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Efficient particle-based estimation of marginal costs in a first-order macroscopic traffic flow model

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Abstract: Marginal costs in traffic networks are the extra costs incurred to the system as the result of extra traffic. Marginal costs are required frequently e.g. when considering system optimal traffic assignment or tolling problems. When explicitly considering spillback in a traffic flow model, one can use a numerical derivative or resort to heuristics to calculate the marginal costs. Numerical derivatives are computationally demanding, restricting its use to simple networks. Heuristic approaches in most cases approximate the marginal costs by only considering the extra costs on the links which are traveled by the extra traffic, excluding the possibly external costs incurred on other links due to spillback. This paper proposes a novel way to estimate the true marginal costs of traffic in a dynamic discrete LWR model which correctly deals with congestion onset, spillback and dissolution. The proposed methodology tracks virtual changes in density through the network by means of particles which travel along with the characteristics of traffic. By using density based cost functions, the virtual changes in density can be directly related to the marginal costs. The computational efficiency of the methodology stems from the fact that only local conditions are considered when propagating the virtual change in density. The paper discusses the methodology and necessary model extensions, provides a numerical validation experiment illustrating the exact detail of the solution by comparison to a numerical derivative and discusses some generalizations

Keywords: Optimization, Dynamic Traffic Assignment, System Optimal, LWR, marginal costs, particle

1. INTRODUCTION

This paper deals with the subject of calculating marginal costs in a macroscopic first-order traffic flow model, without the need to calculate a computationally demanding numerical derivative or use a heuristic approximation which does not correctly deal with spillback. Marginal costs are the extra costs incurred to the whole system as the result of adding extra traffic. They can be separated in direct and external costs. Direct costs are experienced by the extra traffic, whereas external costs are experienced by other traffic in the network. The concept of marginal costs within traffic modelling has first been used in the context of road pricing by (Pigou A.C., 1920). Later on, marginal travel time functions where used for the purpose of system optimal (SO) traffic assignment as first described by (Merchant. D.K. and Nemhauser. G.L., 1978) to attain a system optimal state according to Wardrop's 2nd principle (Wardrop J.G., 1952) with minimal total travel time. In the context of system optimal assignment, most studies use analytical link travel time functions which have desirable mathematical properties but fail to represent the fundamental traffic characteristic of congestion onset, spillback and dissolution, as for example concluded in (Nie X. and Zhang H.M., 2005). For this reason, in (Peeta .S. and Mahmassani. H.S., 1995) the authors use a dynamic simulation model which explicitly models spillback and thus features better traffic realism. When formulating the dynamic SO assignment problem, they find themselves challenged to calculate the marginal path costs. Finding the true global marginal costs in that simulation model would require a numerical derivative and is quoted as: "*… a brute force approach which is computationally in-efficient even on existing high powerful super computers.*". Therefore a heuristic approach is used. The heuristic approach fails to correctly estimate the true marginal costs by only considering the effects of spillback on the links which are travelled by the extra traffic, disregarding other affected links in the network, as the authors say: *"… path marginal's in these experiments are not necessarily global as they are based on local link level marginal travel times"*. Similar reasoning is found in (Ziliaskopoulos. A.K et al., 2004) where again a heuristic approach is chosen.

The motivation for this research is to generate on-line real-time system optimal route guidance advices of which a description can be found in (Zuurbier F.S. et al., 2006). This problem is closely related to dynamic traffic assignment, as described by (Bottom J., 2000). For the representation of traffic, a macroscopic first-order traffic flow model is used, also known as the LWR model (Lighthill M.J. and Whitham G.B., 1955, Richards P.I., 1956). This model has desirable properties such as the minimal required state detail, computational efficiency, a convenient state-space formulation, the ability to be used as a state estimator and can properly represent congestion onset, spillback and dissolution.

This paper presents a computational efficient way to calculate the marginal costs of traffic in the discrete LWR model. The state in such a model can be described by cell densities. Given these cell densities, the cost and marginal cost of traffic per cell can be found using differentiable cell-cost functions. This paper describes a heuristic approach which allows modelling the propagation of a virtual change in density as it travels along with the characteristics of traffic. E.g. downstream (free flow) or upstream (congestion). By allocating this virtual change in density to the traffic cells correctly, as it spreads through the network in time and space, the marginal costs can be found by multiplying the virtual change in density with the marginal cell cost function. By summarizing all marginal costs for all cells to which the virtual change spreads an expression for the global marginal cost of the virtual change in density is found.

Section 2 discusses the first-order dynamic traffic flow model for networks. In Section 3 the cost and marginal cost of traffic in the discrete LWR model is discussed. Section 4 discusses how virtual changes in traffic density can be tracked along with the characteristic of traffic. Section 5 combines differentiable cell-price functions with the virtual changes in density to calculate the marginal state costs. In Section 7 a numerical experiment is presented, illustrating the quality of the proposed methodology and the mild computational expense compared to numerical derivatives. In Section 7, the paper finishes with a series of conclusions and places the methodology in a broader context for optimization problems using the LWR model.

