Non-local fractional boundary value problems with applications to predator-prey models

Fečkan, Michal; Marynets, Kateryna

DOI
10.58997/ejde.2023.58

Publication date
2023

Document Version
Final published version

Published in
Electronic Journal of Differential Equations

Electronic Journal of Differential Equations, Vol. 2023 (2023), No. 58, pp. 1–17.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, https://ejde.math.unt.edu
DOI: 10.58997/ejde.2023.58

NON-LOCAL FRACTIONAL BOUNDARY VALUE PROBLEMS WITH APPLICATIONS TO PREDATOR-PREY MODELS

MICHAL FEČKAN, KATERYNA MARYNETS

ABSTRACT. We study a nonlinear fractional boundary value problem (BVP) subject to non-local multipoint boundary conditions. By introducing an appropriate parametrization technique we reduce the original problem to an equivalent one with already two-point restrictions. Using a notion of Chebyshev nodes and Lagrange polynomials we construct a successive iteration scheme, that converges to the exact solution of the non-local problem for particular values of the unknown parameters, which are calculated numerically.

1. INTRODUCTION

Fractional calculus and fractional differential equations denote a separate and dynamically developing branch in the classical theory of differential equations and dynamical systems. The first mention of fractional calculus is dated by late 1695 and belongs to Leibniz. However a real breakthrough in this field belongs to Riemann and Liouville, who introduced a notion of the Riemann-Liouville fractional integral and differential operators [15, 24]. Later followed also the Caputo type, Hilfer and Hilfer-Prabhakar derivatives, all of which belong to the group of fractional operators with a singular kernel [1, 15, 20]. Due to this singularity differential equations are characterized by a non-local behavior, what makes analysis of the corresponding dynamical systems even more challenging.

In this article we study a differential system with Caputo fractional derivative of order $p \in (0, 1)$. This differential operator is often used in modeling of the real-world problems, because of its property to preserve values of the unknown function and its derivatives, which in turn coincide with the integer order derivatives (for more information on the applications we refer to [6, 15, 21, 22, 23]). This gives a particular advantage in analysis of the corresponding initial value or boundary value problems (IVPs or BVPs). To answer questions about existence and uniqueness of solutions, their boundedness or stability one can apply tools from the operator and fixed point theory, stability and functional analysis. However, less is possible to find an explicit form of the exact solution to the studied problem. The well-known techniques, such as the Laplace, Mellin or Fourier transforms, power series method etc., have already their limitations when studying the linear systems. Since most

2020 *Mathematics Subject Classification.* 34A08, 34K07, 34K28.
Key words and phrases. Caputo derivative; non-local boundary conditions; Chebyshev nodes; approximation of solutions; Lagrange polynomial interpolation; predator-prey model.

Submitted February 23, 2023. Published September 11, 2023.

dynamical models, describing the real-world processes, are of the nonlinear nature, development of efficient approximation techniques is one of the promising directions in this field.

Recently a series of new results in approximation of solutions to the nonlinear fractional differential systems (FDS) of the Caputo type was published. Authors suggested a so-called "successive approximations technique" [19] to construct approximate solutions of the studied systems by incorporating also different types of linear [2, 8, 9, 11] and nonlinear boundary conditions [10, 12]. This approach demonstrates its effectiveness also when dealing with highly nonlinear systems, systems of a general order $p \in (m, m+1)$ or mixed order equations (see discussions in [3, 4, 8, 9]).

Motivated by these results, in this paper we analyze a fractional BVP (FBVP) with a non-local behavior not only in the differential system, but also in the boundary conditions. Moreover, we modify the aforementioned successive approximations technique by coupling it with the polynomial interpolation method [5, 13, 14, 18]. The outline of the paper is the following. In Section 2 we introduce main notations, used throughout the paper, and formulate necessary definitions and statements, needed to prove our results. Section 3 contains the problem formulation and justification of the parametrization techniques for elimination of the non-local boundary conditions. In Section 4 we construct a "classical" and "improved" (polynomial) successive approximations schemes and prove the main results of this paper. And finally, in Section 5 we demonstrate efficiency of the developed method on a particular example of a non-local FBVP describing a predator-prey model with prey refuge.

2. Notation, auxiliary statements and definitions

For a fixed $n \in \mathbb{N}$ and a bounded set $D \subset \mathbb{R}^n$, the following notation apply:

- For any vector-column $x = \operatorname{col}(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $n \times n$ real matrix A operations $|\cdot|$, $=$, $\leq$, $\geq$, max, inf, and sup are understood component-wise;
- I_n is a unit n-dimensional matrix;
- O_n is a zero n-dimensional matrix;
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- $r(A)$ is the maximal (in modulus) eigenvalue of matrix A.

2.1. **Auxiliary lemmas.**

Definition 2.1 ([12])**.** For a set $D \subset \mathbb{R}^n$, closed interval $[a, b] \subset \mathbb{R}$, Caratheodory function $f : [a, b] \times D \to \mathbb{R}^n$ and the n-dimensional square matrix K with non-negative entries, we write $f \in \operatorname{Lip}(K, D)$, if

$$|f(t, u) - f(t, v)| \leq K|u - v| \tag{2.1}$$

holds, for all $\{u, v\} \subset D$ and a.e. $t \in [a, b]$.

Lemma 2.2 ([2])**.** *Let $f(t)$ be a continuous function for $t \in [0, T]$. Then for all $t \in [0, T]$ it holds*

$$\begin{aligned} &\Big|\frac{1}{\Gamma(p)}\Big[\int_0^t (t-s)^{p-1} f(s)ds - \Big(\frac{t}{T}\Big)^p \int_0^T (T-s)^{p-1} f(s)ds\Big]\Big| \\ &\leq \alpha_1(t) \max_{t\in[0,T]} |f(t)|, \end{aligned} \tag{2.2}$$

where

$$\alpha_1(t) = \frac{2t^p}{\Gamma(p+1)}\Big(1-\frac{t}{T}\Big)^p. \tag{2.3}$$

Lemma 2.3 ([2])**.** *Let $\{\alpha_m(t)\}_{m\in\mathbb{Z}_0}$ be a sequence of continuous functions at the interval $[0,T]$ given by*

$$\alpha_m(t) := (\mathcal{I}\alpha_{m-1})(t), \quad m \in \mathbb{N}, \tag{2.4}$$

with

$$\begin{aligned}(\mathcal{I}y)(t) := \frac{1}{\Gamma(p)}\Big[&\int_0^t \Big((t-s)^{p-1} - \frac{t}{T}(T-s)^{p-1}\Big)y(s)ds \\ &+ \frac{t}{T}\int_t^T (T-s)^{p-1}y(s)ds\Big],\end{aligned} \tag{2.5}$$

where $\alpha_0(t) = 1$ and $\alpha_1(t)$ is defined by formula (2.3)*. Then*

$$\alpha_{m+1}(t) \le \frac{T^{mp}}{2^{m(2p-1)}\Gamma^m(p+1)}\alpha_1(t) \le \frac{T^{(m+1)p}}{2^{(m+1)(2p-1)}\Gamma^{m+1}(p+1)}, \tag{2.6}$$

holds for all $m \in \mathbb{Z}_0$.

