THE BERTHING SHIP PROBLEM:

FORCES ON BERTHING STRUCTURES FROM MOVING SHIPS

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SUMMARY

In this report a mathematical model is formulated which is sufficiently accurate to describe quantitatively and qualitatively the behaviour of a ship berthing to a structure equipped with fenders as well as to determine the response of the fenders themselves. In order to achieve this a set of equations is drawn up by which the (coupled) transient motions - in six degrees of freedom - of shiplike bodies can be described adequately.

To this end use is made of the so-called 'impulse response function' - technique, which has as the restriction that the ship-fluid system is supposed to be linear. This approach enables the inclusion of external forces of arbitrary nature; the fluid reactive forces are taken into account by means of the hydrodynamic coefficients which are incorporated in the impulse response functions representing the properties of the linear ship-fluid system.
The influences of a restricted water depth and of a (quay-)wall parallel to the ship can be taken into account.

The 'impulse response function'-technique is applied to the case of a ship with a box-like shape in order to avoid coupling between the respective ship motions. This is, however, not an essential simplification. Since berthing manoeuvres and the ship-fender interactions take place mainly in the horizontal plane, only the sway and yaw motions are considered; the effect of a forward speed is not included. For the case of shallow water with a horizontal bottom and relatively large horizontal dimensions the respective impulse response functions for the sway and yaw motion are calculated from experiments and/or theory.

Using the 'impulse response function'-technique a mathematical model is presented describing both the behaviour of the schematized ship berthing to a structure equipped with one undamped, (non-)linear fender and the behaviour of the fender itself. 'Centric' as well as 'eccentric impacts' are considered.

An extensive experimental verification was carried out by means of model tests on shallow water with relatively large horizontal dimensions. Two water depths were regarded.
The same situations as investigated experimentally were simulated numerically by means of the mathematical model, using thereby the impulse response functions calculated.

The results of calculations and model tests show a very good agreement.

Section I: INTRODUCTION

I.1: GENERAL DESCRIPTION OF THE PROBLEM AND ITS SIMPLIFICATION

Ships are becoming larger and larger. As a consequence berthing facilities have to be newly constructed or adapted to the larger units. Up to now reliable, theoretically founded, design criteria are hardly available. The lack of good design criteria is the prime reason for making researches into the possibilities of an experimental and/or theoretical determination of berthing forces.

Generally a berthing facility consists of one or more elastic elements (fenders) attached to a rigid structure (finger pier, caisson-type jetty, etc.). The fenders absorb the berthing forces and form a protection for ship and berthing structure.

The phenomena occurring during the berthing manoeuvre of a ship are complicated and the fender loads are influenced by a lot of parameters: the geometry and the rigidity of the (hull of the) ship, the elastic properties of the fender(s), the speed of approach, the forces exerted by tugs, wind, current and waves, the mode of motion (in general translation combined with rotation), the keel clearance.
As the maximum permissable berthing force against the side of e.g. a mammoth tanker is distinctly lower than what is acceptable for the berthing structure, the ship is therefore the prevailing factor for fender design.

The behaviour of a berthing ship and the resulting fender loads can be determined either by means of experiments with scale models or by way of an analytical treatment of the phenomenon. Of course a combination of both methods is possible as well.
On the one hand model testing has a few drawbacks. Model tests are expensive and time consuming. The test set-up is complicated; it is essential that the elastic properties of fenders are simulated very carefully and, sometimes, sophisticated facilities are needed

to simulate the relevant environmental conditions. For these reasons test programs are usually restricted to final design configurations and selected conditions which are assumed to be the most critical. Besides, the insight gained from model tests into the fundamentals of the problem remains limited: only the resulting output is measured without yielding much knowledge of the mechanism which causes the output. On the other hand a general analytical treatment of the problem is rather complicated.

This report presents a mathematical model describing the behaviour of a ship berthing to a jetty (or similar facility) and predicting the fender loads in a purely theoretical way, in which all essential features are maintained and which produces qualitative results of sufficient accuracy for most practical applications.

For reasons of completeness the mathematical model to be presented will be generalized to systems with six degrees of freedom. The ship is berthing to a fender mounted on a rigid structure, while the elastic characteristics of the fender may be non-linear and asymmetric.

In addition to the fender loads other external forces upon the ship, such as forces exerted by wind, waves, current, tugs and mooring lines, can be incorporated in the model as well. Within the scope of the mathematical model also the influence of a closed vertical wall (a quay) can be taken into account.

For the specific case of the berthing ship the undermentioned assumptions and simplifications are made.

As berthing manoeuvres and the ship-fender interactions take place mainly in the horizontal plane, only the surge, sway and yaw motions of the ship are of importance; so heaving, rolling and pitching are neglected. Hereby it is assumed implicitly that the heave, roll and pitch motions - which in a way do occur in reality - do not influence the motions in the horizontal plane.

The ship's forward speed is supposed to be zero or negligible. This assumption is justified by the fact that during a berthing operation the forward speed is indeed small or zero.
Only berthing operations on sheltered locations (e.g. harbours) are considered, i.e. the influences of waves, current and wind are not taken into account.
Special attention is paid to the case when shallowness of the water is of dominant importance, for, berthing structures are often located in shallow water. Farther, the bottom is horizontal. The fluid domain is supposed to be relatively large in the horizontal directions; this implies an open (jetty-type) berthing structure.
The vessel is considered as a rigid prismatic body with a rectangular cross-section and a symmetrical distribution of mass. This schematization is justified by the fact that many sea-going vessels and the most inland ships have a more or less box-like shape, being slightly streamlined at bow and stern.
A very important assumption is that the displacements of the ship remain small, for, this makes it possible to regard the problem of the ship motions as linear.
Further it is supposed that the fluid is inviscid and incompressible and moves irrotationally.

With the supposed linearity of the ship-fluid system and under the simplifications mentioned above the problem of a ship berthing to a fender structure now has been reduced greatly. What remains is the formulation of a mathematical model based on the linearity of the ship-fluid system, which is able to describe the force(s) exerted by a schematized ship with a horizontal motion (swaying and yawing) upon some berthing facility at calm shallow water with idealized properties and with relatively large horizontal dimensions.

I.2: SURVEY OF EXISTING STUDIES AND OUTLINE OF THE APPROACH TO BE FOLLOWED

When designing a berthing structure generally an approach is used in which it is assumed that the energy to be absorbed by the fender(s) equals the kinetic energy of the ship. Usually the mode of motion of the ship then consists of a translation - with or without forward speed - combined with a rotation; to include the effect of the entrained water a certain constant added mass (moment of inertia) is introduced (see e.g. refs. [1] through [17]).

This approach, in fact, involves the use of Newton's second law

$$(1^a) \qquad \frac{d}{dt}(m\dot{x}) = f(t) ,$$

describing the motion(s) $x(t)$ of a free floating ship with mass (moment of inertia) m in response to some external force or moment $f(t)$; t represents the time-coordinate. Since m may be regarded as a constant the equations of motion become:

$$(1^b) \qquad m\ddot{x} = f(t) .$$

In the following the concept force(s) has to be understood in a generalized sense meaning force(s) or moment(s). In general the external force f(t) in eqs. $(1^{a,b})$ is composed of:

- forces, e.g. due to waves, varying arbitrarily in time,
- hydrodynamic and hydrostatic restoring forces, which are a function of the motions of the ship,
- (restoring) forces due to the fender system, which are a function of the instantaneous position of the ship.

In the classical theory of ship motions it is common practice to formulate the equations (of motion) as follows:

$$(2) \qquad (m + a)\ddot{x} + b\dot{x} + cx = f(t),$$

where a, b and c are coefficients representing the hydrodynamic and the hydrostatic restoring functions. Eq. (2) is not a real equation of motion in the sense that it relates the variables of the instantaneous motion to the instantaneous values of the exciting forces. On the contrary, eq. (2) merely represents a set of algebraic equations fixing the amplitudes and phases of the (six) oscillations of the ship under the action of an exciting oscillatory force at one specific frequency. Therefore eq. (2) can only be used as a description in the frequency domain of (a) steady oscillatory motion(s), since the hydrodynamic coefficients (a and b) depend on the frequency of motion.

The analytical (and experimental) work on the problem of the berthing of ships, as mentioned in refs. [1] through [17] , in principle is based on eq. (2). Hereby the coefficient a is supposed to be independent of the frequency, while the coefficients b and c are neglected. Eq. (2) then reduces to:

(3) $$(m + a)\ddot{x} = f(t)$$

Eq. (3) can be regarded as a differential equation, representing a set of equations of motion, which is adequate to describe the motion of a body in an infinite fluid; it will yield, however, incorrect results when a free water surface is present. Then a 'memory effect', is introduced, i.e. each occurrence is, in fact, dependent on all preceding occurrences. The hydrodynamic influences are reflected only by the added mass (moment of inertia), which is assumed to be constant during the motion of the ship; the effect of the hydrodynamic damping has not been taken into account. Besides, the choice of a value for the added mass (moment of inertia) is a problem, the more so as it appears from literature (see e.g. refs. [18] through [32]) that the hydrodynamic coefficients are very much dependent on the frequency, especially in shallow water: generally it holds true that the assumption of constant hydrodynamic coefficients cannot be justified. Consequently, to determine the fender forces as a result of the berthing of a ship a time domain description of the behaviour of the moving ship is needed, which makes allowance for the frequency dependency of the fluid reaction forces; or, in other words, a method has to be used in which the hydrodynamic coefficients are taken into account as functions of the frequency. To this end an approach will be followed in which use is made of the so-called 'impulse response function'-technique.

If for any linear system the response k(t) to a unit impulse is known, then the response x(t) of the system to an arbitrary forcing function f(t) is:

$$x(t) = \int_{-\infty}^{t} f(\tau)\, k(t - \tau)\, d\tau \; ; \tag{4}$$

τ represents an integration variable (time).
It will be obvious that the berthing ship as a whole - i.e. the combination of ship, fluid and fender(s) - may not be thought of as a linear system. By isolating the free floating ship in still water, however, the ship and the fluid combined can be regarded as a system for which the assumption of linearity holds true as long as the motions remain small. The external forces then may be linear or non-linear and can be incorporated in the forcing function. The forces exerted somewhere upon the ship will be conceived of as input signals, whereas the motion of the ship will be considered to be the output signal. If the input signal and the output signal are represented by f(t) and x(t), respectively, then - provided the ship-fluid system is linear - they are connected by means of a convolution integral over the entire time history of the forcing function(s) according to eq. (4). There is a theoretical relationship between the equations in the time domain and those in the frequency domain.

In this approach to the berthing ship problem the system of the ship-fluid interaction is regarded as a black box, relating the input and output signals of the system without reflecting the physical processes behind it (see fig. 1).

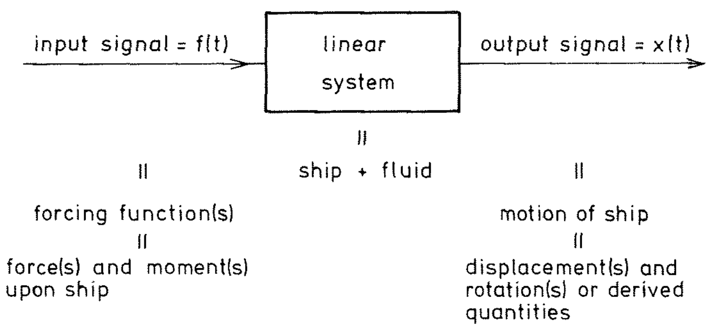

Fig. 1 - Schematic representation of mathematical model.

The external forces may be of arbitrary nature; this implies that besides fender loads also forces exerted upon the ship by wind, waves, current, tugs and mooring lines can be incorporated in the forcing function. It is essential that the effect of the nearness of the bottom on the hydrodynamic coefficients as well as the effects of a quay parallel to the ship can be included.

A good review of an other description of the ship-fluid system in the frequency domain and the time domain, together with the possibilities and difficulties of passing from one to another, is given in refs. [33] and [34]. For practical applications of the approach presented herein is referred to refs. [35] and [31] .

In section II the 'impulse response function'-technique is described in a general mathematical formulation, and its features are discussed. Then the impulse response functions are calculated for

the schematized ship (model) in case of horizontal motions (sway and yaw) at zero forward speed on (shallow) water with relatively large horizontal dimensions.
In Section III the 'impulse response function-technique' is applied to the berthing ship problem. First of all the mathematical model to simulate the berthing of a (schematized) ship to a jetty and to determine the relevant quantities is presented. Use is made of the impulse response functions as calculated in Section II. Then, the results of theoretical and experimental investigations for certain situations are compared and discussed. Possible extensions of the mathematical model to other situations are indicated.
Section IV reviews the main conclusions which can be drawn from the research of the berthing ship problem.

Section II: THE 'IMPULSE RESPONSE FUNCTION'-TECHNIQUE

II.1: THE 'IMPULSE RESPONSE FUNTION'-TECHNIQUE FOR ARBITRARY SHIP MOTIONS WITH SIX DEGREES OF FREEDOM

In this section a general formulation of the ship motion problem is presented in the frequency domain as well as in the time domain. This general formulation is valid for both deep and shallow water; also the effect of the presence of (a) vertical wall(s) can be included.

A theoretical derivation of the formulae is provided.

The two important assumptions made are that the ship behaves as a rigid body and that the motions of the ship remain small. The effects governed by rigid body characteristics and by hydrodynamics must be incorporated separately, since they are controlled by different parameters.

Before starting the formal formulation of the problem three further restrictions are made: firstly, the ship's form is transversely symmetric with respect to the vertical centre plane, longitudinal symmetry is not assumed; secondly, at rest the ship is floating upright in stable equilibrium; and thirdly, the ship is following a straight track at a constant mean forward speed in the above plane of symmetry. In principle these simplifications are not essential to the general formulation of the problem, but they facilitate the formulation greatly. Besides they correspond to what is common practice in naval hydrodynamics.

Analogous to ref. [23] then the following co-ordinate system are introduced:

$O\bar{x}_1\bar{x}_2\bar{x}_3$ = space fixed right-handed system of Cartesian co-ordinates with origin O; $O\bar{x}_1\bar{x}_2$ coincides with the water surface at rest; the vertical $O\bar{x}_3$-axis is positive upwards; the forward speed V of the ship coincides with the positive $O\bar{x}_1$-axis.

$ox_1x_2x_3$ = right-handed Cartesian co-ordinate system parallel with $O\bar{x}_1\bar{x}_2\bar{x}_3$, but translating with the (constant) ship's speed V; at rest the origin o coincides with the ship's centre of gravity G; the longitudinal ox_1-axis is positive in forward direction, the ox_2-axis is positive to port-side, the ox_3-axis is positive upwards.

Gxyz = moving right-handed Cartesian co-ordinate system with origin G and fixed with respect to the ship; Gxz coincides with the longitudinal plane of symmetry of the ship; the Gy-axis is positive to port-side, the Gz-axis is positive upwards.

The relations between the two co-ordinate system are:

$$\bar{x}_1 = x_1 + Vt, \qquad \bar{x}_2 = x_2, \qquad \bar{x}_3 = x_3 + a,$$

where a = distance of G below the plane of the water-line,
t = time co-ordinate.

The motions of the ship now can be represented by the motion variable $x_j(t)$, where j = 1, 2, ..., 6; x_1, x_2 and x_3 stand for the translations surge, sway and heave, while x_4, x_5 and x_6 denote the rotations around the ox_1-axis, the ox_2-axis and the ox_3-axis, respectively. In naval hydrodynamics it is usual to introduce a set of three independent angular displacements, the so-called Eulerian angles, viz.: yawing, being about the absolutely vertical ox_3-axis, pitching around the rotated position of the ox_2-axis, which remains in the horizontal plane, and rolling about the position of the ox_1- axis after the previous two rotations. If only small motion amplitudes are considered, these Eulerian angles coincide with the angular displacements about the space fixed axes (see ref. [23]). The displacements $x_j(t)$ in the six respective directions then are:

$x_1(t)$ = translation in the $\bar{x}_1$-direction = surge motion (positve forwards),

$x_2(t)$ = translation in the $\bar{x}_2$-direction = sway motion (positive to port-side),

$x_3(t)$ = translation in the $\bar{x}_3$-direction = heave motion (positive upwards),

$x_4(t)$ = rotation about the $O\bar{x}_1$-axis = roll motion (positive from deck to starboard-side),

$x_5(t)$ = rotation about the $O\bar{x}_2$-axis = pitch motion (positive with bow moving downwards),

$x_6(t)$ = rotation about the $O\bar{x}_3$-axis = yaw motion (positive with bow moving to port-side).

The combination of ship plus fluid can be conceived of as a(n arbitrary) stable system: the forces exerted somewhere upon the ship are regarded as input signals, whereas the motion of the ship (displacement and rotation or derived quantities) is considered to be the output signal. In consequence of the 'memory effect' associated with the influence(s) of the free surface (and of the vorticity, respectively), it is necessary to represent the transient ship motion - arising from a set of forces - in terms of a convolution integral over the entire time history of the forcing functions. Thus the six components of the motion have to be considered to be of the (general) form (see refs. [36] , [37, 38]):

(5) $$u_j(t) = \int_{-\infty}^{t} k_j\left[f_i(\tau),\, t-\tau\right] d\tau,$$

$$i,j = 1,\ 2,\ \ldots.6 \qquad ,$$

where τ = integration variable (time),

$f_i(t)$ = forcing function in the i-direction = input signal,

k_j = kernel for motion in the j-direction,

$u_j(t)$ = response of the system in the j-direction to the set of input signals $\{f_i(t)\}$ = output signal.

The kernel k_j depends on the set of forcing functions $\{f_i(t)\}$, on the retarded time $t - \tau$, on the geometry of the ship and on the physical properties of the fluid.
If it is allowed to consider the ship and the fluid combined as a stable linear system, eq. (5) changes into a more familiar and simple form (see refs. [36] , [37, 38]):

$$u_j(t) = \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau)\, k_{ij}(t-\tau)\, d\tau = \tag{6}$$

$$= \sum_{i=1}^{6} \int_{0}^{\infty} k_{ij}(\tau)\, f_i(t-\tau)\, d\tau \, , \qquad j = 1, 2, \ldots 6,$$

where $k_{ij}(t)$ = response for the j-direction to a unit pulse (i.e. Dirac function at $t = 0$) in the i-direction = impulse response function (i.r.f.).

On account of the above definition for $k_{ij}(t)$ it holds good that (principle of causality):

$$k_{ij}(t) = 0 \qquad \text{for } t < 0; \tag{7^a}$$

as initial condition it is stated:

$$k_{ij}(0) = 0 \ . \tag{7^b}$$

The i.r.f. $k_{ij}(t)$ is a real function of t which depends on the geometry of the ship as well as on the boundaries of the fluid domain and its physical properties. The matrix $\{k_{ij}(t)\}$ completely characterizes the response to an arbitrary excitation. Apart from convergence of the (convolution) integrals the only assumption required in this is that the ship-fluid system behaves linearly. The input signals need not be linear.

As an example of the necessity for the representation given above, it can be noted that in the case of a captive model which is given a short 'pulse' disturbance and then returned to its original steady restrained condition, an unsteady fluid motion - visible especially in the disturbance of the free surface - and an associated force will persist thereafter, in principle ad infinitum.

On account of several investigations (see e.g. refs. [20, 23, 26, 30, 31]) it can be stated that the ship-fluid system is linear. In addition to the references mentioned a good survey on this point as well as a (comprehensive) description of character and behaviour of the linear ship-fluid system are given in ref. [34] . All (experimental) data indicate that this basic linearity assumption is a good working approximation for small to moderate displacements of real ship forms. Therefore it is hypothesized that the assumption of linearity of the ship-fluid system holds absolutely.

With regard to the fluid idealization the facts point in two directions. While it is practically sure that the restriction to a homogeneous, incompressible fluid, free from surface tension, is not a serious limitation, the viscosity leads to complications. On the one hand, in dealing with (ship) motions it is of great advantage and in most cases necessary to consider the water as inviscid ; it is a logic consequence of the validity of the linearization, for if the viscosity would have a great influence then the linearity would be impaired as well. On the other hand, flow seperation and consequent eddy formation are distinctly perceptible, especially with lateral ship motions. It makes itself primarily felt in additional damping and in a change in the hydrodynamic coefficients which couple the motions mutually.

Consistent with the hypothesis of linearity of the ship-fluid system it is assumed that $|f_i(t)|$ remains bounded. This assumption is closely linked up with the demand for stability of the linear system: if $f_i(t)$ is bounded in time, then $u_j(t)$ will be bounded in time as well.

$k_{ij}(\infty) = \lim_{t\to\infty} k_{ij}(t)$ need not necessarily equal zero; in a damped system which is not unstable $k_{ij}(\infty)$ can be a finite constant.

If the displacement or the rotation acts as output signal, in modes of motion without a restoring force (surge, sway and yaw) the i.r.f. will increase indefinitely as t tends to infinity; this implies that for these modes of motion the ship-fluid system cannot be considered as being stable. For the remaining modes of motion (heave, roll and pitch) it holds good that $k_{ij}(\infty) = 0$, and the ship-fluid system does behave stable.

Looking upon the velocity (or the acceleration) as output signal in modes of motion without a restoring force the i.r.f. will asymptotically approach to a certain finite value as t increases, whereas in the remaining modes of motion $k_{ij}(\infty) = 0$. In this latter case the ship-fluid system then behaves stable in all modes of motion.

As far as the ship motions are concerned therefore distinction can be made between:

a - ship motions with a restoring force (heave, roll and pitch motion),

and

b - ship motions without a restoring force (surge, sway and yaw motion).

Ad a: In this case the ship-fluid system always behaves stable, independent of the fact whether the displacement/rotation or the velocity or the acceleration is conceived of as output signal.

Eq. (6) now can be written as:

$$(6^a) \qquad x_j(t) = \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau)\, k^{*}_{ij}(t-\tau)\, d\tau =$$

$$= \sum_{i=1}^{6} \int_{0}^{\infty} k^{*}_{ij}(\tau)\, f_i(t-\tau)\, d\tau \;, \quad j = 1, 2, \ldots 6$$

(exclusive of j = 1, 2, 6),

where $k^{*}_{ij}(t)$ = i.r.f. based on the displacement/rotation as output signal.

Ad b: In this case the ship-fluid system only behaves stable if the velocity (or the acceleration) is considered to be the output signal. Displacements or rotations can be measured in a much easier way than velocities and accelerations. Besides the calculation of the velocity from displacements/rotations is more accurate than the calculation of the acceleration. Consequently a choice is made in favour of the velocity as output signal.
Eq. (6) then becomes:

$$(6^b) \qquad \dot{x}_j(t) = \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau)\, k_{ij}(t-\tau)\, d\tau =$$

$$= \sum_{i=1}^{6} \int_{0}^{\infty} k_{ij}(\tau)\, f_i(t-\tau)\, d\tau \qquad j = 1, 2, \ldots, 6 ,$$

where $k_{ij}(t)$ = i.r.f. based on the velocity as output signal.
On behalf of the generality of the following dissertation the velocity will be conceived of as output signal throughout.
For an elaborated definition and a further explanation of the concept 'stability' in case of the linear ship-fluid system is referred to Appendix I.
The i.r.f.'s $k^{*}_{ij}(t)$ and $k_{ij}(t)$ are related as follows. According to eq. (6^a) $x_j(t)$ is:

$$x_j(t) = \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau)\, k^{*}_{ij}(t-\tau)\, d\tau \; ;$$

farther it can be derived that:

$$\dot{x}_j(t) = \sum_{i=1}^{6} \frac{d}{dt} \int_{-\infty}^{t} f_i(\tau)\, k^*_{ij}(t-\tau)\, d\tau =$$

$$= \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau) \frac{d}{dt} k^*_{ij}(t-\tau)\, d\tau + \sum_{i=1}^{6} f_i(\tau)\, k^*_{ij}(0);$$

$k^*_{ij}(0) = 0$, so that:

$$\dot{x}_j(t) = \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau) \frac{d}{dt} k^*_{ij}(t-\tau)\, d\tau \quad ;$$

being equivalent with eq. (6^b) this expression yields:

$$k_{ij}(t) = \frac{d}{dt} k^*_{ij}(t). \tag{8}$$

Successively now the two respective cases of ship motions with and ship motions without a restoring force are dealt with. In regard to the behaviour of the i.r.f. at infinity the first case corresponds with $k_{ij}(\infty) = 0$, i.e. $\int_0^\infty |k_{ij}(t)|dt$ does exist, and the latter case corresponds with $k_{ij}(\infty) = \text{constant} \neq 0$, i.e. $\int_0^\infty |k_{ij}(t)|dt$ does not exist.

