Bed-load formulae for non-uniform sediment

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Internal Report no. 4-78

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Contents

1. Introduction

This investigation is carried out in the framework of a research project on morphological computations for large ranges of grainsizes. It is then necessary to describe the transport of each sizefraction seperately.

This paper shows some of the major efforts which have been made to develop a bed-load formula for every fraction of the sediment mixture. The most general form of such a formula reads:

$$s_i = f_i(u, D_1, \ldots, D_i, \ldots D_n, p_1, \ldots, p_i, \ldots, p_{n-1}) \qquad (1)$$

in which: s_i = bed-load rate of fraction i in volume (including pores) per unit time and unit width

u = flow velocity

D_i = diameter of sediment fraction i

p_i = probability of sediment fraction i

n = number of fractions.

All the bed-load formulae per sediment fraction which are discussed here are deduced from one of the classical bed-load formulae for uniform sediment. These formulae, which have an empirical or stochastical-empirical character, are summarized below.

The formula of Kalinske (1947) results from stochastical considerations. In order to calculate the bed-load transport he uses parameters like the particle velocity and the probability of a particle being eroded from the bed.
Because of the turbulent watermovement he assumes a normally distributed water velocity at the bottom.
The bed-load formulae of Einstein (1950) also has a stochastical-empirical character. He assumes a normal distribution of the liftforce working on a sediment particle and includes the intermittent movement of the sediment particles in his considerations.
In both the formula of Kalinske and Einstein appear several correction coefficients which have been determined empirically.
The bed-load formula of Meyer-Peter and Müller (1948) gives the bed-load transport of uniform sediment in the form of a relation between two dimensionless parameters. The constants in this formula have been determined out of many experiments. A general form of this formula is:

$$X = f(Y) \tag{2}$$

in which: X = dimensionless transportparameter
Y = dimensionless flowparameter.

These three bed-load formulae have been adapted more or less by other investigators for heterogeneous sediment.

In Table 1 a review is given of the different investigators who carried out these adaptions.

Basic bed-load formula	Adaption for heterogeneous sediment by	Remarks
Kalinske (1947)	Pantelopulos (1955,1957)	
Einstein (1950)	Ning Chien (1953)	"Large ranges of grain sizes"
Meyer-Peter & Müller (1948)	Egiazaroff (1965)	theoretical analysis of the critical shear stress of a fraction in a sediment mixture
	Antsyferov (1973)	
	Ashida & Michiue (1973)	correction of Egiazaroff's theorie and experimental verification of the bed-load formula.
	Suzuki (1976)	

Table 1

The basic bed-load formulae are compared in Fig. 1. The above-mentioned bed-load formulae for heterogeneous sediment will be treated in the next sections.

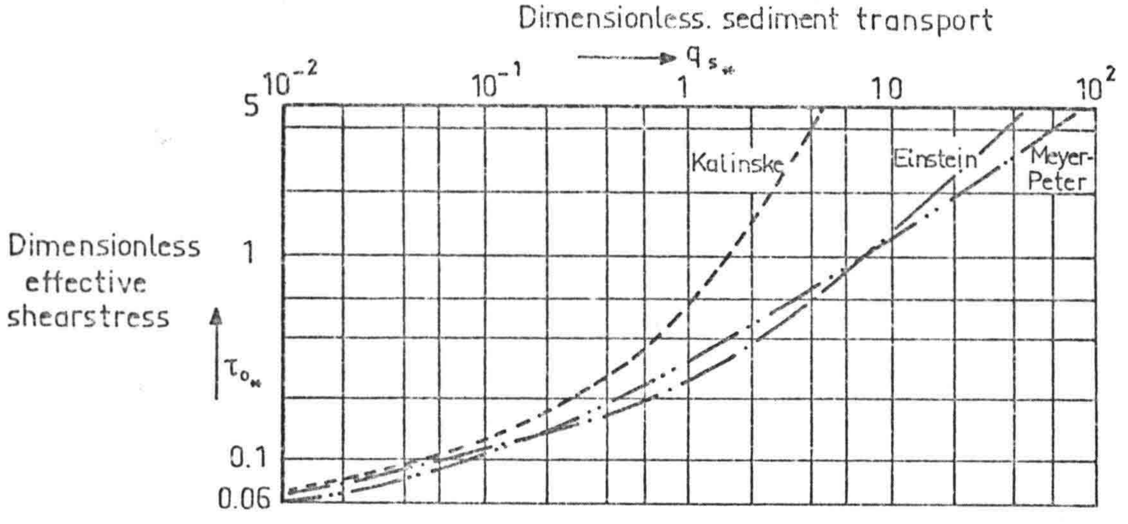

Fig. 1 Comparison of basic bed-load formulae (after Paintal, 1971)

2. Kalinske and Pantelopulos

The bed-load formula of Kalinske (1947) can be derived in a simple way. He considers the toplayer of a plane bed and assumes that the sediment particles are spheres with diameter D. Suppose P = percentage of a bed area occupied by grains. Then $P/(\frac{1}{4}\pi D^2)$ = number of grains per unit of bedarea. According to Kalinske the water velocity near the bottom is normally distributed:

$$f(u) = \frac{1}{2\pi\sigma_u} \cdot \exp\left(-\frac{(u-\bar{u})^2}{2\sigma_u^2}\right)$$

in which: σ_u^2 = variance of the water velocity near the bottom
$\bar{u}$ = mean velocity near the bottom.

The probability of a grain, with critical velocity u_c, of being eroded is:

$$\int_{u_c}^{\infty} f(u) \, . \, du$$

The number of grains that moves per unit of bedarea is then:

$$\frac{P}{\pi D^2/4} \int_{u_c}^{\infty} f(u) \, . \, du$$

Kalinske assumes an empirical relation for the grainvelocity during movement:

$$u_p = \overline{(u - u_c)}_{bottom}$$

The bed-load transport per unit time and width (real volume) becomes:

$$q_s = \frac{\pi D^3}{6} u_p \frac{P}{\pi D^2/4} \int_{u_c}^{\infty} f(u) du$$

$$\boxed{q_s = 2/3 \; P \; D \; \bar{u}_p} \qquad (3)$$

in which: $\bar{u}_p = u_p \int_{u_c}^{\infty} f(u) du$ = "mean grain velocity" (including rest periods)

This expression in combination with the normal distribution can be worked out to a general relationship:

$$\bar{u}p/u = f(\tau_c/\tau_e, \sigma_u/\bar{u}) \qquad (4)$$

in which: τ_c = critical shear stress of the sediment particle
τ_e = effective shear stress acting on the particles.