2.2. **Some results from polynomial interpolation theory.**

Definition 2.4 ([18])**.** *For a given continuous vector-function $y : [0,T] \to \mathbb{R}^n$ and a natural number q, we denote by $L^q y$ the n-dimensional Lagrange polynomials of degree q, such that*

$$(\mathrm{L^q y})(\mathrm{t_i^q}) = \mathrm{y}(\mathrm{t_i^q}), \quad \mathrm{i} = \overline{1, \mathrm{q}+1}, \tag{2.7}$$

where

$$t_i^q = \frac{T}{2}\Big[\cos\frac{(2i-1)\pi}{2(q+1)} + 1\Big], \quad i = \overline{1, q+1} \tag{2.8}$$

are Chebyshev nodes, translated from the interval $(-1,1)$ to $(0,T)$, and

$$L^q y := \mathrm{col}(L^q y_1, L^q y_2, \dots L^q y_n). \tag{2.9}$$

Definition 2.5 ([18])**.** Denote by $\mathcal{P}_q$ a set of all polynomials of degree not higher than q ($q \ge \mathbb{N}$) on $[0,T]$. For any continuous function $y : [0,T] \to \mathbb{R}$, there exists a unique polynomial $p_q^* \in \mathcal{P}_q$, for which

$$\max_{t\in[0,T]} |y(t) - p_q^*| = E_q(y),$$

where

$$E_q(y) := \inf_{p\in\mathcal{P}_q} \max_{t\in[0,T]} |y(t) - p(t)|. \tag{2.10}$$

Here p_q^* is a polynomial of the best uniform approximation of y in $\mathcal{P}_q$, and the number $E_q(y)$ is called the error of the best uniform approximation.

Proposition 2.6 ([17])**.** *For any $q \in \mathbb{N}$ and a continuous function $y : [0,T] \to \mathbb{R}$, the corresponding interpolation polynomial* (2.7) *with the Chebyshev nodes* (2.8) *admits the estimate*

$$|y(t) - (L^q)y(t)| \le \Big(\frac{2}{\pi}\ln q + 1\Big)E_q(t), \quad t \in [0,T]. \tag{2.11}$$

Proposition 2.7 ([16]). *If $y \in C([0,T],\mathbb{R})$ and $q \in \mathbb{N}$, then*

$$E_q(t) \le 6\omega\Big(y; \frac{T}{2q}\Big), \tag{2.12}$$

where

$$\omega\,(y;\delta) := \sup\{|y(t)-y(s)| : \{t,s\} \subset [0,T], |t-s| \le \delta\}, \quad \forall \delta > 0$$

is the modulus of continuity of a continuous function y (for more information we refer to [15]*).*

Definition 2.8 ([18]). A function $y : [0,T] \to \mathbb{R}$ is said to satisfy the Dini-Lipschitz condition if its modulus of continuity has the property

$$\lim_{\delta\to\infty} \omega(y;\delta)\ln\delta = 0.$$

Remark 2.9. Note, that from (2.12) it follows that

$$\lim_{q\to\infty} E_q(y)\ln q = 0, \tag{2.13}$$

for any y satisfying the Dini-Lipschitz condition. Taking into account estimate (2.11), equality (2.13) ensures the uniform convergence of the Lagrange interpolation polynomials at Chebyshev nodes for this class of functions. In particular, every α-Hölder continuous function $y : [0,T] \to \mathbb{R}$ with $\alpha > 0$ satisfies the Dini-Lipschitz condition.

3. Non-local FBVP and its parametrization

Consider a FDS of order $p \in (0,1)$:

$${}_0^C D_t^p x(t) = f(t, x(t)), \quad t \in [0,T],\ x, f \in \mathbb{R}^n, \tag{3.1}$$

subject to the non-local multipoint boundary constraints

$$\phi(x) = \gamma, \tag{3.2}$$

where ${}_0^C D_t^p(\cdot)$ denotes the Caputo fractional differential operator with the lower limit at zero [15], $x : [0,T] \to \mathbb{R}^n$ and $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector-functions, $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous on $[0,T]$ non-linear functional, evaluated at an arbitrary number of points $t_i \in [0,T]$ $(i = \overline{1,n})$, and γ is a given n-dimensional vector.

Together with the BVP (3.1), (3.2) we study the parametrized two-point FBVP

$$\begin{gathered} {}_0^C D_t^p x(t) = f(t, x(t)), \quad t \in [0,T],\ x, f \in \mathbb{R}^n, \\ x(0) = \xi, \quad x(T) = \eta, \end{gathered} \tag{3.3}$$

and a perturbed IVP:

$${}_0^C D_t^p x(t) = f(t, x(t)) + \Delta, \quad t \in [0,T],\ x, f \in \mathbb{R}^n, \tag{3.4}$$

$$x(0) = \xi, \tag{3.5}$$

where the parameters $\xi, \eta \in \mathbb{R}^n$ and the perturbation term Δ are to be defined.

Using the Caputo integral operator [15], solution of the IVP (3.4), (3.5) can be written in an integral form that reads

$$x(t) = \xi + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, x(s))ds + \frac{\Delta t^p}{\Gamma(p+1)}. \tag{3.6}$$

To find the perturbation term Δ we require (3.6) to also satisfy the two-point parametrized boundary conditions (3.3). Direct calculations show that

$$\Delta(\xi,\eta)=\frac{1}{T^p}\Big[\Gamma(p+1)(\eta-\xi)-p\int_0^T(T-s)^{p-1}f(s,x(s))ds\Big]. \tag{3.7}$$

Substitution of (3.7) into (3.6) yields

$$\begin{aligned} x(t)=\Big[1-\big(\frac{t}{T}\big)^p\Big]\xi+\big(\frac{t}{T}\big)^p\eta+\frac{1}{\Gamma(p)}\Big[\int_0^t(t-s)^{p-1}f(s,x(s))ds \\ -\big(\frac{t}{T}\big)^p\int_0^T(T-s)^{p-1}f(s,x(s))ds\Big]. \end{aligned} \tag{3.8}$$

An appropriate choice of the vector parameters ξ,η, substituted into (3.8), leads to the exact solution of the original problem (3.1), (3.2). However, the nonlinearity $f(\cdot,x(\cdot))$ in the right hand-side of FDS (3.1), which depends also on the unknown function $x(\cdot)$, enables the integral calculation in (3.8). But if instead of the exact values of $x(\cdot)$ we consider their approximation, then formula (3.8) leads to an explicit approximate solution of problem (3.1), (3.2) for particular values of ξ and η. In the next section we give a detailed justification of this approximation process.