In this paper, super scripts are used to distinguish parameters or similar variables. Sub scripts are used as indices in time and space. In general parameters are denoted by ϕ with additional super and sub scripts for further specification.

2. DISCRETE LWR MODEL FOR NETWORKS

Within the discrete first-order traffic-flow model (LWR model), time and space are discretized in a step-size Δt [h] and segment or cell length $\Delta \ell_l$ [km] respectively where $l \in \mathbf{l}$ is the index of a link from the set of links $\mathbf{l}$ in the network. The state variable is the cell density r_{lck} [veh/h] for cells with an index $c = 1..\phi_l^{\text{nCells}}$ of link l at discrete time index k where the latter denotes an interval in time $[k\Delta t, (k+1)\Delta t)$. The number of cells ϕ_l^{nCells} on a link is related to the length of the link in the following way $\phi_l^{\text{length}} = \phi_l^{\text{nCells}} \Delta \ell_l$, whereby the cell length $\Delta \ell_l = \phi_l^{\text{free}} \Delta t$ is based on the step size and the free-flow speed ϕ_l^{free} [km/h]. The basic model state-space equation is given by:

$$r_{lc,k+1} = r_{lck} + (f_{lck}^{\text{in}} - f_{lck}^{\text{out}}) \frac{\Delta t}{\Delta \ell_l} \tag{1}$$

Whereby f_{lck}^{in} [veh/h] denotes the flux into a cell c on link l at time k and f_{lck}^{out} [veh/h] the flux out of that cell. The flux determines the change in cell densities and constitutes the flow of traffic over the border of a cell. The fluxes determine the actual change in density per time step. Given the density of a cell, the first-order traffic flow model also assumes a homogeneous cell flow q_{lck} [veh/h] associated with that density. To that end, the so-called fundamental diagram (FD) is used $q_{lck} = \Gamma(r_{lck})$. The Godunov scheme, as first applied to this model in (Lebacque J.P. and Khoshyaran M.M., 2002), determines the cell fluxes, by specifying a demand $D(r_{lck})$ [veh/h] and supply $S(r_{lck})$ [veh/h] function based on the FD in the following way:

$$D(r_{lck}) = \begin{cases} \Gamma(r_{lck}) \text{ if } r_{lck} < \phi_l^{\text{critical}} \\ \phi_l^{\text{capacity}}, \text{ otherwise} \end{cases} \tag{2}$$

And,

$$S(r_{lck}) = \begin{cases} \Gamma(r_{lck}) \text{ if } r_{lck} \geq \phi_l^{\text{critical}} \\ \phi_l^{\text{capacity}}, \text{ otherwise} \end{cases} \tag{3}$$

Whereby the capacity ϕ_l^{capacity} [veh/h] of a link determines the maximum vehicle throughput and the critical density ϕ_l^{critical} [veh/km] indicates the threshold amount of vehicles in a cell from which this throughput capacity starts to deteriorate and congestion effects occur. Given the demand and supply function, the flux is defined as the minimum between both functions, where $f_{lck}^{\text{out}} = \min(D(r_{lck}), S(r_{l,c+1,k}))$ for two adjacent cells. Necessarily the flux out of one cell equals the flux in to the next adjacent cell $f_{l,c-1,k}^{\text{out}} = f_{lck}^{\text{in}}$.

Using this approach, all cells on a link can be updated by (1) except the first and last cell since the flux at these cells is determined differently. These depend on the way the link is located in a network and the route choice of traffic. In order to use the first-order traffic flow model on a network, the concept of a node $n \in \mathbf{n}$, given by an index n from the set of indices $\mathbf{n}$, is used. Nodes are defined as points and do not occupy physical space. They merely handle the exchange of traffic between the last cells of connected in-links and first cells of connected out-links. To that end, let $i \in L_n^{\text{in}}$ denote the index i of an in-link from the set of in-links L_n^{in} of node n. Likewise $j \in L_n^{\text{out}}$ denotes the index of an out-link j from the set of out-links $j \in L_n^{\text{out}}$ of node n. In addition, the route choice of traffic is represented by the split fraction ψ_{ijk} specifying the way in which the aggregated traffic flow is distributed over out-links j when coming from link i on that node at time k. The split fraction is the direct result of the composition of traffic and the route choice it has. In this research, the route choice ψ_{ijk} is determined exogenously.