Combination of Eqs. (3), (4) and an empirical relation for τ_c gives, for different intensities of turbulence, the bedload formula of Kalinske.

Remark: For relatively large τ_e or water velocities and therefore large bedload transports this formula is incorrect, because Kalinske only considers one grainlayer, while in these circumstances more layers can be moving at the same time (Fig. 1).

Pantelopulos (1955, 1957) extends with an identical derivation Kalinske's formula to a bed-load formula per sedimentfraction.
In stead of Eq. (3) he finds for every fraction:

$$\boxed{q_{s_i} = 2/3 \; p(D_i) \; . \; \Delta D_i \; . \; D_i \; . \; \overline{u_p} \; (D_i)} \qquad (5)$$

in which: $p(D_i).\Delta D_i$ = part of a unit bedarea occupied by grains with a diameter between D_i and $D_i + \Delta D_i$

$$\overline{u_p}\ (D_i) = \overline{(u - u_c(D_i))}_{bottom} \cdot \int_{u_c(D_i)}^{\infty} f(u).du$$

In the same way as Kalinske did Pantelopulos finds:

$$\overline{u_p}\ (D_i)/\bar{u} = f(\tau_{c_i}/\tau_{\acute{e}}, \sigma_u/\bar{u}) \qquad (6)$$

Now Pantelopulos only needs an empirical relation for τ_{c_i}, which, however, was not available.

He carries out some experiments and calculations and finds out that the critical shear stress of particles of a certain fraction in a sediment mixture can be entirely different from the critical shear stress of the same particles in the uniform case. (Fig. 2).

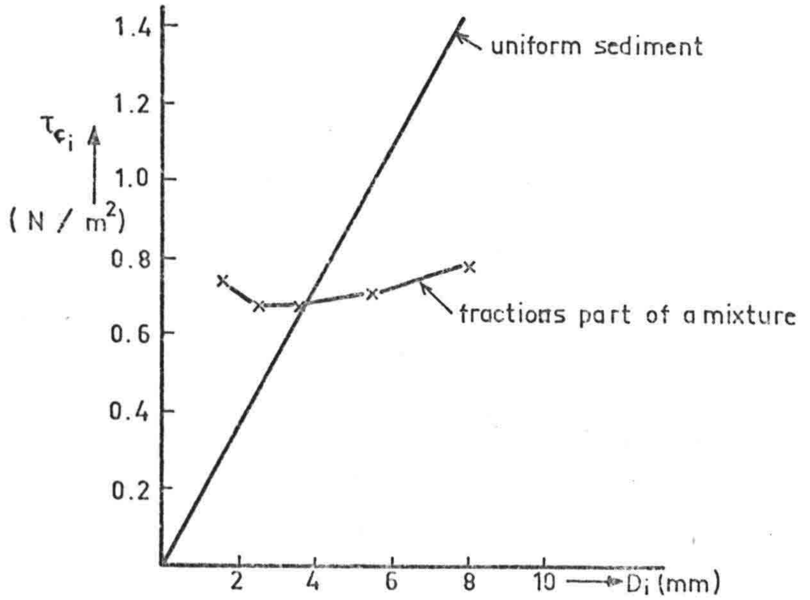

Fig. 2 Critical shear stresses of particles, as well as part of a mixture as in the uniform case (after Pantelopulos, 1957).

It seems that in these experimental circumstances the critical shear stress of the larger particles is decreasing and of the smaller particles is increasing when compared with the uniform case.
The experimental results show that the critical shear stresses of all the fractions nearly have the same value. This value can be estimated by the mean of the values calculated with a formula of Kalinske-White, in which the critical shear stress for uniform grains is linearly dependent of the graindiameter ($\tau_c = a(\rho_s - \rho) g D_i$).
The conclusion that this procedure is correct for every sediment mixture may not be justified. The sediment mixtures used by Pantelopulos are shown in Fig. 3.

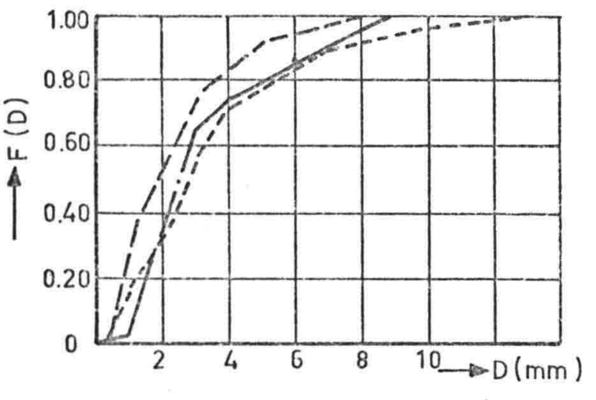

Fig. 3 Sediment mixtures used by Pantelopulos

Remark: In the experiments Pantelopulos did not directly measure the critical shear stress for every fraction. He only measured the bedload transport per fraction and then determined τ_{c_i} in such a way that the bedload formula (Eqs. (5),(6)) gave the right value.

Because of the restricted experimental verification it is hardly possible to draw general conclusions about this bed-load formula per sediment fraction of Pantelopulos.

3. Einstein

The bed-load formula of Einstein (1950) is derived by equaling the number of sediment particles being deposited per unit time and bedarea (n_a) to the number of particles being eroded per unit time and bedarea (n_e).
In the expression of n_d Einstein uses the mean steplength of the particles. This is according to Einstein a constant times the grain diameter ($A_L D_i$). He finds an expression for this steplength by using a parameter p, which is the probability of erosion. If this probability is large as a result of hydraulic conditions the steplength of the particles also will be large. The parameter p can also be found in the expression for n_e.
Next Einstein states that p is also the probability of the liftforce on the particle being larger than its weight (under water):

$$p = \text{probability of } L/W > 1$$

in which: $W = g(\rho_s - \rho)A_2 D_i^3$ = particle weight under water
$L = c_L \cdot \rho \cdot \frac{1}{2} u^2 A_1 D_i^2$ = lift force
c_L = liftcoefficient.

