Low-Rank Covariance Matrix Recovery From Rank-One Measurements
An Analytical Solution

Wang, Peilan; Fang, Jun; Ma, Binyao; Wang, Bin; Leus, Geert

DOI
10.1109/LSP.2025.3584252

Publication date
2025

Document Version
Final published version

Published in
IEEE Signal Processing Letters

Citation (APA)
Wang, P., Fang, J., Ma, B., Wang, B., & Leus, G. (2025). Low-Rank Covariance Matrix Recovery From Rank-One Measurements: An Analytical Solution. *IEEE Signal Processing Letters*, *32*, 2674-2678. https://doi.org/10.1109/LSP.2025.3584252

Low-Rank Covariance Matrix Recovery From Rank-One Measurements: An Analytical Solution

Peilan Wang, *Member, IEEE*, Jun Fang, *Senior Member, IEEE*, Binyao Ma, *Student Member, IEEE*, Bin Wang, and Geert Leus, *Fellow, IEEE*

***Abstract*—In this letter, we propose an analytical solution for recovering a low-rank positive semi-definite (PSD) matrix from its rank-one measurements. We show that by utilizing a set of structured measurement vectors, we can analytically determine the null space of this low-rank PSD matrix. Based on the result, the PSD matrix can be efficiently recovered. Our analysis shows that the proposed method only requires $(N-K)(2K+1)+K^2$ measurements to guarantee exact recovery of the PSD matrix, where N and K respectively denote the dimension and the rank of the PSD matrix. Numerical results show that the proposed method achieves a considerable improvement over existing state-of-the-art methods in terms of both sample complexity and computational efficiency. Specifically, the proposed method helps improve the computational efficiency by an order of magnitude as compared with existing methods.**

***Index Terms*—Low-rank covariance matrix recovery, rank-one measurements, covariance sketching.**

I. Introduction

RECOVERING structured signals from incomplete observations presents a fundamental challenge in signal processing [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. A classical problem of this kind is to estimate a low-rank positive semi-definite (PSD) covariance matrix from its rank-one measurements [8], [9], [10], [11], [12], [13], [14], [15]. Such a problem arises from a variety of applications such as quantum computing [17], [18], [19], cognitive radio [20], optical imaging [21], [22] and radar imaging [23].

Previous works on low-rank PSD matrix recovery can be broadly categorized into two groups [8]: convex relaxation methods and non-convex optimization methods. The convex relaxation methods [4], [5], [6], [7] convert the non-convex optimization problem into a convex one by performing lifting the dimension via nuclear norm minimization. These approaches tend to suffer a high computational complexity, especially in large-scale settings. Many researchers have leveraged the inherent Vandermonde structure or other structural properties of the PSD matrix to further reduce sample and computational complexity [24], [25], [26], [27]. On the other hand, nonconvex approaches, e.g. [9], [10], [16], have shown promising results in both computational and statistical efficiency. But these methods face challenges such as sensitivity to the initialization and risk of getting stuck in local optima.

In contrast to existing works that rely on iterative approaches, we propose an analytical solution for low-rank PSD matrix recovery. The key idea behind the proposed method is to analytically determine the null space of the low-rank PSD matrix by utilizing structured measurement vectors. Based on the derived null space, the PSD matrix can be efficiently recovered. Our analysis indicates that the proposed method is able to provide a substantial improvement over existing state-of-the-art methods in both sample complexity and computational complexity.

II. Problem Formulation

We consider the problem of estimating a low-rank PSD matrix $\boldsymbol{R}\in\mathbb{C}^{N\times N}$ from a few rank-one measurements $\{y_m\}_{m=1}^M$:

$$y_m=\boldsymbol{a}_m^H\boldsymbol{R}\boldsymbol{a}_m=\|\boldsymbol{a}_m^H\boldsymbol{X}\|_2^2,\quad\forall m=1,2,\ldots,M,\tag{1}$$

where $\boldsymbol{a}_m\in\mathbb{C}^N$ represents the mth measurement vector, and $\boldsymbol{R}=\boldsymbol{X}\boldsymbol{X}^H$ with $\boldsymbol{X}\in\mathbb{C}^{N\times K}$ being a full column rank matrix. Note that y_m can also be written as:

$$y_m=\text{tr}(\boldsymbol{R}\boldsymbol{a}_m\boldsymbol{a}_m^H),\tag{2}$$

where $\boldsymbol{a}_m\boldsymbol{a}_m^H$ is a rank-one sensing matrix.

The problem of estimating $\boldsymbol{R}$ from $\{y_m\}_{m=1}^M$ is referred to as rank-one matrix sensing or covariance sketching. This problem is typically formulated as a non-convex optimization problem with the objective of minimizing the rank of $\boldsymbol{R}$ [8]:

$$\begin{aligned}\min_{\boldsymbol{R}}\quad&\text{rank}(\boldsymbol{R})\\ \text{s.t.}\quad&y_m=\text{tr}(\boldsymbol{R}\boldsymbol{a}_m\boldsymbol{a}_m^H),\quad\forall m=1,2,\ldots,M,\\ &\boldsymbol{R}\succeq 0.\end{aligned}\tag{3}$$

However, the rank$(\cdot)$ in the objective function is non-convex, and solving this problem is NP-hard in general. To address this challenge, previous studies proposed a variety of convex relaxation methods, such as nuclear norm minimization and its variants [3]. In addition to the above approach, some other works propose to minimize the following nonconvex least-squares (LS) loss function via a vanilla gradient descent [9], [10]:

$$\min_{\boldsymbol{X}\in\mathbb{C}^{N\times K}}\quad f(\boldsymbol{X}):=\frac{1}{4M}\sum_{m=1}^M\left(y_m-\|\boldsymbol{a}_m^H\boldsymbol{X}\|_2^2\right)^2.\tag{4}$$

For instance, [9] has demonstrated the surprising effectiveness of such a method in solving the rank-one matrix sensing problem.

