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Citation (APA)

de Pagter, B., & Ricker, W. J. (2026). Fourier Multipliers in Homogeneous Banach Spaces. *Integral Equations and Operator Theory*, 98(3), Article 28. <https://doi.org/10.1007/s00020-026-02849-7>

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Fourier Multipliers in Homogeneous Banach Spaces

Ben de Pagter  and Werner J. Ricker

Dedicated to the memory of Rien Kaashoek.

Abstract. Let G be an infinite, compact abelian group, E be a homogeneous Banach space over G and $\mathcal{L}(E)$ be the space of all continuous linear operators from E into itself equipped with the operator norm. Translation operators are isometries in E (by definition) and so the closed subalgebra $\mathfrak{m}(E)$ of $\mathcal{L}(E)$ consisting of those operators which commute with all translations is well defined. It is shown that there exists a contractive projection $\mathcal{Q}$ of $\mathcal{L}(E)$ onto $\mathfrak{m}(E)$ which is positivity preserving. Moreover, every operator $\mathcal{Q}(T) \in \mathfrak{m}(E)$, with $T \in \mathcal{L}(E)$, is induced by a unique Fourier multiplier function $\hat{T} \in \ell^\infty(\Gamma)$, where Γ is the dual group of G . In the setting of the homogeneous Banach spaces $L^p(G)$, for $1 \leq p < \infty$ and G an amenable group, these results are due to W. Arendt and J. Voigt.

Mathematics Subject Classification. Primary 43A15, 43A22, 46E30; Secondary 47B38, 47B65, 47B90.

Keywords. Homogeneous Banach space, Compact abelian group, Translation operator, Fourier multiplier, Order bounded operator.

1. Introduction

The theory of homogeneous Banach spaces over locally compact abelian groups first arose in the seminal paper [21] of G.E. Shilov. Since then, various aspects of the topic have been intensively investigated; see, for example, [7], [10], [17] and the historical comments and references therein.

Infinite, compact abelian groups G form the setting of this article. Our aim is to extend two well known results, for certain Fourier multiplier operators acting in $L^p(G)$, for $1 \leq p < \infty$, due to W. Arendt and J. Voigt whenever G is an amenable group (all compact abelian groups are amenable); see Theorems 1.1 and 3.1 in [1]. Our extensions (see Theorems 5.1 and 5.4 below) pertain to the class of homogeneous Banach spaces over G , a class

which is significantly larger than the spaces $L^p(G)$, $1 \leq p < \infty$. In particular, it includes all translation invariant Banach function spaces over G which have order continuous norm and hence, also encompasses all rearrangement invariant spaces over G with order continuous norm.

Section 2 formulates some relevant preliminaries from harmonic analysis, Banach function spaces and the theory of Bochner integrals. Homogeneous Banach spaces E over G are discussed in Sect. 3, together with certain aspects concerning translation invariant Banach function spaces over G . Sect. 4 is devoted to the theory of Fourier multiplier functions defined on the dual group Γ of G and the corresponding multiplier operators in $\mathcal{L}(E)$ that they induce, which necessarily commute with all translation operators in E ; see Lemma 4.5. Conversely (cf. Lemma 4.7), if an operator $T \in \mathcal{L}(E)$ commutes with all translations, then it is necessarily induced by a Fourier multiplier function on Γ . In the final Section 5 we formulate and prove the extension of the results of Arendt and Voigt alluded to above. Some applications and examples are also presented.

2. Preliminaries

Throughout this article, G is an infinite, compact abelian group equipped with normalized Haar measure μ . The Borel σ -algebra of G is denoted by $\mathcal{B}(G)$. The space of all $\mathbb{C}$ -valued Borel measurable functions on G , with the usual identification of μ -a.e. functions, is denoted by $L^0(G)$ and, for each $1 \leq p \leq \infty$, $L^p(G)$ is the usual L^p -space on G with respect to μ .

The space of all regular, complex Borel measures on G is denoted by $M(G)$ and is equipped with the variation norm $\|\cdot\|_{M(G)}$, that is, $\|\lambda\|_{M(G)} = |\lambda|(G)$, for $\lambda \in M(G)$, where $|\lambda|$ is the variation measure of λ . The dual group of G is denoted by Γ ; see, for example, Section 1.2.1 of [18]. For $x \in G$ and $\gamma \in \Gamma$ we write $\gamma(x) = (x, \gamma)$. Since we frequently consider Γ as a subset of $C(G)$, we shall write the group operation in Γ as *multiplication*, that is, if $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1 \gamma_2 \in \Gamma$ is given by

$$(x, \gamma_1 \gamma_2) = (x, \gamma_1)(x, \gamma_2), \quad x \in G.$$

Here $C(G)$ is the Banach space of all continuous, $\mathbb{C}$ -valued functions on G equipped with the sup-norm. The identity element of Γ is, of course, $\mathbf{1} = \chi_G$. Moreover,

$$(x, \gamma^{-1}) = \overline{(x, \gamma)} = (-x, \gamma), \quad x \in G, \quad \gamma \in \Gamma.$$

The discrete group Γ is countable if and only if G is metrizable; see Theorem 2.2.6 in [18].

Given any measure $\lambda \in M(G)$, its Fourier-Stieltjes transform $\hat{\lambda} : \Gamma \rightarrow \mathbb{C}$ is defined by

$$\hat{\lambda}(\gamma) = \int_G \overline{(x, \gamma)} d\lambda(x), \quad \gamma \in \Gamma.$$

Then $\hat{\lambda} \in \ell^\infty(\Gamma)$ and $\|\hat{\lambda}\|_\infty \leq \|\lambda\|_{M(G)}$ for every $\lambda \in M(G)$; see Section 1.3.3 of [18]. If $f \in L^1(G)$ and $\lambda = f \cdot \mu$ (that is, λ is the measure $\lambda(A) = \int_A f d\mu$,

for $A \in \mathcal{B}(G)$), then its Fourier-Stieltjes transform $\hat{\lambda}$ is also denoted by $\hat{f}$ and is called the Fourier transform of f , that is,

$$\hat{f}(\gamma) = \int_G f(x) \overline{(x, \gamma)} d\mu, \quad \gamma \in \Gamma.$$

By the Riemann-Lebesgue lemma, the Fourier transform $\hat{f} \in c_0(\Gamma)$ for all $f \in L^1(G)$. For each $f \in L^1(G)$, the support of $\hat{f}$ is denoted by

$$\text{supp}(\hat{f}) = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\} \subseteq \Gamma, \quad (1)$$

which is necessarily a countable subset of Γ as $\hat{f} \in c_0(\Gamma)$.

Denote by $\tau(G)$ the linear span of the dual group Γ formed in the Banach space $C(G)$, that is, $\tau(G)$ consists of all functions on G which are of the form $f = \sum_{j=1}^n \alpha_j \gamma_j$, where $\gamma_j \in \Gamma$ and $\alpha_j \in \mathbb{C}$, for $j = 1, \dots, n$, and $n \in \mathbb{N}$. The elements of $\tau(G)$ are referred to as *trigonometric polynomials* on G . Note that in this case the support of $\hat{f}$ is given by

$$\text{supp}(\hat{f}) = \{\gamma_j : \alpha_j \neq 0\}. \quad (2)$$

According to Appendix A16 in [18], the Banach space $C(G)$ is separable if and only if G is metrizable.