In finding an expression for the liftforce Einstein introduces a number of correction coefficients and figures. The roughness of the bedsurface and the frequency of a certain fraction are important parameters. The most important coefficient is the hiding factor ξ, which compensates the liftforce for the phenomenon of "hiding of smaller particles behind larger ones".
Because of earlier experimental results Einstein assumes that the liftforce is fluctuating according to a normal distribution.
This seems to contradict the assumption of Kalinske that the water velocity near the bottom is normally distributed. The bed-load formula of Einstein which already gives the bedload transport per fraction can be written as follows:

$$\boxed{\frac{q_{s_i}}{\sqrt{\Delta g D_i^3}} = \frac{p_i}{A_x} \frac{p}{1-p}} \qquad (7)$$

For the probability of erosion p Einstein finds the expression:

$$\boxed{p = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_x \psi_x - 1/\eta_o}^{B_x \psi_x + 1/\eta_o} \exp(-t^2)dt} \qquad (8)$$

According to Einstein A_x, B_x and η_o are universal constants, which have to be determined empirically. As a result of experiments with uniform sediment he finds A_x = 27,0 and B_x = 0,156. For η_o he finds: η_o = 0,5. The dimensionless flow parameter ψ_x is defined as:

$$\psi_x = \xi \, Y \, (\beta/\beta_x)^2 \psi \tag{9}$$

in which: $\psi = \frac{\Delta D_i}{Ri}$

with:
- i = energy slope
- R = hydraulic radius with respect to the grain

and:
- Y = pressure correction in transition smooth-rough
- $\beta/\beta_x = ({}^{10}\log 10{,}6)/({}^{10}\log 10{,}6X/\Delta')$
- X = characteristic grain size of the mixture
- Δ' = apparent roughness diameter.

Einstein and Ning Chien (1953) carry out experiments with "large-range" mixtures (= mixtures with a large range of grainsizes). In this way they find empirically modified values of the hiding factor ξ.
An important effect on ξ is caused by a phenomenon called surface segregation. This is the accumulation of coarse particles underneath the fine grains. Einstein and Ning Chien also find alternative fine and coarse layers in the bed which they call bed stratification.
A disadvantage of Einstein's formula is the complex form in which it appears after combination of Eqs. (7), (8) and (9). The formula includes many correction coefficients which have to be found in different figures. Further only the hiding factor is corrected for large-range mixtures. The other coefficients A_x, B_x, η_o, Y and Δ' are determined in experiments with uniform sediment.
In the next chapter the simplest form of a bedload formula per size fraction is given, which can be derived from the Einstein formula.

4. Basic-hypothesis

Under certain conditions a basic hypothesis can be derived from Einstein's bedload formula. Expression (7) can be written as:

$$q_{s_i} = p_i \frac{\sqrt{\Delta g D_i^3}}{A_x} \frac{p}{1-p}$$

or

$$s_i = \frac{q_{s_i}}{C_o} = p_i \underbrace{\frac{\sqrt{\Delta g D_i^3}}{C_o . A_x} \frac{p}{1-p}}_{I}$$

in which: $C_o = 1 - \varepsilon_o$ with ε_o = porosity of sediment.

With the assumption that the underlined term I is only a function of D_i and u (but not of p_i) the basic hypothesis results. If $p_i = 1$ term I is equal to the sediment transport s_i of fraction i in the uniform case. In a general form this basic hypothesis sounds:

$$\boxed{s_i = f_i(u, D_1, \ldots, D_i, \ldots D_n, p_1, \ldots, p_i, \ldots p_{n-1}) = p_i . f'(u, D_i)} \qquad (10)$$

This simplified bedload formula per fraction assumes that the bedload transport of fraction i is a linear function of the probability of fraction i. It also means that the different fractions are transported independently. In Fig. 4 this basic hypothesis is shown in case of two sediment fractions and a constant flow velocity u.

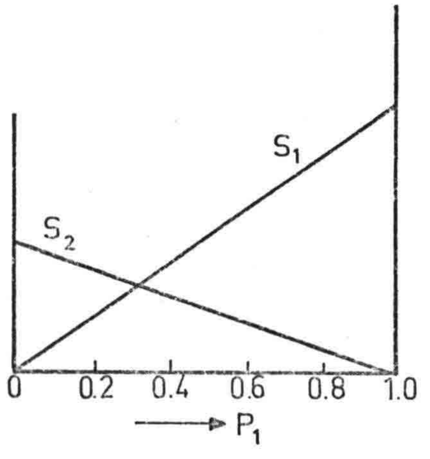

Fig. 4 Basic hypothesis for two fractions

Despite the simplification of this formula Antsyferov (1973) uses it in experiments with heterogeneous sediment (0.1 - 5.0 mm). According to the experimental results he substitutes for the function $f'(u,D_i)$ a formula similar to the transport formula of Engelund and Hansen.
Combination of Eq. (10) with the E.H.-formula gives:

$$s_i = p_i[0.084 \frac{\sqrt{g}}{\Delta^2} (\mu R_b i)^{5/2} \frac{1}{D_i}]$$

The dimensionless form becomes:

$$\boxed{\frac{s_i}{\sqrt{\Delta g D_i^3}} = p_i \times 0.084 \left(\frac{\mu R_b i}{\Delta D_i}\right)^{5/2}} \qquad (11)$$

in which: $\mu = (C_t^2/g)^{2/5}$

C_t = Chézy-coefficient of bedforms and grains.

The total shear stress multiplied by μ gives the effective shear stress (= the part of the total shear stress working on the grains).

Using this formula there are two possibilities:

(i) The righthandside of Eq. (11) except p_i is independent of p_i; Expression (11) is in agreement with the basic hypothesis.

(ii) Factor μ is a function of p_i. A change in the composition of the bed (p_i) may change the bedform and the roughness of the grains and hence μ will be influenced. The form of this function $\mu(p_i)$ is unknown. Formula (11) then deviates from the basic hypothesis.

The basic hypothesis can also be combined with a typical bedload formula like Meyer-Peter and Müller (1948). This formula gives (like the formula of Engelund and Hansen) a relation between two dimensionless parameters:

$$X = 13.3 \ (Y^{-1} - 0.047)^{3/2}$$

in which: $X = s/\sqrt{\Delta g D^3}$ = sediment transportparameter

$Y = \Delta D/\mu R_b i$ = flow parameter

The inverse value of Y is sometimes called: the dimensionless effective shear stress: $\tau_{e*} = Y^{-1}$.

According to Meyer-Peter and Müller the value 0.047 must not be interpreted as a dimensionless critical shear stress. Factor μ is now defined as:

$$\mu = (C_t/C_g)^{3/2}$$

in which: C_g = Chézy-coefficient of the grains.

