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Correlation-Function Techniques: An analysis of their performance and capabilities

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A. INTRODUCTION

The interest in correlation-function techniques shown by a number of groups working at the RWS is mainly due to the two following reasons:

1. BOUNDARY-VALUES DETERMINATION:

Most mathematical models of physical systems need as input information the values of certain physical quantities along the geometrical boundary of the model and do so for long periods of time. For example, two-dimensional models for tide and water-quality prediction require as input the values of water heights and/or velocities, concentrations of salt and pollutants along the boundary of the region under study. Obviously, it is impossible to actually measure these physical quantities along the whole boundary, even for short periods of time. The best one can do is to measure them at, say, N stations on the boundary and try to interpolate or deduce from these measurements the values of the physical quantities at other positions on the boundary. The number of stations, N, should be large enough so that the model has all the required information and the interpolation and computation errors are kept sufficiently small. On the other hand one should choose the positions of the stations and the "interpolation" procedure in such a way that N, the number of stations required, is as small as possible.

Correlation-function techniques offer one possible "interpolation" procedure for the determination of boundary values. Once the correlation function between two positions (i.e. stations) has been experimentally determined, it is sufficient to measure the relevant quantities at one station (the "reference" station) for a period of time, in order to know also the values at the second station for, roughly speaking, the same period of time. Moreover, the correlation functions between different quantities (e.g. water levels and velocities) at the same station can be determined and consequently the measurement of only one physical parameter at one station could allow for the calculation of more parameters at a number of stations.

2. PREDICTION:

Once the correlation-function between the two stations has been experimentally determined, it may be that it shows the existence of an appreciable time-lag between the stations. If such is the case, then the measurements at one station could be used to predict the values at the other station. Relevant examples could be: the propagation of a storm from the English coast towards the Dutch coast or the propagation of a tidal wave from a rivermouth up-stream.

For these reasons, preliminary studies were undertaken some time ago at RWS (see e.g. report P-5299, Sept. 1974 by S.K. Liu, J.J. Leendertse and J. Voogt and DDWT-77.122 by M. de Ras). As part of these studies, correlation functions between a number of stations on the Dutch shelf of the North Sea, were determined. If one looks at the results that were then obtained, one discovers some surprises. In particular, the transfer functions are not smooth functions of the frequency but display a pronounced "structure", i.e. peaks and dips (see fig. 1). This is not what one would expect from our knowledge of hydrodynamics and from the dimensions of the North Sea. We do not expect resonances to appear in the frequency range from zero to a couple of cycles per day. Moreover, most of the peaks and dips seem to appear at the frequencies characteristic of the tide, i.e. one, two or four cycles per day. This cannot be a coincidence. (For definitions of transfer and other function, see Section D).

The present study was undertaken in order to clarify the meaning of the correlation functions as were obtained in these previous studies, to improve, if possible, on these former calculations and to establish criteria that could make these results and techniques more useful.

B. SOME COMMENTS ABOUT DIFFERENT TERMINOLOGIES AND APPROACHES TO THE PROBLEM

Some books on statistics and statistical analysis of signals, devote little space to cause-effect relationships between varables. This has led some engineers and statisticians to a slightly blurred terminology and sometimes even to misunderstandings. For example, they will call any correlation function between two variables "response function", even when these variables are not causally related. A more consequent terminology would be to use "response function" only when one variable is the physical cause of the other variable; under these conditions, the response function satisfies well-defined mathematical properties.

In the above-mentioned report by Liu, Leendertse and Voogt, equation 3 defines the response function H(t) as follows:

$$y(t) = \int_{0}^{\infty} ds\ H(s)\ x(t-s).$$

Since the integration is never over negative values of s, y(t) can be affected only by x(t') with $t' \leq t$, i.e. by the past history of x(t). Therefore in this case it is totally consistent to say that y(t) is 'the effect', x(t) 'the cause' and H(t) 'the response function'. Unfortunately, Liu, Leendertse and Voogt apply this definition to the case when y(t) and x(t) are water levels at two different positions, i.e. to two variables which do not satisfy a cause-effect relationship. If such is the case, the integration over s in the above formula has to run from minus infinity to plus infinity. Liu, Leendertse and Voogt took into account this correction but they continued to call H(t) "the response function".

Another point which is worth mentioning is that many people working with mathematical models, e.g. of tidal movement in a finite area, tend to forget the physical forces responsible for the phenomenon: the "input" for these models are the boundary conditions (e.g. water levels and velocities on the boundary) and there is a tendency to identify this "input" with the real, physical causes leading to the occurrence of the tides. This identification can also lead to misunderstandings and loss of insight.

C. CALCULATIONS DONE WITH FRANOM AND DTBAPP

As mentioned before, one does not expect peaks to show up in the response and transfer functions of the North Sea; at the frequencies under consideration they should be smooth functions. In order to verify the results obtained previously and to get some experience with the programmes, new calculations of correlation functions were undertaken. As input data we used the measurements of bottom-pressure recorders lying on a line between Cromer (England) and Den Helder. These recordings were made every 30 minutes for a period of approximately 15 days, from August 22nd through September 9th, 1972. The results obtained with the FRANOM package of programmes can be seen in fig. 1 to 16. One observes again strong peaks of the transfer function at tidal frequencies (see fig. 11 and 14). Notice that the coherency function approaches unity only at astronomical (tidal) frequencies (fig. 15).

A preliminary analysis of these results led to the idea that one is in fact measuring two different correlation functions associated with two different generating mechanisms: an astronomical one (gravitation) and a meteorological one (wind- sea friction, etc.). For this reason we split each input series in two: one containing only astronomical components, the other one containing the rest, which we call the meteorological component. The series so generated were analyzed separately. The results are presented in fig. 17 to 46. No noticeable improvement was observed of, for example, the coherency between the meteorological components at different stations (see fig. 45).

Since the amplitudes of the meteorological components are much smaller than those of the (main) tidal components, we considered the possibility that the low values of the coherency were due to large errors in the meteorological components. For this reason more extensive and accurate calculations were undertaken. This time we used water-levels from the station B.G.II and velocity measurements from the station Noord-Banjaard , which were made every 30 minutes from July 19th through October 24th, 1974. There were some 'blank' periods in these series which were filled in with the corresponding HATYAN programma. The first two months were used to determine the correlation functions by means of the DTBAPP package and these correlations were then used to 'predict' the third month (water levels at B.G.II as input, velocity at Noord-Banjaard as output). Moreover, using HATYAN, the tidal components were determined for the first two months and in this way each series of measurements was split in two: an astronomical series and a meteorological one. The results are shown in fig.47. to. 74.

The astronomical component is "predicted" rather accurately, however, the coherency between meteorological components did not improve.

In the next sections we will present a theoretical analysis which shows that it is impossible to define meaningful and/or useful correlation functions between meteorological components. For the sake of clarity and precision, we quote the definitions used by Liu, Leendertse and Voogt;

Response function H(t)

$$y(t) = \int_0^\infty ds\ H(s)\ x(t-s) \quad \text{or} \quad y(t) = \int_{-\infty}^\infty ds\ H(s)\ x(t-s),$$

cross-spectrum $P_{yx}(\omega)$

$$P_{yx}(\omega) = \int_{-\infty}^{\infty} dt \ \gamma_{yx}(t) \ \exp\ (it\omega),$$

auto-spectrum $P_{xx}(\omega)$

$$P_{xx}(\omega) = \int_{-\infty}^{\infty} dt \ \gamma_{xx}(t) \ \exp\ (it\omega),$$

where e.g.

$$\gamma_{yx}(t) = \frac{1}{2T} \int_{-T}^{T} ds \ x(s) y(s+t),$$

and coherency function (squared)

$$\Omega^2_{xy}(\omega) = |P_{xy}(\omega)|^2 / P_{xx}(\omega) \ P_{yy}(\omega).$$

D. PHYSICAL AND MATHEMATICAL BACKGROUND

D1. RESPONSE THEORY, THE LINEAR CASE

Let us consider a system which can be described by a set of variables $\{v\}$ and whose evolution in time, is totally and uniquely determined by a differential equation.

$$\frac{\partial}{\partial t} v(t) = M(v) + F(t), \qquad (1)$$

where F(t) is some external force acting on the system. The set of variables $\{v\}$ can be discrete (e.g. the position and velocity of a particle) or continuous (e.g. the velocity field of a fluid as a function of position). In the last case M will in general contain space derivatives of v, i.e. $\partial v/\partial x$, $\partial^2 v/\partial x^2$, etc. For the time being, we will limit ourselves to strictly linear systems, i.e.

$$M(\lambda v) = \lambda M(v), \tag{2}$$

where λ is an arbitrary number. In this case, we can write eq.(1) as

$$\frac{\partial v}{\partial t} = Mv + F(t), \tag{3}$$

where M is some operator independent of v. The most general solution to eq.(3) is

$$v(t) = R(t)v(0) + \int_0^t ds\ R(t-s)F(s), \tag{4a}$$

with

$$R(t) \equiv \exp(Mt), \tag{4b}$$

and v(0) is the initial condition, i.e. the given value of v at some initial time t=0. R(t) is usually called the memory Kernel (or operator). The fact that the integration in eq.(4) extends up to s=t, is a reflexion of causality, i.e. only past causes, F(s) with $s<t$, can affect events at time t. Substituting this last expression for v(t) in eq.(3), shows that this is, in effect, the required solution.

Example

Let us say that we are dealing with the problem of (molecular or gradient-type) diffusion of a substance in a one-dimensional fluid at rest. Then the time evolution of the solute concentration c(x,t) is given by

$$\frac{\partial}{\partial t} c(x,t) = D \frac{\partial^2}{\partial x^2} c(x,t) + F(x,t), \tag{5}$$

where D is the (molecular) diffusion coefficient and F(x,t) is a distribution of sources and sinks which is given or controlled from outside the system. We have then

$$c(x,t) = R(t)c(x,0) + \int_0^t ds\ R(t-s)F(x,s), \tag{6a}$$

with

$$R(t) \equiv \exp(tD\partial^2/\partial x^2), \tag{6b}$$

We need now an explicit expression for the memory operator (6b). For any integrable function f(x) one has that

$$\exp(tD\partial^2/\partial x^2)f(x) = (2\pi Dt)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dy \, \exp\left[-(x-y)^2/2Dt\right] f(y). \qquad (7)$$

This identity can be proven, e.g., by replacing $f(x)$ by its Fourier expansion. It should be stressed that the explicit form of the memory operator $R(t)$ is strongly influenced by the spatial boundary conditions on $f(x)$. In the above example the boundary conditions are $f(x) \to 0$ sufficiently fast for $|x| \to \infty$, since $f(x)$ had to be integrable. See section D5.

D2. FINITE MEMORY

For most physical systems, the state at a certain time is not determined by past facts which took place (infinitely) long ago. Mathematically, this is expressed by

$$R(t)v(0) \longrightarrow 0 \qquad \text{for} \quad t \gg \tau, \qquad (8)$$

where τ is a characteristic time associated with M (and possibly with $v(0)$ too). Then one has

$$v(t) \to \int_{-\infty}^{t} ds \, R(t-s)F(s) \qquad \text{for } t \gg \tau. \qquad (9a)$$

If the system has been 'started' so long ago that all actual measurements are independent of the initial condition $v(0)$ then, for all practical purposes, one can write

$$v(t) = \int_{-\infty}^{t} ds \, R(t-s)F(s). \qquad (9b)$$

D3. RESPONSE FUNCTION

Fourier-transformation of eq.(9b) over time gives

$$v(\omega) = R(\omega) \; F \; (\omega), \qquad (10)$$

where e.g $R(\omega) \equiv \int_{-\infty}^{\infty} dt \, \exp\,(i\omega t) \; R(t)$.

The function $R(\omega)$ is called the response function. Since the integration in eq.(9b) runs up to s=t, one can take $R(t)=0$ for all $t<0$. This implies that if $R(\omega)$ has any pole on the complex-ω plane, then these poles have to lie on the lower half plane ($\mathrm{Im}\ \omega < 0$). When $R(t)$ is an operator in real space, i.e. when it contains space derivatives, it is convenient and common practice to take the Fourier-transform both over time <u>and</u> space; the resulting memory function $R(k,\omega)$ is then a function of wave-vector k and frequency ω. For example, in the above case of molecular diffusion one obtains

$$R(k,\omega) = [Dk^2 - i\omega]^{-1},$$

$$v(k,\omega) = R(k,\omega)\ F(k,\omega)$$

where for example

$$F(k,\omega) = \int dx \int dt\ F(x,t) \exp[i(\omega t - kx)].$$

I.e. $R(k,\omega)$ has a pole in the lower half-plane at $\omega = -iDk^2$. Moreover, since all physical quantities are real, one has that

$$R^*(-k^*, -\omega^*) = R(k,\omega),$$

where the asterisk denotes complex conjugation.

D4. TRANSFER FUNCTIONS

Let us consider two particular system variables, say v_1 and v_2. For each one of them an equation analogous to eq.(10) holds, i.e.

$$v_1(\omega) = R_1(\omega)\ F(\omega) \quad \text{and} \tag{11a}$$

$$v_2(\omega) = R_2(\omega)\ F(\omega), \tag{11b}$$

where $R_1(\omega)$ and $R_2(\omega)$ are the corresponding response functions. It follows straightforwardly that

$$\begin{aligned} v_1(\omega) &= R_1(\omega)\ R_2^{-1}(\omega)\ v_2(\omega) \\ &\equiv C_{12}(\omega)\ v_2(\omega). \end{aligned} \tag{12}$$

The function $C_{12}(\omega)$ is called the transfer function between variables 1 and 2. Notice that it is not symmetric in 1 end 2. Notice also that $C_{12}(\omega)$ may have poles anywhere in the complex-ω plane, since the possible zeros of $R_2(\omega)$ in the upper half-plane, i.e. Im $\omega>0$, become poles of $C_{12}(\omega)$. This is the mathematical expression of the physical fact that v_1 and v_2 are not causally connected, they are only correlated due to the fact that both are causally connected to the same external force F. Going back from frequency to time representation, the analytical properties of $C_{12}(\omega)$ imply that

$$v_1(t) = \int_{-\infty}^{\infty} ds\ C_{12}(t-s)v_2(s). \tag{13}$$

The fact that the time integration in eq.(13) extends from past (s<t) to future (s>t) is again a reflexion of the absence of a cause-and-effect relationship between $v_1(t)$ and $v_2(t)$: the values of v_1 at a certain time t_o may be correlated with, but not caused by, the values of v_2 at later times $t_o + \Delta$, $\Delta > 0$.

When the system variables and the external force are position dependent, i.e. $v \equiv v(x,t)$, then it is impossible to define a transfer function between variables at two different positions : for homogeneous systems what one should do is to introduce a transfer $C_{12}(k,\omega)$ between the space-Fourier transforms $v_1(k,\omega)$ and $v_2(k,\omega)$ at the same wave-vector k since now

$$v_1(x,\omega) = \int dy\ R_1(x-y,\omega)F(y,\omega) \quad \text{and} \quad v_2(x,\omega) = \int dy\ R_2(x-y,\omega)F(y,\omega)$$

and therefore

$$v_1(k,\omega) = R_1(k,\omega)F(k,\omega) \quad \text{and} \quad v_2(k,\omega) = R_2(k,\omega)F(k,\omega).$$

This limitation can be overcome only when the external force takes very special forms. Two of these particular forms are

a) A point force at position x_o, i.e.

$$F(x,t) = \phi(t)\delta(x-x_o), \tag{14}$$

where $\delta(x)$ is the (one-dimensional) Dirac delta function satisfying

$$\delta(x) = 0 \text{ for } x \neq 0 \text{ and } \int_{-\infty}^{\infty} dx\ \delta(x) = 1.$$

In this particular case one has

$$F(k,\omega) = \phi(\omega)\ \exp\ (-ikx_o), \tag{15}$$

therefore

$$v(x,\omega) = \int dk\ \exp\left[ik(x-x_o)\right]\ R(k,\omega)\ \phi(\omega)$$

$$\equiv R(x-x_o,\omega)\ \phi(\omega), \tag{16a}$$

and similarly

$$v(y,\omega) = R(y-x_o,\omega)\ \phi(\omega). \tag{16b}$$

Consequently, it is possible to write

$$v(x,\omega) = R(x-x_o,\omega)\ R^{-1}(y-x_o\omega)\ v(y,\omega)$$

$$\equiv C_{xy}(x_o,\omega)\ v(y,\omega). \tag{17}$$

Notice that the response and transfer functions depend upon x_o, the location of the external force. Therefore $C_{xy}(x_o,\omega)$ will be useful only if the external force acts <u>always</u> at the same position x_o (e.g. the discharge of a river into a lake or bay).

b) Position-independent force, i.e.

$$F(x,t) = \phi(t) \tag{18}$$

Then for all x

$$v(x,\omega) = v(k=0,\omega)\ \phi(\omega) \tag{19}$$

and trivially

$$C_{xy}(\omega) \equiv 1. \tag{20}$$

D5. NON-HOMOGENEOUS SYSTEMS

In the preceding sections we have considered unbounded, homogeneous systems. However, the systems we would like to understand, e.g. the sea not far from the coast, estuaries, etc., are bounded, non-homogeneous and anisotropic. For these (linear) systems the most general causal relationship will be expressed by

$$v(x,t) \equiv \int_{-\infty}^{t} ds \int dy \; R(x,y,t-s) \; F(y,s), \qquad (21)$$

where F(x,t) is the position-dependent external force and the space integration extends over the whole volume of the system. From eq.(21) it follows that, just like in the homogeneous case, it is impossible to define, for arbitrary forces F, a transfer function between variables at two different positions. This can be done only for special forces F. In particular

a) Point force at x_o, as given by eq.(14), then

$$v(x,\omega) = R(x,x_o,\omega) \; \phi(\omega) \qquad (22)$$

and consequently the transfer function between variables at positions x and y is given by

$$v(x,\omega) = R(x,x_o,\omega) \; R^{-1}(y,x_o,\omega) \; v(y,\omega)$$

$$\equiv C_{xy}(x_o,\omega) \; v(y,\omega). \qquad (23)$$

Notice that the response and transfer functions depend upon x_o, the location of the external force. Therefore the same limitations as those discussed in D4.a. apply in this case.

b) Position-independent force, as given by eq.(18). In this case

$$v(x,\omega) = \int dy \; R(x,y,\omega) F(\omega) \qquad (24)$$

and consequently

$$C_{xy}(\omega) = \int dz \; R(x,z,\omega) \; \left[\int dz \; R(y,z,\omega)\right]^{-1}. \qquad (25)$$

D6. MORE GENERAL TYPES OF EXTERNAL FORCES

Up till now we have considered only the simple coupling to the external forces as expressed by eq.(3). In many practical situations, however, one has to deal with more complicated types of coupling. For example, when studying the dynamics of velocity fields $\vec{v}(\vec{x},t)$ and surface elevation $h(\vec{x},t)$ in the sea, the external forces acting on the system are:

a) The gravitational attraction exerted by the moon and the sun moving on their astronomical orbits with respect to the earth. This leads to a time dependent force which has the form of eq.(3)

b) Meteorological forces, in particular the wind stress acting on the sea surface due to a difference between the wind velocity $\vec{w}(\vec{x},t)$ and the sea surface velocity $\vec{v}(\vec{x},t)$. This force is usually approximated by an expression of the following type

$$\vec{F}_w = C_w(\vec{w}-\vec{u})|\vec{w}-\vec{u}|/H, \qquad (26)$$

where C_w is a kind of friction or drag coefficient, $\vec{u}$ is a depth averaged sea velocity and H is the sea depth. The essential difference between eq.(3) and the last equation is that in the latter one the <u>system variables</u> $\vec{u}$ and H show up. This calls for a completely different mathematical analysis and leads to non-linear effects which, in general, cannot be neglected (similar in nature to the 'secular effects' of classical dynamics). Consequently, it is impossible to define (linear) response and transfer functions as was done in the previous sections.

<u>Example</u>

The simplest example would be

$$\frac{dv}{dt} = -\gamma v + c(t)v + f(t), \qquad (27)$$

where both c(t) and f(t) are external forces. The corresponding solution is

$$v(t) = R(t,0)v(0) + \int_{-\infty}^{t} d\tau\ R(t,\tau)f(\tau), \qquad (28a)$$

with

$$R(t,\tau) \equiv \exp \int_{\tau}^{t} ds[c(s)-\gamma] . \qquad (28b)$$

If the system is stable then one has that $R(t,\tau)\to 0$ for $t \gg \tau$. From these results it is clear that the response is strongly non-linear in c(t), moreover since it is the time integral of c(t) that matters, it is not sufficient that c(t) is small in order to linearize in c(t). In frequency space, while it may be true that $R(\omega_v,\omega_f)$ is peaked around $(\omega_v - \omega_f) = 0$, $R(\omega_v,\omega_f)$ does not vanish for $\omega_v - \omega_f \neq 0$. Consequently many different input frequencies ω_f contribute to the output response at frequency ω_v. Therefore it is impossible to introduce simple transfer functions and eliminate the external forces f(t) and c(t) as was done in sections D4 and D5. Formula (28) explains also why the noise spectrum has peaks at the tidal freqeencies for which the coherency is highest, $\Omega \simeq 1$. This apparent paradox was already noticed by De Ras (DDWT-77.122). Needless to say, real situations are even more complicated than the example presented above. As it was mentioned at the beginning of this section, the coupling to external forces appearing in the equations describing the dynamics of the sea, not only contains system variables (the sea velocity) but these variables appear non-linearly.

D7. NON-LINEAR EFFECTS

The wind-stress term mentioned in the previous section is not the only non-linear effect affecting the water movement. One also has the advective term $(\vec{v} \cdot \vec{\nabla}\, \vec{v})$ and the bottom friction which is similar in form to the wind-stress, eq.(26). In particular in shallow water, when the sea depth H is small, these non-linear terms become rather strong and generate higher harmonics of the fundamental tidal frequencies.

Mathematically this is expressed through the non-linear response functions R_n, which allow to write

$$v(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{t} ds_1 \int_{-\infty}^{t} ds_2 \, ... \int_{-\infty}^{t} ds_n \, R_n(t-s_1, t-s_2, ..., t-s_n) F(s_1) .... F(s_n), \quad (29a)$$

or

$$v(\omega) = \sum_{n=1}^{\infty} R_n(\omega_1, \omega_2, ..., \omega_n) \, F(\omega_1) .... F(\omega_n), \quad (29b)$$

where

$$\omega = \sum_{i=1}^{n} \omega_i \quad \text{for all n.}$$

As mentioned in the previous section, this makes impossible the introduction of transfer functions and elimination of the external forces. A simple example would be as follows:

$$v_1(\omega) = R_1(\omega)\ F(\omega) + R_3(\omega,\omega,-\omega)\ F^2(\omega)\ F^*(\omega) + \ldots, \tag{30a}$$

$$v_2(\omega) = \tilde{R}_1(\omega)\ F(\omega) + \tilde{R}_3(\omega,\omega,-\omega)\ F^2(\omega)\ F^*(\omega) + \ldots, \tag{30b}$$

which would lead to

$$v_2(\omega) = C_{21}(\omega)\ v_1 + \Omega_{21}(\omega,\omega,-\omega)\ v_1^2(\omega)\ v_1^*(\omega) + \ldots \tag{31}$$

an expression which is not useful.

D8: NOISE, FLUCTUATIONS AND COHERENCY

In real physical systems one always has to deal with external and internal sources of noise which can never be totally eliminated. Under such circumstances, it is more convenient and logical to work with averages and other well defined statistical properties. Therefore one introduces the average correlation between two variables $v_1(t)$ and $v_2(t)$ as follows:

$$P_{12}(\tau) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\ v_1(t)\ v_2(t-\tau). \tag{32}$$

Replacing into this expression $v_1(t)$ and $v_2(t)$ by their Fourier expansions and using eq.(10), one obtains

$$P_{12}(\tau) = \int d\omega \exp(i\omega\tau) R_1(-\omega) R_2(\omega) F_1(-\omega) F_2(\omega) \tag{33a}$$

or

$$P_{12}(\omega) = R_1(\omega) R_2(-\omega) F_1(\omega) F_2(-\omega), \tag{33b}$$

where

$$F_i(\omega) = F(\omega) + n_i(\omega), \qquad i=1,2, \tag{33c}$$

$n_i(\omega)$ being the random force acting on the variable i. These random forces are assumed to be totally uncorrelated with $F(\omega)$ and between themselves. It follows then that the relation between the average correlation functions and the transfer functions defined in the previous sections is

$$C_{12}(\omega) \equiv R_1(\omega)\ R_2^{-1}(\omega) = P_{12}(\omega)\ P_{22}^{-1}(\omega)\ \left[1 + |n_2(\omega)/F(\omega)|^2\right]. \qquad (34)$$

Assuming that the noise is much smaller than the non-random component of the external force, a very good approximation to the real transfer function will be given by

$$H_{12}(\omega) = P_{12}(\omega)\ P_{22}^{-1}(\omega). \qquad (35)$$

A measure of the noise at frequency ω can be given by the coherency function, defined as follows

$$|\Omega_{12}(\omega)|^2 \equiv |P_{12}(\omega)|^2\ \ P_{22}^{-1}(\omega)\ P_{11}^{-1}(\omega)$$

$$= \left[1 + |n_1(\omega)/F(\omega)|^2\right]^{-1} \left[1 + |n_2(\omega)/F(\omega)|^2\right]^{-1} \leq 1. \qquad (36)$$

This function is equal to 1 if and only if the forces acting on 1 and 2 are identical, i.e. when there is no noise source between 1 and 2. One also has, from eq.(34), that

$$H_{12}(\omega) \leq C_{12}(\omega), \qquad (37)$$

the equality being satisfied if and only if there is no noise. The properties which where proven for $C_{12}(\omega)$ in sections D4 and D5 and the serious limitations discussed in sections D6 and D7 can be easily translated into similar properties of $H_{12}(\omega)$. The notation for the functions $H_{12}(\omega)$, $P_{ij}(\omega)$ and $|\Omega_{12}(\omega)|^2$ comes very close to the one used by Liu, Leendertse and Voogt.

E. CONCLUSIONS

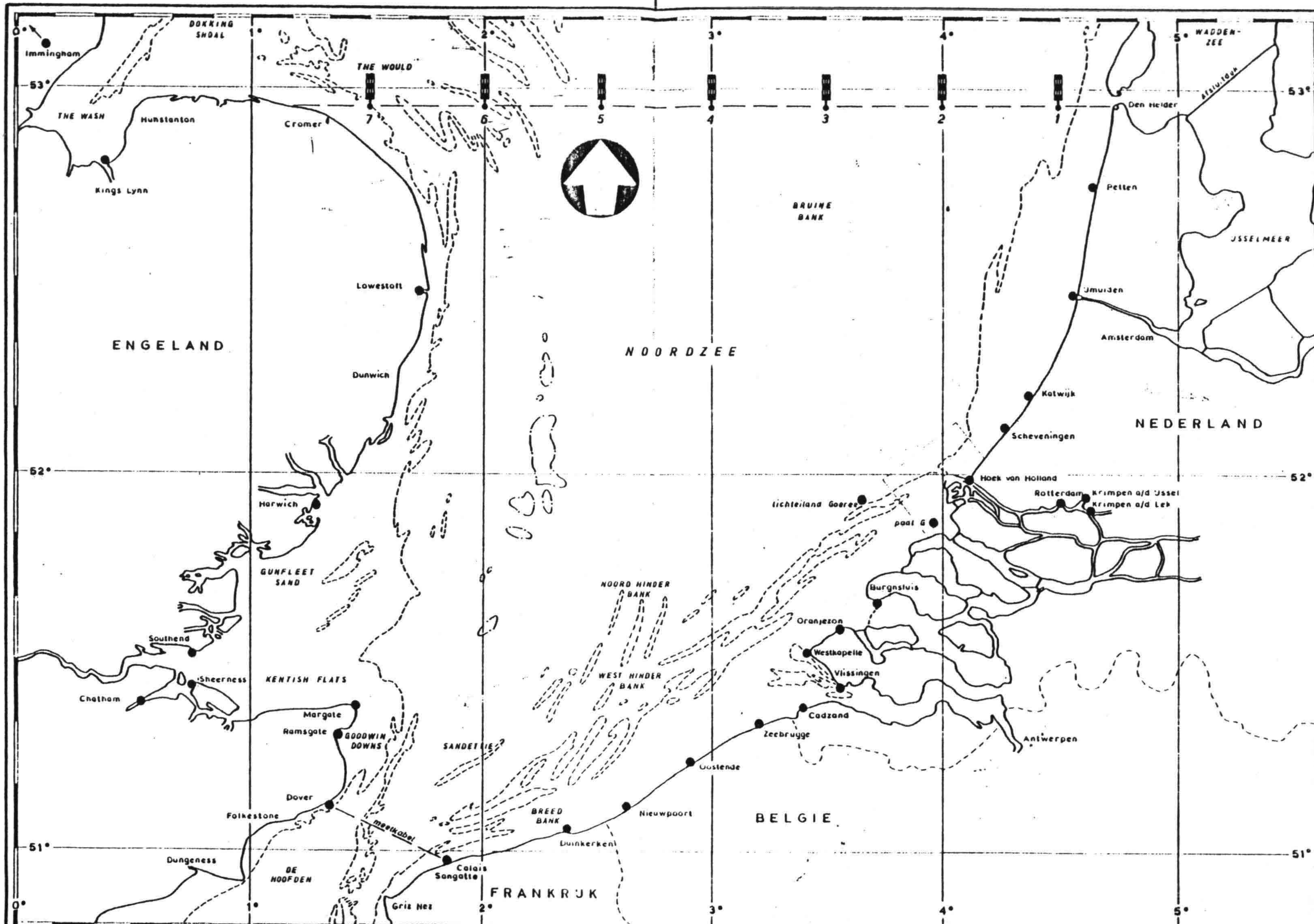
In order to decide when and to what extent correlation function techniques can be a useful working tool, one should first analyse the physical laws governing the phenomenon under consideration.

In the specific case of water levels and velocities at different sea stations, the physical analysis leads to the following conclusions:

1. Correlation function techniques will be useful in the computation or prediction of the fundamental tidal components and their lowest harmonics.
2. The "meteorological component", i.e. non-tidal part, of the water movement cannot be computed or predicted by means of these techniques.
3. The usefulness of this method decreases rapidly with increasing order of the tidal harmonics.

All these conclusions are confirmed by the numerical calculations. In particular the second conclusion is strikingly verified when one computes the correlation function between e.g. water level and one velocity component both measured at the same position; it turns out that even in this case the meteorological components show a negligible coherency function, i.e. $\Omega^2 << 1$. See fig. 75.

Other problems which can be successfully approached with correlation function techniques are, for example, discharges at fixed positions (run-off, cooling-water, etc.) into a basin or bay.


Immingham
DOKKING SHOAL
THE WASH
Hunstanton
Kings Lynn
Cromer
THE WOULD
Lowestoft
Dunwich
ENGELAND
Harwich
GUNFLEET SAND
Southend
Sheerness
Chatham
KENTISH FLATS
Margate
Ramsgate
GOODWIN DOWNS
Dover
Folkestone
Dungeness
DE HOOFDEN
meetkabel
Calais
Sangatte
Griz Nez
FRANKRIJK
SANDETTIE
BREED BANK
Duinkerken
Nieuwpoort
Oostende
Zeebrugge
Cadzand
BELGIE
Antwerpen
NOORDZEE
BRUINE BANK
NOORD HINDER BANK
WEST HINDER BANK
lichteiland Goeree
paal G
Burgsluis
Oranjezon
Westkapelle
Vlissingen
Hoek van Holland
Rotterdam
Krimpen a/d IJssel
Krimpen a/d Lek
Scheveningen
Katwijk
IJmuiden
Amsterdam
Petten
Den Helder
Afsluitdijk
IJSSELMEER
WADDEN-ZEE
NEDERLAND
1
2
3
4
5
6
7
0°
1°
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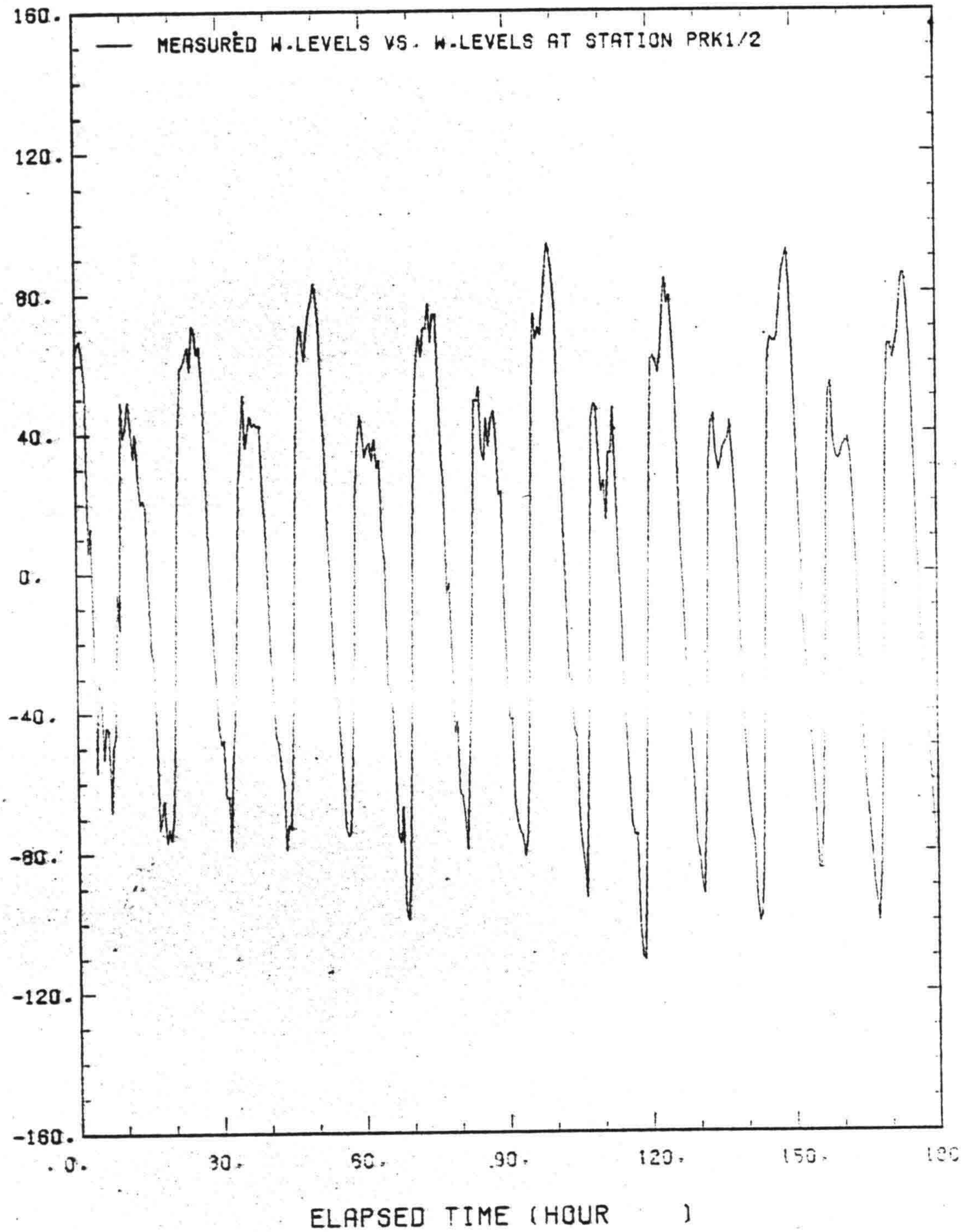
3°
4°
5°
51°
52°
53°

Figures 2 to 16:

Water-level measurements at stations 1 and 2 on a line from Den Helder to Cromer.

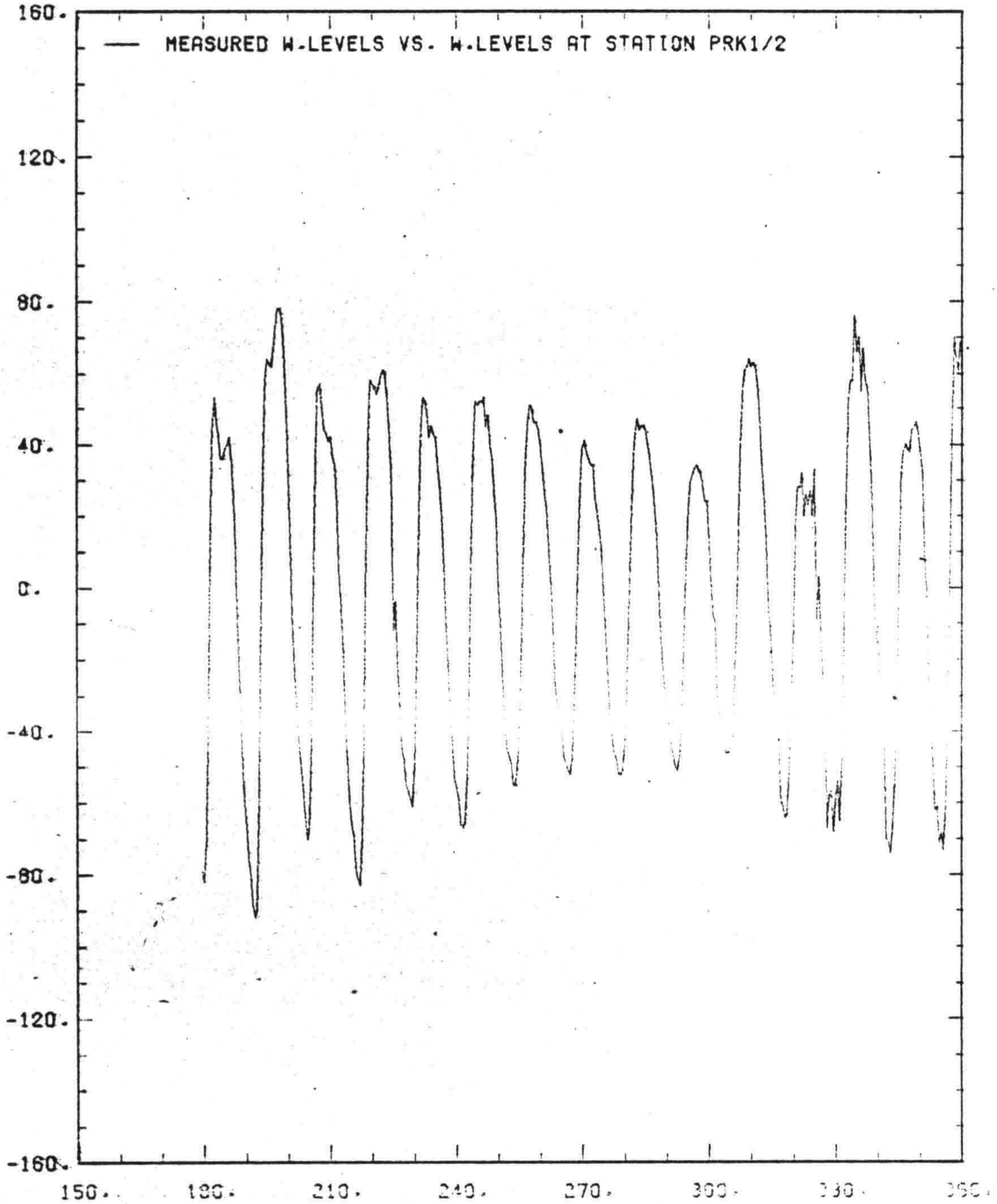


X SERIES
(GIVEN)

PART 1 OF 2

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

RAND CORPORATION
77/11/29 20:56:10



X SERIES
(GIVEN)