II.1.a: SHIP MOTIONS WITH A RESTORING FORCE: $k_{ij}(\infty) = 0$

It is supposed that $f_i(t)$ has the characteristic of a harmonic (force) excitation in the i-direction with form:

$$f_i(t) = f_{ia}\, e^{i(\omega t + \phi_i)} , \tag{9}$$

where f_{ia} = amplitude of the harmonic (force) excitation in the i-direction,

ω = circular frequency,

ϕ_i = phase angle of the harmonic (force) excitation in the i-direction,

$i = \sqrt{-1}$

Substitution of this expression into eq. (6^a) yields for $\dot{x}_j(t)$:

$$(10^a) \qquad \dot{x}_j(t) = f_{ia} \{K_{ij}^{(c)}(\omega) - i\,K_{ij}^{(s)}(\omega)\}\, e^{i(\omega t + \phi_i)} ,$$

where

$$(11^a) \qquad K_{ij}^{(c)}(\omega) = \int_{-\infty}^{\infty} k_{ij}(\tau)\, \cos(\omega\tau)\, d\tau$$

and

$$(11^b) \qquad K_{ij}^{(s)}(\omega) = \int_{-\infty}^{\infty} k_{ij}(\tau)\, \sin(\omega\tau)\, d\tau$$

represent the Fourier cosine transform and the Fourier sine transform of $k_{ij}(t)$, respectively (see refs. [39, 40, 41]). Hereby it should be borne in mind that $k_{ij}(t) = 0$ for $t \leqq 0$ (i.e. eqs.($7^{a,b}$)). The i.r.f. $k_{ij}(t)$ is related to $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$ by the inverse Fourier transforms (see refs.[39, 40, 41]):

$$(12^{a,b}) \qquad k_{ij}(t) = \frac{2}{\pi} \int_0^{\infty} K_{ij}^{(c)}(\omega)\, \cos(\omega t)\, d\omega =$$

$$= \frac{2}{\pi} \int_0^{\infty} K_{ij}^{(s)}(\omega)\, \sin(\omega t)\, d\omega.$$

The relation between $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$ is unique: if one of the two functions is known, then the other can be determined by means of eqs. ($12^{a,b}$) and ($11^{a,b}$).

An other way of representing eq. (10^{a}) is:

$$\frac{\dot{x}_j(t)}{f_i(t)} = K_{ij}(i\omega) ,$$

where $K_{ij}(i\omega)$ is a complex quantity given as:

$$K_{ij}(i\omega) = K_{ij}^{(c)}(\omega) - i\,K_{ij}^{(s)}(\omega) , \tag{13}$$

with $K_{ij}^{(c)}(\omega) = \mathrm{Re}\left[K_{ij}(i\omega)\right]$ and $K_{ij}^{(s)}(\omega) = -\mathrm{Im}\left[K_{ij}(i\omega)\right]$;

the symbolic notation Re [...] and Im [...] means 'real part of' and 'imaginary part of', respectively.

$K_{ij}(i\omega)$ represents the harmonic transfer function for the j-direction in response to a (harmonic force) excitation in the i-direction; or in other words, $K_{ij}(i\omega)$ is the frequency response function (f.r.f.). On account of eq. (13) and refs. [37 through 41] it can be stated in a more general way that the f.r.f. and the i.r.f. are related by a Fourier transform:

$$K_{ij}(i\omega) = \int_{-\infty}^{\infty} k_{ij}(\tau)\, e^{-i\omega\tau}\, d\tau, \tag{14^{a}}$$

or

$$k_{ij}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{ij}(i\omega)\, e^{i\omega t}\, d\omega. \tag{14^{b}}$$

By eq. (10^{a}) it can be seen that $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$ are the respective amplitudes of the in-phase and out-of-phase components of the response in the j-direction to a harmonic forcing function - with unit amplitude and circular frequency ω - in the i-direction. In this context eq. (10^{a}) can be written as:

(10^b) $$\dot{x}_j(t) = f_{ia}\sqrt{\{K_{ij}^{(c)}(\omega)\}^2 + \{K_{ij}^{(s)}(\omega)\}^2}\; e^{i\{\omega t + \phi_i + \theta_j(\omega)\}},$$

where $$\tan\{\theta_j(\omega)\} = \frac{K_{ij}^{(s)}(\omega)}{K_{ij}^{(c)}(\omega)}.$$

This expression shows that the response of the linear ship-fluid system to a harmonic (force) excitation with unit amplitude has the amplitude

$$\sqrt{\{K_{ij}^{(c)}(\omega)\}^2 + \{K_{ij}^{(s)}(\omega)\}^2}$$

and follows the excitation by the phase

$$\arctan\{K_{ij}^{(s)}(\omega) / K_{ij}^{(c)}(\omega)\}.$$

The behaviour of the linear ship-fluid system in the frequency domain can be described by (see refs. [33, 34]):

(15) $$\sum_{j=1}^{6}\{(m_{jk} + a_{jk})\ddot{x}_j + b_{jk}\dot{x}_j + c_{jk}x_j\} = f_k(t),$$

where
m_{jk} = inertia matrix (i.e. generalized mass) of the ship,
$a_{jk} = a_{jk}(\omega)$ = hydrodynamic coefficient of the mass term in the k-equation as a result of motion in the j-direction,
$b_{jk} = b_{jk}(\omega)$ = hydrodynamic coefficient of the damping force in the k-equation as a result of motion in the j-direction,
c_{jk} = hydrostatic restoring coefficient in the k-equation as a result of a static displacement in the j-direction at zero forward speed,

$f_k(t)$ = external exciting harmonic force upon the ship in the k-direction; $f_k(t) = 0$ for $k \neq i$.

For the case under consideration with $k_{ij}(\infty) = 0$ it holds good that $c_{jk} > 0$. By substitution of eqs. (10^a) and (9) into eq. (15) a system of equations can be formed for the unknown functions $a_{jk}(\omega)$ and $b_{jk}(\omega)$; the coefficients c_{jk} are supposed to be known, e.g. as being determined from static measurements. In order to determine the unknown hydrodynamic coefficients it is necessary to consider the responses to excitations in each of the modes of motion seperately. If the in-phase and out-of-phase components of the responses are seperated, then enough equations are obtained to determine the hydrodynamic coefficients.

N.B. Generally the following can be remarked. If the ship has a forward speed, then there are 72 unknown function $a_{jk}(\omega)$ and $b_{jk}(\omega)$, which formally all are present, except in those cases where the modes of motion are uncoupled. At zero forward speed it holds good that $a_{jk}(\omega) = a_{kj}(\omega)$ and $b_{jk}(\omega) = b_{kj}(\omega)$, so that in this latter case the number of unknown functions formally amounts to 42 (see further ref. [23]).

In principle the hydrodynamic coefficients can be determined from the set of i.r.f.'s $\{k_{ij}(t)\}$; therefore they contain no information which is not derivable from these functions.

The response for a given frequency, as determined by the pair of functions

$$\sqrt{\{K_{ij}^{(c)}(\omega)\}^2 + \{K_{ij}^{(s)}(\omega)\}^2} \quad \text{and} \quad \arctan\{K_{ij}^{(s)}(\omega)/K_{ij}^{(c)}(\omega)\} \text{ in eq. } (10^b)$$

or alternatively by the pair of functions $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$ in eq. (10^a), represents a mapping in the frequency domain of the unit response function, which is defined in the time domain. Since by means of eqs. ($10^{a,b}$) it is permitted to pass from either domain to the other , the two representations (in frequency and time domain) of the linear ship-fluid system are completely equivalent.

II.1.b: SHIP MOTIONS WITHOUT A RESTORING FORCE: $k_{ij}(\infty)$ = constant $\neq 0$

The integrals in eqs. (6^b), ($11^{a,b}$) and (14^a) have to be convergent. As $|f_i(t)|$ is supposed to be bounded in time, these conditions are fulfilled only if $\int_{-\infty}^{\infty} |k_{ij}(t)|dt$ does exist; considering the behaviour of the i.r.f. at infinity this should imply that $k_{ij}(\infty) = 0$.
However, the case in which $k_{ij}(t)$ approaches some non-zero but finite limit as t tends to infinity can be treated too.
If $\lim_{t\to\infty} k_{ij}(t) = k_{ij}(\infty)$ = constant $\neq 0$, the ordinary Fourier transform of $k_{ij}(t)$ does not exist. This difficulty can be overcome if in such a case use is made of the generalized function theory (see refs. [40, 41]).
The i.r.f. $k_{ij}(t)$ can be written as:

$$k_{ij}(t) = \{k_{ij}(t) - k_{ij}(\infty)\, U(t)\} + k_{ij}(\infty)\, U(t) \;,$$

where $\quad U(t)$ = unit step function = $\begin{cases} 0 \text{ for } t < 0 \\ \frac{1}{2} \text{ for } t = 0 \\ 1 \text{ for } t > 0 \;. \end{cases}$

The Fourier transform of $\{k_{ij}(t) - k_{ij}(\infty)\, U(t)\}$ does exist:

$$\int_{-\infty}^{\infty} \{k_{ij}(\tau) - k_{ij}(\infty)\, U(\tau)\}e^{-i\omega\tau}d\tau = -\frac{1}{i\omega}\, e^{-i\omega\tau}\{k_{ij}(\tau) - k_{ij}(\infty)\, U(\tau)\}\Big|_{-\infty}^{\infty} +$$

$$+ \frac{1}{i\omega}\int_{-\infty}^{\infty} \{\dot{k}_{ij}(\tau) - k_{ij}(\infty)\;\delta(\tau)\}e^{-i\omega\tau}d\tau =$$

$$= \frac{1}{i\omega}\int_{-\infty}^{\infty} \dot{k}_{ij}(\tau)e^{-i\omega\tau}d\tau - \frac{1}{i\omega}\, k_{ij}(\infty),$$

where $\quad \delta(t)$ = delta function or Dirac funtion.

For $t < 0$ is $k_{ij}(t) = \dot{k}_{ij}(t) = 0$, so that it can be written:

$$\int_{-\infty}^{\infty} \{k_{ij}(\tau) - k_{ij}(\infty)\ U(\tau)\} e^{-i\omega\tau}\ d\tau = \frac{1}{i\omega}\ \{ \int_{0}^{\infty} \dot{k}_{ij}(\tau) e^{-i\omega\tau} d\tau - k_{ij}(\infty)\} .$$

N.B. For $\omega = 0$ one obtains:

$$\lim_{\omega \to 0} \int_{0}^{\infty} \dot{k}_{ij}(\tau)\ e^{-i\omega\tau} d\tau = \int_{0}^{\infty} \dot{k}_{ij}(\tau)\ d\tau = k_{ij}(\infty) - k_{ij}(0) = k_{ij}(\infty) .$$

From the generalized function theory (see refs. [40, 41]) it is known that:

$$\int_{-\infty}^{\infty} U(\tau)\ e^{-i\omega\tau}\ d\tau = \frac{1}{i\omega} + \pi\,\delta(\omega) \quad .$$

The Fourier transform of $k_{ij}(t)$ then takes the following form:

$$K_{ij}(i\omega) = \int_{-\infty}^{\infty} k_{ij}(\tau) e^{-i\omega\tau}\ d\tau = \int_{0}^{\infty} k_{ij}(\tau)\ e^{-i\omega\tau}\ d\tau =$$

$$= \pi\ \delta(\omega)\ k_{ij}(\infty) + \frac{1}{i\omega} \int_{0}^{\infty} \dot{k}_{ij}(\tau)\ e^{-i\omega\tau} d\tau \quad .$$

This expression contains a singularity for $\omega = 0$; if this singularity is excluded $K_{ij}(i\omega)$ changes into:

$$K_{ij}(i\omega) = \frac{1}{i\omega} \int_{0}^{\infty} \dot{k}_{ij}(\tau)\ e^{-i\omega\tau} d\tau, \qquad \omega \neq 0,$$

which by means of eqs. (13) and ($11^{a,b}$) can be written as:

$$K_{ij}(i\omega) = \int_0^\infty k_{ij}(\tau)\cos(\omega\tau)\,d\tau - i\int_0^\infty k_{ij}(\tau)\sin(\omega\tau)\,d\tau =$$
$$= -\frac{i}{\omega}\left\{\int_0^\infty \dot{k}_{ij}(\tau)\cos(\omega\tau)d\tau - i\int_0^\infty \dot{k}_{ij}(\tau)\sin(\omega\tau)\,d\tau\right\},$$

or otherwise:

$$K_{ij}^{(c)}(\omega) - i\,K_{ij}^{(s)}(\omega) = -\frac{i}{\omega}\dot{K}_{ij}^{(c)}(\omega) - \frac{1}{\omega}\dot{K}_{ij}^{(s)}(\omega),$$

where $\dot{K}_{ij}^{(s)}(\omega)$ and $\dot{K}_{ij}^{(c)}(\omega)$ are the Fourier sine transform and the Fourier cosine transform of $\dot{k}_{ij}(t)$, respectively (see refs. [39, 40, 41]). From this it follows finally:

$$(16^{a,b})\qquad K_{ij}^{(c)}(\omega) = -\frac{1}{\omega}\dot{K}_{ij}^{(s)}(\omega),\quad K_{ij}^{(s)}(\omega) = \frac{1}{\omega}\dot{K}_{ij}^{(c)}(\omega)\qquad \text{with } \omega \neq 0.$$

If use is made of these expressions for $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$, then eqs. $(10^{a,b})$ and (13) keep their validity setting beforehand in eq. (9) $f_{ia} = 1$ and $\phi_i = 0$.
It has to be noted that $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$ in the case under consideration are no longer Fourier transforms of $k_{ij}(t)$, because these do not exist. Nevertheless an inverse Fourier transform is still possible. To that end the following expression is considered:

$$\int_0^\infty \{k_{ij}(\tau) - k_{ij}(\infty)\}\cos(\omega\tau)\,d\tau =$$
$$= \frac{1}{\omega}\{k_{ij}(\tau) - k_{ij}(\infty)\}\sin(\omega\tau)\Big|_0^\infty - \frac{1}{\omega}\int_0^\infty \dot{k}_{ij}(\tau)\sin(\omega\tau)d\tau =$$
$$= -\frac{1}{\omega}\dot{K}_{ij}^{(s)}(\omega) = K_{ij}^{(c)}(\omega),$$

or, $K_{ij}^{(c)}(\omega)$ is the Fourier cosine transform of $\{k_{ij}(t) - k_{ij}(\infty)\}$; for the inverse Fourier transform it can be written (see refs.

[39, 40, 41]):

$$(17) \qquad k_{ij}(t) = k_{ij}(\infty) + \frac{2}{\pi} \int_0^\infty K_{ij}^{(c)}(\omega) \cos(\omega t)\, d\omega.$$

When $k_{ij}(\infty) = 0$ eq. (17) is identical with eq. (12^a).
In an analogous way as above it can be considered:

$$\int_0^\infty \{k_{ij}(\tau) - k_{ij}(\infty)\} \sin(\omega\tau)\, d\tau =$$

$$= -\frac{1}{\omega} \{k_{ij}(\tau) - k_{ij}(\infty)\} \cos(\omega\tau)\Big|_0^\infty + \frac{1}{\omega} \int_0^\infty \dot{k}_{ij}(\tau) \cos(\omega\tau)\, d\tau =$$

$$= -\frac{1}{\omega} \{ k_{ij}(\infty) + \frac{1}{\omega} \int_0^\infty \dot{k}_{ij}(\tau) \cos(\omega\tau)\, d\tau = \frac{1}{\omega} \{ \dot{K}_{ij}^{(c)}(\omega) - k_{ij}(\infty)\} =$$

$$= K_{ij}^{(s)}(\omega) - \frac{1}{\omega} k_{ij}(\infty),$$

or, $K_{ij}^{(s)}(\omega) - \frac{1}{\omega} k_{ij}(\infty)$ is the Fourier sine transform of $\{k_{ij}(t) - k_{ij}(\infty)\}$; for the inverse Fourier transform it holds good that (see refs. [39, 40, 41]):

$$k_{ij}(t) = k_{ij}(\infty) + \frac{2}{\pi} \int_0^\infty \{K_{ij}^{(s)}(\omega) - \frac{1}{\omega} k_{ij}(\infty)\} \sin(\omega t)\, d\omega =$$

$$= k_{ij}(\infty) \{1 - \frac{2}{\pi} \int_0^\infty \frac{\sin(\omega t)}{\omega}\, d\omega\} + \frac{2}{\pi} \int_0^\infty K_{ij}^{(s)}(\omega) \sin(\omega t)\, d\omega =$$

$$= \frac{2}{\pi} \int_0^\infty K_{ij}^{(s)}(\omega) \sin(\omega t)\, d\omega,$$

since $\displaystyle\int_0^\infty \frac{\sin(\omega t)}{\omega}\, d\omega = \frac{\pi}{2}$ for $t > 0$.

Therefore, eq. (12^b) holds generally even when $k_{ij}(\infty)$ = constant = $\neq 0$.
If $K_{ij}^{(c)}(\omega)$ and $K_{ij}^{(s)}(\omega)$ are known, it is not difficult to determine whether $k_{ij}(\infty)$ equals zero or not. According to eq. (12^b) $k_{ij}(\infty)$ can be written as:

$$k_{ij}(\infty) = \lim_{t\to\infty} \frac{2}{\pi} \int_0^\infty K_{ij}^{(s)}(\omega)\ \sin(\omega t)\ d\omega.$$

Substitution of eq. (16^b) into this expression yields:

$$k_{ij}(\omega) = \lim_{t\to\infty} \frac{2}{\pi} \int_0^\infty \dot{K}_{ij}^{(c)}(\omega)\ \frac{\sin(\omega t)}{\omega}\ d\omega.$$

If a function $f(\omega)$ on an interval $(0,a)$ satisfies the Dirichlet conditions, then it holds good for positive values of a that (see refs. [39, 42]):

$$\lim_{t\to\infty} \int_0^a f(\omega)\ \frac{\sin(\omega t)}{\omega}\ d\omega = \frac{1}{2}\pi f(0^+)\ , \qquad a > 0.$$

Making use of this theorem $k_{ij}(\infty)$ changes into:

(18) $$k_{ij}(\infty) = \dot{K}_{ij}^{(c)}(0^{(+)}) = \lim_{\omega\to 0} \omega\ K_{ij}^{(s)}(\omega).$$

The complete set of i.r.f.'s forms a so-called i.r.f. matrix, which in principle can be determined experimentally.

For the above case with $k_{ij}(\infty)$ = constant $\neq 0$ it applies that in eq. (15) - i.e. the description in the frequency domain of the linear ship-fluid system - $c_{jk} = 0$.

II.2.: THE 'IMPULSE RESPONSE FUNCTION'-TECHNIQUE FOR THE CASE OF UNCOUPLED SHIP MOTIONS

A special case arises when the modes of motion of the ship are uncoupled.
For uncoupled ship motions eq. (6^b) can be written as:

(19) $$\dot{x}_i(t) = \int_{-\infty}^{t} f_i(\tau)\; k_{ii}(t-\tau)\; d\tau = \int_{0}^{\infty} k_{ii}(\tau)\; f_i(t-\tau)\; d\tau, \qquad i = 1, 2, \ldots 6.$$

This description of the linear ship-fluid system in the time domain is equivalent to its description in the frequency domain according to (see refs. [37, 38]):

(20) $$\mathcal{F}\{\dot{x}_i(t)\} = \mathcal{F}\{f_i(t)\}\;\; \mathcal{F}\{k_{ii}(t)\} \;,$$

where $$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(\tau)\; e^{-i\omega\tau}\; d\tau$$

represents the Fourier transform of the function f(t). Eq. (20) can be derived by taking the Fourier transform of eq. (19). Eq. (20) is only then a meaningful expression, if the respective Fourier transforms of $\dot{x}_i(t)$, $f_i(t)$ and $k_{ii}(t)$ in a general sense - i.e. thinking in terms of the generalized function theory (see refs. [40, 41]) - do exist.

N.B. Naturally it is also possible to take the Laplace transform of eq. (19) - this yields a similar expression as eq. (20) - and, subsequently, to make use of the Laplace transforms of $\dot{x}_i(t)$, $f_i(t)$ and $k_{ii}(t)$.

Generally the Fourier transforms in eq. (20) are complex functions with real and imaginary parts, which both depend on the circular frequency ω.

On account of eq. (13) - whether or not combined with eqs. ($16^{a,b}$) - and of refs. [37, 38] it can be stated generally that:

$$K_{ii}(i\omega) = \mathcal{F}\{k_{ii}(t)\} \quad ,$$

in other words, the f.r.f. and the i.r.f. are related by a Fourier transform:

$$(21^a) \qquad K_{ii}(i\omega) = \int_{-\infty}^{\infty} k_{ii}(\tau)\, e^{-i\omega\tau}\, d\tau$$

or

$$(21^b) \qquad k_{ii}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{ii}(i\omega)\, e^{i\omega t}\, d\omega \quad .$$

In accordance with the definition of the i.r.f., $k_{ii}(t)$ is a real function of t for which it holds good that:

$$(22) \qquad k_{ii}(t) = 0 \qquad \text{for} \qquad t \leqq 0.$$

With regard to eq. (21^a) and eq. (21^b) (again) distinction has to be made between ship motions with a restoring force and ship motions without a restoring force; these two cases correspond with $k_{ii}(\infty) = 0$ and $k_{ii}(\infty) = \text{constant} \neq 0$, respectively.