Let $f \in L^1(G)$ and $\lambda \in M(G)$. The convolution $f * \lambda \in L^1(G)$ of f and λ is defined by

$$(f * \lambda)(x) = \int_G f(x - y) d\lambda(y), \quad \mu\text{-a.e. } x \in G, \quad (3)$$

(see Section 1.3 of [18]) and satisfies $(f * \lambda)^\wedge = \hat{f} \cdot \hat{\lambda}$ pointwise on Γ .

We will have the occasion to use Bochner integrals over G . For the definition and basic properties of the Bochner integral we refer the reader to Chapter II in [4]. Let $(X, \|\cdot\|_X)$ be a Banach space with dual Banach space X^* . The value of a functional $x^* \in X^*$ at an element $x \in X$ is denoted by $\langle x, x^* \rangle$. Let $\lambda \in M(G)^+$ be a positive measure and $F : G \rightarrow X$ be Bochner λ -integrable. The Bochner integral of F with respect to λ is denoted by $\int_G^{(B)} F d\lambda \in X$. Recall, in particular, that

$$\left\| \int_G^{(B)} F d\lambda \right\|_X \leq \int_G \|F\|_X d\lambda \quad (4)$$

and that

$$\left\langle \int_G^{(B)} F d\lambda, x^* \right\rangle = \int_G \langle F, x^* \rangle d\lambda, \quad x^* \in X^*. \quad (5)$$

Let $\lambda \in M(G)$, not necessarily positive. By the Radon-Nikodym theorem, there exists a $\mathcal{B}(G)$ -measurable function $h : G \rightarrow \mathbb{C}$ such that $|h(x)| = 1$, for $x \in G$, and $\lambda = h|\lambda|$ (see, for example, [19], Theorem 6.12). Let X be a Banach space and $F : G \rightarrow X$ be Bochner $|\lambda|$ -integrable. Then

the function $hF : G \rightarrow X$ is also Bochner $|\lambda|$ -integrable. Therefore, we can define the Bochner λ -integral of F by setting

$$\int_G^{(B)} F d\lambda = \int_G^{(B)} hF d|\lambda|.$$

Using (4) and (5), it is readily verified that

$$\left\| \int_G^{(B)} F d\lambda \right\|_X \leq \int_G \|F\|_X d|\lambda| \quad (6)$$

and that

$$\left\langle \int_G^{(B)} F d\lambda, x^* \right\rangle = \int_G \langle F, x^* \rangle d\lambda, \quad x^* \in X^*. \quad (7)$$

The following fact will be used.

Lemma 2.1. *Let X be a Banach space, $\lambda \in M(G)$ and $F : G \rightarrow X$ be continuous. Then F is Bochner $|\lambda|$ -integrable over G and so $\int_G^{(B)} F d\lambda \in X$ exists.*

Proof. For each $x^* \in X^*$ the scalar-valued function $\langle F, x^* \rangle$ defined on G is continuous and hence, Borel measurable. Therefore, the function F is weakly $|\lambda|$ -measurable; see p. 41 of [4]. Furthermore, the range $F(G)$ of F is a compact subset of X and so $F(G)$ is separable. Accordingly, it follows from Pettis's measurability theorem (see Theorem II.1.2 in [4]) that F is strongly $|\lambda|$ -measurable. Furthermore, since $F(G)$ is a bounded subset of X , it is clear that $\int_G \|F\|_X d|\lambda| < \infty$. Via Theorem II.2.2 of [4] we can conclude that F is Bochner $|\lambda|$ -integrable. This completes the proof of the lemma. $\square$

We end this section by recalling the definition of Banach function spaces (briefly, B.f.s.) over G .

Definition 2.2. A Banach function space over G is an ideal $E \subseteq L^0(G)$ (that is, E is a linear subspace of $L^0(G)$ with the property that $f \in E$ whenever $f \in L^0(G)$ satisfies $|f| \leq |g|$ for some $g \in E$) equipped with a norm $\|\cdot\|_E$ such that:

- (i) it follows from $f, g \in E$ with $|f| \leq |g|$ that $\|f\|_E \leq \|g\|_E$;
- (ii) the normed space $(E, \|\cdot\|_E)$ is complete.

For the general theory of B.f.s.' we refer to Chapter 15 in [22]. A special class consists of the rearrangement invariant B.f.s.' Recall, for any function $f \in L^0(G)$, that the decreasing rearrangement $f^* : [0, 1] \rightarrow [0, \infty]$ of $|f|$ is defined by

$$f^*(t) = \inf \{s \geq 0 : \mu(\{x \in G : |f(x)| > s\}) \leq t\}, \quad t \in [0, 1].$$

The function f^* is decreasing, right-continuous and is equimeasurable with $|f|$. For properties of decreasing rearrangements we refer to Section II.1 in [2] and Section II.2 in [11].

A B.f.s. $(E, \|\cdot\|_E)$ over G is called *rearrangement invariant* if it follows from $f \in E$, $g \in L^0(G)$ and $f^* = g^*$ that $g \in E$ and $\|g\|_E = \|f\|_E$. Examples of rearrangement invariant B.f.s.' include the L^p -spaces, for $1 \leq p \leq \infty$, Orlicz spaces, Marcinkiewicz spaces and Lorentz spaces; see, for example, [11], Chapter II and [2], Chapter 2, as well as [3].

3. Homogeneous Banach Spaces

As before, G is an infinite, compact abelian group with normalized Haar measure μ . For each $y \in G$ the translation operator $\tau_y : L^0(G) \rightarrow L^0(G)$ is the linear map defined by

$$(\tau_y f)(x) = f(x - y), \quad x \in G, \quad f \in L^0(G).$$

The following definition is motivated by Chapter I, Sect. 2 in [10].

Definition 3.1. A *homogeneous Banach space* over G is a Banach space $(E, \|\cdot\|_E)$ which is a linear subspace of $L^1(G)$ satisfying the following three conditions.

- (H-1) The natural inclusion $E \subseteq L^1(G)$ is continuous.
- (H-2) If $f \in E$ and $y \in G$, then $\tau_y f \in E$ with $\|\tau_y f\|_E = \|f\|_E$.
- (H-3) For each $f \in E$ we have $\lim_{y \rightarrow 0} \|f - \tau_y f\|_E = 0$.

Note that condition (H-2) implies that each translation operator $\tau_y : E \rightarrow E$ is a linear isometry. Furthermore, condition (H-3) implies, for each $f \in E$, that the map $y \mapsto \tau_y f$ is continuous from G into E .

The classical examples of homogeneous Banach spaces are the B.f.s.' $L^p(G)$, for $1 \leq p < \infty$. It can be shown that the space $L^\infty(G)$ is *not* homogeneous (since we assume that G is infinite). For the circle group $G = \mathbb{T}$ this is easily verified.

Let E be a B.f.s. over G . If E satisfies condition (H-2), then E is called a *translation invariant B.f.s.* over G ; see [14], [15], [16] for various aspects of such spaces. Any translation invariant B.f.s. E over G satisfies $E \subseteq L^1(G)$ with a continuous embedding (see Proposition 3.4 in [15]) and so E satisfies condition (H-1). However, such a B.f.s. need not be a homogeneous B.f.s. (consider, for example, the space $L^\infty(G)$). Actually, a translation invariant B.f.s. E over G satisfies (H-3) if and only if E has order continuous norm (briefly, o.c.-norm), that is, $\|f_n\|_E \downarrow 0$ for every sequence $(f_n)_{n=1}^\infty$ in E satisfying $f_n(x) \downarrow 0$ for μ -a.e. $x \in G$; see Proposition 3.8 in [15] and Theorem 7.11 in [16]).