PART 2 OF 2

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

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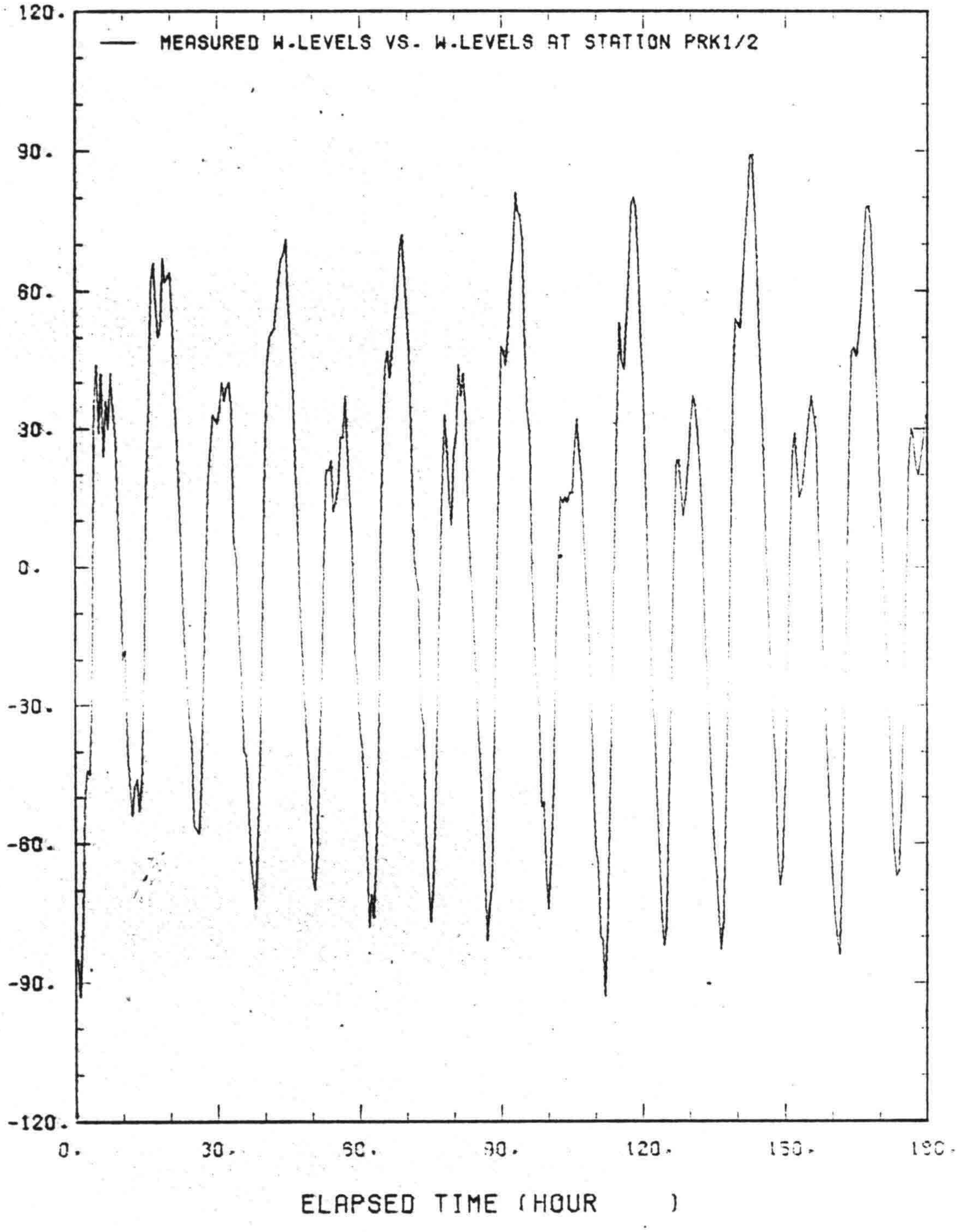
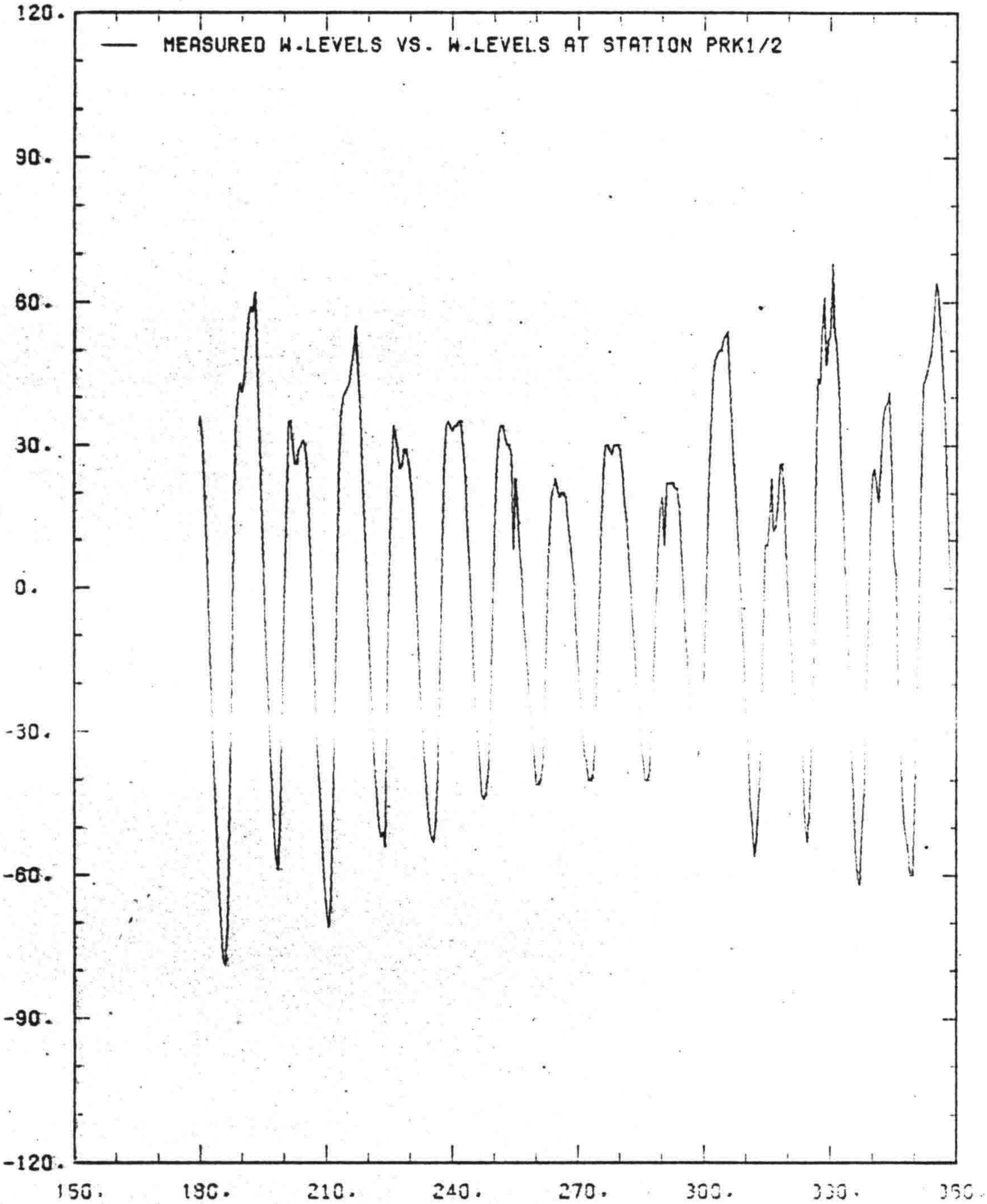


Y SERIES
(GIVEN)

PART 1 OF 2

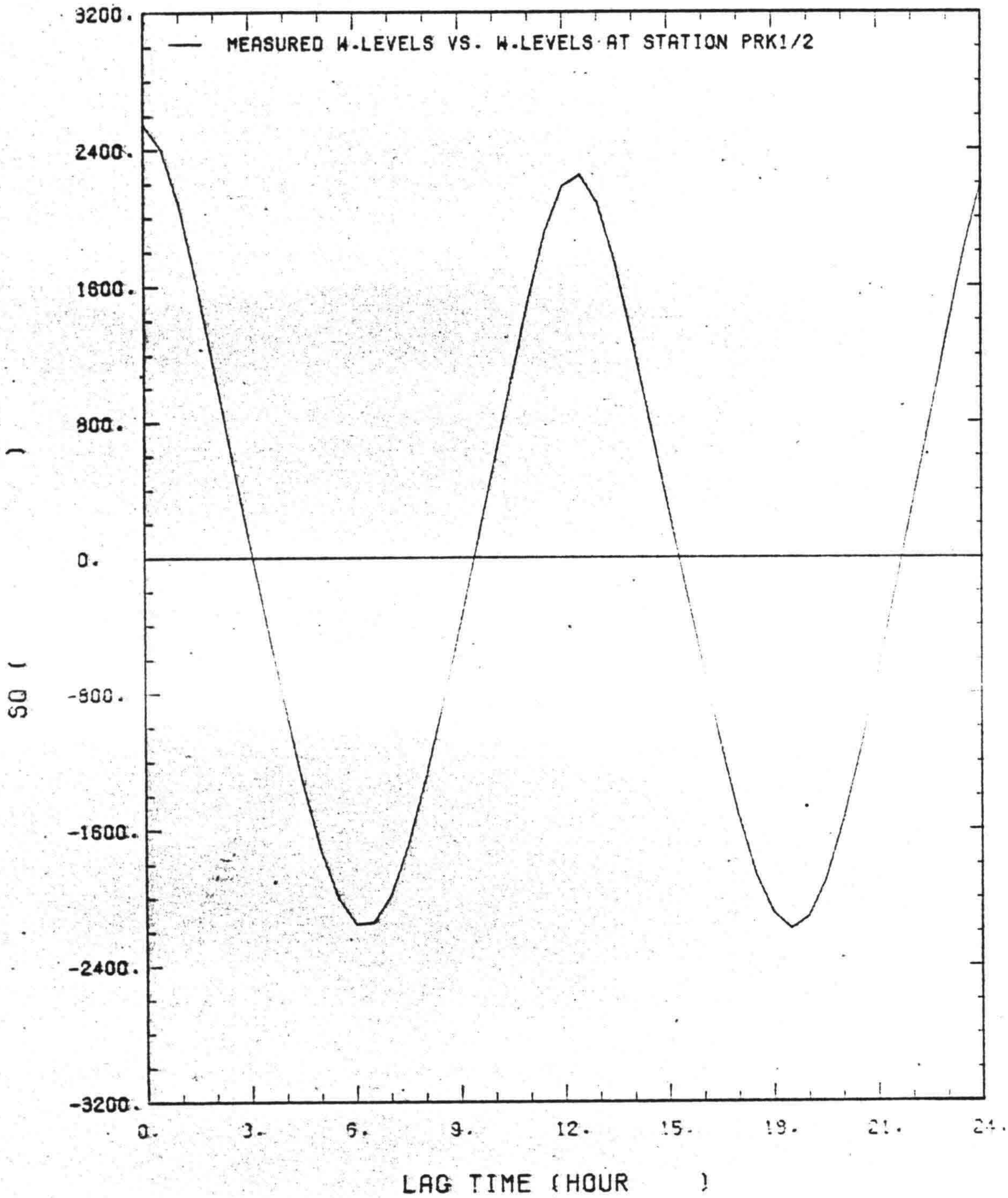


Y SERIES
(GIVEN)

PART 2 OF 2

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

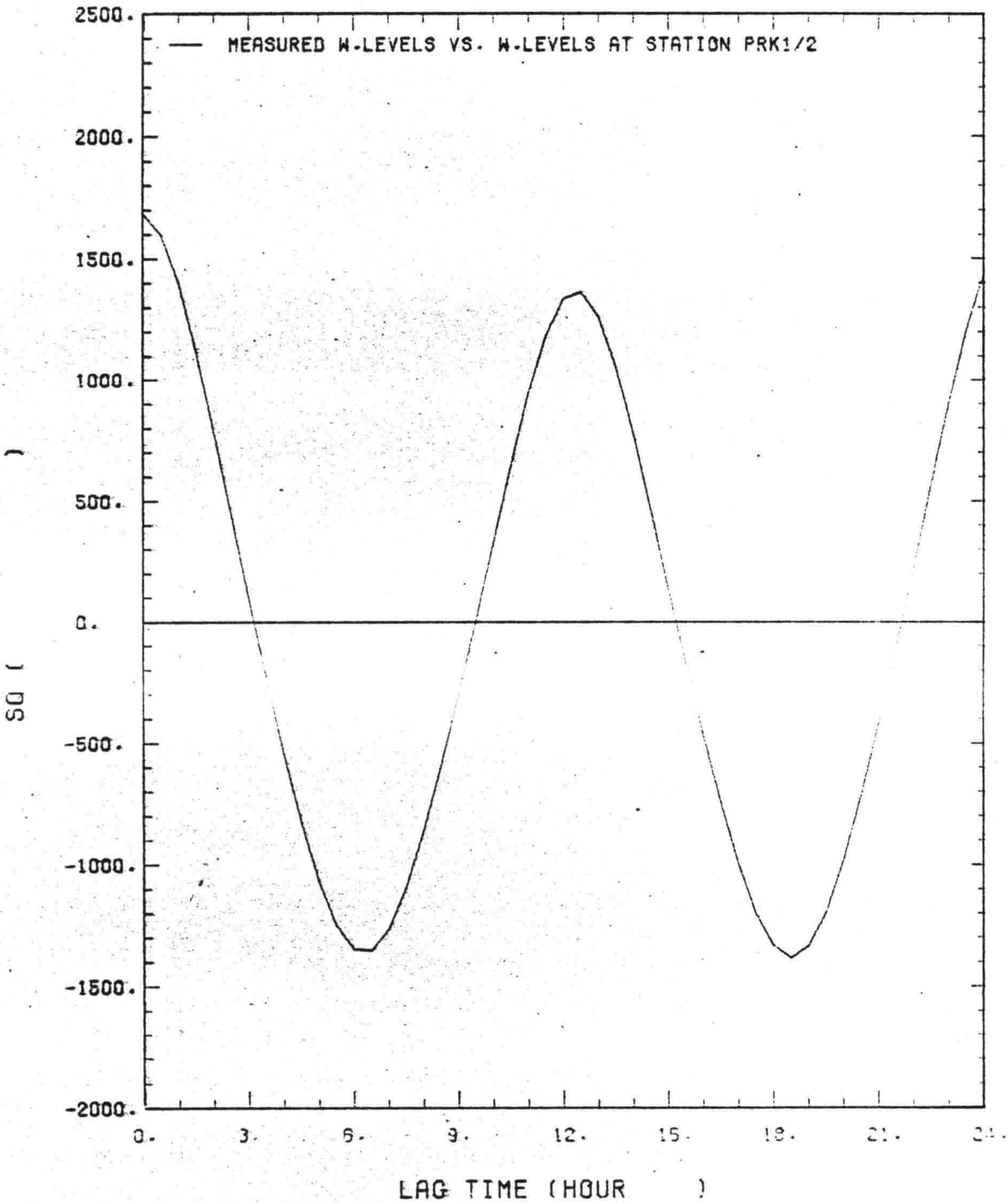
RAND CORPORATION
77/11/29 20:56:10



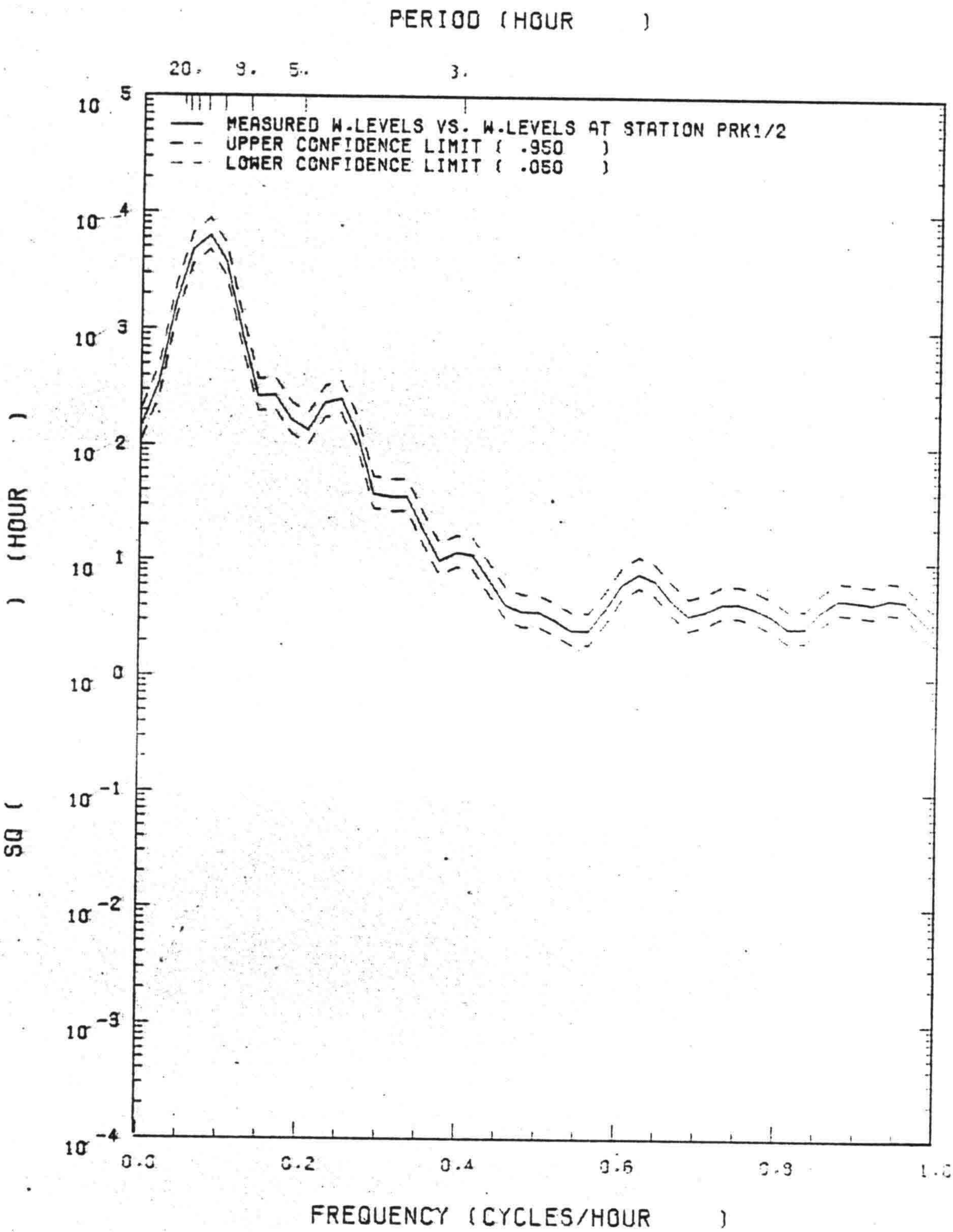
AUTO-COVARIANCE
OF THE X SERIES

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

RAND CORPORATION
77/11/29 20:58:10



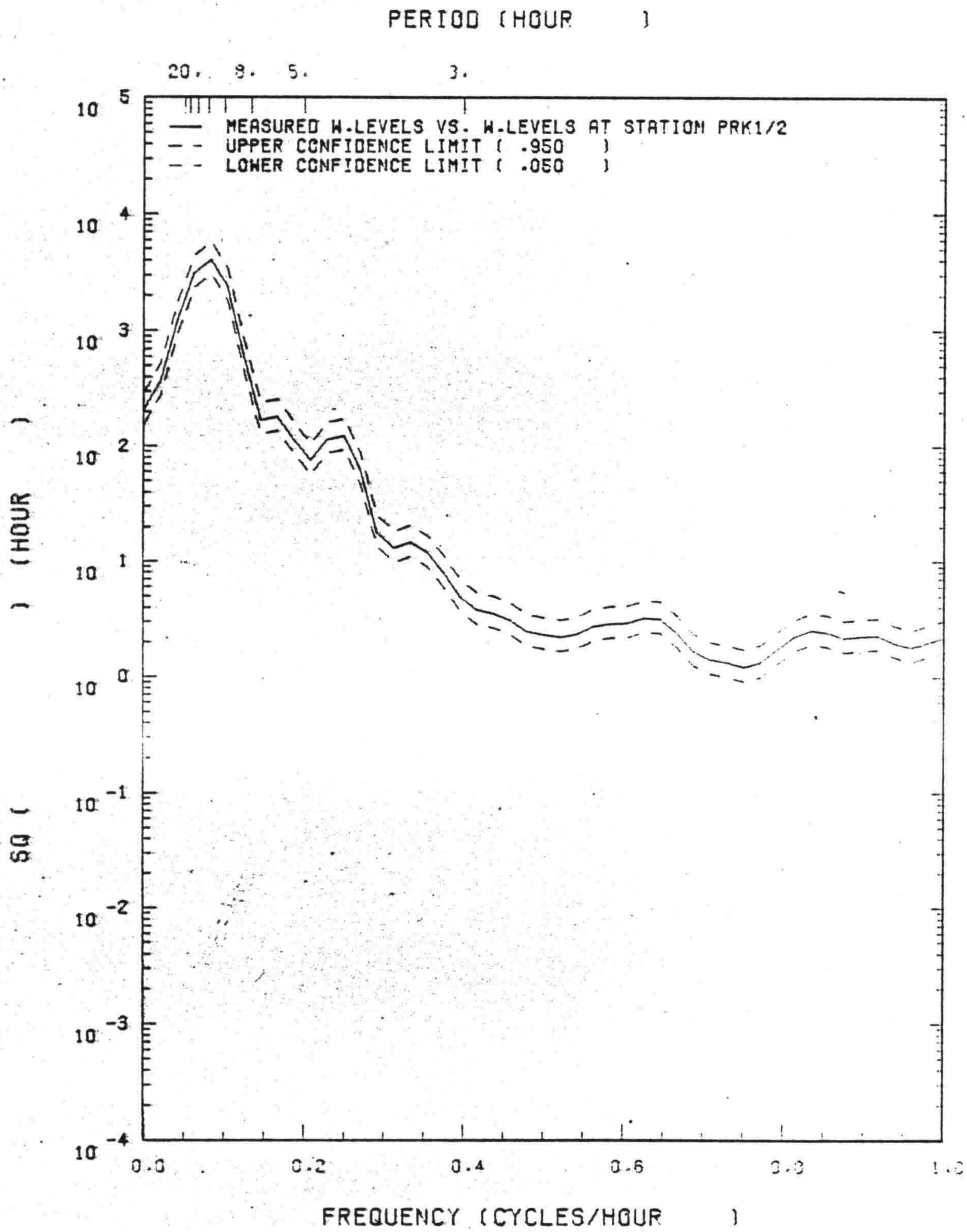
AUTO-COVARIANCE
OF THE Y SERIES



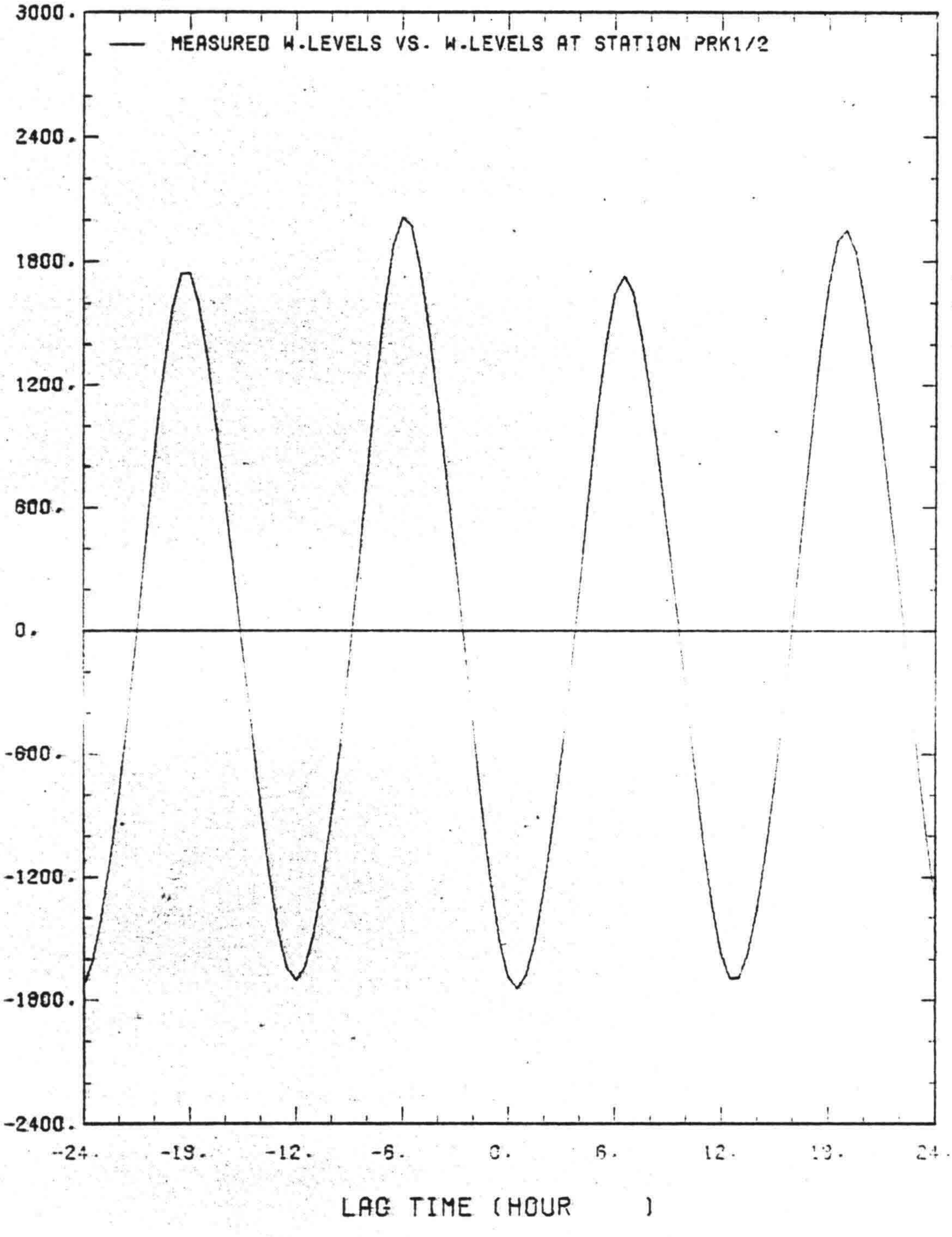
COMPUTED POWER SPECTRAL ESTIMATES
OF THE X SERIES

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

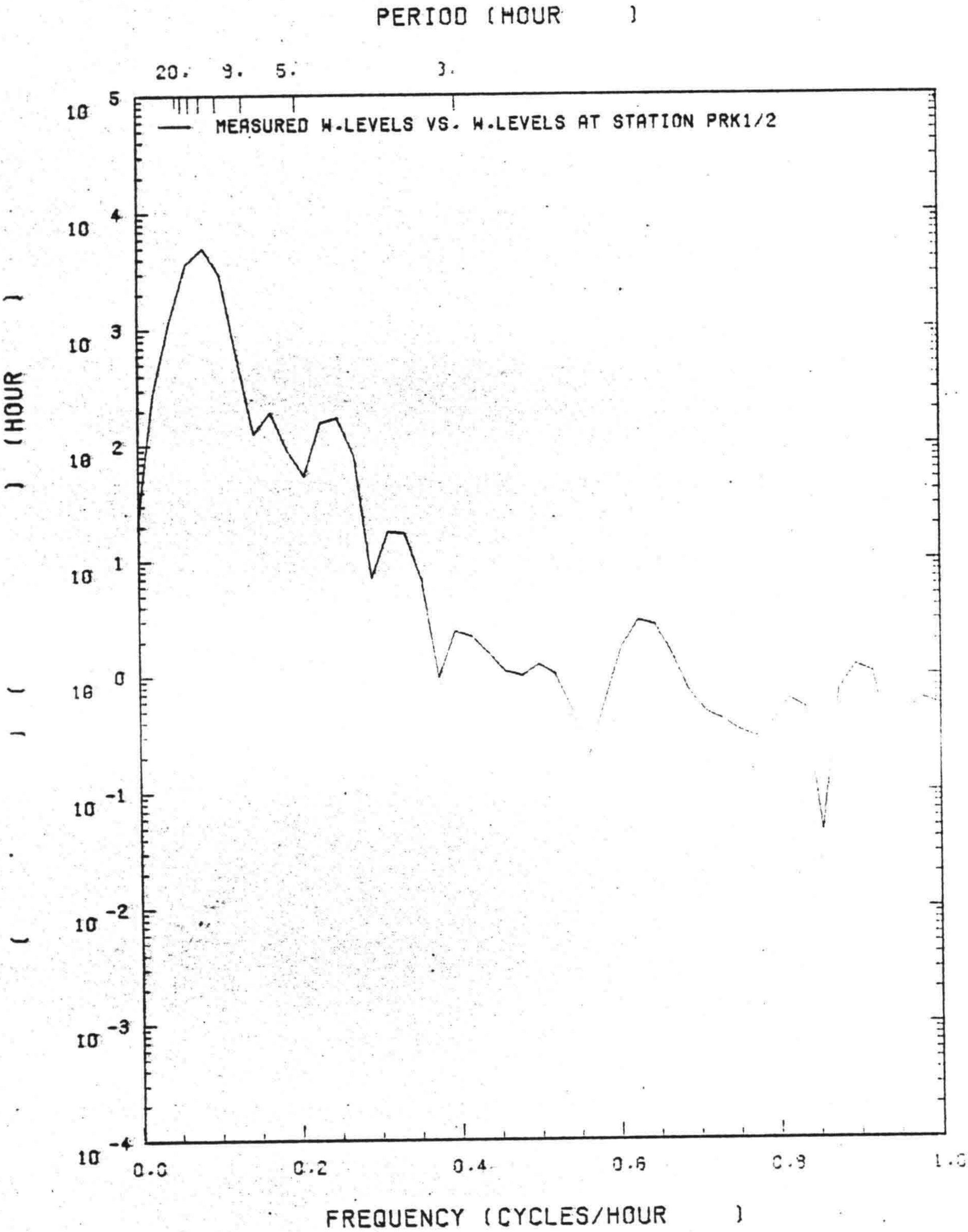
RAND CORPORATION
77/11/29 20:58:10



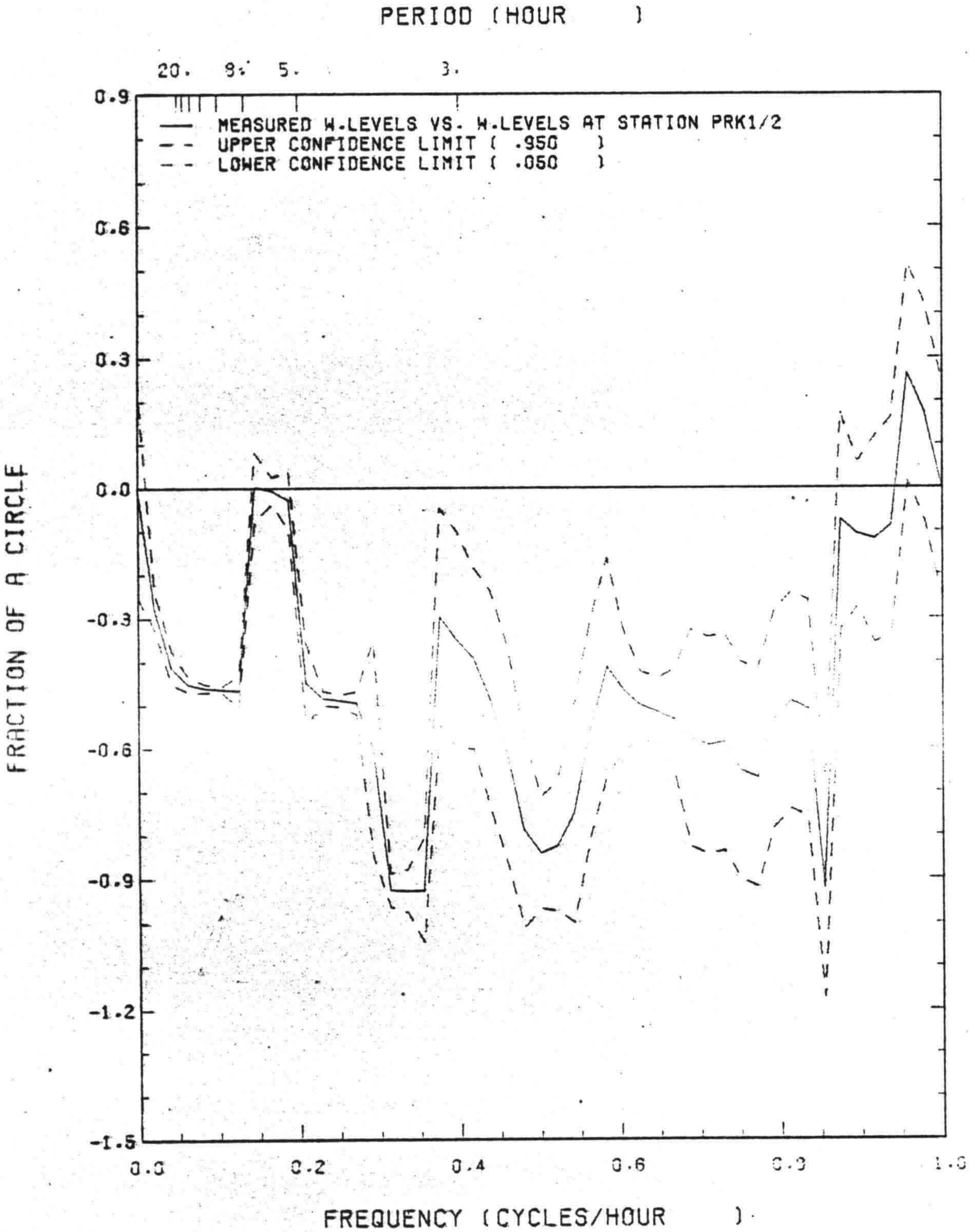
COMPUTED POWER SPECTRAL ESTIMATES
OF THE Y SERIES



CROSS-COVARIANCE
OF THE X SERIES AND THE Y SERIES
(NEGATIVE AND POSITIVE LAG)



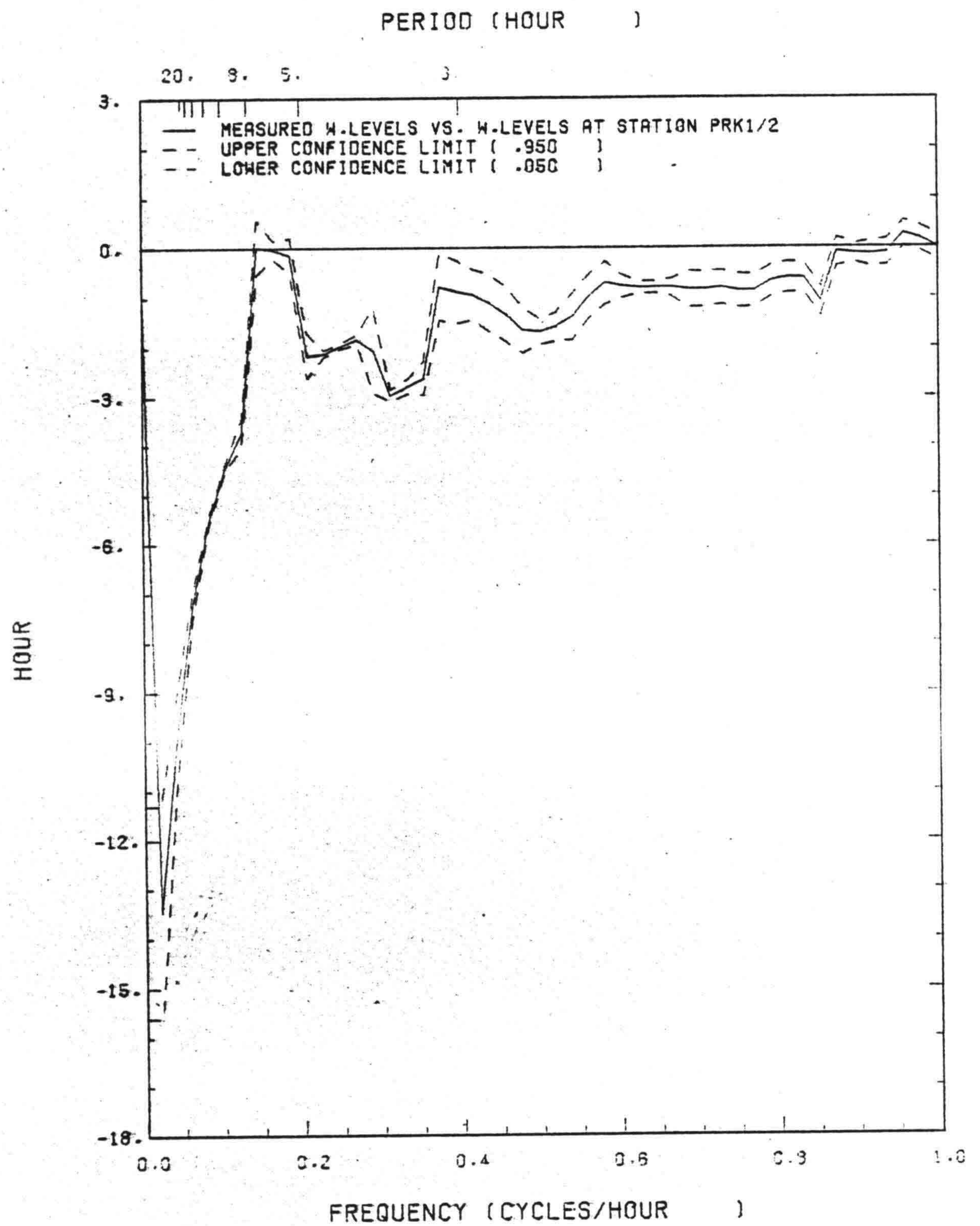
AMPLITUDE OF THE CROSS-SPECTRUM
OF THE X SERIES AND THE Y SERIES



PHASE SPECTRUM
OF THE X SERIES AND THE Y SERIES
(IN FRACTION OF A CIRCLE)

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

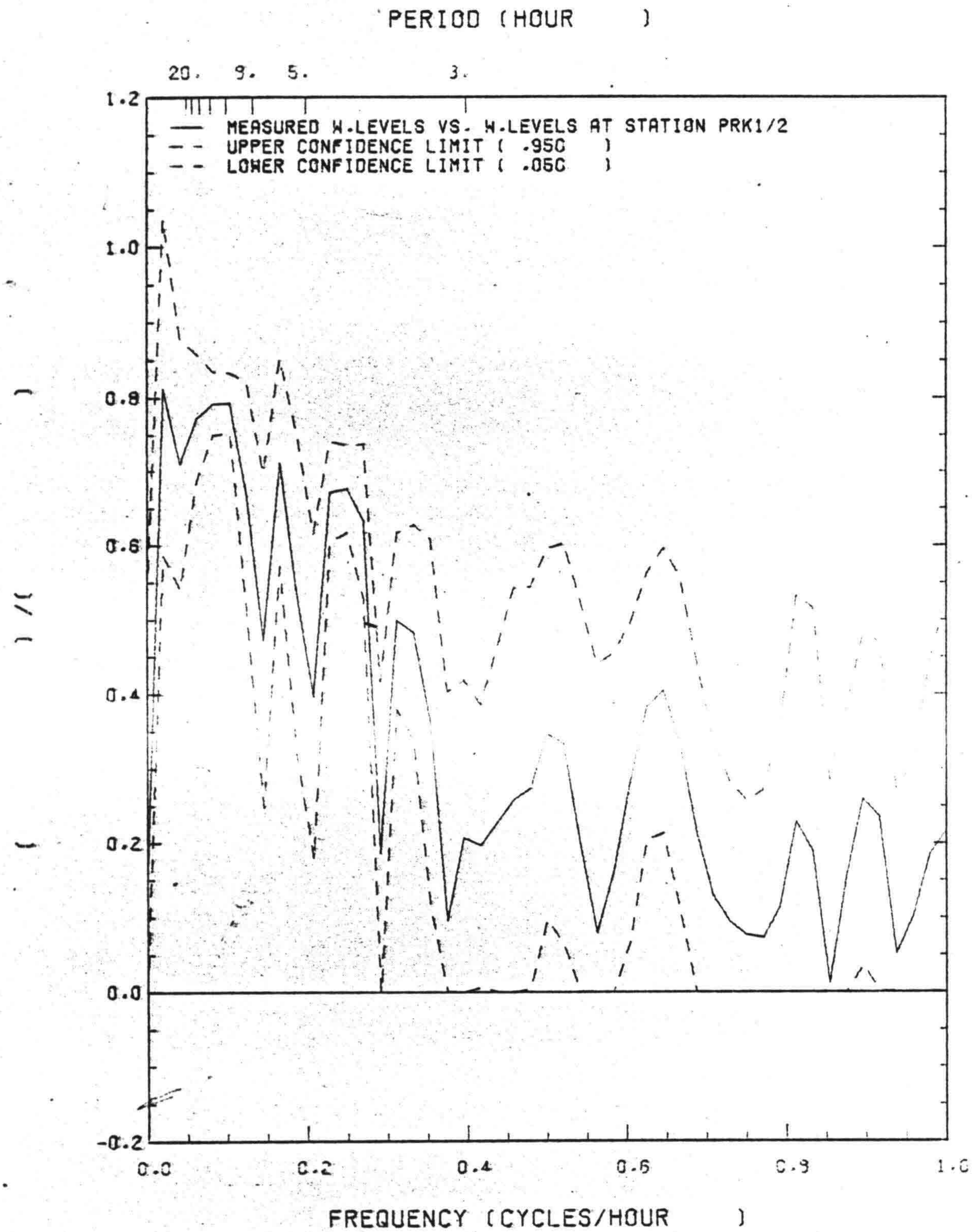
RAND CORPORATION
77/11/29 20:56:10



PHASE SPECTRUM
OF THE X SERIES AND THE Y SERIES
(IN HOUR)

