S$, for all

$t \in [0,T]$ and $\omega \in \Omega$ we have

$$\begin{aligned}|(F(t,\omega,x))(s) - (F(t,\omega,y))(s)| &= |f(t,\omega,s,x(s)) - f(t,\omega,s,y(s))| \\ &\le L_f^R |x(s) - y(s)| \\ &\le L_f^R \|x-y\|_{L^\infty(S)} \\ &\lesssim_\eta L_f^R \|x-y\|_{E_\eta}.\end{aligned}$$

Also, by the second part of (C1), for almost all $s \in S$, for all $t \in [0,T]$ and $\omega \in \Omega$ we have

$$|(F(t,\omega,0))(s)| = |f(t,\omega,s,0)| < \sup_{t,s,\omega} |f(t,\omega,s,0)| < \infty.$$

Combing the above results we see that F is well-defined and locally Lipschitz. In a similar way one shows that F has linear growth (see (A2)) if (C2) holds. The same arguments work for G.

Since $L^\infty(S) \hookrightarrow L^p(S) = E$ we may consider F as an E-valued mapping. It follows from the Pettis measurability theorem that for all $x \in E_\eta$, $(t,\omega) \mapsto F(t,\omega,x)$ is strongly measurable in E and adapted.

To model the term $g(t,x,u(t,s))\frac{\partial w(t,s)}{\partial t}$, let $H := L^2(S)$ and let W_H be a cylindrical Brownian motion. Define the multiplication operator function $\Gamma : [0,T] \times \Omega \times E_\eta \to \mathcal{L}(H)$ as

$$(\Gamma(t,\omega,x)h)(s) := (G(t,\omega,x))(s)h(s), \quad s \in S.$$

Then Γ is well-defined, because for all $t \in [0,T]$, $\omega \in \Omega$ we have $G(t,\omega,x) \in L^\infty(S)$.

Now let $\theta_B > \theta_B' > \frac{d}{4m}$ be such that $\theta_B + \eta < \frac12$ and (10.3) holds for the pair $(\theta_B, 2)$. Define $(-A)^{-\theta_B}B : [0,T] \times \Omega \times E_\eta \to \gamma(H,E)$ as

$$(-A)^{-\theta_B}B(t,\omega,x)h = i(-A)^{-\theta_B}G(t,\omega,x)h,$$

where $i : H^{2m\theta_B',2}(S) \to L^p(S)$ is the inclusion operator. This is well-defined, because $(-A)^{-\theta_B} : H \to H^{2m\theta_B',2}(S)$ is a bounded operator and therefore by the right-ideal property and Corollary 2.2 it follows that

$$\|i(-A)^{-\theta_B}\|_{\gamma(H,E)} \le \|(-A)^{-\theta_B}\|_{\mathcal{L}(L^2(S),H^{2m\theta_B',2}(S))} \|i\|_{\gamma(H^{2m\theta_B',2}(S),L^p(S))} < \infty.$$

Moreover, B is locally Lipschitz. Indeed, fix $x,y \in E_\eta$ and let

$$R := \max\{\operatorname*{ess\,sup}_{s\in S}|x(s)|, \operatorname*{ess\,sup}_{s\in S}|y(s)|\} < \infty.$$

It follows from the right-ideal property that

$$\begin{aligned}&\|i(-A)^{\theta_B}(B(t,\omega,x) - B(t,\omega,y))\|_{\gamma(H,E)} \\ &\quad\le \|i(-A)^{-\theta_B}\|_{\gamma(H,E)}\|\Gamma(t,\omega,x) - \Gamma(t,\omega,y)\|_{\mathcal{L}(H)} \\ &\quad\le \|i(-A)^{\theta_B}\|_{\gamma(H,E)}\|G(t,\omega,x) - G(t,\omega,y)\|_{L^\infty(S)} \\ &\quad\le \|i(-A)^{\theta_B}\|_{\gamma(H,E)} L_g^R \|x-y\|_{L^\infty(S)} \\ &\quad\lesssim_{a,p} \|i(-A)^{\theta_B}\|_{\gamma(H,E)} L_g^R \|x-y\|_{E_\eta}.\end{aligned}$$

In a similarly way one shows that B has linear growth. Notice that B is H-strongly measurable and adapted by the Pettis measurability theorem.

If $p \in [2,\infty)$, then E has type 2 and it follows from Lemma 5.2 that $(-A)^{-\theta_B}B$ is locally L^2_γ-Lipschitz and B has linear growth in the sense of (A3) if (C2) holds.