The bedload formula per fraction becomes:

$$\boxed{\frac{s_i}{\sqrt{\Delta g D_i^3}} = p_i \left[13.3 \ \left(\frac{\mu R_b i}{\Delta D_i} - 0.047\right)^{3/2}\right]} \qquad (12)$$

Just as before there are two possibilities:
(i) μ is not a function of p_i (basic hypothesis)
(ii) μ is a function of p_i.

The second possibility - probably the best one - has the disadvantage that the relation $\mu(p_i)$ is unknown.

Remarks

1. In practice the factor μ is partly used to match measurements and formula.
2. The next chapters will show that other investigators do not correct via μ or τ_{e^*} $(= \frac{\mu R b i}{\Delta D})$ but via the constant 0.047.
3. This bedload formula per fraction Eq. (12) has not been verified experimentally.

5. Egiazaroff's theory

Starting from physical considerations Egiazaroff (1965) derives an expression for the dimensionless critical shear stress of a grain ($D = D_i$) which is part of a mixture ($D_m = \sum_i p_i D_i$).
First of all he derives the same parameter for uniform sediment (Egiazaroff, 1957).
He considers the equilibrium of forces working on a spherical grain which is on the threshold of movement (Fig. 5).

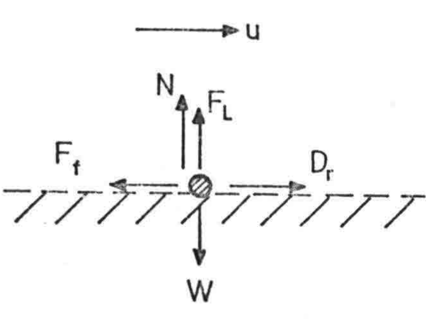

Fig. 5 Forces working on a spherical sedimentgrain on the threshold of movement.

Here: Dr = Dragforce

F_L = Liftforce

W = Weight of the particle under water

F_f = Friction force

N = Normal force = $W - F_L$.

According to Egiazaroff:

$$Dr = f \cdot N = f(W - F_L) = F_f \tag{13}$$

in which: f = friction coëfficient.

Equation (13) can be written as:

$$\frac{\pi D^2}{4} \cdot c_D \cdot \tfrac{1}{2} \rho u_o^2 = f\{\frac{\pi D^3}{6} g(\rho_s - \rho) - \frac{\pi D^2}{4} c_L \tfrac{1}{2} \rho u_o^2\} \tag{14}$$

in which: c_D = drag coefficient

c_L = lift coefficient

u_o = water velocity near the grain.

Egiazaroff introduces a factor ξ_o:

$$\xi_o = u_o/u \tag{15}$$

in which u = mean flow velocity.

For the critical shear stress he writes:

$$\tau_c = \lambda_o \cdot \tfrac{1}{2} \rho u^2 \tag{16}$$

Combination of Eqs. (14), (15) and (16) gives an expression for the dimensionless critical shear stress τ_{c_x}:

$$\tau_{c_x} = \frac{\tau_c}{g(\rho_s - \rho)D} = \frac{2}{3} \frac{f}{1 + f.c_L/c_D} \frac{\lambda_o}{c_D.\xi_o^2}$$

Egiazaroff neglects the liftforce by stating that:

$$f.c_L/c_D << 1$$

which gives as a result:

$$\tau_{c_*} = \frac{2}{3}\,\frac{\lambda_o}{c_D . \xi_o^2} \tag{18}$$

He assumes a logarithmic velocity profile near the bottom in the (vertical) z-direction:

$$u_o(z) = u_* \; 5.75 \; {}^{10}\log(30.2\; z/k_s)$$

in which: u_* = shear velocity
k_s = Nikuradse sand roughness = D.

Egiazaroff defines the bottom (z = 0) by assuming that for complete turbulence (c_D = 0.4) the dimensionless critical shear stress is equal to the Shields-value (τ_{c_*} = 0.06). He finds then z = 0.63 D as the point of application of the forces on the particle.

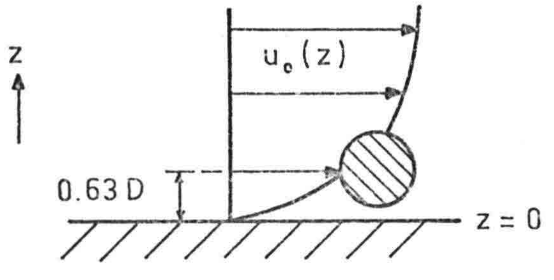

Fig. 6 Point of application of the forces on the particle.

Now Egiazaroff extends his theory to non-uniform particles. Important assumptions are:

(i) The point of application is now z = 0.63 D_i

(ii) The value of k_s is equal to the mean grain diameter of the mixture: $k_s = D_m$.

For complete turbulence Egiazaroff's result is:

$$\boxed{\tau_{c_{*i}} = \frac{\tau_{ci}}{(\rho_s - \rho) g D_i} = \frac{0{,}1}{({}^{10}\log\; 19 D_i/D_m)^2}} \tag{19}$$

in which: $\tau_{c_{*i}}$ = critical dimensionless shear stress of fraction i.

Egiazaroff verifies this relationship with experimental results of Pantelopulos (1957, see page 5) and others.

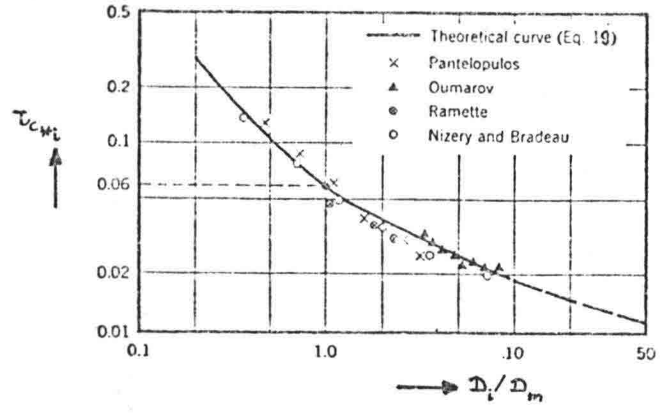

Fig. 7 Experimental verification of equation (19)

Egiazaroff concludes that experimental results and Eq. (19) are in good agreement.