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

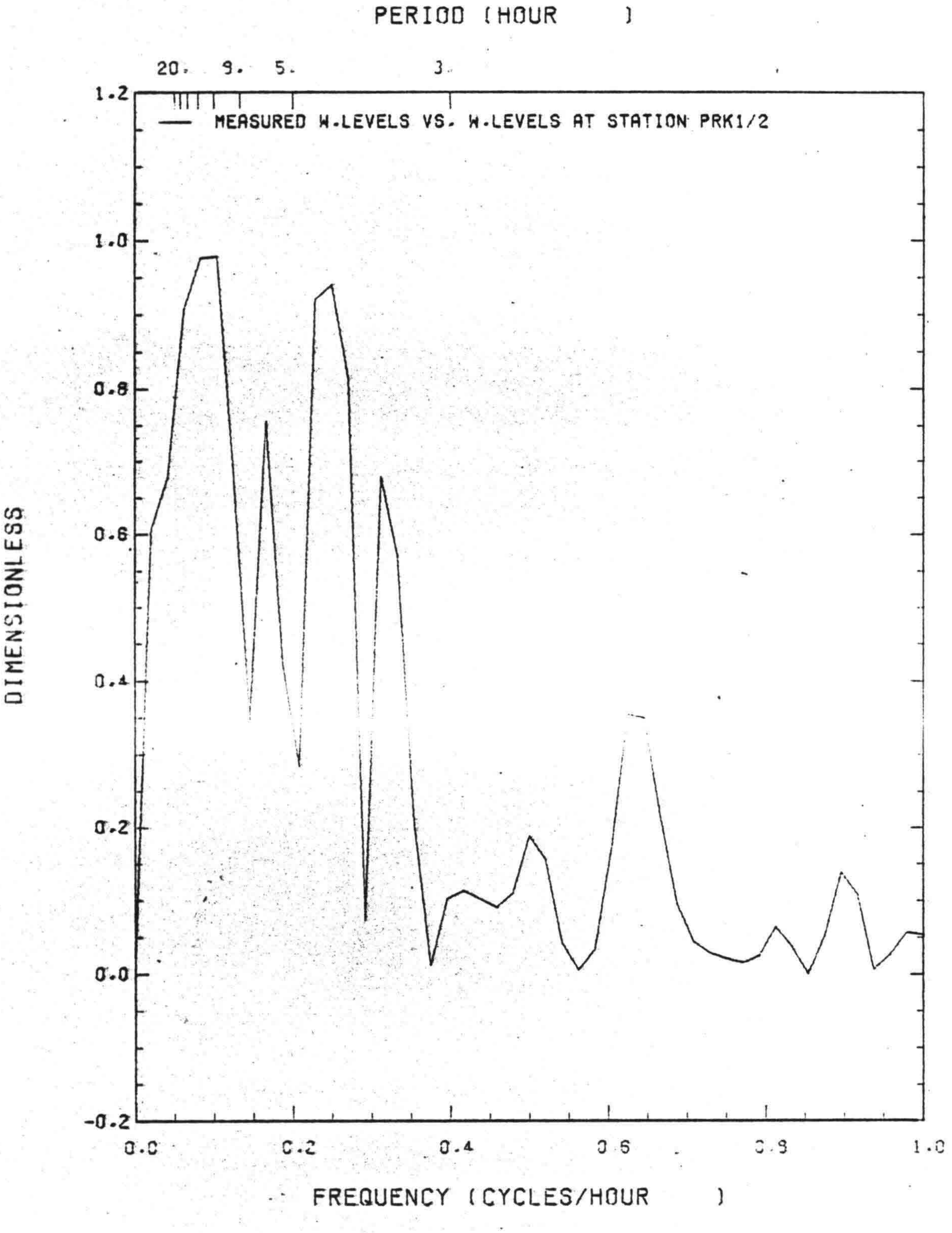
RAND CORPORATION
77/11/29 20:56:18



AMPLITUDE
OF THE FREQUENCY RESPONSE FUNCTION
OF THE X SERIES AND THE Y SERIES

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

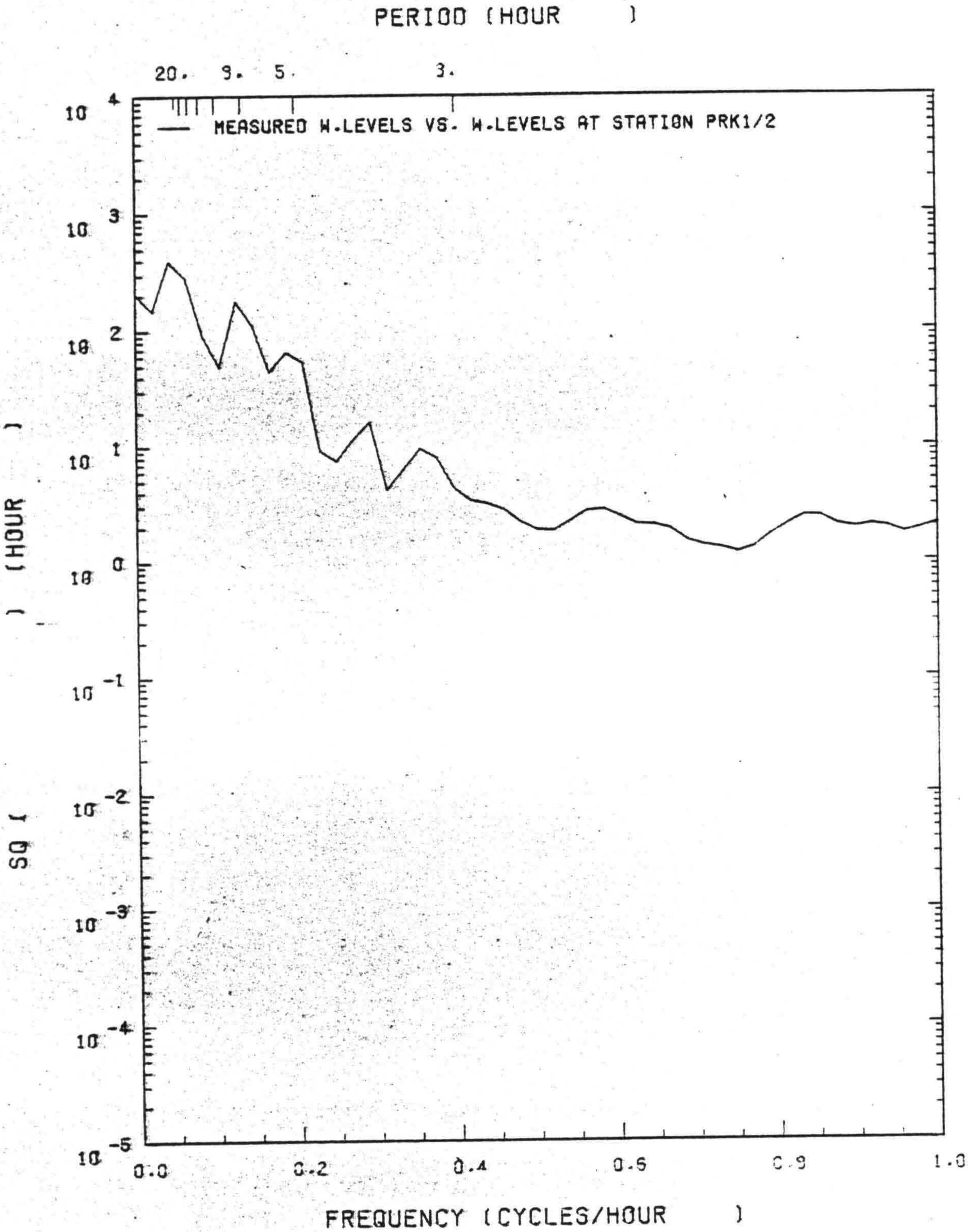
RAND CORPORATION
77/11/29 20:56:10



SQUARED COHERENCY
OF THE X SERIES AND THE Y SERIES

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

RAND CORPORATION
77/11/29 20:56:10



SPECTRUM OF NOISE
OF THE X SERIES AND THE Y SERIES

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

RAND CORPORATION
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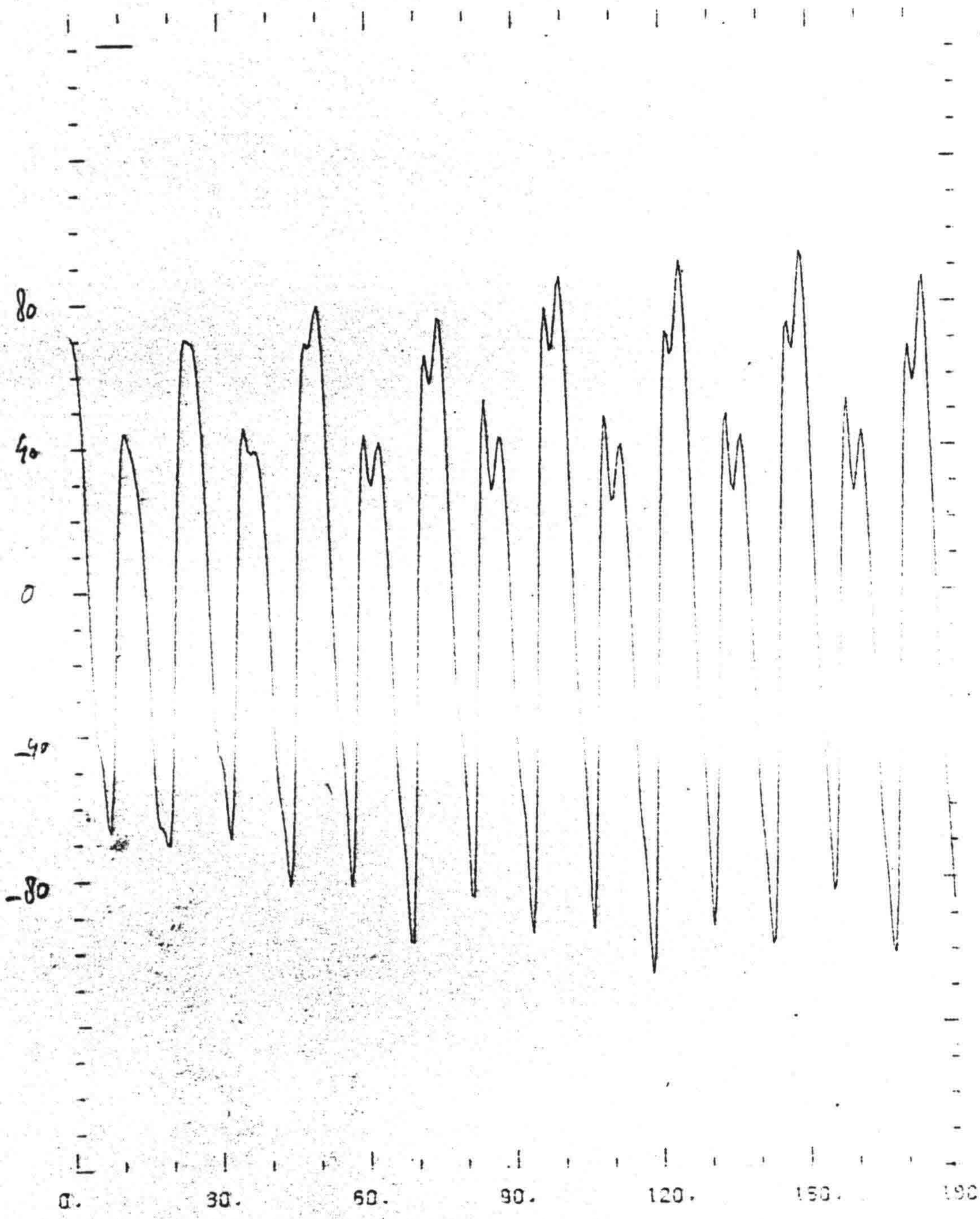
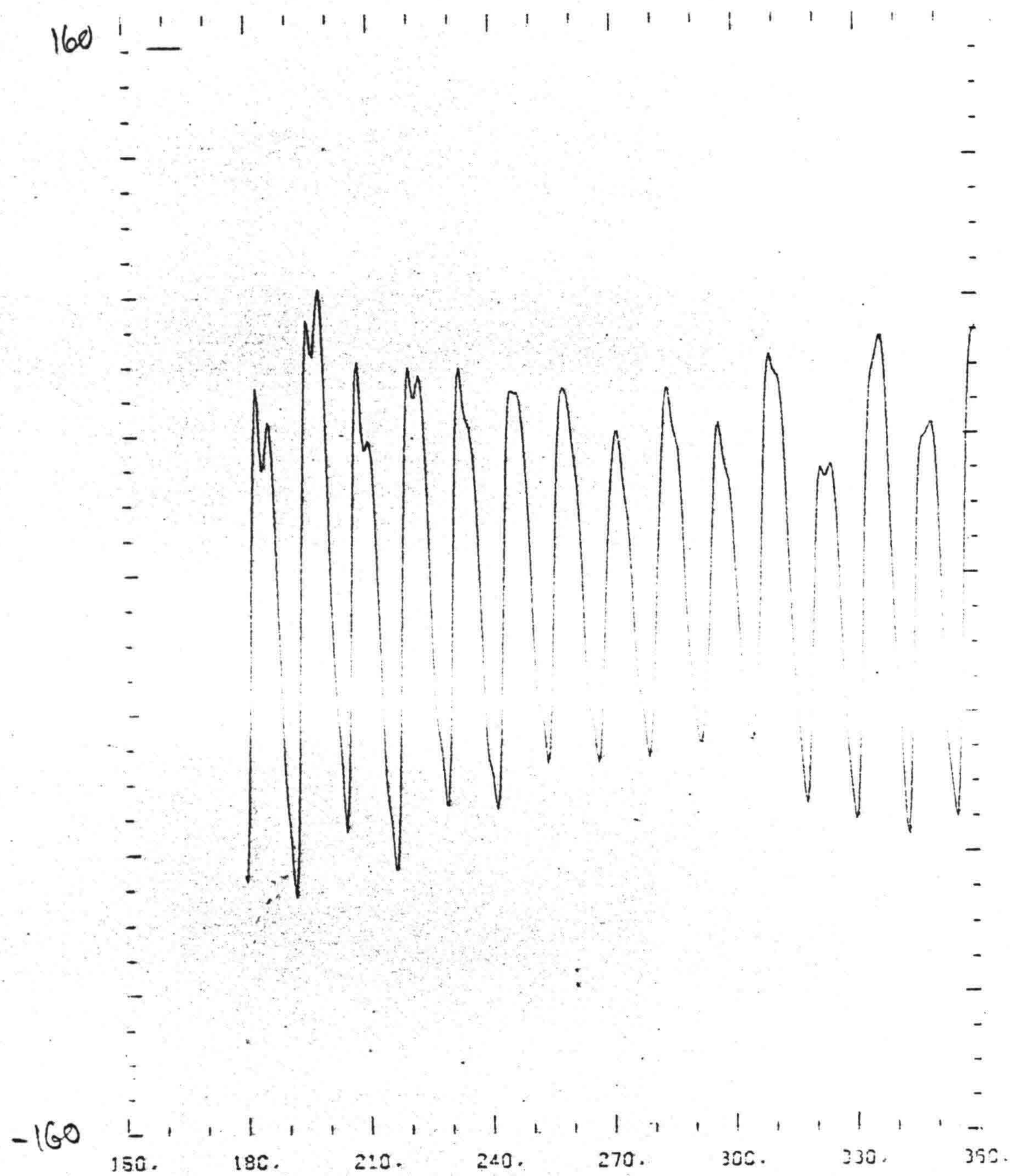
Figures 17 to 31:

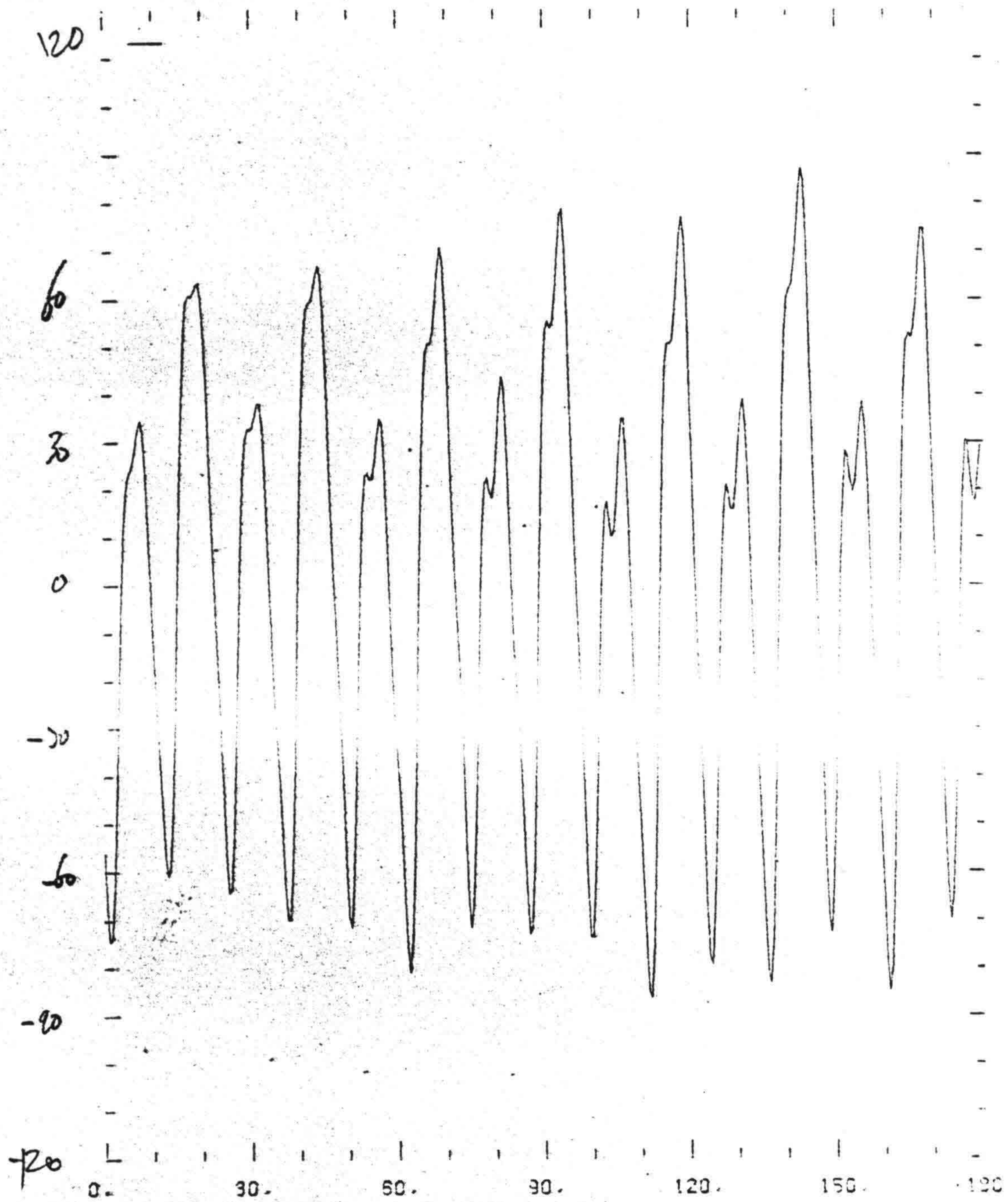
Astronomical components at stations 1 and 2.
Text and scale of each plate should be identical to the corresponding plate in the previous series.



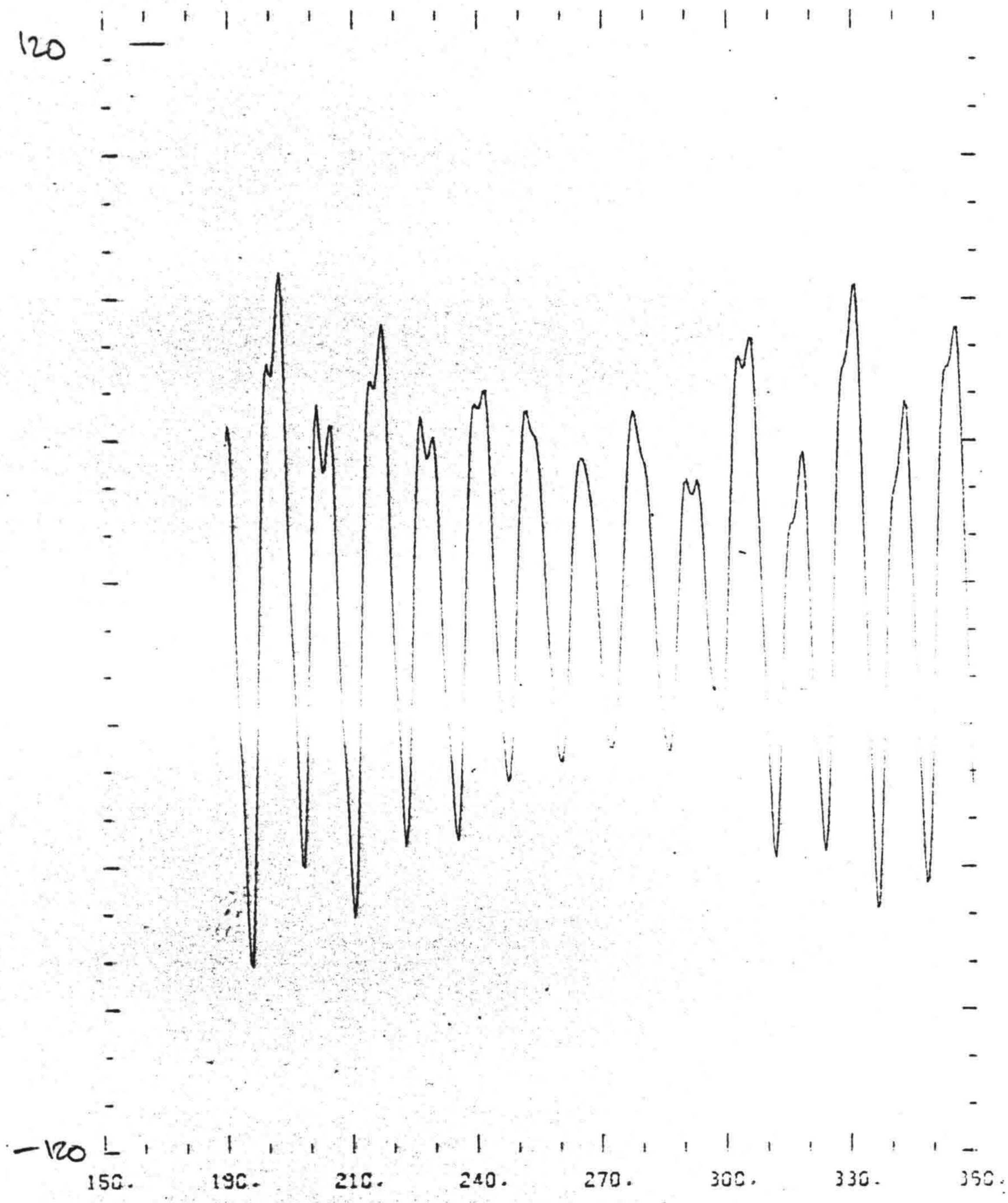
PART 1 OF 2



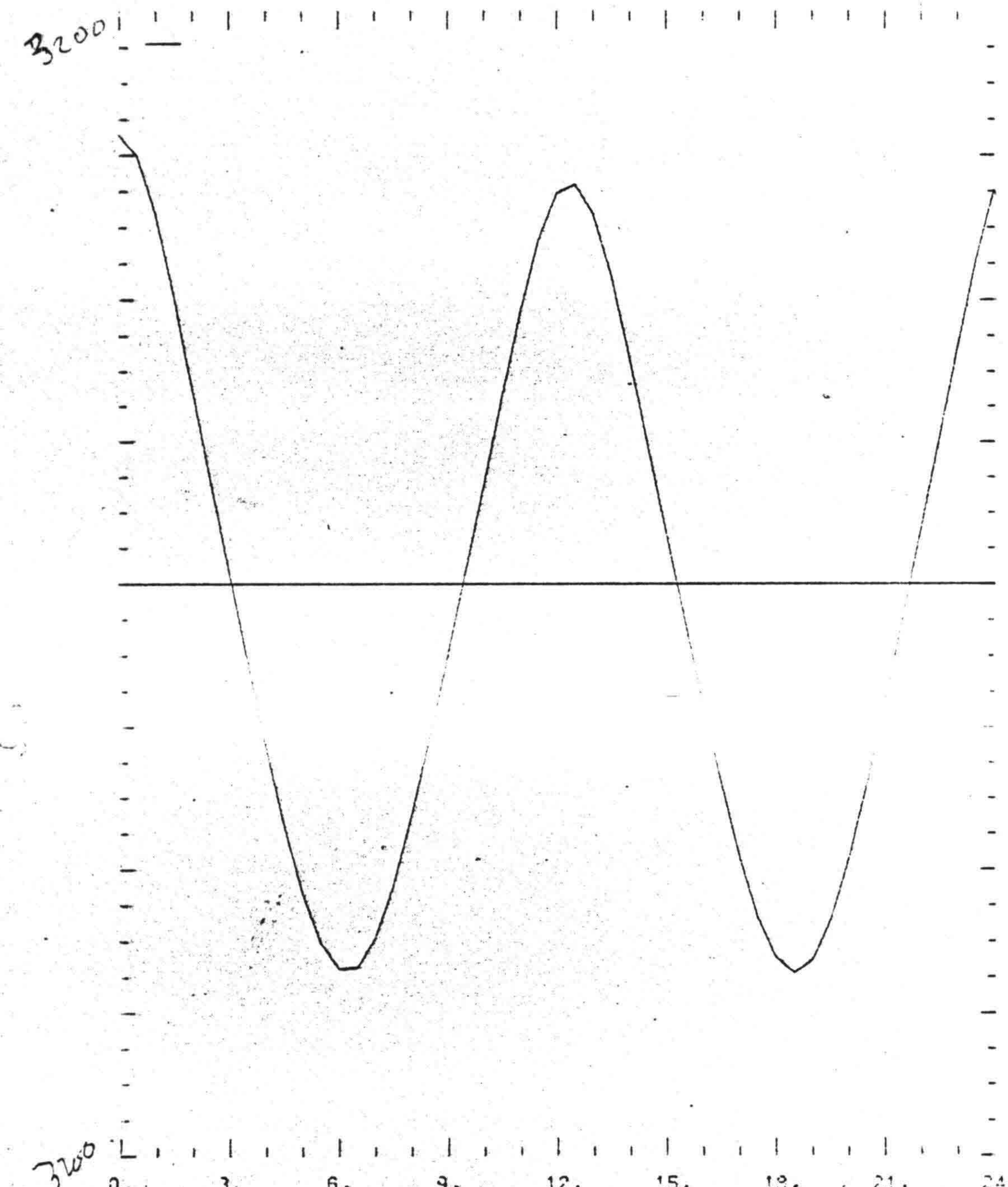
PART 2 OF 2



PART 1 OF 2



PART 2 OF 2


3200
-3200
0.
3.
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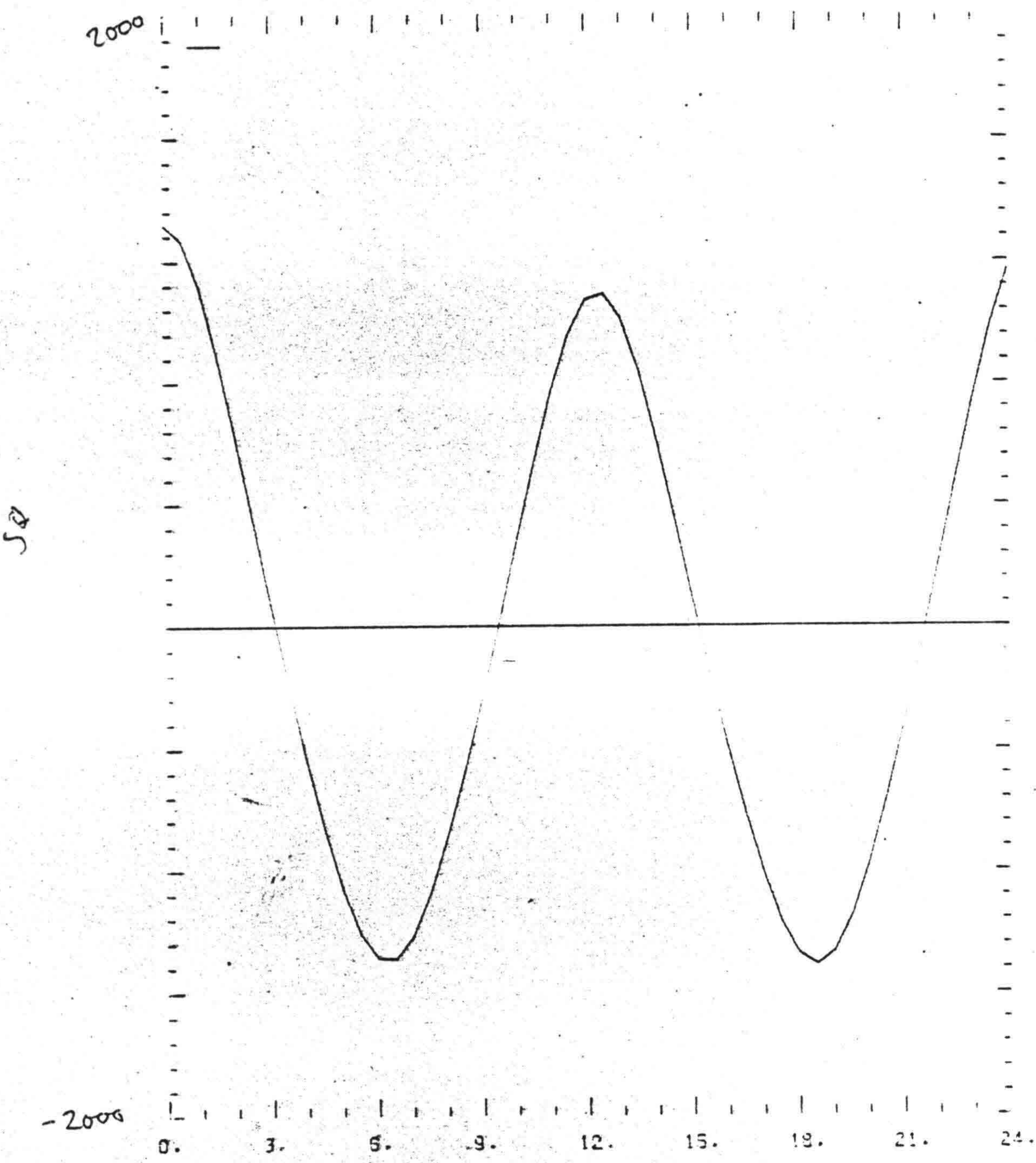
18.
21.
24


2000
-2000
0.
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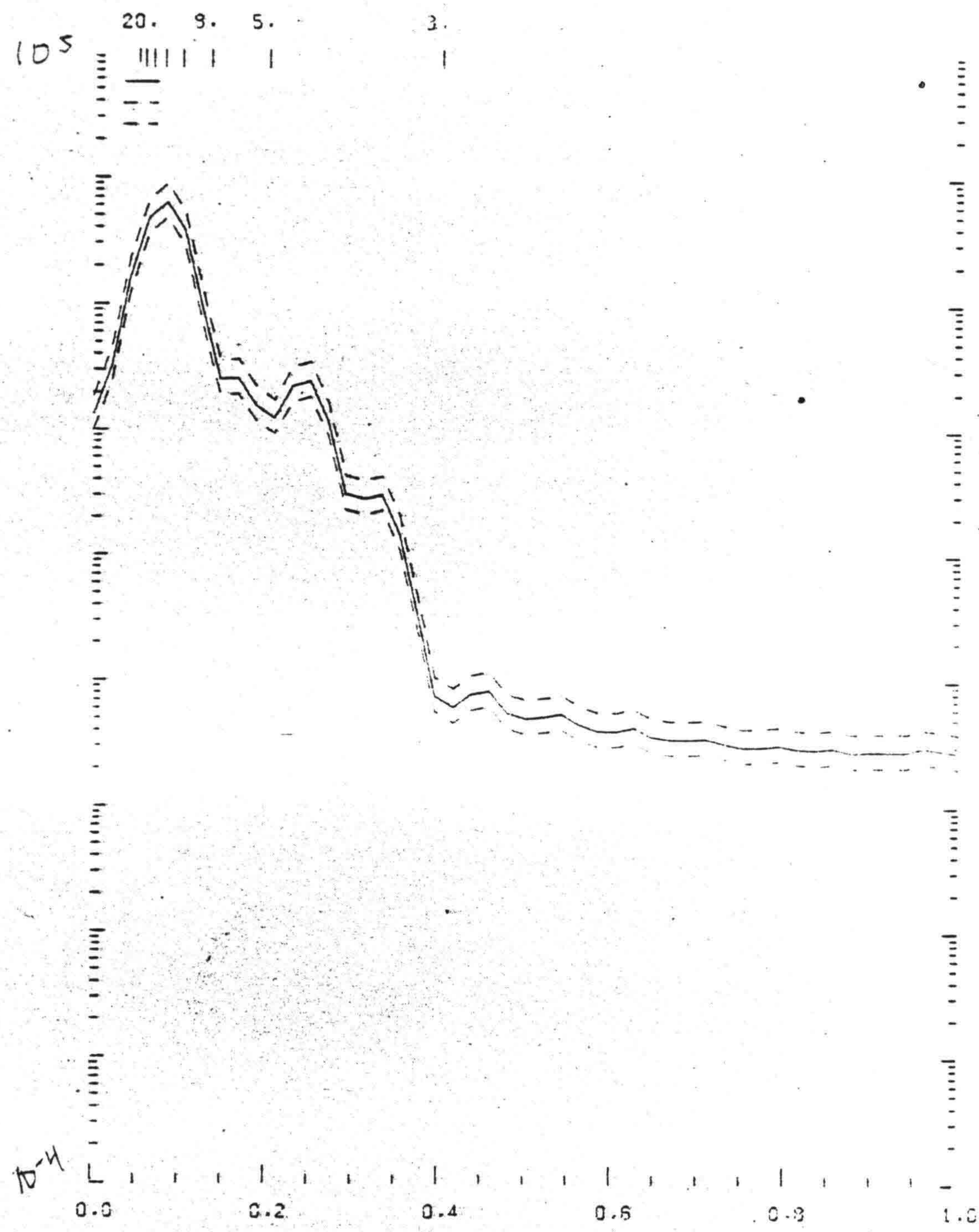

20.
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3.
10^5
10^-4
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0.2
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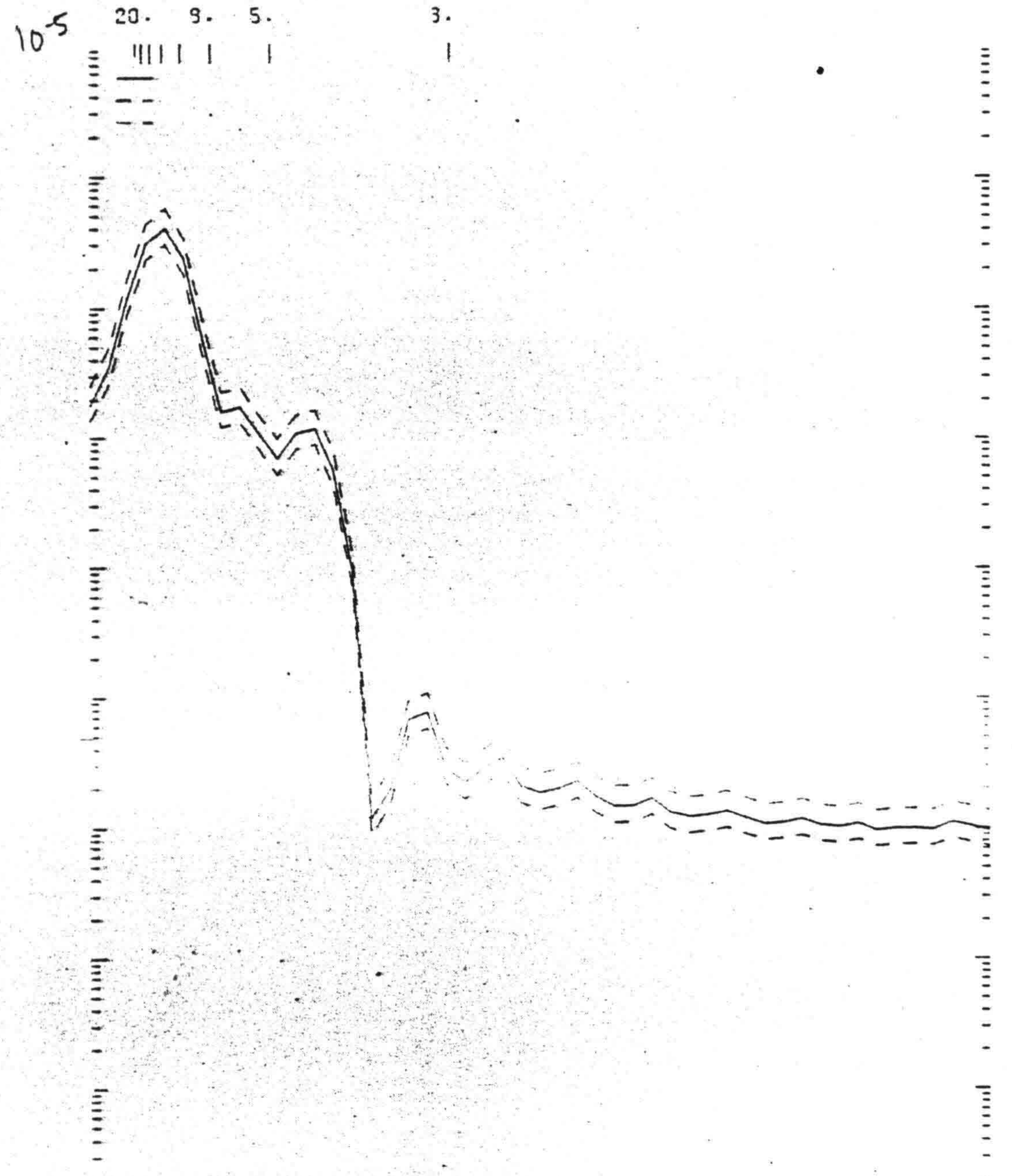

10-5
20.
9.
5.
3.
10-4
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0.2
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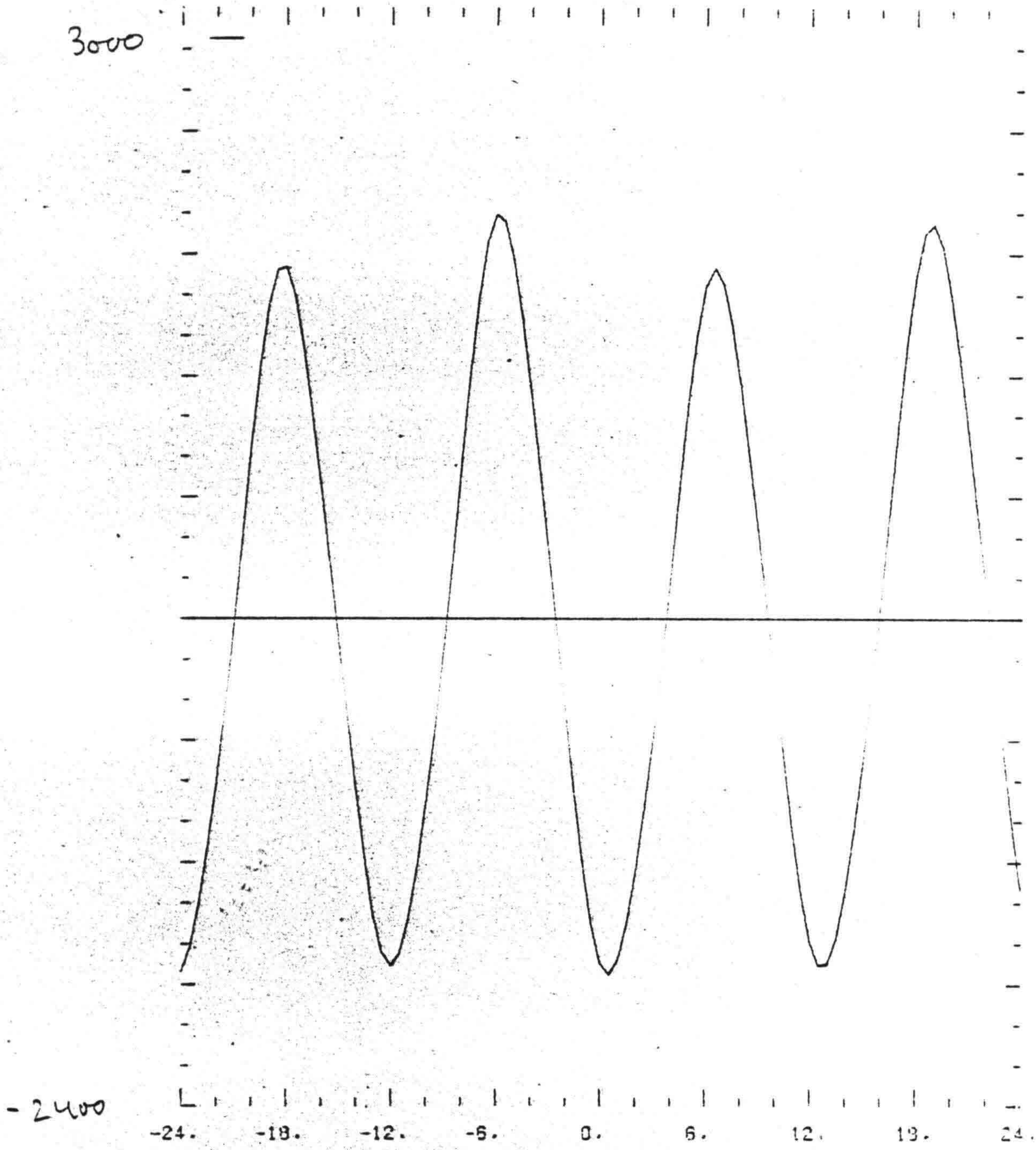
3000
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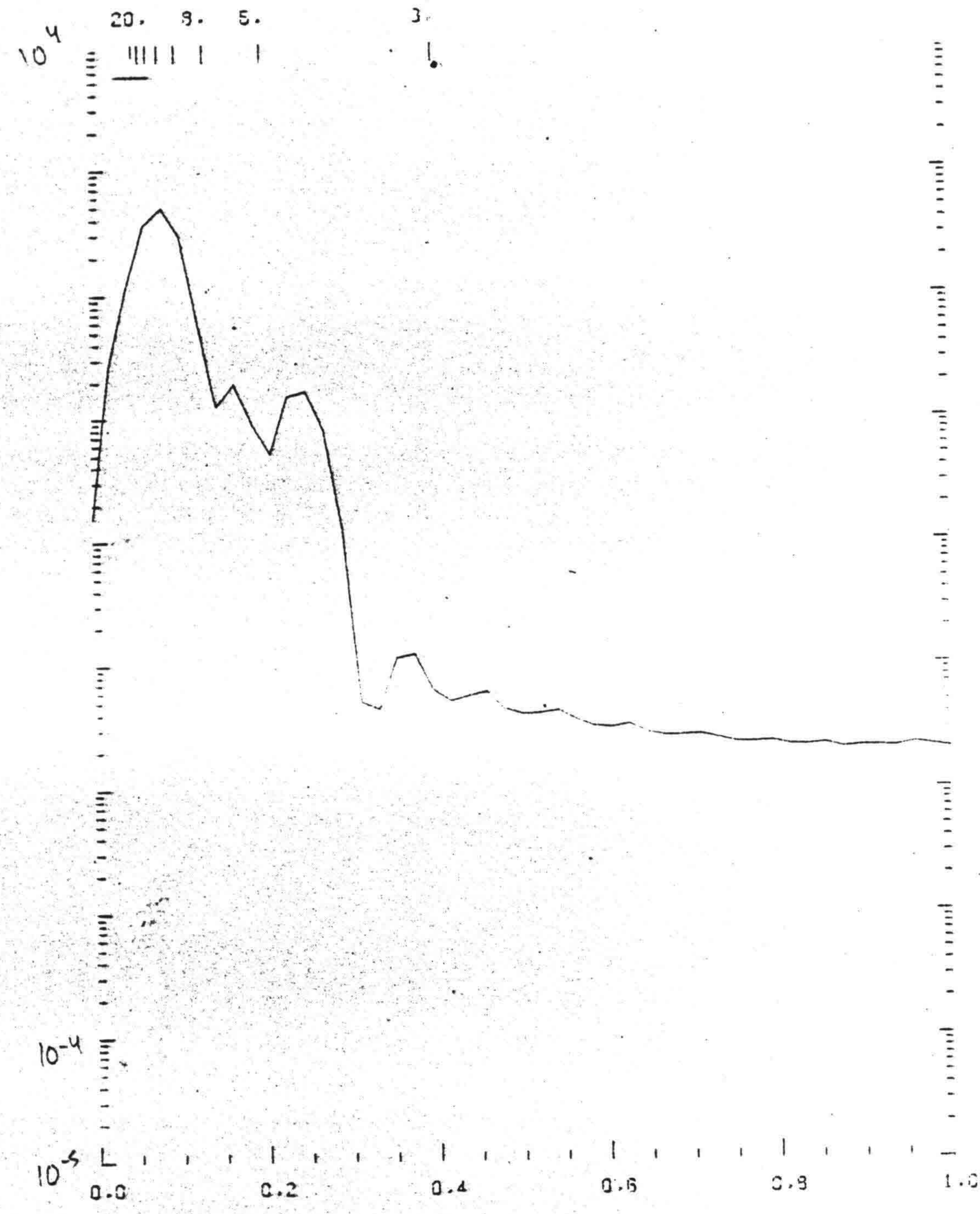