II.2.a: UNCOUPLED SHIP MOTIONS WITH A RESTORING FORCE: $k_{ii}(\infty) = 0$

In this case the ordinary Fourier transform of $k_{ii}(t)$ does exist and eq. (21^a) remains valid. The f.r.f. $K_{ii}(i\omega)$ can be written as:

$$(23) \qquad K_{ii}(i\omega) = \int_{0}^{\infty} k_{ii}(\tau)\, e^{-i\omega\tau}\, d\tau = \text{Re}\left[K_{ii}(i\omega)\right] + i\,\text{Im}\left[K_{ii}(i\omega)\right] \quad ,$$

where $$\mathrm{Re}\left[K_{ii}(i\omega)\right] = K_{ii}^{(c)}(\omega) = \int_0^\infty k_{ii}(\tau)\cos(\omega\tau)\,d\tau =$$

$$= \text{even function of } \omega,$$

$$\text{i.e. } K_{ii}^{(c)}(\omega) = K_{ii}^{(c)}(-\omega),$$

$$\mathrm{Im}\left[K_{ii}(i\omega)\right] = -K_{ii}^{(s)}(\omega) = -\int_0^\infty k_{ii}(\tau)\sin(\omega\tau)\,d\tau =$$

$$= \text{odd function of } \omega,$$

$$\text{i.e. } K_{ii}^{(s)}(\omega) = -K_{ii}^{(s)}(-\omega).$$

With these expressions eq. (21^b) becomes:

$$k_{ii}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{K_{ii}^{(c)}(\omega)\cos(\omega t) + K_{ii}^{(s)}(\omega)\sin(\omega t)\}\,d\omega +$$

$$+ \frac{i}{2\pi}\int_{-\infty}^{\infty}\{K_{ii}^{(c)}(\omega)\sin(\omega t) - K_{ii}^{(s)}(\omega)\cos(\omega t)\}d\omega;$$

$k_{ii}(t)$ is a real function of t, $K_{ii}^{(c)}(\omega)$ and $\cos(\omega t)$ are even functions of ω, and $K_{ii}^{(s)}(\omega)$ and $\sin(\omega t)$ are odd functions of ω; therefore:

$$\int_{-\infty}^{\infty}K_{ii}^{(c)}(\omega)\sin(\omega t)\,d\omega = \int_{-\infty}^{\infty}K_{ii}^{(s)}(\omega)\cos(\omega t)\,d\omega = 0,$$

so that,

$$k_{ii}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}K_{ii}^{(c)}(\omega)\cos(\omega t)\,d\omega +$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}K_{ii}^{(s)}(\omega)\sin(\omega t)\,d\omega =$$

$$= \frac{1}{\pi} \int_0^\infty K_{ii}^{(c)}(\omega)\cos(\omega t)\, d\omega +$$

$$+ \frac{1}{\pi} \int_0^\infty K_{ii}^{(s)}(\omega)\sin(\omega t)\, d\omega.$$

Since the first term in the right-hand member of this expression is an even function of t and the second term an odd function, while at the same time $k_{ii}(t) = 0$ for $t < 0$, it must hold good that:

$$(24^{a,b}) \qquad k_{ii}(t) = \frac{2}{\pi} \int_0^\infty K_{ii}^{(c)}(\omega)\cos(\omega t)\, d\omega =$$

$$= \frac{2}{\pi} \int_0^\infty K_{ii}^{(s)}(\omega)\sin(\omega t)\, d\omega.$$

II.2.b: UNCOUPLED SHIP MOTIONS WITHOUT A RESTORING FORCE: $k_{ii}(\infty) = \text{constant} \neq 0$

When $k_{ii}(\infty) = \text{constant} \neq 0$ the ordinary Fourier transform of $k_{ii}(t)$ does not exist. Using the generalized function theory it can be derived that (see Section II.1.b):

$$(25^{a}) \qquad K_{ii}(i\omega) = \pi\, \delta(\omega)\, k_{ii}(\infty) + \frac{1}{i\omega} \int_0^\infty \dot{k}_{ii}(\tau)\, e^{-i\omega\tau}\, d\tau,$$

or, with $\omega \neq 0$:

$$(25^{b}) \qquad K_{ii}(i\omega) = \frac{1}{i\omega} \int_0^\infty \dot{k}_{ii}(\tau)\, e^{-i\omega\tau}\, d\tau =$$

$$= \mathrm{Re}\left[K_{ii}(i\omega)\right] + i\, \mathrm{Im}\left[K_{ii}(i\omega)\right],$$

where
$$\mathrm{Re}\left[K_{ii}(i\omega)\right] = K_{ii}^{(c)}(\omega) = -\frac{1}{\omega}\dot{K}_{ii}^{(c)}(\omega) =$$

$$= -\frac{1}{\omega}\int_0^\infty \dot{k}_{ii}(\tau)\sin(\omega\tau)\,d\tau =$$

$$= \text{even function of } \omega,$$

$$\mathrm{Im}\left[K_{ii}(i\omega)\right] = -K_{ii}^{(s)}(\omega) = -\frac{1}{\omega}\dot{K}_{ii}^{(c)}(\omega) =$$

$$= -\frac{1}{\omega}\int_0^\infty \dot{k}_{ii}(\tau)\cos(\omega\tau)\,d\tau =$$

$$= \text{odd function of } \omega.$$

Since $\dot{k}_{ii}(t)$ again is a real function of t with $\dot{k}_{ii}(t) = 0$ for $t < 0$, and since it holds good that

$$\omega\,\mathrm{Re}\left[K_{ii}(i\omega)\right] = \omega K_{ii}^{(c)}(\omega) = -\int_0^\infty \dot{k}_{ii}(\tau)\sin(\omega\tau)\,d\tau =$$

$$= \text{odd function of } \omega,$$

$$-\omega\,\mathrm{Im}\left[K_{ii}(i\omega)\right] = \omega K_{ii}^{(s)}(\omega) = \int_0^\infty \dot{k}_{ii}(\tau)\cos(\omega\tau)\,d\tau =$$

$$= \text{even function of } \omega,$$

the following expression(s) can be derived in a completely analogous way as in Section II.2.a:

$$(26^{a,b})\qquad \dot{k}_{ii}(t) = \frac{2}{\pi}\int_0^\infty \omega K_{ii}^{(s)}(\omega)\cos(\omega t)\,d\omega =$$
$$= -\frac{2}{\pi}\int_0^\infty K_{ii}^{(c)}(\omega)\sin(\omega t)\,d\omega.$$

Integration of eq. (26^a) with respect to t yields:

$$k_{ii}(t) = \frac{2}{\pi} \int_0^\infty K_{ii}^{(s)}(\omega) \sin(\omega t)\, d\omega + \text{'constant of integration'},$$

where 'constant of integration' = 0 since $k_{ii}(0) = 0$.
Similarly integration of eq. (26^b) with respect to t gives:

$$k_{ii}(t) = \frac{2}{\pi} \int_0^\infty K_{ii}^{(c)}(\omega) \cos(\omega t)\, d\omega + \text{'constant of integration'},$$

supposing that the integral

$$\int_0^\infty \mathrm{Re}\, [K_{ii}(i\omega)]\, d\omega = \int_0^\infty K_{ii}^{(c)}(\omega)\, d\omega$$

converges absolutely, it holds an account of the lemma of Riemann-Lebesgue (see refs. [42, 43]):

$$\lim_{t\to\infty} \int_0^\infty K_{ii}^{(c)}(\omega) \cos(\omega t)\, d\omega = 0,$$

so that

$$\lim_{t\to\infty} k_{ii}(t) = k_{ii}(\infty) = \text{constant}.$$

Consequently $k_{ii}(t)$ can be written as:

$$(27^{a,b}) \qquad k_{ii}(t) = k_{ii}(\infty) + \frac{2}{\pi} \int_0^\infty K_{ii}^{(c)}(\omega) \cos(\omega t)\, d\omega =$$

$$= \frac{2}{\pi} \int_0^\infty K_{ii}^{(s)}(\omega) \sin(\omega t)\, d\omega,$$

where - according to eq. (18) - $k_{ii}(\infty)$ has the form:

$$k_{ii}(\infty) = \dot{K}_{ii}^{(c)}(0^{(+)}) = \lim_{\omega \to 0} \omega K_{ii}^{(s)}(\omega). \tag{28}$$

For a physical interpretation and an explanation of the behaviour of the i.r.f. at infinity in case of uncoupled ship motions without a restoring force is referred to Appendix II.

II.3: THE IMPULSE RESPONSE FUNCTION FOR UNCOUPLED HORIZONTAL SHIP MOTIONS AT ZERO FORWARD SPEED, AS DETERMINED FROM THE HYDRODYNAMIC COEFFICIENTS: $k_{ii}(t)$ for $i = 1, 2, 6$.

From the foregoing it is obvious that the main interest concerns the i.r.f.: for the determination of transient ship motions the i.r.f. must be known.
In this section the i.r.f. is determined for the (simplified) case of uncoupled ship motions in the horizontal plane. This implies that only the surge, sway and yaw motions are taken into account; so heaving, rolling and pitching are neglected. Further the ship's forward speed is supposed to be zero.

In the case of uncoupled ship motions the description in the frequency domain of the linear ship-fluid system reads (see eq. (15)):

$$\{m_{ii} + a_{ii}(\omega)\}\ddot{x}_i + b_{ii}(\omega)\,\dot{x}_i + c_{ii}x_i = f_i(t), \tag{29}$$

$$i = 1, 2, \ldots 6,$$

where $c_{ii} = 0$ for $i = 1, 2, 6.$

Since in the following only motions in the horizontal plane - i.e. motions without restoring force - are considered, eq. (29) takes the form:

(30) $$\{m_{ii} + a_{ii}(\omega)\}\ddot{x}_i + b_{ii}(\omega)\,\dot{x}_i = f_i(t),\ i = 1,\ 2,\ 6,$$

while with respect to the i.r.f. it applies that:

$$k_{ii}(\infty) = \text{constant} \neq 0.$$

For the f.r.f. $K_{ii}(i\omega)$ it then can be written (see refs. [37, 38]):

(31) $$K_{ii}(i\omega) = \frac{1}{\{m_{ii} + a_{ii}(\omega)\}i\omega + b_{ii}(\omega)},$$

with (see eq. (25^b)):

(32^a) $$\mathrm{Re}\left[K_{ii}(i\omega)\right] = K_{ii}^{(c)}(\omega) = \frac{b_{ii}(\omega)}{\{m_{ii} + a_{ii}(\omega)\}^2\omega^2 + b_{ii}^2(\omega)}$$

and

(32^b) $$\mathrm{Im}\left[K_{ii}(i\omega)\right] = -\,K_{ii}^{(s)}(\omega) = \frac{-\omega\{m_{ii} + a_{ii}(\omega)\}}{\{m_{ii} + a_{ii}(\omega)\}^2\omega^2 + b_{ii}^2(\omega)}.$$

From eqs. ($32^{a,b}$) and ($27^{a,b}$) it follows that the i.r.f.'s for uncoupled horizontal ship motions (at zero speed of advance) can be determined if the hydrodynamic coefficient $a_{ii}(\omega)$ and $b_{ii}(\omega)$ are known functions of ω.

In the case under consideration the uncoupling of the ship motions is materialized by schematizing the ship to a rigid prismatic body with a rectangular cross-section and a symmetrical distribution of mass. Besides, in case of shallow water the uncoupling of the motions requires a horizontal bottom. When a closed wall is present the ship motions are only uncoupled if one of the horizontal body axes of the (schematized) ship is parallel to the wall.
From a physical point of view the hydrodynamic coefficients $a_{ii}(\omega)$ and $b_{ii}(\omega)$ must be even functions of ω; this is affirmed

mathematically by eqs. ($32^{a,b}$) where $K_{ii}^{(c)}(\omega)$ is an even function of ω and $K_{ii}^{(s)}(\omega)$ an odd function. For (very) small values of ω $a_{ii}(\omega)$ and $b_{ii}(\omega)$ - in case of surge, sway and yaw motions at zero speed of advance - then can be represented by:

$$\left.\begin{aligned} &(33^a) \quad a_{ii}(\omega) = a_{ii}(0) + a_{ii}^{(2)}\omega^2 + \text{terms of higher order in } \omega \\ &(33^b) \quad b_{ii}(\omega) = \qquad\qquad b_{ii}^{(2)}\omega^2 + \text{terms of higher order in } \omega \end{aligned}\right\} \text{for } \omega\to 0,$$

respectively,

where $a_{ii}^{(n)}$ = coefficient of term with order n in power series development for $a_{ii}(\omega)$,

$b_{ii}^{(n)}$ = coefficient of term with order n in power series development for $b_{ii}(\omega)$.

In case of (very) great values of ω $a_{ii}(\omega)$ and $b_{ii}(\omega)$ can be approximated by (see Appendix III):

$$\left.\begin{aligned} &(34^a) \quad a_{ii}(\omega) \simeq a_{ii}(\infty) = p_i \\ &(34^b) \quad b_{ii}(\omega) = \frac{q_i}{\omega^3} \end{aligned}\right\} \text{for } \omega \to \infty,$$

respectively,

where p_i = constant in approximative expression for $a_{ii}(\omega)$ in case $\omega \to \infty$,

q_i = constant in approximative expression for $b_{ii}(\omega)$ in case $\omega \to \infty$;

p_i is dependent on the mode of motion as well as the water depth, q_i is also dependent on the mode of motion but independent of the water depth. For ship motions on (shallow) water with unrestricted horizontal dimensions eqs. ($33^{a,b}$) and ($34^{a,b}$) apply in any case (see ref. [30] and Appendix III). When a closed vertical wall is present, which is parallel to one of the horizontal body axes of the ship, the validity of eqs. ($33^{a,b}$) and (34^a) seems to be

confirmed by ref. [31].
Starting from eq. (28) and making use of eqs. (32^b) and ($33^{a,b}$) $k_{ii}(\infty)$ can be written as:

$$k_{ii}(\infty) = \lim_{\omega\to 0} \omega K_{ii}^{(s)}(\omega) = \lim_{\omega\to 0} \frac{\omega^2\{m_{ii} + a_{ii}(\omega)\}}{\{m_{ii} + a_{ii}(\omega)\}^2 \omega^2 + b_{ii}^2(\omega)} =$$

$$= \lim_{\omega\to 0} \frac{\omega^2\{m_{ii} + a_{ii}(0) + a_{ii}^{(2)}\omega^2 + \dots\}}{\{m_{ii} + a_{ii}(0) + a_{ii}^{(2)}\omega^2 + \dots\}^2\omega^2 + \{b_{ii}^{(2)}\omega^2 + \dots\}^2} =$$

$$= \frac{1}{m_{ii} + a_{ii}(0)} \; .$$

The fact that this expression for $k_{ii}(\infty)$ is independent of b_{ii}, is caused by the parabolic behaviour of $b_{ii}(\omega)$ near by the point $\omega=0$.
N.B. Since it holds good that

$$k_{ii}(\infty) = \frac{1}{m_{ii} + a_{ii}(0)} = \text{constant} \neq 0,$$

the linear ship-fluid system indeed does behave stable in the case under consideration (see Appendix I).

With $k_{ii}(\infty)$ known, substitution of eqs. ($32^{a,b}$) into eqs. ($27^{a,b}$) yields the following expression(s) for $k_{ii}(t)$:

$$(35^a) \quad k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} + \frac{2}{\pi}\int_0^\infty \frac{b_{ii}(\omega)\cos(\omega t)}{\{m_{ii} + a_{ii}(\omega)\}^2\omega^2 + b_{ii}^2(\omega)}\, d\omega =$$

$$(35^b) \quad = \frac{2}{\pi}\int_0^\infty \frac{\omega\{m_{ii} + a_{ii}(\omega)\}\sin(\omega t)}{\{m_{ii} + a_{ii}(\omega)\}^2\omega^2 + b_{ii}^2(\omega)}\, d\omega$$

$\Bigg\}$ for $t > 0$;

(22) $$k_{ii}(t) = 0 \quad \text{for } t \leqq 0.$$

N.B. If the hydrodynamic coefficients $a_{ii}(\omega)$ and $b_{ii}(\omega)$ were constants - i.e. quantities independent of the circular frequency ω - then eq. (35^a) can be solved analytically, yielding:

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}}\{1 + e^{-t\frac{b_{ii}}{m_{ii} + a_{ii}}}\} \quad \text{for } t > 0,$$

where a_{ii} and b_{ii} have to be conceived of as constant quantities. By means of eqs. (9) and (10^b) it can be obtained from eq. (30):

(36^a) $$m_{ii} + a_{ii}(\omega) = \frac{1}{\omega}\frac{K_{ii}^{(s)}(\omega)}{\{K_{ii}^{(c)}(\omega)\}^2 + \{K_{ii}^{(s)}(\omega)\}^2} \quad ,$$

(36^b) $$b_{ii}(\omega) = \frac{K_{ii}^{(c)}(\omega)}{\{K_{ii}^{(c)}(\omega)\}^2 + \{K_{ii}^{(s)}(\omega)\}^2} \quad ;$$

these same expressions can also be derived from eqs. (32^a) and (32^b).

N.B. Using eq. (32^a) combined with eqs. ($33^{a,b}$) and ($34^{a,b}$) it can be shown in a simple way that the integral

$$\int_0^\infty K_{ii}^{(c)}(\omega)\, d\omega$$

indeed converges absolutely.

From the foregoing it is obvious that the derivation of an expression for $k_{ii}(t)$ - even in case $k_{ii}(\infty)$ = constant $\neq 0$ - does not present any problem, if use is made of eq. (24^b): for, eq. (24^b) is identical to eq. (27^b). For the case of uncoupled horizontal ship motions (with zero speed of advance) eq. (24^b) as well as eq. (27^b) lead directly to the expression for $k_{ii}(t)$ as given in eq. (35^b).

In addition to the method already dealt with, there are two further methods to derive an expression for $k_{ii}(t)$ as given in eq. (35^a). These methods, in which use is made of Laplace transforms, are not specifically different but they are more direct. For an explanation in this is referred to Appendix IV.

II.4: EXAMPLES OF THE IMPULSE RESPONSE FUNCTION IN CASE OF SHALLOW WATER WITH UNRESTRICTED HORIZONTAL DIMENSIONS: $k_{ii}(t)$ for i = 2, 6

As pointed out already the determination of transient ship motions requires knowledge of the behaviour of the i.r.f.'s. With regard to the question whether the i.r.f.'s have to be determined theoretically or experimentally, in general the following can be remarked.

The respective descriptions of the linear ship-fluid system in the time domain and the frequency domain are completely equivalent. Both methods of description can be used in order to define the response to transient disturbances; there is no specific advantage attached to either of them. If the linear ship-fluid system has been formulated mathematically the i.r.f.'s or the f.r.f.'s can be evaluated, but if this is not possible they can also be determined in an experimental way.

The f.r.f.'s can be determined experimentally using a harmonically varying input signal. The measured output signal contains only one single frequency due to the linearity of the

ship-fluid system. Therefore it is sufficiently characterized by its amplitude and phase, which are represented in the f.r.f. in a complex way.
Since the i.r.f. and the f.r.f. are related by means of a Fourier transform, a mere determination of the f.r.f. is sufficient. Direct determination of an i.r.f., however, would be far more efficient than direct determination of a f.r.f. In the first case a few experiments, using a pulse or/and an arbitrary function of time, are sufficient, whereas in the second case many tests have to be carried out in order to find the f.r.f. over a sufficiently long interval of the frequency. In this context, by way of example, refs. [44] and [45] may be mentioned: in ref. [44] the f.r.f.'s for heave and pitch are determined by means of transient (force) pulse tests, in ref. [45] the i.r.f.'s for sway and yaw are calculated from measured f.r.f.'s; both references concern mainly ships with non-zero forward speed. Compared with transient pulse tests experiments to determine f.r.f.'s are much easier, since the pulse technique presents more specific problems and demands a higher degree of accuracy of the measuring equipment. For these reasons the choice in favour of a determination of the f.r.f. - what actually amounts to direct determination of the hydrodynamic coefficients as functions of the frequency - is obvious. Consequently, if the hydrodynamic coefficients are known along a frequency range which is sufficiently large, then the corresponding i.r.f.'s can be determined making use of the expressions derived in the preceding section(s).

In ref. [30] the hydrodynamic coefficients of a ship (model) were determined in case of swaying and yawing at zero speed of advance. The water was calm (no waves, no current) and shallow and had relatively large horizontal dimensions; the bottom was horizontal. Farther, the ship (model) was schematized to a rigid prismatic body with a rectangular cross-section and a symmetrical distribution of mass (see also Section II.3.a), so that any coupling between sway and yaw motion did not exist.

The results for $a_{ii}(\omega)$ and $b_{ii}(\omega)$ in case i = 2, 6 as presented in ref. [30] now will be used for the numerical calculation of the two i.r.f.'s $k_{22}(t)$ and $k_{66}(t)$.
The main particulars of the schematized ship used in the model are given in Table 1.

Table 1. Main particulars of ship model.

length (on the water-line)	L	2.438	m
beam	B	0.375	m
draught	D	0.150	m
volume of displacement	L.B.D	0.1371	m^3
area of cross-section	B.D	0.056	m^2
water-line area	L.B	0.924	m^2
lateral plane area	L.D	0.366	m^2
block coefficient		1.000	
centre of gravity (with respect to frame 10)		0	m
centre of gravity in height (with respect to keel point)		0.140	m
mass for horizontal (surge and sway) motion	m_{11}, m_{22}	137.24	kg
mass-moment of inertia around Gz-axis	m_{66}	50.99	kg m^2
radius of gyration with respect to Gz-axis		0.610	m

Since shallowness of the water is of dominant importance for the hydrodynamic coefficients, two water depths were chosen, viz. h = 0.200 m and h = 0.175 m.
Farther in the following is:

ρ = specific mass density of fluid = 1000 kg m^{-3},

g = acceleration due to gravity = 9.81 m s^{-2}.

In ref. [30] the hydrodynamic coefficients were determined both in an experimental and in an analytical way. As a consequence of the restricted possibilities of the experimental facilities the hydrodynamic coefficients could only be measured in a limited frequency range. Therefore an analytical determination of the hydrodynamic coefficients was necessary, not only to check the measured results but also to obtain information concerning $a_{ii}(\omega)$ and $b_{ii}(\omega)$ (i = 2, 6) along a sufficiently long frequency range.
As in ref. [30] the theoretical results for the hydrodynamic coefficients were derived using strip theory (i.e. a two-dimensional approach), three-dimensional effects, such as the circulation around 'bow' and 'stern' cannot be taken into account. This is the main cause of the discrepancy in ref. [30] between the theoretical and the experimental results for the hydrodynamic coefficients in the lower frequency range. In case of higher circular frequencies the theory is sufficiently accurate - also on shallow water - to determine the hydrodynamic coefficients in a (two-dimensional) stripwise manner (see further ref. [30] and Appendix III).
Since the hydrodynamic coefficients for the lower circular frequencies have a relatively greater influence on the behaviour of the i.r.f. than those for the higher circular frequencies, especially in the lower frequency range $a_{ii}(\omega)$ and $b_{ii}(\omega)$ (i = 2, 6) must be known as accurate as possible.
The respective i.r.f.'s for the sway motion and the yaw motion are calculated making use of eq. (35^a). For an outline and an elucidation of the method used for the numerical calculation of eq. (35^a) reference is made to Appendix V.

II.4.a: THE IMPULSE RESPONSE FUNCTION FOR THE SWAY MOTION: $k_{22}(t)$

The hydrodynamic coefficients for the case of pure swaying at zero forward speed, as determined theoretically and experimentally in ref. [30] , are given in dimensionless form ($a_{22}(\rho LBD)^{-1}$, $b_{22}(\rho LBD)^{-1}(B/g)^{\frac{1}{2}}$ in figs. $2^{a,b}$ and $3^{a,b}$ as functions of the dimensionless circular frequency $\omega(B/g)^{\frac{1}{2}}$ with the dimensionless water depth h/D as a parameter. In these figs. the hydrodynamic coefficients for the sway motion are represented for that part of the frequency range, for which also experimental results are available; the hydrodynamic coefficients in case of higher circular frequencies are given in ref. [30]. The figs. $2^{a,b}$ and $3^{a,b}$ show that in case of low circular frequencies a discrepancy exists between the theoretical (i.e. two-dimensional) and the experimental (i.e. three-dimensional) results for the hydrodynamic sway coefficients. As a consequence, for the low circular frequencies the hydrodynamic sway coefficients calculated two-dimensionally have been adapted to the (three-dimensional) experimental values; for the higher frequencies they are maintained. To avoid a possible non-linear distortion of the f.r.f. (i.e. the hydrodynamic coefficients) this local adaptation of theory to experiment is based on the smallest amplitudes of the harmonically oscillating sway motion.

Starting from the hydrodynamic coefficients $a_{22}(\omega)$ and $b_{22}(\omega)$ as given in figs $2^{a,b}$ and $3^{a,b}$ and farther in ref. [30] , the i.r.f. for the sway motion $k_{22}(t)$ can then be calculated making use of eq. (35^a) (see also Appendix V). The results are presented in dimensionless form ($\rho LBD\ k_{22}$) in figs. 4 and 5 as functions of the dimensionless time $t(g/B)^{\frac{1}{2}}$ with the dimensionless water depth h/D as a parameter. The figs. 4 and 5 each show three curves:

- the dot and dash line represents $k_{22}(t)$ as calculated from hydrodynamic coefficients determined theoretically (i.e. two-dimensionally) along the whole frequency range;
- the full line represents $k_{22}(t)$ as calculated from hydrodynamic

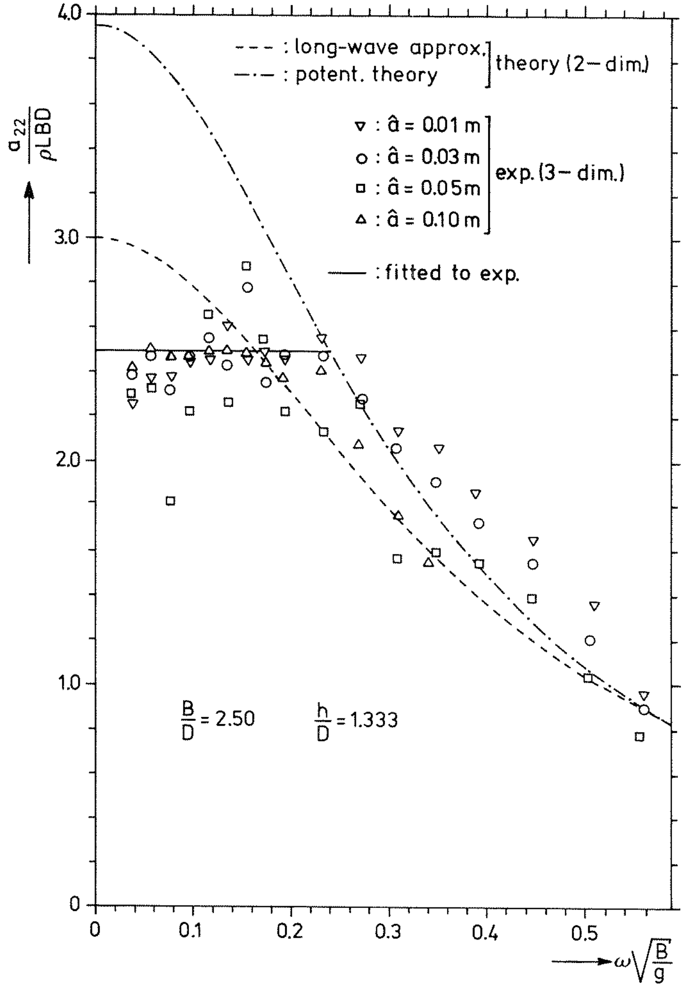

Fig. 2[a] - Added mass for swaying motion ($h/D = 1.333$).