Despite previous efforts over past years, there still lacks an analytical solution to solve this challenging problem. Also, as pointed out in [9], the required sample complexity that

Received 22 March 2025; revised 17 June 2025; accepted 23 June 2025. Date of publication 30 June 2025; date of current version 14 July 2025. The associate editor coordinating the review of this article and approving it for publication was Prof. Alexander Jung. *(Corresponding author: Jun Fang.)*

Peilan Wang, Jun Fang, Binyao Ma, and Bin Wang are with the National Key Laboratory of Wireless Communications, University of Electronic Science and Technology of China, Chengdu 611731, China (e-mail: JunFang@uestc.edu.cn).

Geert Leus is with the Department of Microelectronics, Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: g.j.t.leus@tudelft.nl).

Digital Object Identifier 10.1109/LSP.2025.3584252

guarantees recovery of the PSD matrix is in the order of $\mathcal{O}(NK^4\log(N))$, which scales exponentially with the rank K and thus incurs an excessive number of samples even for a moderate value of K.

In this paper, by devising a set of structured measurement vectors $\{\boldsymbol{a}_m\}_m^M$, we show that $\boldsymbol{R}$ can be exactly recovered via an analytical method without involving any iterative steps. In addition, the proposed method needs only $(N-K)(2K+1)+K^2$ rank-one measurements, which enjoys a much lower sample complexity than existing methods.

III. Proposed Method

To recover $\boldsymbol{R}$ from its rank-one measurements, we first aim to obtain its column space, i.e. col$(\boldsymbol{R})$. Note that it is challenging to directly recover col$(\boldsymbol{R})$ from its rank-one measurements $\{y_m\}_{m=1}^M$. To address this challenge, we, instead, recover the null space of $\boldsymbol{R}$ first, which is the key idea of the proposed method. After the null space of $\boldsymbol{R}$ is obtained, its column space can be easily determined since the orthogonal complement of the null space is the column space of $\boldsymbol{R}$.

A. Recovering null$(\boldsymbol{R})$

According to the rank-nullity theorem, the dimension of the null space of $\boldsymbol{R}$ is given by

$$\dim(\text{null}(\boldsymbol{R})) = N - \text{rank}(\boldsymbol{R}) = N - K.$$

In other words, our goal is to identify $N-K$ linearly independent vectors that span the null space of $\boldsymbol{R}$.

Let $\{\boldsymbol{a}_k^\star\}_{k=1}^{N-K}$ represent the vectors spanning the null space of $\boldsymbol{R}$, such that

$$\boldsymbol{a}_k^{\star H}\boldsymbol{R}\boldsymbol{a}_k^\star = 0, \quad k = 1,2,\ldots,N-K. \tag{5}$$

In the following, we discuss how to obtain a set of vectors spanning the null space of $\boldsymbol{R}$. We first randomly generate an arbitrary $N\times N$ unitary matrix

$$\boldsymbol{V} = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \ldots\ \boldsymbol{v}_N] \in \mathbb{C}^{N\times N}. \tag{6}$$

Let $\boldsymbol{V}_0$ denote a matrix constructed by the first K columns of $\boldsymbol{V}$:

$$\boldsymbol{V}_0 = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \ldots\ \boldsymbol{v}_K] \in \mathbb{C}^{N\times K}. \tag{7}$$

Based on $\boldsymbol{V}_0$, we define a set of matrices as follows:

$$\boldsymbol{V}_k = [\boldsymbol{V}_0\ \boldsymbol{v}_{K+k}] \in \mathbb{C}^{N\times(K+1)}, \quad k = 1,2,\ldots,N-K. \tag{8}$$

We show that each vector $\boldsymbol{a}_k^\star$ which satisfies (5) can be written as a linear combination of the columns of $\boldsymbol{V}_k$, i.e.,

$$\boldsymbol{a}_k^\star = \boldsymbol{V}_k\boldsymbol{w}_k^\star, \tag{9}$$

where $\boldsymbol{w}_k^\star$ is a nonzero vector. The rationale behind (9) can be explained as follows. Let $\boldsymbol{B}\in\mathbb{C}^{N\times K}$ be a matrix such that col$(\boldsymbol{B})$ = col$(\boldsymbol{R})$. Define

$$\boldsymbol{T}_k \triangleq \boldsymbol{B}^H\boldsymbol{V}_k \in \mathbb{C}^{K\times(K+1)}. \tag{10}$$

Since $K < K+1$, there must exist a nonzero vector $\boldsymbol{w}_k^\star$ such that $\boldsymbol{T}_k\boldsymbol{w}_k^\star = 0$, which implies that $\boldsymbol{B}^H\boldsymbol{V}_k\boldsymbol{w}_k^\star = 0$. Hence, $\boldsymbol{a}_k^\star = \boldsymbol{V}_k\boldsymbol{w}_k^\star$ satisfies $\boldsymbol{B}^H\boldsymbol{a}_k^\star = 0$, thus meeting the condition in (5). Note that $\boldsymbol{V}_k$ is a full column rank matrix. Therefore, $\boldsymbol{a}_k^\star = \boldsymbol{V}_k\boldsymbol{w}_k^\star$ must be a nonzero vector for any nonzero $\boldsymbol{w}_k^\star$.