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10^-4
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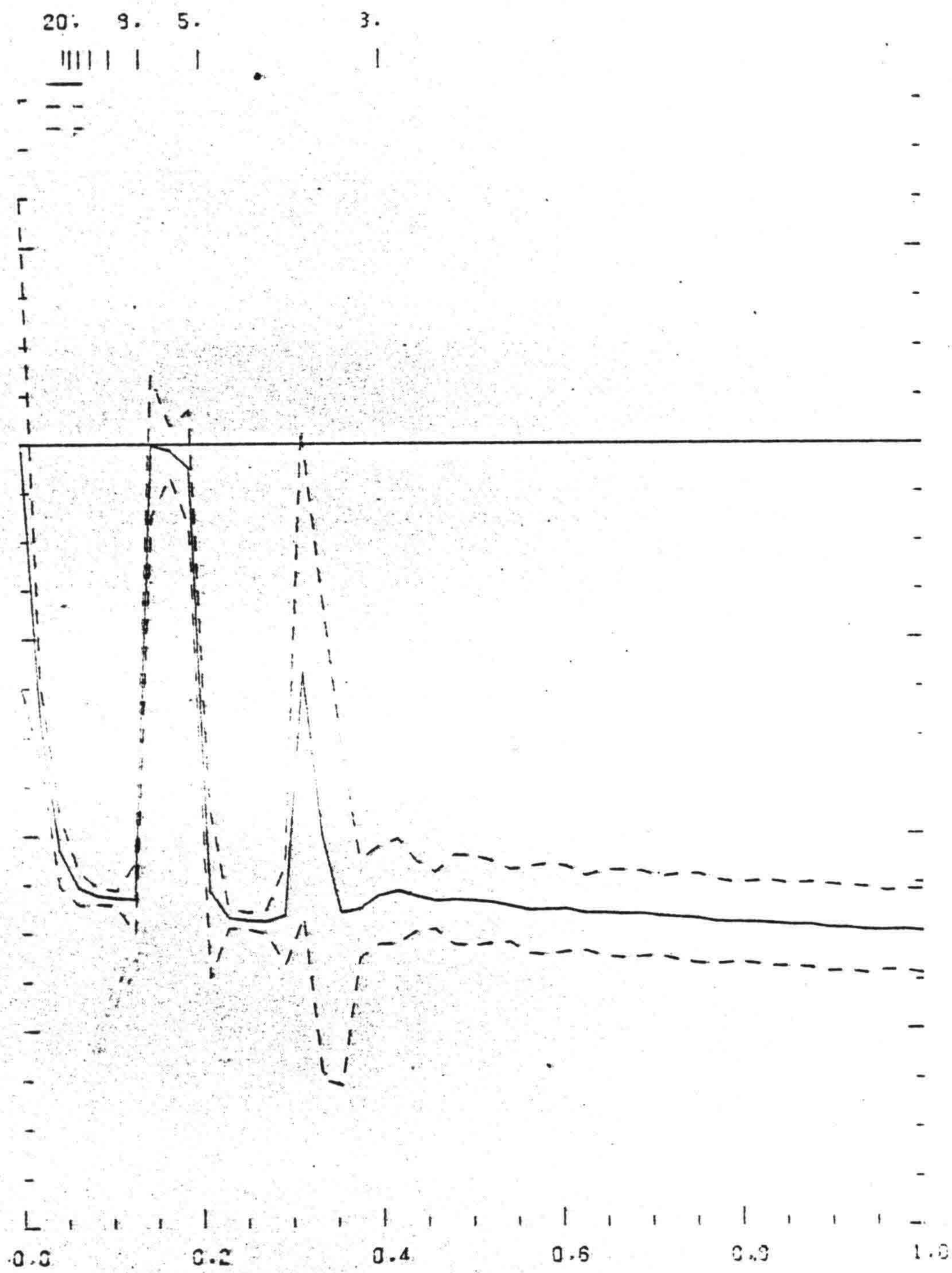

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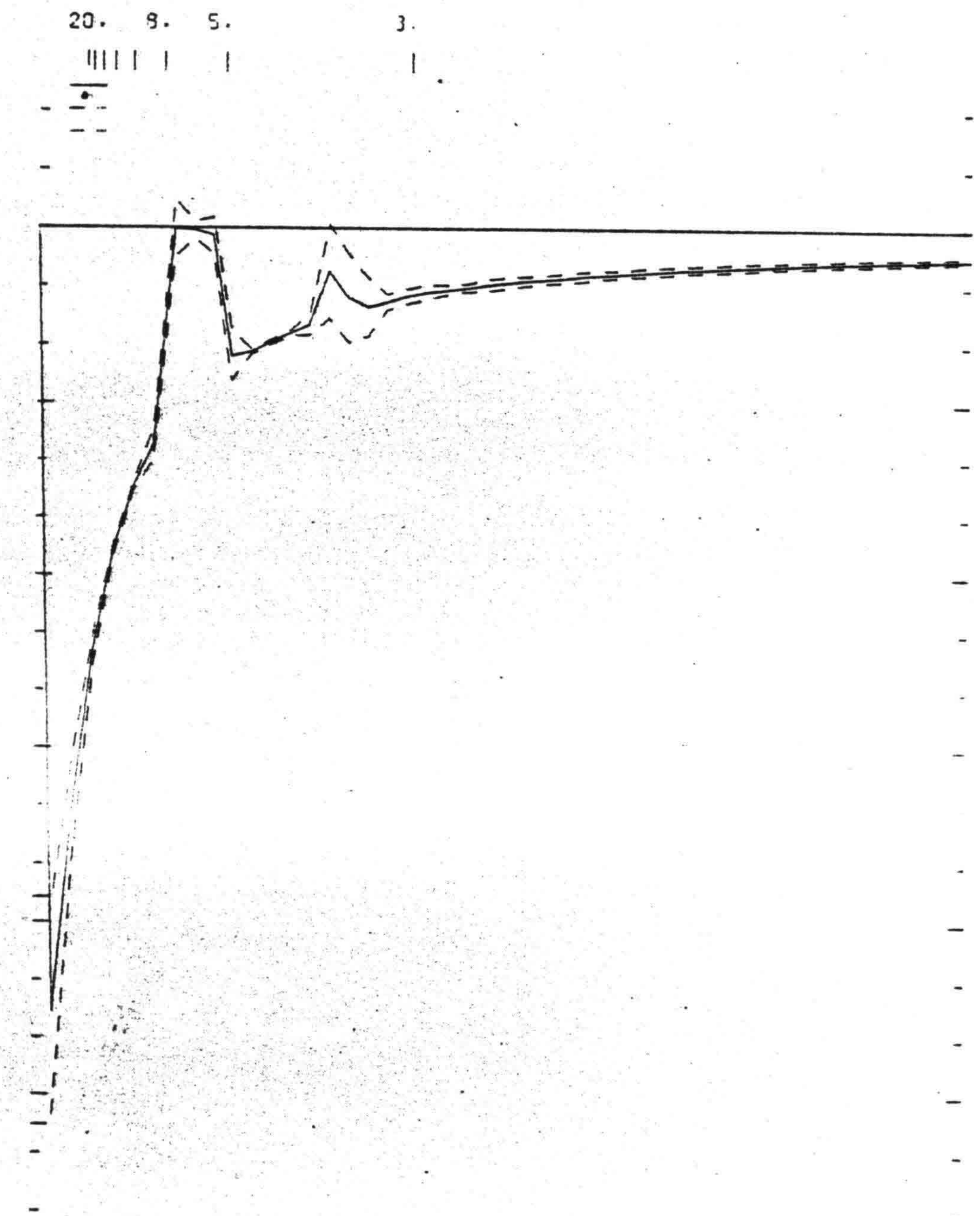
20. 8. 5. 3.
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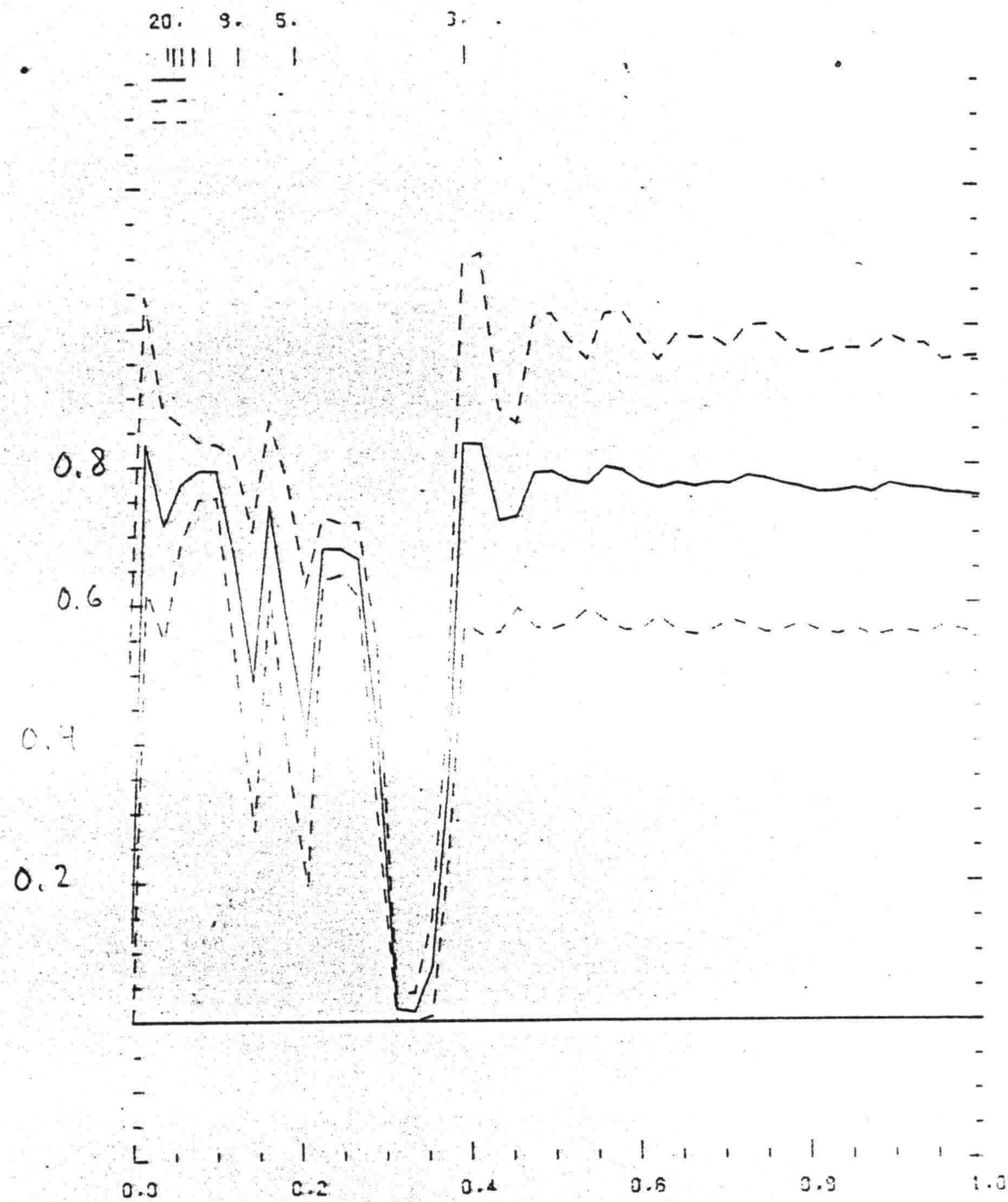
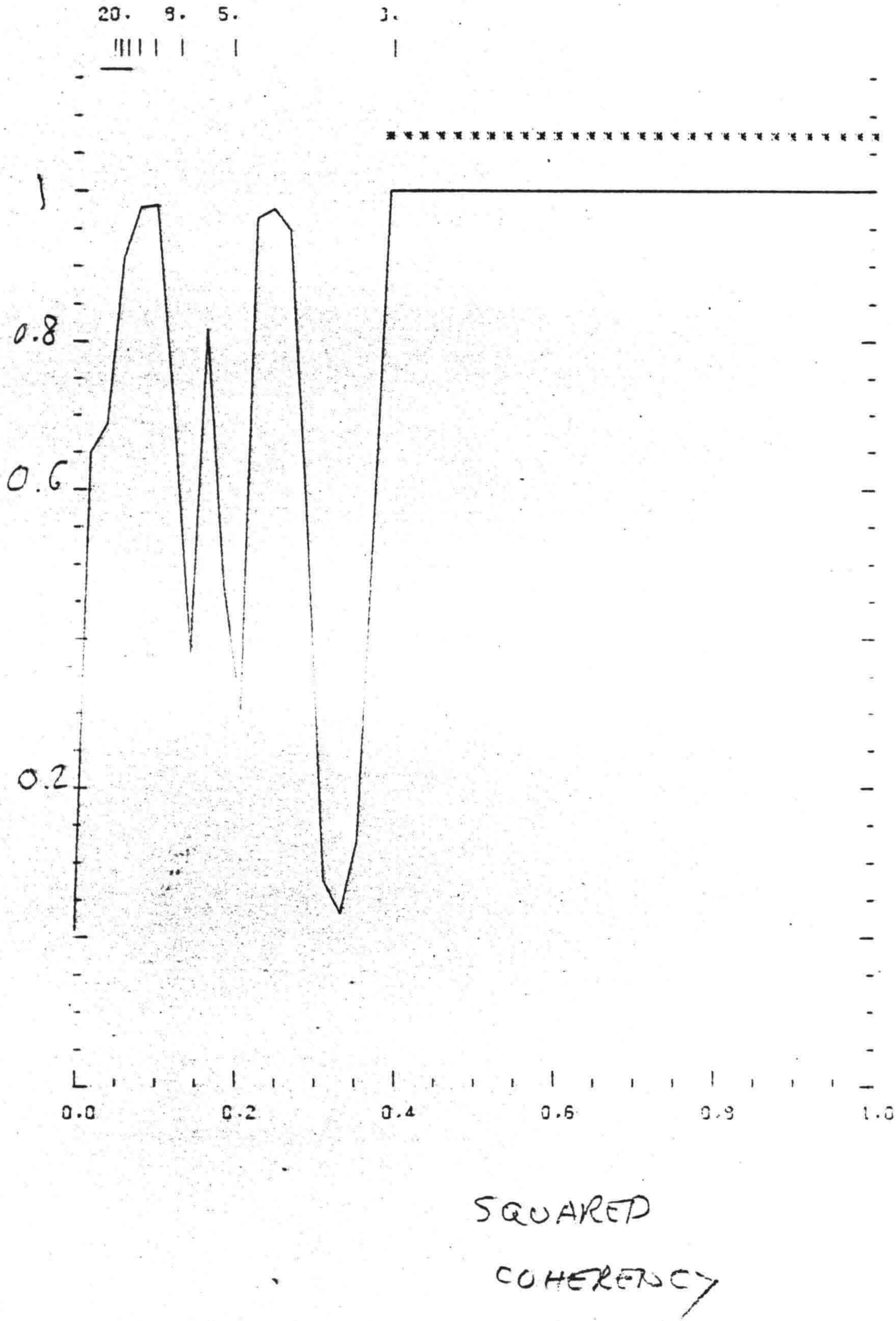


AMPLITUDE FREQUENCY

RESPONSE



20.
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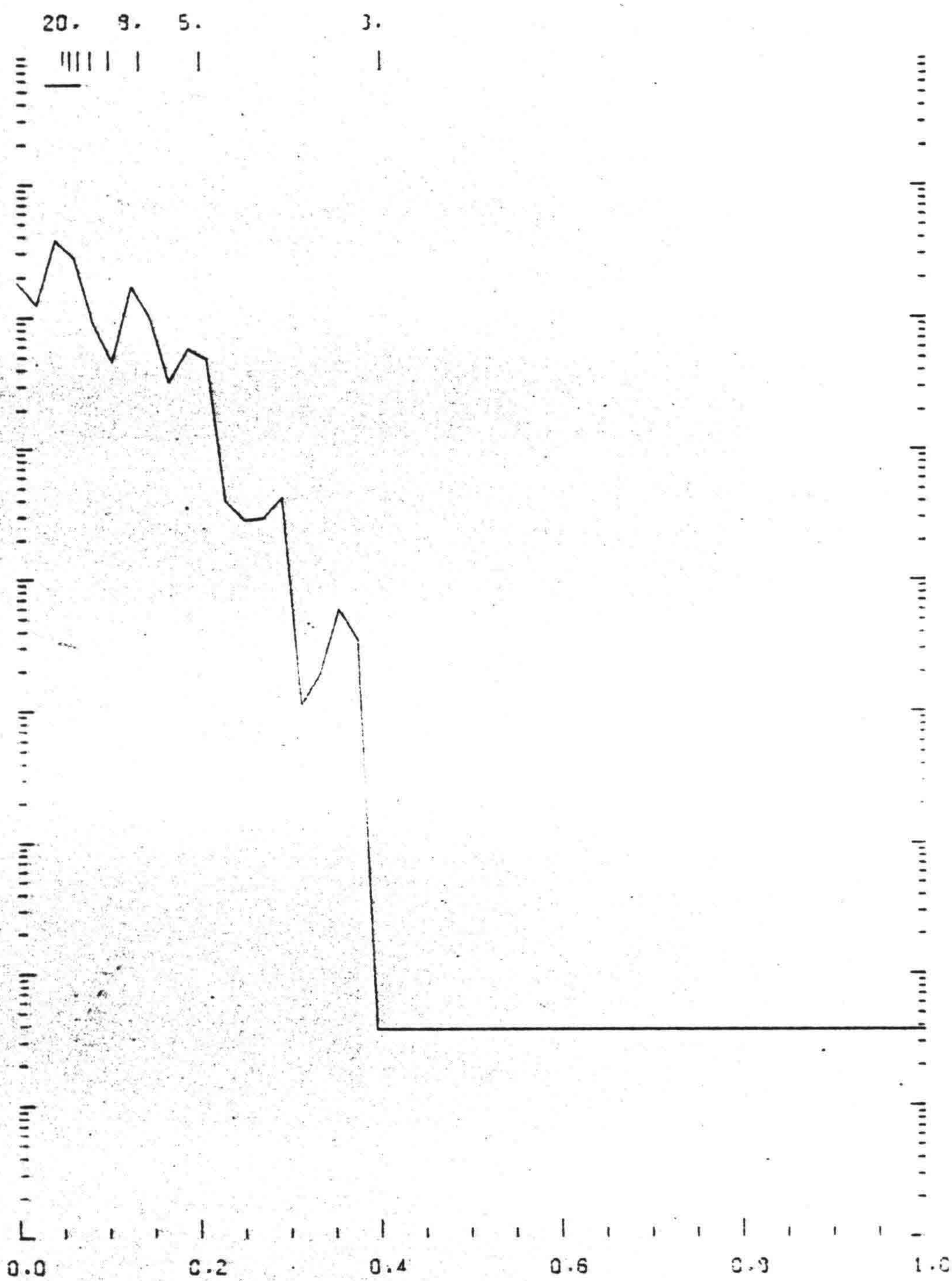
SQUARED COHERENCY

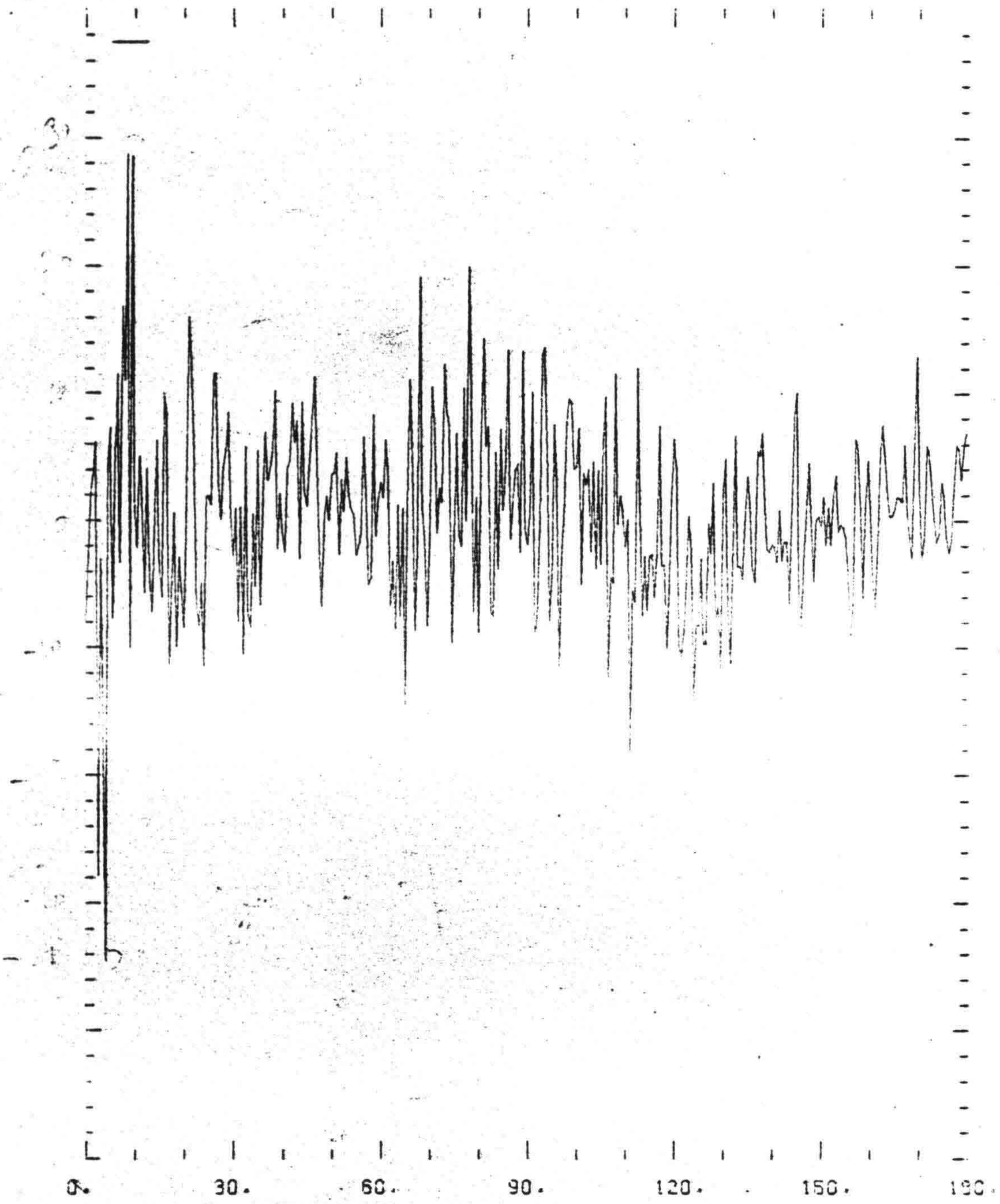


NOISE

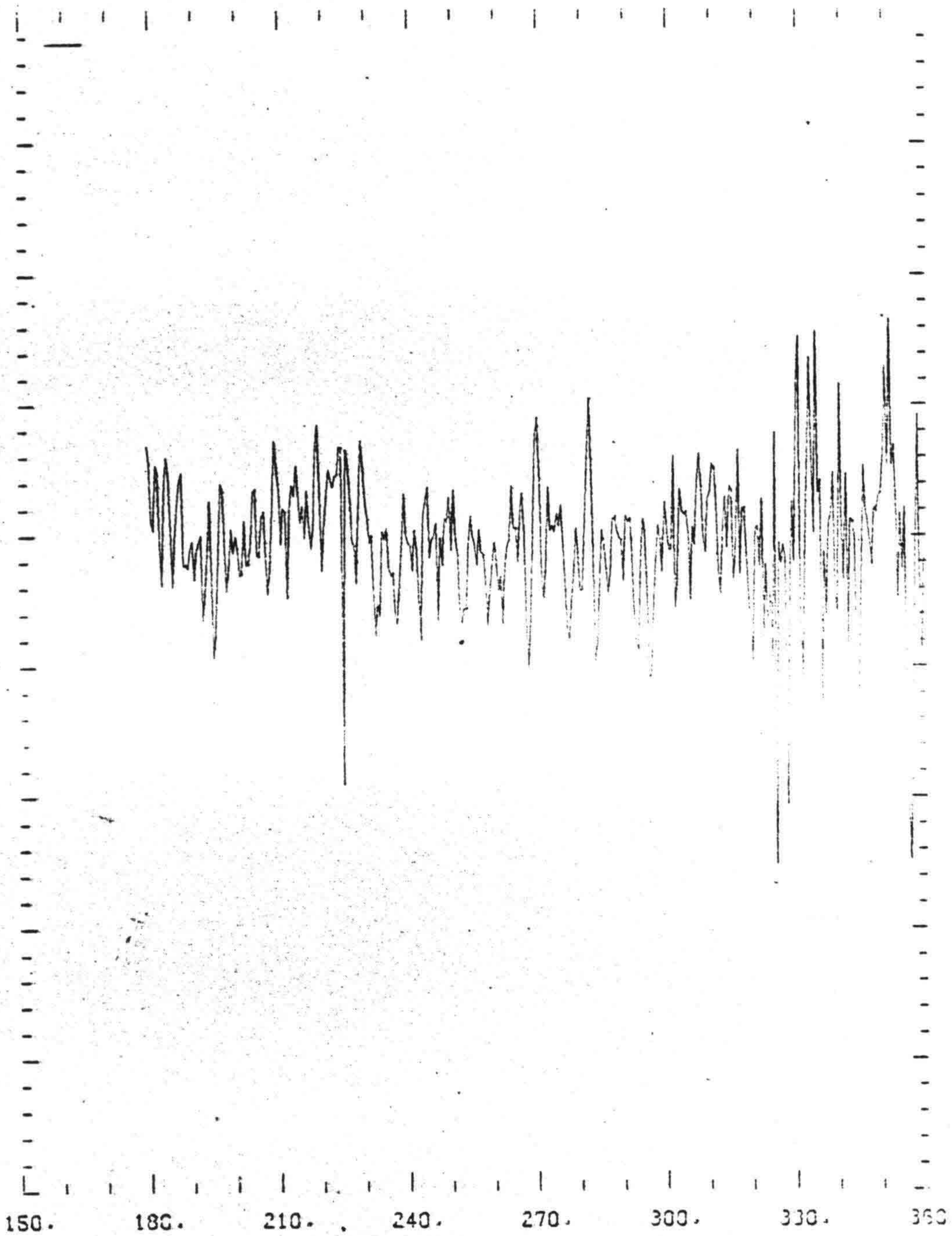
SPECTRUM

Figures 32 to 46

Meteorological components at stations 1 and 2.
The text of each plate should be identical to the corresponding plate in the first (and second) series.



PART 1 OF 2



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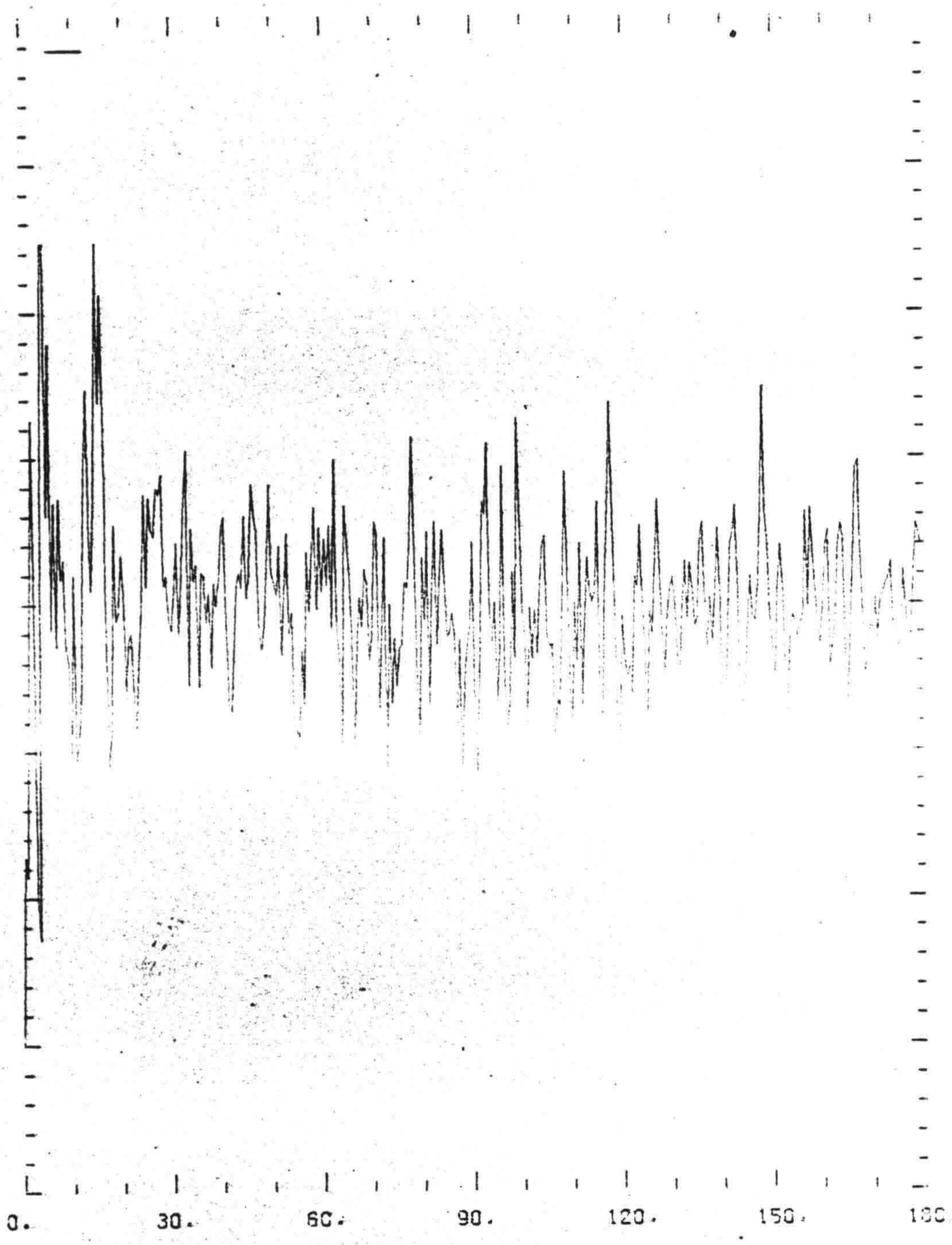
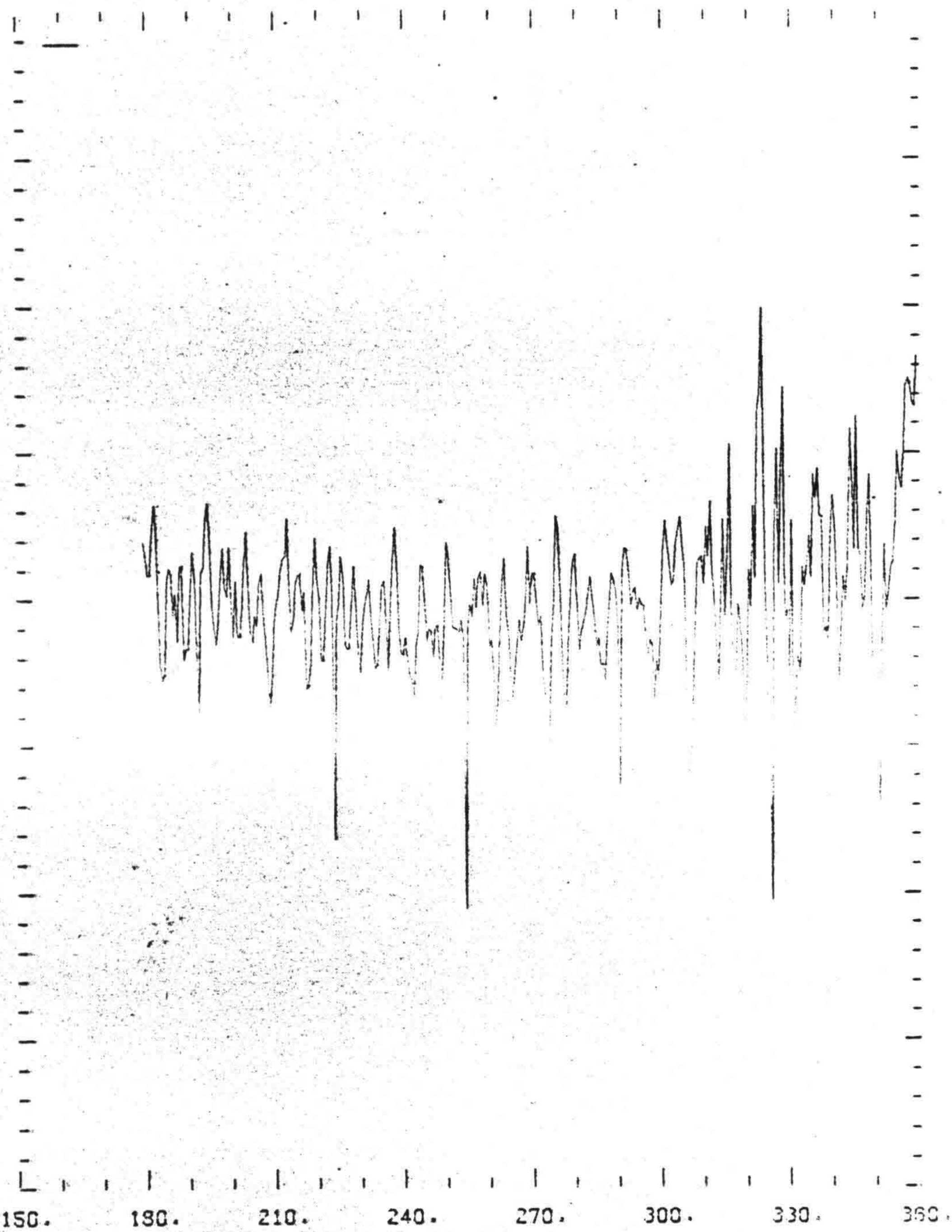
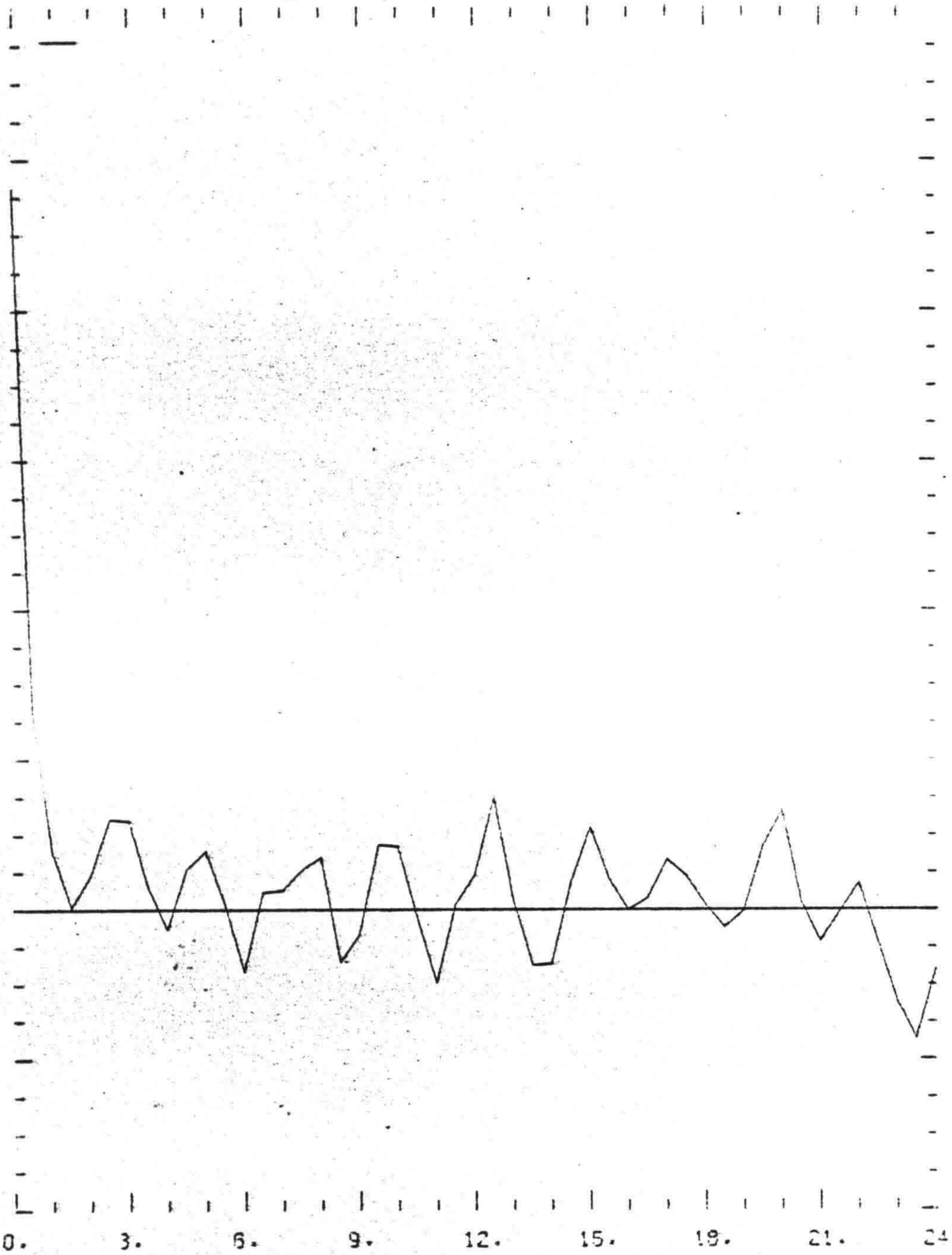


PART 1 OF 2



PART 2 OF 2



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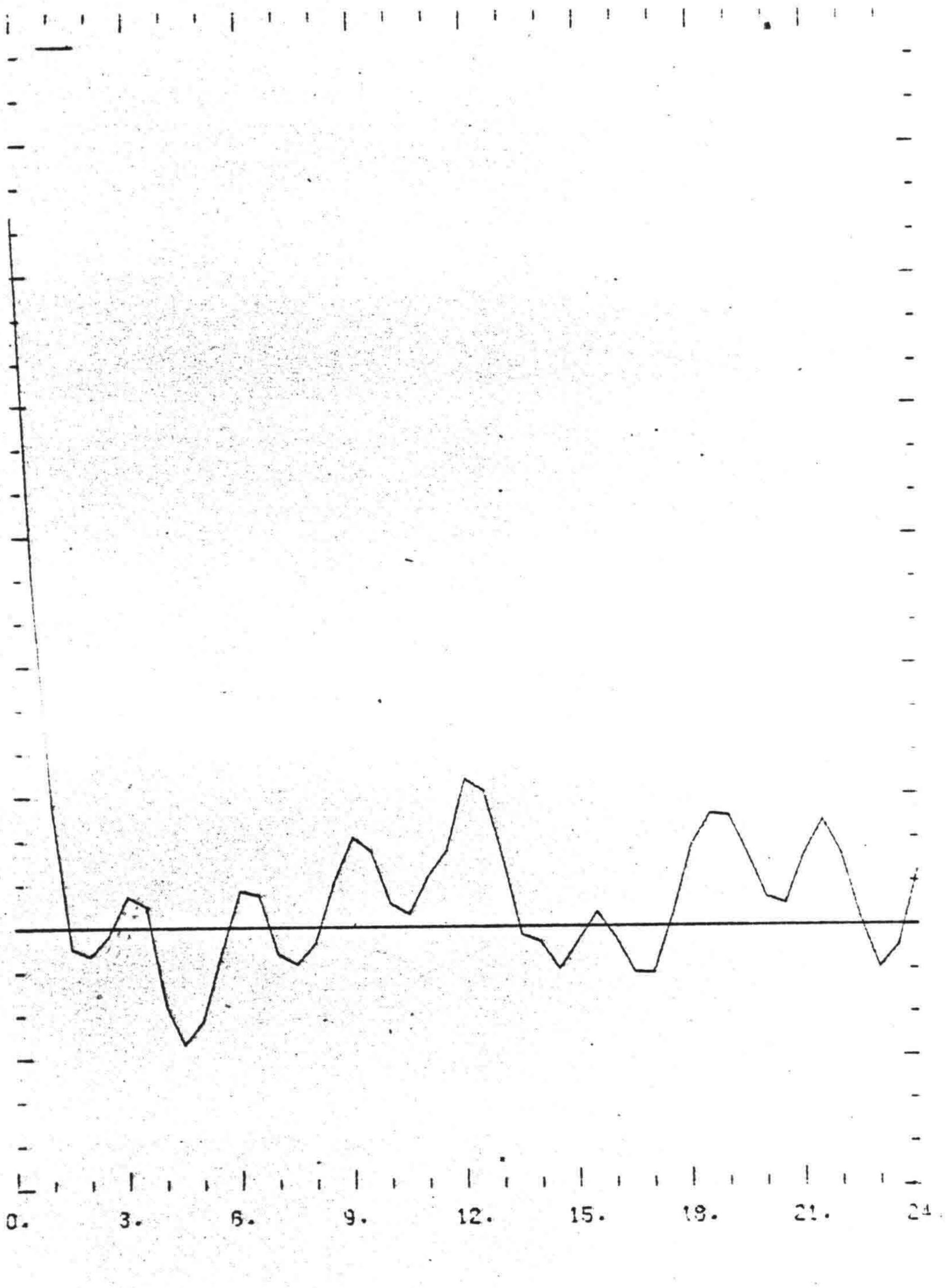

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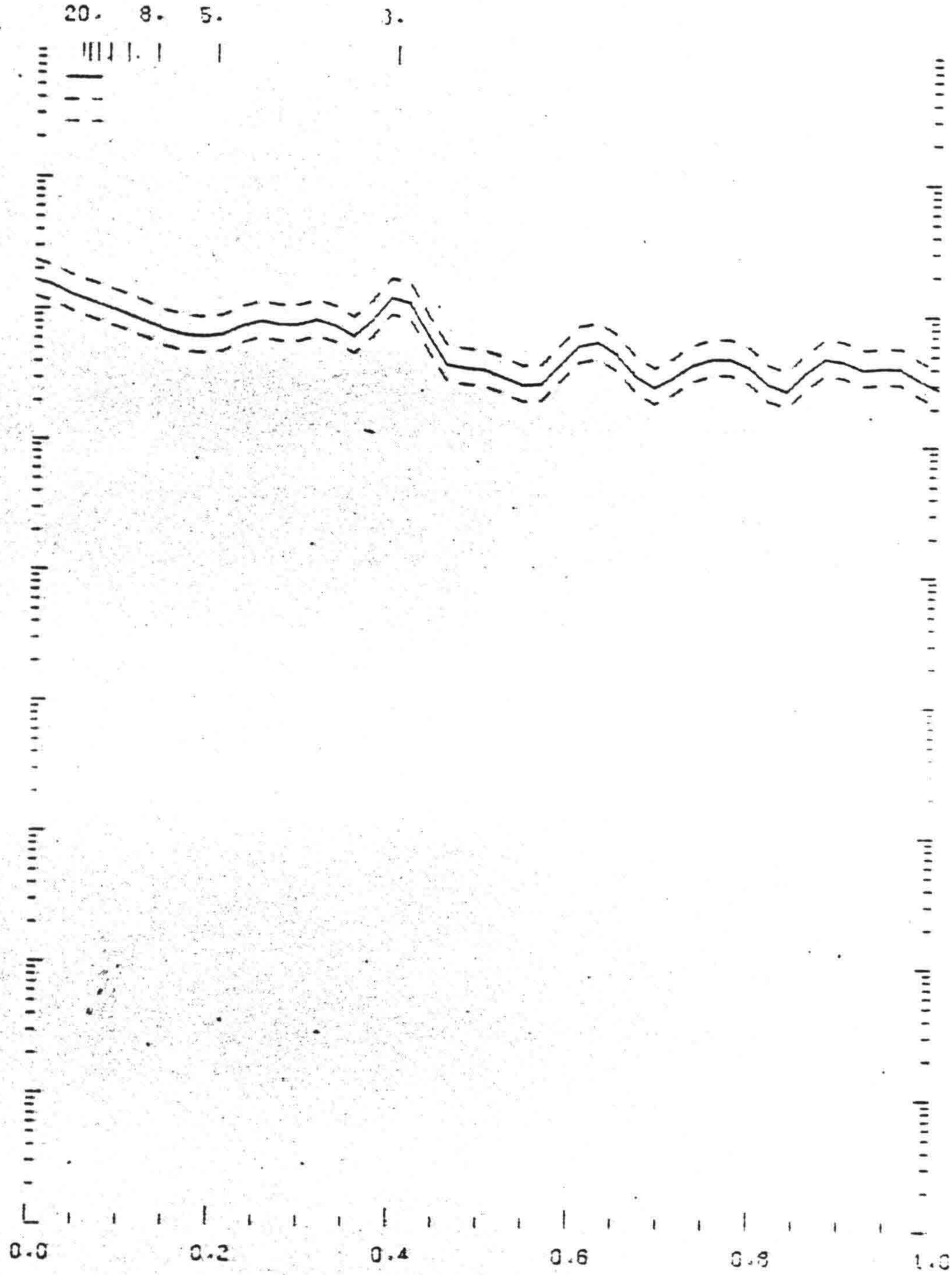