In case $p \in (1,2)$ the above result holds as well. This may be deduced from the previous case. Indeed, for each n define $(-A)^{-\theta_B} B_n : [0,T] \times \Omega \times E_\eta \to \gamma(H,E)$ as $(-A)^{-\theta_B} B_n(x) = (-A)^{-\theta_B} B(x)$ for all $\|x\|_{E_\eta} \leq n$ and $(-A)^{-\theta_B} B_n(x) = (-A)^{-\theta_B} B_n\big(\frac{nx}{\|x\|}\big)$ otherwise. Define $(-A)^{-\theta_B} B_n^\infty : [0,T] \times \Omega \times L^\infty(S) \to \gamma(H,H)$ as $(-A)^{-\theta_B} B_n^\infty(x) = (-A)^{-\theta_B} B_n(x)$. Replacing E with $L^2(S)$ in the above calculation it follows that B_n^∞ is a Lipschitz function uniformly on $[0,T] \times \Omega$. Since H has type 2, $(-A)^{-\theta_B} B_n^\infty$ is L^2_γ-Lipschitz. Fix a finite Borel measure μ on $(0,T)$ and fix $\phi_1, \phi_2 \in L^2_\gamma((0,T),\mu;E_\eta)$. Since $H \hookrightarrow E$ continuously, it follows that

$$\begin{aligned}
&\|(-A)^{-\theta_B}(B_n(t,\omega,\phi_1) - B_n(t,\omega,\phi_2))\|_{\gamma(L^2((0,T),\mu;H),E)} \\
&\qquad \leq C\|(-A)^{-\theta_B}(B_n^\infty(t,\omega,\phi_1) - B_n^\infty(t,\omega,\phi_2))\|_{\gamma(L^2((0,T),\mu;H),H)} \\
&\qquad \leq C\|\phi_1 - \phi_2\|_{L^2((0,T),\mu;L^\infty(S))} \\
&\qquad \leq C\|\phi_1 - \phi_2\|_{L^2((0,T),\mu;E_\eta)},
\end{aligned}$$

where C also depends on n. In a similarly way one shows that B has linear growth in the sense of (A3) if g has linear growth.

If $u_0 \in H^{2m\beta,p}_{\{B_j\}}(S)$ almost surely, where $\beta \in (\frac{d}{2mp}, \frac{1}{2} - \frac{d}{4m}]$ is such that (10.3) holds for the pair (β,p), then $\omega \mapsto u_0(\cdot,\omega) \in H^{2m\beta,p}_{\{B_j\}}(S) = E_\beta$ is strongly $\mathcal{F}_0$-measurable. This follows from the Pettis measurability theorem.

(1): It follows from Theorem 8.1 with η, θ_B as above and with $\eta + \theta_B < \frac{1}{2}$ and $\theta_F = 0$, $\tau = p \wedge 2$ that there is a unique maximal local mild solution $(U(t))_{t\in[0,\varrho)}$ in $V^{0,\text{loc}}_{\alpha,r}([0,\varrho) \times \Omega; E_\eta)$ for all $\alpha > 0$ and $r > 2$ satisfying $\eta + \theta_B < \alpha < \frac{1}{2} - \frac{1}{r}$. In particular U has almost all paths in $C([0,\varrho), E_\eta)$. Now take $u(t,\omega,s) := U(t,\omega)(s)$ to finish the proof of (1).

(2): Let $\delta = \eta > \frac{d}{2mp}$ and $\lambda \geq 0$ be such that $\lambda + \delta < \frac{1}{2} - \frac{d}{4m}$. Choose $\theta_B > \frac{d}{4m}$ such that $\lambda + \delta < \frac{1}{2} - \theta_B$. It follows from Theorem 8.1 that almost surely, $U - Su_0 \in C^\lambda_{\text{loc}}([0,\varrho(\omega)); H^{2m\delta,p}_{\{B_j\}}(S))$. First consider the case that $(\frac{1}{2} - \frac{d}{4m}, p)$ satisfies (10.3). Since $u_0 \in H^{m-\frac{d}{2},p}_{\{B_j\}}(S) = E_{\frac{1}{2}-\frac{d}{4m}} \subseteq E_\delta$ almost surely and $\lambda + \delta < \frac{1}{2} - \frac{d}{4m}$ we have $Su_0 \in C^\lambda([0,T]; H^{2m\delta,p}_{\{B_j\}}(S))$ almost surely. Therefore, almost all paths of U are in $C^\lambda_{\text{loc}}([0,\varrho(\omega)); H^{2m\delta,p}_{\{B_j\}}(S))$. In the case $(\frac{1}{2} - \frac{d}{4m}, p)$ does not satisfy (10.3) one can redo above argument with $\frac{1}{2} - \frac{d}{4m} - \epsilon$ for $\epsilon > 0$ small. This proves (2).