We now discuss how to find the vector $\boldsymbol{w}_k^\star$ given a pre-specified $\boldsymbol{V}_k$ for each k. To this objective, we substitute $\boldsymbol{a}_k^\star = \boldsymbol{V}_k\boldsymbol{w}_k^\star$ into (5), which yields:

$$\begin{aligned} 0 = \boldsymbol{a}_k^{\star H}\boldsymbol{R}\boldsymbol{a}_k^\star &= (\boldsymbol{w}_k^\star)^H\boldsymbol{V}_k^H\boldsymbol{R}\boldsymbol{V}_k\boldsymbol{w}_k^\star \\ &= (\boldsymbol{w}_k^\star)^H\boldsymbol{G}_k\boldsymbol{w}_k^\star, \quad k = 1,2,\ldots,N-K, \end{aligned} \tag{11}$$

where $\boldsymbol{G}_k \triangleq \boldsymbol{V}_k^H\boldsymbol{R}\boldsymbol{V}_k \in \mathbb{C}^{(K+1)\times(K+1)}$ is a reduced-dimensional PSD matrix. Typically, we have rank$(\boldsymbol{G}_k) = K$ with probability 1 due to the random generation of $\boldsymbol{V}$ [28].

From (11), we see that once $\boldsymbol{G}_k$ is recovered, $\boldsymbol{w}_k^\star$ can be determined by solving $\boldsymbol{G}_k\boldsymbol{w}_k = \boldsymbol{0}$, and subsequently, $\boldsymbol{a}_k^\star = \boldsymbol{V}_k\boldsymbol{w}_k^\star$ can be obtained. Thus, our goal now becomes recovering the matrix $\boldsymbol{G}_k$. Since $\boldsymbol{G}_k$ has a much lower dimension than that of $\boldsymbol{R}$, recovering $\boldsymbol{G}_k$ is considerably more favorable in terms of sample complexity as compared to recovering $\boldsymbol{R}$.

We now discuss how to recover $\boldsymbol{G}_k$. For each k, we first construct a set of structured measurement vectors $\boldsymbol{a}_{k,m} = \boldsymbol{V}_k\boldsymbol{w}_{k,m}$ for $m = 1,2,\ldots,\bar{M}$, where $\boldsymbol{w}_{k,m}\in\mathbb{C}^{K+1}$ is a randomly generated vector. We see that each measurement vector $\boldsymbol{a}_{k,m}$ lies in the range space of $\boldsymbol{V}_k$. This is the reason why we say $\boldsymbol{a}_{k,m}$ is a structured measurement vector. The corresponding rank-one measurement $y_{k,m}$ is thus given by

$$\begin{aligned} y_{k,m} &= \boldsymbol{a}_{k,m}^H\boldsymbol{R}\boldsymbol{a}_{k,m} = \boldsymbol{w}_{k,m}^H\boldsymbol{G}_k\boldsymbol{w}_{k,m} \\ &= (\boldsymbol{w}_{k,m}^T\otimes\boldsymbol{w}_{k,m}^H)\text{vec}(\boldsymbol{G}_k) = \tilde{\boldsymbol{w}}_{k,m}^H\boldsymbol{g}_k, \end{aligned} \tag{12}$$

where $\tilde{\boldsymbol{w}}_{k,m} \triangleq \boldsymbol{w}_{k,m}^*\otimes\boldsymbol{w}_{k,m}$, $\otimes$ denotes the Kronecker product, and $\boldsymbol{g}_k \triangleq \text{vec}(\boldsymbol{G}_k)$ is the vectorized form of $\boldsymbol{G}_k$. By stacking the measurements $\{y_{k,m}\}_{m=1}^{\bar{M}}$, we obtain

$$\begin{aligned} \boldsymbol{y}_k \triangleq \begin{bmatrix} y_{k,1} \\ y_{k,2} \\ \vdots \\ y_{k,\bar{M}} \end{bmatrix} &= \begin{bmatrix} \tilde{\boldsymbol{w}}_{k,1}^T \\ \tilde{\boldsymbol{w}}_{k,2}^T \\ \vdots \\ \tilde{\boldsymbol{w}}_{k,\bar{M}}^T \end{bmatrix}\boldsymbol{g}_k \\ &= \boldsymbol{W}_k\boldsymbol{g}_k. \end{aligned} \tag{13}$$

The above linear system has $(K+1)^2$ unknown variables. By randomly generating $\bar{M}\geq(K+1)^2$ columns $\{\boldsymbol{w}_{k,m}\}_{m=1}^{\bar{M}}$, we have almost surely that rank$(\boldsymbol{W}_k) = (K+1)^2$. Then, the unknown vector $\boldsymbol{g}_k$ can be recovered via a simple least-squares approach, i.e.,

$$\hat{\boldsymbol{g}}_k = (\boldsymbol{W}_k^H\boldsymbol{W}_k)^{-1}\boldsymbol{W}_k^H\boldsymbol{y}_k. \tag{14}$$

Once $\boldsymbol{g}_k$ is estimated, the matrix $\boldsymbol{G}_k$ can be recovered accordingly.

After $\boldsymbol{G}_k$ is estimated, we can obtain $\boldsymbol{a}_k^\star = \boldsymbol{V}_k\boldsymbol{w}_k^\star$ by solving $\boldsymbol{G}_k\boldsymbol{w}_k^\star = \boldsymbol{0}$. By repeating the above process for each $k = 1,\ldots,N-K$, we can obtain a set of vectors $\{\boldsymbol{a}_k^\star\}$ with each vector lying in the null space of $\boldsymbol{R}$. One thing that remains to be shown is that these $N-K$ vectors are linearly independent and constitute the basis for the null space of $\boldsymbol{R}$. We have the following result concerning this problem.