20. 8. 5. 3.
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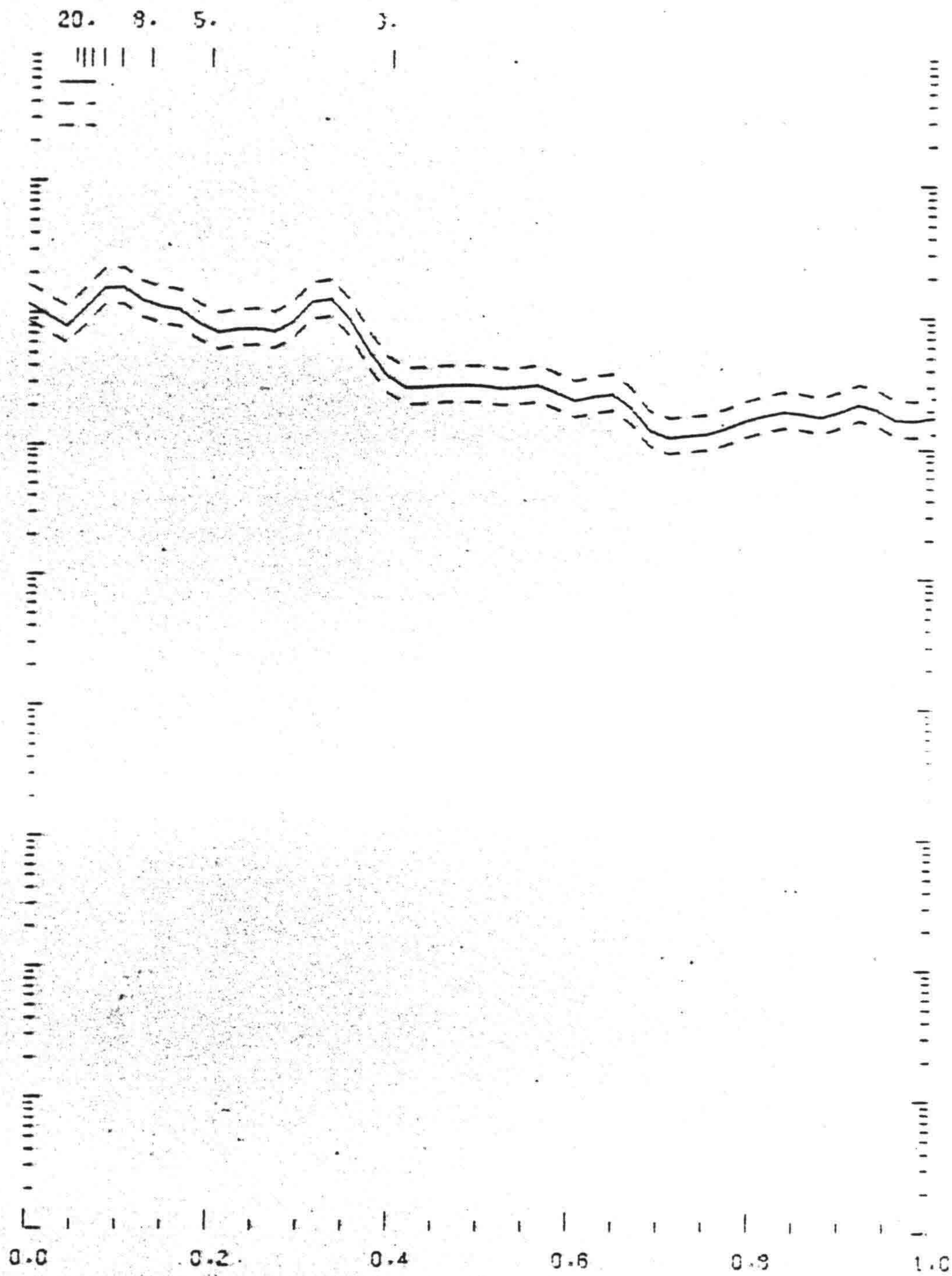

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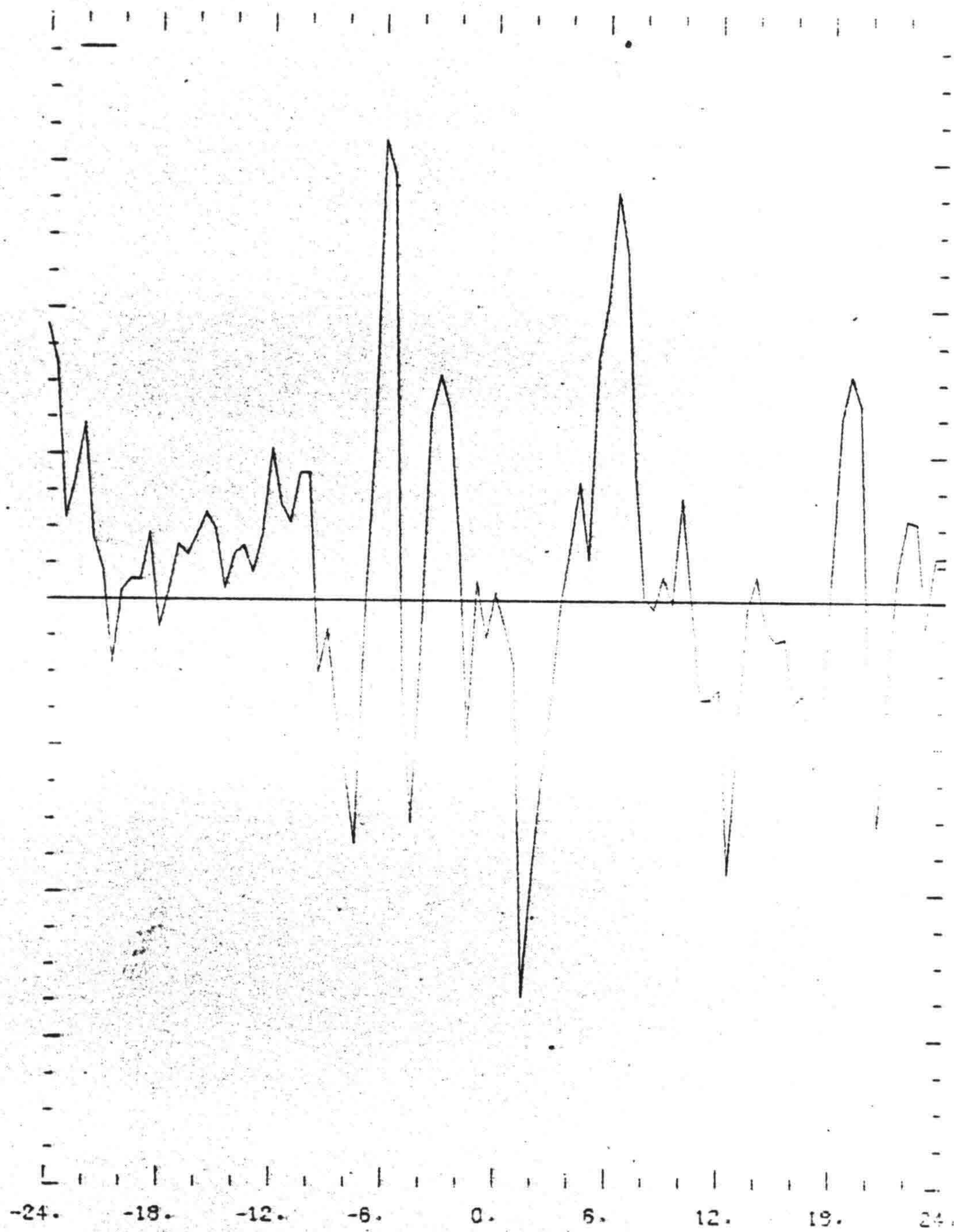

-24.
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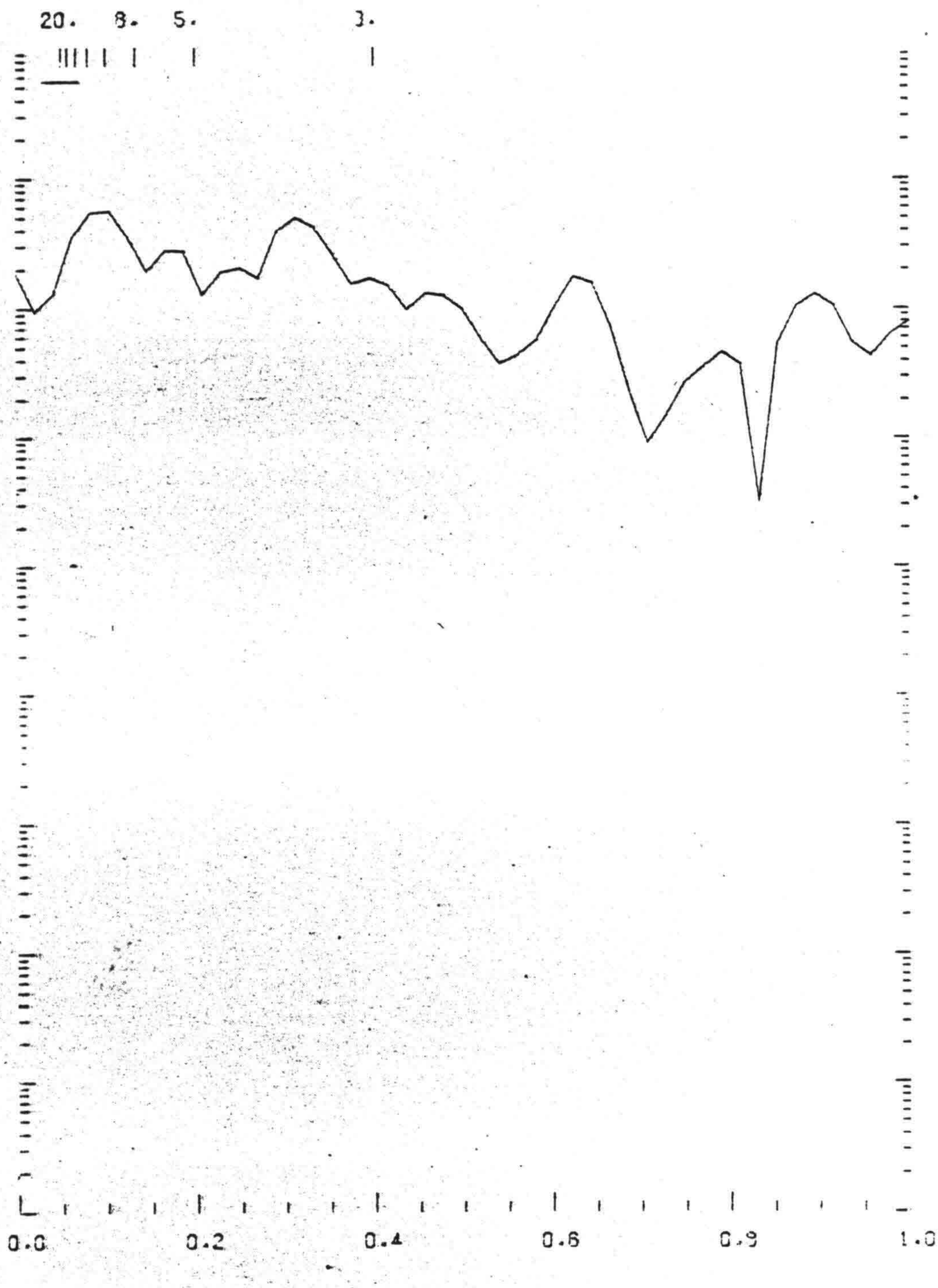
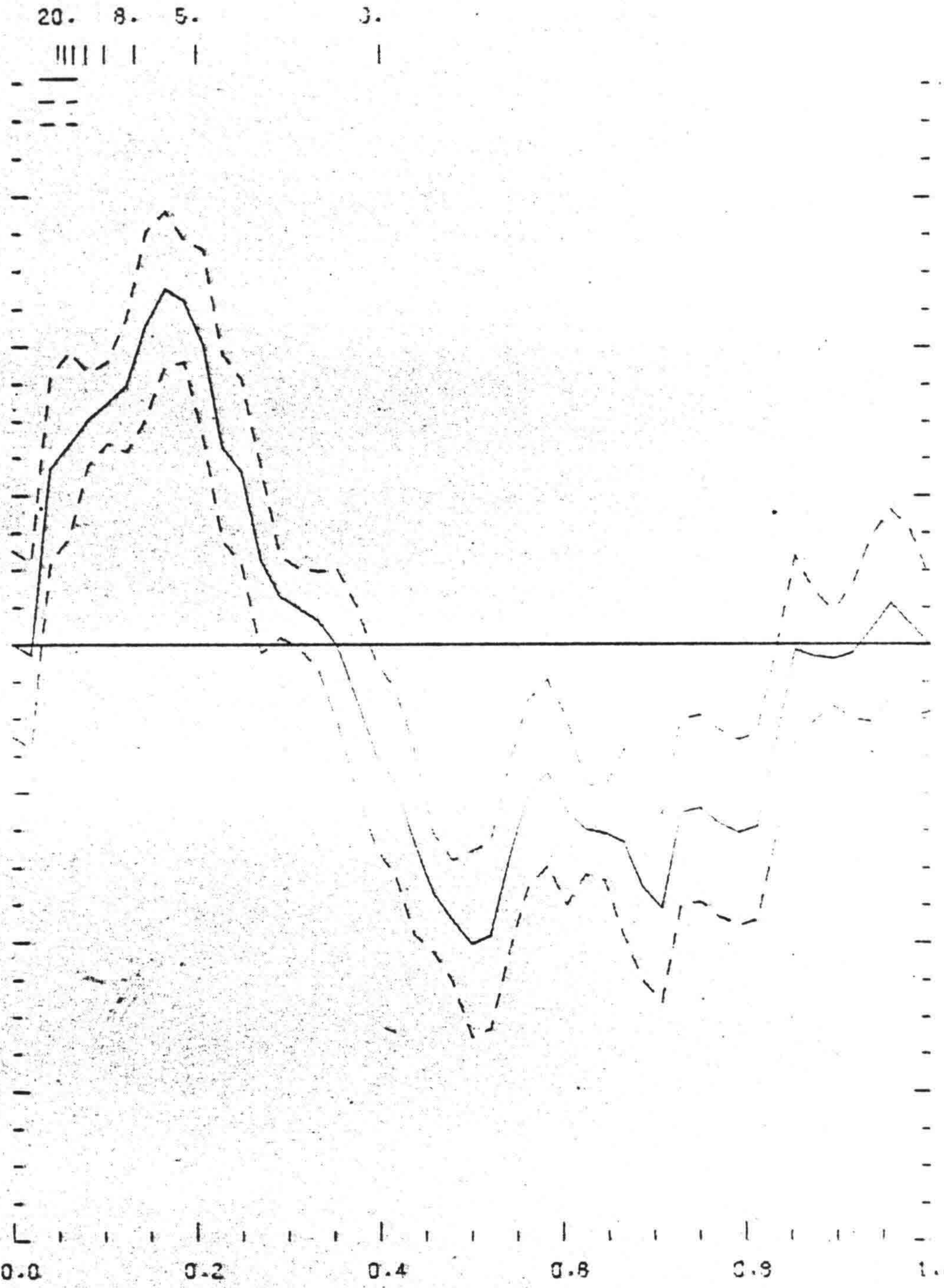

20. 8. 5. 3.
0.0 0.2 0.4 0.6 0.8 1.0



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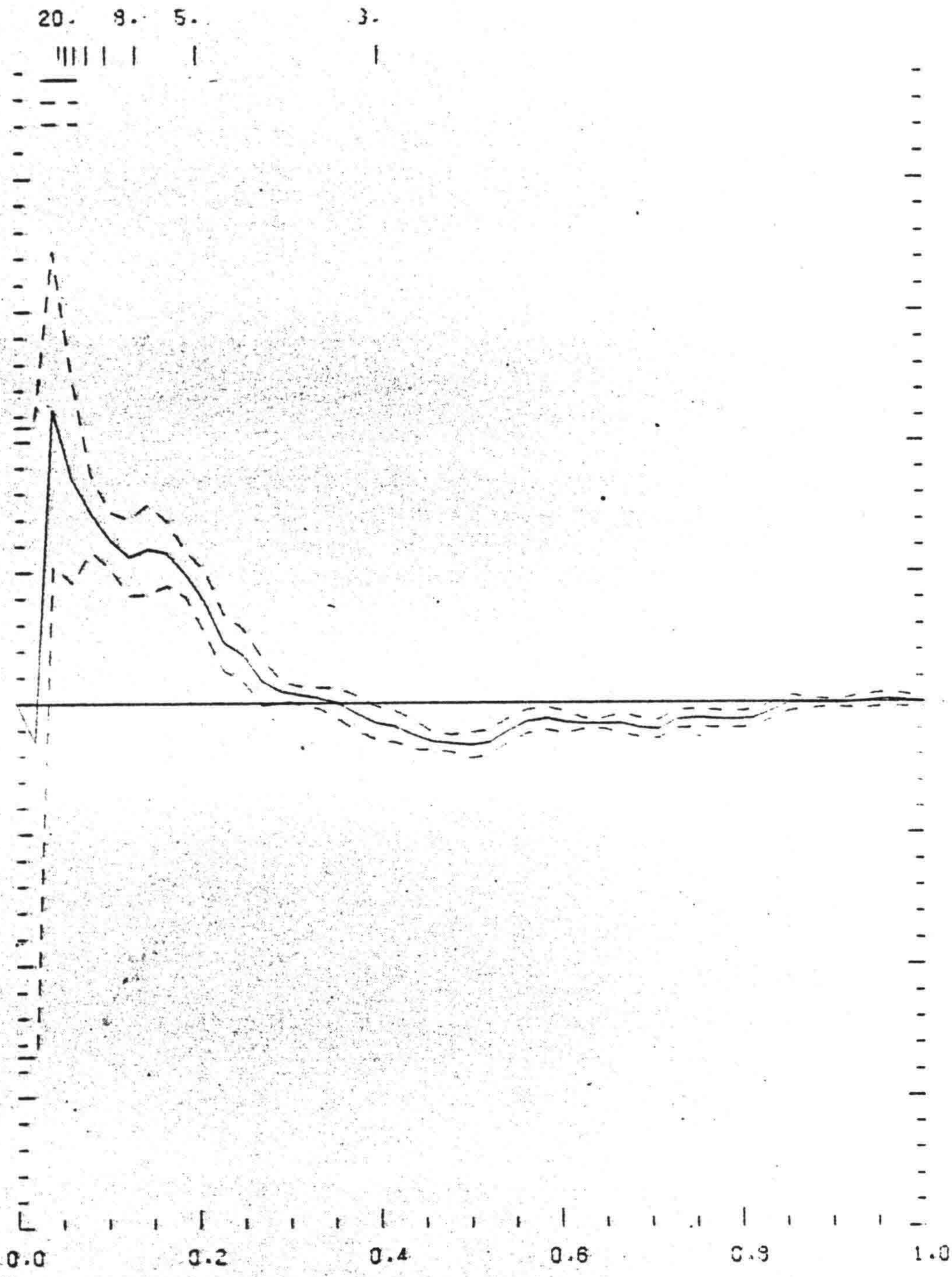

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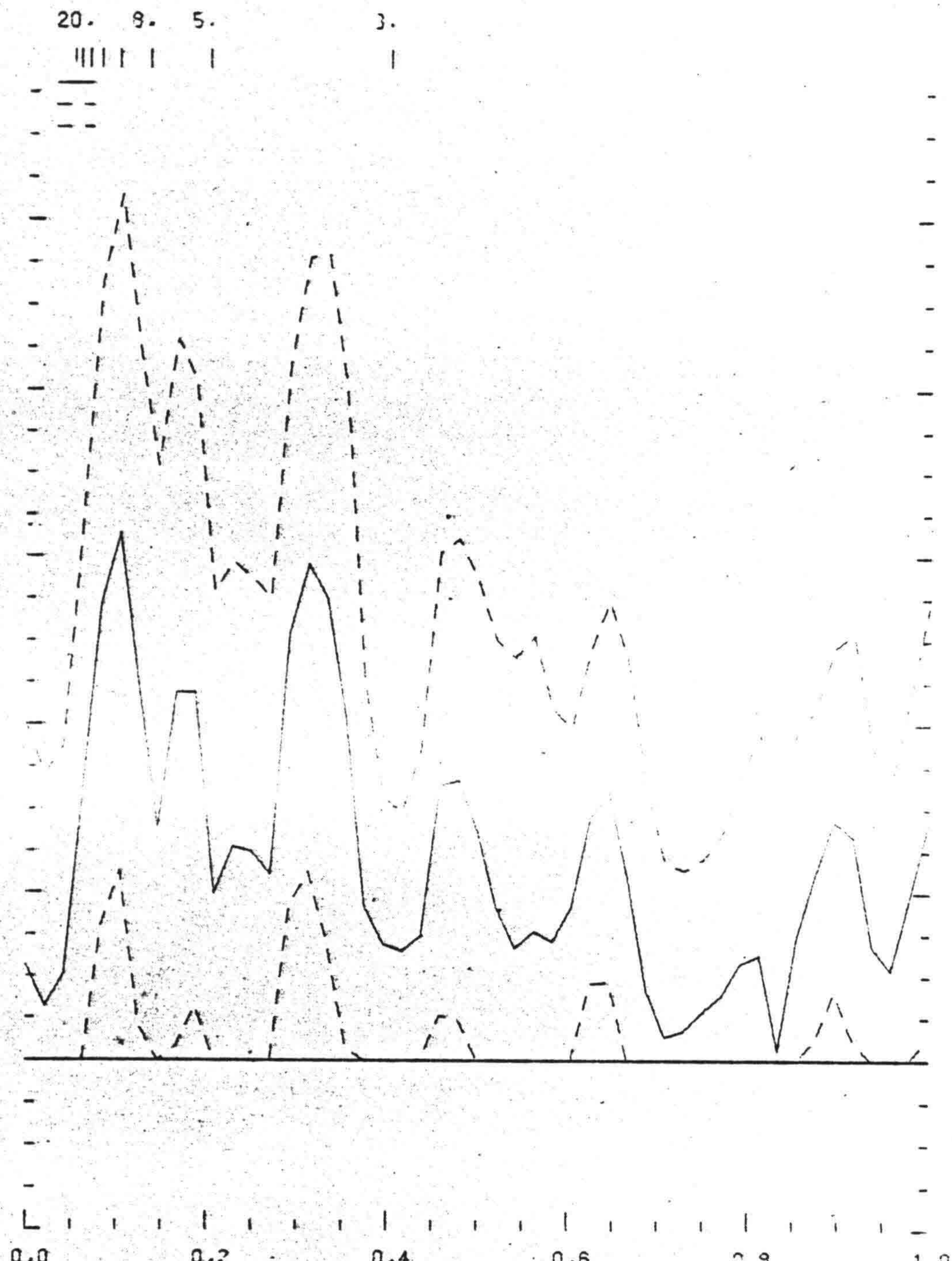
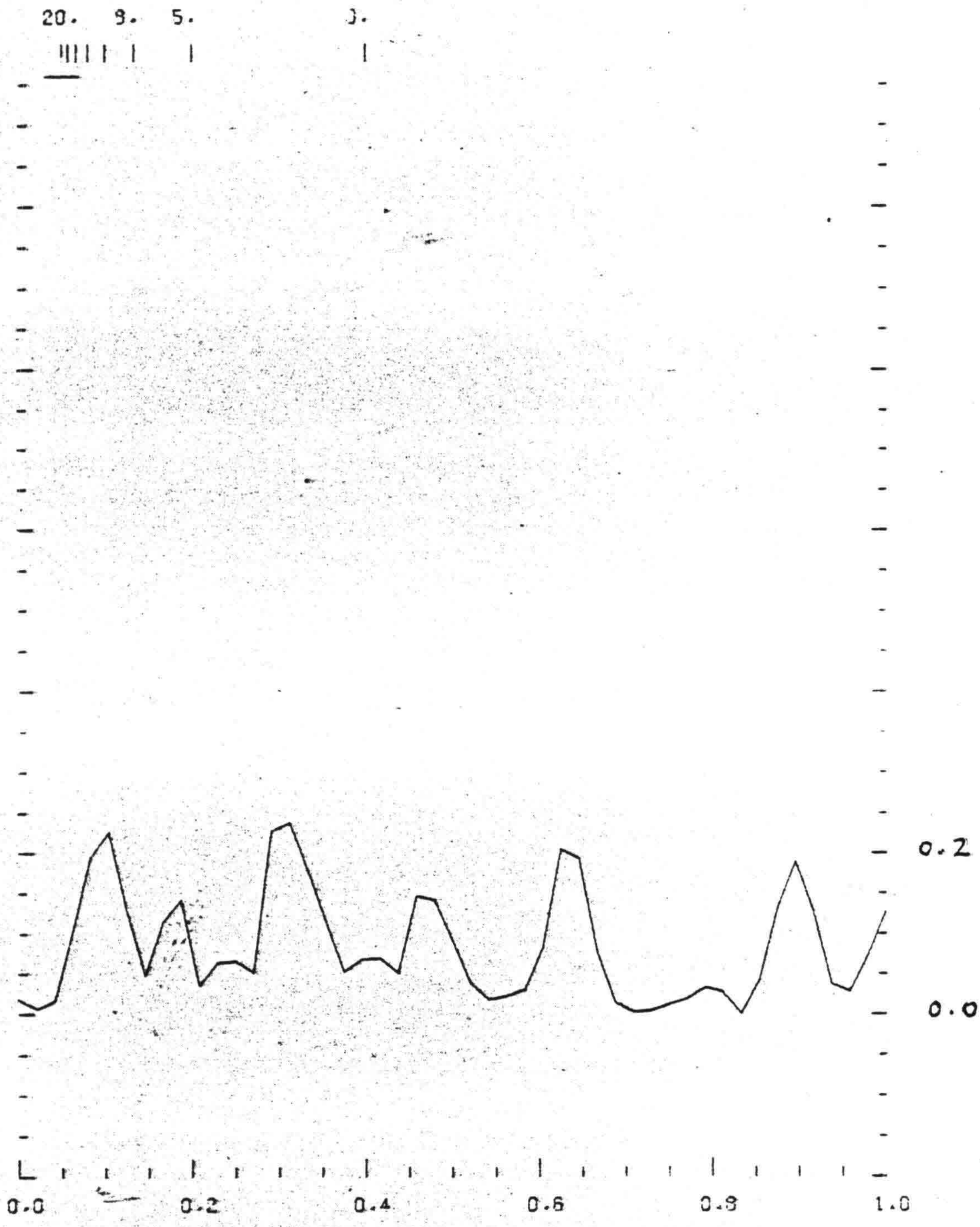


AMPLITUDE FREQUENCY RESPONSE



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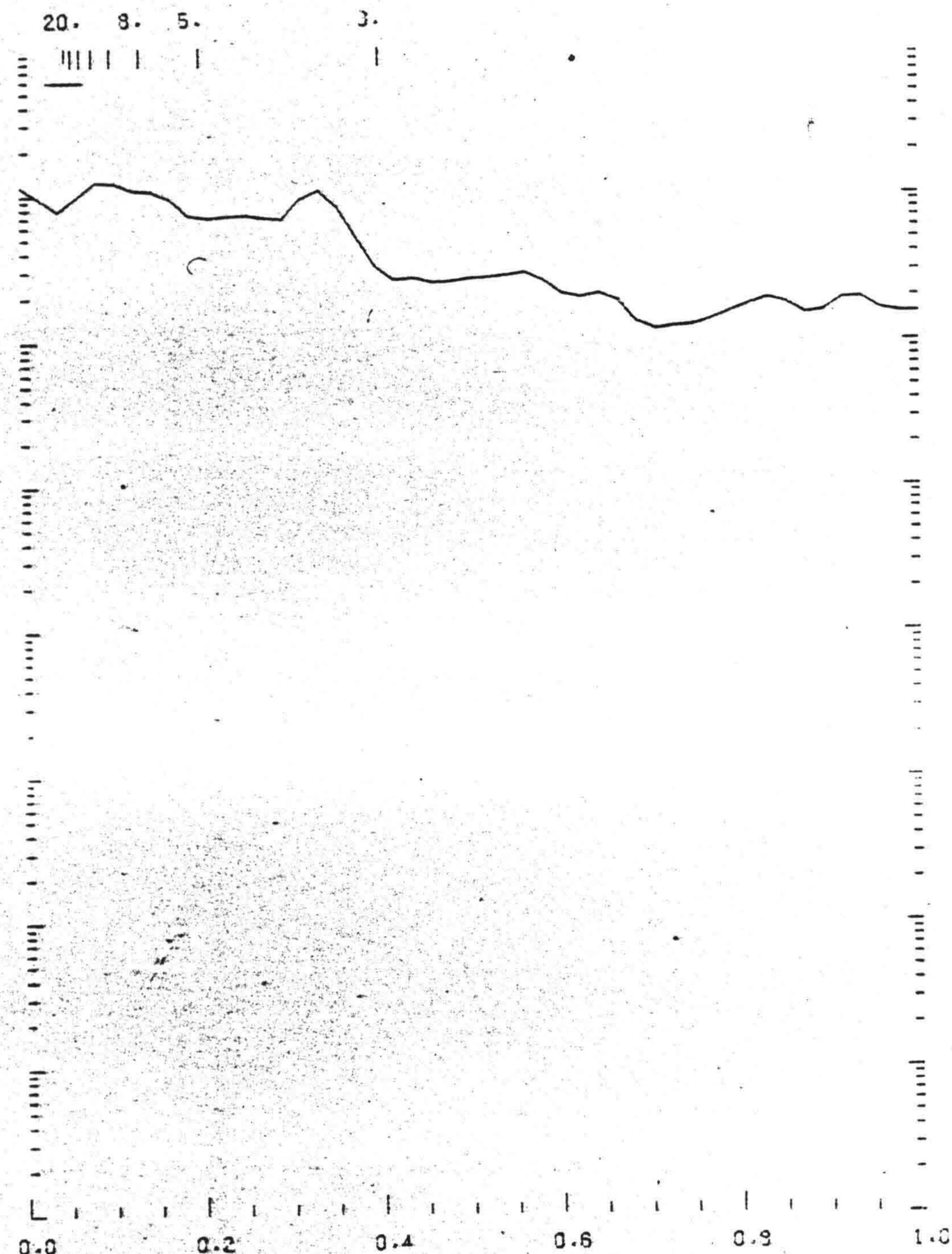

20.
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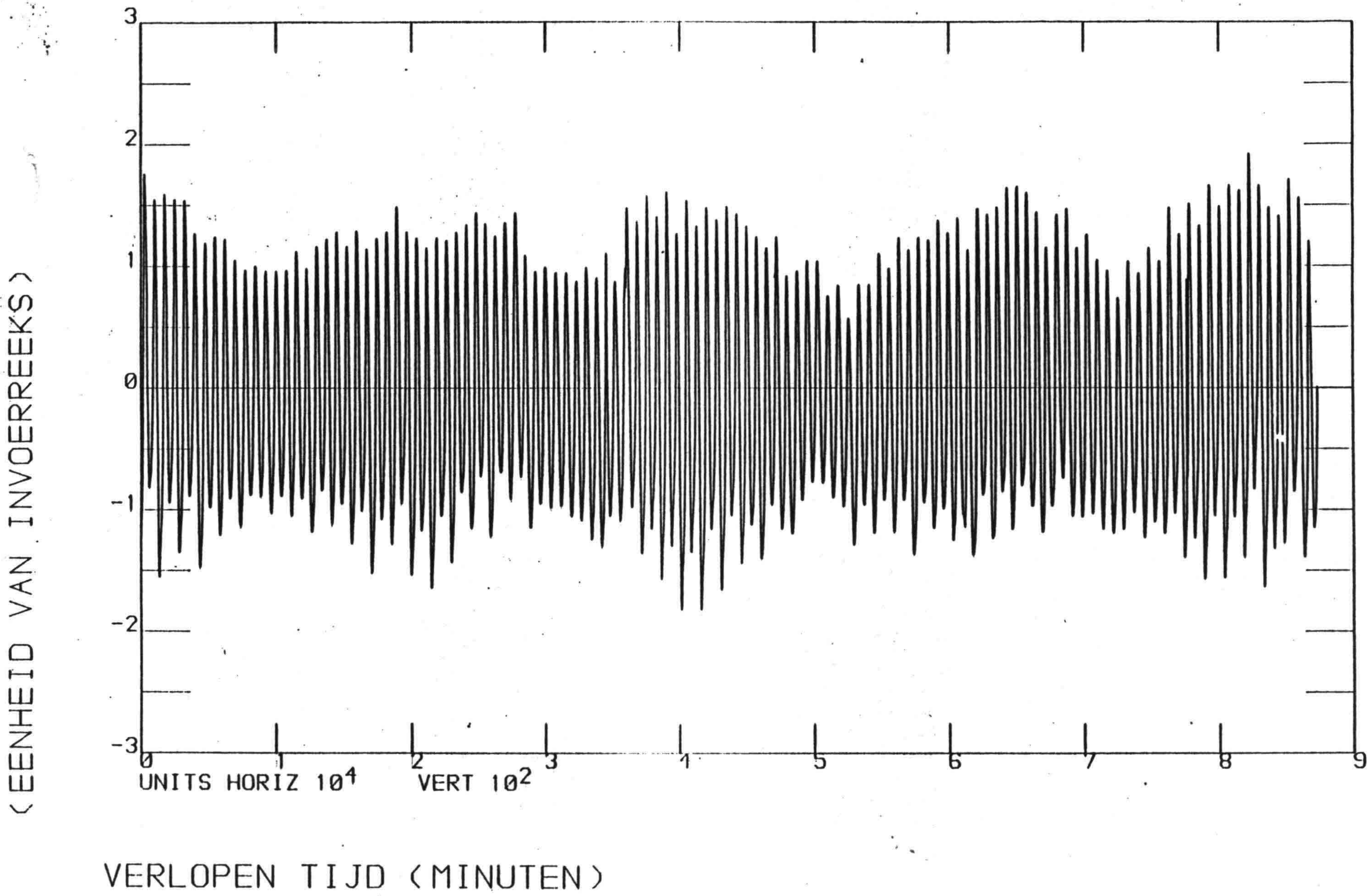
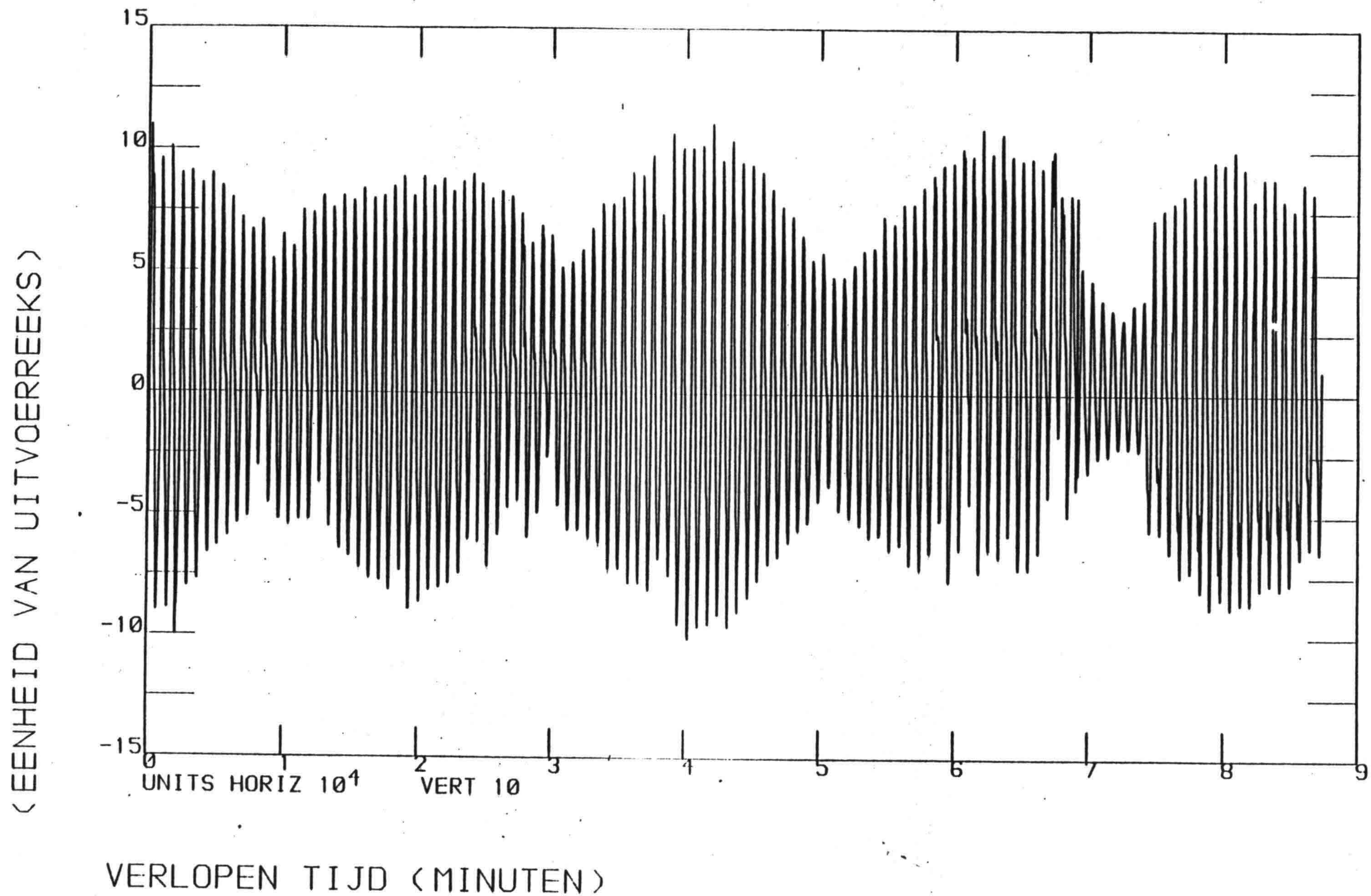
Figures 47 to 60

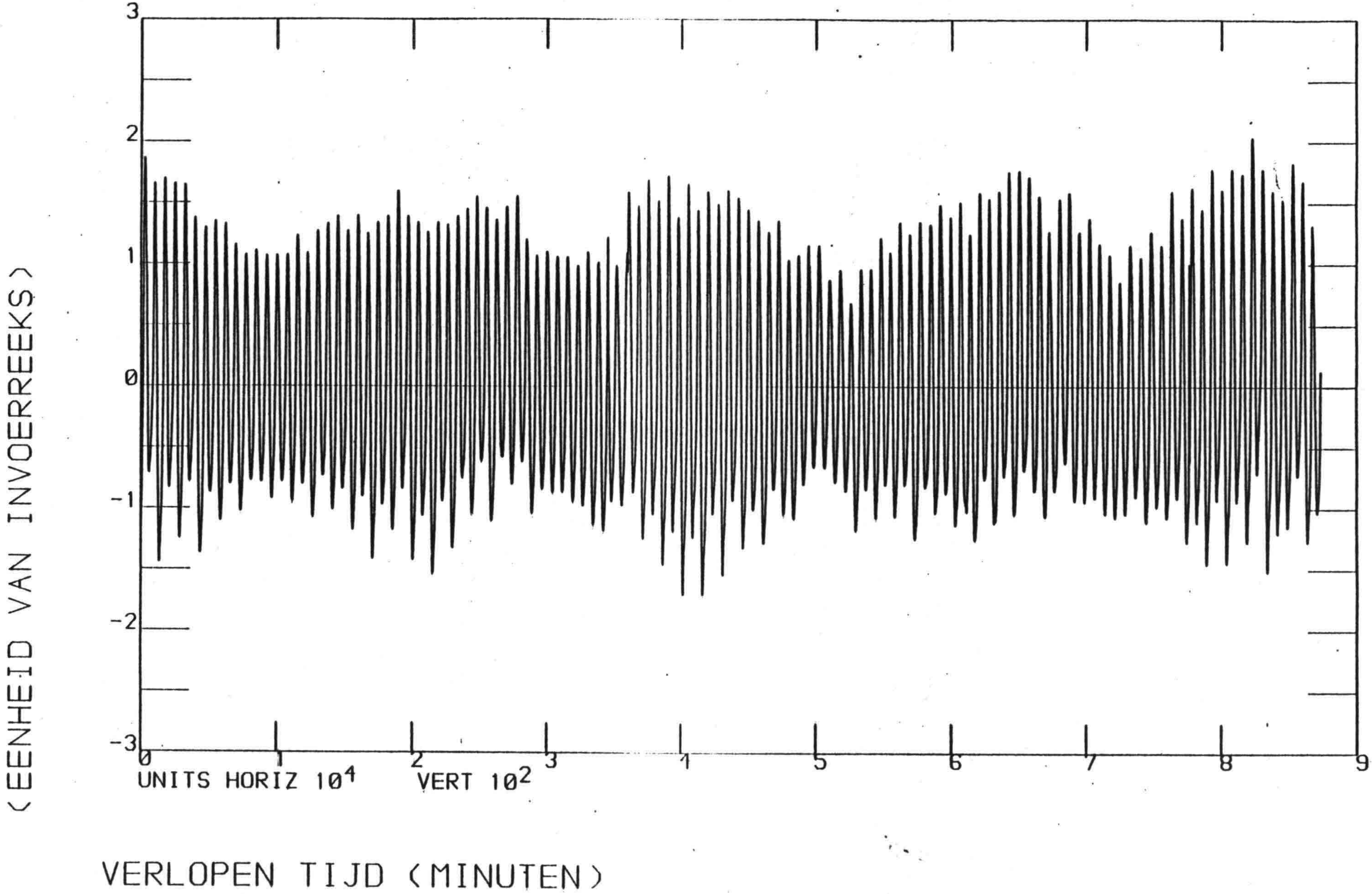
Measurements of water-levels at B.G.II and Noord-Banjaard.