4. Successive approximations technique and its modification

We assume, that BVP (3.1), (3.2) satisfies the following conditions:

(A1) f is bounded: there exists $M>0$ such that

$$|f(t,x)|\le M,\quad \forall(t,x)\in G_f:=[0,T]\times\mathcal{D}_\rho, \tag{4.1}$$

and $f\in\operatorname{Lip}(K,\mathcal{D}_\rho)$.

(A2) The set $\mathcal{D}$ is defined by closed and bounded sets $\mathcal{D}_\xi$ and $\mathcal{D}_\eta\subset\mathbb{R}^n$ as

$$\mathcal{D}:=\{(1-\theta)\xi+\theta\eta:\xi\in\mathcal{D}_\xi,\eta\in\mathcal{D}_\eta,\theta\in[0,1]\}. \tag{4.2}$$

Additionally, for any $\rho\in\mathbb{R}^n_+$ we define a componentwise ρ-neighborhood of the set $\mathcal{D}$ by

$$\mathcal{D}_\rho:=\Xi_\rho(\mathcal{D}), \tag{4.3}$$

where

$$\begin{gathered}\Xi_\rho(\mathcal{D}):=\cup_{\xi\in\mathcal{D}}\Xi_\rho(\xi),\\ \Xi_\rho(\xi):=\{\xi\in\mathbb{R}^n:|\xi-z|\le\rho\}\text{for all }\xi.\end{gathered}$$

(A3) The spectral radius of the matrix

$$Q:=\frac{KT^p}{2^{2p-1}\Gamma(p+1)} \tag{4.4}$$

satisfies

$$r(Q)<1. \tag{4.5}$$

Under conditions (A1)–(A3), we construct a classical and a modified iteration processes for approximation of solutions of the original non-local BVP (3.1), (3.2), prove their convergence and derive a determining system for calculation of the numerical values of the unknown parameters ξ, η, introduced in (3.3).

4.1. **Classical numerical-analytic method and its convergence.** Let us connect with the exact solution of the parametrized FBVP (3.1), (3.3) a sequence of functions in an operator form

$$
\begin{gathered}
x_0(t,\xi,\eta) := \Big[1-\big(\frac{t}{T}\big)^p\Big]\xi + \big(\frac{t}{T}\big)^p\eta, \\
x_m(t,\xi,\eta) := x_0(t,\xi,\eta) + (\Lambda N_f x_{m-1}(\cdot,\xi,\eta))(t), \quad t\in[0,T],\ m\in\mathbb{N},
\end{gathered} \tag{4.6}
$$

where Λ stands for a linear operator, acting on the space $C([0,T])$, defined by

$$
(\Lambda y)(t) := \frac{1}{\Gamma(p)}\int_0^t \Big((t-s)^{p-1}\Big[y(s) - \frac{1}{T^p}\int_0^T (T-\tau)^{p-1}y(\tau)d\tau\Big]\Big)ds, \tag{4.7}
$$

for all $t\in[0,T]$ and N_f is the Nemytskii operator generated by the nonlinearity f in the right hand-side of (3.1),

$$
(N_f y)(t) := f(t,y(t)), \quad t\in[0,T]. \tag{4.8}
$$

Similarly to the results in [9, 10, 12], where authors studied FDSs with the two-point linear and nonlinear boundary conditions, one can prove a uniform convergence of the sequence (4.6) to a parametrized limit function $x_\infty(\cdot,\xi,\eta)$ and its relation to the exact solution $x(t)$ of the original FBVP (3.1), (3.2).

Theorem 4.1. *Assume, that the FBVP* (3.1), (3.3) *satisfies conditions* (A1)–(A3). *Then for each fixed* $(\xi,\eta)\in\mathcal{D}_\xi\times\mathcal{D}_\eta$*:*

(1) *The sequence of functions* (4.6) *are continuous and satisfy the two-point parametrized boundary conditions*
$$
x_m(0,\xi,\eta)=\xi, \quad x_m(T,\xi,\eta)=\eta.
$$
(2) *The sequence of functions* (4.6) *for* $t\in[0,T]$ *converges uniformly as* $m\to\infty$ *to a parameter-dependent limit function*
$$
x_\infty(t,\xi,\eta) = \lim_{m\to\infty} x_m(t,\xi,\eta). \tag{4.9}
$$
(3) *The limit function* (4.9) *satisfies the parametrized boundary conditions*
$$
x_\infty(0,\xi,\eta)=\xi, \quad x_\infty(T,\xi,\eta)=\eta.
$$
(4) *The limit function* (4.9) *is a unique solution to the integral equation*
$$
x_m(t,\xi,\eta) := x_0(t,\xi,\eta) + (\Lambda N_f x_{m-1}(\cdot,\xi,\eta))(t), \quad t\in[0,T],
$$
i.e., it is a unique solution on $t\in[0,T]$ *of the Cauchy problem for a perturbed system of FDEs,*
$$
{}_0^C D_t^p x(t) = f(t,x(t)) + \Delta(\xi,\eta), \quad t\in[0,T], \quad x,f\in\mathbb{R}^n, \tag{4.10}
$$
$$
x(0)=\xi, \tag{4.11}
$$
where $\Delta:\mathcal{D}_\xi\times\mathcal{D}_\eta\to\mathbb{R}^n$ *is the mapping defined by*
$$
\Delta(\xi,\eta) := \frac{1}{T^p}\Big[\Gamma(p+1)(\eta-\xi) - p\int_0^T (T-s)^{p-1}N_f x(s)ds\Big]. \tag{4.12}
$$
(5) *The following error estimate holds:*
$$
|x_\infty(t,\xi,\eta) - x_m(t,\xi,\eta)| \le \frac{T^p}{2^{2p-1}\Gamma(p+1)}Q^m(I_n-Q)^{-1}M, \tag{4.13}
$$
where M *and* Q *are defined by* (4.1) *and* (4.4) *respectively.*

Consider now a Cauchy problem

$$ {}_0^C D_t^p x(t) = f(t, x(t)) + \mu, \quad t \in [0, T], \tag{4.14} $$

$$ x(0) = \xi, \tag{4.15} $$

where $\mu \in \mathbb{R}^n$, which we call a control parameter, and $\xi \in \mathcal{D}_\xi$.

Theorem 4.2. *Let $\xi \in \mathcal{D}_\xi$, $\eta \in \mathcal{D}_\eta$ and $\mu \in \mathbb{R}^n$ be given vectors. Assume that all conditions of Theorem 4.1 are satisfied for the FDS* (3.1).