Every rearrangement invariant B.f.s. over G is, in particular, translation invariant (but, not conversely; see Example 3.2 in [15]). Indeed, $(\tau_y f)^* = f^*$ for all $f \in L^0(G)$ and $y \in G$. Consequently, a rearrangement invariant B.f.s. E over G is a homogeneous Banach space if and only if E has o.c.-norm.

Another well known example of a homogeneous Banach space is

$$A(G) = \left\{ f \in C(G) : \|f\|_{A(G)} = \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| < \infty \right\};$$

see, for example, Ch. 9 in [9] and, for G the circle group $\mathbb{T}$, Section I.6 in [10]. The space $A(G)$ is in fact a Banach algebra (with respect to the pointwise operations) which is isometrically isomorphic to the convolution algebra $\ell^1(\Gamma)$. For the case $G = \mathbb{T}$ the space $A(\mathbb{T})$ is sometimes called the Wiener algebra. It is known (cf. Lemma 37.3 and Theorem 37.4 in [9]) that

$$A(G) \subsetneq C(G) \subsetneq L^\infty(G).$$

Evidently, $A(G)$ is *not* a B.f.s. over G . As noted before, Γ is countable if and only if G is metrizable.

To verify that $A(G)$ is a homogeneous Banach space, let $f \in A(G)$. Then $f = \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \gamma$ as an absolutely convergent series in $C(G)$ and so

$$\|f\|_1 \leq \|f\|_\infty \leq \sum_{\gamma \in \Gamma} \|\hat{f}(\gamma) \gamma\|_\infty = \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)| = \|f\|_{A(G)}.$$

Hence, $A(G)$ is continuously included in $L^1(G)$. Furthermore, for every $f \in L^1(G)$, in particular, for every $f \in A(G)$, we have that

$$(\tau_y f)^\wedge(\gamma) = \overline{(y, \gamma)} \hat{f}(\gamma), \quad \gamma \in \Gamma, \quad y \in G, \quad (8)$$

from which it is clear that condition (H-2) is also satisfied. Finally, using (8) once again, a routine argument shows that $A(G)$ satisfies (H-3) as well. Hence, $A(G)$ is a homogeneous Banach space over G .

4. Fourier Multipliers

Let E be a homogeneous Banach space over G . The space of all bounded linear operators from E into itself is denoted by $\mathcal{L}(E)$ and is equipped with the operator norm. An operator $T \in \mathcal{L}(E)$ is called an E -multiplier operator if $T\tau_y = \tau_y T$ for all $y \in G$. The space of all such operators is denoted by $\mathfrak{m}(E)$, which is a closed linear subspace of $\mathcal{L}(E)$. The following lemma exhibits a large class of E -multiplier operators.

Lemma 4.1. *Let E be a homogeneous Banach space over G and $\lambda \in M(G)$. Then $f * \lambda \in E$, for all $f \in E$, and the linear map $T_\lambda : E \rightarrow E$, defined by $T_\lambda f = f * \lambda$, for $f \in E$, satisfies $T_\lambda \in \mathfrak{m}(E)$ with $\|T_\lambda\| \leq \|\lambda\|_{M(G)}$.*

Proof. Fix $f \in E$. As observed before, the convolution $f * \lambda \in L^1(G)$, as defined in (3), exists. Define the function $F : G \rightarrow E$ by $F(y) = \tau_y f$, for $y \in G$. It follows from (H-2) and (H-3) that F is well defined and continuous on G . Lemma 2.1 implies that the Bochner integral $\int_G^{(B)} \tau_y f d\lambda = \int_G^{(B)} F d\lambda \in E$ exists.

For each $g \in L^\infty(G)$, define the linear functional $\varphi_g \in L^1(G)^*$ by $\langle h, \varphi_g \rangle = \int_G h g d\mu$, for $h \in L^1(G)$. Since the embedding of E into $L^1(G)$ is continuous, it is clear that the restriction of φ_g to E belongs to E^* . Noting that

$$\int_G \int_G |(\tau_y f)(x)| |g(x)| d\mu(x) d|\lambda|(y) \leq \|g\|_\infty \|f\|_1 |\lambda|(G) < \infty,$$

we can use (7) and apply Fubini's theorem to compute

$$\begin{aligned} \left\langle \int_G^{(B)} F(y) d\lambda(y), \varphi_g \right\rangle &= \int_G \langle F(y), \varphi_g \rangle d\lambda \\ &= \int_G \left(\int_G (F(y))(x) g(x) d\mu(x) \right) d\lambda(y) \\ &= \int_G \left(\int_G (\tau_y f)(x) d\lambda(y) \right) g(x) d\mu(x) \\ &= \int_G (f * \lambda)(x) g(x) d\mu(x) = \langle f * \lambda, \varphi_g \rangle. \end{aligned}$$

Since $\int_G^{(B)} F d\lambda \in E \subseteq L^1(G)$ and $L^\infty(G)$ separates the points of $L^1(G)$, we can conclude that $f * \lambda = \int_G^{(B)} F d\lambda \in E$.

Define the linear map $T_\lambda : E \rightarrow E$ by $T_\lambda f = f * \lambda$, for $f \in E$. It follows from (6) that

$$\begin{aligned} \|T_\lambda f\|_E &= \left\| \int_G^{(B)} F(y) d\lambda(y) \right\|_E \leq \int_G \|F(y)\|_E d|\lambda|(y) \\ &= \int_G \|\tau_y f\|_E d|\lambda|(y) = \int_G \|f\|_E d|\lambda|(y) = \|f\|_E \|\lambda\|_{M(G)} \end{aligned}$$

and hence, $T_\lambda \in \mathcal{L}(E)$ with $\|T_\lambda\| \leq \|\lambda\|_{M(G)}$. Finally, it is now routine to check that $T_\lambda \in \mathfrak{m}(E)$. $\square$

In particular, it follows from Lemma 4.1 that $f * g \in E$ for all $f \in E$ and $g \in L^1(G)$. Let $\gamma \in \Gamma$, the dual group of G . Then $\gamma \in C(G) \subseteq L^1(G)$ and so $f * \gamma \in E$ for all $f \in E$. It is readily verified that $f * \gamma = \hat{f}(\gamma)\gamma$. Hence,

$$\hat{f}(\gamma)\gamma = f * \gamma \in E, \quad f \in E, \quad \gamma \in \Gamma. \quad (9)$$

It follows from (9) that $\gamma \in E$ whenever there exists a function $f \in E$ satisfying $\hat{f}(\gamma) \neq 0$. Defining

$$\Gamma_E = \{\gamma \in \Gamma : \gamma \in E\},$$

it is now clear that

$$\Gamma_E = \bigcup_{f \in E} \text{supp}(\hat{f}). \quad (10)$$

Note that $\Gamma_E \neq \emptyset$ whenever $E \neq \{0\}$.