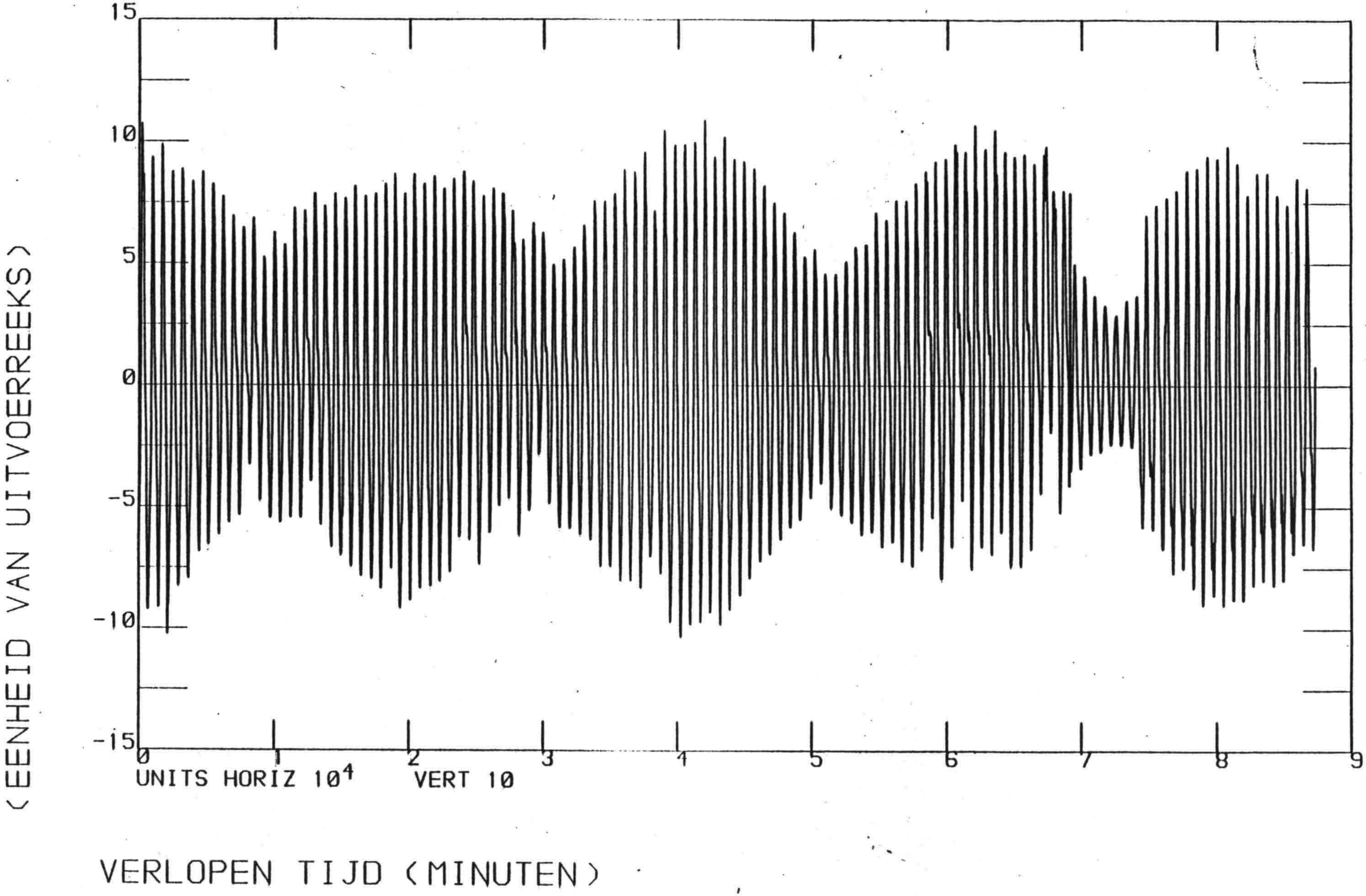
GEGEVEN INVOERREEKS



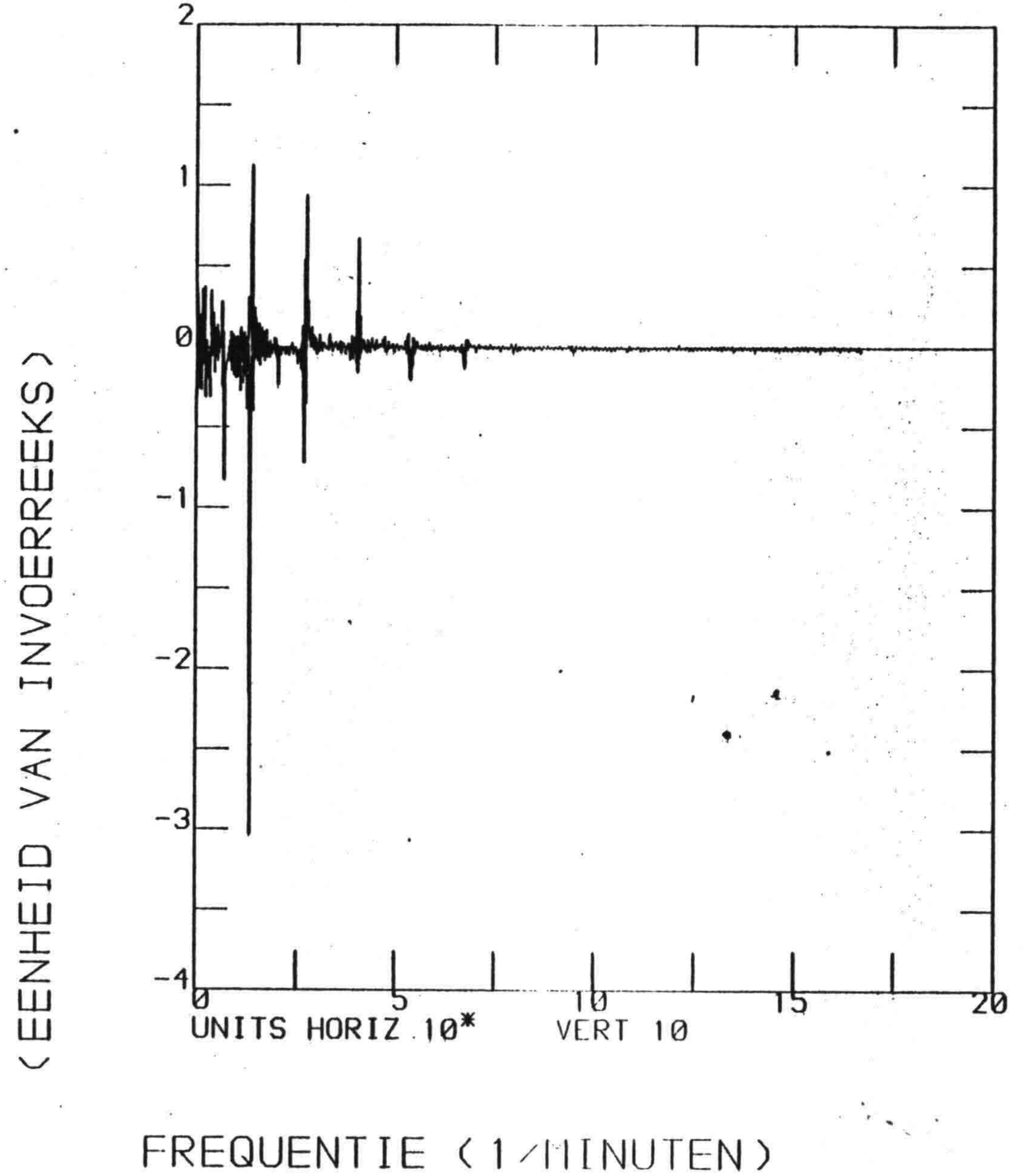
GEGEVEN UITVOERREEKS



GEMODIFICEERDE INVOERREEKS



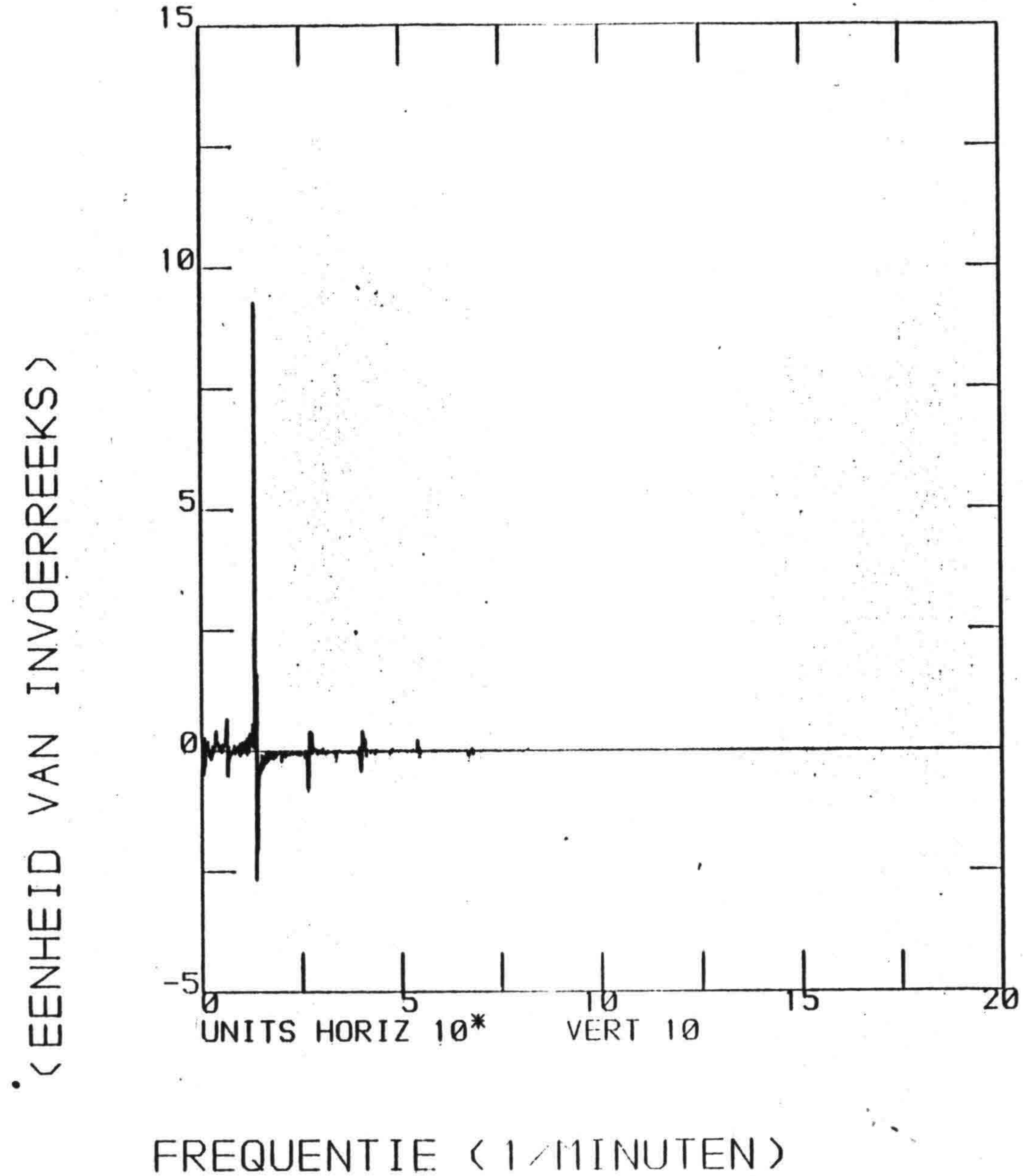
GEMODIFICEERDE UITVOERREEKS



COSINUSCOEFFICIENTEN VAN
GEMODIFICEERDE INVOERREEKS

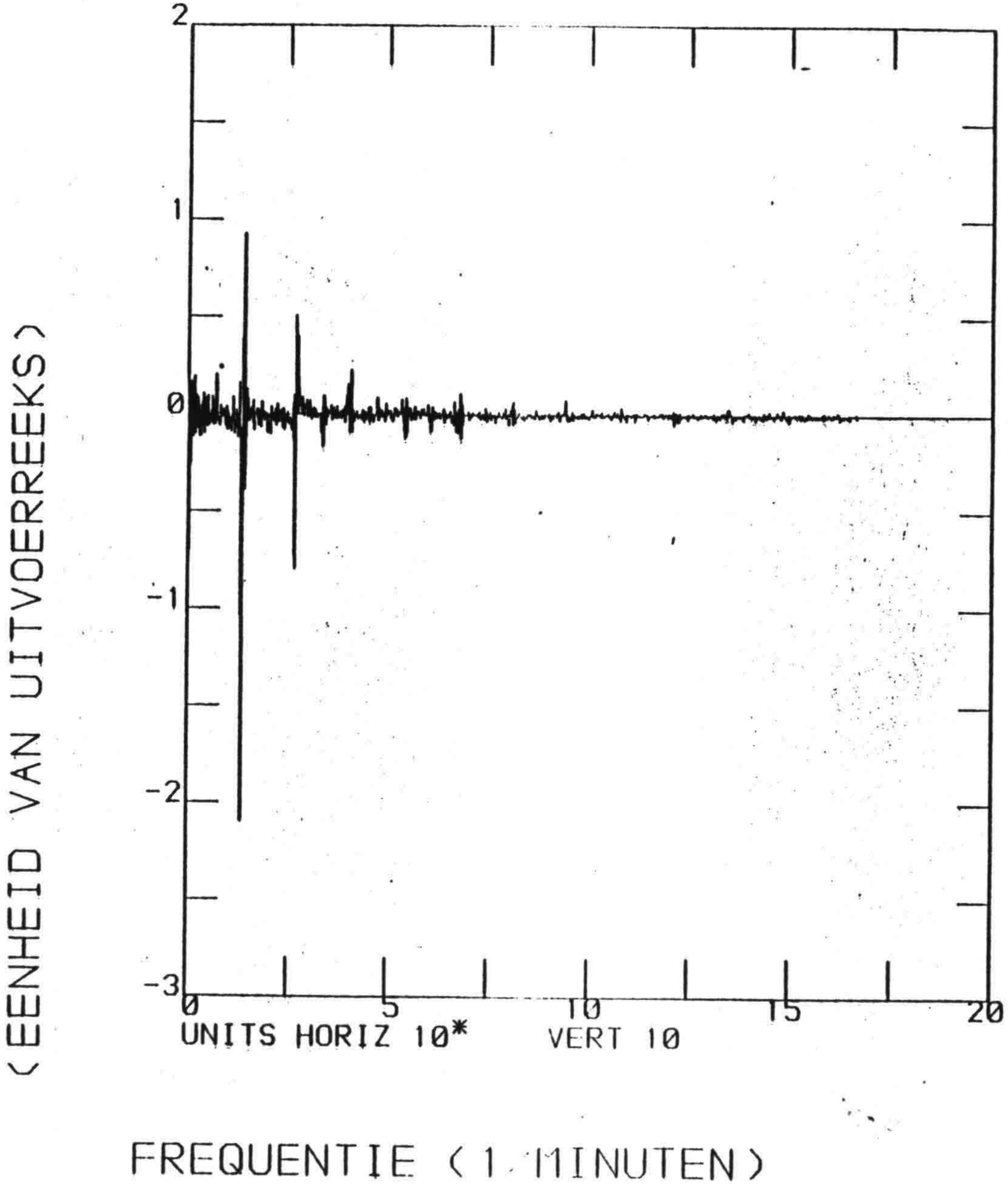
(1)

1

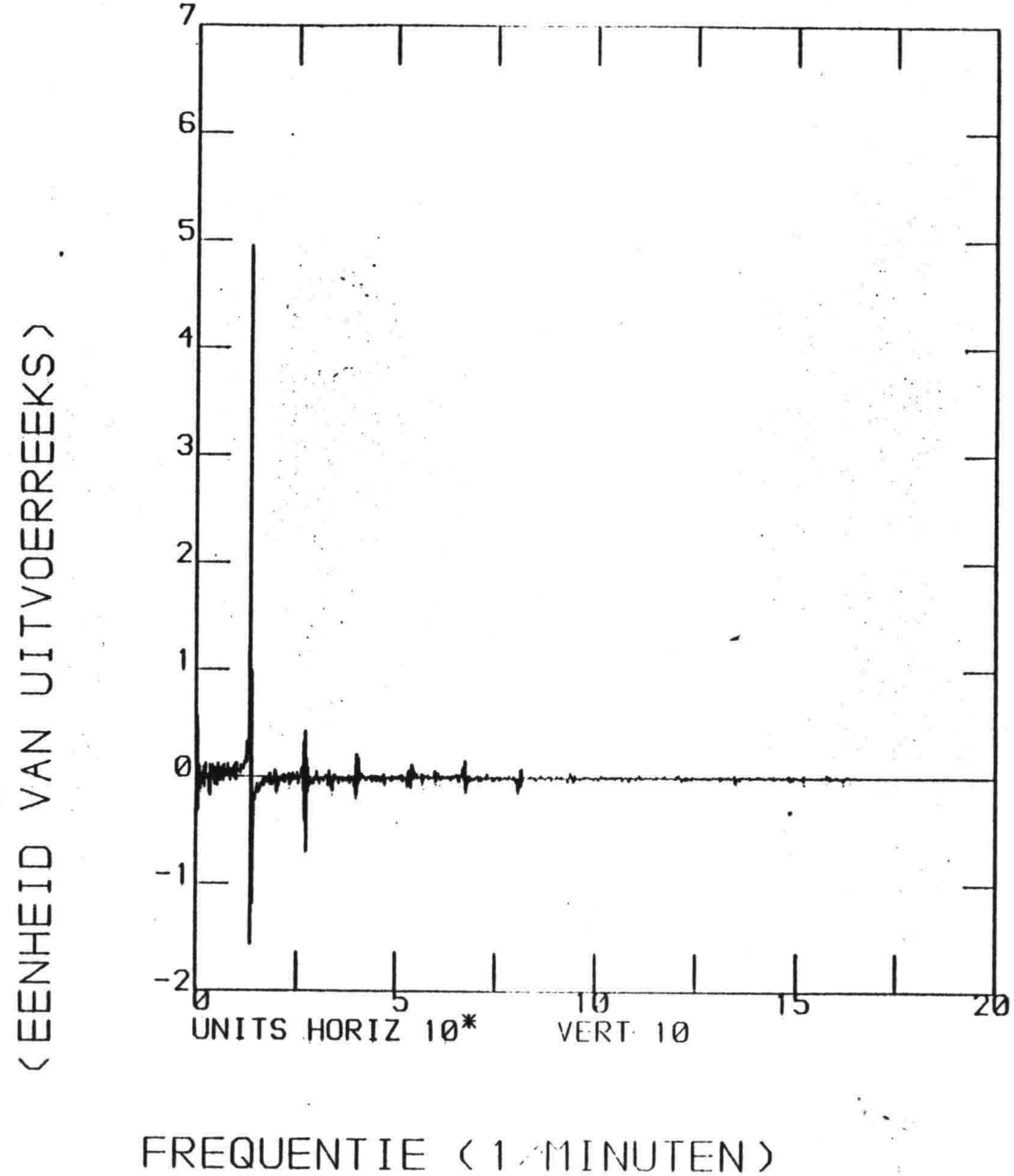


SINUSCOEFFICIENTEN VAN
GEMODIFICEERDE INVOERREEKS

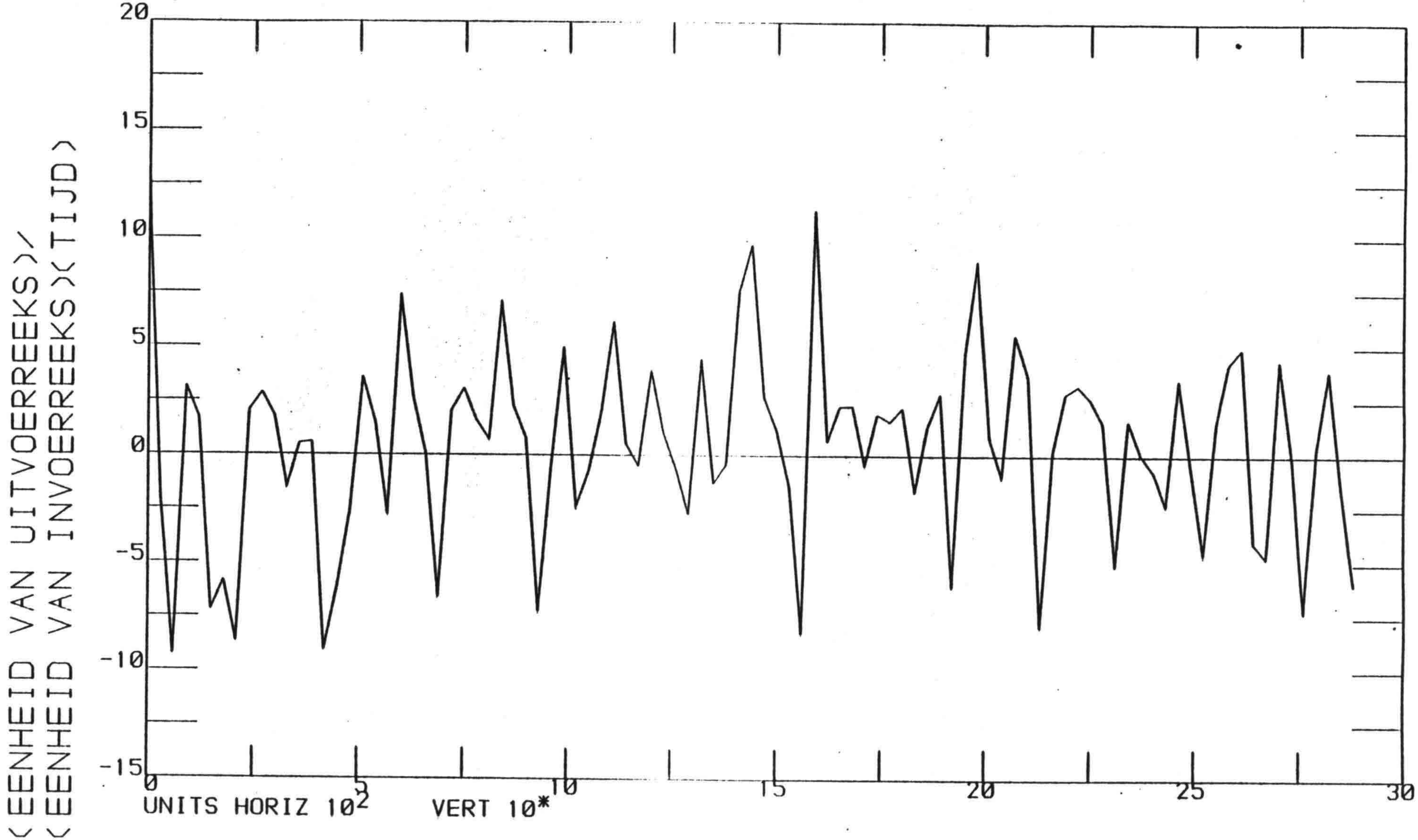
(S)



COSINUSCOEFFICIENTEN VAN
GEMODIFICEERDE UITVOERREEKS

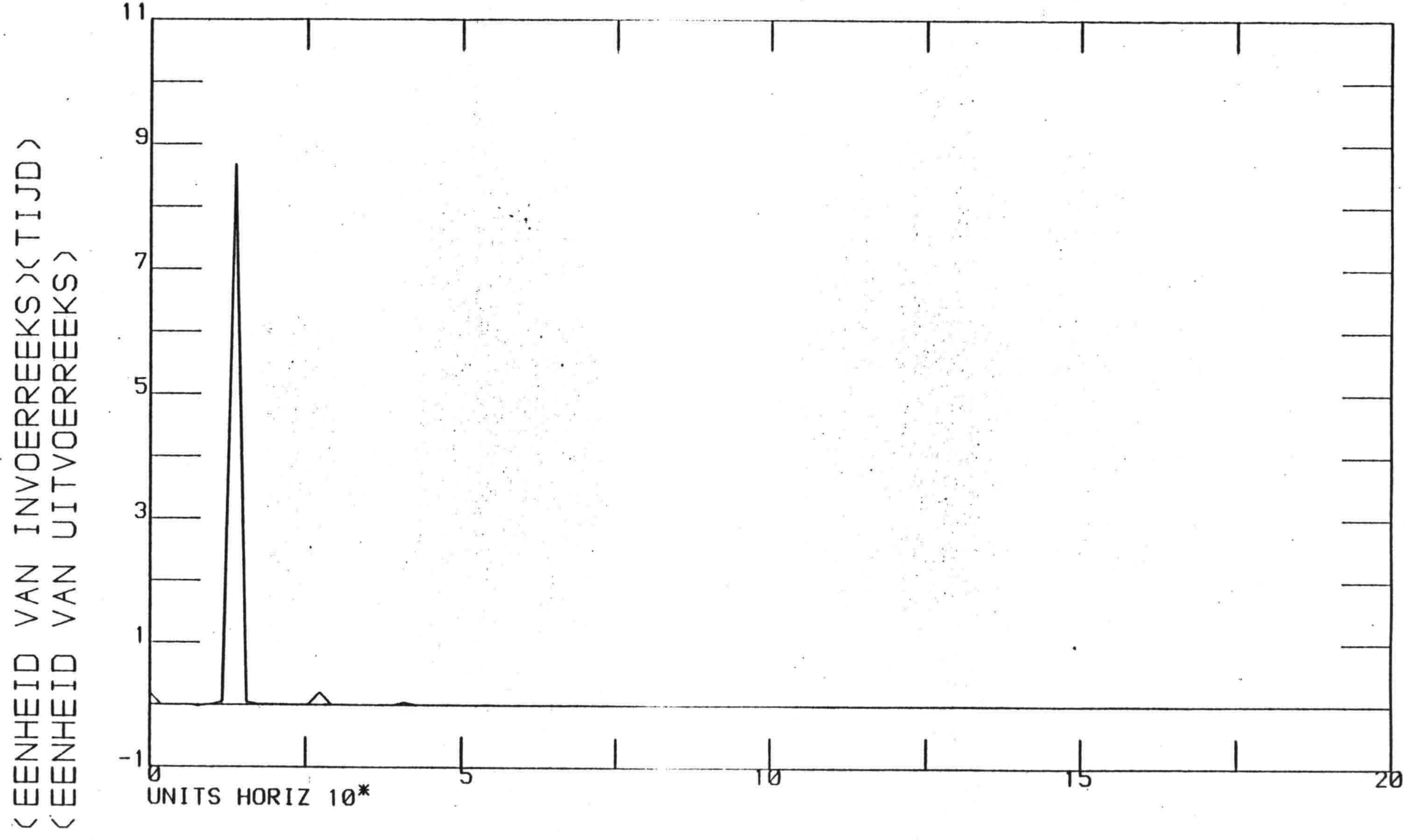


SINUSCOEFFICIENTEN VAN
GEMODIFICEERDE UITVOERREEKS



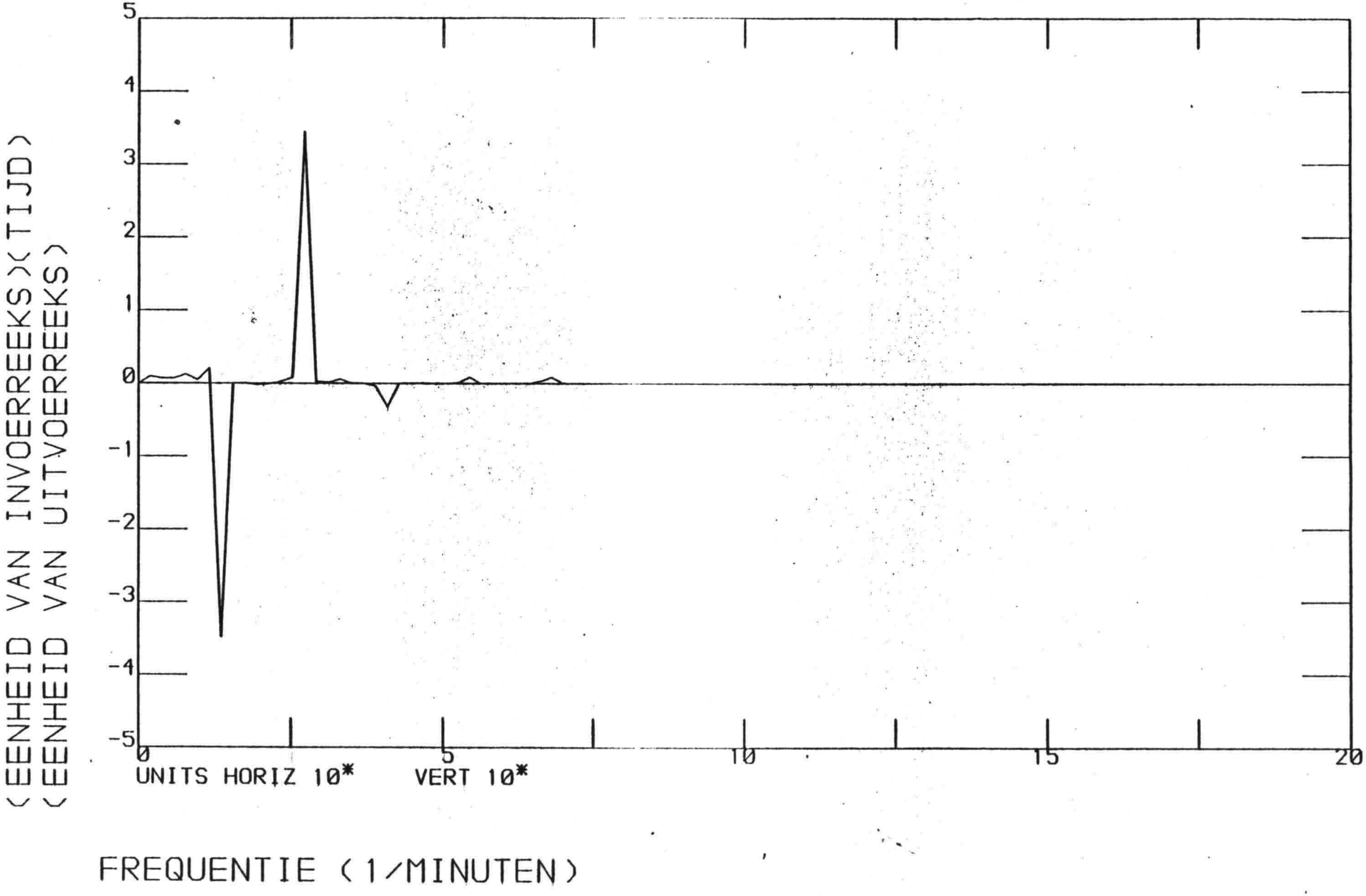
TIJDSVERLOOP (MINUTEN)

IMPULSRESPONSFUNCTIE VAN DE IN- EN UITVOERREEKS

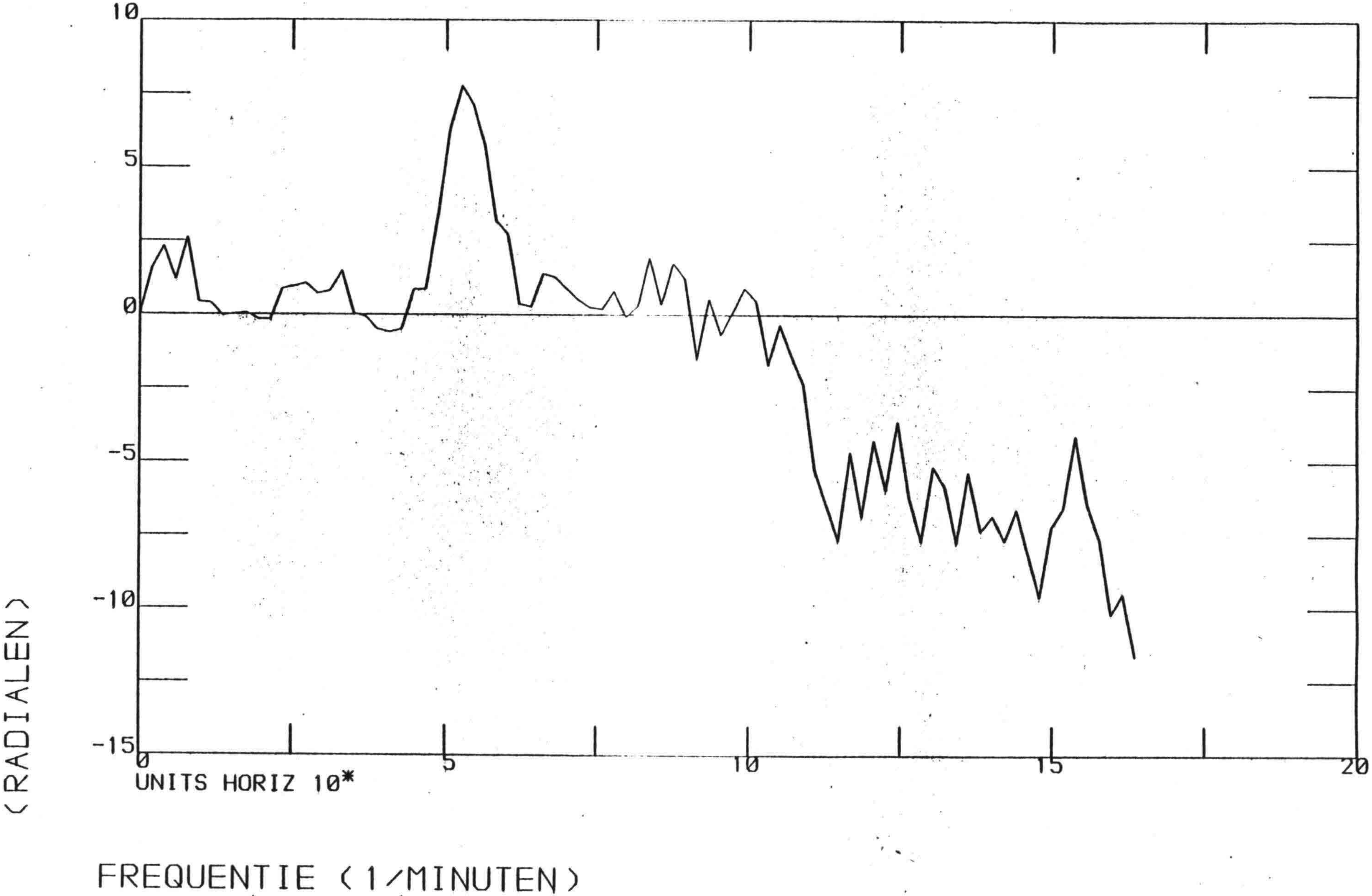


FREQUENTIE (1/MINUTEN)

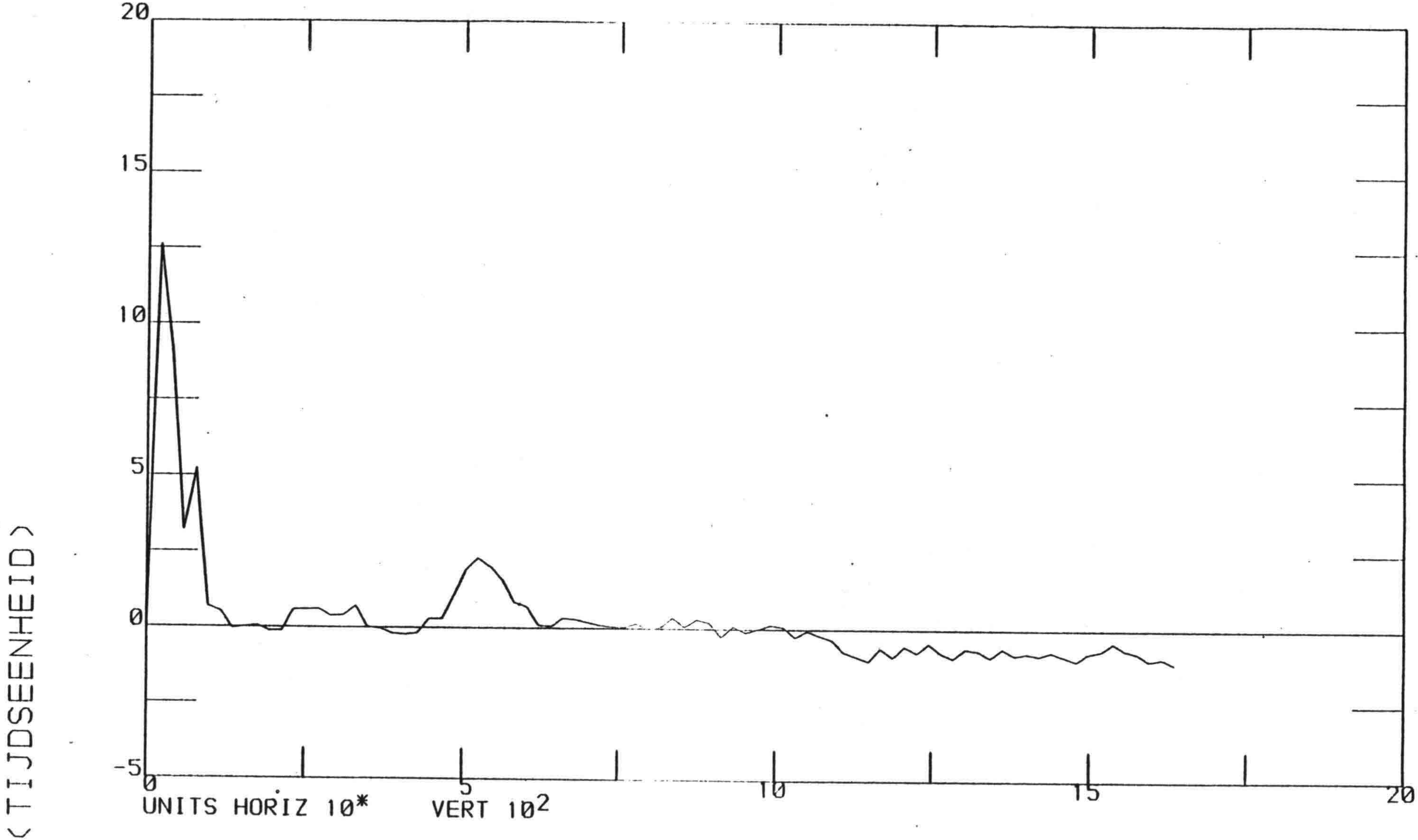
COSPECTRUM VAN DE IN- EN DE UITVOERREEKS



QUADRATUURSPECTRUM VAN DE IN- EN DE UITVOERREEKS

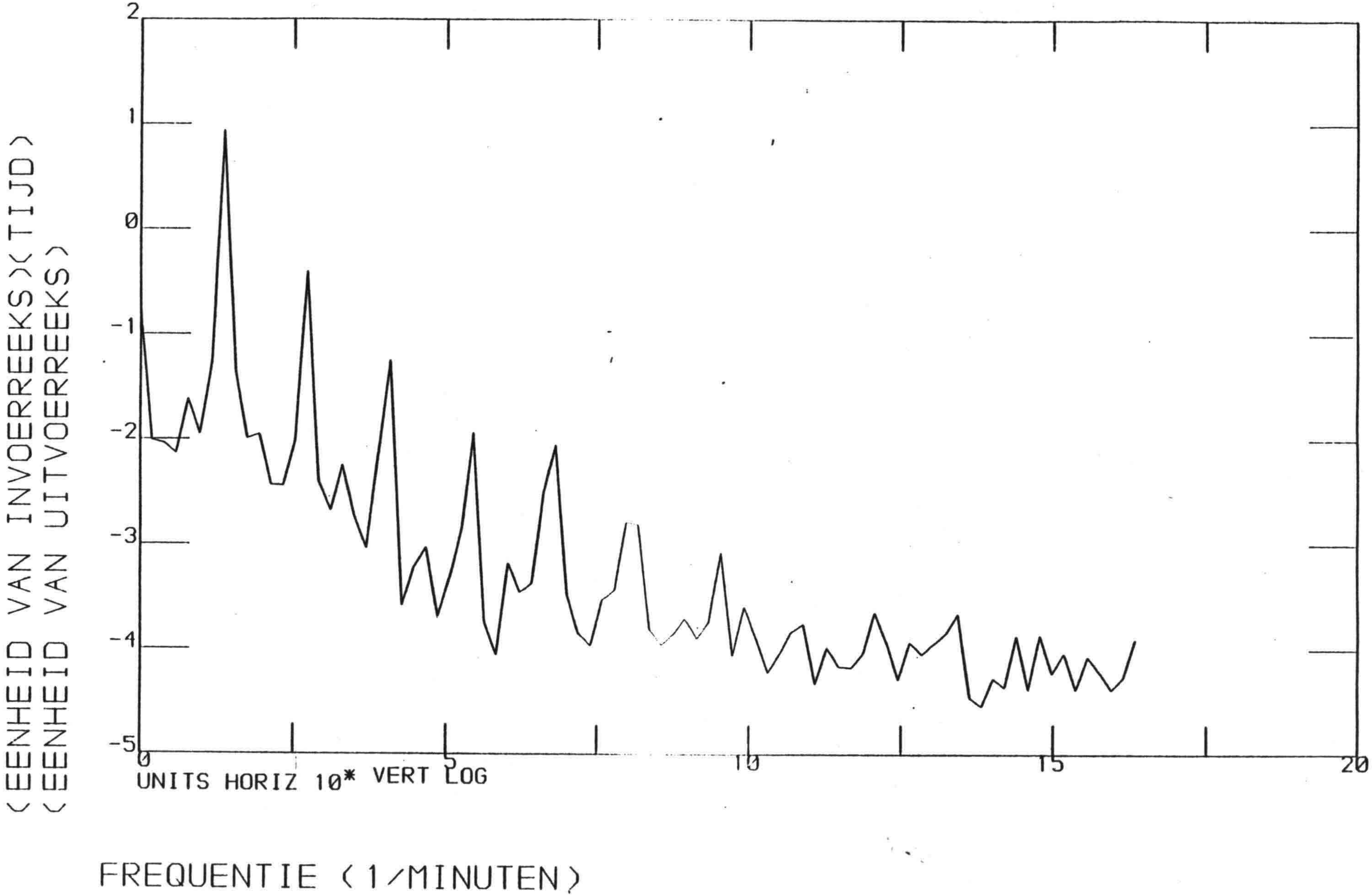


FASESPECTRUM VAN DE IN- EN UITVOERREEKS IN RADIALEN

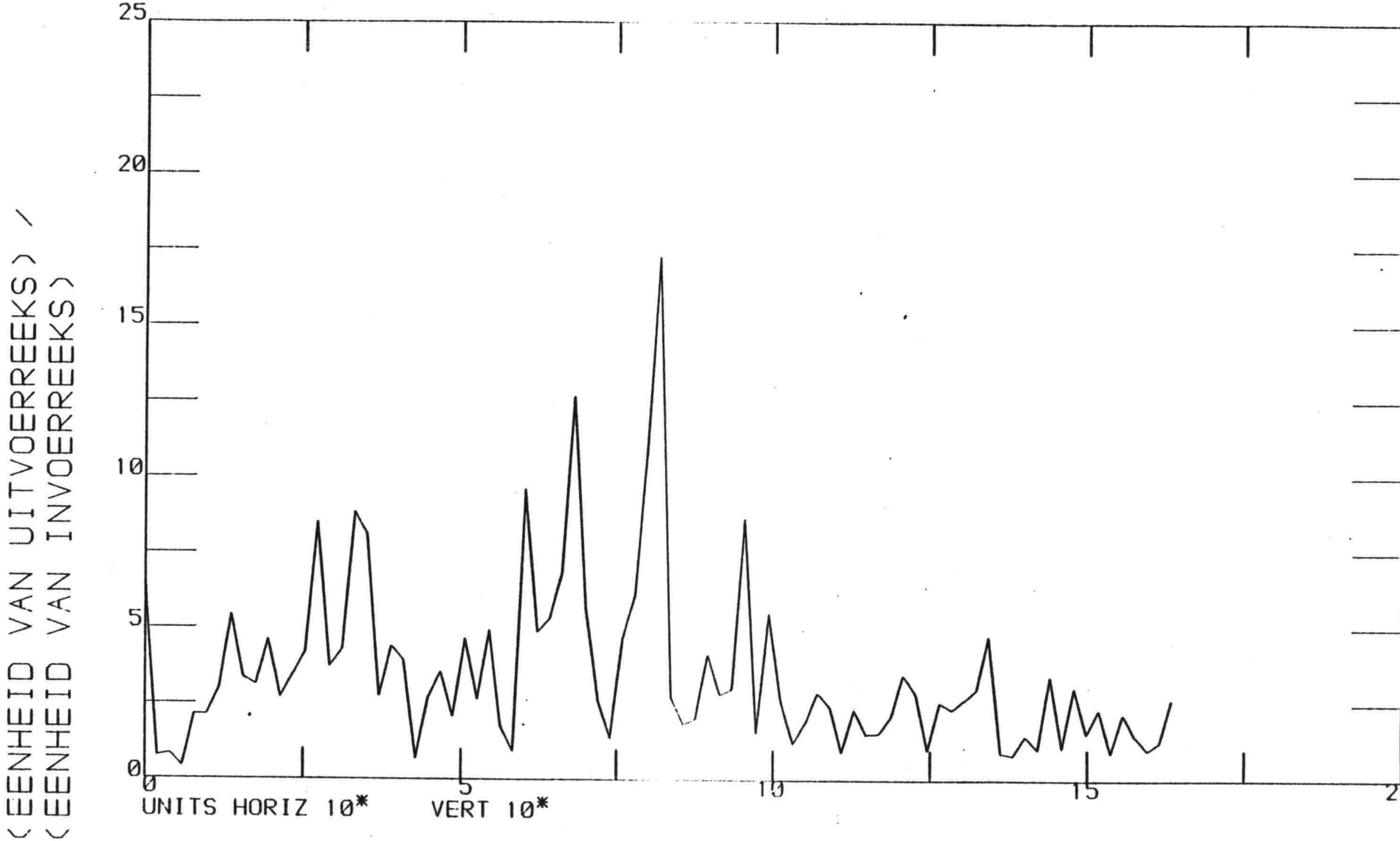


FREQUENTIE (1/MINUTEN)