Then the solution $x = x(\cdot, \xi, \eta, \mu)$ of IVP (4.14), (4.15) *also satisfies boundary conditions* (3.3) *if and only if*

$$ \mu = \Delta(\xi, \eta), \tag{4.16} $$

where $\Delta(\xi, \eta)$ is given by (4.12)*, and in this case*

$$ x(t, \xi, \eta, \mu) = x_\infty(t, \xi, \eta), \text{ for all } t \in [0, T]. \tag{4.17} $$

Theorem 4.3. *Assume that BVP* (3.1), (3.2) *satisfies conditions* (A1)–(A3). *Then $x_\infty(\cdot, \xi^*, \eta^*)$ is a solution to the FDS* (3.1) *with nonlinear boundary conditions* (3.2) *if and only if the point (ξ^*, η^*) is a solution to the determining system*

$$ \begin{gathered} \Delta(\xi^*, \eta^*) = \frac{1}{T^p}\Big[\Gamma(p+1)(\eta - \xi) - p\int_0^T (T-s)^{p-1} N_f x_\infty(s, \xi, \eta) ds\Big] = 0, \\ \phi(x_\infty) = \gamma, \end{gathered} \tag{4.18} $$

where $\Delta(\xi, \eta)$ is given by (4.12) *and the second equation comes from the non-local boundary condition* (3.2).

Since the proofs of Theorems 4.1-4.3 overlap with their analogues in [9, 12] and do not contain any new techniques, we leave them to the reader. In the next section we modify iterations (4.6) by additionally interpolating them via the Lagrange polynomials, constructed at the Chebyshev nodes. One of the main advantages of this approach is in the polynomial form of the approximate solution that is easier to analyze further.

4.2. **Polynomial interpolation method and its convergence.** Assuming that the FBVP (3.1), (3.3) satisfies conditions (A1)–(A3), let us fix a natural number q and introduce a modified approximation scheme:

$$ \begin{gathered} u_0^q(t, \xi, \eta) := L^q x_0(t, \xi, \eta), \\ u_m^q(t, \xi, \eta) := u_0^q(t, \xi, \eta) + (\Lambda^q N_f u_{m-1}^q(\cdot, \xi, \eta))(t), \quad t \in [0, T],\ m \in \mathbb{N}, \end{gathered} \tag{4.19} $$

with

$$ (\Lambda^q y)(t) := \frac{1}{\Gamma(p)} L^q\Big(\int_0^t \Big((t-s)^{p-1}\Big[y(s) - \frac{1}{T^p}\int_0^T (T-\tau)^{p-1} y(\tau) d\tau\Big]\Big) ds\Big), \tag{4.20} $$

where $t \in [0, T]$, $x_0(t, \xi, \eta)$ is defined by (4.6), and $L^q(\cdot)$ are the Lagrange polynomials (2.9) satisfying a component-wise property (2.7).

Similarly to (2.9), for any continuous vector-function $y : [0, T] \to \mathbb{R}^n$ we put

$$ E_q y := \operatorname{col}(E_q y_1, E_q y_2, \dots, E_q y_n). $$

Additionally, if $D \subset \mathbb{R}^n$ is a closed domain and $f : [0, T] \times D \to \mathbb{R}^n$, then

$$ l_{q,D}(f) := \Big(\frac{2}{\pi}\ln q + 1\Big) \sup_{p \in \mathcal{P}_{q,D}} E_q(N_f p), \tag{4.21} $$

where

$$\mathcal{P}_{q,D} := \{y : y \in \mathcal{P}_q^n, y([0,T]) \subset D\}, \quad \mathcal{P}_q^n := \underbrace{\mathcal{P}_q \times \cdots \times \mathcal{P}_q}_{n}.$$

Let us prove convergence of the iterative scheme (4.19) and establish connection between the original approximations (4.6) and the modified sequences of functions (4.19).

Theorem 4.4. *Let there exist a non-negative vector*

$$\rho_q := \frac{T^p M}{2^{p-1}\Gamma(p+1)}\Big[\frac{12}{q^p}\Big(\frac{2}{\pi}\ln q + 1\Big) + 1\Big], \tag{4.22}$$

such that for all fixed $y \in \mathcal{D}_{\rho_q}$ it holds that

$$f \in \mathrm{Lip}(K, \mathcal{D}_{\rho_q}).$$

Then, for all fixed $(\xi,\eta) \in \mathcal{D}_\xi \times \mathcal{D}_\eta$ and $q \in \mathbb{N}$ we have:

(1) *For any $m \in \mathbb{N}_0$, function $u_m^q(t,\xi,\eta)$ in (4.19) is a vector polynomial of degree q having values in $\mathcal{D}_{\rho_q}$ and satisfying the parametrized boundary conditions*

$$\lim_{q\to\infty} u_m^q(0,\xi,\eta) = \xi, \quad \lim_{q\to\infty} u_m^q(T,\xi,\eta) = \eta.$$

(2) *The sequences of functions $\{u_m^q(t,\xi,\eta)\}$ and $\{u_\infty^q(t,\xi,\eta)\}$ converge uniformly to their corresponding limit functions, as $m,q\to\infty$:*

$$\begin{gathered} u_\infty^q(t,\xi,\eta) := \lim_{m\to\infty} u_m^q(t,\xi,\eta), \\ u_\infty(t,\xi,\eta) = x_\infty(t,\xi,\eta) := \lim_{q\to\infty} u_\infty^q(t,\xi,\eta) \end{gathered} \tag{4.23}$$

for each $t \in [0,T]$ and $m \in \mathbb{N}_0$, where $x_\infty(t,\xi,\eta)$ is defined by (4.9).

(3) *The limit function $u_\infty(t,\xi,\eta)$ satisfies the parametrized two-point boundary constraints*

$$u_\infty(0,\xi,\eta) = \xi, \quad u_\infty(T,\xi,\eta) = \eta.$$

(4) *The estimate*

$$\begin{gathered} |x_\infty(t,\xi,\eta) - u_m^q(t,\xi,\eta)| \le \frac{T^p}{2^{2p-1}\Gamma(p+1)} Q^m (I_n - Q)^{-1} M \\ +(I_n - Q)^{-1} Big(l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)} M\Big) + Q^{m+1} l_q(x_0) \end{gathered} \tag{4.24}$$

holds, for all $t \in [0,T]$ and $m \in \mathbb{N}_0$, where $x_\infty(t,\xi,\eta)$ and $u_m^q(t,\xi,\eta)$ are given by (4.9) and (4.19) respectively.

Proof. Let us fix values of parameters $\xi \in \mathcal{D}_\xi$ and $\eta \in \mathcal{D}_\eta$ and show that

$$\{u_m^q(t,\xi,\eta) : t \in [0,T]\} \subset \mathcal{D}_{\rho_q}. \tag{4.25}$$

From the first expression in (4.19) it follows that (4.25) holds for $m = 0$. Let us now prove (4.25) for all $m \ge 1$. For this purpose we first derive some intermediate estimates.