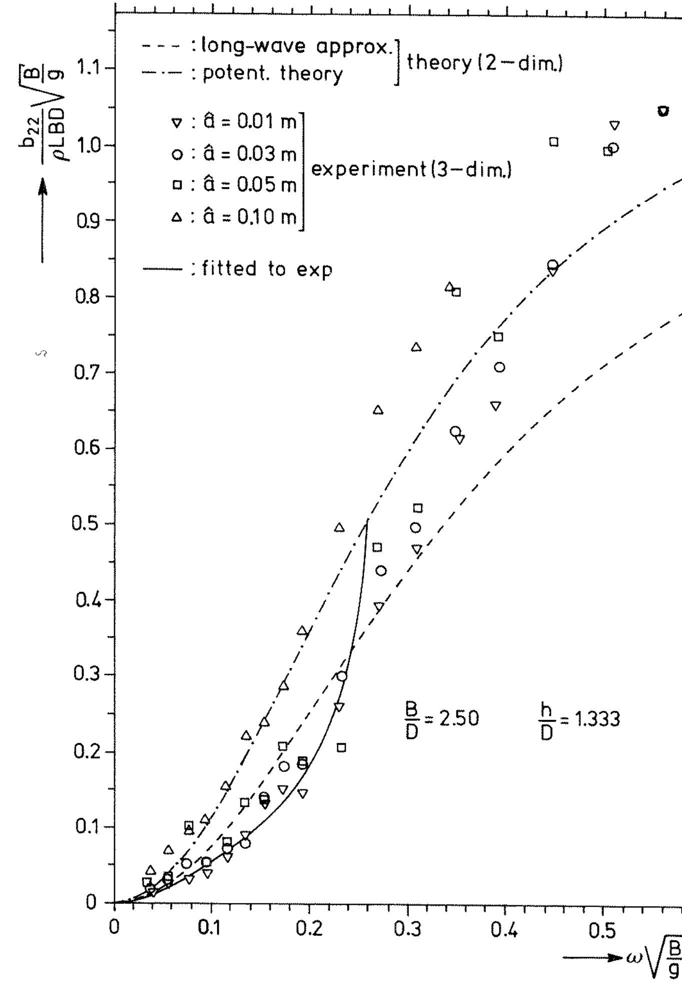

Fig. 2[b] - Sway damping force coefficient ($h/D = 1.333$).

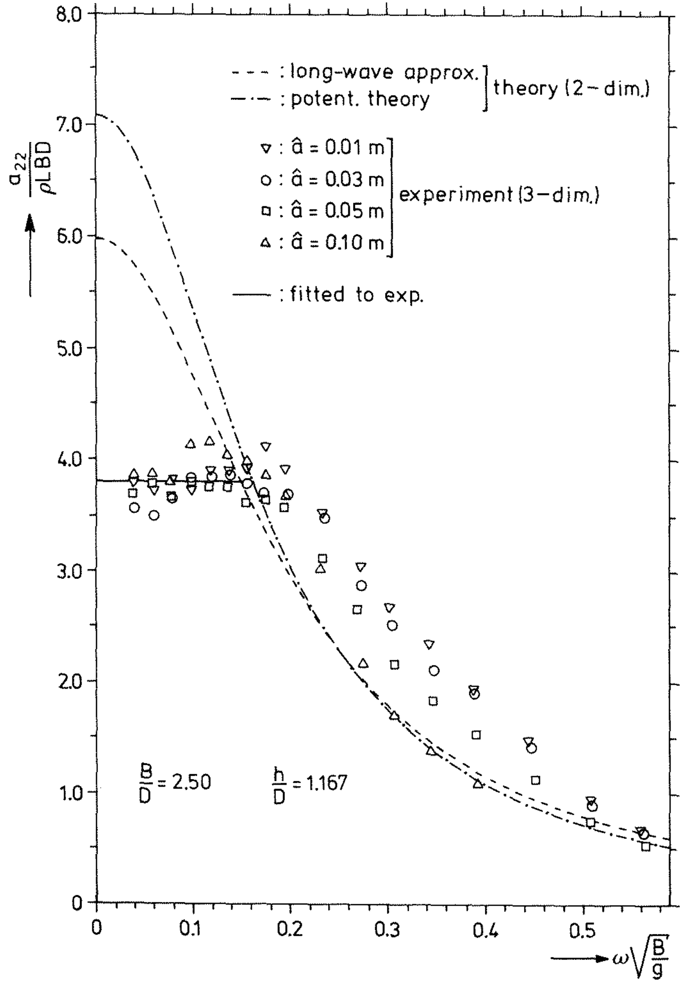

Fig. 3[a] - Added mass for swaying motion (h/D = 1.167).

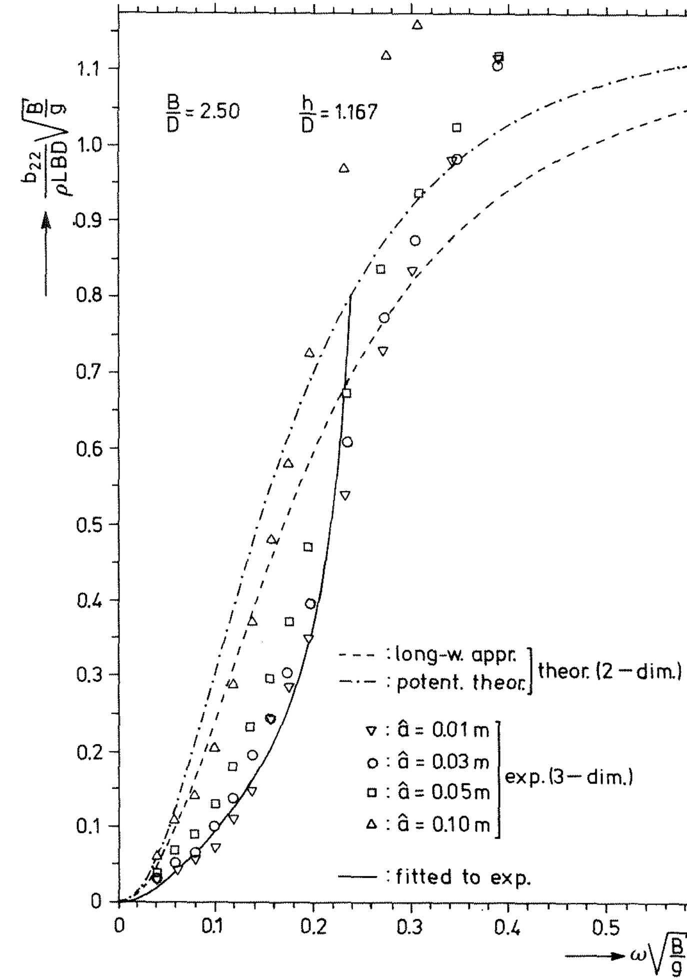

Fig. 3[b] - Sway damping force coefficient (h/D = 1.167).

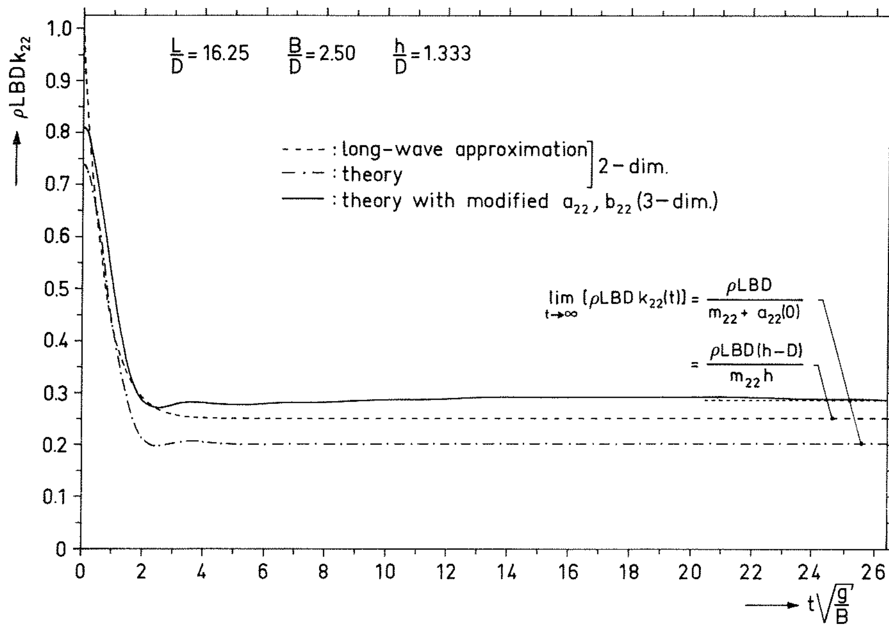

Fig. 4 - Impulse response function for sway motion (h/D =1.333).

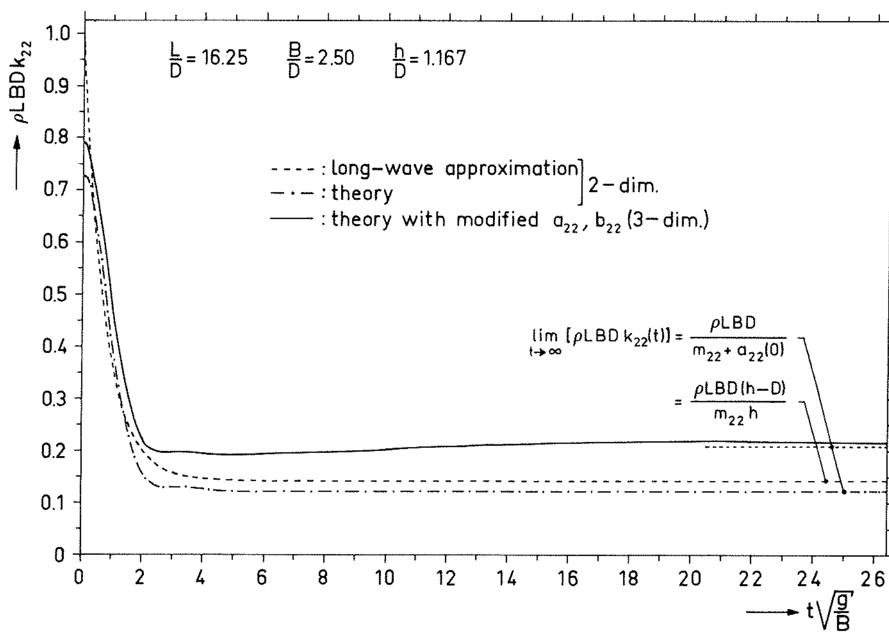

Fig. 5 - Impulse response function for sway motion (h/D = 1.167).

coefficients which in case of higher frequencies were determined theoretically (i.e. two-dimensionally) and in case of low frequencies were adapted to experimental (i.e. three-dimensional) values;

- the broken line represents $k_{22}(t)$ as calculated from an expression which can be derived analytically using a long-wave approximation for the motion of the water; this long-wave approximation is basically two-dimensional (see Appendix VII).

From figs. 4 and 5 it can be seen that the i.r.f. $k_{22}(t)$ approximates rather quickly to a constant value as t increases; this means that in the convolution integral eq. (19) - representing for i = 2 the motion of the ship in the sway direction - much emphasis is laid on the very near past of the time history of the forcing function.

II.4.b: THE IMPULSE RESPONSE FUNCTION FOR THE YAW MOTION: $k_{66}(t)$

The hydrodynamic coefficients for the case of pure yawing at zero forward speed, as determined theoretically and experimentally in ref. [30], are given in dimensionless form ($a_{66}(\frac{1}{12}L^2\rho LBD)^{-1}$, $b_{66}(\frac{1}{12}L^2\rho LBD)^{-1}(B/g)^{\frac{1}{2}}$) in figs. $6^{a,b}$ and $7^{a,b}$ as functions of the dimensionless circular frequency $\omega(B/g)^{\frac{1}{2}}$ with the dimensionless water depth h/D as parameter. In these figs. the hydrodynamic coefficients for the yaw motion are represented along that part of the frequency range, for which also experimental values are available; for the hydrodynamic coefficients in case of higher circular frequencies is referred to ref. [30]. The figs. $6^{a,b}$ and $7^{a,b}$ show that the theoretical (i.e. two-dimensional) and the experimental (i.e. three-dimensional) results for the hydrodynamic yaw coefficients do not agree along the frequency range considered. As a consequence in the low frequency range the hydrodynamic yaw coefficients calculated two-dimensionally have been adapted to the (three-dimensional) experimental values; they are maintained for the higher frequencies. To avoid a possible non-linear distortion of the f.r.f. (i.e. the hydrodynamic coefficients) this local adaptation of theory to experiment is based on the smallest amplitudes of the harmonically

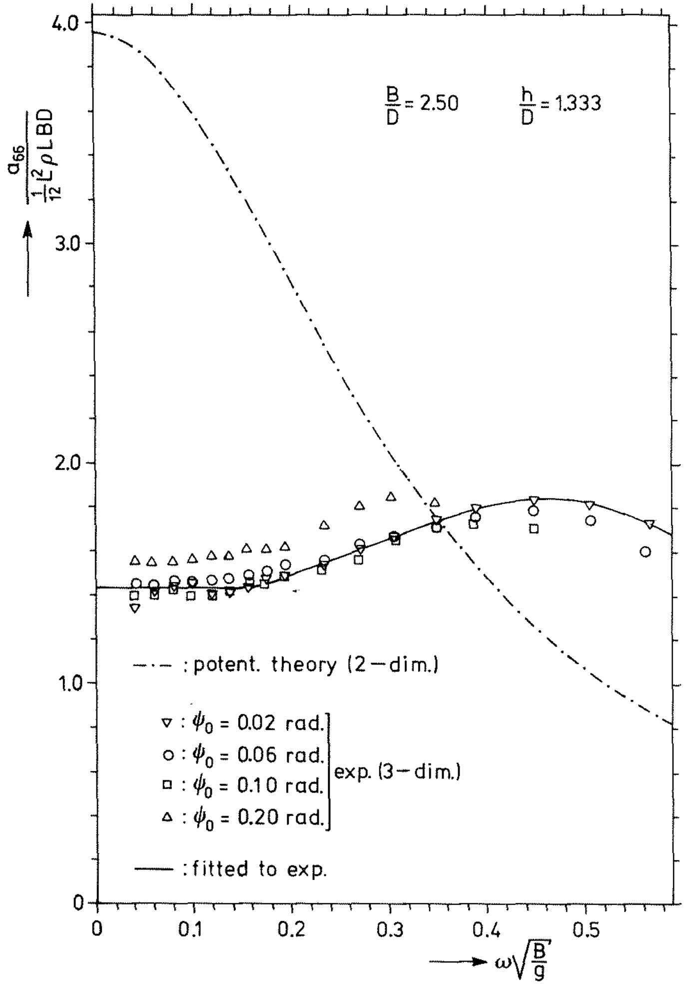

Fig. 6[a] - Added mass-moment of inertia for yawing motion (h/D = 1.333).

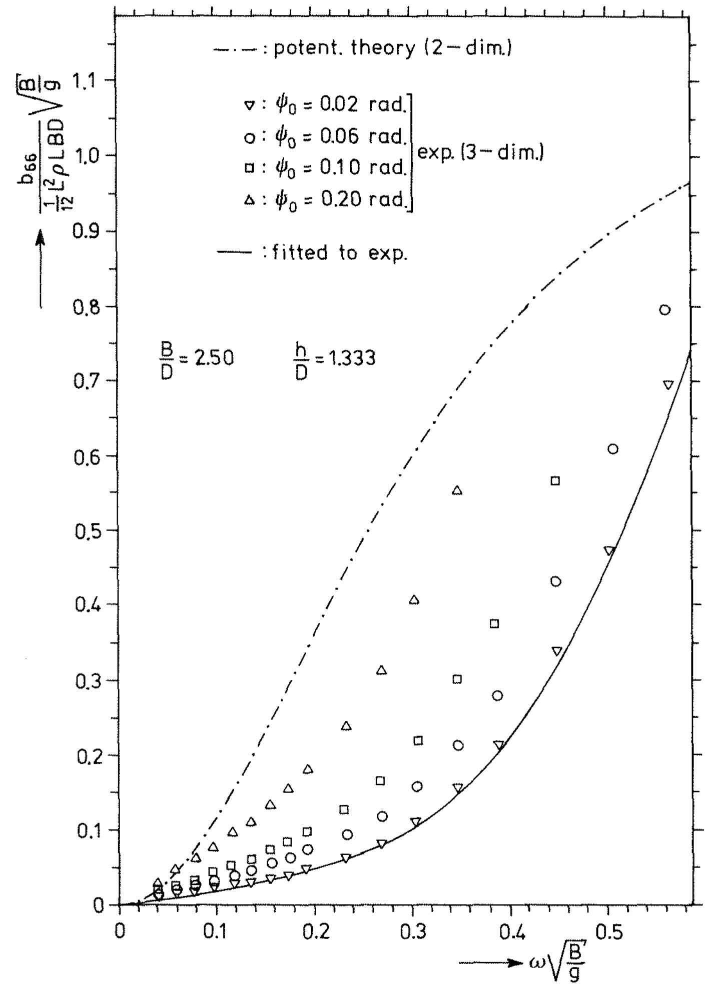

Fig. 6[b] - Yaw damping moment coefficient (h/D = 1.333).

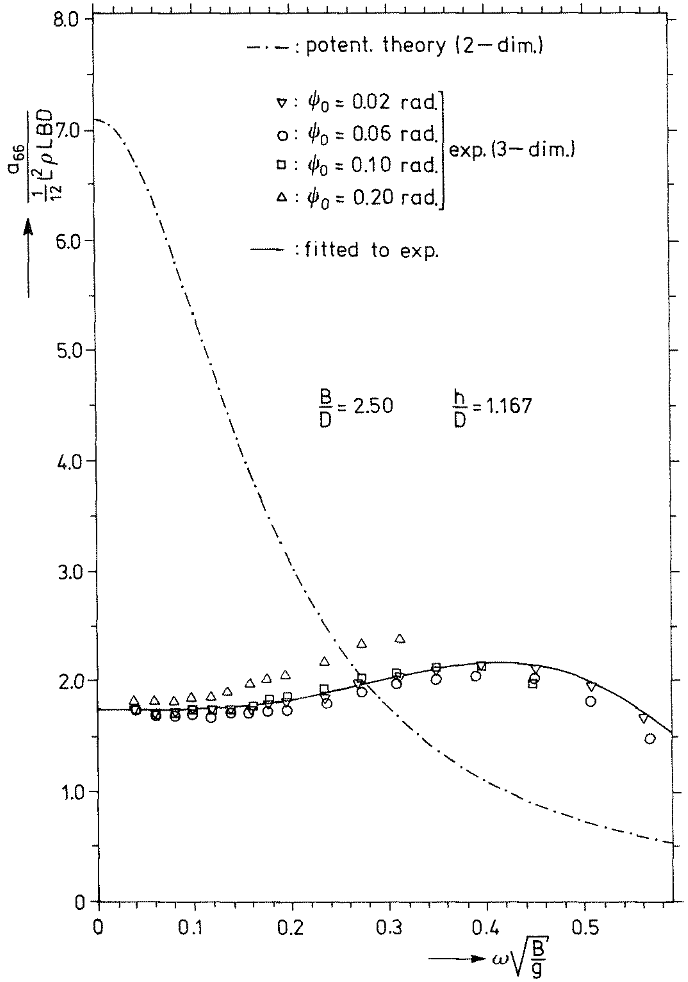

Fig. 7a - Added mass-moment of inertia for yawing motion (h/D = 1.167).

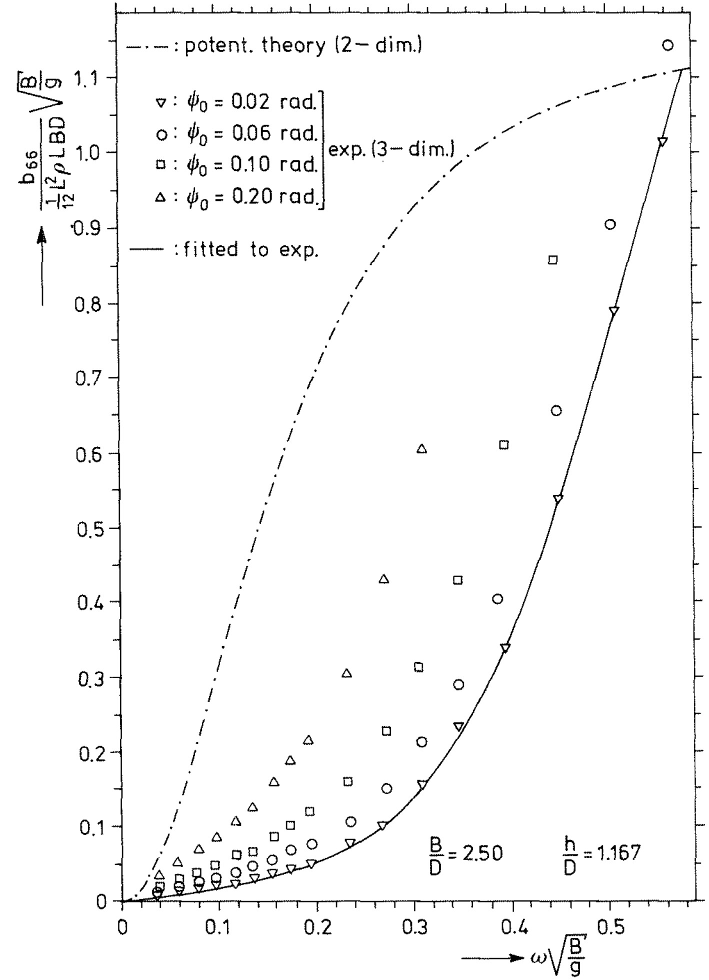

Fig. 7b - Yaw damping moment coefficient (h/D = 1.167).

oscillating yaw motion.
Concerning the values of the added mass-moment of inertia for the yawing motion $a_{66}(\omega)$ and the yaw damping moment coefficient $b_{66}(\omega)$ as actually used the following additional remarks have to be made. In case h/D = 1.333, for $\omega(B/g)^{\frac{1}{2}} \leqq 0.567$ (i.e. $\omega \leqq 2.9\ s^{-1}$) $a_{66}(\omega)$ as well as $b_{66}(\omega)$ have been adapted to their corresponding experimental values; for $2.9 \leqq \omega \leqq 5.9\ s^{-1}$ $a_{66}(\omega)$ was extrapolated by means of a straight line - according to $a_{66}(\omega) = -34.32 + 217.22$ -, which was faired into the curve at $\omega = 2.9\ s^{-1}$; for $\omega > 5.9\ s^{-1}$, it was applied $a_{66}(\omega) = \frac{1}{12} L^2 a_{22}(\omega)$ (see Appendix III); for $2.9 \leqq \omega \leqq 3.3\ s^{-1}$ $b_{66}(\omega)$ was extrapolated by means of a straight line - according to $b_{66}(\omega) = 235.36\ \omega - 436.89$ - which was faired into the curve at $\omega = 2.9\ s^{-1}$; for $\omega > 3.3\ s^{-1}$ it was applied $b_{66}(\omega) = \frac{1}{12} L^2 b_{22}(\omega)$ (see Appendix III). In case h/D = 1.167, for $\omega(B/g)^{\frac{1}{2}} = 0.567$ $a_{66}(\omega)$ as well as $b_{66}(\omega)$ have been adapted to their corresponding values; for $2.9 \leqq \omega \leqq 3.9\ s^{-1}$ $a_{66}(\omega)$ was extrapolated by means of a straight line - according to $a_{66}(\omega) = -88.26\,\omega + 367.75$ -, which was faired into the curve at $\omega = 2.9\ s^{-1}$; for $\omega > 3.9\ s^{-1}$ it was applied $a_{66}(\omega) = \frac{1}{12} L^2 a_{22}(\omega)$ and for $\omega > 2.9\ s^{-1}$ $b_{66}(\omega) = \frac{1}{12} L^2 b_{22}(\omega)$ (see Appendix III).