(3), (4): This follows from Theorems 8.1 and parts (3), (4) of 8.1. □

Remark 10.4. The above approach also works for systems of equations.

Laplacian in L^p. Let S be a open subset (not necessarily bounded) of $\mathbb{R}^d$. Consider the following perturbed heat equation with Dirichlet boundary values:

$$\begin{aligned}
\frac{\partial u}{\partial t}(t,s) &= \Delta u(t,s) + f(t,s,u(t,s)) \\
&\qquad + \sum_{n\geq 1} b_n(t,s,u(t,s))\,\frac{\partial W_n(t)}{\partial t}, && s \in S,\ t \in (0,T], \\
u(t,s) &= 0, && s \in \partial S,\ t \in (0,T], \\
u(0,s) &= u_0(s), && s \in S.
\end{aligned}$$

The functions $f, b_n : [0,T] \times \Omega \times S \times \mathbb{R} \to \mathbb{R}$ are jointly measurable, and adapted in the sense that for each $t \in [0,T]$, $f(t,\cdot)$ and $b_n(t,\cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}_S \otimes \mathcal{B}_\mathbb{R}$-measurable.

We assume that $(W_n)_{n\geq 1}$ is a sequence of independent standard Brownian motions on Ω and $u_0 : S\times\Omega \to \mathbb{R}$ is an $\mathcal{B}_S \otimes \mathcal{F}_0$-measurable initial value condition. We say that $u : [0,T]\times\Omega\times S \to \mathbb{R}$ is a *solution* of (10.4) if the corresponding functional analytic model (SCP) has a mild solution U and $u(t,s,\omega) = U(t,\omega)(s)$.

Let $p \in [1,\infty)$ be fixed and let $E := L^p(S)$. It is well-known that the Dirichlet Laplacian Δ_p generates a uniformly exponentially stable and analytic C_0-semigroup $(S_p(t))_{t\geq 0}$ on $L^p(S)$, and under a regularity assumption on ∂S one can identify $\mathcal{D}(\Delta_p)$ as $W^{2,p}(S)\cap W^{1,p}_0(S)$. Consider the following p-dependent condition:

(C) There exist constants L_f and L_{b_n} such that

$$|f(t,\omega,s,x) - f(t,\omega,s,y)| \leq L_f|x-y|,$$
$$|b_n(t,\omega,s,x) - b_n(t,\omega,s,y)| \leq L_{b_n}|x-y|,$$

for all $t\in[0,T]$, $\omega\in\Omega$, $s\in S$, and $x,y\in\mathbb{R}$. Furthermore, f satisfies the boundedness condition

$$\sup \|f(t,\omega,\cdot,0)\|_{L^p(S)} < \infty,$$

where the supremum is taken over all $t\in[0,T]$ and $\omega\in\Omega$, and the b_n satisfy the following boundedness condition: for all finite measures μ on $(0,T)$,

$$\sup \Big\| \Big(\int_0^T \sum_{n\geq 1} |b_n(t,\omega,\cdot,0)|^2 \, d\mu(t)\Big)^{\frac12} \Big\|_{L^p(S)} < \infty,$$

where the supremum is taken over all $\omega\in\Omega$.

Theorem 10.5. *Let S be an open subset of $\mathbb{R}^d$ and let $p\in[1,\infty)$. Assume that condition* (C) *holds with $\sum_{n\geq 1} L^2_{b_n} < \infty$. If $u_0 \in L^p(S)$ almost surely, then for all $\alpha>0$ and $r>2$ such that $\alpha < \frac12 - \frac1r$, the problem* (10.4) *has a unique solution $U\in V^0_{\alpha,r}([0,T]\times\Omega; L^p(S))$. Moreover, for all $\lambda\geq 0$ and $\delta\geq 0$ such that $\lambda+\delta<\frac12$ there is a version of U such that almost surely, $t\mapsto U(t) - S_p(t)u_0$ belongs to $C^\lambda([0,T];[L^p(S),\mathcal{D}(\Delta_p)]_\delta)$.*

Remark 10.6. Under regularity conditions on ∂S and for $p\in(1,\infty)$ one has

$$[L^p(S), D(\Delta_p)]_\delta = \big\{x\in H^{2\delta,p}(S) : x = 0 \text{ on } \partial S \text{ if } 2\delta - \frac{1}{p} > 0\big\}$$

provided $\delta\in(0,1)$ is such that $2\delta - \frac1p \neq 0$.