By putting $v := N_f u_{m-1}^q$ we compute

$$|(\Lambda v)(t) - (\Lambda^q v)(t)| = |(\Lambda v)(t) - (L^q)(\Lambda v)(t)| \le 6\Big(\frac{2}{\pi}\ln q + 1\Big)\omega\Big(\Lambda v; \frac{T}{2q}\Big) \tag{4.26}$$

Next we evaluate the difference $|\Lambda v(t_2) - \Lambda v(t_1)|$, for $t_1, t_2 \in [0, T]$, $t_1 < t_2$:

$$
\begin{aligned}
&|\Lambda v(t_2) - \Lambda v(t_1)| \\
&\le \Big|\frac{1}{\Gamma(p)}\int_0^{t_2}(t_2-s)^{p-1}v(s)ds - \frac{1}{\Gamma(p)}\int_0^{t_1}(t_1-s)^{p-1}v(s))ds\Big| \\
&\quad + \Big|\frac{t_2^p - t_1^p}{T^p}\int_0^T (T-\tau)^{p-1}v(\tau)d\tau\Big|ds \\
&\le \frac{1}{\Gamma(p)}\Big|\int_{t_1}^{t_2}(t_2-s)^{p-1}v(s)ds\Big| + \frac{1}{\Gamma(p)}\Big|\int_0^{t_1}\big((t_2-s)^{p-1} - (t_1-s)^{p-1}\big)v(s)ds\Big| \\
&\quad + \frac{t_2^p - t_1^p}{\Gamma(p+1)}\|v\|_0 \\
&\le 2\frac{(t_2-t_1)^p + (t_2^p - t_1^p)}{\Gamma(p+1)}\|v\|_0 \\
&\le 4\frac{(t_2-t_1)^p}{\Gamma(p+1)}\|v\|_0,
\end{aligned}
$$

since

$$t_2^p - t_1^p \le (t_2 - t_1)^p.$$

This leads to the following estimate of the term $\omega(\Lambda v; \frac{T}{2q})$ in (4.26):

$$\omega\Big(\Lambda v; \frac{T}{2q}\Big) \le 4\frac{T^p}{2^p q^p \Gamma(p+1)}\|v\|_0.$$

Consequently, (4.26) gives

$$\|\Lambda v - \Lambda^q v\|_0 \le 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}\|v\|_0, \tag{4.27}$$

that is

$$\|\Lambda - \Lambda^q\| \le 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}.$$

Now, from the iterative formula (4.19), estimates (4.27) and (2.2), we obtain

$$
\begin{aligned}
|u_m^q - u_0^q| &= |\Lambda^q N_f u_{m-1}^q| \le |\Lambda^q N_f u_{m-1}^q - \Lambda N_f u_{m-1}^q| + |\Lambda N_f u_{m-1}| \\
&= |(\Lambda^q - \Lambda)N_f u_{m-1}| + |\Lambda N_f u_{m-1}| \\
&\le 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M + \frac{T^p}{2^{p-1}\Gamma(p+1)}M \\
&= \frac{T^p M}{2^{p-1}\Gamma(p+1)}\Big[\frac{12}{q^p}\Big(\frac{2}{\pi}\ln q + 1\Big) + 1\Big] = \rho_q.
\end{aligned} \tag{4.28}
$$

Taking into account definition (4.3) for the set $\mathcal{D}_{\rho_q}$, inequality (4.28) proves that (4.25) holds, for all $m \in \mathbb{N}_0$.

To prove that iterations (4.19) converge to (4.9) as q tends to infinity let us estimate a difference,

$$
\begin{aligned}
&|x_{m+1} - u_{m+1}^q| \\
&\le |x_0 - u_0^q| + |\Lambda N_f x_m - \Lambda^q N_f u_m^q| \\
&\le |x_0 - u_0^q| + |\Lambda(N_f x_m - N_f u_m^q)| + |(\Lambda - \Lambda^q)N_f u_m^q| \\
&\le l_q(x_0) + K\mathcal{I}(|x_m - u_m^q|) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M,
\end{aligned} \tag{4.29}
$$

where we used inequality (4.27), notation (4.21) and Lemma 2.2 for

$$|\Lambda(N_f x_m - N_f u_m^q)| \le \mathcal{I}(|N_f x_m - N_f u_m^q|) \le K\mathcal{I}(|x_m - u_m^q|).$$

The method of mathematical induction leads to the following results:

- for $m = 0$:

$$\begin{aligned}|x_1 - u_1^q| &\le l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M + K l_q(x_0)\alpha_1(t)\\ &\le l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M + Q l_q(x_0);\end{aligned}$$

- for $m = 1$:

$$\begin{aligned}|x_2 - u_2^q| &\le l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M\\ &\quad + K\Big\{l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M\Big\}\alpha_1(t) + K^2 l_q(x_0)\alpha_2(t)\\ &\le Q^0\Big\{l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M\Big\}\\ &\quad + Q\Big\{l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M\Big\} + Q^2 l_q(x_0);\end{aligned}$$

- and thus, for a general m:

$$|x_{m+1} - u_{m+1}^q| \le (I_n - Q)^{-1}\Big(l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M\Big) + Q^{m+1} l_q(x_0).$$

Coupling the estimates above with inequality (4.13), we obtain

$$\begin{aligned}&|x_\infty(t,\xi,\eta) - u_m^q(t,\xi,\eta)|\\ &= |x_\infty(t,\xi,\eta) - x_m(t,\xi,\eta)| + |x_m(t,\xi,\eta) - u_m^q(t,\xi,\eta)|\\ &\le \frac{T^p}{2^{2p-1}\Gamma(p+1)}Q^m(I_n - Q)^{-1}M\\ &\quad + (I_n - Q)^{-1}\Big(l_q(x_0) + 24\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M\Big) + Q^{m+1} l_q(x_0).\end{aligned}$$

For large q from the second term in the inequality above we obtain

$$\lim_{q\to\infty}\Big(\frac{2}{\pi}\ln q + 1\Big)\frac{T^p}{2^p q^p \Gamma(p+1)}M = 0,$$

assuming conditions (A1)–(A3) and Proposition 2.6 hold. Thus, (4.19) is convergent to (3.8) as $m \to \infty$ and its limit for $q \to \infty$ is near the solution $x(t) = x_\infty(t, \xi^*, \eta^*)$ of the original equation. This completes the proof. □

Since, according to Theorem 4.4, the limit functions of the sequences $x_m(\cdot,\xi,\eta)$ and $u_m^q(\cdot,\xi,\eta)$ converge to the same limit, Theorem 4.3 holds also for the iteration process (4.19) and reads as follows.