Starting from the hydrodynamic coefficients $a_{66}(\omega)$ and $b_{66}(\omega)$, as given in figs. $6^{a,b}$ and $7^{a,b}$ and farther above and in ref. [30], the i.r.f. for the yaw motion $k_{66}(t)$ then can be calculated making use of eq. (35^a) (see also Appendix V). The results are presented in dimensionless form ($m_{66}\ k_{66}$) in figs. 8 and 9 as functions of the dimensionless time $t(g/B)^{\frac{1}{2}}$ with the dimensionless water depth h/D as a parameter. The figs. 8 and 9 each show two curves:

- the dot and dash line represents $k_{66}(t)$ as calculated from hydrodynamic coefficients determined theoretically (i.e. two-dimensionally) along the whole frequency range;
- the full line represents $k_{66}(t)$ as calculated from hydrodynamic coefficients which in case of low frequencies were adapted to experimental (i.e. three-dimensional) values and further were

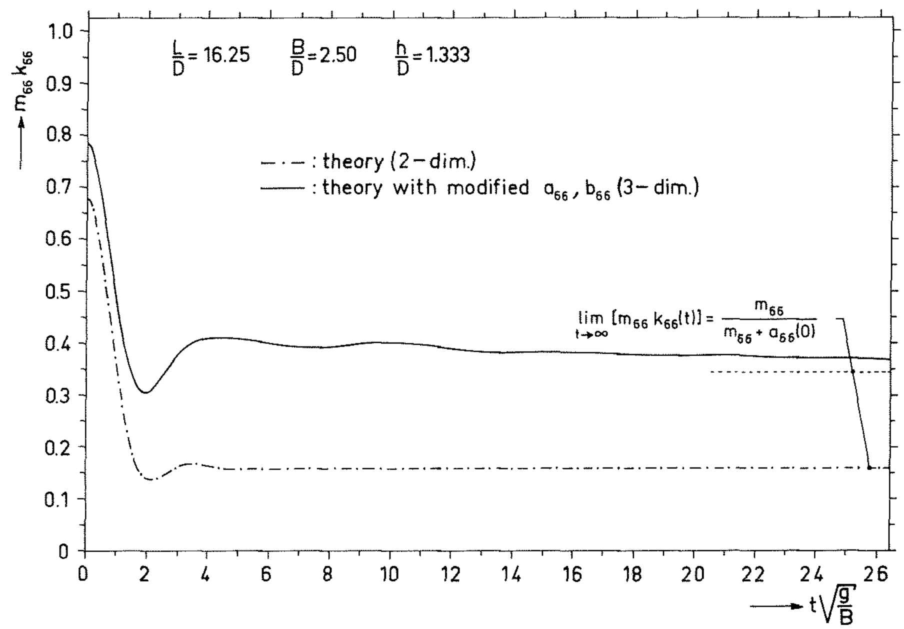

Fig. 8 - Impulse response function for yaw motion (h/D = 1.333).

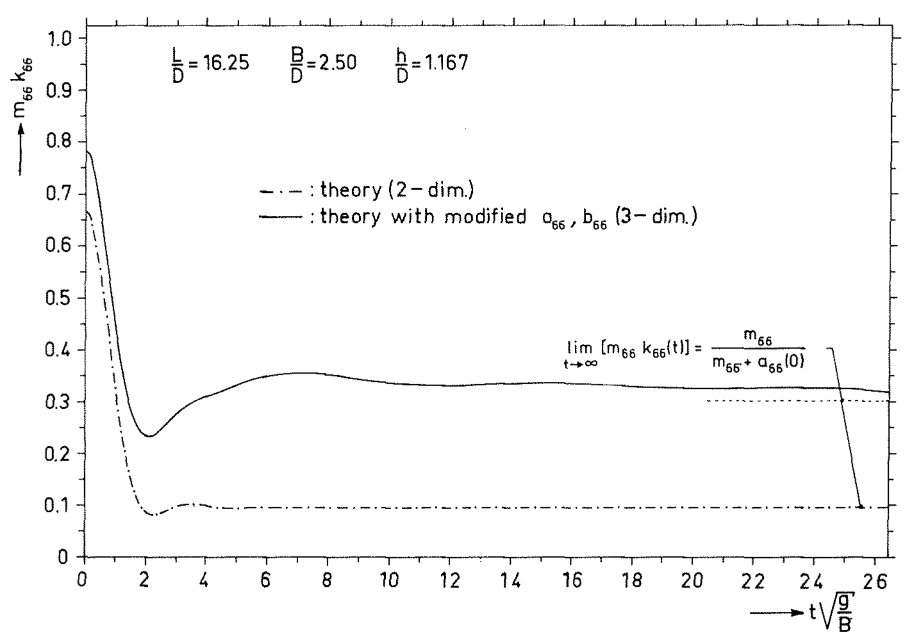

Fig. 9 - Impulse response function for yaw motion (h/D = 1.167).

extrapolated linearly, and which in case of higher frequencies were determined theoretically (i.e. two-dimensionally).

From figs. 8 and 9 it can be seen that the i.r.f. $k_{66}(t)$, just like $k_{22}(t)$, approximates rather quickly to a constant value as t increases; this means that in the convolution integral eq. (19) - representing for i = 6 the motion of the ship in the yaw direction - much emphasis is laid on the very near past of the time history of the forcing function.

SECTION III: APPLICATION OF THE 'IMPULSE RESPONSE FUNCTION'-TECHNIQUE TO THE BERTHING SHIP PROBLEM

III.1: GENERAL CONSIDERATIONS

In this section the 'i.r.f.'-technique is applied to the berthing ship problem. First of all the mathematical model to simulate the berthing of a (schematized) ship to a jetty and to determine the fender loads is presented. Then, the results of theoretical and experimental investigations for certain situations are compared and discussed.

For the sake of convenience the assumptions and simplifications concerning the case of the berthing ship - as presented and discussed in Section I - are recapitulated.
The displacements of the ship are assumed to remain small. Only the motions in the horizontal plane are considered, particularly the sway and yaw motions. The ship's forward speed is supposed to be zero. The vessel is schematized to a rigid prismatic body with a rectangular cross-section and a symmetrical distribution of mass. Further, it is assumed that the fluid is inviscid and incompressible and moves irrotationally; this assumption of fluid idealization is, however, not essential. The fluid domain is supposed to be relatively large in the horizontal directions; this implies an open berthing structure. The water depth may be arbitrary and the bottom is horizontal.
The influences of waves, current and wind are not taken into account.

The theoretical as well as the experimental investigations of the berthing ship problem are carried out with the same schematized ship (model) and for the same water depths as described in Section II.4. The fender loads and the ship trajectories are determined for various values of the (lateral) speed of approach, the fender elasticity and the water depth.
For reasons of clearness a choice is made for a simple berthing facility, viz. an open (jetty-type) berthing structure with one single fender.

III.2: OUTLINE OF THE MATHEMATICAL MODEL; NUMERICAL SOLUTION

Consider the schematized ship berthing to an open jetty equipped with one single fender. The characteristics of the fender are assumed to be represented by an undamped spring with a horizontal line of action situated in the plane of the water surface at rest. The mass of the fender is supposed to be small with respect to the mass of the ship. The frictional force between the hull of the ship and the fender is neglected.

N.B. As only horizontal motions are involved, the position of the ship's centre of gravity G in height is of no importance. Consequently, for the sake of simplicity, G is assumed to be situated in the free water surface at rest.

Initially, i.e. before the first contact between ship and fender, the ship moves laterally towards the berth with a constant speed of approach v_A and without rotation. The first contact between ship and fender is supposed to take place at point of time $t = 0$. Then the line of action of the fender is perpendicular to the longitudinal axis of symmetry of the ship; its initial distance to the ship's centre of gravity G is denoted by e_0. Since the constant forward speed of the ship V equals zero, the $ox_1x_2x_3$-co-ordinate system is space fixed. At $t = 0$ the $ox_1x_2x_3$-co-ordinate system is assumed to coincide with the moving, ship fixed, Gxyz-co-ordinate system. When $e_0 \neq 0$, at $t = 0$ the ship starts rotating, so that for $t > 0$ the motion of the ship consists of a translation in the sway direction and a rotation around the ox_3-axis. The co-ordinates of the ship's centre of gravity G at point of time t during the contact between ship and fender are indicated by $x_{1G}(t)$ and $x_{2G}(t)$; the angle of rotation of the ship's longitudinal axis of symmetry around the ox_3-axis then is $x_6(t) = \psi(t)$. The co-ordinates of the point of the fender are $x_{1f}(t) = -e_0$ for all t, and $x_{2f}(t)$ with $x_{2f}(t) = \frac{1}{2}B$ for $t \leq 0$. The impression of the fender is denoted by $\Delta x_{2f}(t) = x_{2f}(t) - \frac{1}{2}B$. With the position of the ship given by $x_{1G}(t)$, $x_{2G}(t)$

and its orientation by $\psi(t)$ the impression of the fender can be expressed as:

$$\Delta x_{2f}(t) = x_{2G} - \frac{1}{2} B \{1 - \cos(\psi)\} + \{x_{1f} - x_{1G} + \frac{1}{2} B \sin(\psi)\} \tan(\psi) , \tag{37}$$

$$\Delta x_2 f(t) \geqq 0 .$$

For a definition sketch see fig. 10.

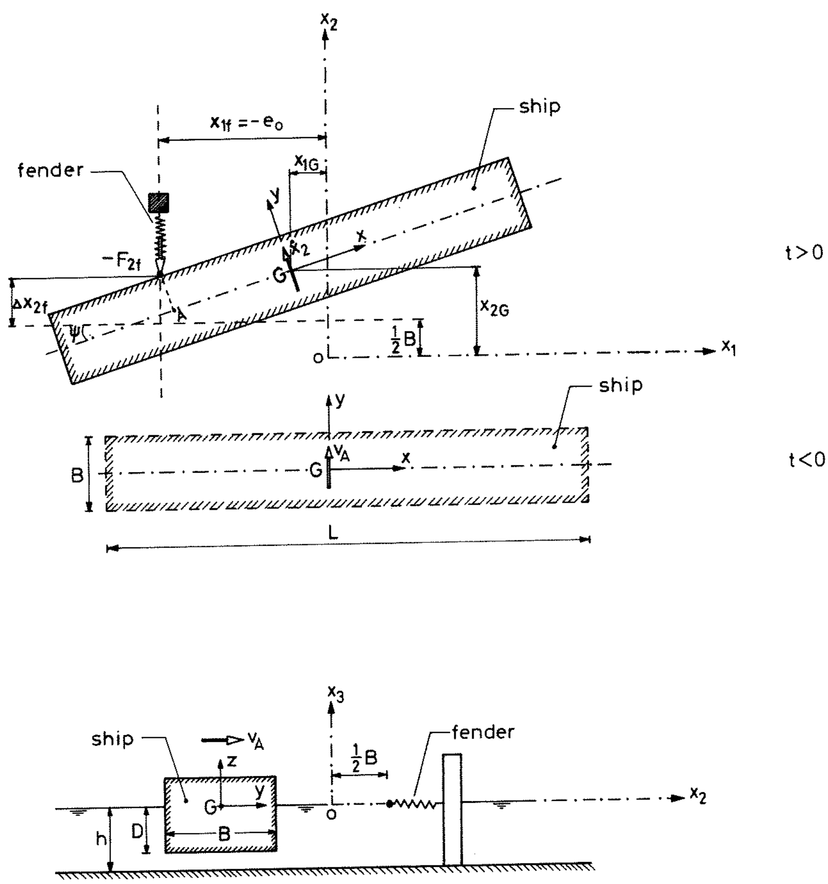

Fig. 10 - Definition sketch: plan and cross-section.

The relation between the impression of the fender $\Delta x_{2f}(t)$ at a certain point of time $t > 0$ and the corresponding reaction force in the fender $F_{2f}(t)$ can be represented as:

$$F_{2f}(t) = f(\Delta x_{2f}) \qquad \text{for } t \geqq 0 \;. \tag{38}$$

The resulting force and moment, as acting in and about the ship's centre of gravity, then become:

$$\left.\begin{aligned} &(39^a) \qquad f_2(t) = F_{2f} \cos(\psi) \\ &(39^b) \qquad f_6(t) = -\,\overline{AG}\;\; F_{2f} \cos(\psi) \end{aligned}\right\} \; t \geqq 0 \;,$$

respectively,

where $\overline{AG} = \dfrac{x_{1G} - x_{1f}}{\cos(\psi)} - \dfrac{1}{2} B \tan(\psi)$

with $x_{1f} = -\,e_0, \quad \psi < \dfrac{\pi}{2} \quad , \quad |\overline{AG}| \leqq \dfrac{1}{2} L \quad .$

According to eq. (19) the description of the linear ship-fluid system in the time domain in case of uncoupled sway and yaw motions reads as follows:

$$\dot{x}_i(t) = \int_{-\infty}^{t} f_i(\tau)\; k_{ii}(t - \tau)\; d\tau, \qquad i = 2,\, 6; \tag{40}$$

the forcing functions $f_i(t)$ are the input signals of the linear ship-fluid system, the velocities $\dot{x}_i(t)$ from which the translation and the rotation can be determined are the output signals. Since the forcing functions $f_2(t)$ and $f_6(t)$, as acting during the contact between ship and fender, are functions of the displacement(s) of the ship as well as of the impression of the fender, eqs. (37), (38), $(39^{a,b})$ and (40) combined form a closed loop system; eq. (40) represents a set of two integro-differential equations. Then, if the time history of the forcing functions $f_i(t)$ before touching the fender is also known, it is possible to determine fender loads and ship trajectories, provided the relevant i.r.f.'s are known. From the expression for the i.r.f.'s in case of uncoupled horizontal motions at zero forward speed - eq. (35^a) - and eq. (40) it follows that the time history of the forcing functions for $t \leq 0$ can be chosen such that the ship has got a constant lateral speed of approach and a zero rotational velocity at the first instant of contact between ship and fender, i.e. $\dot{x}_2(0) = v_A$, $\dot{x}_6(0) = 0$. Eq. (40) then changes into:

$$(41^a) \qquad \dot{x}_2(t) = v_A + \int_0^t f_2(\tau)\, k_{22}(t-\tau)\, d\tau,$$

$$(41^b) \qquad \dot{x}_6(t) = \int_0^t f_6(\tau)\, k_{66}(t-\tau)\, d\tau;$$

$f_2(t)$ has to be interpreted as the component of the reaction force in the fender, which - during the successive positions of the ship - acts upon the ship's centre of gravity G and has a direction perpendicular to the ship's longitudinal plane of symmetry (see also eq. (39^a)); likewise, $\dot{x}_2(t)$ is the velocity of G in the sway direction; $f_6(t)$ is the moment of $f_2(t)$ around the Gz-axis (see also eq. (39^b)) and $\dot{x}_6(t)$ is the rotational velocity of the ship.
The eqs. (37), (38), ($39^{a,b}$) and ($41^{a,b}$) combined now have to be solved (numerically); naturally this can only be done if the fender characteristics as well as the i.r.f.'s $k_{22}(t)$ and $k_{66}(t)$ are known. The initial values of the problem are:

$$(42) \qquad \begin{cases} \dot{x}_2(0) = v_A, & \dot{x}_6(0) = \dot{\psi}(0) = 0, & \\ x_{1G}(0) = 0, & x_{2G}(0) = 0, & x_6(0) = \psi(0) = 0, \\ x_{2f}(0) = \frac{1}{2}B, & f_2(0) = 0, & f_6(0) = 0 . \end{cases}$$

The numerical solution of the set of eqs. (37), (38), ($39^{a,b}$)

and ($41^{a,b}$) with initial conditions (42) is carried through according to the following procedure. Suppose that the (mathematical) simulation of the schematized ship berthing to the jetty has arrived to the point of time t. Δt is the time increment applied, so the above set of equations has to be solved for the point of time $t + \Delta t$. First of all the velocities for $t + \Delta t$ are predicted:

$$\dot{x}_2(t + \Delta t) = \dot{x}_2(t), \quad \dot{x}_6(t + \Delta t) = \dot{x}_6(t).$$

Subsequently the new orientation and the new position of the ship are determined by numerical integration of the velocities, applying the trapezoidal rule:

$$\psi(t+\Delta t) = \psi(t) + \frac{\dot{x}_6(t) + \dot{x}_6(t + \Delta t)}{2} \Delta t,$$

$$x_{1G}(t + \Delta t) = x_{1G}(t) - \Delta t \frac{\dot{x}_2(t) + \dot{x}_2(t + \Delta t)}{2} \sin\{\bar{\psi}(t + \Delta t)\},$$

$$x_{2G}(t + \Delta t) = x_{2G}(t) + \Delta t \frac{\dot{x}_2(t) + \dot{x}_2(t + \Delta t)}{2} \cos\{\bar{\psi}(t + \Delta t)\},$$

where $\bar{\psi}(t + \Delta t) = \dfrac{\psi(t) + \psi(t + \Delta t)}{2}$,

representing the mean value of $\psi(t)$ on the interval of time considered. For time $t + \Delta t$ eq. (37) then yields the displacement of the fender $\Delta x_{2f}(t + \Delta t)$ and eq. (38) the fender force $F_{2f}(t + \Delta t)$; the resulting force and moment, as acting in and about the ship's centre of gravity G - $f_2(t + \Delta t)$ and $f_6(t + \Delta t)$, respectively - can be predicted by means of eqs. ($39^{a,b}$). Now the time history of the forcing functions is known until the time $t + \Delta t$; therefore the convolution integrals in eqs. (41^a) and (41^b) can be calculated. The numerical integration of these convolution integrals is carried out by means of the trapezoidal rule, using a time increment equal to the time step Δt. In doing so one obtains new, corrected, values for the velocities at $t + \Delta t$. Finally these corrected velocities are compared with the corresponding values predicted at the beginning of the calculation. If the respective differences are acceptable (i.e. in case the predicted and the calculated velocities at $t + \Delta t$ are in satisfactory agreement), the calculation continues for the next time step; if not, the calculation is repeated with the new corrected velocities $\dot{x}_2(t + \Delta t)$ and $\dot{x}_6(t + \Delta t)$ (predictor-corrector method, c.q. iteration procedure). The criterion for the continuation of the calculation for the next time step is based on the absolute value of the difference between the predicted and the calculated velocities (expressed in $m\ s^{-1}$): it is assumed that this absolute value has to be smaller than 10^{-7}. The calculation is finished when the ship loses the contact with the fender; this is the case when $\Delta x_{2f}(t + \Delta t)$ becomes zero.

For the case of a berthing operation in which $x_{1f} = -e_0 = 0$ (i.e. a 'centric impact') to a linear fender, a criterion can be derived for the convergence of the computational scheme (see Appendix VI), viz.:

(43) $$\Delta t < 2\sqrt{\frac{1}{c_0 k_{22}(0^+)}} ,$$

where c_0 = spring rate of linear fender.

III.3: EXAMPLES OF CALCULATED BERTHING OPERATIONS AND THEIR EXPERIMENTAL VERIFICATION

In order to examine the adequacy of the mathematical model for the simulation of berthing operations under conditions as described in Sections III.1 and III.2, an extensive experimental program was carried out to analyse the behaviour of a (schematized) ship berthing to a(n open) jetty equipped with one single fender.

Afterwards typical test situations were selected for the numerical simulation to see whether the observed phenomena could be reproduced by means of the mathematical model.
Two kinds of fenders were considered: a linear fender represented by

$$(44^a) \qquad F_{2f}(t) = \begin{cases} 0 & \text{for } \Delta x_{2f}(t) < 0, \\ -c_0 \, \Delta x_{2f}(t) & \text{for } \Delta x_{2f}(t) \geqq 0, \end{cases}$$

and a non-linear fender represented by

$$(44^b) \qquad F_{2f}(t) = \begin{cases} 0 & \text{for } \Delta x_2 f(t) < 0 \,, \\ -c_1 \, \Delta x_{2f}(t) & \text{for } 0 \leqq \Delta x_{2f}(t) \leqq d_{sc}, \\ -c_1 \, \Delta x_{2f}(t) - c_2 \, \{\Delta x_{2f}(t) - d_{sc}\} & \text{for } \Delta x_{2f}(t) \geqq d_{sc}, \end{cases}$$

where c_0 = spring rate of linear fender,
c_1, c_2 = respective spring rates of the two linear springs which combined form the non-linear fender,
d_{sc} = initial distance (i.e. at rest) between the two linear spring elements of the non-linear fender.

A distinction can be made between two kinds of berthing operations, viz. berthing operations in which $e_0 = 0$ ('centric impacts') and berthing operations in which $e_0 \neq 0$ ('eccentric impacts').

III.3.a: DESCRIPTION OF THE EXPERIMENTAL SET-UP

The experimental study was executed with a schematized ship model berthing to an open jetty equipped with one single fender, in water with respective depths amounting to 1.333 and 1.167 times the draught of the vessel. The schematized ship model and the water depths are the same as described in Section II.4 and ref. [30]. The lay-out of the test set-up and the conditions under which the operations took place, correspond with the situation as described in Sections III.1 and III.2.

In the experiments the following quantities were measured as functions of the time: the displacement of the fender (and therefore the fender load), the position of the ship's centre of gravity G and the angle of rotation of the ship's longitudinal plane of symmetry.

The model tests were carried out in the Laboratory of Fluid Mechanics of the Delft University of Technology, department of Civil Engineering. The experimental facility was situated in the middle of a rectangular basin with relatively large horizontal dimensions - effective length = 33.15 m, effective breadth = 13.95 m - and a horizontal bottom.

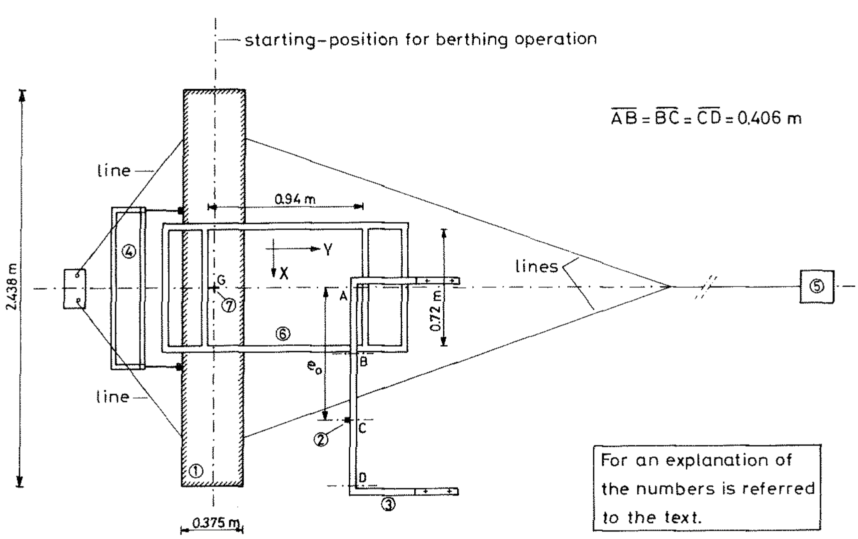

Fig. 11 - Plan of berthing lay-out .

The test arrangement consisted of the following principal parts (the numbers refer to fig. 11):

1 - the schematized ship model;

2 - the fender;

3 - an open structure, fixed to the bottom of the basin, which acted as support for the fender; the value of e_0 could be varied by moving the fender along the structure (for instance to the places indicated by A, B, C and D);

4 - an open structure, fixed to the bottom of the basin, to fasten the ship in a fixed position when at rest; this fixed position acted as starting-position for the berthing operation;

5 - a facility to give the ship model the proper constant lateral speed of approach;

6 - a 'position follower' to measure the 'X/Y-co-ordinates' of the ship's centre of gravity G;

7 - a facility mounted on the bottom of the ship to measure the angle of rotation of the ship's longitudinal plane of symmetry.

Fig. 12^a shows a general view of the test arrangement.
The longitudinal plane of symmetry of the ship model in its starting-position (i.e. when at rest) coincided with the breadthwise axis of symmetry of the basin. The trajectory of the ship's centre of gravity G before the contact between ship and fender coincided with the lengthwise axis of symmetry of the basin.
The 'position follower' was mounted horizontally on a frame which was adjustable in height. In principle it was a mechanical X/Y-recorder.