Remarks:

1. Comparison of $\tau_{c_{x_i}}$ (calculated, Egiazaroff) and $\tau_{c_{x_i}}$ (measured, Pantelopulos (1957) shows that Egiazaroff must have multiplied the measured values with a factor 5 to get a good agreement. Egiazaroff does not mention this and it is unclear whether the definition of the "threshold of movement" has anything to do with this.
2. The way Egiazaroff chose the dimensionless variables along the axis in Fig. 7 includes the danger of a spurious correlation. A better comparison takes place in Fig. 8; here the non-dimensionless values of τ_{c_i} are shown as a function of D_i in the particular case of the experiments of Pantelopulos (1957).

In Fig. 8 it is also possible to compare the differences between Egiazaroff's calculated and Pantelopulos' measured values of τ_{c_i} with those in the uniform case.

These last values are determined in two ways:

1. calculated according Kalinske-White (see page 5)
2. measured by Pantelopulos (1957).

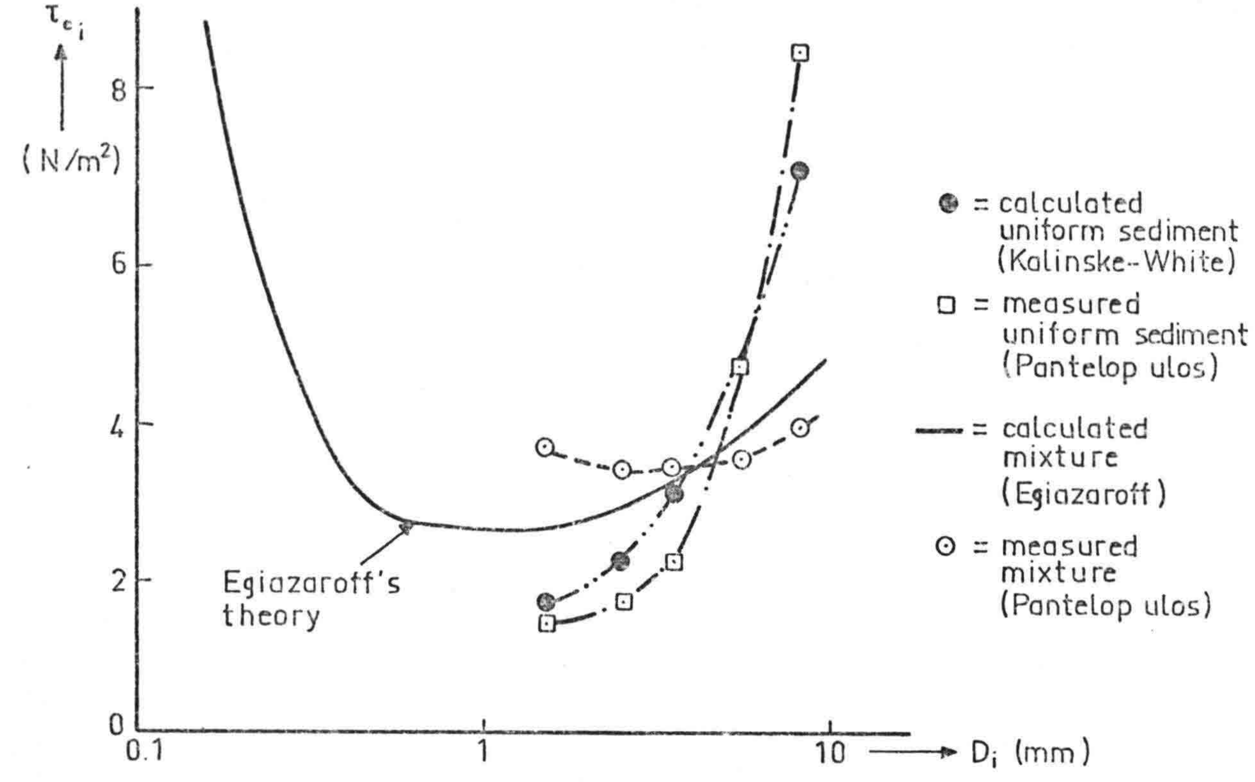

Fig. 8 Comparison between measured and calculated values of τ_{c_i} for non-uniform and uniform sediment

The factor 5, mentioned above, has been used for all the values in Fig. 8, except for the theoretical ones of Egiazaroff. It can be seen in this figure that Egiazaroff's theory has only been tested in a restricted area. Egiazaroff's line is steeper than the nearly constant measured values of Pantelopulos (per fraction).
However, the smaller steepness in the non-uniform case seems to be a general trend which is found in the measurements of Pantelopulos as well as in the calculated values of Egiazaroff.

The next step of Egiazaroff is the substitution of the new expression (19) in his own sediment transport formula. The result is:

$$\boxed{\frac{q_s}{q\sqrt{i}} = K.\xi_o \frac{\left(\frac{R_b . i}{\Delta D} - \tau_{c*_i}\right)}{\tau_{c*_i}}} \qquad (20)$$

This formula which shows some resemblance with the formula of Meyer-Peter and Müller (M-P & M) has not been verified (experimentally or with river measurements) by Egiazaroff.

6. Ashida & Michiue using Egiazaroff's theory

Ashida & Michiue (A & M, 1973) develop a bedload formula per fraction by combination of their own bedload formula (uniform sediment), the theory of Egiazaroff and the basic hypothesis.
This bedload formula for uniform sediment is as follows:

$$\frac{q_s}{\sqrt{\Delta g D^3}} = 17\ \tau_{e_x}^{3/2}\ (1 - \tau_{c_x}/\tau_x)(1 - u_{c_x}/u_x) \qquad (21)$$

in which: $\tau_x = \tau/(\rho_s - \rho)gD = \rho u_x^2/(\rho_s - \rho)gD$

τ = total shear stress on the bed

$\tau_e = \mu\tau$ = effective shearstress

τ_c = critical shearstress of the grains

This formula also shows a large resemblance with the bedloadformula of M-P & M. The bedload formula per fraction can be found by multiplying the righthand side of Eq. (21) by p_i (basic hypothesis) and by substituting for τ_{c_x} the theoretical expression of Egiazaroff:

$$\frac{q_{s_i}}{\sqrt{\Delta g D_i^3}} = 17 \cdot p_i \cdot \tau_{e_x}^{3/2}\ (1 - \tau_{c_{x_i}}/\tau_x)(1 - u_{c_x}/u_x) \qquad (22)$$

But first A & M verify Egiazaroff's theory. From the results of only four experiments they find a good agreement except in the range $D_i/D_m < 0.4$. In this area they give a correction on Egiazaroff's theory based on only one measurement (Fig. 9).