Proof. We check the conditions of Theorem 7.1 (for $p=1$ we use the extensions of our results to UMD$^-$ spaces described in Section 9, keeping in mind that L^1-spaces have this property). It was already noted that (A1) is fulfilled. Let $E := L^p(S)$ and define $F : E\to E$ as $F(t,x)(s) := f(t,s,x(s))$. One easily checks that F satisfies (A2) with $\theta_F = \eta = 0$. Let $H := l^2$ with standard unit basis $(e_n)_{n\geq 1}$, and let $B : [0,T]\times\Omega\times E \to \mathcal{L}(H,E)$ be defined as $(B(t,\omega,x)e_n)(s) := b_n(t,\omega,s,x(s))$.

Then for all finite measures μ on $(0,T)$ and all $\phi_1, \phi_2 \in \gamma(L^2((0,T),\mu;H),E)$,

$$\begin{aligned}
&\|B(\cdot,\phi_1) - B(\cdot,\phi_2)\|_{\gamma(L^2((0,T),\mu;H),E)} \\
&\eqsim_p \Big\| \Big(\int_0^T \sum_{n\geq 1} |b_n(t,\cdot,\phi_1(t)(\cdot)) - b_n(t,\cdot,\phi_2(t)(\cdot))|^2 \, d\mu(t) \Big)^{\frac12} \Big\|_E \\
&\leq \Big\| \Big(\int_0^T \sum_{n\geq 1} L_{b_n}^2 |\phi_1(t)(\cdot) - \phi_2(t)(\cdot)|^2 \, d\mu(t) \Big)^{\frac12} \Big\|_E \\
&\eqsim_p L\|\phi_1 - \phi_2\|_{\gamma(L^2(0,T),\mu),E)},
\end{aligned}$$

where $L = (\sum_{n\geq 1} L_{b_n}^2)^{\frac12}$. Moreover,

$$\|B(\cdot,0)\|_{\gamma(L^2(0,T),\mu;H),E))} \eqsim_p \Big\| \Big(\int_0^T \sum_{n\geq 1} |b_n(t,\cdot,0)|^2 \, d\mu(t) \Big)^{\frac12} \Big\|_E < \infty.$$

From these two estimates one can obtain (A3). □

Appendix A. Measurability of stochastic convolutions

In this appendix we study progressive measurability properties of processes of the form

$$t \mapsto \int_0^t \Phi(t,s) \, dW_H(s)$$

where Φ is a two-parameter process with values in $\mathcal{L}(H,E)$.

Proposition A.1. *Let E be a UMD space. Assume that $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathcal{L}(H,E)$ is H-strongly measurable and for each $t \in \mathbb{R}_+$, $\Phi(t,\cdot)$ is adapted and has paths in $\gamma(L^2(\mathbb{R}_+;H),E)$ almost surely. Then the process $\zeta : \mathbb{R}_+ \times \Omega \to E$,*

$$\zeta(t) = \int_0^t \Phi(t,s) \, dW_H(s),$$

has a version which is adapted and strongly measurable.

Proof. It suffices to show that ζ has a strongly measurable version $\tilde{\zeta}$, the adaptedness of $\tilde{\zeta}$ being clear. Below we use strong measurability for metric spaces as in [43].

Let $L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$ denote the closure of all adapted strongly measurable processes which are almost surely in $\gamma(L^2(\mathbb{R}_+;H),E)$. Note that by [32] the stochastic integral mapping extends to $L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$.