The 'position follower' consisted of a carriage and a routing carrier. The carriage - measuring the motions of G in the 'Y-direction' - moved along two parallel, horizontal shafts, had a span of 0.70 m and could cover a distance of 0.90 m. The carriage was composed of two parallel, horizontal shafts along which the

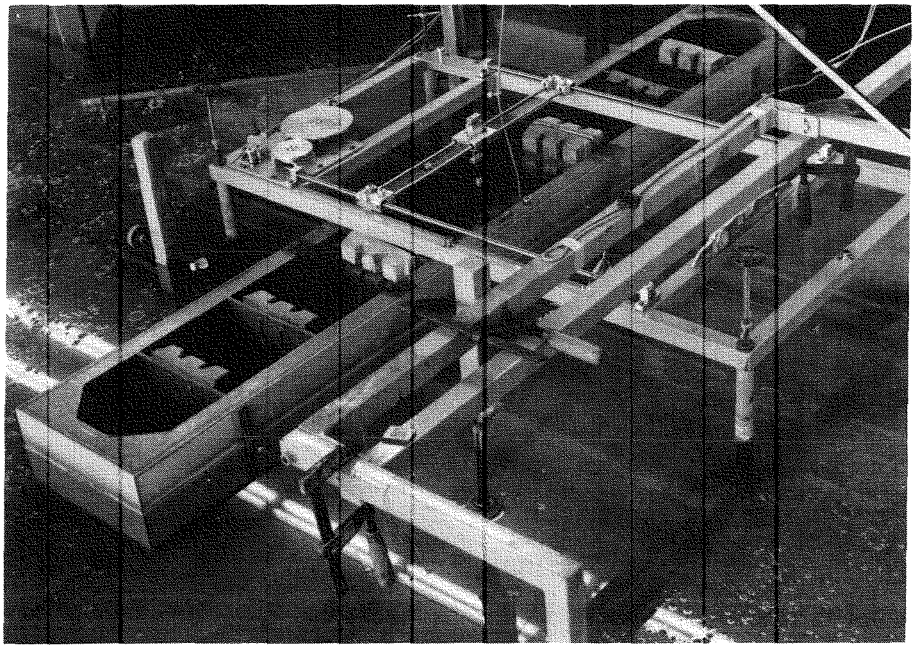

Fig. 12^a - General view of experimental set-up .

routing carrier could move; the routing carrier measured the motions of G in the 'X-direction' and could cover a distance of 0.60 m. The alignment of the carriage was such that the 'X-' and 'Y-directions' were orthogonal. The direction of motion of the carriage was chosen parallel to the lateral speed of approach of the ship. The 'X-' and 'Y-co-ordinates' were measured by means of two independent string-driven potentiometers of high precision, one for each axis (see figs. 13^a and 13^b).
The horizontal motions of the ship's centre of gravity G were transferred to the routing carrier via a shaft. This shaft was connected with the routing carrier in such a way that it - while in upright position - only could move vertically without restraint,

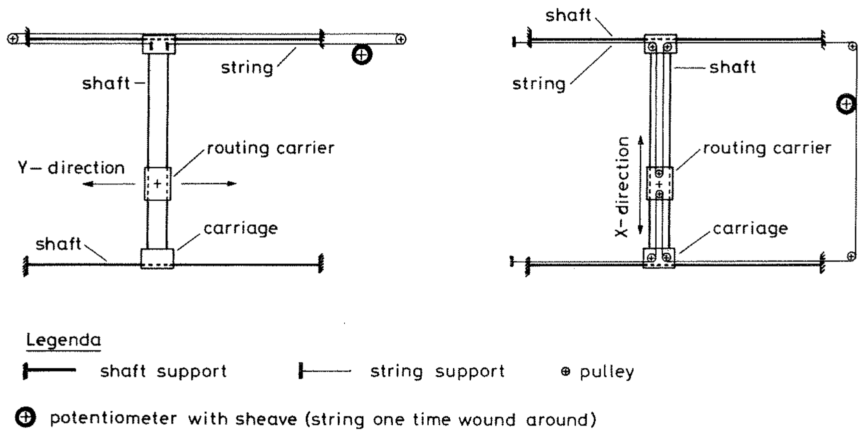

Fig. 13^a - Plan of carriage (= 'Y'-)axis string driven potentiometer circuit.

Fig. 13^b - Plan of routing carrier (= 'X'-) axis string driven potentiometer circuit.

and rotation (around its lengthwise axis) with respect to the carrier was impossible. By a universal joint, situated in the plane of the water-line just above the ship's centre of gravity G, this vertically movable, non-rotatable shaft was coupled to a second shaft which coincided with the Gz-axis of the moving, ship fixed Gxyz-co-ordinate system. This second shaft was my means of a gear-wheel transmission - mounted on the bottom of the ship - connected with a precision potentiometer, by which the angle of rotation of the ship's longitudinal plane of symmetry could be measured. In order to prevent that during a berthing operation too vehement roll motions - if any - yet were transferred to the routing carrier, c.q. carriage, the absolutely vertical shaft (i.e. the upper one) was supported elastically with respect to the sides of the ship (see fig. 12^b).

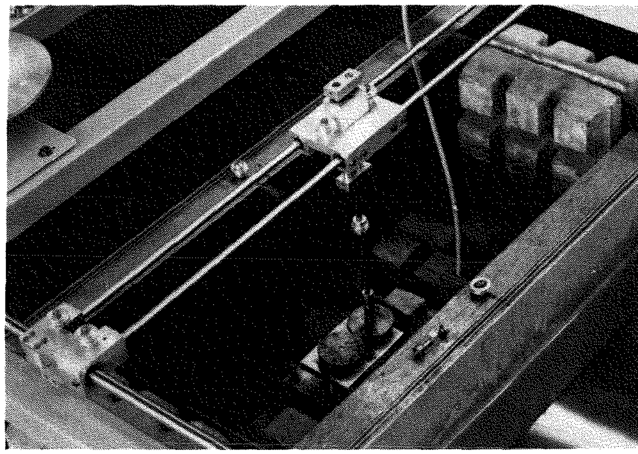

Fig. 12[b] - Connection between ship model and 'position follower'.

A measuring arrangement of this type implied that the ship model was allowed to heave, to roll and to pitch without any restraint, whereas the motions in the horizontal plane (i.e. translations as well as rotation) could be measured without being influenced.
The 'position follower' was constructed as light and rigid as possible. The friction in the moving parts was minimized by applying eminent materials, such as precision ball-bearings, ball bushing constructions, special extruded and hardened shafts, etc.
According to Section II.4 the mass of the ship model, as based on the volume of displacement, amounted to 137.10 kg, whereas the mass for horizontal motions as used in the tests and the calculations, m_{22}, was 137.24 kg. This difference was caused by the presence of the 'position follower' (carriage with routing carrier), which contributed to the (moving) mass of the ship. Otherwise, the contribution of the mass of the 'position follower' to that of the ship model - including frictional effects in the moving parts - could be considered as negligible (less than 0.5 per cent., i.e.

within the accuracy of the measurements.

To give the ship model the proper constant speed of approach two equal horizontal forces were applied to the fore and aft end of the ship model such that rotational motions did not arise. These forces were exerted by a weight connected to the ship model via lines and pulleys. At the beginning of a test the ship model was released from its starting-position at a distance of about 0.50 m from the fender. Then it was accelerated gradually until the distance to the fender was about 0.10 m, at which moment the weight reached a cantilever . Till the fender was touched the only external force acting on the ship model was the fluid resistance. It appeared that in this phase the lateral speed of approach remained almost constant. The following constant lateral speeds of approach were applied in the tests: circa 0.01 m s^{-1}, 0.02 m s^{-1} and 0.03 m s^{-1}. Their actual values were determined by (numerical) differentiation of the displacement of the ship's centre of gravity G in the 'Y-direction' as measured by the 'position follower'.

Several fenders were used. The elasticity of these fenders was simulated by means of two or more undamped leaf springs, as shown in fig. 12^c and fig. 12^d. The frictional force between the hull of the ship and the fender was minimized by using a (small) horizontal wheel which was fitted on a precision ball-bearing at the extreme end of the fender. The fender was attached to its supporting structure in such a way that this horizontal wheel was situated in the water surface at rest and the line of action of the fender was perpendicular to the longitudinal plane of symmetry of the ship when approaching laterally. The reaction forces (or strictly speaking the impressions) of the fender were measured by means of strain gauge transducers. The own mass of the fender could be neglected with respect to the mass of the ship model. The natural period of the respective fenders was many times smaller than the length of time of their impression.

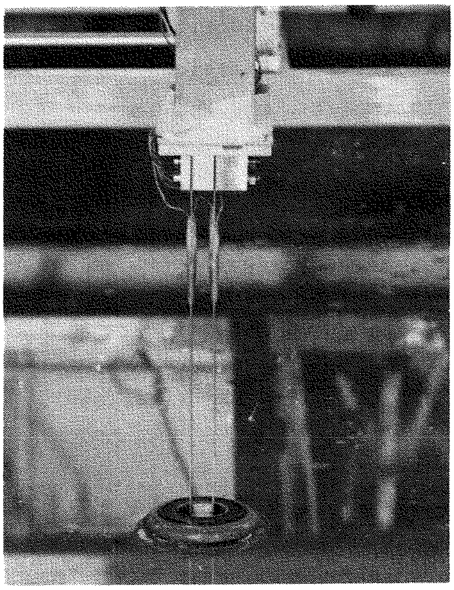

Fig. 12[c] - Simulation of linear fender.

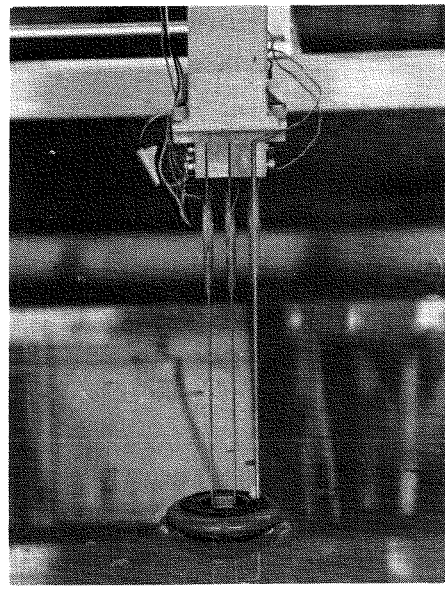

Fig. 12[d] - Simulation of non-linear fender.

The 'centric impacts' were carried out with three linear fenders (c_0 = 2146 kg s^{-2}; c_0 = 1373 kg s^{-2}; c_0 = 576 kg s^{-2}) and with one non-linear fender (c_1 = 625 kg s^{-2}, c_2 = 1108 kg s^{-2}, d_{sc} = 0.664 $\times$ 10^{-2} m). The 'eccentric impacts' were carried out with one linear fender (c_0 = 637 kg s^{-2}) for three values of e_0 (e_0 = 0.406 m; e_0 = 0.813 m; e_0 = 1.219 m).

All signals were recorded simultaneously on paper chart.

III.3.b: CALCULATION OF BERTHING OPERATIONS

For the numerical simulation of the berthing operations those test situations were selected from the experiments, which were in agreement with the conditions and the situation as described in Sections III.1 and III.2. This implied that the calculations were carried out for the same schematized ship (model), for the same constant lateral speed of approach, for the same water depths, for the same fenders and for the same values of e_0 as in the tests. Since the schematized ship model and the water depths were the same as described in Section II.4, in the numerical calculations use could be made of the relevant i.r.f.'s as determined at that place.

In the numerical simulation of the berthing operations the following quantities were calculated as functions of the time (see also fig. 10):

the velocity of the ship's centre of gravity G in the sway direction, $\dot{x}_2(t)$;

the rotational velocity of the ship, $\dot{x}_6(t) = \dot{\psi}(t)$;

the co-ordinates of the ship's centre of gravity G, $x_{1G}(t)$ and $x_{2G}(t)$;

the angle of rotation of the ship's longitudinal axis of symmetry around the ox_3-axis, $x_6(t) = \psi(t)$;

the impression of the fender, $\Delta x_{2f}(t)$;

the reaction force in the fender, $F_{2f}(t)$;

in the relevant cases these quantities were determined only for the length of time, during which there was contact between ship and fender.

The results of the calculations showed that generally $x_{1G}(t)$ was very small with respect to $x_{2G}(t)$; in all cases considered $x_{1G}(t)$ remained smaller than 0.4×10^{-3} m. Besides, the values of $x_{1G}(t)$ as determined experimentally fell within the accuracy of the measurements. For these reasons $x_{1G}(t)$ further is left out of consideration.

According to criterion (43), yielding a condition for the convergence of the computational scheme in case of a 'centric impact' against a linear fender, the time step of the calculations, Δt, - for the situations considered - has to be smaller than 0.4 s.

On the one side the computing time roughly is linearly proportional to the inverse of the time step Δt. On the other hand, systematic calculations with varying time step have shown that the accuracy of the calculations decreases with increasing values of Δt. To arrive at an accuracy as great as possible all calculations were carried out with a time step $\Delta t = 0.01$ s.

N.B. Using a long-wave approximation for the motion of the water, the expressions describing the berthing of the schematized ship (model) to a linear fender can be derived analytically (see Appendix VII).

III.3.c: PRESENTATION AND DISCUSSION OF RESULTS

In this section the most representative results of the tests are given together with the corresponding results of the calculations.

Since the berthing operation of the (schematized) ship to a certain fender can be described completely by the reaction force in the fender and the position and orientation of the ship during its contact with the fender, in the following only $F_{2f}(t)$, $x_{2G}(t)$ and $\psi(t)$ will be considered. In order to bring about a 'collapse of data' these quantities are represented in dimensionless form by

$\dfrac{F_{2f}}{v_A \sqrt{c_0 M_0}}$ = dimensionless reaction force in the (linear) fender,

$\dfrac{x_{2G}}{v_A} \sqrt{\dfrac{c_0}{M_0}}$ = dimensionless translation of the ship's centre of gravity G during the contact between ship and (linear) fender,

$\dfrac{\psi m_{66}}{v_A e_0 M_0} \sqrt{\dfrac{c_0}{M_0}}$ = dimensionless angle of rotation of the ship's longitudinal axis of symmetry during the contact between ship and (linear) fender, where $\frac{1}{M_0} = \frac{e_0^2}{m_{66}} + \frac{1}{\rho LBD}$;

M_0 has to be interpreted as the reduced or effective mass of the ship (model) for horizontal motion. The results are presented as functions of the dimensionless time

$$t\sqrt{\frac{c_0}{M_0}} \quad .$$

The respective expressions with which $F_{2f}(t)$, $x_{2G}(t)$, $\psi(t)$ and t are made dimensionless can be determined analytically by solving the problem of the schematized ship berthing to a linear fender for the case of motion in an ideal medium to the neglect of the hydrodynamic effects.

The dimensionless representation above of the relevant quantities applies to the case of the ship berthing to a linear fender. For the case of the ship berthing to the non-linear fender c_0 has to be replaced by c_1.

The parameters which further play a part in the presentation of the experimental and theoretical results are the (dimensionless) water depth, the (dimensionless) characteristics of the fender, the (dimensionless) initial distance of the line of action of the fender to the ship's centre of gravity G and - for the tests - the (dimensionless) constant lateral speed of approach.

In addition to the experimental results which are plotted as centred symbols, the figures to be presented each show three curves representing the theoretical results (see also Sections II.4.a and II.4.b):

- the dot and dash line represents the results as calculated by means of i.r.f.'s which have to be considered as two-dimensional;
- the full line represents the results as calculated by means of i.r.f.'s which can be considered as three-dimensional;
- the broken line represents the results as determined (analytically)

by making use of a long-wave approximation for the motion of the water (see Appendix VII); these results are basically two-dimensional.

Therefore, the theoretical results as given by the dot and dash lines and the broken lines have to be considered as two-dimensional, whereas the theoretical results as given by the full lines have to be considered as three-dimensional.

Two kinds of berthing operations were investigated, viz. berthing operations in which $e_0 = 0$ ('centric impacts') and berthing operations in which $e_0 \neq 0$ ('eccentric impacts').

III.3.c.1: CENTRIC IMPACTS: $e_0 = 0$

Since in case of a 'centric impact' $\psi(t) = 0$, $x_{2G}(t) = \Delta x_{2f}(t)$ and $F_{2f}(t) = f(\Delta x_{2f})$, only the results for $F_{2f}(t)$ have to be presented.

Figs. 14 through 19 show $F_{2f}\, v_A^{-1} (c_0 M_0)^{-1/2}$ versus $t(c_0/M_0)^{1/2}$ for the case of a linear fender, with the dimensionless water depth h/D and the dimensionless fender characteristic $c_0(\rho g D^2)^{-1}$ as parameters. As could be expected the calculated fender forces are proportional to the constant lateral speed of approach. Farther, it can be seen from these figures that in case of increasing fender stiffness the (maximum value of the) fender force also increases, whereas the length of time of the contact between ship and fender decreases and the point of time at which the fender force reaches its maximum occurs earlier. In case of a greater water depth - at constant fender stiffness - the (maximum value of the) fender force as well as the length of time of the contact between ship and fender is smaller, while the point of time at which the fender force reaches its maximum occurs earlier.

Figs. 20 through 23 show $F_{2f}\, v_A^{-1} (c_1 M_0)^{-1/2}$ versus $t(c_1/M_0)^{1/2}$ for the case of the non-linear fender, with the dimensionless water depth

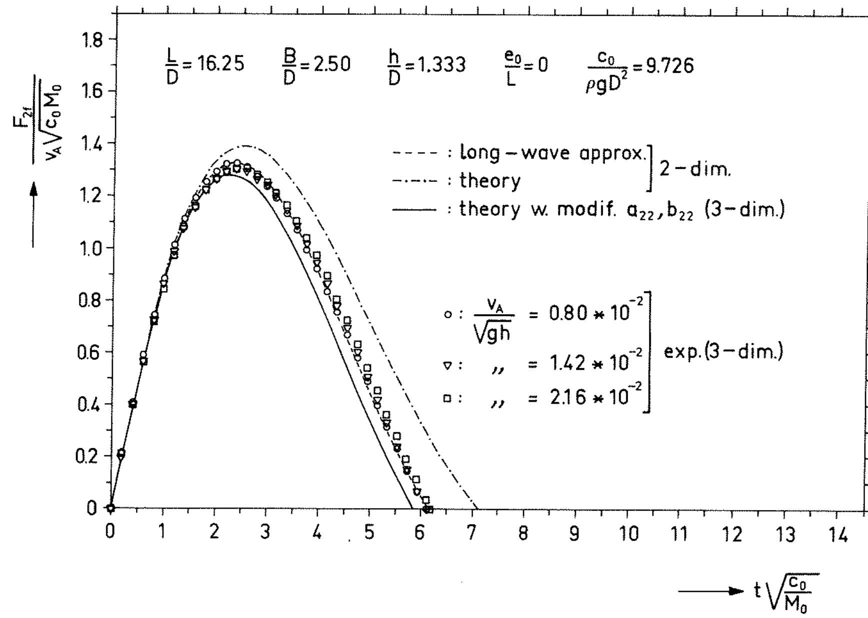

Fig. 14 - Time history of fender force: linear fender, centric impact (h/D = 1.333).

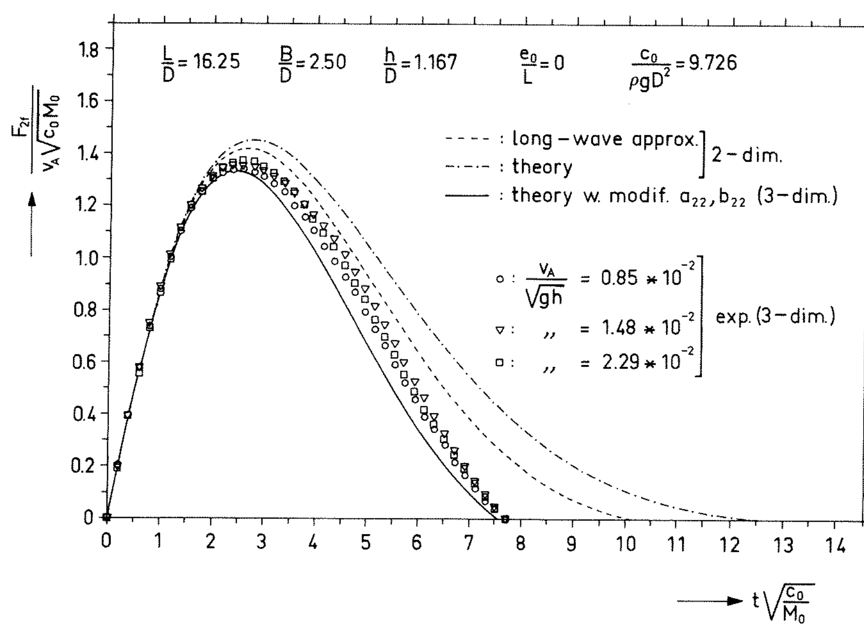

Fig. 15 - Time history of fender force: linear fender, centric impact (h/D = 1.167).

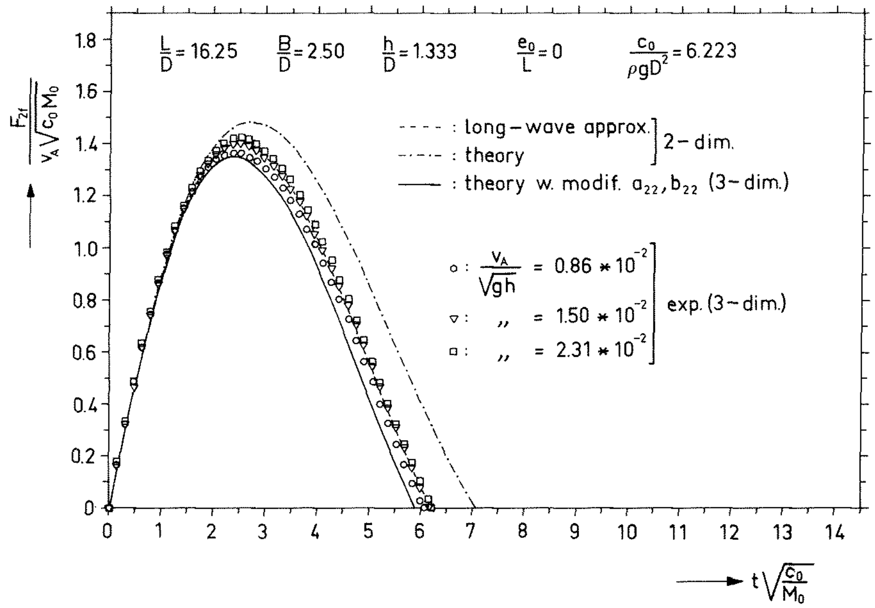

Fig. 16 - Time history of fender force: linear fender, centric impact (h/D = 1.333).

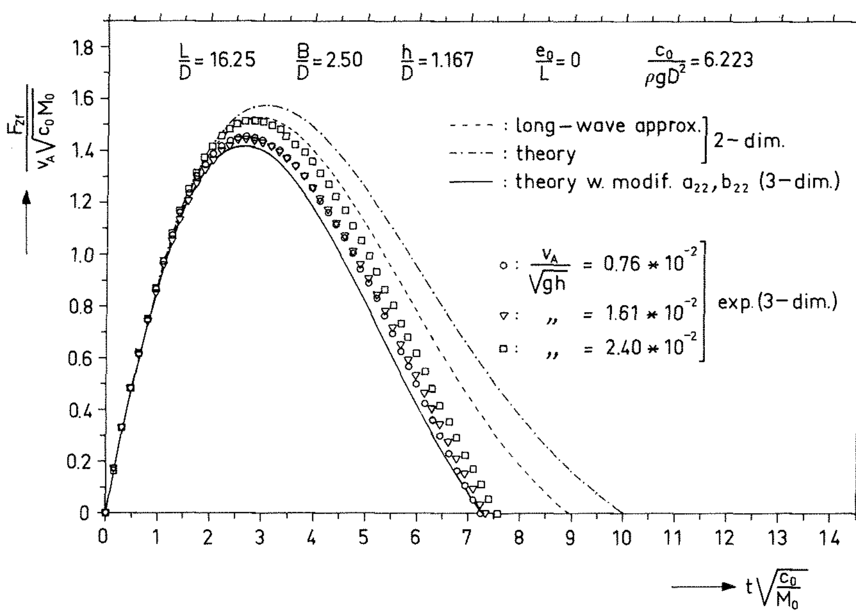

Fig. 17 - Time history of fender force: linear fender, centric impact (h/D = 1.167).

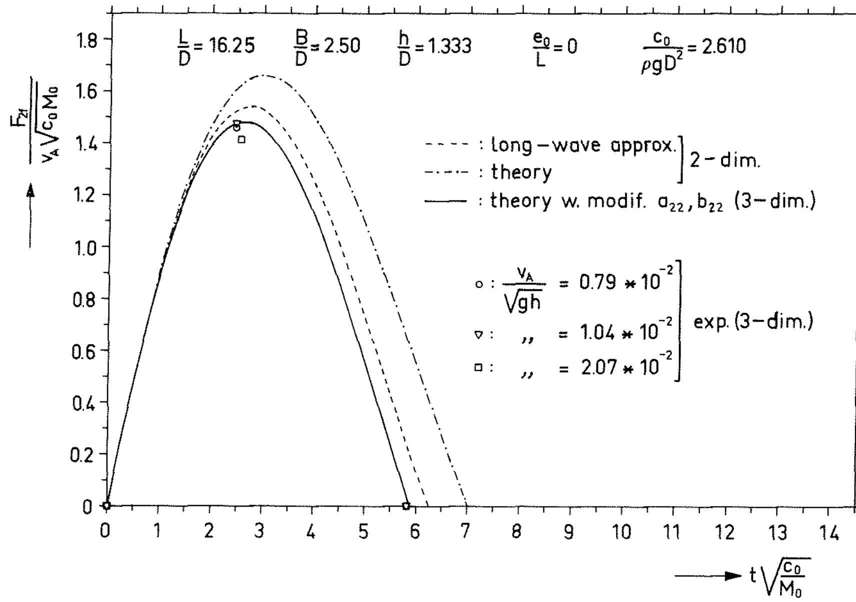

Fig. 18 - Time history of fender force: linear fender, centric impact (h/D = 1.333).