The set of all trigonometric polynomials on G which belong to E is denoted by $\tau_E(G)$, that is,

$$\tau_E(G) = \tau(G) \cap E.$$

Using (2) and (10) it is readily verified that $\tau_E(G)$ is equal to the linear span of Γ_E in $C(G)$. Furthermore, it follows from (9) that $f * h \in \tau_E(G)$ for all $f \in E$ and $h \in \tau(G)$. For later reference we formulate the following observation as a lemma.

Lemma 4.2. *Let E be a homogeneous Banach space over G . Then $\tau_E(G)$ is dense in E .*

Proof. According to Theorem (28.53) of [9] there exists an approximate identity $\{h_\alpha\} \subseteq \tau(G)$ for $L^1(G)$ such that $h_\alpha \geq 0$ with $\|h_\alpha\|_1 = \int_G h_\alpha d\mu = 1$ for all α and $\lim_\alpha \|h_\alpha \chi_{G \setminus U}\|_\infty = 0$ for each neighbourhood U of $0 \in G$. As observed prior to the lemma, $h_\alpha * f \in \tau_E(G)$ for each $f \in E$ and each α . Using the stated properties of $\{h_\alpha\}$ it is routine to check that $f * h_\alpha \rightarrow_\alpha f$ in E for each $f \in E$; cf. the proof of Lemma I.2.2 in [10]. Hence, we can conclude that $\tau_E(G)$ is dense in E . $\square$

Since $|\gamma| = \chi_G$, for every $\gamma \in E$, given a translation invariant B.f.s. $E \neq \{0\}$ over G , it follows that $\Gamma_E \neq \emptyset$ if and only if $\chi_G \in E$, if and only if $\tau_E(G) = \tau(G)$, if and only if $L^\infty(G) \subseteq E$. There exist translation invariant B.f.s.' over (every) G such that $\Gamma_E = \emptyset$; see Example 3.6 in [15]. In view of the observation after (10), such a space E is not a homogeneous Banach space over G .

Let $\{\gamma_j\}_{j=1}^n$ be any finite subset of Γ and define $E = \text{span}\{\gamma_j : 1 \leq j \leq n\}$ in $C(G)$. Then E is a (finite dimensional) homogeneous Banach space over G with respect to any of the norms $\|\cdot\|_p$, for $1 \leq p \leq \infty$. The fact that $\tau_y f \in E$, for all $y \in G$ and $f \in E$, follows from the identity

$$\tau_y \gamma = \overline{(y, \gamma)} \gamma, \quad \gamma \in \Gamma. \quad (11)$$

In particular, this example shows that homogeneous Banach spaces need not be dense in $L^1(G)$. It follows from (2), for such spaces E , that $\Gamma_E = \{\gamma_j\}_{j=1}^n$, which shows that $\Gamma_E \subsetneq \Gamma$ (hence, also $C(G) \not\subseteq E$) in general. If $f \in E$, then it need not follow that $\bar{f}$ or $|f|$ belongs to E (consider $G = \mathbb{T}$ and $E = \text{span}\{z\}$ in $C(\mathbb{T})$, for example).

The homogeneous Banach spaces $C^{(n)}(\mathbb{T})$, for $n \in \mathbb{N}$, and $\text{lip}_\alpha(\mathbb{T})$, for $0 < \alpha < 1$ (see pp. 14-16 in [10]) show that Γ_E need not be a norm bounded subset of E .

Let $E \neq \{0\}$ be a homogeneous Banach space over G . A function $\varphi : \Gamma \rightarrow \mathbb{C}$ is called a *Fourier E -multiplier function* if $\text{supp}(\varphi) \subseteq \Gamma_E$ and φ has the property that, for every $f \in E$, there exists $g \in E$ satisfying $\hat{g} = \varphi \hat{f}$. By property (H-1) in Definition 3.1 and injectivity of the Fourier transform on $L^1(G)$, it follows that such a function g is unique; we denote it by $M_\varphi^E f$. The so defined linear map $M_\varphi^E : E \rightarrow E$ is also continuous, as follows via the Closed Graph Theorem (using the continuity of the Fourier transform from E into $c_0(\Gamma)$; see property (H-1)). Hence, $M_\varphi^E \in \mathcal{L}(E)$. A routine calculation, using the identity $\hat{\gamma} = \chi_{\{\gamma\}}$, shows that $M_\varphi^E \gamma = \varphi(\gamma) \gamma$ for every $\gamma \in \Gamma_E$. Since $\gamma \neq 0$, this implies that $|\varphi(\gamma)| \leq \|M_\varphi^E \gamma\|_E / \|\gamma\|_E$ for all $\gamma \in \Gamma_E$. But, $\text{supp}(\varphi) \subseteq \Gamma_E$ and so we can conclude that

$$\|\varphi\|_\infty \leq \|M_\varphi^E\|. \quad (12)$$

The collection of all Fourier E -multiplier functions is denoted by $\mathcal{M}_E(\Gamma)$, which is clearly a linear subspace of $\ell^\infty(\Gamma)$. Actually, $\mathcal{M}_E(\Gamma)$ is a *sub-algebra* of $\ell^\infty(\Gamma)$. Indeed, let $\varphi, \psi \in \mathcal{M}_E(\Gamma)$. The identity $\text{supp}(\varphi\psi) = \text{supp}(\varphi) \cap \text{supp}(\psi)$ implies that $\text{supp}(\varphi\psi) \subseteq \Gamma_E$. Moreover, given $f \in E$, there exist functions $g, h \in E$ satisfying $\hat{g} = \varphi \hat{f}$ and $\hat{h} = \psi \hat{g}$. Then $\hat{h} = \varphi \psi \hat{f}$. Accordingly, $\varphi\psi \in \mathcal{M}_E(\Gamma)$. It follows from (10) that χ_{Γ_E} is the unit element of $\mathcal{M}_E(\Gamma)$, corresponding to the multiplier operator $M_{\chi_{\Gamma_E}}^E = I$.

For $E = \text{span}\{\gamma_j : 1 \leq j \leq n\}$ with $\{\gamma_j\}_{j=1}^\infty \subseteq \Gamma$ a finite set, it is clear that

$$\mathcal{M}_E(\Gamma) = \left\{ \sum_{j=1}^n \alpha_j \chi_{\{\gamma_j\}} : \alpha_j \in \mathbb{C}, 1 \leq j \leq n \right\}.$$

The homogeneous Banach space $A(G)$ given prior to (8) satisfies $\mathcal{M}_{A(G)}(\Gamma) = \ell^\infty(\Gamma)$. Indeed, fix $\varphi \in \ell^\infty(\Gamma)$. Given $f \in A(G)$, it is clear that $\varphi \hat{f} \in \ell^1(\Gamma)$.

Recalling that $A(G)$ is isometrically isomorphic with $\ell^1(\Gamma)$ via the Fourier transform, it follows that there exists a unique $g \in A(G)$ satisfying $\hat{g} = \varphi \hat{f}$. Note that actually $g = \sum_{\gamma \in \Gamma} \varphi(\gamma) \hat{f}(\gamma) \gamma$ as an absolutely convergent series in $A(G)$. Hence, $\varphi \in \mathcal{M}_{A(G)}(\Gamma)$. Note also that $\|M_\varphi^{A(G)}\| = \|\varphi\|_\infty$.