Proposition 1: Given the matrices $\{\boldsymbol{V}_k\}_{k=1}^{N-K}$ constructed in (8), the vectors $\{\boldsymbol{a}_k^\star\}_{k=1}^{N-K}$ obtained via the above scheme are linearly independent with probability one.

Proof: See Appendix A. □

According to the above proposition, we know that the column space of the matrix $\boldsymbol{A} \triangleq [\boldsymbol{a}_1^\star\ \boldsymbol{a}_2^\star\ \ldots\ \boldsymbol{a}_{N-K}^\star]$ spans the null space of $\boldsymbol{R}$, i.e. null$(\boldsymbol{R})$ = col$(\boldsymbol{A})$. Thus we can obtain the column space of $\boldsymbol{R}$ by finding the null space of $\boldsymbol{A}^H$.

B. Recovering $\boldsymbol{R}$

Denote $\boldsymbol{B}\in\mathbb{C}^{N\times K}$ as the recovered matrix whose columns span the column space of $\boldsymbol{R}$. Since col$(\boldsymbol{B})$ = col$(\boldsymbol{R})$, the PSD $\boldsymbol{R}$ can be represented as

$$\boldsymbol{R} = \boldsymbol{B}\boldsymbol{S}\boldsymbol{B}^H, \tag{15}$$

Algorithm 1: Proposed NSL-based Rank-One Matrix Sensing Method.

Input:
A set of matrices $\{\boldsymbol{V}_k\}_{k=1}^{N-K}$ satisfying (8); measurement vectors $\{\boldsymbol{a}_{k,m} = \boldsymbol{V}_k \boldsymbol{w}_{k,m}\}_{k=1,m=1}^{N-K,\bar{M}}$ and associated measurements $\{y_{k,m}\}_{k=1,m=1}^{N-K,\bar{M}}$; measurement vectors $\{\boldsymbol{a}_l\}_{l=1}^{L}$ and associated measurements $\{y_l\}_{l=1}^{L}$ [1];
Step 1: Recover col($\boldsymbol{R}$)
for $k = 1$ **to** $N-K$ **do**
  Construct $\boldsymbol{W}_k$ based on (13);
  Recover $\boldsymbol{g}_k$ via (14) and subsequently recover $\boldsymbol{G}_k$;
  Obtain $\boldsymbol{a}_k^\star = \boldsymbol{V}_k \boldsymbol{w}_k^\star$, where $\boldsymbol{w}_k^\star$ is solved via $\boldsymbol{G}_k \boldsymbol{w}_k^\star = \boldsymbol{0}$;
end for
Construct $\boldsymbol{A} \triangleq [\boldsymbol{a}_1^\star, \ldots, \boldsymbol{a}_{N-K}^\star]$ and obtain the matrix $\hat{\boldsymbol{B}}$.
Step 2: Recover $\boldsymbol{R}$
Recover $\boldsymbol{S}$ via the LS method using the measurements in (16);
Reconstruct $\boldsymbol{R} = \boldsymbol{B}\boldsymbol{S}\boldsymbol{B}^H$ as in (15).
Output: Recovered matrix $\boldsymbol{R}$.

where $\boldsymbol{S} \in \mathbb{C}^{K\times K}$ is a small PSD matrix. Now, our goal is to recover the unknown small-dimensional matrix $\boldsymbol{S}$ in order to reconstruct a high-dimensional $\boldsymbol{R}$.

To achieve this, let $\boldsymbol{a}_l$, for $l = 1, 2, \ldots, L$, represent the lth measurement vector. The corresponding measurement is given by

$$\begin{aligned} y_l &= \boldsymbol{a}_l^H \boldsymbol{R} \boldsymbol{a}_l = \boldsymbol{a}_l^H \boldsymbol{B}\boldsymbol{S}\boldsymbol{B}^H \boldsymbol{a}_l \\ &= \boldsymbol{c}_l^H \boldsymbol{S} \boldsymbol{c}_l, \quad l = 1, 2, \ldots, L, \end{aligned} \tag{16}$$

where $\boldsymbol{c}_l \triangleq \boldsymbol{B}^H \boldsymbol{a}_l$. Similarly, a least-squares method can be utilized to recover $\boldsymbol{S} \in \mathbb{C}^{K\times K}$. It can be readily verified that a minimum of $L \geq K^2$ measurements is required to recover $\boldsymbol{S}$. Note that we can reuse the previous rank-one measurements $\{y_{k,m}\}_{k,m}$ to recover $\boldsymbol{S}$, provided that there exist at least K^2 linear independent vectors $\{\tilde{\boldsymbol{c}}_l \mid \tilde{\boldsymbol{c}}_l \triangleq \boldsymbol{c}_l \otimes \boldsymbol{c}_l^*\}_{l=1}^{K^2}$. Thus no extra measurements are required. For clarity, we summarize the proposed method in Algorithm 1.

IV. Sample and Computational Complexity Analysis

In this section, we analyze the sample and computational complexity of the proposed method.

A. Sample Complexity Analysis

We see that the proposed method involves two steps. In the first step, we need to recover a set of matrices $\{\boldsymbol{G}_k \in \mathbb{C}^{(K+1)\times(K+1)}\}_{k=1}^{N-K}$, which requires $(N-K)\bar{M} \geq (N-K)(K+1)^2$ rank-one measurements. Such a sample complexity can be further reduced by reusing some of the measurements.