Let $G \subseteq \mathbb{R}_+ \times \Omega$ be the set of all (t,ω) such that $\Phi(t,\cdot,\omega) \in \gamma(L^2(\mathbb{R}_+;H),E)$. Since Φ is H-strongly measurable, we have $G \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{A}$. Moreover, letting $G_t = \{\omega \in \Omega : (t,\omega) \in G\}$ for $t \in \mathbb{R}_+$, we have $\mathbb{P}(G_t) = 1$ and therefore $G_t \in \mathcal{F}_0$. Define the H-strongly measurable function $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathcal{B}(H,E)$ as $\Psi(t,s,\omega) := \Phi(t,s,\omega)\mathbf{1}_{[0,t]}(s)\mathbf{1}_G(t,\omega)$. It follows from [32, Remark 2.8] that the map $\mathbb{R}_+ \times \Omega \ni (t,\omega) \to \Psi(t,\cdot,\omega) \in \gamma(L^2(\mathbb{R}_+;H),E)$ is strongly measurable. Hence, the map $\mathbb{R}_+ \ni t \to \Psi(t,\cdot) \in L^0(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$ is strongly measurable. Since it takes values in $L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$ it follows from an approximation argument that it is strongly measurable as an $L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$-valued map. Since the elements which are represented by an adapted step process are dense in $L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$, it follows from [43, Proposition 1.9] that we can find a sequence of processes $(\Psi_n)_{n\geq 1}$, where each $\Psi_n : \mathbb{R}_+ \to L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))$ is a countably valued simple function of the form

$$\Psi_n = \sum_{k\geq 1} \mathbf{1}_{B^n_k}\Phi^n_k, \quad \text{with } B^n_k \in \mathcal{B}_{\mathbb{R}_+} \text{ and } \Phi^n_k \in L^0_{\mathbb{F}}(\Omega;\gamma(L^2(\mathbb{R}_+;H),E)),$$

such that for all $t\in\mathbb{R}_+$ we have $\|\Psi(t)-\Psi_n(t)\|_{L^0(\Omega;\gamma(L^2(\mathbb{R}_+;H),E))} \leq 2^{-n}$, where with a slight abuse of notation we write

$$\|\xi\|_{L^0(\Omega;F)} := \mathbb{E}(\|\xi\|_F \wedge 1)$$

keeping in mind that this is not a norm. Notice that by the Chebyshev inequality, for a random variable $\xi:\Omega\to F$, where F is a normed space, and $\varepsilon\in(0,1]$, we have

$$\mathbb{P}(\|\xi\|_F > \varepsilon) = \mathbb{P}((\|\xi\|_F\wedge 1) > \varepsilon) \leq \varepsilon^{-1}\|\xi\|_{L^0(\Omega;F)}.$$

It follows from [32, Theorems 5.5 and 5.9] that for all $t\in\mathbb{R}_+$, for all $n\geq 1$ and for all $\varepsilon,\delta\in(0,1]$,

$$\mathbb{P}\Big(\Big\|\int_{\mathbb{R}_+} \Psi(t,s)-\Psi_n(t,s)\,dW_H(s)\Big\| > \varepsilon\Big) \leq \frac{C\delta^2}{\varepsilon^2} + \frac{1}{\delta 2^n}.$$

Taking $\varepsilon\in(0,1]$ arbitrary and $\delta=\frac{1}{n}$, it follows from the Borel-Cantelli lemma that for all $t\in\mathbb{R}_+$,

$$\mathbb{P}\Big(\bigcap_{N\geq 1}\bigcup_{n\geq N}\Big\{\Big\|\int_{\mathbb{R}_+} \Psi(t,s)-\Psi_n(t,s)\,dW_H(s)\Big\| > \varepsilon\Big\}\Big) = 0.$$

Since $\varepsilon\in(0,1]$, was arbitrary, we may conclude that for all $t\in\mathbb{R}_+$, almost surely,

$$\zeta(t,\cdot) = \int_{\mathbb{R}_+} \Psi(t,s)\,dW_H(s) = \lim_{n\to\infty}\int_{\mathbb{R}_+} \Psi_n(t,s)\,dW_H(s).$$

Clearly,

$$\int_{\mathbb{R}_+} \Psi_n(\cdot,s)\,dW_H(s) = \sum_{k\geq 1}\mathbf{1}_{B^n_k}(\cdot)\int_{\mathbb{R}_+}\Phi^n_k(s)\,dW_H(s)$$

has a strongly $\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{F}_\infty$-measurable version, say $\zeta_n:\mathbb{R}_+\times\Omega\to E$. Let $C\subseteq\mathbb{R}_+\times\Omega$ be the set of all points (t,ω) such that $(\zeta_n(t,\omega))_{n\geq 1}$ converges in E. Then $C\in\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{F}_\infty$ and we may define the process $\tilde{\zeta}$ as $\tilde{\zeta}=\lim_{n\to\infty}\zeta_n\mathbf{1}_C$. It follows that $\tilde{\zeta}$ is strongly $\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{F}_\infty$-measurable and for all $t\in\mathbb{R}_+$, almost surely, $\tilde{\zeta}(t,\cdot)=\zeta(t,\cdot)$. $\square$

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