Given the demand function (2), the demand at the end of a link i is determined as $D_{ik} = D(r_{lck})$ where $c = \phi_l^{\mathrm{nCells}}$ and $l = i$. This demand is directed over the out-links j according to the aggregated route choice ψ_{ijk}. As a result, the total demand for an out-link j is found by $D_{jk} = \sum_{i\in L_n^{\mathrm{in}}} D(r_{lck})\psi_{ijk}$. This demand is directed to the first cell of a link j, whereby the supply of this cell $S_{jk} = S(r_{lck})$ is determined by the supply function (3) where $c = 1$ and $l = j$. Given both D_{jk} and S_{jk}, the flux into the first cell of link j is defined as $F_{jk} = \min(D_{jk}, S_{jk})$. Given this flux, the reduction $M_{jk} = F_{jk} / D_{jk}$ defines the ratio between what 'wants' to flow and what 'can' flow due to congestion per out-link j. Traffic flowing out of link i to link j does so at the maximum rate as determined by the most heavily congested out-link to ensure there is no overtaking. To that end, the node congestion multiplier $M_{ijk} = M_{j_i^{\mathrm{r}}k}$ specifies the reduction for each in- and out-link pair. Whereby j_i^{r} is the representative direction for link i and defined as: $M_{j_i^{\mathrm{r}}k} = \min(M_{jk}) \quad \forall \quad D_{ik}\psi_{ijk} > 0$.

Given the multiplier, the flux $f_{jk}^{\mathrm{in}} = \sum_{j\in L_n^{\mathrm{out}}} D_{ik} M_{ijk} \psi_{ijk}$ into the first cell of a link j and that out of the last cell $f_{ik}^{\mathrm{out}} = \sum_{i\in L_n^{\mathrm{in}}} D_{ik} M_{ijk} \psi_{ijk}$ can be determined in order to update the state space equation in (1).

Given the network, origins $o \in \mathbf{o}$ with index o from the set of indices $\mathbf{o}$ are defined as in-links to a node with an exogenous determined demand function D_{ok} [veh/h] specifying the in-flow to a network. And likewise, destinations $d \in \mathbf{d}$ with index d from the set of destinations $\mathbf{d}$ are defined as out-links of a node with a pre specified supply function S_{dk} [veh/h].

This concludes the description of the first-order traffic flow model for aggregated traffic flow on a network consisting of links, nodes, origins and destinations. The next section will discuss the cost associated with traffic in this network.

3. THE COST OF TRAFFIC

The cost of traffic is defined at the cell level by a cost function $C(r_{lck})$ based on the density r_{lck} for link l, cell $c \in 1..\phi_l^{\mathrm{nCells}}$ at time k. The cost function can for example express the cost associated with congestion or emission in that cell. The state vector $\mathbf{r}_k$ is a vector of all cell densities for all links $r_{lck} \in \mathbf{r}_k$ at time k. When performing an assignment from an initial time k_0 for a horizon of H [-] steps, until step $k_h = k_0 + H$, the total cost associated with the trajectory of states $\mathbf{r}_{k_0}..\mathbf{r}_{k_h}$ is defined as the summation of all cell costs in that trajectory.

The background of this research is route guidance. Route guidance advices are represented by φ_{ak_t} [veh/h] and denote a flow of vehicles advised to take route $a \in \mathbf{a}$ from a set of considered routes $\mathbf{a}$, at time k_t during the optimization horizon $k_0 < k_t < k_h$. For the purpose of routing control, we are interested in finding the change in total system costs corresponding to a change in routing which is written as:

$$\sum_{k=k_t}^{k_h} \sum_{l\in\mathbf{l}} \sum_{c=1}^{\phi_l^{\mathrm{nCells}}} \frac{\Delta C(r_{lck})}{\Delta\varphi_{ak_t}} \tag{4}$$

Due to a change in routing $\Delta\varphi_{ak_t}$ changes in cell costs $\Delta C(r_{lck})$ starting from the initial time of routing k_t until the end of the horizon k_h will occur due to changes in density Δr_{lck}. By replacing the exact change Δ with an infinitesimal change d and using the chain rule, the argument of (4) can be rewritten as:

$$\frac{\Delta C(r_{lck})}{\Delta\varphi_{ak_t}} = \frac{dC(r_{lck})}{dr_{lck}} \frac{dr_{lck}}{d\varphi_{ak_t}} \tag{5}$$

This notation is interesting since it brings us to the core of the solution, which is to separate the problem of calculating the marginal costs into two sub problems. The first is to calculate the marginal cell price functions and the second is to track the change in traffic density through the network as the result of the routing action. The first sub-problem is solved when assuming a differentiable cost function with respect to the state variable density. The remaining sub problem is to track a change in density through the network in the discrete LWR model. This is discussed in the next section.

4. TRACKING CHANGES IN DENSITY

In order to calculate the marginal costs according to (5), it is not necessary to actually change the density in the model (as would be the case when using a numerical derivative). By considering it infinitesimally small, it is only necessary to determine how and where this virtual change in density Δr spread through the network, which is discussed next.

Assume a change in routing $\Delta\varphi$ occurs somewhere in the network at time k_t, resulting in a change in density Δr

downstream of where the routing instruction was given. Let's assume this change in density Δr will initially travel downstream under free-flow conditions. In this situation, the change in system costs is incurred by Δr, the so-called direct costs.