FASESPECTRUM VAN DE IN- EN UITVOERREEKS IN TIJDSEENHEDEN



AMPLITUDE VAN DE KRUISSPECTRALE
DICHTHEIDSFUNCTIE VAN DE IN- EN UITVOERREEKS



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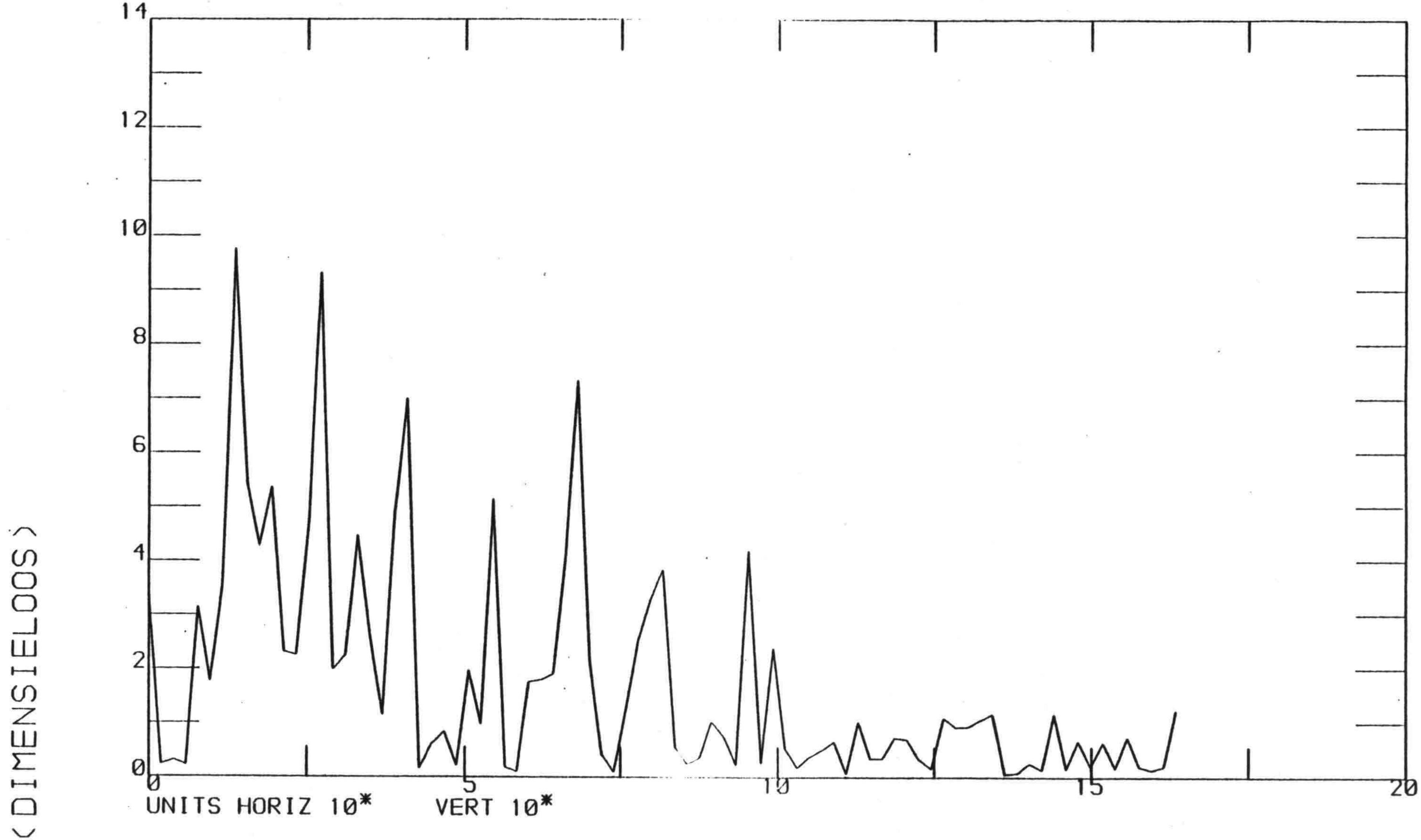
VAN DE IN- EN DE UITVOERREEKS

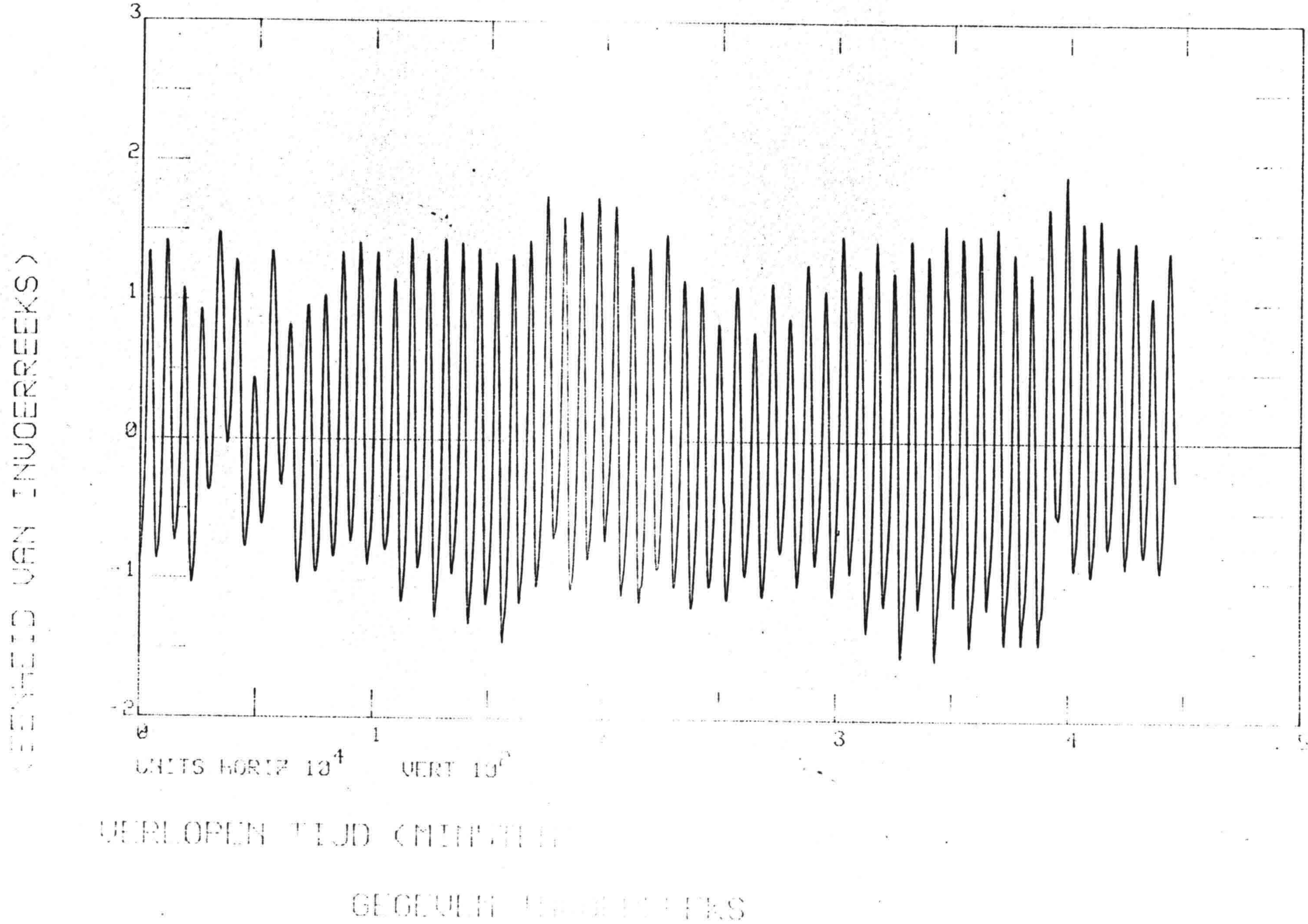


FREQUENTIE (1/MINUTEN)

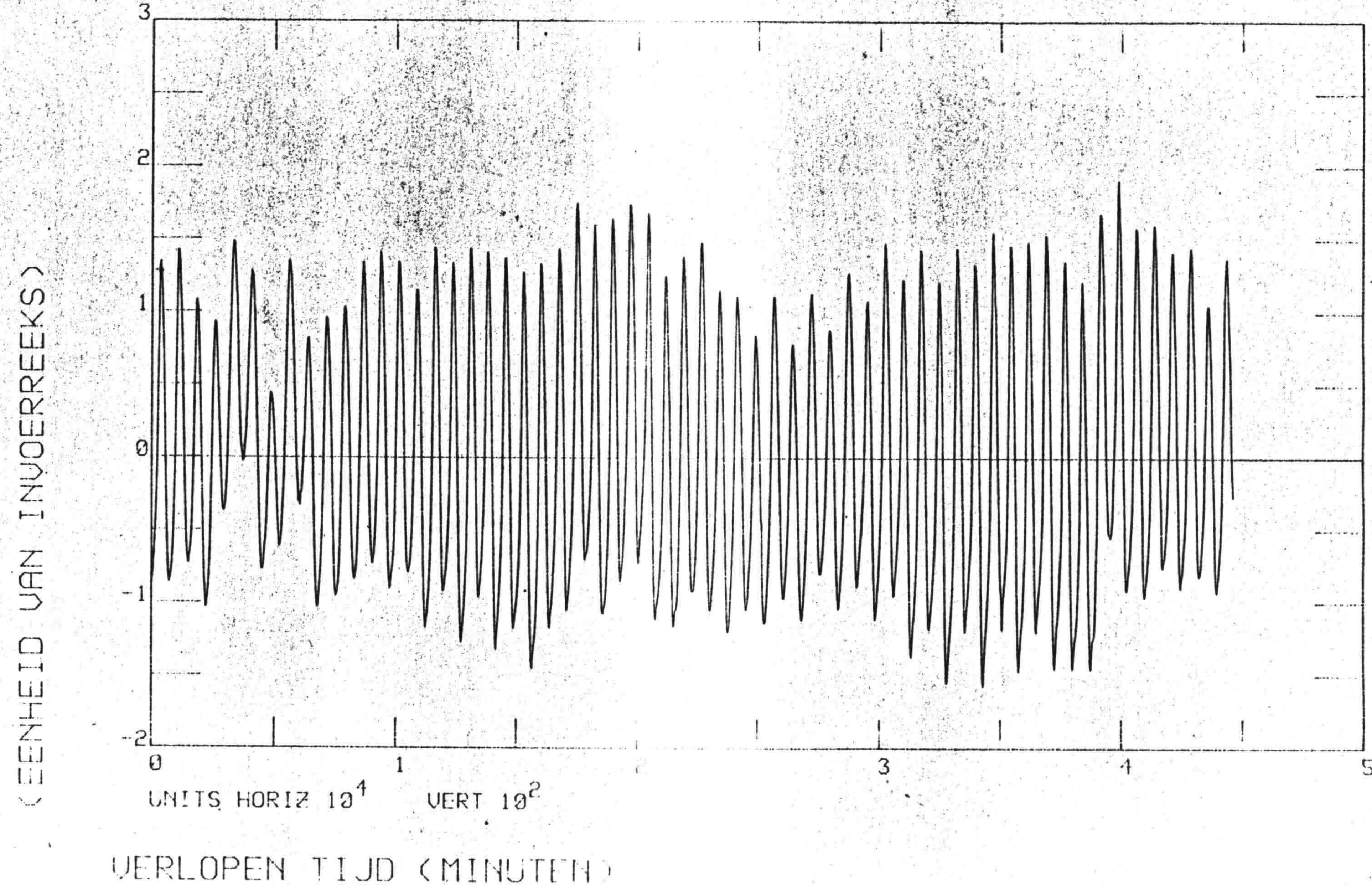
COHERENTIEKWADRAAT VAN DE IN- EN UITVOERREEKS

Figures 61 to 74

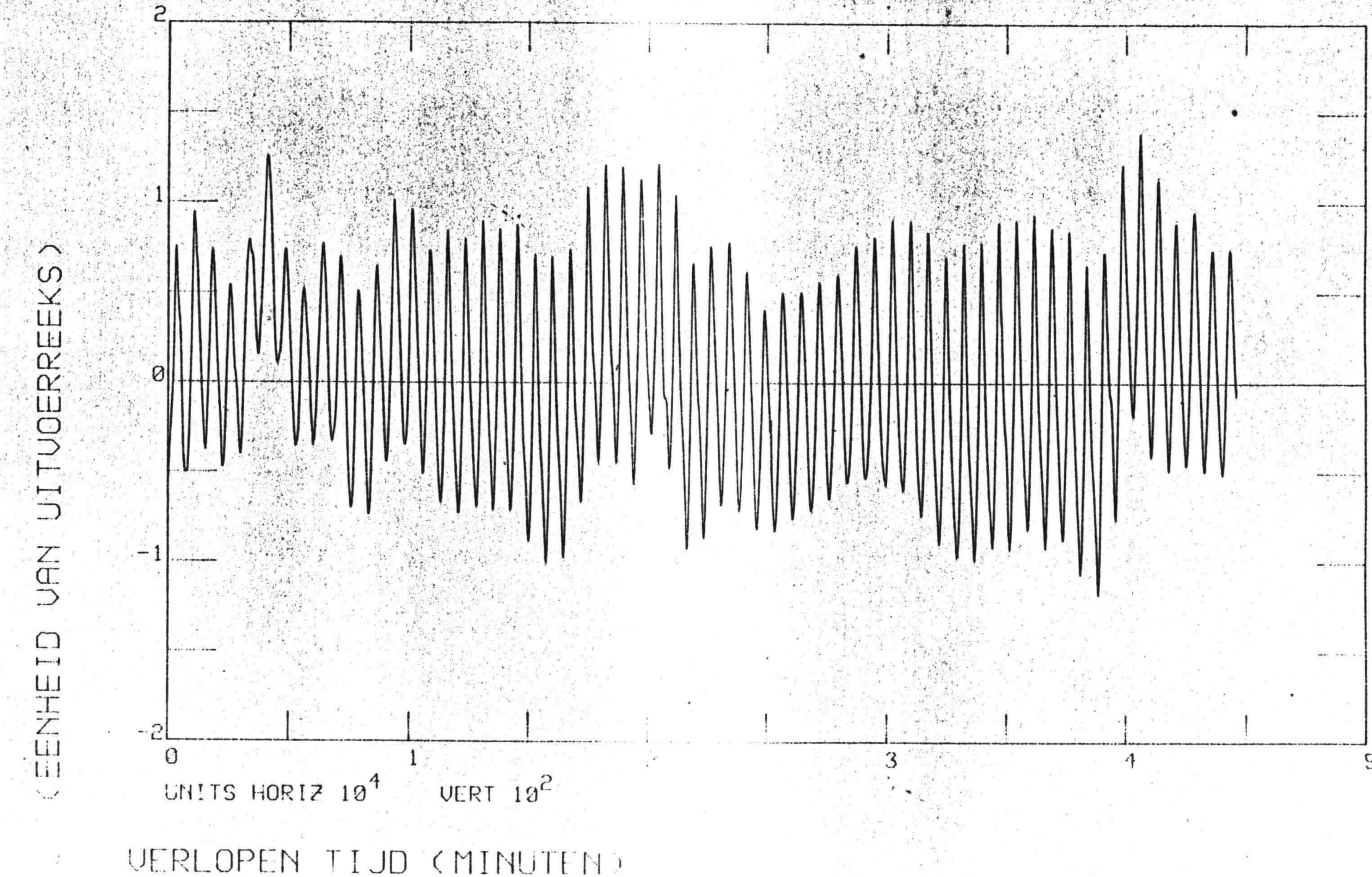
Astronomical components at B.G.II and N.B.



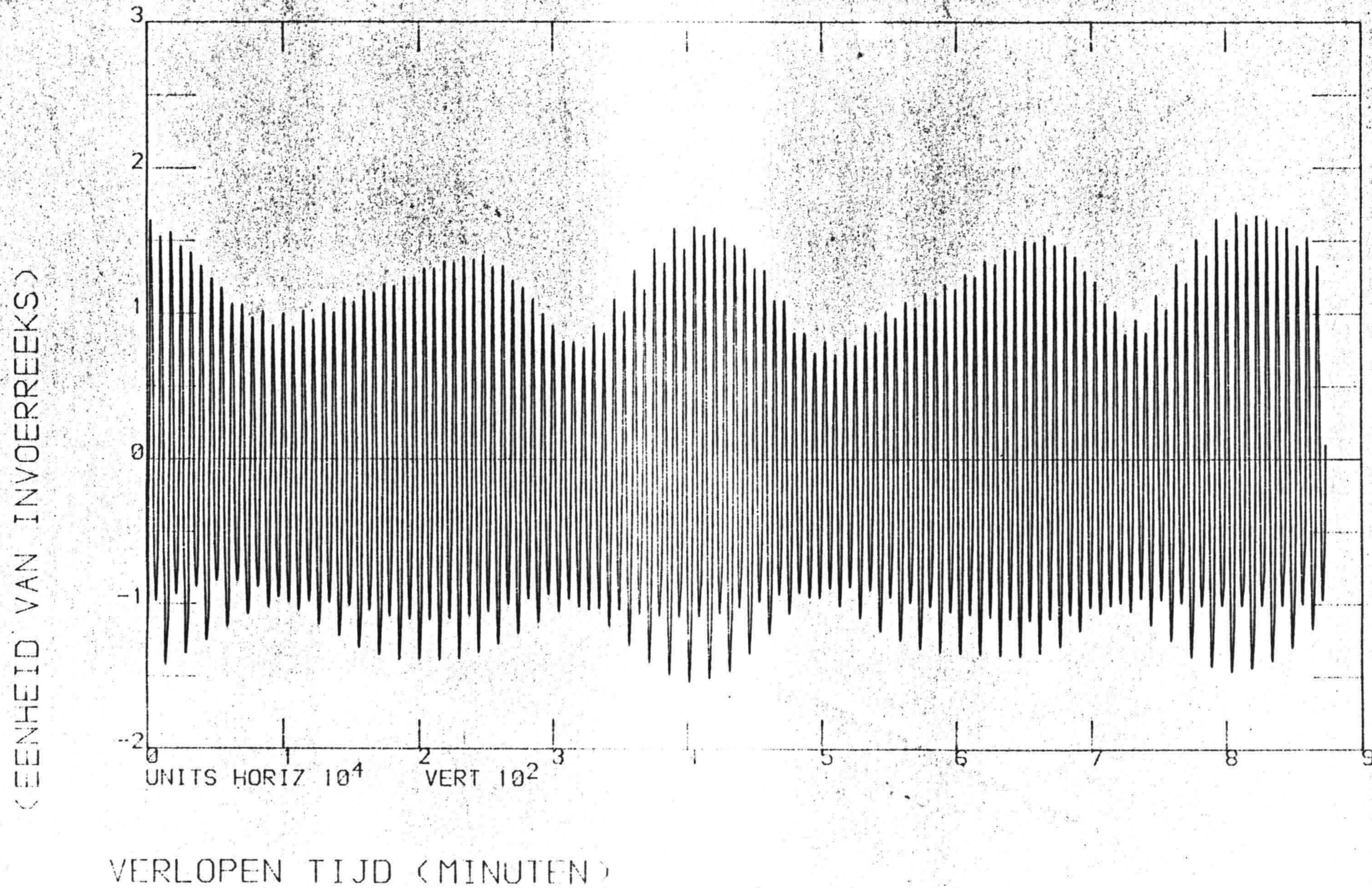
EENHEID VAN INVOERREEKS)
3
2
1
0
-1
-2
0
1
3
4
UNITS HORIZ 10^4 VERT 10^0
VERLOPEN TIJD (MIN
GEGEVEN



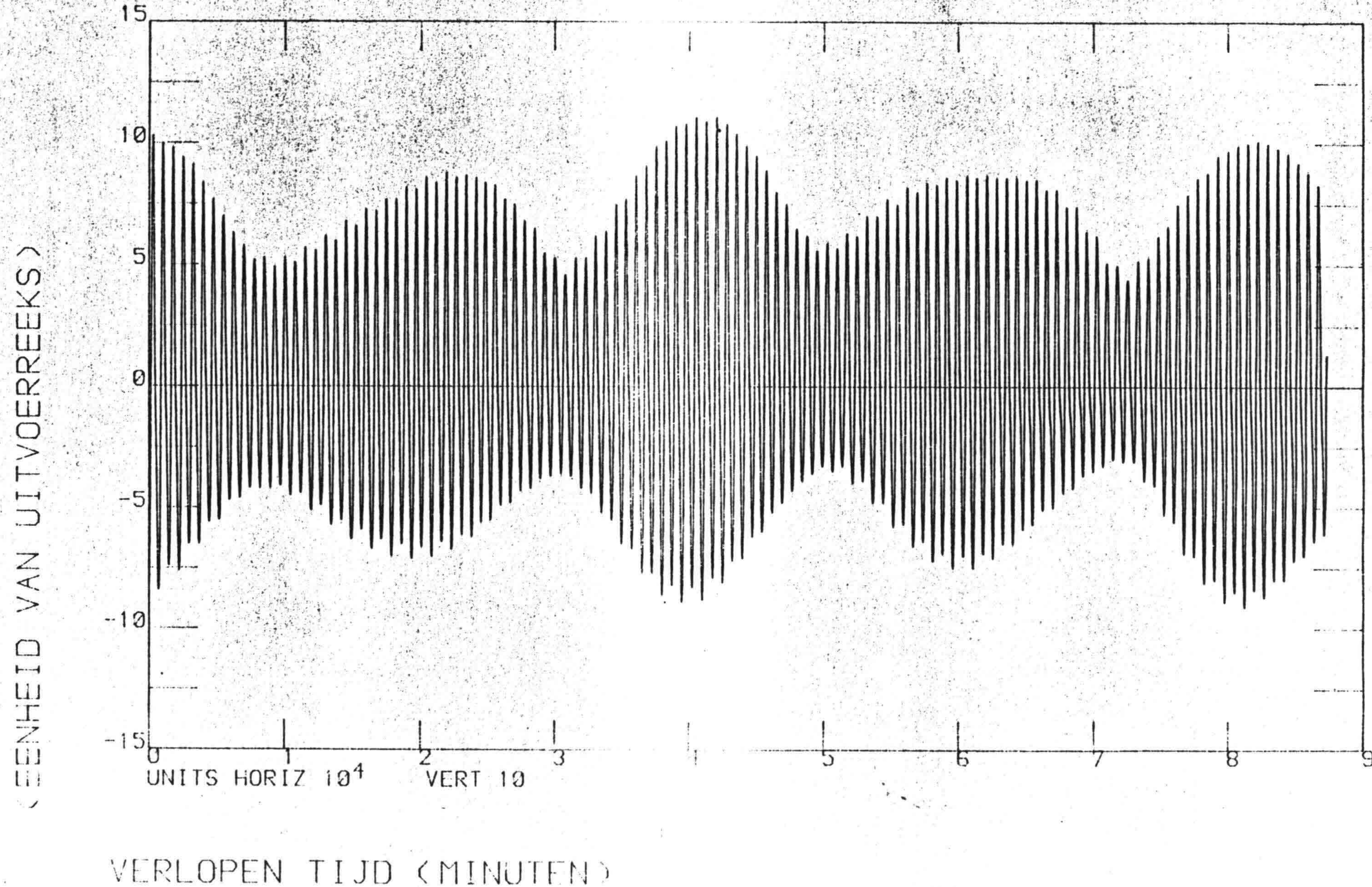
GEMODIFICEERDE INVOERREEKS



GECONVOLUEERDE UITVOERREEKS MET AANGEPAST GEMIDDELDE



GEGEVEN INVOERREEKS



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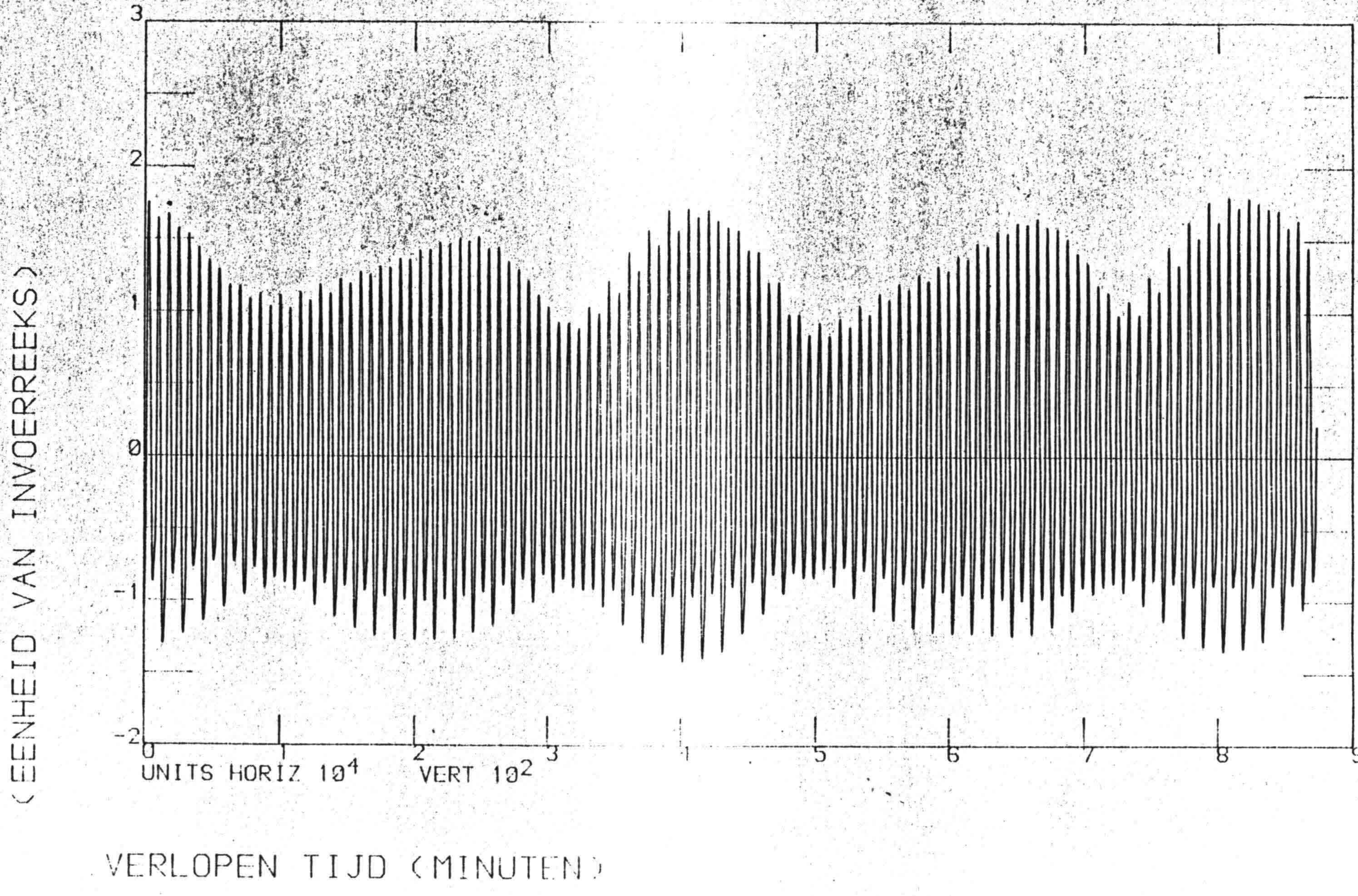

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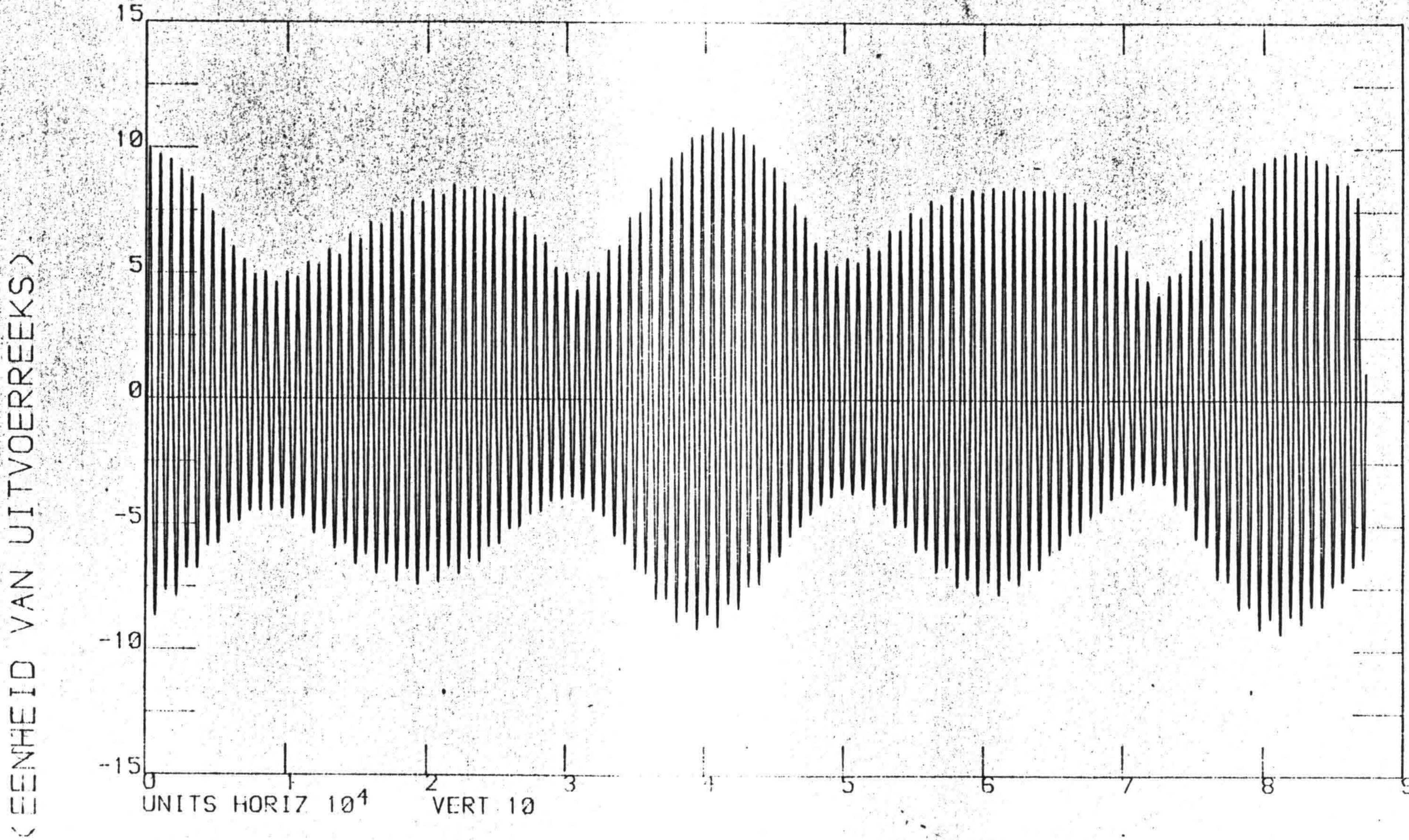

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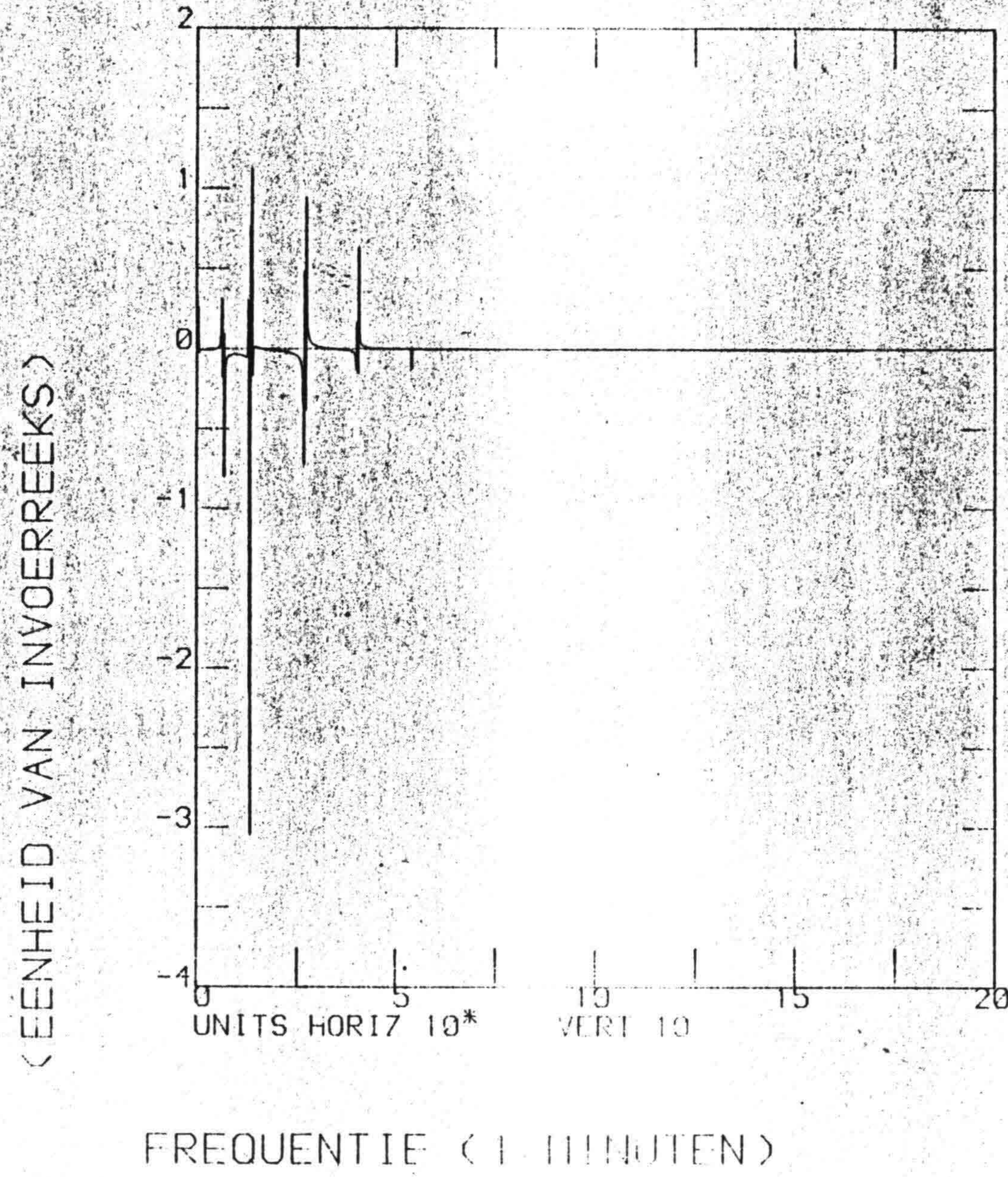
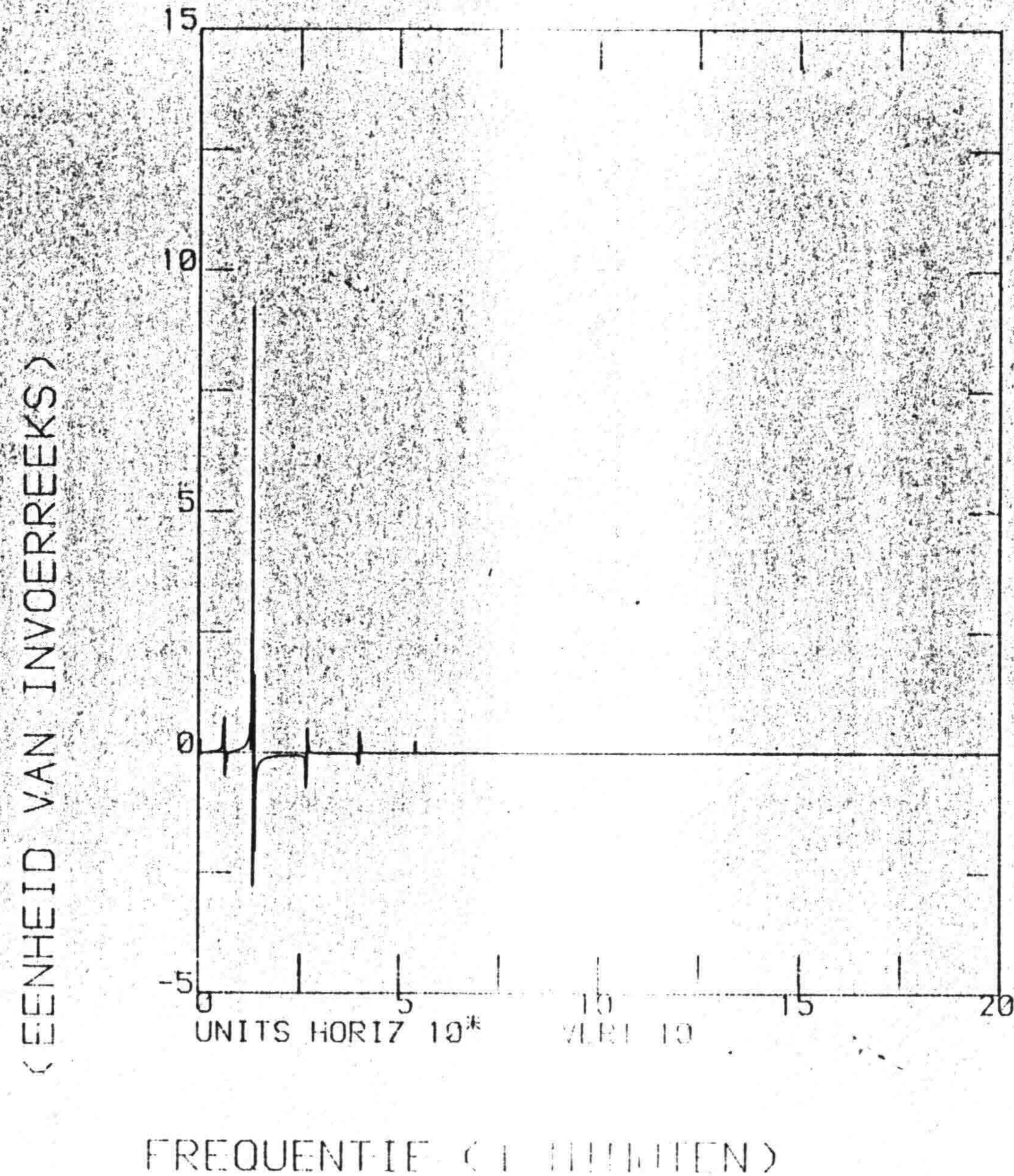
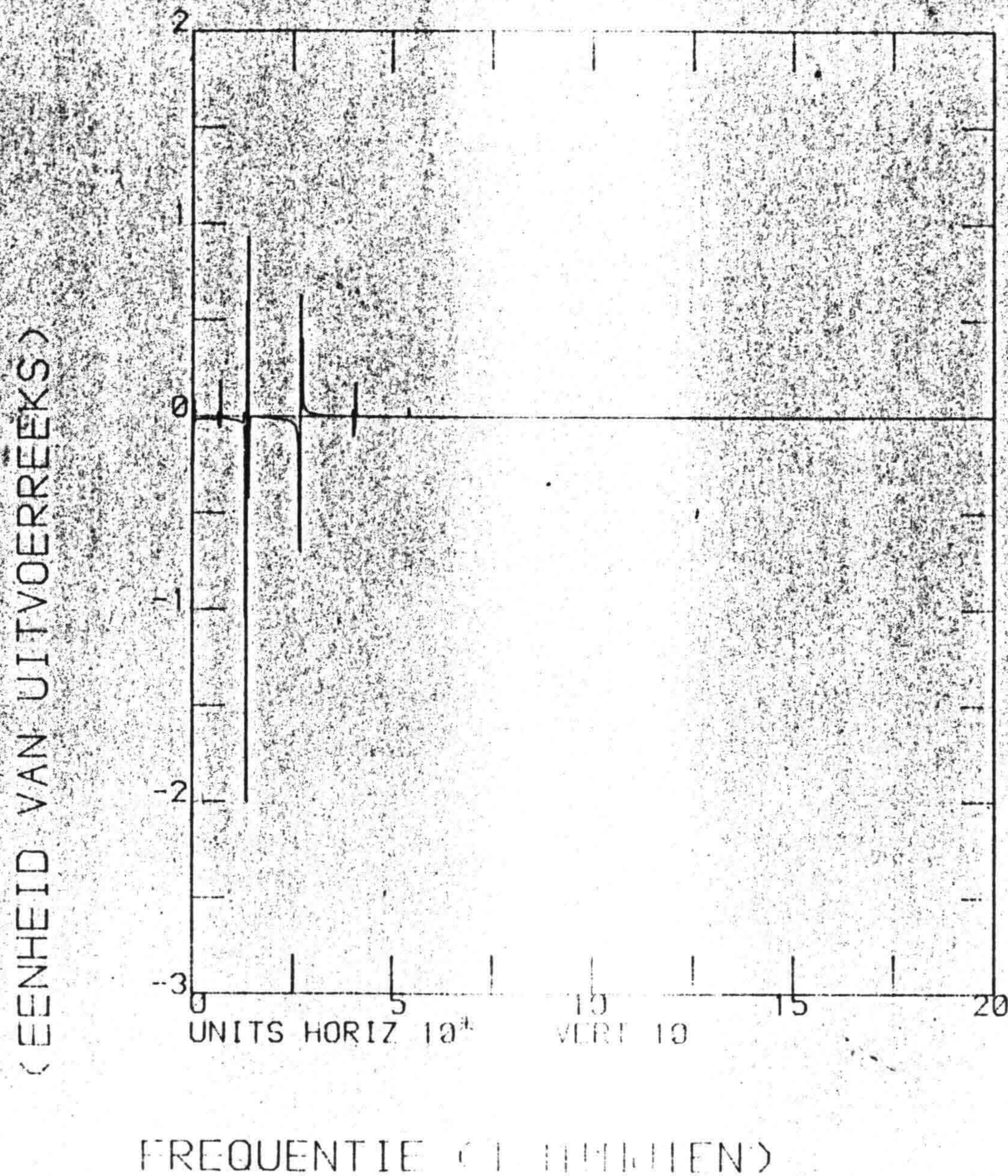