To investigate the minimum number of measurements required, let $\bar{M}_{\min} = (K+1)^2$ and

$$\boldsymbol{w}_{k,m} = [w_m^1 \ w_m^2 \ \ldots \ w_m^K \ 0]^T = [\boldsymbol{r}_m^T \ 0]^T \in \mathbb{C}^{K+1}. \tag{17}$$

The associated rank-one measurement $y_{k,m}$ becomes

$$\begin{aligned} y_{k,m} &= \boldsymbol{w}_{k,m}^H \boldsymbol{G}_k \boldsymbol{w}_{k,m} = \boldsymbol{r}_m^H \boldsymbol{V}_0^H \boldsymbol{R} \boldsymbol{V}_0 \boldsymbol{r}_m, \\ &= (\boldsymbol{r}_m^T \otimes \boldsymbol{r}_m^H)\,\mathrm{vec}(\boldsymbol{V}_0^H \boldsymbol{R} \boldsymbol{V}_0) = \tilde{\boldsymbol{r}}_m^H \tilde{\boldsymbol{q}}_0, \end{aligned} \tag{18}$$

where $\tilde{\boldsymbol{r}}_m = \boldsymbol{r}_m^* \otimes \boldsymbol{r}_m \in \mathbb{C}^{K^2\times 1}$ and $\tilde{\boldsymbol{q}}_0 = \mathrm{vec}(\boldsymbol{V}_0^H \boldsymbol{R} \boldsymbol{V}_0)$. It is readily seen that the measurements in (18) are independent of the index k, which implies that they can be shared in recovering all matrices $\{\boldsymbol{G}_k\}_{k=1}^{K}$. Next, we further determine how many measurements can be reused.

Let M_0 denote the number of shared measurements. Since $\boldsymbol{w}_{k,m} = [\boldsymbol{r}_m^T \ 0]^T$, we have $\tilde{\boldsymbol{w}}_{k,m} = [\tilde{\boldsymbol{r}}_m^T \ \boldsymbol{0}^T]^T$ and the resulting matrix $\boldsymbol{W}_k \in \mathbb{C}^{\bar{M}_{\min}\times\bar{M}_{\min}}$ in (13) is a square matrix that can be rewritten as

$$\begin{aligned} \boldsymbol{W}_k &= \begin{bmatrix} \tilde{\boldsymbol{r}}_1^T & \boldsymbol{0} \\ \tilde{\boldsymbol{r}}_2^T & \boldsymbol{0} \\ \vdots & \vdots \\ \tilde{\boldsymbol{r}}_{M_0}^T & \boldsymbol{0} \\ \tilde{\boldsymbol{w}}_{k,M_0+1}^T(1:K^2) & \tilde{\boldsymbol{w}}_{k,M_0+1}^T(K^2+1:\bar{M}_{\min}) \\ \vdots & \vdots \\ \tilde{\boldsymbol{w}}_{k,\bar{M}_{\min}}^T(1:K^2) & \tilde{\boldsymbol{w}}_{k,\bar{M}_{\min}}^T(K^2+1:\bar{M}_{\min}) \end{bmatrix}, \\ &= \begin{bmatrix} \boldsymbol{W}^{(0)} & \boldsymbol{0} \\ \boldsymbol{C}_k & \boldsymbol{D}_k \end{bmatrix} \end{aligned} \tag{19}$$

where $\tilde{\boldsymbol{w}}_{k,m}(i:j)$ denotes the vector formed by the elements between indices i and j, $\boldsymbol{W}^{(0)}$ is the submatrix consisting of the first $M_0 \times K^2$ entries of $\boldsymbol{W}_k$, and $\boldsymbol{C}_k \in \mathbb{C}^{(\bar{M}_{\min}-M_0)\times K^2}$ and $\boldsymbol{D}_k \in \mathbb{C}^{(\bar{M}_{\min}-M_0)\times(\bar{M}_{\min}-K^2)}$ denote the associated submatrices of $\boldsymbol{W}_k$.

By devising $\boldsymbol{r}_m \in \mathbb{C}^{K\times 1}$, we can construct at most K^2 linearly independent columns $\{\tilde{\boldsymbol{r}}_m \in \mathbb{C}^{K^2\times 1}\}$. If $M_0 = K^2$, the matrix $\boldsymbol{W}^{(0)}$ becomes a square matrix, we can easily assure $\mathrm{rank}(\boldsymbol{W}_k) = \bar{M}_{\min} = (K+1)^2$ by devising $\boldsymbol{D}_k$ to satisfy $\mathrm{rank}(\boldsymbol{D}_k) = \bar{M}_{\min} - K^2 = (K+1)^2 - K^2 = 2K+1$. On the other hand, if $M_0 > K^2$, then there is no guarantee that the resulting square matrix $\boldsymbol{W}_k$ will be full rank. Therefore, the maximum number of rank-one measurements that can be shared in recovering $\{\boldsymbol{G}_k\}_{k=1}^{K}$ is equal to $M_0 = K^2$.

As for the second step, the measurements obtained in the first step can be directly reused for the second step. Consequently, the minimum number of rank-one measurements required by the proposed method is $M_{min} = (N-K)\bar{M}_{\min} - (N-K-1)M_0 = (N-K)(2K+1) + K^2$.