When at some point the change Δr reaches a cell c for which it holds that the downstream cell $c+1$ is congested, the change in density will not travel further downstream. This can be verified by considering the flux out f_{lck}^{out} of the cell c as described in Section 2. The flux out is in this case determined by the downstream cell supply function $S(r_{l,c+1,k})$ and does not change due to the presence of the extra traffic Δr. The change is momentary stalled at cell c. However, if cell c reaches a density which is over–critical, at some point the supply function $S(r_{lck}) < D(r_{l,c-1,k})$ will become smaller than the demand function of the upstream cell. When this occurs, the initial change in density Δr will start to change the density of cell $c-1$ resulting in external costs to the system.

The above discussed principle is used to determine how the virtual change in density Δr will propagate in the network. The methodology to do so is by introducing particles into the simulation which travel along with the characteristics of travel. Each particle has a vector of virtual densities. This vector describes the exact distribution of the initial change in density relative to the current position of the particle. The way in which the current state of the system effects the change in virtual density is captured by this vector. Compared to a numerical derivative (or co-state equation) only a few local calculations are required to determine the change in virtual densities and the current position of the particle.

Let a particle $z \in \mathbf{z}_{ak_t}$ be given an index z from the set of indices $\mathbf{z}_{ak_t}$ of particles which are used to track changes in density for route a at time k_t. A particle has a current link index l_{zk}, a current cell index c_{zk}, a downstream length left l_{zk}^{left} [km] and a vector ξ_{zhk} [veh/km] of virtual densities with length $h = 1..\phi_z^{\text{trail}}$. This vector captures the exact distribution of the virtual change in density Δr relative to the current position of the particle as will be described in the second part of this Section.

First the way in which the particle travels along with the traffic flow in down- and upstream direction is discussed. When travelling downstream, the particle is assumed to travel along with the characteristic speed of traffic. The characteristic speed $c(r_{lck})$ [veh/h] is the speed with which small perturbations in density travel in the LWR model. This speed is found by the derivative of the flow-density relationship $c(r_{lck}) = d\Gamma(r_{lck})/dr_{lck}$. Given this speed, at each time step a distance $dl = c(r_{lck})\Delta t$ can be travelled. Let l_{zk}^{left} [km] be the remaining length left until the end of the current cell $c = c_{zk}$. If at a time k it holds that $l_{zk}^{\text{left}} - dl < 0$ the particle will propagate to the next downstream cell $c_{z,k+1} = c_{zk} + 1$ and the new length left becomes $l_{z,k+1}^{\text{left}} = l_{zk}^{\text{left}} - dl + \Delta\ell_l$.

When a particle moves into a cell c_{zk} which is congested (over critical $r_{lck} > \phi_l^{\text{crit}}$), the particle will start moving upstream. It will do so in a manner that ensures that the particle is always located in the congested supply cell. So given the current position $c = c_{zk}$, the particle will move to cell $c_{z,k+1} = c_{zk} - 1$ if the following condition is met:

$$D(r_{l,c-2,k}) > S(r_{l,c-1,k}) \qquad (6)$$

By the time the particle has reached the second cell $c = 2$, it is checked whether or demand D_{jk} at the node for link $j = l_{zk}$ at which the particle is on, exceeds that of the supply of the link S_{jk}, which is the supply of the first cell. If this is the case, the particle moves to the first cell and the change will start to propagate upstream over the node. Se also Fig 1.

At this point, the movement of the particle in the down- and upstream direction has been discussed. What is discussed next is how the initial change in density Δr, as the result of a routing action $\Delta\varphi$ which was once all located in one cell at time k_t, will be distributed over the cells relative to the current position of the particle and captured by the vector of virtual densities ξ_{zhk}.

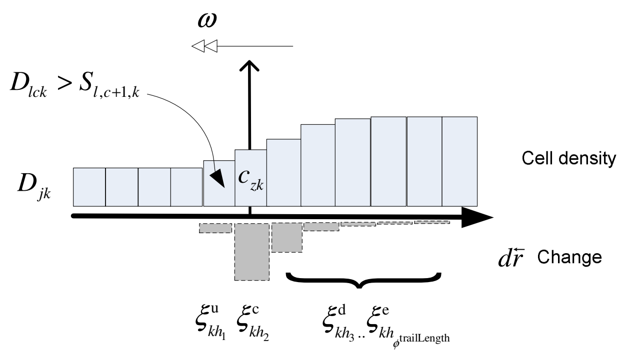

Fig 1 Upstream propagation of a particle on a link and definition of the vector of virtual densities

Surrounding the current position of the particle is the vector of virtual densities ξ_{zhk} [veh/km]. The elements of this vector describe the exact amount of virtual density for a cell, relative to the current cell position c_{zk} of the particle. Whereby superscripts are added to distinguish between individual elements in the vector.