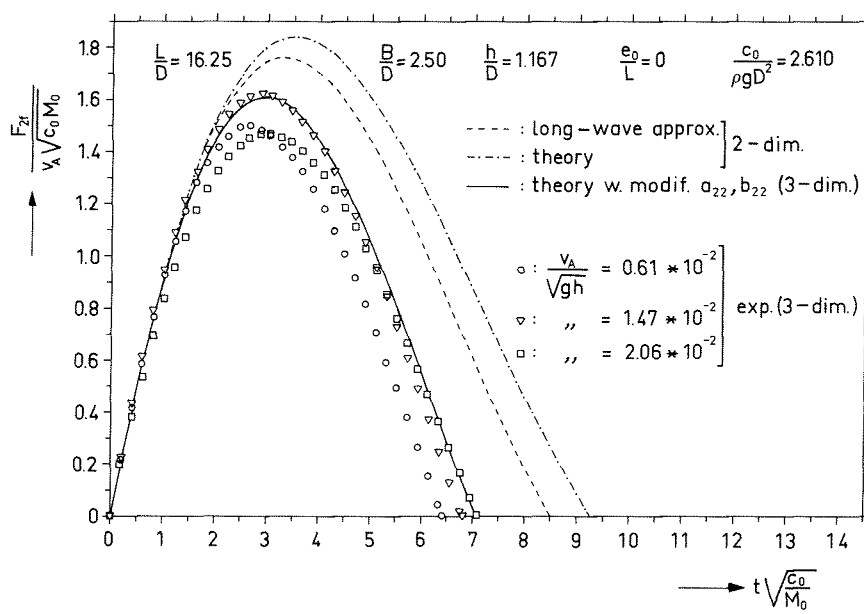

Fig. 19 - Time history of fender force: linear fender, centric impact (h/D = 1.167).

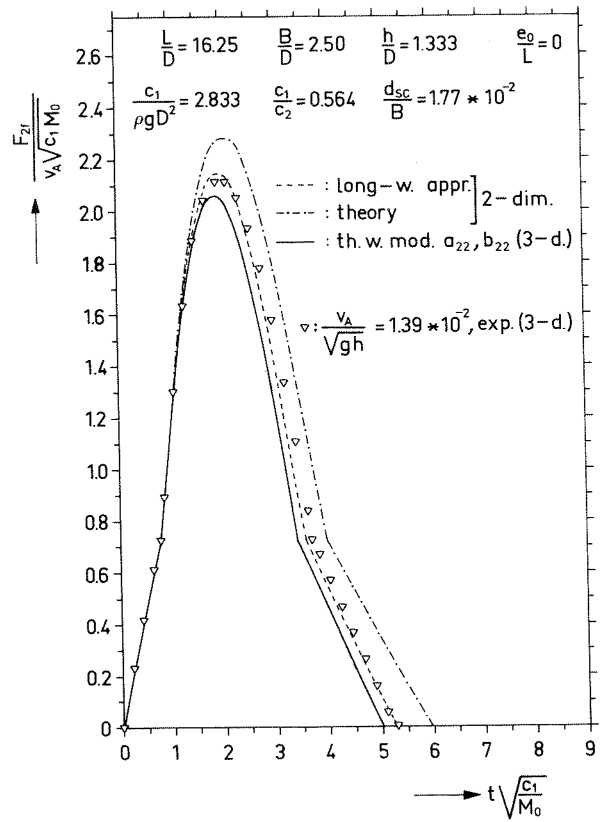

Fig. 20 - Time history of fender force: non-linear fender, centric impact ($h/D = 1.333$).

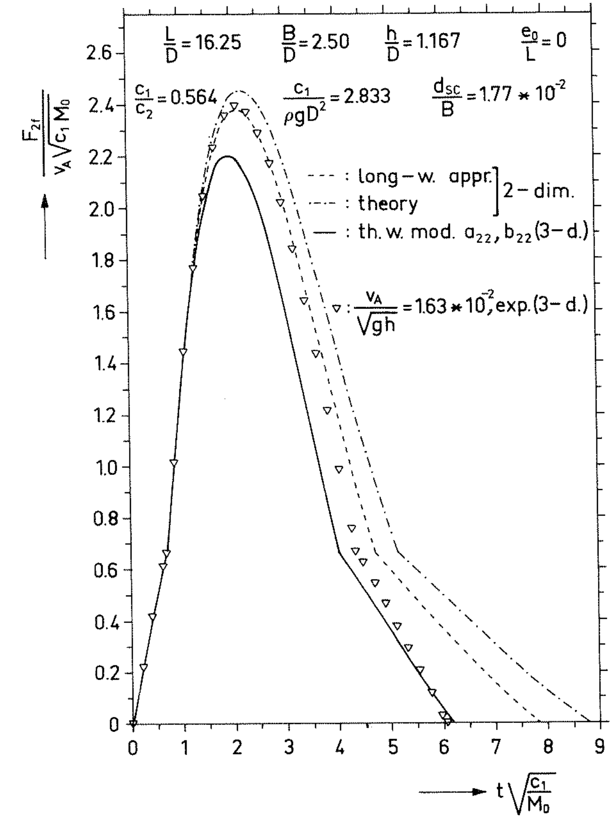

Fig. 21 - Time history of fender force: non-linear fender, centric impact ($h/D = 1.167$).

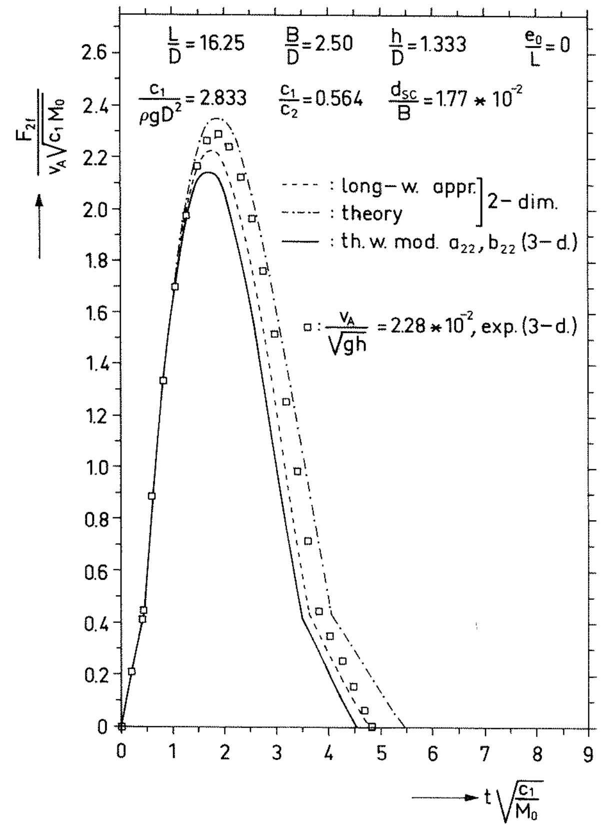

Fig. 22 - Time history of fender force: non-linear fender, centric impact (h/D = 1.333 .

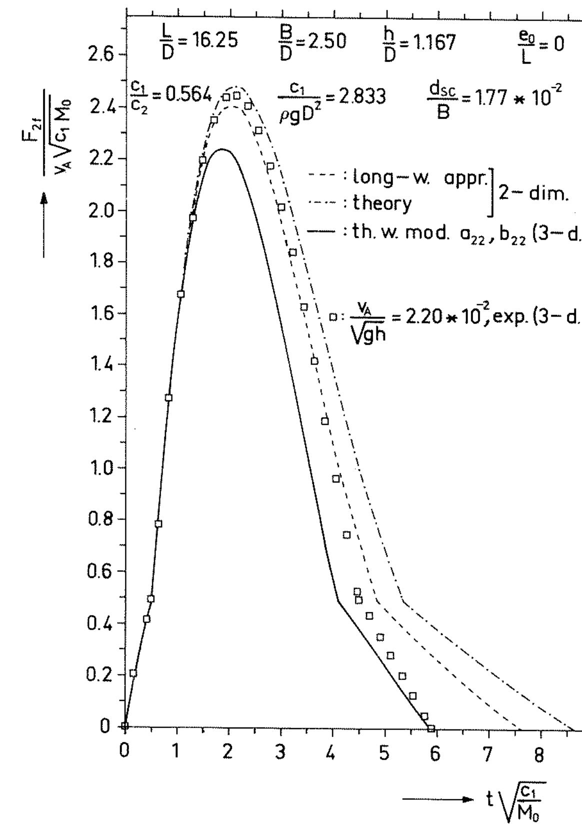

Fig. 23 - Time history of fender force: non-linear fender, centric impact (h/D = 1.167).

h/D and the dimensionless constant lateral speed of approach $v_A(gh)^{-1/2}$ as parameters. The dimensionless fender characteristics are represented by $c_1(\rho g D^2)^{-1}$, c_1/c_2 and d_{sc}/B. From figs. 20 through 23 it appears that a greater lateral speed of approach causes an increase of the (maximum value of the) fender force and a decrease of the length of time of the contact between ship and fender; likewise, the point of time at which the fender force reaches its maximum occurs earlier in case of a greater lateral speed of approach. Farther, in case of increasing water depth the (maximum value of the) fender force as well as the length of time of the contact between ship and fender decrease, while the point of time at which the fender force reaches its maximum occurs earlier.

The total amount of energy E absorbed by a fender with linear behaviour is given by

$$E = \int_0^{(\Delta x_{2f})_{max}} F_{2f}(t)\; d(\Delta x_{2f}) = \frac{1}{2}\, c_0\; (\Delta x_{2f})^2_{max},$$

where $(\Delta x_{2f})_{max}$ = maximum impression of the (linear) fender.
By means of this expression the influence of the fender stiffness in case of a linear fender on the absorption of energy can be represented. However, problems arise when the fender is infinitely stiff (i.e. $c_0 \to \infty$) and when the fender is infinitely soft (i.e. $c_0 = 0$). The kinetic energy of the schematized ship at the first moment of contact between ship and fender in case of a constant lateral speed of approach v_A - this implies $\omega = 0$ - is:

$$\frac{1}{2}\,\{\rho LBD + a_{22}(0)\}\; v_A^2$$

In case $c_0 \to \infty$, during the impact the total kinetic energy of the schematized ship is transferred to the '(linear) fender' in a length of time $\Delta t = 0$; the (hydrodynamic) damping then can be neglected.

As small lengths of time correspond with high (circular) frequencies - $\omega = \infty$ is predominant -, the total amount of energy absorbed by the '(linear) fender' in case $c_0 \to \infty$ becomes:

$$\frac{1}{2} c_0 (\Delta x_{2f})^2_{max} \longrightarrow \frac{1}{2} \{\rho LBD + a_{22}(\infty)\} v_A^2 \qquad \text{for } c_0 \to \infty .$$

In case $c_0 = 0$ the presence of the fender is not palpable. During the 'impact' the kinetic energy of the schematized ship is transferred to the '(linear) fender' in a length of time $\Delta t \to \infty$; the (hydrodynamic) damping does not play any part since the lateral speed of the ship does not change: v_A is maintained. The energy 'absorbed by the (linear) fender' equals the kinetic energy of the ship at the first moment of 'contact' between ship and fender, and becomes:

$$\frac{1}{2} c_0 (\Delta x_{2f})^2_{max} \longrightarrow \frac{1}{2} \{\rho LBD + a_{22}(0)\} v_A^2 \qquad \text{for } c_0 \downarrow 0.$$

Figs. 24 and 25 show the dimensionless absorbed energy, $c_0 (\Delta x_{2f})^2_{max} (\rho LBD)^{-1} v_A^2$, versus the dimensionless fender characteristic $c_0 (\rho g D^2)^{-1}$, with the dimensionless water depth h/D as parameter. In these figures the total amount of absorbed energy, $\frac{1}{2} c_0 (\Delta x_{2f})^2_{max}$, was made dimensionless with the kinetic energy as possessed by the schematized ship before and during the first contact between ship and fender in case of the absence of water, viz. $\frac{1}{2} \rho LBD v_A^2$. From figs. 24 and 25 it can be seen that - at constant water depth - a stiff fender absorbs less energy to stop the ship than a soft fender. This effect is caused by the greater wave radiation in case of a stiffer fender. Farther, in case of a smaller water depth the total amount of energy as absorbed by the (linear) fender increases.

Generally it can be stated that the agreement between theory and experiment is amply satisfactory. Notably by means of the theory adapted to the three-dimensional situation the (maximum values of the) fender forces as well as the lengths of time of the contact

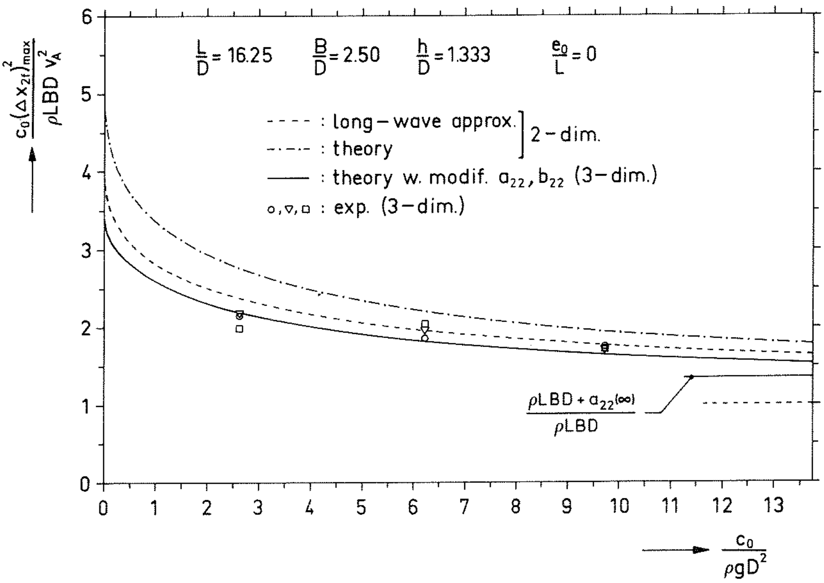

Fig. 24 - Influence of fender elasticity on absorbed energy: linear fender, centric impact (h/D = 1.333).

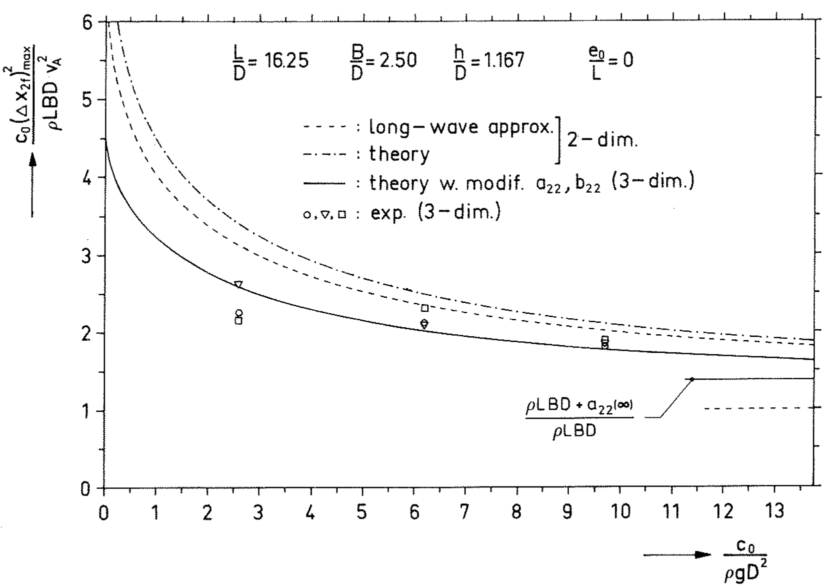

Fig. 25 - Influence of fender elasticity on absorbed energy: linear fender, centric impact (h/D = 1.167).

between ship and fender, and the points of time at which the fender forces reach their maxima are predicted very well. This applies for both water depths investigated.
For a first estimation of the maximum value of the fender force use can be made of the long-wave approximation or - with less accuracy - of the two-dimensional theory. The same holds in case of a first evaluation of both the lengths of time of the contact between ship and fender and the points of time at which the fender forces reach their maxima. For the smaller water depth the long-wave approximation as well as the two-dimensional theory do not provide a very good prediction of the length of time of the contact between ship and fender.

In the mathematical model viscous effects have not been taken into account. Since in model tests these effects are overestimated, it may be concluded from the good agreement between calculated and measured results that the viscosity of the fluid does not influence the fender forces significantly.

Although the combination of purely lateral speed of approach and 'centric impact' not often will occur, the experimental and theoretical results demonstrate that the mathematical model presented provides a good foundation for the determination of the forces exerted by a moving ship on some (open) berthing structure, equipped with linear or non-linear fenders.

III.3.c.2: ECCENTRIC IMPACTS: $e_0 \neq 0$

In case of an 'eccentric impact' the results for $F_{2f}(t)$ as well as the results for $x_{2G}(t)$ and $\psi(t)$ have to be presented.

Figs. 26 through 33 show $F_{2f}v_A^{-1}(c_0M_0)^{-1/2}$, $x_{2G}v_A^{-1}(c_0/M_0)^{1/2}$ and $\psi m_{66}(v_Ae_0M_0)^{-1}(c_0/M_0)^{1/2}$ versus $t(c_0/M_0)^{1/2}$ for the case of a linear fender, with the dimensionless water depth h/D and the dimensionless initial distance of the line of action of the fender to the ship's centre of gravity, e_0/L, as parameters. The dimensionless

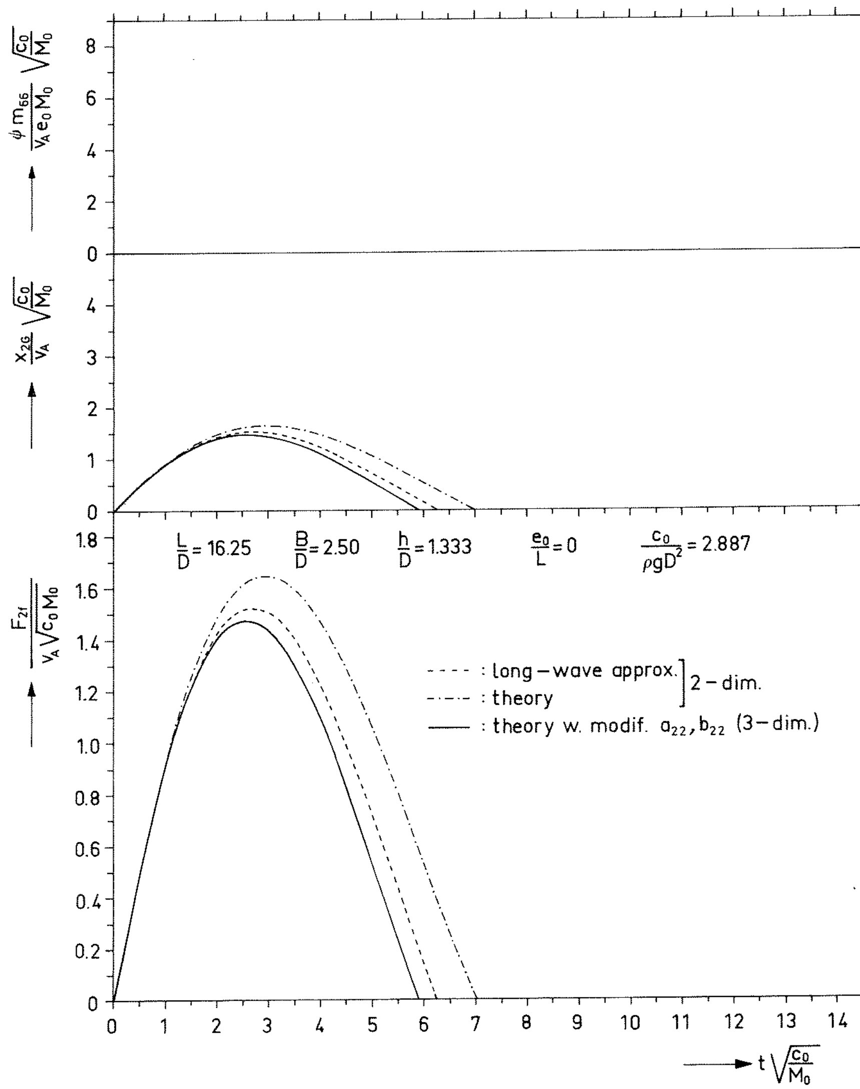

Fig. 26 - Time histories of fender force, translation of G and angle of rotation: linear fender, centric impact (h/D = 1.333).

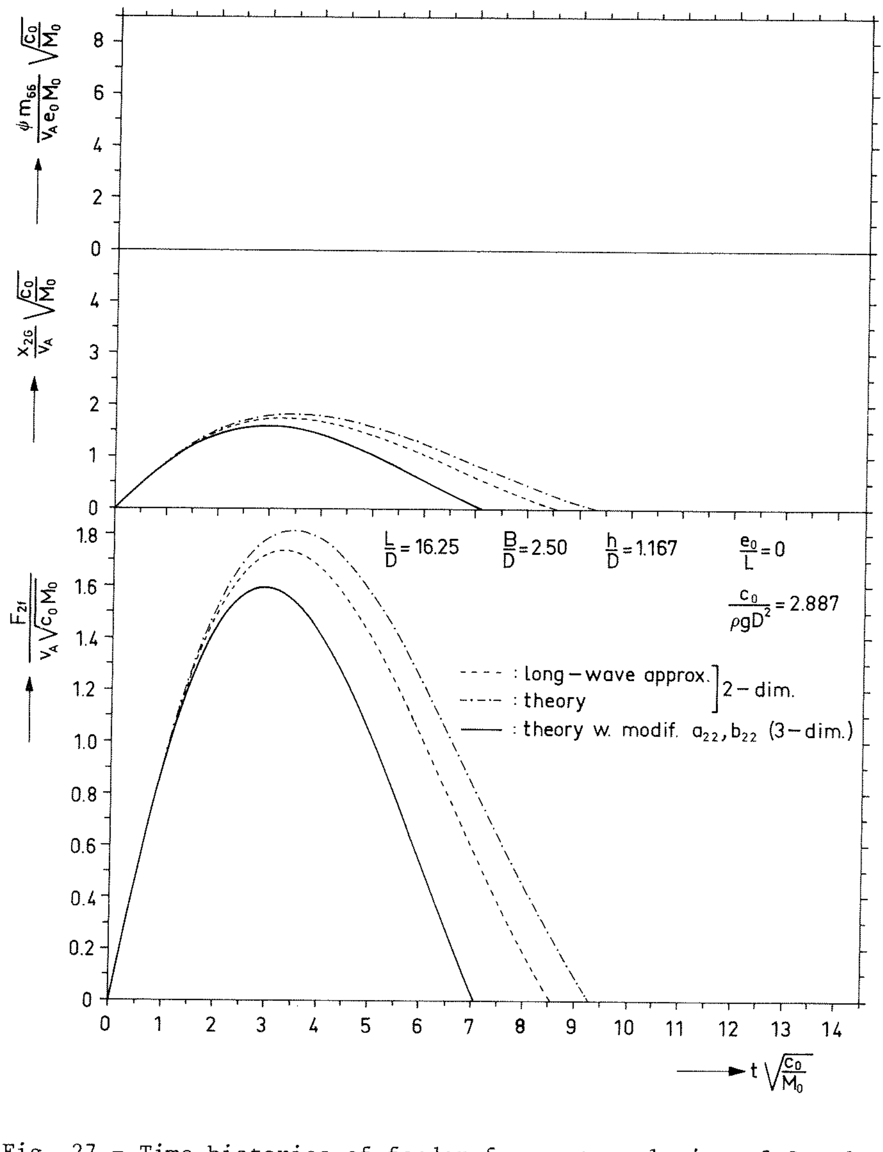

Fig. 27 - Time histories of fender force, translation of G and angle of rotation: linear fender, centric impact (h/D = 1.167).