B. Computational Complexity Analysis

Next, we analyze the computational complexity of the proposed method. The major computational tasks include recovering $\{\boldsymbol{G}_k\}_{k=1}^{N-K}$ and $\tilde{\boldsymbol{S}}$, as well as obtaining $\mathrm{null}(\boldsymbol{A}^H)$. As for the recovery of $\boldsymbol{G}_k$, the dominant operation is the calculation of $\boldsymbol{W}_k^H \boldsymbol{y}_k$ in (14), which incurs a computational complexity of $\mathcal{O}(\bar{M}(K+1)^2)$. Similarly, the computational complexity in recovering $\tilde{\boldsymbol{S}}$ is at the order of $\mathcal{O}(LK^2)$. To obtain $\mathrm{null}(\boldsymbol{A}^H)$, a singular value decomposition needs to be performed, which incurs a computational complexity at the order of $\mathcal{O}(NK^2)$, considering the fact that $N \gg K$ [29]. Thus, the overall computational complexity of the proposed method is $\mathcal{O}((N-K)\bar{M}(K+1)^2 + LK^2 + NK^2)$.

[1]These measurements can be come from $\{y_{k,m}\}_{k=1,m=1}^{N-K,\bar{M}}$ as long as they satisfy the linear independence constraints required to recover $\boldsymbol{S}$.

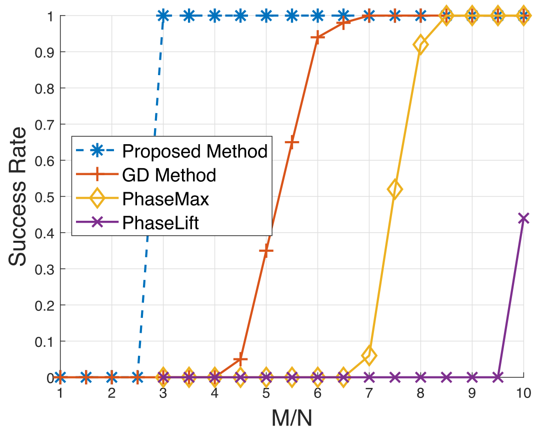

Fig. 1. Success Rate versus M/N when N=100.

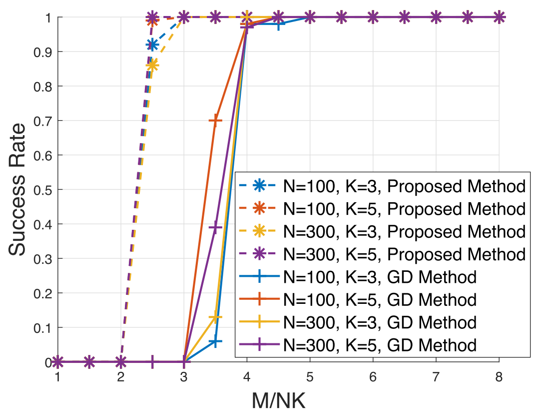

Fig. 2. Success Rate versus M/NK under different choices of (N, K).

V. Simulation Results

In this section, we demonstrate the effectiveness of the proposed method through numerical results. We consider a low-rank PSD matrix $\boldsymbol{R} = \boldsymbol{X}\boldsymbol{X}^H$, where entries of $\boldsymbol{X} \in \mathbb{R}^{N\times K}$ are independent and identically distributed (i.i.d.) random variables following a Gaussian distribution $\mathcal{N}(0, \frac{1}{N})$. The relative estimation error is defined as $\epsilon \triangleq \|\hat{\boldsymbol{R}} - \boldsymbol{R}\|_F / \|\boldsymbol{R}\|_F$ where $\hat{\boldsymbol{R}}$ is the recovered PSD matrix.

We first consider the case of $K = 1$. In this case, the problem reduces to a phase retrieval problem. Classical methods for phase retrieval include phaseMax [15] and phaseLift [14], with the sample complexity respectively given by $\mathcal{O}(N)$ and $\mathcal{O}(N \log N)$. Also, for the phase retrieval problem, the gradient descent (GD) method [9] has a sample complexity of $\mathcal{O}(N \log N)$, which achieves a state-of-the-art performance. We use the metric of success rate to evaluate the performance of respective methods. In each trial, if the relative estimation error ϵ is less than 1×10^{-5}, the trial is considered successful. Fig. 1 depicts the success rates of respective methods as a function of the ratio M/N. Results are averaged over 500 trials. From Fig. 1, we see that our proposed method can recover the PSD matrix using only $3N$ measurements, which presents a clear advantage over existing methods which require at least $8N$ measurements to ensure successful matrix recovery.

In Fig. 2, we plot the success rates of our proposed method and the GD method under different choices of (N, K). We see that our proposed method requires about $2NK \sim 3NK$ measurements to recover the low-rank PSD matrix, which is consistent with our theoretical analysis. The GD method requires more measurements (about $5NK$) than our proposed method for recovery of the PSD matrix. Fig. 3 reports the average running time of respective algorithms as a function of the matrix dimension N. Given $N = 200$, $K = 2$, the proposed method only requires 0.04s to exactly recover the PSD matrix while the GD method needs nearly 1s to converge to a critical point. We see that the proposed method is much more computationally efficient than the GD method since it admits an analytical solution.