Let ξ^{u}_{zhk} represent the amount of virtual density located at the upstream cell of the current position of the particle, and so $c = c_{zk} - 2 + h$ where $h = 1$. The element ξ^{c}_{zhk} is positioned at the current cell $h = 2$ and represents the amount of virtual density there. Whereas all elements ξ^{d}_{zhk} are positioned downstream of the current location $2 < h \le \phi^{\mathrm{trail}}_{z}$.

So, right after time k_t when a virtual change in routing $\Delta\varphi_{ak_t}$ resulted in a virtual change in density Δr on some cell c on a link l, particle is created on that exact position, $c = c_{ck}$ and $l = l_{zk}$, with all virtual density located in the element $\xi^{\mathrm{c}}_{zhk} = \Delta r$ and all other elements zero. At that point the virtual change in density is directly attributable to the change in routing $\xi^{\mathrm{c}}_{zhk} = \Delta r_{lck} \big/ \Delta\varphi_{ak_t}$. When travelling downstream in free flow, all virtual density is presumed to remain located at the current position of the particle. This assumption is valid if changes in density travel at free-flow speed. This is the case when considering a triangular fundamental diagram. When assuming other types of diagrams rarefaction effects may occur. Rarefaction is the opposite of compression and results in an acceleration fan. As a result a small part of the change in density will travel faster compared to the characteristic speed and reach the end of a node before the particle does. This amount of density is small and so is its contribution to the marginal costs. As a result rarefaction is ignored.

When the particle enters a congested cell, and (6) holds, the virtual change in density Δr will spillback from cell $c = c_{zk}$ to upstream cell $c-1$. This change is captured by the rate of change ϑ_l. The rate of change is found by considering the derivative of (1) which yields:

$$\frac{dr_{l,c-1,k+1}}{d\varphi_{ak_t}} = \frac{dr_{l,c-1,k}}{dr_{lck}} \frac{dr_{lck}}{d\varphi_{ak_t}} = \vartheta_l \frac{dr_{lck}}{d\varphi_{ak_t}} \tag{7}$$

This rate of change is in turn dependant on the derivative of the out-flux of a cell which is determined by the supply of the downstream cell when congested, and so:

$$\vartheta_l = \frac{-\Delta t}{\Delta \ell_l} \frac{df^{\mathrm{out}}_{l,c-1,k}}{dr_{lck}} \quad \text{and} \quad \frac{df^{\mathrm{out}}_{l,c-1,k}}{dr_{lck}} = \frac{dS(r_{lck})}{dr_{lck}} \tag{8}$$

The vector of virtual densities captures the change in density $\xi_{zhk} = dr_{lck} \big/ d\varphi_{ak_t}$ due to the change in routing and by substitution in (7) it can be rewritten as:

$$\xi_{zhk} = \vartheta_l \, \xi_{z,h-1,k} \tag{9}$$

This relationship can be used to determine the changes in the diffusion weights at each step by rewriting (9) as a state space equation which results in:

$$\xi_{zh,k+1} = \alpha(1-\vartheta_l)\xi_{zhk} + \beta\vartheta_l \, \xi_{z,h-1,k} \tag{10}$$

With the exceptions that for $h = 1$, $\alpha = 0$ which is 1 otherwise, and for $h = \phi^{\mathrm{trail}}$ $\beta = 0$ which is 1 otherwise.

If the particle moves one cell upward, the elements of the virtual densities vector must be shifted in order to correctly represent the amount of virtual density per cell. This is because the vector is always defined relative to the current position of the particle. Shifting entails to set $\xi_{z,h+1,k^*} = \xi_{zhk}$ where k^* is an intermediate step, $\xi^{\mathrm{u}}_{zhk^*} = 0$ and summarizing all residual virtual densities in the last element by $\xi^{\mathrm{d}}_{zhk^*} = \xi^{\mathrm{d}}_{zhk} + \xi^{\mathrm{d}}_{z,h-1,k}$ where $h = \phi^{\mathrm{trail}}$ to preserve the total amount of virtual density surrounding the particle.

At this point, the propagation of a virtual change in density on a link has been discussed. Next, the propagation of virtual changes in densities at a node is discussed.