Remark 4.3. Let E be a homogeneous Banach space over G and $\varphi : \Gamma \rightarrow \mathbb{C}$ be a function with the property that, for every $f \in E$, there exists $g \in E$ satisfying $\hat{g} = \varphi \hat{f}$. Then $\text{supp}(\varphi \chi_{\Gamma_E}) \subseteq \Gamma_E$ and so $\varphi \chi_{\Gamma_E} \in \mathcal{M}_E(\Gamma)$. Note also that Lemma 4.1 implies that

$$\{\hat{\lambda} \chi_{\Gamma_E} : \lambda \in M(G)\} \subseteq \mathcal{M}_E(\Gamma),$$

since $(f * \lambda)^\wedge = \hat{\lambda} \cdot \hat{f}$ for all $\lambda \in M(G)$ and all $f \in E$.

Remark 4.4. Let $\varphi \in \mathcal{M}_E(\Gamma)$ and let $S \in \mathcal{L}(E)$ be the operator $S = M_\varphi^E$. For each $\gamma \in \Gamma_E$ we have that $(S\gamma)^\wedge = \varphi \hat{\gamma} = \varphi \chi_{\{\gamma\}}$. Consequently, the function φ can be recovered from the operator S via the formula

$$\varphi(\gamma) = \begin{cases} (S\gamma)^\wedge(\gamma) & \text{if } \gamma \in \Gamma_E \\ 0 & \text{if } \gamma \in \Gamma \setminus \Gamma_E \end{cases}. \quad (13)$$

Let $\varphi \in \mathcal{M}_E(\Gamma)$. If $y \in G$ and $f \in L^1(G)$, then $(\tau_y f)^\wedge(\gamma) = \overline{\gamma(y)} \hat{f}(\gamma)$ for all $\gamma \in \Gamma$. Consequently, using (10), we find that

$$(\tau_y M_\varphi^E f)^\wedge(\gamma) = (M_\varphi^E \tau_y f)^\wedge(\gamma) = \overline{\gamma(y)} \varphi(\gamma) \hat{f}(\gamma), \quad \gamma \in \Gamma, \quad f \in E,$$

which implies that $\tau_y M_\varphi^E = M_\varphi^E \tau_y$ for all $y \in G$, that is, $M_\varphi^E \in \mathfrak{m}(E)$. Note that the estimate (12) implies that the linear, multiplicative map $\varphi \mapsto M_\varphi^E$ from $\mathcal{M}_E(\Gamma)$ into $\mathfrak{m}(E)$ is injective. In Lemma 4.7 below we will see that this map is also surjective.

Summarizing the above discussion yields the following result.

Lemma 4.5. *Let $E \neq \{0\}$ be a homogeneous Banach space over G . Every $\varphi \in \mathcal{M}_E(\Gamma)$ belongs to $\ell^\infty(\Gamma)$ and there exists an operator $M_\varphi^E \in \mathfrak{m}(E)$ satisfying the inequality (12) and*

$$(M_\varphi^E f)^\wedge = \varphi \hat{f}, \quad f \in E.$$

Moreover, $\mathcal{M}_E(\Gamma)$ is a subalgebra of $\ell^\infty(\Gamma)$ with unit element χ_{Γ_E} .

Remark 4.6. Let $\varphi \in \mathcal{M}_E(\Gamma)$. If $f \in \tau_E(G)$ is given by $f = \sum_{j=1}^n c_j \gamma_j$, with $c_j \in \mathbb{C}$ and $\gamma_j \in \Gamma_E$, for $j = 1, \dots, n$, then

$$M_\varphi^E f = \sum_{j=1}^n \varphi(\gamma_j) c_j \gamma_j.$$

Indeed, apply the identity $M_\varphi^E \gamma = \varphi(\gamma) \gamma$ for all $\gamma \in \Gamma_E$ and use the linearity of M_φ^E .

The converse of Lemma 4.5 is also valid.

Lemma 4.7. *Let $E \neq \{0\}$ be a homogeneous Banach space over G and $T \in \mathfrak{m}(E)$. Then there exists a unique function $\varphi \in \mathcal{M}_E(\Gamma)$ such that $T = M_\varphi^E$.*

Proof. Fix $\gamma \in \Gamma_E$. It follows from (11) that $\tau_y T\gamma = T\tau_y \gamma = \overline{\gamma(y)} T\gamma$ for all $y \in G$. Taking Fourier transforms yields, via (8), that

$$\overline{\alpha(y)} (T\gamma)^\wedge(\alpha) = \overline{\gamma(y)} (T\gamma)^\wedge(\alpha), \quad \alpha \in \Gamma, y \in G.$$

For each $\alpha \in \Gamma \setminus \{\gamma\}$ there exists $y \in G$ with $\alpha(y) \neq \gamma(y)$, from which it follows that $\text{supp}((T\gamma)^\wedge) \subseteq \{\gamma\}$. Hence, for every $\gamma \in \Gamma_E$, there is $\varphi(\gamma) \in \mathbb{C}$ satisfying $(T\gamma)^\wedge = \varphi(\gamma) \chi_{\{\gamma\}} = \varphi(\gamma) \hat{\gamma}$. For $\gamma \in \Gamma \setminus \Gamma_E$ we set $\varphi(\gamma) = 0$. Then $\text{supp}(\varphi) \subseteq \Gamma_E$ and

$$(Tf)^\wedge = \varphi \hat{f}, \quad f \in \tau_E(G). \quad (14)$$

The density of $\tau_E(G)$ in E (see Lemma 4.2) and the continuity of the Fourier transform from E into $c_0(\Gamma)$ can then be used, together with (14), to deduce that $(Tf)^\wedge = \varphi \hat{f}$ for all $f \in E$. Hence, $\varphi \in \mathcal{M}_E(\Gamma)$ and $T = M_\varphi^E$.

The uniqueness of the function φ has already been observed in the discussion prior to Lemma 4.5. The proof is thereby complete. $\square$

Note that Lemma 4.7 implies that the operator algebra

$$\mathfrak{m}(E) = \{M_\varphi^E : \varphi \in \mathcal{M}_E(\Gamma)\}$$

and so $\mathfrak{m}(E)$ is commutative for every homogeneous Banach space E over G . Indeed, it is evident that $M_\varphi^E M_\psi^E = M_{\varphi\psi}^E = M_\psi^E M_\varphi^E$ for all $\varphi, \psi \in \mathcal{M}_E(\Gamma)$.

Remark 4.8. Let G be any infinite, compact abelian group. If E is a Banach space of functions on G which only satisfies (H1) and (H2), then there may exist operators $T \in \mathfrak{m}(E)$ which are *not* of the form M_ψ^E for any $\psi \in \mathcal{M}_E(\Gamma)$. Indeed, consider $E = L^\infty(G)$ and let $\varphi \in L^\infty(G)$ be an *invariant mean* with the properties stated in Theorem 3.4 of [20]. In particular, φ is *not* given by integration over G with respect to normalized Haar measure μ on G . Define $T \in \mathcal{L}(L^\infty(G))$ to be the rank-1 projection given by $Tf = \langle f, \varphi \rangle \chi_G$, for $f \in L^\infty(G)$, in which case $T \in \mathfrak{m}(L^\infty(G))$. Also, the function $\psi = \chi_{\{1\}}$ on Γ belongs to $\mathcal{M}_{L^\infty(G)}(\Gamma)$ and the corresponding multiplier operator $M_\psi^{L^\infty(G)} \in \mathfrak{m}(L^\infty(G))$ is given by

$$M_\psi^{L^\infty(G)} f = \left(\int_G f d\mu \right) \chi_G = f * \chi_G, \quad f \in L^\infty(G).$$

A direct calculation shows that the condition $TM_\psi^{L^\infty(G)} = M_\psi^{L^\infty(G)}T$ would imply that $\langle f, \varphi \rangle = \int_G f d\mu$ for all $f \in L^\infty(G)$, which is a contradiction. This shows that the algebra $\mathfrak{m}(L^\infty(G))$ is not commutative. Consequently, the result of Lemma 4.7 is not valid for $E = L^\infty(G)$.