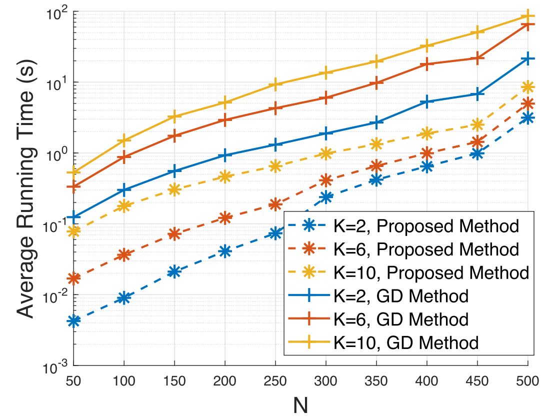

Fig. 3. Average running Time (s) versus various combinations of (N, K).

VI. Conclusion

In this letter, we proposed an analytical solution to solve the challenging low-rank PSD matrix recovery problem from its rank-one measurements. The proposed method requires a total number of $(N - K)(2K + 1) + K^2$ measurements for matrix recovery, which is significantly smaller than the number of measurements required by existing methods. Numerical results were provided to illustrate the efficiency of the proposed method.

Appendix

The basic idea is to prove that the rank of $\boldsymbol{A}$ is equal to $N - K$. To show this, we first rewrite

$$\boldsymbol{a}_k^\star = \boldsymbol{V}_k \boldsymbol{w}_k^\star = \boldsymbol{V}_0 \boldsymbol{u}_k + x_k \boldsymbol{v}_{K+k}, k = 1, \ldots, N - K, \quad (20)$$

where $\boldsymbol{w}_k^\star = [\boldsymbol{u}_k^H \;\; x_k^*]^H$. Typically, we have $x_k \neq 0$, $\forall k$ with probability 1 [28]. Define

$$\bar{\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{\Gamma} = [\tilde{\boldsymbol{a}}_1^\star \; \tilde{\boldsymbol{a}}_2^\star \; \ldots \; \tilde{\boldsymbol{a}}_{N-K}^\star], \quad (21)$$

where $\Gamma = \text{diag}(1/x_1, 1/x_2, \ldots, 1/x_{N-K})$, $\tilde{\boldsymbol{a}}_k^\star = \boldsymbol{V}_0 \tilde{\boldsymbol{u}}_k + \boldsymbol{v}_{K+k}$, and $\tilde{\boldsymbol{u}}_k = \boldsymbol{u}_k / x_k$.

Clearly, we have $\text{rank}(\boldsymbol{A}) = \text{rank}(\bar{\boldsymbol{A}})$. Defining $\boldsymbol{T} = \bar{\boldsymbol{A}}^H \bar{\boldsymbol{A}}$, we can calculate $\boldsymbol{T}$ as

$$\boldsymbol{T} = \begin{bmatrix} \|\tilde{\boldsymbol{u}}_1\|_2^2 + 1 & \tilde{\boldsymbol{u}}_1^H \tilde{\boldsymbol{u}}_2 & \ldots & \tilde{\boldsymbol{u}}_1^H \tilde{\boldsymbol{u}}_{N-K} \\ \tilde{\boldsymbol{u}}_2^H \tilde{\boldsymbol{u}}_1 & \|\tilde{\boldsymbol{u}}_2\|_2^2 + 1 & \ldots & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\boldsymbol{u}}_{N-K}^H \tilde{\boldsymbol{u}}_1 & \ldots & \ldots & \|\tilde{\boldsymbol{u}}_{N-K}\|_2^2 + 1 \end{bmatrix}.$$

This can be recast as

$$\boldsymbol{T} \overset{(a)}{=} \tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^H + \boldsymbol{I} \overset{(b)}{=} \boldsymbol{P}_w(\boldsymbol{\Sigma}_w + \boldsymbol{I})\boldsymbol{P}_w^H, \quad (22)$$

where in (a) we define $\tilde{\boldsymbol{U}} \triangleq [\tilde{\boldsymbol{u}}_1 \; \tilde{\boldsymbol{u}}_2 \; \ldots \; \tilde{\boldsymbol{u}}_{N-K}]$; in (b) we express the eigen-value decomposition $\tilde{\boldsymbol{U}}\tilde{\boldsymbol{U}}^H = \boldsymbol{P}_w \boldsymbol{\Sigma}_w \boldsymbol{P}_w^H \succeq 0$. As a result, we have $\text{rank}(\boldsymbol{T}) = \text{rank}(\boldsymbol{\Sigma}_w + \boldsymbol{I}) = N - K = \text{rank}(\bar{\boldsymbol{A}}) = \text{rank}(\boldsymbol{A})$. This concludes the proof.

REFERENCES

[1] Z.-L. Shi, X. P. Li, W. Li, T. Yan, J. Wang, and Y. Fu, "Robust low-rank matrix recovery as mixed integer programming via ℓ_0-Norm optimization," *IEEE Signal Process. Lett.*, vol. 30, pp. 1012–1016, 2023.

[2] X. P. Li, Z.-L. Shi, Q. Liu, and H. C. So, "Fast robust matrix completion via entry-wise ℓ_0-Norm minimization," *IEEE Trans. Cybern.*, vol. 53, no. 11, pp. 7199–7212, Nov. 2023.

[3] A. Pramanik, H. K. Aggarwal, and M. Jacob, "Deep generalization of structured low-rank algorithms (Deep-SLR)," *IEEE Trans. Med. Imag.*, vol. 39, no. 12, pp. 4186–4197, Dec. 2020.

[4] E. J. Candes, T. Strohmer, and V. Voroninski, "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," *Commun. Pure Appl. Math.*, vol. 66, no. 8, pp. 1241–1274, Aug. 2013.