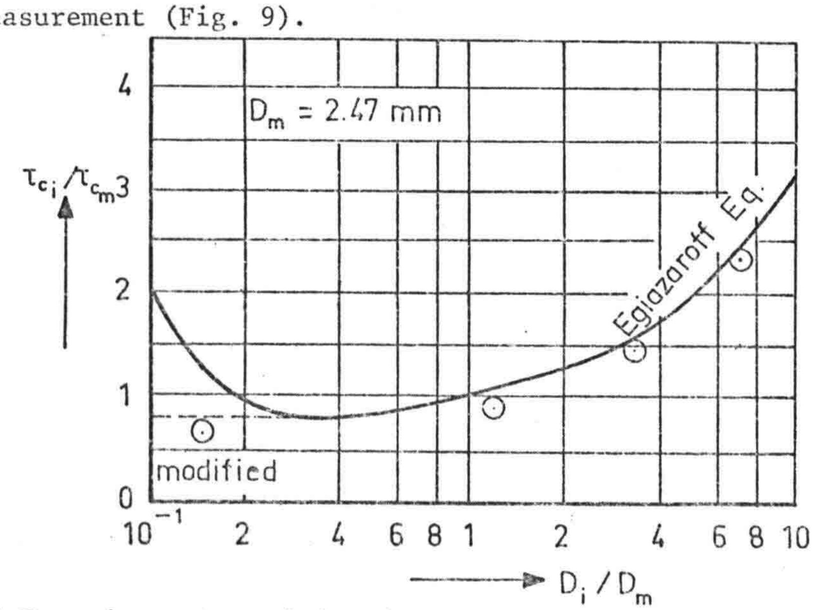

Fig. 9 Experimental verification of Egiazaroff's theory (A & M, 1973)

The final form of the bedload formula per fraction (A & M, 1973) is:

$$\frac{q_{s_i}}{\sqrt{\Delta g D_i^3}} = 17 \cdot p_i \cdot \tau_{e_x}^{3/2} (1 - \tau_{c_{x_i}}/\tau_x)(1 - u_{c_{x_i}}/u_x) \qquad (23)$$

in which: $\tau_{c_{x_i}} = \dfrac{0,1}{({}^{10}\log 19\, D_i/D_m)^2}$ for $D_i/D_m \geq 0.4$

$\tau_{c_{x_i}} \approx 0,0519\, D_m/D_i$ for $D_i/D_m < 0.4$

Remarks:

1. The verification of Egiazaroff's theory with four experiments and the correction of it on the basis of only one measurement seems arbitrary. Moreover A & M only use one grainsize mixture (D_m = 2.47 mm; grainsize distribution see Fig. 10 - 1) so that it seems impossible to draw general conclusions from these experiments.
2. Figure 10-2 shows a comparison between these experiments, those of Pantelopulos (factor 5) and the calculated critical shearstresses according to Egiazaroff's theory (D_m as a parameter).
   Especially the steep part of Egiazaroff's theory (small D_i) is verified insufficiently.

In the same paper A & M verify the new bedload formula per fraction , Eq. (23). The experimental conditions were chosen in such a way that there existed no bedforms. This means that the total shear stress τ is equal to the effective shearstress τ_e. In Fig. 11 a comparison between formula and experiments is shown.

Ashida & Michiue conclude from this figure that the bedload transport per fraction is sufficiently described by Eq. (23) except for the larger fractions ($D_i/D_m > 1$); in this area they recommend further investigations.