Theorem 4.5. *Let the non-local BVP* (3.1), (3.2) *satisfy conditions* (A1)–(A3). *Then $u_\infty(\cdot,\xi^*,\eta^*)$ is a solution to the FDS* (3.1) *with non-local boundary conditions* (3.2), *if and only if the point (ξ^*,η^*) is a solution to the determining system*

$$\Delta^q(\xi^*,\eta^*) = 0, \tag{4.30}$$

$$\phi(u_\infty) = \gamma, \tag{4.31}$$

where

$$\Delta^q(\xi^*,\eta^*) = \frac{1}{T^p}\Big[\Gamma(p+1)(\eta-\xi) - p\int_0^T (T-s)^{p-1} N_f u_\infty(\cdot,\xi,\eta))(s)ds\Big],$$

and the second equation comes from the boundary condition (3.2)*.*

Remark 4.6. (1) For practical reasons, instead of the exact system (4.30), (4.31) we introduce an approximate determining system of the form

$$\begin{aligned} \Delta_m^q(\xi,\eta) &= 0, \\ \phi(u_m^q) &= \gamma, \end{aligned} \tag{4.32}$$

where the map $\Delta_m^q : D_\xi \times D_\eta \to \mathbb{R}^n$ is defined as

$$\Delta_m^q(\xi,\eta) := \frac{1}{T^p}\Big[\Gamma(p+1)(\eta-\xi) - p\int_0^T (T-s)^{p-1} N_f^q u_m(\cdot,\xi,\eta)(s)ds\Big].$$

(2) On every iteration step m, solutions $(\bar\xi_m,\bar\eta_m)$ of the system (4.32) stand for the m-th approximation to their exact values (ξ^*,η^*). By plugging $(\bar\xi_m,\bar\eta_m)$ into the sequence of functions (4.19) we obtain the m-th approximation to the exact solution of the parametrized problem (3.1), (3.3), or of the original non-local problem (3.1), (3.2). This solution is then given by

$$X_m^q(t) = u_m^q(t,\bar\xi_m,\bar\eta_m).$$

5. Predator-prey model with prey refuge

Predator-prey models are often used in modeling of biological and ecological dynamical systems. They are usually written in the form of systems of ordinary differential equations and are driven by a set of parameters that characterize a physical system under consideration.

Recently the predator-prey models were modified to incorporate also a memory effect. For this purpose the integer order derivatives were replaced by their fractional analogues. In this Section we apply the iteration technique, developed earlier, to find an approximate solution to a fractional order predator-prey model with prey refuge. For details about this model we refer the reader to [7].

We consider a dynamical system

$$\begin{aligned} {}_0^C D_t^p x(t) &= rx\left(1-\frac{x}{k}\right) - c(1-m)xy, \\ {}_0^C D_t^p y(t) &= ec(1-m)xy - dy, \end{aligned} \tag{5.1}$$

for $t\in[0,T]$ and $p\in(0,1)$, where t stands for time, $x(t)$, $y(t)$ correspond to the predator and prey densities, and r,k,m,e,d are positive constant parameters, driving the system.

Under the set of physically relevant parameter values $T=4/5$, $r=1.2$, $k=40$, $c=1$, $d=0.4$, $e=0.2$, $m=0.1$, and $p=0.98$, the FDS (5.1) is written as

$$\begin{aligned} {}_0^C D_t^{0.98} x(t) &= 1.2x\left(1-\frac{x}{40}\right) - 0.9xy, \\ {}_0^C D_t^{0.98} y(t) &= 0.18xy - 0.4y, \end{aligned} \tag{5.2}$$

for $t\in[0,4/5]$.

We aim to construct an approximate solution to system (5.2) that satisfies the non-local boundary conditions

$$\begin{gathered} x^2(0) - x\big(\frac{4}{5}\big)y(0) = 2.54, \\ y^2\big(\frac{4}{5}\big)x(0) = 2.38 \end{gathered} \tag{5.3}$$

in the domain

$$\mathcal{D}_\rho = \{(x,y) : 2.2 \le x \le 3.1,\ 1 \le y \le 1.5\}. \tag{5.4}$$

The parametrized constraints (3.3) in this case have the form:

$$\begin{gathered} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} := \xi, \\ \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} := \eta \end{gathered} \tag{5.5}$$

with

$$\begin{gathered} \xi \in \mathcal{D}_\xi := \{\xi \in \mathbb{R}^2 : 2.2 \le \xi \le 2.5\}, \\ \eta \in \mathcal{D}_\eta := \{\eta \in \mathbb{R}^2 : 0.5 \le \eta \le 1.5\}. \end{gathered}$$

Calculations show that conditions (A1)–(A3) hold with

$$\begin{gathered} M = \begin{bmatrix} 0.6417 \\ 0.237 \end{bmatrix}, \quad K = \begin{bmatrix} 1.736 & 2.79 \\ 0.27 & 0.958 \end{bmatrix}, \\ \rho^q = \begin{bmatrix} 1.819303410 \\ 0.1649011626 \end{bmatrix}, \quad r\Big(Q = \frac{KT^p}{2^{2p-1}\Gamma(p+1)}\Big) < 0.96. \end{gathered}$$

Let us now use a modified version of the numerical-analytic method (4.6), where the Lagrange polynomial interpolation (LPI) scheme is applied to obtain approximate solutions to the problem in the form of the q-th order polynomials. For this purpose we fix the order of interpolation at $q = 4$ and compute the Chebyshev nodes (2.8),

$$\begin{gathered} t_1 = 0.7804226065, \quad t_2 = 0.6351141009, \quad t_3 = 0.4, \\ t_4 = 0.1648858991, \quad t_5 = 0.0195773935. \end{gathered}$$

On every iteration step we find numerical values of the unknown parameters as roots of the approximate determining system (4.32). These values are given in Table 1 and they determine approximate solutions to the problem (5.2), (5.3).

TABLE 1. Numerical values of the unknown parameters for $m = 0, 1, 2, 3$

Approx.	Numerical value of ξ	Approx.	Numerical value of η
$\xi_{1,0}$	2.303032253	$\eta_{1,0}$	2.928500653
$\xi_{1,1}$	2.308072335	$\eta_{1,1}$	2.971104555
$\xi_{1,2}$	2.308081308	$\eta_{1,2}$	2.970853059
$\xi_{1,3}$	2.308127313	$\eta_{1,3}$	2.971164378
$\xi_{2,0}$	0.9445718795	$\eta_{2,0}$	1.016462968
$\xi_{2,1}$	0.9389864677	$\eta_{2,1}$	1.015336111
$\xi_{2,2}$	0.9390368098	$\eta_{2,2}$	1.015339844
$\xi_{2,3}$	0.9390134110	$\eta_{2,3}$	1.015328673

Below we also give explicit forms of these solutions and plot the estimates of our computations. On the zero-th iteration step we obtain the following approximation to the exact solution of the FBVP (5.2), (5.3):

$$\begin{aligned} X_0^q(t) &= -0.01956900249t^4 + 0.0549367687t^3 - 0.0646567198t^2 \\ &\quad + 0.654212958t + 2.303545749, \\ Y_0^q(t) &= -0.00224924845t^4 + 0.00631442252t^3 - 0.00743160017t^2 \\ &\quad + 0.0751949749t + 0.9446309006. \end{aligned}$$

A comparison of the left and right hand-sides of the system (5.2) is given on Figure 1.