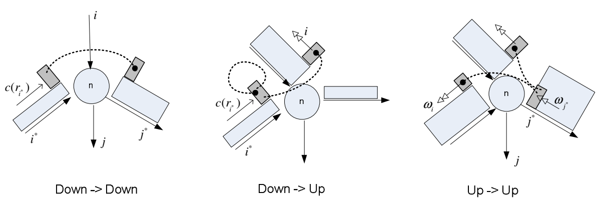

Fig 2 Propagation of virtual changes in density at a node

When a particle reaches the end of an in-link i at time k and wants to travel to out-link j since $\psi_{ijk} = 1$, with $M_{ijk} = 1$ (no congestion), the change as a whole is propagated to the next link. Therefore $l_{z,k+1} = j$ and $c_{z,k+1} = 1$. All weights remain unchanged and only $\xi^{\mathrm{c}}_{zhk} > 0$. This is called propagation of the type down-down. Whenever a particle z travelling downstream is at the end of a link $l_z = i^*$ and encounters congestion $M_{ijk} < 1$, the change Δr will start to propagate to the other inlinks $i \in L^{\mathrm{in}}_{n}$ of the to-node, which is called propagation of the type down-up. If a particle travelling upstream and has reached the begin cell of a link $l_z = j^*$ it will start to propagate the change Δr to the in-links $i \in L^{\mathrm{in}}_{n}$ of the from-node which is called propagation of the type up-

up. All propagations of change are depicted in Fig 2. Changes of the type down-up and up-up are captured by the rates of change ϑ_{i^*ik} and ϑ_{j^*ik}. These rates are determined in a similar way by evaluating the derivative of the state space equation (1) for the begin and end cells of a link. At these cells the fluxes f_{jk}^{in} and f_{ik}^{out} are determined by the multiplier M_{ijk} which is in turn based on the Godunov demand and supply functions of the cells. Let the density r_{ik} be that of the last cell of in-link i and likewise r_{jk} that of the first cell of out-link j. The relation between a change in density at r_{i^*} or r_{j^*} and its effect on r_i is given by:

$$\frac{dr_{i,k+1}}{d\varphi_{ak_t}} = \vartheta_{i^*ik}\frac{dr_{i^*k}}{d\varphi_{ak_t}} \text{ and } \frac{dr_{i,k+1}}{d\varphi_{ak_t}} = \vartheta_{j^*ik}\frac{dr_{j^*k}}{d\varphi_{ak_t}} \tag{11}$$

Whereby the rates of change are given by:

$$\vartheta_{i^*ik} = \left(\frac{dD_{ik}}{dr_{i^*k}}\sum_{j\in L_n^{\text{out}}}\psi_{ijuk}M_{ijk} + D_{ik}\sum_{j\in L_n^{\text{out}}}\psi_{ijuk}\frac{\partial M_{ijk}}{\partial r_{i^*k}}\right)\frac{-\Delta t}{\Delta\ell_i} \tag{12}$$

And,

$$\vartheta_{j^*ik} = \frac{-\Delta t}{\Delta\ell_i}D_{ik}\sum_{j\in L_n^{\text{out}}}\psi_{ijk}\frac{\partial M_{ijk}}{\partial r_{j^*k}} \tag{13}$$

These rates express how changes in density at the last cell of a link i^* or the first cell of a link j^* will propagate upstream at the node to other inlinks. More specifically, the change is propagated to the representative elements of the virtual densities vector of particles on the in-links of the node. The representative element is the element which represents the virtual density on the last of in-link i and is denoted by: ξ_{zhk}^{r}. If the particle moves to an upstream, the representative element changes consequently.

If a particle z^* arrives (upstream travel on a link j^*) at the beginning of a link $l_{z^*} = j^*$, then propagation of the type up-up will occur. As a result virtual density from the element $\xi_{z^*hk}^{\text{c}}$ of the arriving particle, will propagate to the representative elements ξ_{zhk}^{r} of new particles z created on the in-links $i\in L_n^{\text{in}}$ of this node.

The amount of virtual density being propagated each time step is determined by the rate of change ϑ_{j^*ik} and the amount of density at the current cell of the initial arriving particle , similar to (7). At each time k the change in virtual density to in-link i becomes $\Delta\xi_{ik}^{\text{up}} = \vartheta_{j^*ik}\xi_{z^*hk}^{\text{c}}$.

When a particle z^* arrives at the node at the end of a link $l_{z^*} = i^*$ (it was moving downstream) and encounters congestion, virtual density will be propagated to other in-links $i\in L_n^{\text{in}}$ where $i \neq i^*$. The initial arriving particle z^* will have its virtual density located in the current cell $\xi_{z^*hk}^{\text{c}} = \xi_{z^*hk}^{\text{r}}$ which equals the representative element at the end of the link and propagation of the type down-up will occur.

Propagation of the type down-up always occurs, since it will also occur between virtual densities at the representative elements of new particles created on the in-links. The total change in virtual density at time k from links i^* to a specific link i and $i^* \neq i$ is found by $\Delta\xi_{ik}^{\text{down}} = \sum_{i^*\in L_n^{\text{in}}}\vartheta_{i^*ik}\xi_{zhk}^{\text{r}}$.