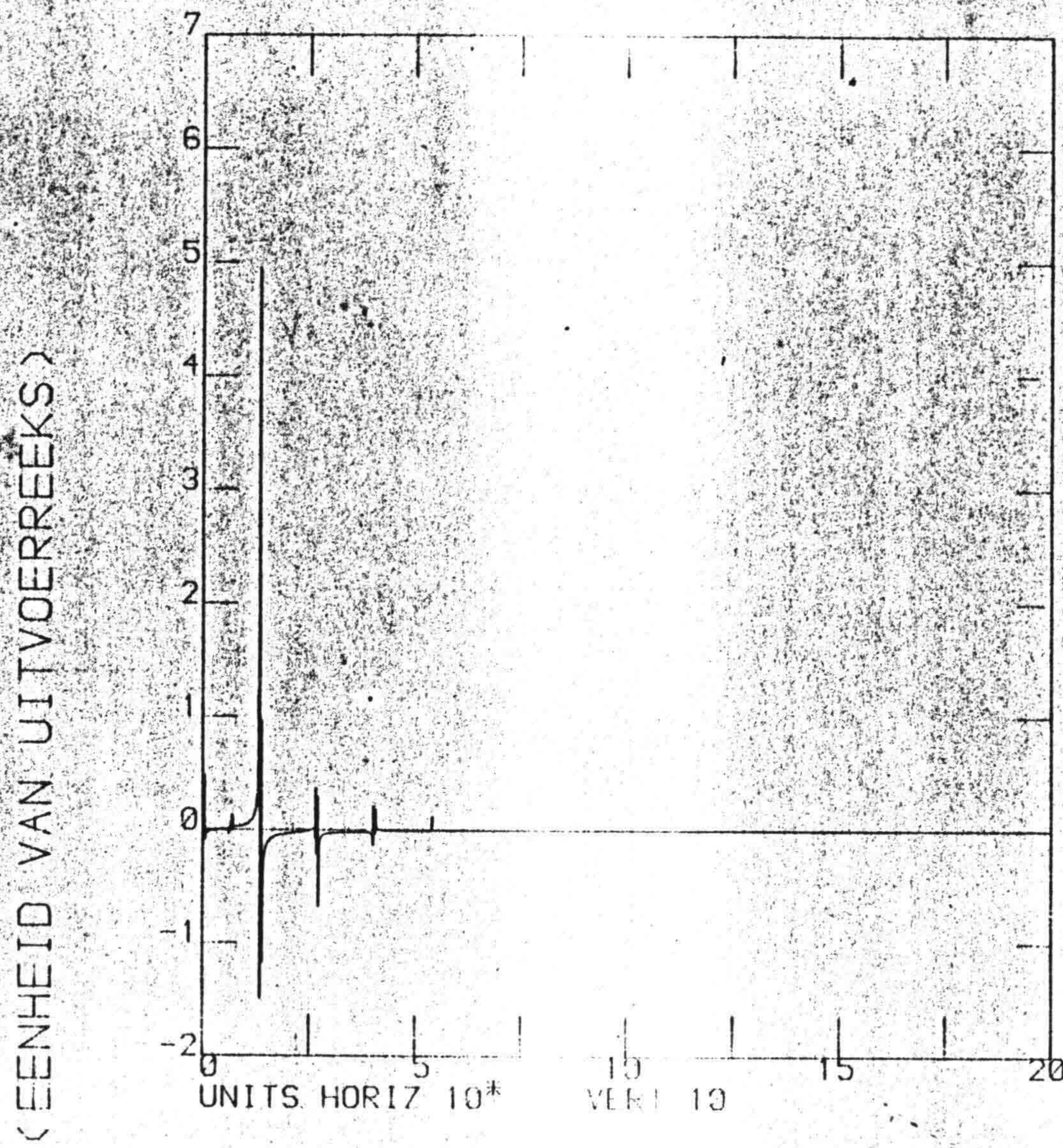
COSINUSCOEFFICIENTEN VAN
GEMODIFICEERDE INVOERREEKS



SINUSCOEFFICIENTEN VAN
GEMODIFICEERDE INVOERREEKS



COSINUSCOEFFIC[illegible] VAN
GEMODIFICEERD[illegible] [illegible]OERREEKS



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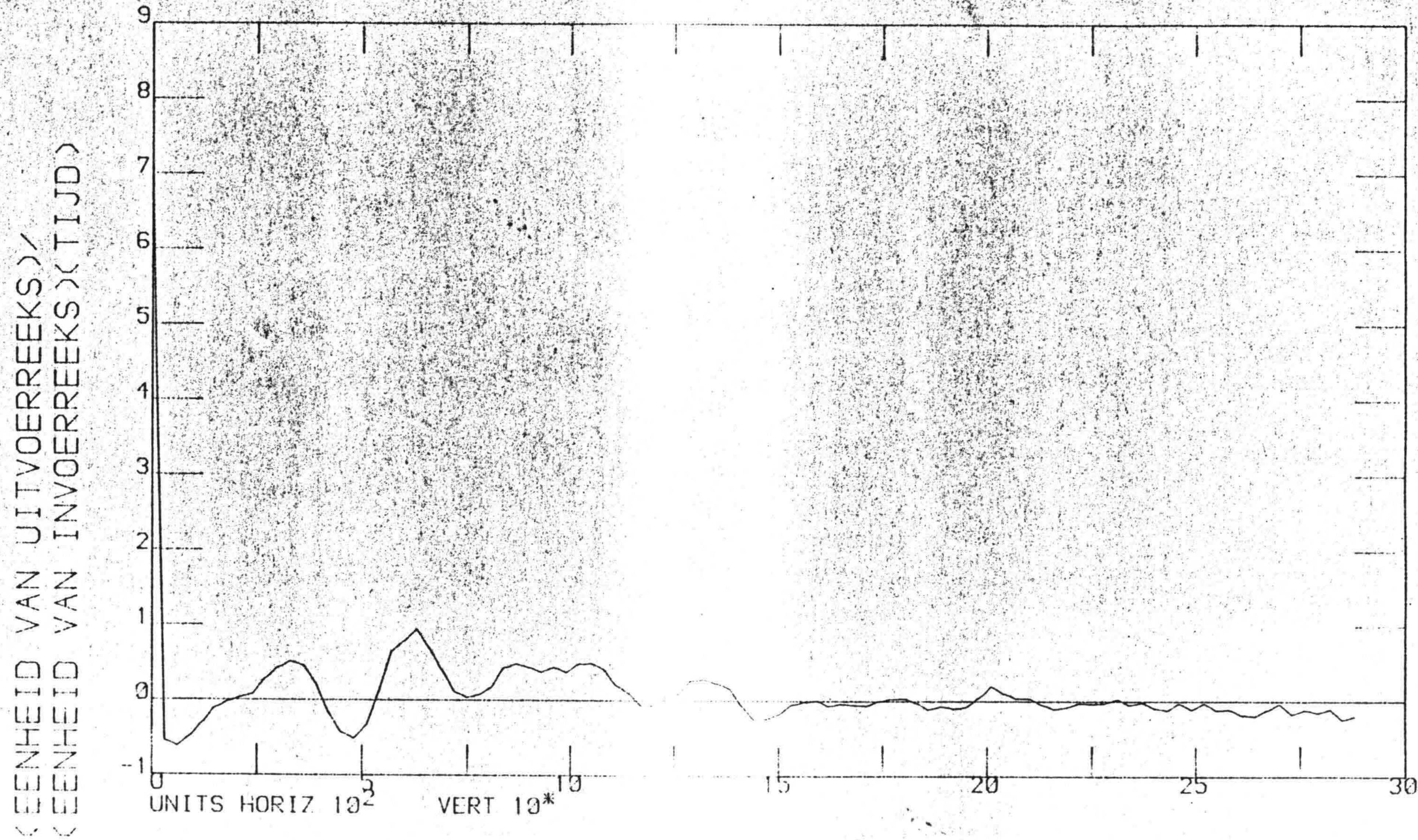
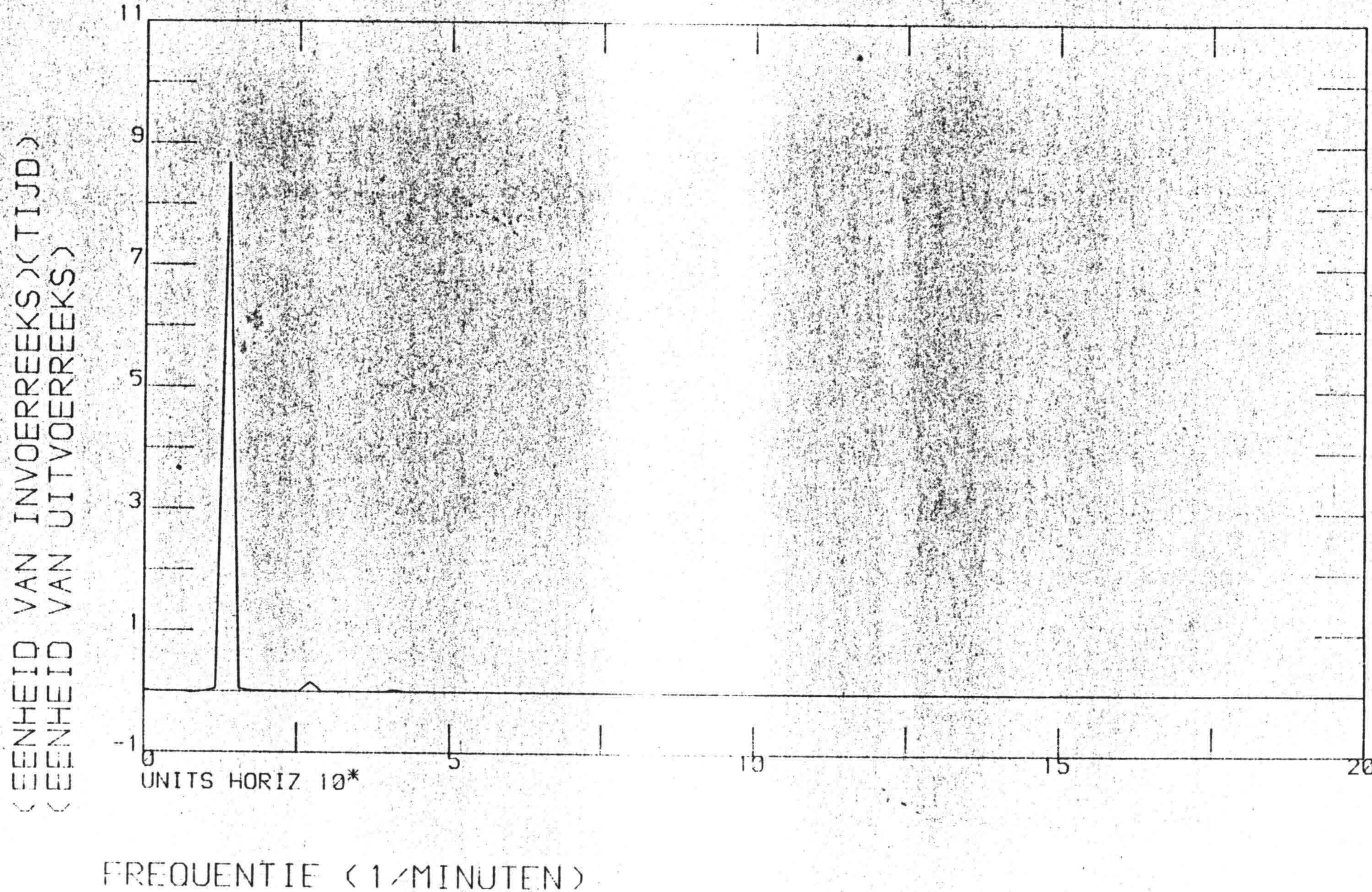
GEMODIFICEERD[illegible] [illegible]VOERREEKS

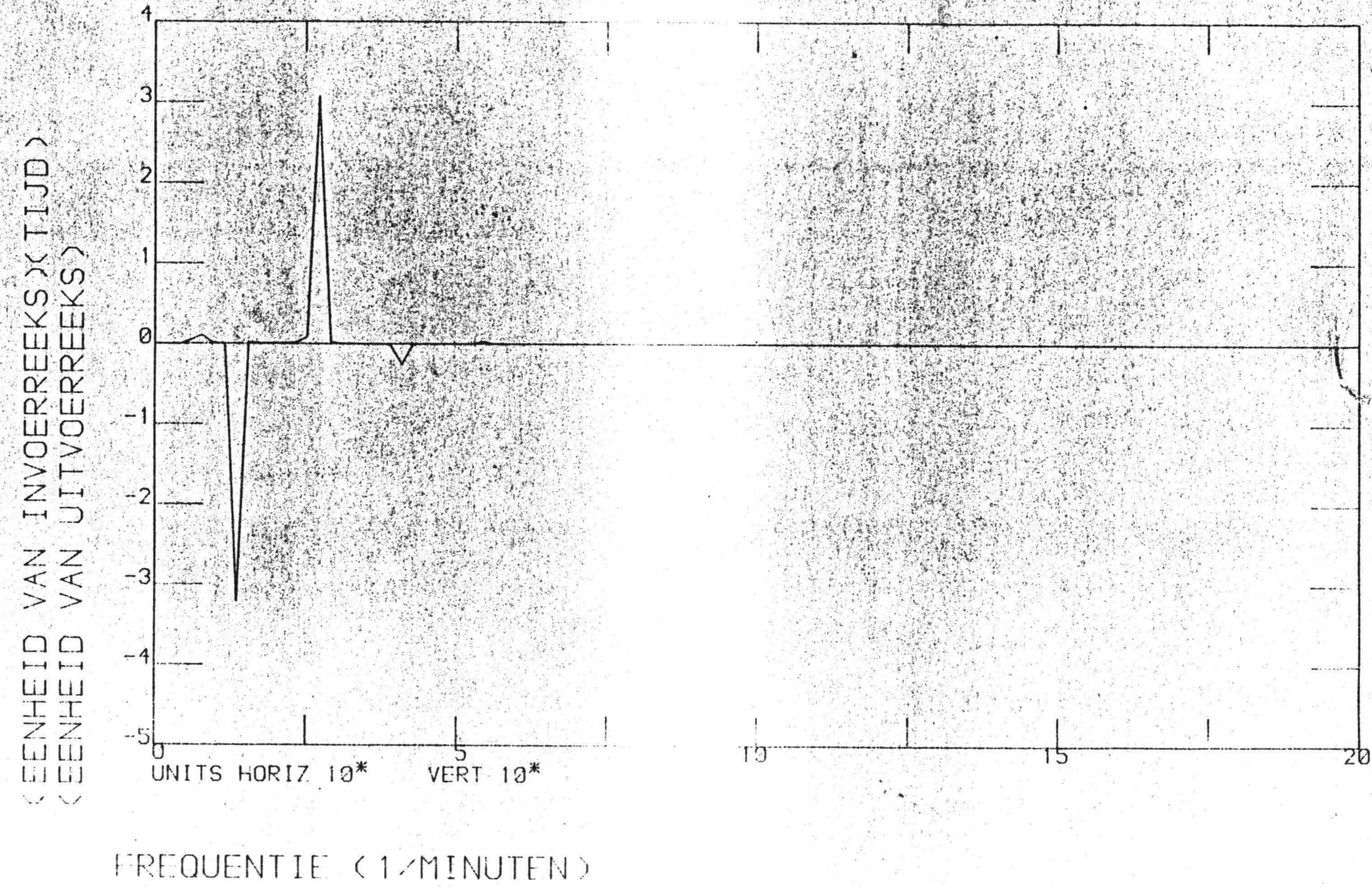


TIJDSVERLOOP (MINUTEN)

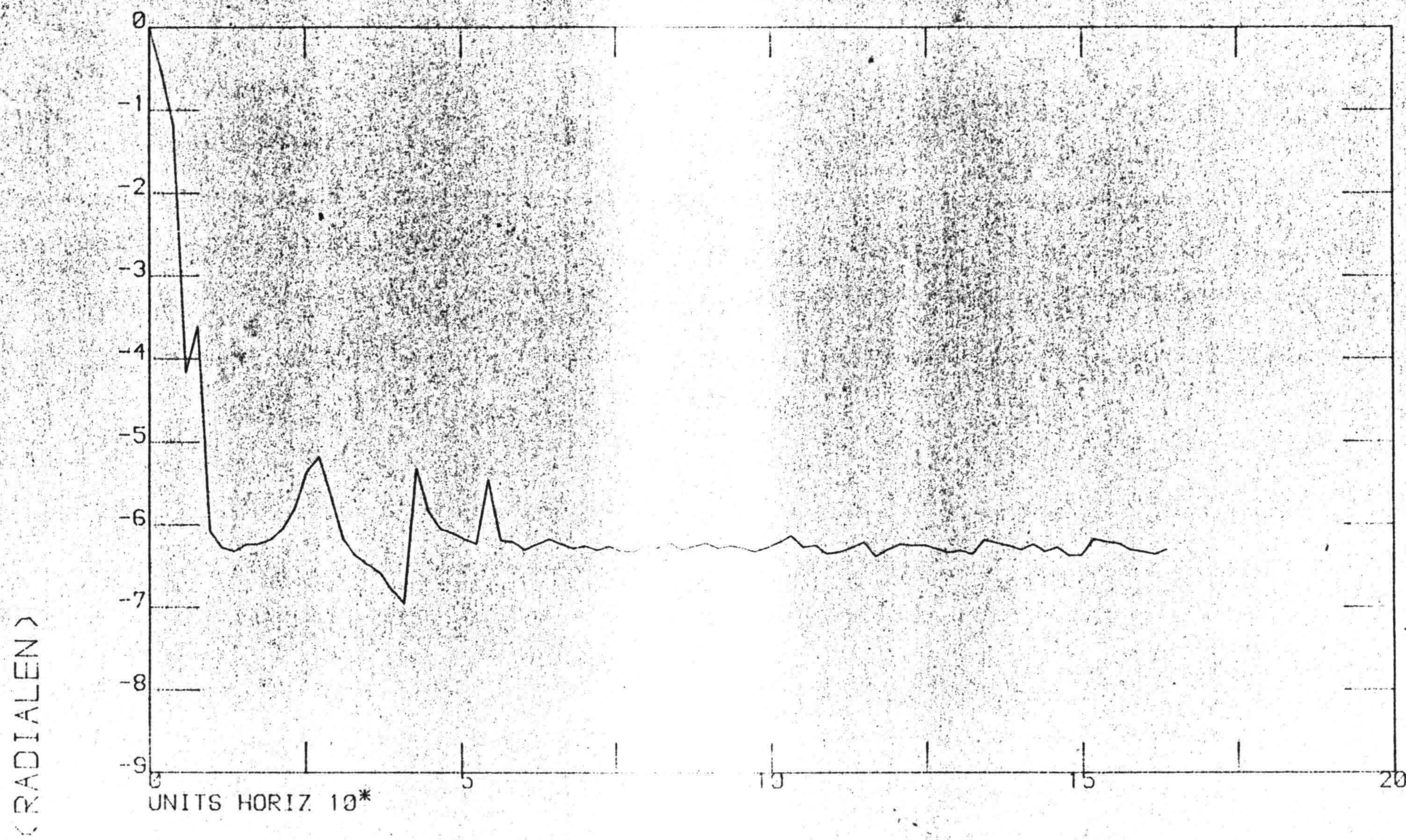
IMPULSRESPONSFUNCTIE VAN DE IN- EN UITVOERREEKS



COSPECTRUM VAN DE IN- EN DE UITVOERREEKS

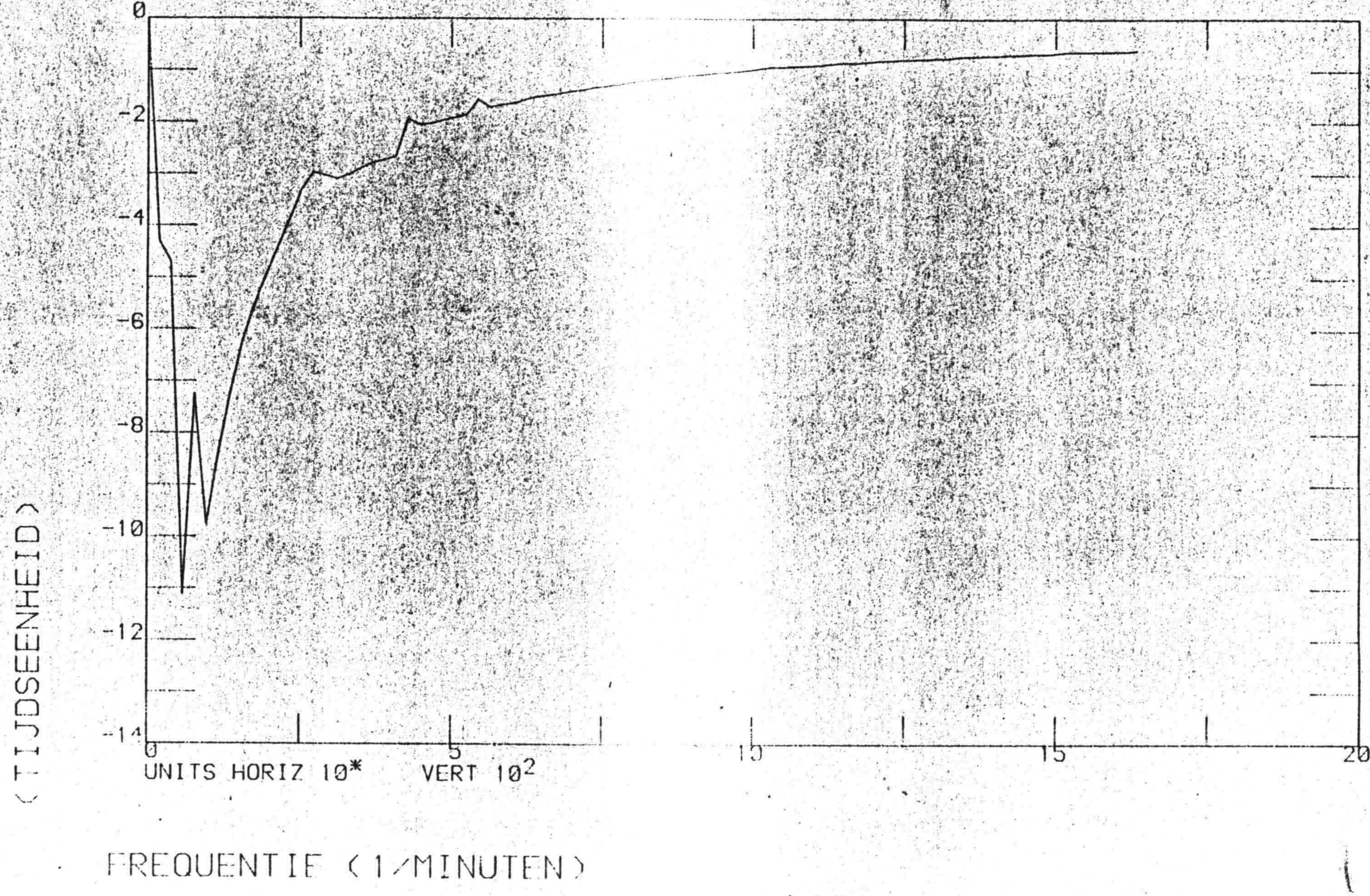


QUADRATUURSPECTRUM VAN DE [illegible] EN DE UITVOERREEKS

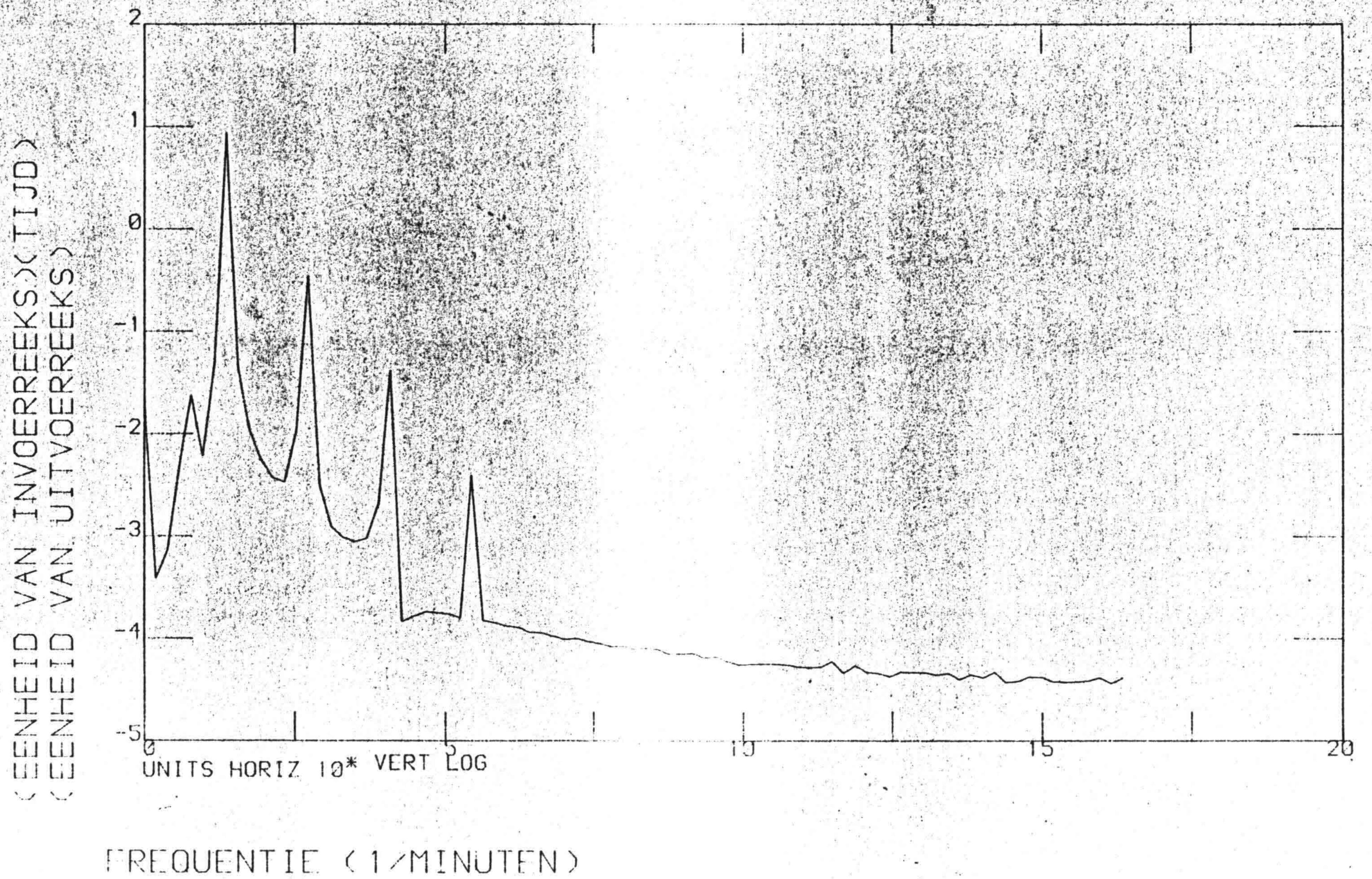


FREQUENTIE (1/MINUTEN)

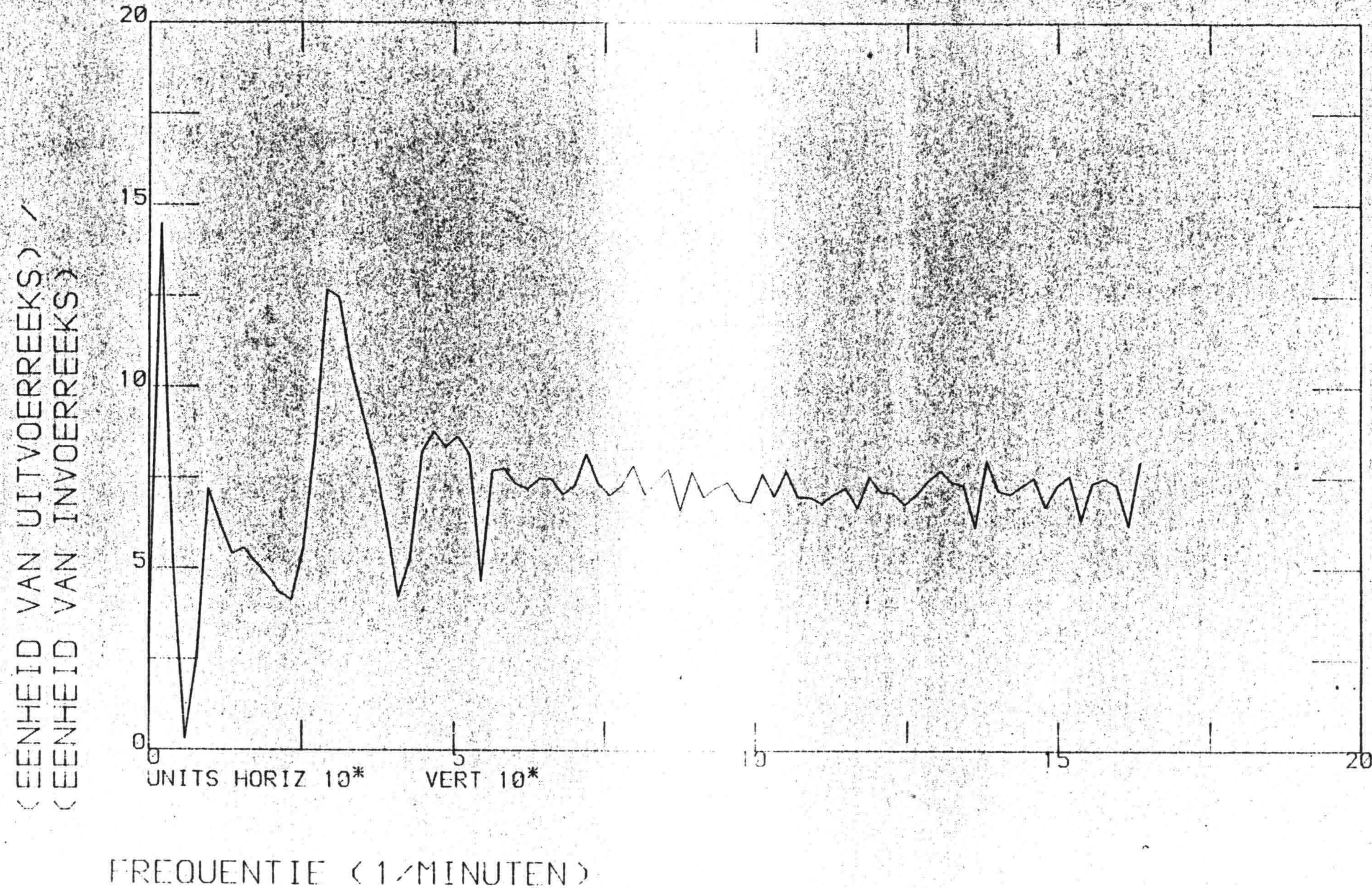
FASESPECTRUM VAN DE IN- EN UITVOERREEKS IN RADIALEN



FASESPECTRUM VAN DE IN- [illegible]VOERREEKS IN TIJDSEENHEDEN



AMPLITUDE VAN DE KRUISSPECTRALE
DICHTHEIDSFUNCTIE VAN DE IN- EN UITVOERREEKS



AMPLITUDE VAN DE FREQU[illegible]RESPONSFUNCTIE
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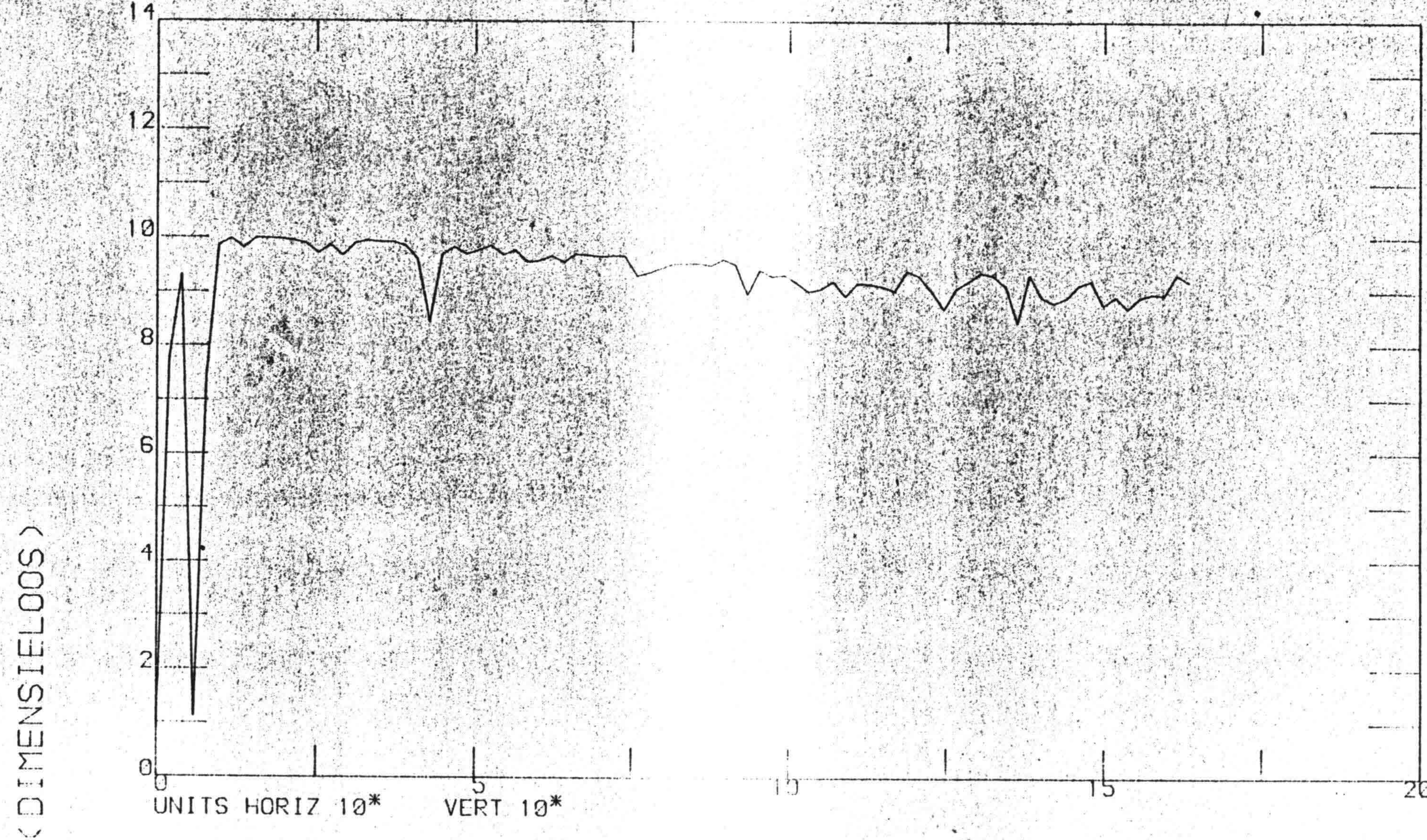
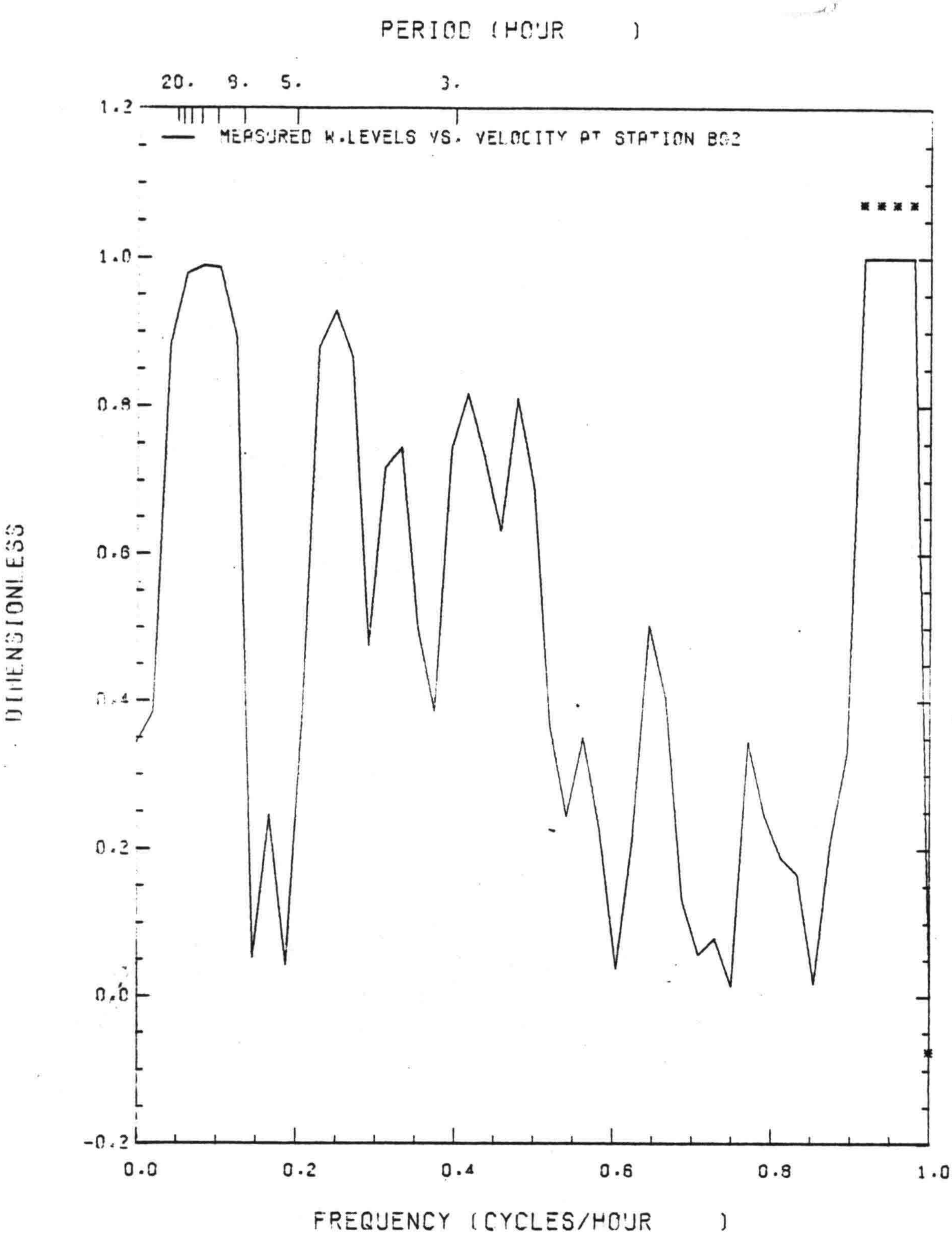


FREQUENTIE (1/MINUTEN)

COHERENTIEKWADRAAT VAN DI III EN UITVOERREEKS

Figure 75

Coherency between water-levels and the East-West velocity component at B.G.II.



SQUARED COHERENCY
OF THE X SERIES AND THE Y SERIES

CROSS-SPECTRAL ANALYSIS
CROSS SYSTEM VERSION 2

RAND CORPORATION
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