5. The Projection of $\mathcal{L}(E)$ onto $\mathfrak{m}(E)$

As before, G is an infinite, compact abelian group with normalized Haar measure μ . Given a homogeneous Banach space E over G , let E^+ denote the cone of all $f \in E$ such that $f(x) \geq 0$ for μ -a.e. $x \in E$. Considering $E = \text{span}\{z\}$ in $C(\mathbb{T})$ shows that $E^+ = \{0\}$ is possible. An operator $T \in \mathcal{L}(E)$ is

called *positive* if $T(E^+) \subseteq E^+$. Observe that each translation operator τ_y , for $y \in G$, is positive.

Since compact abelian groups are amenable, for such compact groups G the following result is an extension of Theorem 1.1 of [1] to a significantly larger class of spaces than $L^p(G)$ for $1 \leq p < \infty$.

Theorem 5.1. *Let $E \neq \{0\}$ be a homogeneous Banach space over G . For each $T \in \mathcal{L}(E)$ define the function $\hat{T} : \Gamma \rightarrow \mathbb{C}$ by*

$$\hat{T}(\gamma) = \begin{cases} (T\gamma)^\wedge(\gamma) & \text{if } \gamma \in \Gamma_E \\ 0 & \text{if } \gamma \in \Gamma \setminus \Gamma_E \end{cases}. \quad (15)$$

Then $\hat{T} \in \mathcal{M}_E(\Gamma)$ and the linear map $\mathcal{Q} : \mathcal{L}(E) \rightarrow \mathcal{L}(E)$, given by

$$\mathcal{Q}(T) = M_{\hat{T}}^E, \quad T \in \mathcal{L}(E),$$

is a contractive projection of $\mathcal{L}(E)$ onto $\mathfrak{m}(E)$, preserving positivity.

Proof. Fix $T \in \mathcal{L}(E)$. For each $f \in E$, the E -valued function $\Phi_f : x \mapsto \tau_{-x}T\tau_x f$ is continuous on G and hence, by Lemma 2.1, it is Bochner μ -integrable. Define the linear map $S_T : E \rightarrow E$ by

$$S_T f = \int_G^{(B)} \Phi_f(x) d\mu(x), \quad f \in E. \quad (16)$$

Using estimate (4), it follows that $S_T \in \mathcal{L}(E)$ as

$$\|S_T f\|_E \leq \int_G \|\Phi_f(x)\|_E d\mu(x) \leq \int_G \|T\| \cdot \|f\|_E d\mu(x) = \|T\| \cdot \|f\|_E,$$

for each $f \in E$. In particular, $\|S_T\| \leq \|T\|$.

To see that $S_T \in \mathfrak{m}(E)$, fix $y \in G$. Then, for $f \in E$, Theorem II.2.6 of [4] and the translation invariance of μ yields

$$\begin{aligned} \tau_y S_T f &= \tau_y \int_G^{(B)} \tau_{-x} T \tau_x f d\mu(x) = \int_G^{(B)} \tau_{y-x} T \tau_x f d\mu(x) \\ &= \int_G^{(B)} \tau_{-x} T \tau_{x+y} f d\mu(x) = S_T \tau_y f, \end{aligned}$$

which shows that $\tau_y S_T = S_T \tau_y$. We can conclude that $S_T \in \mathfrak{m}(E)$. Furthermore, whenever $T \in \mathfrak{m}(E) \subseteq \mathcal{L}(E)$, we note that $\Phi_f(x) = Tf$ for all $f \in E$ and all $x \in G$ and hence, (16) implies that $S_T = T$. Defining $\mathcal{Q}(T) = S_T$, for $T \in \mathcal{L}(E)$, it is now clear that $\mathcal{Q} : \mathcal{L}(E) \rightarrow \mathcal{L}(E)$ is a contractive projection in $\mathcal{L}(E)$ with range $\mathfrak{m}(E)$.

It follows from Lemma 4.7, for each $T \in \mathcal{L}(E)$, that there exists a unique Fourier multiplier function $\hat{T} \in \mathcal{M}_E(\Gamma)$ such that $S_T = M_{\hat{T}}^E$. Next we establish formula (15). To this end recall, for each $g \in L^\infty(G)$, that the linear functional $\varphi_g \in E^*$ is defined by $\langle f, \varphi_g \rangle = \int_G fg d\mu$, for $f \in E$ (see the proof of Lemma 4.1). A direct computation shows that $\tau_y^* \varphi_g = \varphi_{\tau_{-y}g}$ for all $g \in L^\infty(G)$ and $y \in G$, where $\tau_y^* : E^* \rightarrow E^*$ is the Banach space adjoint operator of τ_y .

Let $T \in \mathcal{L}(E)$ be fixed. It follows from formula (13), applied to the operator $S = S_T$, that the function $\hat{T}$ is given by

$$\hat{T}(\gamma) = (S_T \gamma)^\wedge(\gamma) = \int_G (S_T \gamma)(x) \bar{\gamma}(x) d\mu(x) = \langle S_T \gamma, \varphi_{\bar{\gamma}} \rangle \quad (17)$$

for all $\gamma \in \Gamma_E$ (and $\hat{T}(\gamma) = 0$ for $\gamma \in \Gamma \setminus \Gamma_E$). Using (16) in combination with (5), we find that

$$\langle S_T \gamma, \varphi_{\bar{\gamma}} \rangle = \int_G \langle \tau_{-x} T \tau_x \gamma, \varphi_{\bar{\gamma}} \rangle d\mu(x), \quad \gamma \in \Gamma_E. \quad (18)$$

It follows from (11), for all $x \in G$ and $\gamma \in \Gamma_E$, that

$$\begin{aligned} \langle \tau_{-x} T \tau_x \gamma, \varphi_{\bar{\gamma}} \rangle &= \langle T \tau_x \gamma, \tau_{-x}^* \varphi_{\bar{\gamma}} \rangle = \langle T \tau_x \gamma, \varphi_{\tau_x \bar{\gamma}} \rangle \\ &= \overline{\gamma(x)} \gamma(x) \langle T \gamma, \varphi_{\bar{\gamma}} \rangle = \langle T \gamma, \varphi_{\bar{\gamma}} \rangle = (T \gamma)^\wedge(\gamma). \end{aligned} \quad (19)$$

A combination of (17), (18) and (19) yields the formula (15).