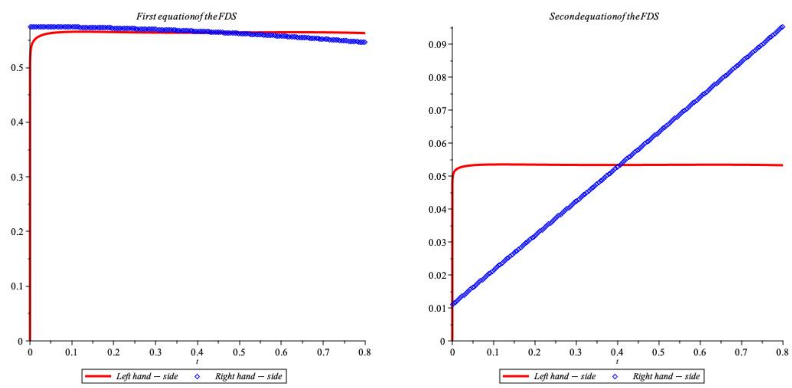

Figure 1. Comparison of the left and right hand-sides of (5.2) in the zero-th approximation (LPI method)

Computations show that the first approximation to the exact solution of the FBVP (5.2), (5.3) is given by a system of polynomials:

$$\begin{aligned} X_1^q(t) &= -0.02112540606t^4 + 0.04090489985t^3 - 0.0802533673t^2 \\ &\quad + 0.722904954t + 2.308639591, \\ Y_1^q(t) &= 0.00073293971t^4 + 0.0000320947t^3 + 0.05954191818t^2 \\ &\quad + 0.0160367660t + 0.9389937391. \end{aligned}$$

A comparison of the left and right hand-sides of the system (5.2) in the first approximation is given on Figure 2.

Note, that the higher order approximations show an even better accuracy of our computations. These results can be seen on Figures 3 and 4.

The error functions calculated on the second and the third iterations for the LPI method, are plotted on Figures 5 and 6.

We additionally depict all 4 approximations to the exact solution of FBVP (5.2), (5.3) (see Figure 7), from which the convergence behavior of our approximations follows straightaway.

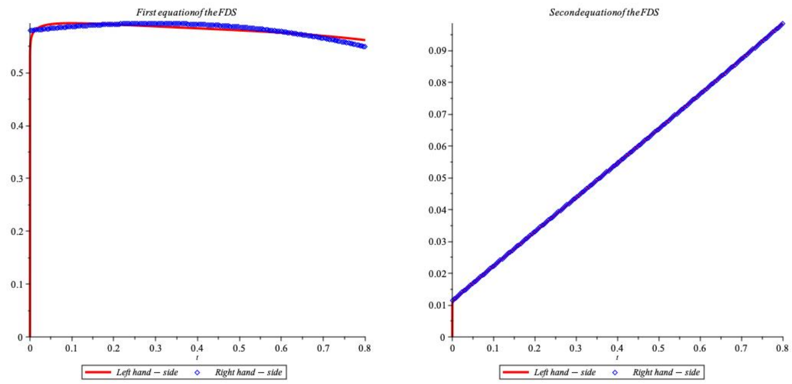

FIGURE 2. Comparison of the left and right hand-sides of (5.2) in the first approximation (LPI method)

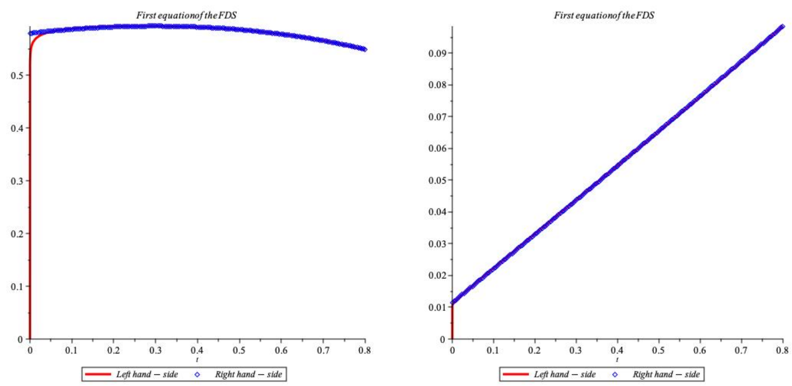

FIGURE 3. Comparison of the left and right hand-sides of (5.2) in the second approximation (LPI method)

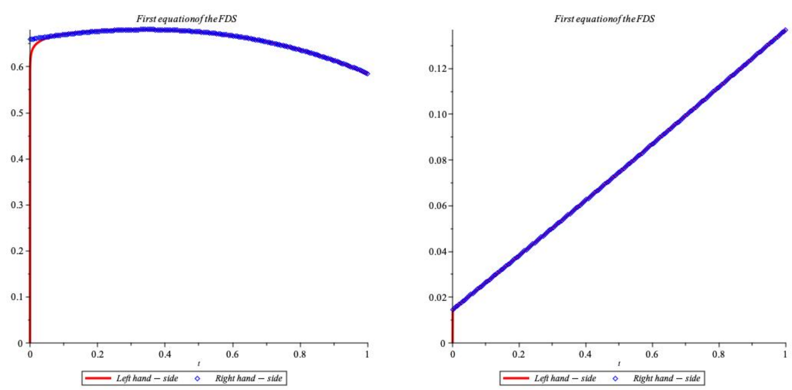

FIGURE 4. Comparison of the left and right hand-sides of (5.2) in the third approximation (LPI method)

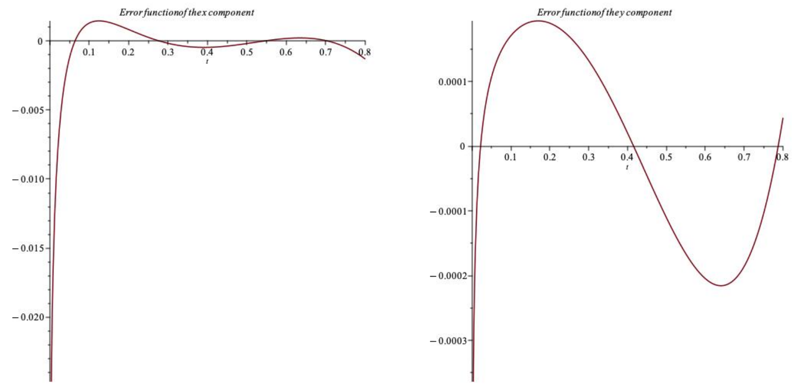

FIGURE 5. Error functions in the second approximation (using the LPI method)