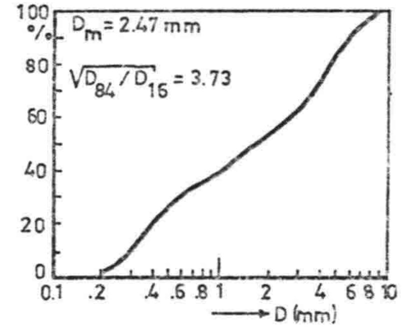

Fig. 10-1

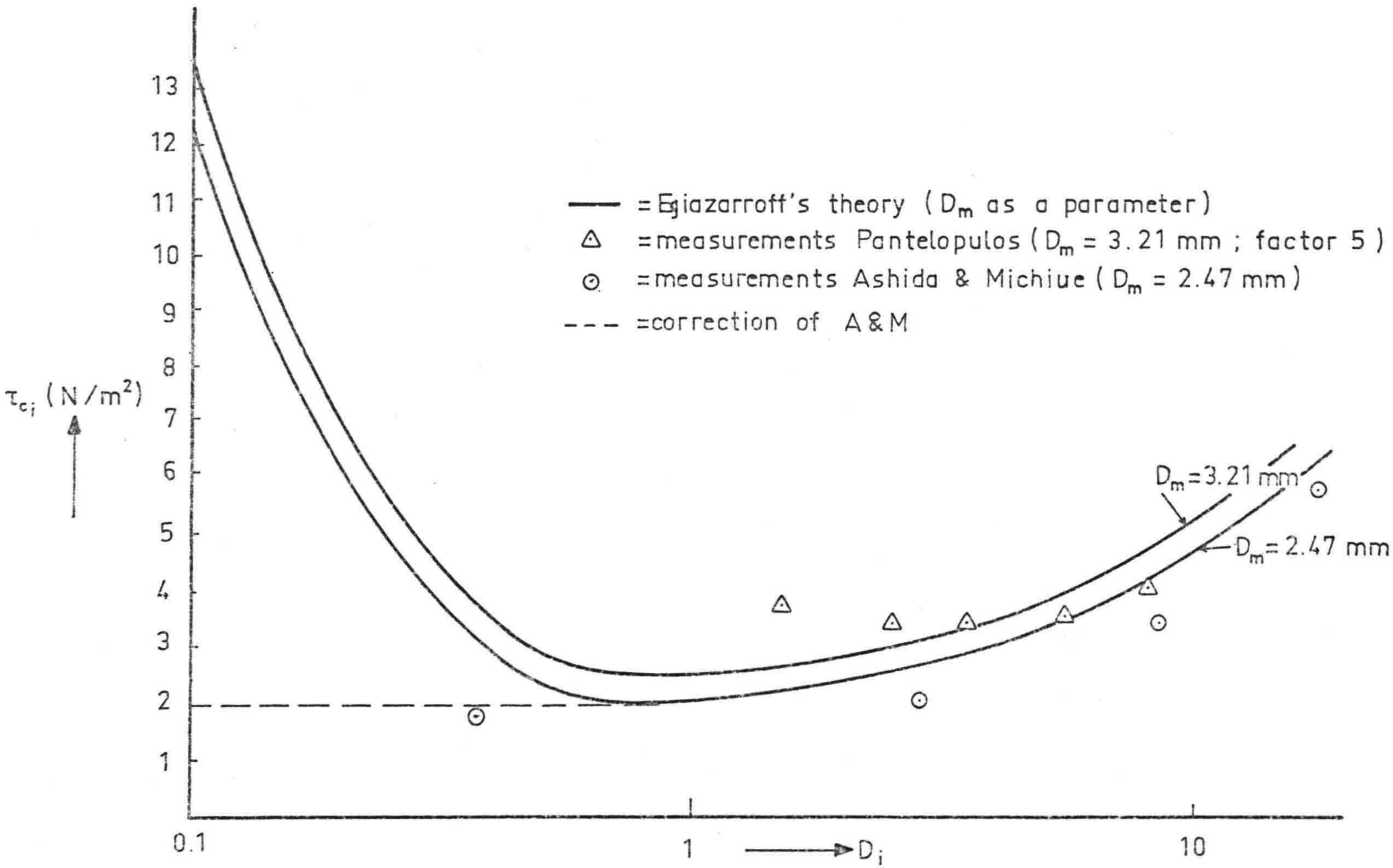

Fig. 10-2

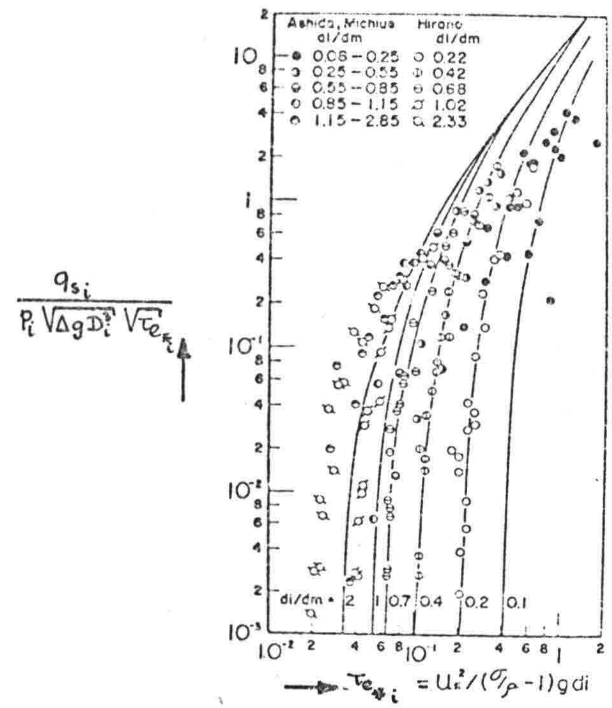

Fig. 11 Comparison between the bedload formula per fraction of A & M and experimental results.

7. Suzuki using Egiazaroff's theory

Suzuki (1976) follows the principle ideas of A & M; The difference is that he does not use the bedload formula (uniform sediment) of A & M but that of M-P & M:

$$\frac{q_s}{\sqrt{\Delta g D^3}} = 8\left(\frac{\mu R_b i}{\Delta D} - 0.047\right)^{3/2} \tag{24}$$

The correction for non-uniform sediment does not take place via the dimensionless effective shear stress τ_{e_x} (or via μ), but via the value 0.047. He replaces this value by a constant times the dimensionless critical shear stress per fraction from Egiazaroff. ($A \cdot \tau_{c_{x_i}}$). The constant A is chosen in such a way that substituting $D_i = D_m$ in this expression gives 0.047 again. According to the basic hypothesis he multiplies the righthand side of Eq. (24) by p_i. The bedload formula per fraction becomes now:

$$\boxed{\begin{aligned} \frac{q_{s_i}}{\sqrt{\Delta g D^3}} &= 8 \cdot p_i \cdot \left(\frac{\mu R_b{}^i}{\Delta D_i} - 0.78\, \tau_{c_{x_i}}\right)^{3/2} \\ \text{in which: } \tau_{c_{x_i}} &= \frac{0.1}{({}^{10}\log 19\, D_i/D_m)^2} \end{aligned}} \tag{25}$$

Suzuki carries out some experiments but is forced to use different values of μ for the different fractions. Obviously, the correction for non-uniform sediment via Egiazaroff's theory is insufficient.
However, the number of experiments is too small to get a real verification of Eq. (25).

Remark:
Suzuki carries out two experiments, each with two sediment fractions (D_1 = 0.6 mm, D_2 = 1.0 mm).
In both experiments he has to use a larger μ-value for fraction 1 (smaller particles) and a smaller μ-value for fraction 2 to get agreement between formula (25) and experimental results. This means that the bedload transport of fraction 1 is larger and of fraction 2 is smaller than according to Eq. (25). In both experiments bedforms were present.

8. Summary

Some of the major conclusions of this investigation are that the number of available concepts for a bedload formula per sediment fraction is small and that generally there is a lack of experimental verification.
The stochastical-empirical approaches (Pantelopulos, Einstein) are, because of the large number of correction coefficients and figures, more complicated than the empirical formulas (M-P & M, A & M, E & H).
The basic-hypothesis is the simplest transport formula per fraction. It is assumed that the different fractions move independently and that the factor μ (M-P & M and E & H) is not a function of p_i. Antsyferov uses the basic-hypothesis in combination with a formula similar to that of Engelund and Hansen.
The formula of Kalinske-Pantelopulos is not right in case of large shear stresses. An experimental verification is missing. Pantelopulos carried out some experiments in which he determined critical shear stresses per fraction (in a sediment mixture); these values appeared to be nearly constant, in contrast with those for the uniform case. The bedload formula

of Einstein is already written in the form of a bedload formula per fraction. However, some of the correction coefficients and constants are determined in conditions with uniform sediment. For large-range mixtures Ning-Chien gives a correction for the hiding-factor ξ and he investigates effects as surface segregation and bedstratification. Egiazaroff derives a theoretical expression for the critical shear stress per fraction $\tau_{c_{x_i}}$.
In the derivation remain some uncertainties:

1. The friction coefficient f is taken equal to unity.
2. The neglection of the liftforce.

Egiazaroff verifies his theory with experiments of Pantelopulos and others. A question which arises is: Why did Egiazaroff multiply Pantelopulos' results with a factor 5? The general trend of Egiazaroff's theory appears to be in agreement with Pantelopulos' measurements.
Egiazaroff substitutes his theoretical expression in his own transport formula. A verification with experiments of river measurements did not take place.
Ashida & Michiue give a correction of Egiazaroff's theory; this seems arbitrary because it is based on only one measurement. They combine their own bedload formula with the basic hypothesis and the corrected theory of Egiazaroff. Experimental results are in reasonable agreement with the new bedload formula per fraction, except for $D_i/D_m > 1$.
Despite of some uncertainties, which are still present in this formula, it has some advantages:

1. It has been verified experimentally
2. It is written in a relatively simple analytical form.
3. It takes into account the mutual influences of the different fractions.