Finally, let $T \in \mathcal{L}(E)$ be a positive operator. If $g \in L^\infty(G)^+$ and $h \in E^+$, then $\langle h, \varphi_g \rangle \geq 0$. So, fix $f \in E^+$. Then $\tau_{-x} T \tau_x f \geq 0$ for all $x \in G$. Hence, for each $g \in L^\infty(G)^+$, we have that

$$\langle \mathcal{Q}(T) f, \varphi_g \rangle = \int_G \langle \tau_{-x} T \tau_x f, \varphi_g \rangle d\mu(x) \geq 0,$$

which implies that $\mathcal{Q}(T) f \geq 0$. This shows that $\mathcal{Q}(T)$ is a positive operator. The proof is thereby complete. $\square$

Remark 5.2. Consider $E = L^2(\mathbb{T})$, where $G = \mathbb{T}$ is the circle group, identified with the interval $[0, 2\pi)$ equipped with normalized Haar measure $(2\pi)^{-1} dx$. For each $n \in \mathbb{Z}$, define the function $e_n(x) = e^{inx}$, for $x \in [0, 2\pi)$. Then $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis in the Hilbert space E and so every operator $T \in \mathcal{L}(E)$ can be represented by an infinite matrix with respect to $\{e_n\}_{n \in \mathbb{Z}}$. Let $T \in \mathcal{L}(E)$. Then the decomposition of T with respect to the projection $\mathcal{Q}$ of Theorem 5.1 corresponds to the decomposition of the matrix of T into its diagonal and off diagonal parts.

As an application, recall *Bochner's theorem* which states, for a function $\varphi : \Gamma \rightarrow \mathbb{C}$, that $\varphi = \hat{\lambda}$ for some measure $\lambda \in M(G)^+$ if and only if φ is positive definite; see, for example, Section 1.4 of [18]. Since $C(G)$ is a homogeneous Banach space over G , by Theorem 5.1 an equivalent statement is that $\varphi = \hat{T}$ for some positive operator $T \in \mathcal{L}(C(G))$. Indeed, by Theorem 5.1 we have, for each $R \in \mathcal{L}(C(G))$, that $\mathcal{Q}(R) \in \mathfrak{m}(C(G))$ and hence, $\hat{R} = \hat{\lambda}$ for some $\lambda \in M(G)$; see, for example, Theorem 3.3.2 in [12]. The positivity of T ensures that $\mathcal{Q}(T)$ is also positive (cf. Theorem 5.1), that is, $f * \lambda = \mathcal{Q}(T) f \geq 0$ for all $f \in C(G)^+$. Accordingly, $\int_G f(-x) d\lambda(x) = (f * \lambda)(0) \geq 0$ for all $f \in C(G)^+$. Via the Riesz representation theorem this implies that $\lambda \in M(G)^+$.

For our final result, let $E \neq \{0\}$ be a translation invariant B.f.s. over G with o.c.-norm. Then E is a homogeneous Banach space over G ; see the

discussion after Definition 3.1. Furthermore, $L^\infty(G) \subseteq E$; see Proposition 3.8 in [15]. In particular, $\Gamma_E = \Gamma$ and $\tau(G) \subseteq E$.

A linear operator $T : E \rightarrow E$ is called *order bounded* if T maps order bounded subsets of E onto order bounded sets, that is, for every $u \in E^+$ there exists $v \in E^+$ such that $|Tf| \leq v$ whenever $f \in E$ satisfies $|f| \leq u$. Evidently, every positive operator is order bounded. The collection of all order bounded linear operators is denoted by $\mathcal{L}_b(E)$ and it has the structure of a Dedekind complete, complex Riesz space (see Section 92 of [23]), as E itself is a Dedekind complete complex Banach lattice. In particular, each $T \in \mathcal{L}_b(E)$ is a linear combination of at most four positive operators (in other words, every $T \in \mathcal{L}_b(E)$ is a *regular operator* on E). Since every positive, linear operator on a Banach lattice is necessarily continuous (see, for example, Theorem 83.12 in [23]), it follows that $\mathcal{L}_b(E) \subseteq \mathcal{L}(E)$.

Recall from Lemma 4.1 that, for each $\lambda \in M(G)$, the operator $T_\lambda \in \mathcal{L}(E)$ of convolution with λ exists. The following fact will be used.

Lemma 5.3. *Let $E \neq \{0\}$ be a translation invariant B.f.s. over G with o.c.-norm. Then*

$$\mathcal{L}_b(E) \cap \mathfrak{m}(E) = \{T_\lambda : \lambda \in M(G)\}.$$

Proof. Let $\lambda \in M(G)$. Then λ is a linear combination of at most four positive measures from $M(G)^+$. Consequently, T_λ is a linear combination of at most four positive operators from $\mathfrak{m}(E)$. Hence, $T_\lambda \in \mathcal{L}_b(E) \cap \mathfrak{m}(E)$. This shows that $\{T_\lambda : \lambda \in M(G)\} \subseteq \mathcal{L}_b(E) \cap \mathfrak{m}(E)$.

For the proof of the converse implication, first observe that $\mathfrak{m}_b(E) = \mathcal{L}_b(E) \cap \mathfrak{m}(E)$ is a (complex) vector sublattice of $\mathcal{L}_b(E)$; see Proposition 5.3 of [16]. Therefore, each operator $T \in \mathfrak{m}_b(E)$ is a linear combination of at most four positive operators in $\mathfrak{m}_b(E)$.

Fix $T \in \mathfrak{m}_b(E)^+$. It follows from Lemma 4.7 that there exists a unique $\varphi \in M_E(\Gamma)$ such that $T = M_\varphi^E$. Let $f \in \tau(G) = \tau_E(G)$ be given by $f = \sum_{j=1}^n c_j \gamma_j$, with $c_j \in \mathbb{C}$ and $\gamma_j \in \Gamma$ for $1 \leq j \leq n$. Since

$$\begin{aligned} |f|^2 &= f\bar{f} = \left(\sum_{j=1}^n c_j \gamma_j \right) \left(\sum_{k=1}^n \overline{c_k} \gamma_k^{-1} \right) \\ &= \sum_{j,k=1}^n c_j \overline{c_k} \gamma_j \gamma_k^{-1} \end{aligned}$$

belongs to $\tau(G)$, Remark 4.6 implies that

$$T(|f|^2) = \sum_{j,k=1}^n \varphi(\gamma_j \gamma_k^{-1}) c_j \overline{c_k} \gamma_j \gamma_k^{-1}. \quad (20)$$

Using that T is positive and $|f|^2 \geq 0$ in E , it follows that $T(|f|^2) \geq 0$ in E (that is, μ -a.e. on G). Since the function (20) is continuous on G , this implies that

$$\sum_{j,k=1}^n \varphi(\gamma_j \gamma_k^{-1}) c_j \overline{c_k} (x, \gamma_j \gamma_k^{-1}) \geq 0, \quad x \in G.$$

Selecting $x = 0$ yields

$$\sum_{j,k=1}^n \varphi(\gamma_j \gamma_k^{-1}) c_j \overline{c_k} \geq 0.$$

This holds for all choices of $c_1, \dots, c_n \in \mathbb{C}$, all $\gamma_1, \dots, \gamma_n \in \Gamma$ and all $n \in \mathbb{N}$. Hence, the function $\varphi : \Gamma \rightarrow \mathbb{C}$ is positive definite. It follows from Bochner's Theorem (see, for example, Theorem 1.4.3 in [18] or Theorem 4.18 in [8]) that there exists $\lambda \in M(G)^+$ such that $\varphi = \hat{\lambda}$. This suffices for the proof of the lemma. $\square$

Let $B(\Gamma) = \{\hat{\lambda} : \lambda \in M(G)\}$. Its closure in $\ell^\infty(\Gamma)$ is denoted by $\overline{B(\Gamma)}$. For compact abelian groups G the next fact is an extension of Theorem 3.1 in [1].