Given the the changes $\Delta\xi_{ik}^{\text{up}}$ and $\Delta\xi_{ik}^{\text{down}}$ the total change in virtual densities at the end of each time step can be determined by:

$$\xi_{zh,k+1}^{\text{c}} = \xi_{zhk}^{\text{c}} - \sum_{i\in L_n^{\text{in}}}\Delta\xi_{ik}^{\text{up}} \quad \text{where } l_z = j^* \tag{14}$$

And

$$\xi_{zh,k+1}^{\text{r}} = (1-\sum_{l\in L_n^{\text{in}}}\vartheta_{i^*lk})\xi_{zhk}^{\text{r}} + \Delta\xi_{ik}^{\text{in}} + \Delta\xi_{ik}^{\text{out}} \quad \text{where } l_z = i \tag{15}$$

and $i^* = i$

The exact value of the rate of change ϑ_l , ϑ_{i^*ik} and ϑ_{j^*ik} depends on the chosen form of the fundamental diagram $\Gamma(r_{lck})$ which is left implicit, since it does not change the methodology.

This concludes the discussion of the methodology. Particles are created which travel along with the characteristics of travel on a link. Surrounding the particle is a vector which represents virtual densities; the virtual change in density due to the virtual change in routing. When encountering congestion, expressions can be derived which relate the rate of change between elements in this vector. This rate is determined by evaluating the derivative of the state equation with respect to an upstream cell, either on the same link or on a connected in-link of a node.

5. ESTIMATING MARGINAL COSTS

The marginal cost associated with a particle is found by multiplying the vector of virtual densities with the marginal cell costs. Let the $z \in \mathbf{z}_{ak_t}[k]$ denote the collection of all particle indices which are in use at time k to track the initial change in density Δr due to the initial route change $\Delta\varphi_{ak_t}$

at time k_t (when new particles are created they are added to the same set). The marginal state cost at time k is found by:

$$\sum_{z \in \mathbf{z}_{ak_t}} \sum_{h=1}^{\phi^{\text{trail}}} \xi_{zhk} \Delta C(r_{lck}) \text{ and } c = c_{zk} - 2 + h \tag{16}$$

6. RESULTS

To illustrate the quality of the above approach, an experiment is carried out which compares the numerical estimated marginal state costs with that of the approximated marginal state costs using the heuristic. To that end, the network in Fig 3 is used.

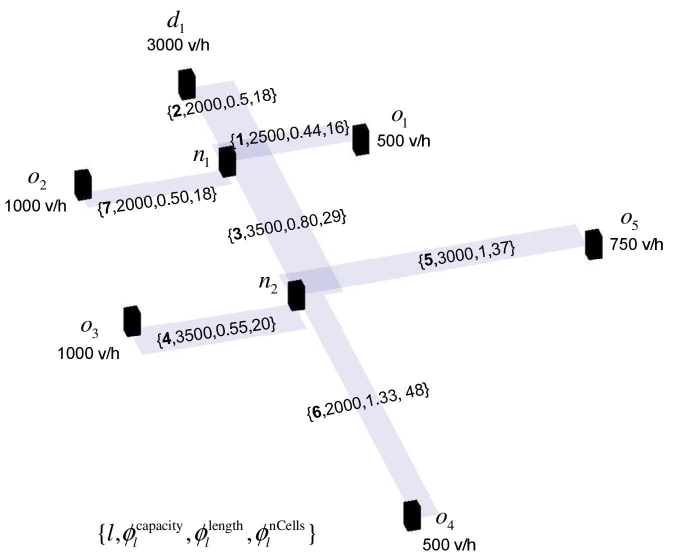

Fig 3 Validation network for estimating marginal costs

The network consists of seven links and eight nodes, relevant link information is displayed in the figure. Destination d_1 has a limited capacity of $S_{d_1 k} = 3000$ [veh/h]. Origins have demand D_{ok} as plotted above them. As a result, spillback will occur and eventually reach every origin. The step size is $\Delta t = 1$ and $\varphi_l^{\text{free}} = 100$ for all links. The experiment starts with an empty network and an initial assignment of 150 steps. From $k_0 = 150$ the network will be evaluated to $k_h = k_0 + H$ where $H = 2000$. The marginal costs are evaluated when travelling from o_1 to d_1 on the route with links [1,2]. A fundamental diagram of the type Smulders (Smulders S.A., 1989) is used and a fixed trail-length of $\phi^{\text{trail}} = 10$. The cell-price function is chosen as $C(r_{lck}) = \frac{1}{2}(r_{lck})^2$ with marginal price $\Delta C(r_{lck}) = r_{lck}$.

Given this setup, a numerical derivative is calculated by comparing the state costs between two network assignments of which one actually has a unit of $\Delta r = 1$ [veh/km] added at k_0 to the first cell $c = 1$ of link $l = 1$. The state cost is found as the summation of all cell costs for all cells and links at a time k.

The marginal costs are approximated using the proposed methodology by creating a particle z at link $l_{zk_0} = 1$ and cell $c_{zk} = 1$ with initial virtual density $\xi_{zhk}^c = \Delta r$ and all other elements zero.