[5] S. Gu, L. Zhang, W. Zuo, and X. Feng, "Weighted nuclear norm minimization with application to image denoising," in *Proc. IEEE Conf. Comput. Vis. pattern Recognit.*, Columbus, OH, USA, 2014, pp. 2862–2869.

[6] R. Kueng, H. Rauhut, and U. Terstiege, "Low rank matrix recovery from rank one measurements," *Appl. Comput. Harmon. Anal.*, vol. 42, no. 1, pp. 88–116, Jan. 2017.

[7] R. Zhou, J. Ying, and D. P. Palomar, "Covariance matrix estimation under low-rank factor model with nonnegative correlations," *IEEE Trans. Signal Process.*, vol. 70, pp. 4020–4030, 2022.

[8] Y. Chi, Y. M. Lu, and Y. Chen, "Nonconvex optimization meets low-rank matrix factorization: An overview," *IEEE Trans. Signal Process.*, vol. 67, no. 20, pp. 5239–5269, Oct. 2019.

[9] Y. Li, C. Ma, Y. Chen, and Y. Chi, "Nonconvex matrix factorization from rank-one measurements," in *Proc. 22nd Int. Conf. Artif. Intell. Statist.*, 2019, pp. 1496–1505.

[10] Y. Li, C. Ma, Y. Chen, and Y. Chi, "Nonconvex matrix factorization from rank-one measurements," *IEEE Trans. Inf. Theory*, vol. 67, no. 3, pp. 1928–1950, Mar. 2021.

[11] X. Xu, Y. Shen, Y. Chi, and C. Ma, "The power of preconditioning in overparameterized low-rank matrix sensing," in *Proc. Int. Conf. Mach. Learn.*, 2023, pp. 38611–38654.

[12] L. Qin, Z. Song, and R. Zhang, "A general algorithm for solving rank-one matrix sensing," in *Proc. 27th Int. Conf. Artif. Intell. Statist.*, S. Dasgupta, S. Mandt, and Y. Li, Eds., May 2024, vol. 238, pp. 757–765.

[13] D. Stóger and Y. Zhu, "Non-convex matrix sensing: Breaking the quadratic rank barrier in the sample complexity," 2024, *arXiv:2408.13276*.

[14] E. J. Candès, T. Strohmer, and V. Voroninski, "PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming," *Commun. Pure Appl. Math.*, vol. 66, no. 8, pp. 1241–1274, Aug. 2013.

[15] T. Goldstein and C. Studer, "PhaseMax: Convex phase retrieval via basis pursuit," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2675–2689, Apr. 2018.

[16] Y. Li, Y. Sun, and Y. Chi, "Low-rank positive semidefinite matrix recovery from corrupted rank-one measurements," *IEEE Trans. Signal Process.*, vol. 65, no. 2, pp. 397–408, Jan. 2017.

[17] A. Kalev, R. L. Kosut, and I. H. Deutsch, "Quantum tomography protocols with positivity are compressed sensing protocols," *NPJ Quantum Inf.*, vol. 1, no. 1, pp. 1–6, Dec. 2015.

[18] L. Tian, J. Lee, S. B. Oh, and G. Barbastathis, "Experimental compressive phase space tomography," *Opt. Exp.*, vol. 20, no. 8, pp. 8296–8308, 2012.

[19] R. Kueng, H. Rauhut, and U. Terstiege, "Low rank matrix recovery from rank one measurements," *Appl. Comput. Harmon. Anal.*, vol. 42, no. 1, pp. 88–116, Jan. 2017.

[20] A. Aubry, A. Maio, and L. Pallotta, "A geometric approach to covariance matrix estimation and its applications to radar problems," *IEEE Trans. Signal Process.*, vol. 66, no. 4, pp. 907–922, Feb. 2018.

[21] R. W. Harrison, "Phase problem in crystallography," *J. Opt. Soc. America A*, vol. 10, no. 5, pp. 1046–1055, May 1993.

[22] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao, and M. Segev, "Phase retrieval with application to optical imaging: A contemporary overview," *IEEE Signal Process. Mag.*, vol. 32, no. 3, pp. 87–109, May 2015.

[23] H. An, J. Wu, K. C. Teh, Z. Sun, and J. Yang, "Geosynchronous spaceborn airborne bistatic SAR imaging based on fast low-rank and sparse matrices recovery," *IEEE Trans. Geosci. Remote Sens.*, vol. 60, pp. 1–14, 2022.

[24] D. D. Ariananda and G. Leus, "Compressive wideband power spectrum estimation," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4775–4789, Sep. 2012.

[25] D. Romero, D. D. Ariananda, Z. Tian, and G. Leus, "Compressive covariance sensing: Structure-based compressive sensing beyond sparsity," *IEEE Signal Process. Mag.*, vol. 33, no. 1, pp. 78–93, Jan. 2016.

[26] O. Mehanna and N. D. Sidiropoulos, "Frugal sensing: Wideband power spectrum sensing from few bits," *IEEE Trans. Signal Process.*, vol. 61, no. 10, pp. 2693–2703, May 2013.

[27] H. Fu and Y. Chi, "Quantized spectral compressed sensing: Cramer-Rao bounds and recovery algorithms," *IEEE Trans. Signal Process.*, vol. 66, no. 12, pp. 3268–3279, Jun. 2018.

[28] X. Feng and Z. Zhang, "The rank of a random matrix," *Appl. Math. Computation*, vol. 185, no. 1, pp. 689–694, Feb. 2007.

[29] X. Li, S. Wang, and Y. Cai, "Tutorial: Complexity analysis of singular value decomposition and its variants," 2019, *arXiv:1906.12085*.