Theorem 5.4. *Let $E \neq \{0\}$ be a translation invariant B.f.s. over G with o.c.-norm and $\overline{\mathcal{L}_b(E)}$ denote the operator norm closure of $\mathcal{L}_b(E)$ in $\mathcal{L}(E)$. Whenever $T \in \overline{\mathcal{L}_b(E)}$ the Fourier E -multiplier function $\hat{T}$, as defined in Theorem 5.1, satisfies $\hat{T} \in \overline{B(\Gamma)}$.*

Proof. Let $T \in \overline{\mathcal{L}_b(E)}$. Then there exists a sequence $(T_n)_{n=1}^\infty$ in $\mathcal{L}_b(E)$ such that $\|T - T_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let $\mathcal{Q} : \mathcal{L}(E) \rightarrow \mathcal{L}(E)$ be the contractive projection onto $\mathfrak{m}(E)$ given in Theorem 5.1. Since $\mathcal{Q}$ preserves positivity, it follows, for each $n \in \mathbb{N}$, that $\mathcal{Q}(T_n) \in \mathcal{L}_b(E) \cap \mathfrak{m}(E)$ and so, by Lemma 5.3, there exists $\lambda_n \in M(G)$ satisfying $\hat{T}_n = \hat{\lambda}_n$. Furthermore, the estimate (12) implies that

$$\|\hat{T} - \hat{\lambda}_n\|_\infty \leq \|M_{\hat{T} - \hat{T}_n}^E\| = \|\mathcal{Q}(T - T_n)\| \leq \|T - T_n\|, \quad n \in \mathbb{N},$$

and hence, $\|\hat{T} - \hat{\lambda}_n\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. The proof is thereby complete. $\square$

We conclude with some examples. Let E be any rearrangement invariant B.f.s. with o.c.-norm over the circle group $\mathbb{T}$. Then E is a homogeneous Banach space over $\mathbb{T}$; see the discussion at the end of Section 3. The *lower and upper Boyd indices* of E are denoted by $\underline{\alpha}_E$ and $\overline{\alpha}_E$, respectively, and always satisfy $0 \leq \underline{\alpha}_E \leq \overline{\alpha}_E \leq 1$; see Sections III.5 and III.6 of [2] for the definition and properties of these indices. According to a theorem of D.W. Boyd, the *Hilbert transform* H is a continuous linear operator on E (and hence, $H \in \mathfrak{m}(E)$) if and only if $0 < \underline{\alpha}_E \leq \overline{\alpha}_E < 1$; see Theorem III.6.10 and Corollary III.6.11 in [2]. Recall that H is the multiplier operator corresponding to the Fourier multiplier function

$$\varphi(n) = -i \operatorname{sgn}(n), \quad n \in \mathbb{Z},$$

with $\operatorname{sgn}(0) = 0$; see [2], Lemma III.6.7. Since $\lim_{n \rightarrow \infty} \varphi(n) \neq \lim_{n \rightarrow -\infty} \varphi(n)$ and both limits exist, it follows from Corollary 3.8 of [1] that $\varphi \notin \overline{B(\mathbb{Z})}$. Consequently, Theorem 5.4 implies that $H \notin \overline{\mathcal{L}_b(E)}$.

Further examples of a similar nature to the Hilbert transform occur for certain elements from the space $\operatorname{bv}(\mathbb{Z})$ of all functions $\varphi : \mathbb{Z} \rightarrow \mathbb{C}$ of bounded variation. Indeed, functions $\varphi \in \operatorname{bv}(\mathbb{Z})$ have the property that both limits $\lim_{n \rightarrow \infty} \varphi(n)$ and $\lim_{n \rightarrow -\infty} \varphi(n)$ always exist; see Proposition 8 (i) in [13]. Moreover, Stečkin's theorem (cf. Theorem 6.3.5 in [6]) ensures that $\operatorname{bv}(\mathbb{Z}) \subseteq \bigcap_{1 < p < \infty} \mathcal{M}_{L^p(\mathbb{T})}(\mathbb{Z})$. Again Corollary 3.8 of [1] implies, whenever

$\lim_{n \rightarrow \infty} \varphi(n) \neq \lim_{n \rightarrow -\infty} \varphi(n)$, that $\varphi \notin \overline{B(\mathbb{Z})}$ and hence, Theorem 3.1 of [1] (see also Theorem 5.4 above) ensures that the corresponding multiplier operator $M_\varphi^{L^p(\mathbb{T})} \notin \overline{\mathcal{L}_b(L^p(\mathbb{T}))}$, for $1 < p < \infty$.

The finite dimensional, homogeneous Banach spaces presented after Lemma 4.2 are surely not dense in $L^1(G)$. Infinite dimensional examples also exist. Indeed, for each $1 \leq p < \infty$, define

$$\mathcal{H}^p = \left\{ f \in L^p(\mathbb{T}) : \hat{f}(n) = 0 \text{ for all } n \in \mathbb{Z} \text{ with } n < 0 \right\}$$

equipped with the norm from $L^p(\mathbb{T})$; cf. Section 3.2 of [5]. The space $\mathcal{H}^p$ consists of all the boundary values of functions in the space H^p of analytic functions on the open unit disc in $\mathbb{C}$; see Theorem 3.4 in [5]. As is well known, the space $\mathcal{H}^p$ is a closed subspace of $L^p(\mathbb{T})$ and so it is a Banach space which is continuously included in $L^1(\mathbb{T})$. Using (8), it is clear that property (H-1) in Definition 3.1 is satisfied. Conditions (H-2) and (H-3) are also satisfied since $L^p(\mathbb{T})$ has these properties. So, we can conclude that $\mathcal{H}^p$ is a homogeneous Banach space over $\mathbb{T}$. Note that $\mathbb{Z}_{\mathcal{H}^p} = \mathbb{Z}^+$, where $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. Evidently, $\mathcal{H}^p$ is infinite dimensional and not dense in $L^1(\mathbb{T})$.

We note, for every $1 < p < \infty$, that $\mathcal{H}^p$ is actually a proper, complemented subspace of $L^p(\mathbb{T})$. Indeed, as is well known, the *Riesz projection* $R : L^p(\mathbb{T}) \rightarrow L^p(\mathbb{T})$, which is the multiplier operator corresponding to the Fourier multiplier function $\chi_{\mathbb{Z}^+}$, is a continuous projection in $L^p(\mathbb{T})$ onto $\mathcal{H}^p$. Let $P \in \mathcal{L}(L^p(\mathbb{T}))$ denote the rank-1 projection given by $Pf = \hat{f}(0) \chi_{\mathbb{T}}$, for $f \in L^p(\mathbb{T})$. Note that $P \in \mathfrak{m}(E) \cap \mathcal{L}_b(E)$. Since $R = \frac{1}{2}(I + P + iH)$, where H is the Hilbert transform as in the previous example, it is also clear that $R \notin \overline{\mathcal{L}_b(L^p(\mathbb{T}))}$.

Data Availability There are no data related to this publication.

Declarations

Ethical statement The authors did not receive support from any organization for the submitted work.

Competing interests The authors have no competing interests to declare that are relevant to the content of this article.

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Received: November 24, 2025.

Accepted: May 28, 2026.