Suzuki combines in the same way as A & M did the bedload formula of M-P & M, the basic-hypothesis and the theoretical expression of Egiazaroff (no correction). The resulting formula is insufficiently experimentally verified. In the two experiments of Suzuki it was found necessary to use different factors μ for both fractions, to get the calculated bedload transport per fraction in agreement with the measured one.

In Table 2 a summary is given of the bedload formulas per fraction which have been treated in this report.

Investigator	Sediment transport formula per fraction	Remarks
"Basic Hypothesis"	$s_i = q_{s_i}/C_o = p_i \times f_i'(u,D_i)$	Large simplification; independent movement of the different fractions
Antsyferov (1973)	$s_i/\sqrt{\Delta g D_i^3} = p_i \times 0.084\ (\mu R_b i/\Delta D_i)^{5/2}$	Similar to Engelund and Hansen. Two possibilities: (i) $\mu \neq \mu(p_i)$ - "basic hypothesis" (ii) $\mu = \mu(p_i)$
	$s_i/\sqrt{\Delta g D_i^3} = p_i \times 13.3\ (\frac{\mu R_b i}{\Delta D_i} - 0.047)^{3/2}$	Similar to Meyer-Peter and Müller (i) $\mu \neq \mu(p_i)$ (ii) $\mu = \mu(p_i)$
Einstein (1950)	$q_{s_i}/\sqrt{\Delta g D_i^3} = \frac{p_i}{A_x} \cdot \frac{p}{p-1}$ with: $p = 1 - \frac{1}{\sqrt{\pi}} \cdot \int_{-B_x\psi_x - 1/\eta_o}^{B_x\psi_x + 1/\eta_o} \exp(-t^2)dt$ $\psi_x = \xi \cdot Y \cdot (\beta/\beta_x)^2 \cdot \psi$	Gives in his original form directly the bedload transport per fraction. Many correction coefficients. Only the hiding-factor ξ is adapted to large-range mixtures (Ning-Chien, 1953).
Pantelopulos (1955)	$q_{s_i} = 2/3\ p(D_i) \cdot \Delta D_i . D_i . \bar{u}_p(D_i)$ with: $\overline{u_p}(D_i)/\bar{u} = f(\tau_{c_i}/\tau_e, \sigma_u/\bar{u})$	Similar to the concept of Kalinske. Incorrect for relatively large transports No analytical expression for τ_{c_i}
Egiazaroff (1965)	$q_{s_i}/q\sqrt{i} = K\ \xi_o \frac{(\frac{R_b i}{\Delta D_i} - \tau_{c_{x_i}})}{\tau_{c_{x_i}}}$ with: $\tau_{c_{x_i}} = \frac{\tau_{c_i}}{(\rho_s - \rho)gD_i} = \frac{0.1}{(^{10}\log 19\ D_i/D_m)^2}$	Egiazaroff's own transport formula with an analytical expression for τ_{c_i}. Only an experimental verification of $\tau_{c_{x_i}}$ with results of Pantèlopulos and others.
Ashida and Michiue (1973)	$q_{s_i}/\sqrt{\Delta g D_i^3} = p_i \times 17\ \tau_{e_x}^{3/2}\ (1 - \tau_{c_{x_i}}/\tau_x)(1 - u_{c_{x_i}}/u_x)$ with $\tau_{c_{x_i}}$ = according to Egiazaroff	Large resemblance with formula of M-P & M Experimental verification of $\tau_{c_{x_i}}$ as well as the total formula.
Suzuki (1976)	$q_{s_i}/\sqrt{\Delta g D_i^3} = p_i \times 8\ (\frac{\mu R_b i}{\Delta D_i} - 0.78\ \tau_{c_{x_i}})^{3/2}$ with $\tau_{c_{x_i}}$ according to Egiazaroff	Similar to M-P & M. Unsufficient experimental verification

Table 2

Literature

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Main Symbols

a	waterdepth	$[LT^{-1}]$
C_g	Chézy-coefficient of the grains	$[L^{\frac{1}{2}}T^{-1}]$
C_t	total Chézy-coefficient	$[L^{\frac{1}{2}}T^{-1}]$
D_i	grain diameter of sediment fraction i	$[L]$
D_m	mean grain diameter of a sediment mixture ($= \sum_i p_i D_i$)	$[L]$
i_b	bottom slope	$[-]$
p_i	probability of sedimentfraction i	$[-]$
q_s	sedimenttransport in volume (real) per unit time and width	$[L^2T^{-1}]$
q_{s_i}	sediment transport of fraction i in volume (real) per unit time and width	$[L^2T^{-1}]$
R_b	hydraulic radius	$[L]$
s	sedimenttransport in volume (including pores) per unit time and width	$[L^2T^{-1}]$
s_i	sedimenttransport of fraction i in volume (including pores) per unit time and width	$[L^2T^{-1}]$
u	mean flow velocity	$[LT^{-1}]$
$\bar{u}_p$	mean grain velocity (including restperiods)	$[LT^{-1}]$
X	transport parameter ($= s/\sqrt{\Delta g D^3}$)	$[-]$
Y	flow parameter ($= \Delta D/\mu R_b i_b$)	$[-]$
Δ	relative density ($(\rho_s - \rho)/\rho$)	$[-]$
μ	bedformfactor	$[-]$
τ	total shear stress on the bed	$[ML^{-1}T^{-2}]$
τ_c	critical shearstress of sediment	$[ML^{-1}T^{-2}]$
τ_e	effective shear stress on the bed	$[ML^{-1}T^{-2}]$
τ_x	dimensionless shear stress ($=\tau/(\rho_s - \rho)gD$)	$[-]$