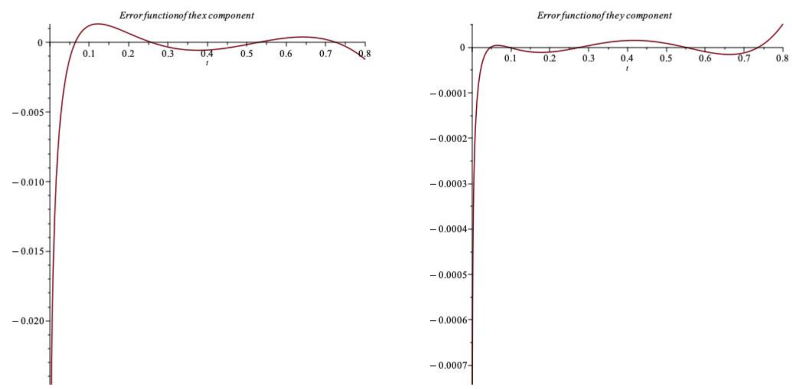

FIGURE 6. Error functions in the third approximation (LPI method)

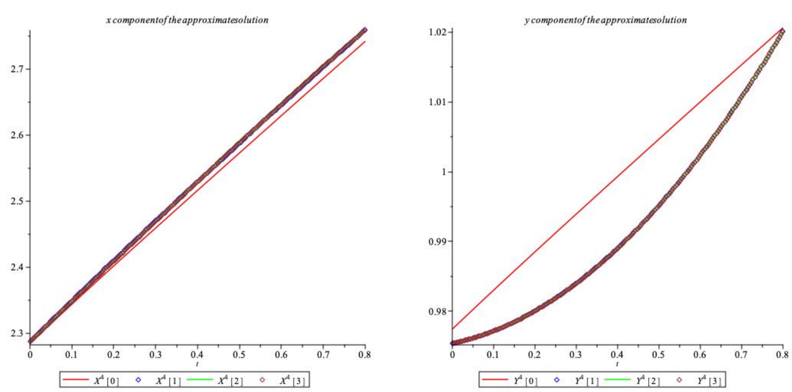

FIGURE 7. Approximations to the exact solution of (5.2), (5.3) for $m = 0, 1, 2, 3$, obtained by the LPI method

6. Conclusions

To summarize, we want to mention the main advantages of using the LPI method and possibilities for future developments:

(1) it is easier to analyse the behaviour of polynomial solutions;
(2) increasing the number of nodes q improves the speed of convergence;
(3) the method is easy to implement using mathematical software;
(4) one could think of interpolation of functions under the integrals as well. However, this approach needs other than Lagrange fundamental polynomials due to a necessity to control the growth of the polynomial for large q.

Acknowledgments. This work is partially supported by the Slovak Grant Agency VEGA No. 2/0127/20 and No. 1/0084/23, and by the National Scholarship Program of the Slovak Republic.

References

[1] K. Diethelm; *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer, Berlin, 2010.
[2] M. Fečkan, K. Marynets; Approximation approach to periodic BVP for fractional differential systems, *Eur. Phys. J.:* Special Topics **226** (2017), 3681–3692.
[3] M. Fečkan, K. Marynets; Approximation approach to periodic BVP for mixed fractional differential systems, *J. Comput. Appl. Math.*, **339** (2018), 208–217.
[4] M. Fečkan, K. Marynets, J. R. Wang; Periodic boundary value problems for higher order fractional differential systems, *Math. Methods Appl. Sci.*, **42** (2019), 3616–3632.
[5] A. A. Kilbas, S. A. Marzan; Cauchy problem for a differential equation with Caputo derivative, *Fract. Calc. Appl. Anal.*, **7(3)** (2004), 297–321.
[6] A. Kilbas, H. Srivastava, J. Trujillo; *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, The Netherlands, 2006.
[7] H. L. Li, L. L. Zhang, C. Hu et al.; Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge, *J. Appl. Math. Comput.* **54** (2017), 435–449.
[8] K. Marynets; On one interpolation type fractional boundary–value problem, *Axioms: SI Fractional Calculus, Wavelets and Fractals*, **9(1)** (2020), 1–21.
[9] K. Marynets, D. Pantova; Approximation approach to the fractional BVP with the Dirichlet type boundary conditions. *Differ. Equ. Dyn. Syst.* (2022).
[10] K. Marynets, D. Pantova; Successive approximations and interval halving for fractional BVPs with integral boundary conditions, *J. of Comp. and Appl. Math.*, (2023), 1–20.
[11] K. Marynets; Solvability analysis of a special type fractional differential system, *Comput. Appl. Math.* (2019), 13p.
[12] K. Marynets; Successive approximations technique in study of a nonlinear fractional boundary value problem, *Mathematics* **9(7)** (2021), 1–19.
[13] I. P. Natanson; *Constructive function theory. Vol. III. Interpolation and approximation quadratures*, New York: Frederick Ungar Publishing Co., 1965.
[14] G. M. Phillips; *Interpolation and approximation by polynomials*, CMS Books in Mathematics, Vol. 14, Springer-Verlag, New York, 2003.
[15] I. Podlubny; *Fractional differential equations*, Academic Press, New York, 1999.
[16] T. J. Rivlin; *An introduction to the approximation of functions*, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-Lindon, 1969.
[17] T. J. Rivlin; *The Chebyshev polynomials*, Wiley-Interscience, NY-London-Syndey, 1974.
[18] A. Rontó, M. Rontó, N. Shchobak; Parametrization for boundary value problems with transcendental non-linearities using polynomial interpolation, *Electron. J. Qual. Theory Differ. Equ.*, **(2018)** No. 59 (2018), 1–22.
[19] M. I. Ronto, A. M. Samoilenko; *Numerical-analytic methods in the theory of boundary value problem*, World Scientific Publishing Co., Inc., River Edge, 2000.
[20] T. Sandev, Ž. Tomovski, J. L. A. Dubbeldam at al.; Generalized diffusion-wave equation with different memory kernels, *J. Phys. A: Math. Theor.*, **52(1)** (2019), 1–23.
[21] T. Sandev, Ž. Tomovski; *Fractional Equations and Models*, Springer Nature, 2019.

[22] Ž. Tomovski, J. L. A. Dubbeldam, J. Korbel; Applications of Hilfer-Prabhakar operator to option pricing financial model, *Fract. Calc. Appl. Anal.*, **23(4)** (2020), 996–1012.
[23] Ž. Tomovski; Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator, *Nonlinear Anal., Theory Methods Appl.*, **75(7)** (2012), 3364–3384.
[24] Y. Zhou; *Basic Theory of Fractional Differential Equations*, World Scientifc, Singapore (2014).

Michal Fečkan
Department of Mathematical Analysis and Numerical Analysis, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia.
Institute of Mathematics, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
Email address: `michal.feckan@fmph.uniba.sk`

Kateryna Marynets
Delft Institute of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4 2628CD Delft, Netherlands
Email address: `K.Marynets@tudelft.nl`