The experiment compares the marginal state cost at time $k_0 + k$ between the numerical derivative and the particle approximation. The results are displayed in Fig 4.

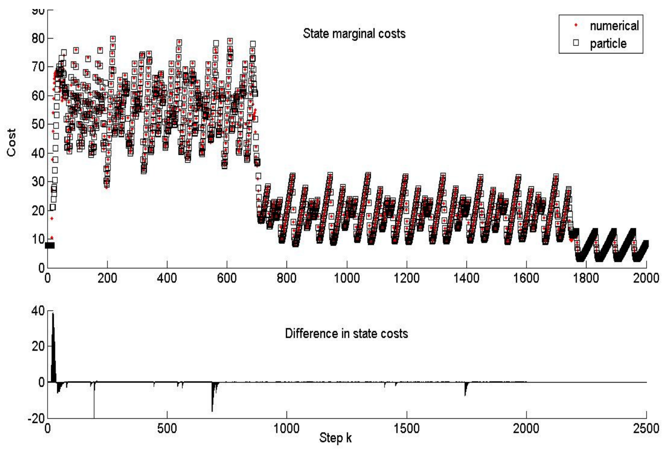

Fig 4 Comparison of numerical and particle derived marginal state costs

Due to the chosen topology and origin sizes the following will happen. First propagation of type down-up will occur at node n_1 at step $k = k_0 + 20$. As a result three particles are created on links [1, 3, 7] which after some steps have summed virtual densities vector of [0, 0.64, 0.36] [km/h] respectively. Then, at step $k = k_0 + 195$ the particle which was created on link 3 will reach node n_2 and start propagating up-up. As a result, three new particles are formed on links [4, 5, 6] with eventual summed virtual densities vector of [0.32, 0.19, 0.13]. From that point on, the particles at links [7, 4, 5] will reach the origins $\{o_2, o_3, o_5\}$ at steps $k = k_0 + \{687, 704, 1742\}$ respectively and exit the network.

As can be verified from Fig 4, the estimated marginal state costs which are drawn by the black squares almost exactly coincide with the numerical estimated state costs drawn by the red dots. When looking at the bottom part, it can be seen that an error is made at the beginning when congestion is first encountered. This is due to ignoring the acceleration fan. In addition, when the particles reach the origins an error is made as well. This is due to the rate of change which is not well defined for an origin link due to the absence of a fundamental diagram there. Overall, the proposed methodology approximates the numerical derivative with an error < 0.004.

The experiment also shows the importance of estimating the external costs correctly. In this experiment, direct costs are only made for the first 20 steps. The external costs however, linger on for 1980 steps and are huge compared to the direct costs. Which supports the result in (Kuwahara. M., 2007) where the marginal cost are shown to be more closely related to the duration of congestion.

The example also shows how heuristic approaches which only consider the marginal costs incurred on the links which are travelled by the extra traffic, will severely underestimate the external costs. Heuristic approaches would only consider the external costs on links [1, 2] whereas the overall contribution of links [7, 3, 5, 4, 6] in this example is far greater.

In addition, since particles can be tracked back all the way to the origin, hidden congestion can be quantified and directly attributed to the control action responsible. This is information which is not available in other heuristic approaches

Given the small scale of the network a single run of H steps would takes only $\tau = 0.08$ [s] for a normal network. When simultaneously estimating the marginal costs of extra traffic from all origins, the computation time increased to $\tau^* = 0.14$ seconds. In order to do this using numerical derivatives would have required $(5+1)\tau$ runs rendering the particle approach faster. Due to the small scale of the experiment, results are expected to increase for larger networks. In addition, future research will be aimed at increasing the efficiency further by aggregating particles which are travelling upstream with the same characteristic which can reduce the number of needed particles.

There are situations in which the approximated costs differ slightly from the numerical costs. These situations occur due to the explicit ignoring of rarefaction and the accompanying acceleration fan. As a result, a small part of the initial downstream travelling change will arrive at the link-end on an earlier time compared to the particle. In the numerical derivative, this early arriving change due to rarefaction will already start being propagated at the node. When i) propagation of the type down-up occurs, and ii) node conditions are still changing (link demand and supply), a small difference can be found between the numerical derivative and particle approximation. Fortunately, when estimating the marginal costs on a later time this error is likely to disappear since node conditions will have stabilized or congestion may have spilled back onto the link preventing the change from ever reaching the node.

This concludes the validation experiment. The estimated marginal costs correctly take into account all external costs due to spillback without the need to explicitly calculate a numerical derivative.

7. CONCLUSION

The paper presented a novel and efficient way of estimating external marginal costs due to spillback in the discrete LWR model correctly.

The presented methodology can be generalized to other traffic control problems aimed at system optimization. As long as the effect of a change in control Δu_{k_t} at time k_t can be expressed in an accompanying change in density Δr_{lck} on a link l for cell c at time $k = k_t + 1$.

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