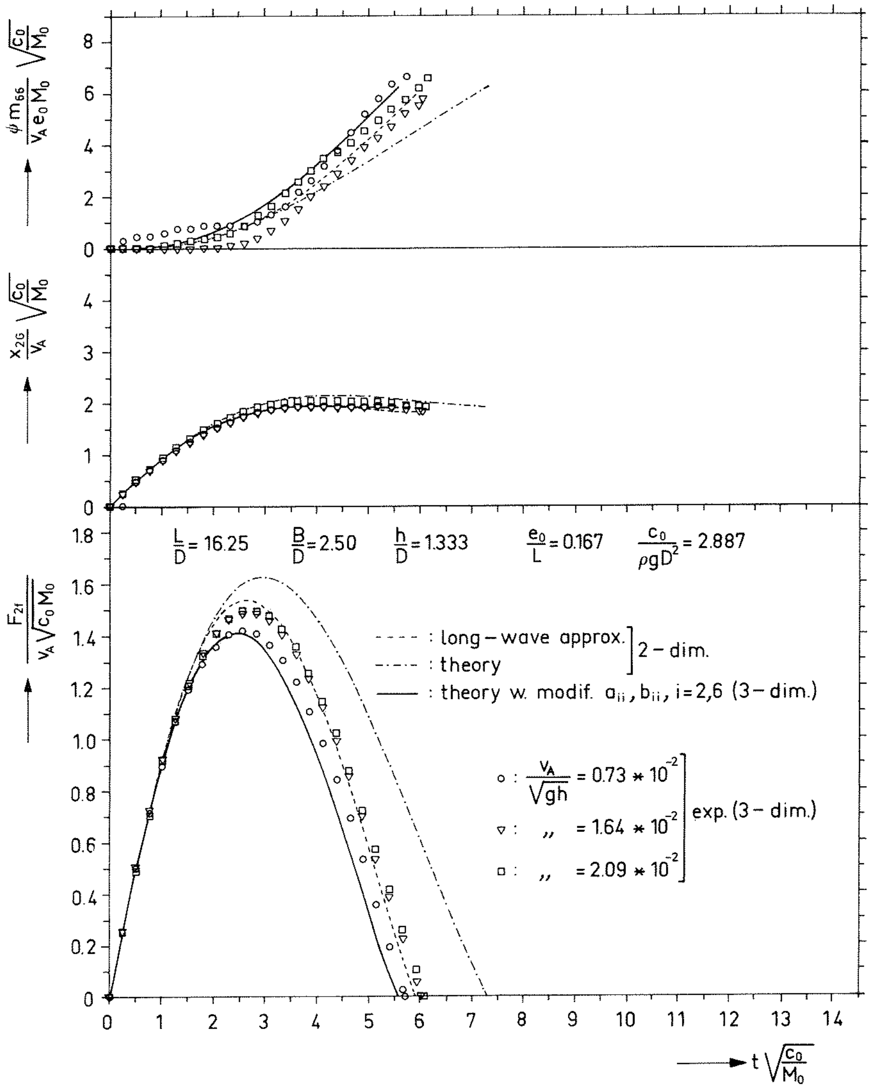

Fig. 28 - Time histories of fender force, translation of G and angle of rotation: linear fender, eccentric impact (h/D = 1.333).

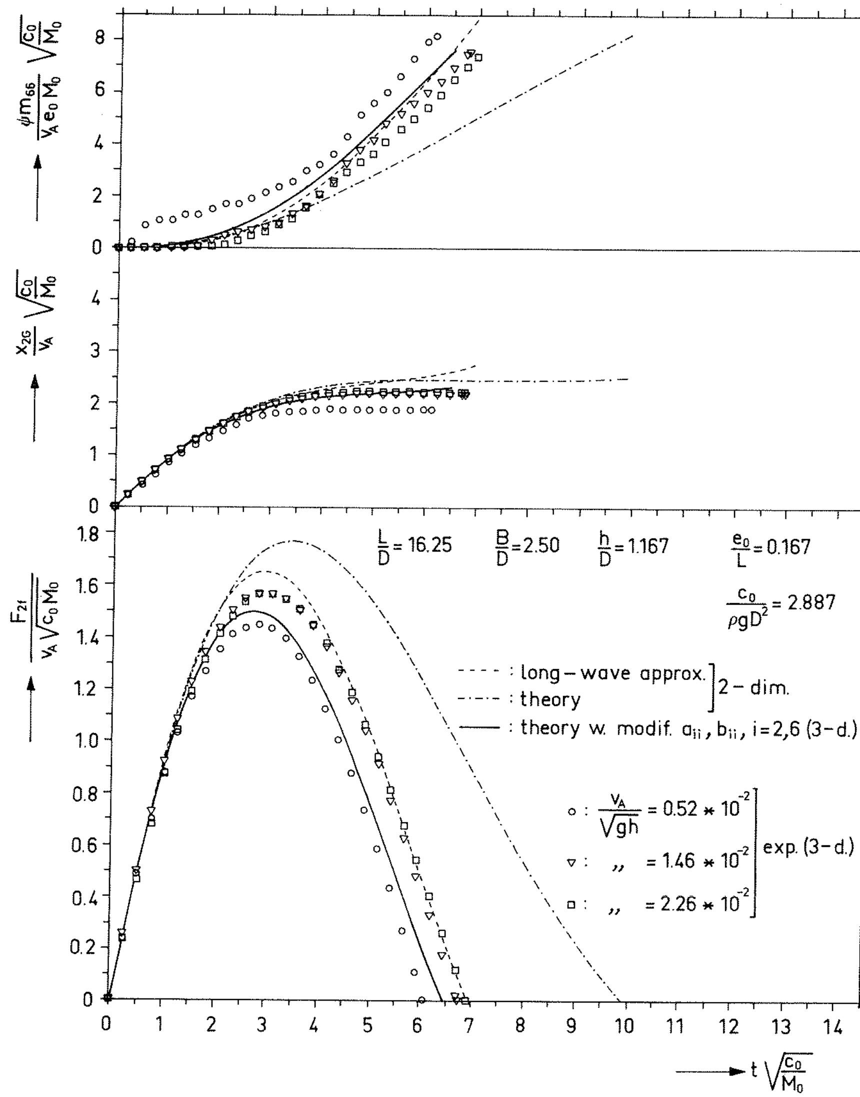

Fig. 29 - Time histories of fender force, translation of G and angle of rotation: linear fender, eccentric impact (h/D = 1.167).

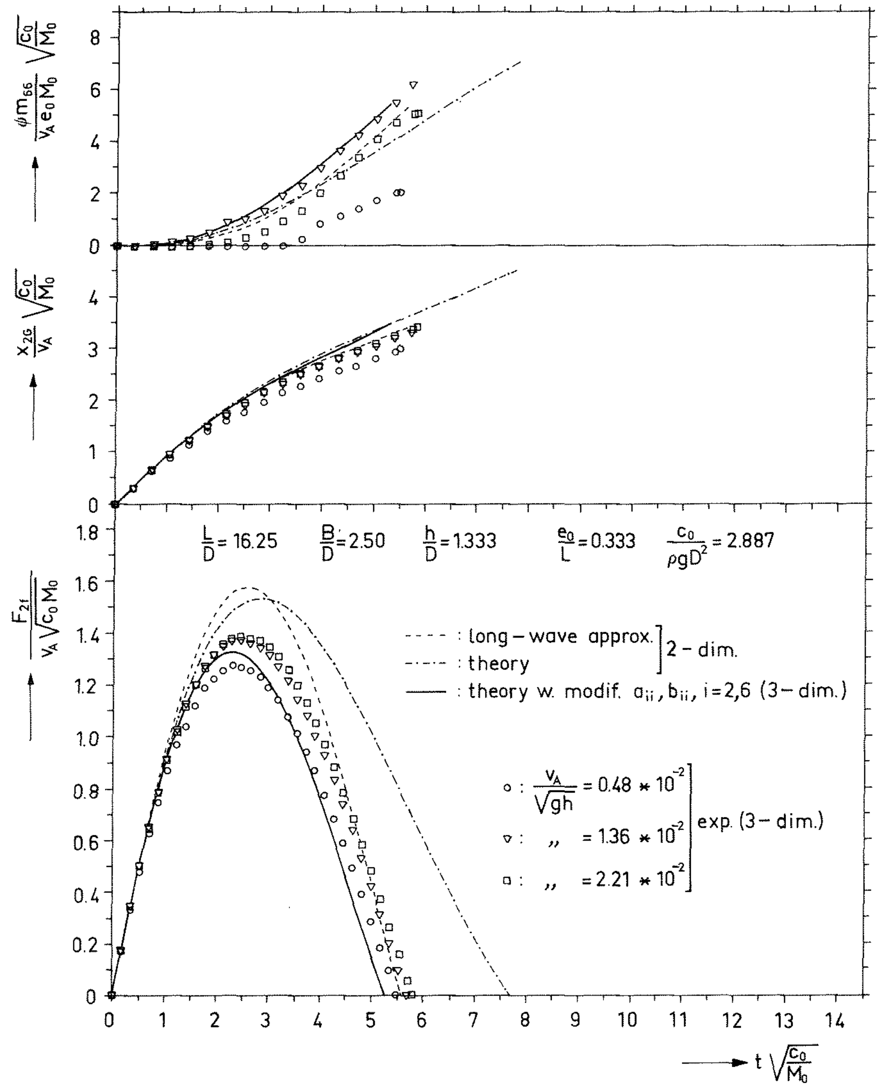

Fig. 30 - Time histories of fender force, translation of G and angle of rotation: linear fender, eccentric impact (h/D = 1.333).

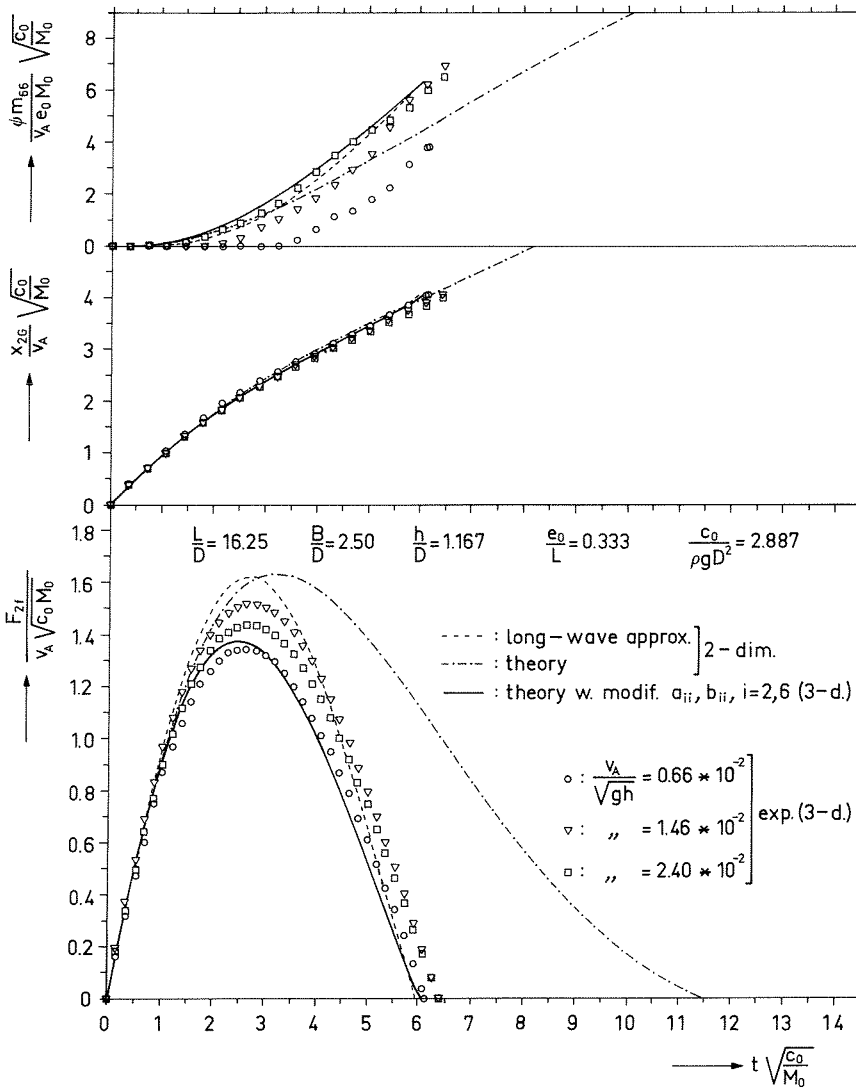

Fig. 31 - Time histories of fender force, translation of G and angle of rotation: linear fender, eccentric impact (h/D = 1.167).

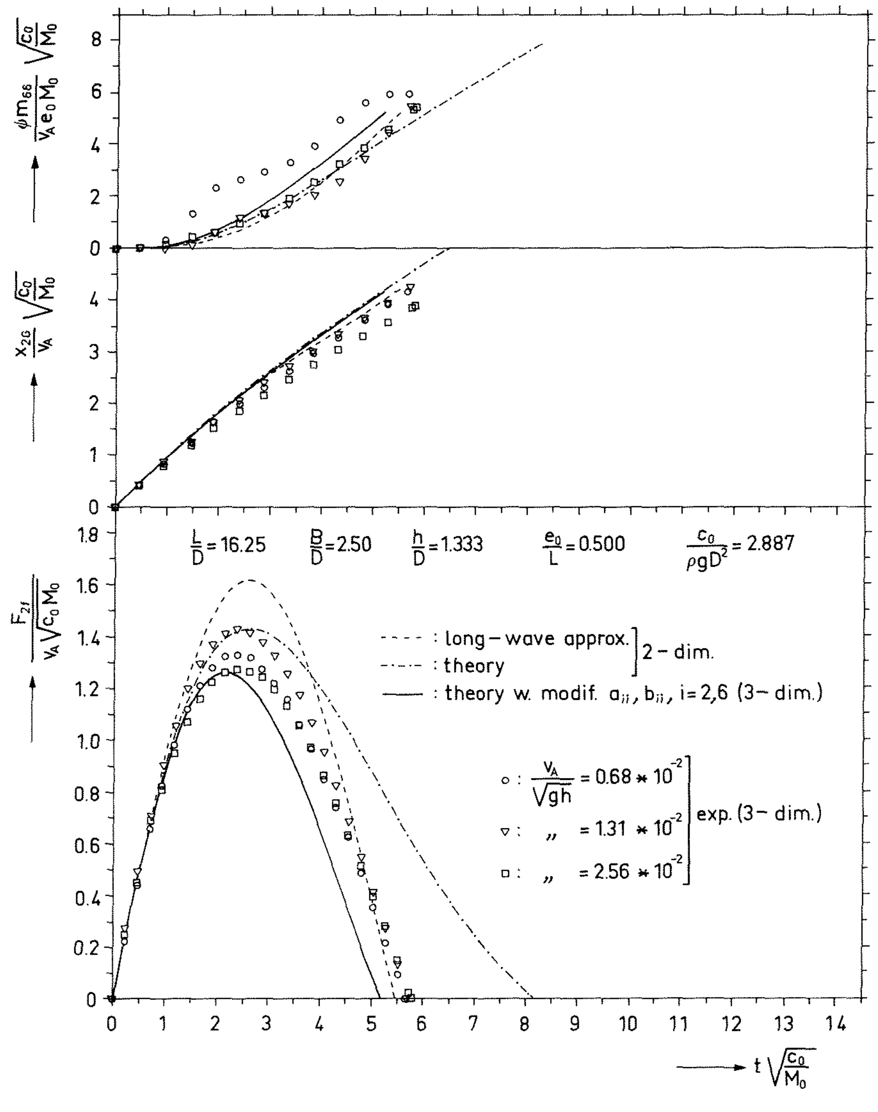

Fig. 32 - Time histories of fender force, translation of G and angle of rotation: linear fender, eccentric impact (h/D = 1.333).

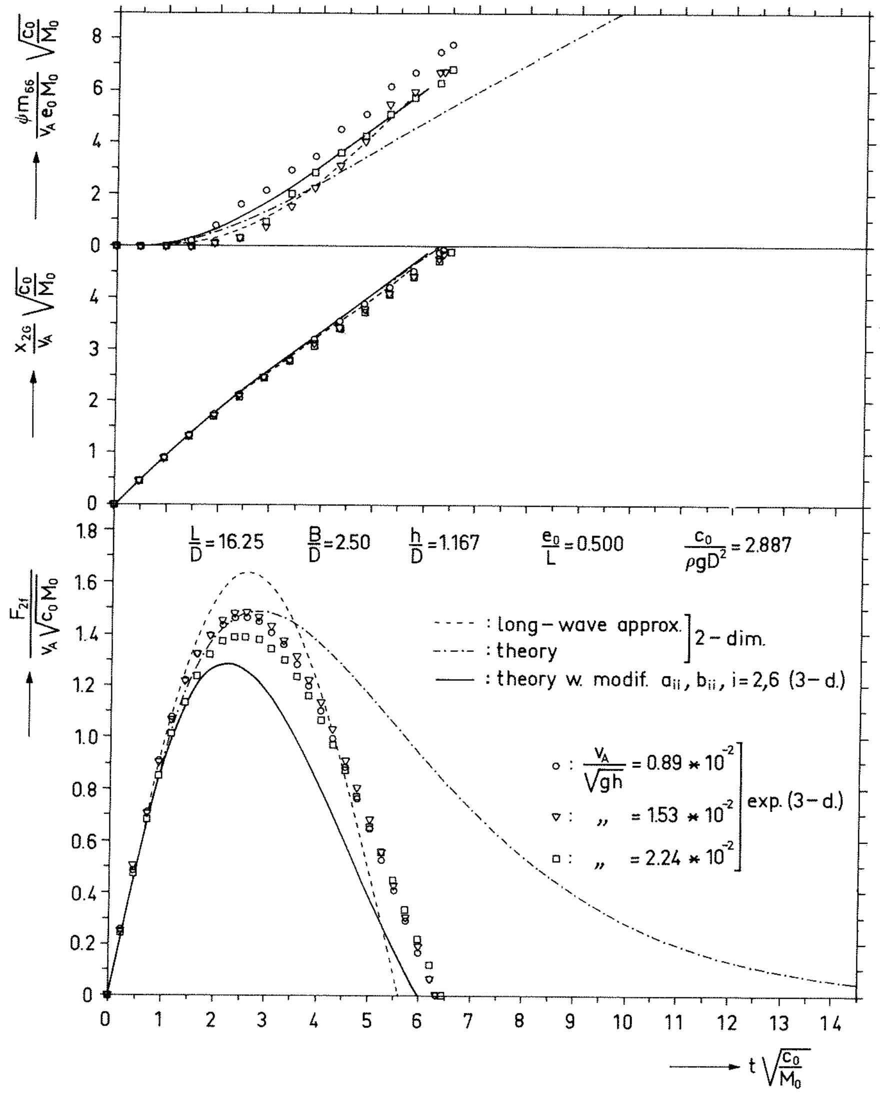

Fig. 33 - Time histories of fender force, translation of G and angle of rotation: linear fender, eccentric impact (h/D = 1.167).

fender characteristic is represented by $c_0(\rho g D^2)^{-1}$. Although for the case $e_0 = 0$ experimental results are not available, for the sake of completeness the calculated results are presented (figs. 26 and 27). As expected the calculated fender forces, the calculated translations of the ship's centre of gravity G and the calculated angles of rotation of the ship's longitudinal axis of symmetry can be considered to be proportional to the constant lateral speed of approach.
From figs. 26 through 33 it can be seen that in case of increasing value of e_0 the (maximum value of the) fender force as well as the length of time of the contact between ship and fender decreases, while the point of time at which the fender force reaches its maximum occurs earlier. This general trend does not apply to the length of time of the contact between ship and fender as calculated by means of the two-dimensional theory for the case with $e_0/L = 0.500$ and $h/D = 1.167$: the gradual sagging of the branch of the curve on the right of the maximum value of the fender force has to be imputed to numerical causes. Farther, in case of a greater water depth the (maximum value of the) fender force as well as the length of time of the contact between ship and fender is smaller, and the point of time at which the fender force reaches its maximum value occurs earlier. Froms figs. 26 through 33 it also appears that in case of a greater water depth the maximum value of the translation of the ship's centre of gravity G as well as the total angle of rotation of the ship's longitudinal axis of symmetry is smaller; mainly this is due to the influence of the shorter length of time of the contact between ship and fender in case of a greater water depth.
In case of increasing value of e_0 it is not possible to describe in general terms the trend of the maximum value of the translation of G and the total angle of rotation of the ship's longitudinal axis of symmetry: this is a consequence of the fact that the values of these both quantities are influenced by the length of time of the contact between ship and fender. So far as the maximum value of the translation of the ship's centre of gravity G is concerned, the

results as calculated by means of the long-wave approximation and the theory adapted to the three-dimensional situation show - for both water depths - the same trend. Farther, all calculated results for the total angles of rotation of the ship's longitudinal axis of symmetry show the same features, for both water depths; due to numerical causes the result calculated by means of the two-dimensional theory for the case with $e_0/L = 0.500$ and $h/D = 1.167$ has to be excluded.
The test results for $\psi(t)$ at the smallest speed of approach may show (locally) some discrepancies with respect to the test results at the higher speeds of approach. This is caused by the relative weak signals combined with tolerance(s) in the gear-wheel transmission which transfers the angle of rotation of the ship's longitudinal plane of symmetry to the recording potentiometer. These tolerances have a greater influence on the measured results as the signals are weaker.

Generally it can be stated that the agreement between theory and experiment is satisfactory. Notably by means of the theory adapted to the three-dimensional situation the (maximum values of the) fender forces as well as the lengths of time of the contact between ship and fender, and the points of time at which the fender forces reach their maxima are predicted well. The same holds good with respect to the maximum values of the translation of the ship's centre of gravity G and the total angles of rotation of the ship's longitudinal axis of symmetry. This applies for both water depths investigated.
For a first estimation of the maximum value of the fender force use can be made of the long-wave approximation or - with less accuracy - of the two-dimensional theory. The same holds in case of a first valuation of both the lengths of time of the contact between ship and fender and the points of time at which the fender forces reach their maxima. For the smaller water depth the two-dimensional theory does not provide a good prediction of the length of time of the contact between ship and fender. Farther, by means of the long-wave

approximation a rather reliable estimation can be made for the maximum value of the translation of the ship's centre of gravity G as well as for the total angle of rotation of the ship's longitudinal axis of symmetry.

From the satisfactory agreement between calculated and measured results it may be concluded that - just as in the case of the centric impacts - the viscosity of the fluid does not influence significantly the relevant quantities which play a part in case of eccentric impacts.

The experimental and theoretical results demonstrate that the mathematical model presented provides a good foundation for the description and the determination of the relevant quantities which figure in the problem of a ship berthing to some (open) structure equipped with fenders.

III.3.d: QUALITATIVE ANALYSIS OF RESULTS

From the results of the investigations as presented in the preceding Section III.3.c it may be concluded that the typical behaviour of a (schematized) ship berthing to a(n open) structure equipped with fenders as well as the response of the fenders themselves can be predicted by the (complicated) mathematical model as described in this report. Generally it holds that both the qualitative and the quantitative agreement between theory and experiment is satisfactory: especially the theory adapted to the three-dimensional situation yields results which differ (very) little from the measured values; for a first estimation of the relevant quantities figuring in the berthing ship problem use can be made of the relatively simple long-wave approximation.

Although the (quantitative) agreement between the results of the mathematical model and those of the physical model in most cases is very reasonable, some discrepancies remain. It is not clear whether these differences are due to experimental errors or due to limitations of the mathematical model.

Experimental errors may be caused by:

- imperfections in the test set-up, such as the restricted horizontal dimensions of the rectangular basin in which the tests were carried out, small differences in height from the horizontal position of that part of the bottom of the basin covered by the motions of the ship model, damping and dynamic effects in the fender, frictional effects between ship model and fender, flexibility of the 'berthing structure', a lateral speed of approach which is not exactly a constant, the possibility that the line of action of the fender is not precisely perpendicular to the longitudinal axis of symmetry of the ship at the first contact between ship and fender, deviations of the intended values of e_0, dynamic effects, damping, friction and tolerance(s) in both the 'position follower' and the facility to measure the angle of rotation of the ship's longitudinal plane of symmetry;
- measuring errors as a result of the limited accuracy of the electronically measuring and recording equipment;
- evaluation errors due to the process of converting analogue signals recorded on paper chart to proper figures.

The magnitude of this first category of errors is hard to estimate, but the total error due to measuring and evaluation inaccuracies is valued at less than five per cent. An exception to this is formed by the error in the measurement of the angle of rotation $\psi(t)$, which may be greater than five per cent. in consequence of mechanical imperfections of the measuring facility (see Section III.3.c.2).

- The 'i.r.f.'-technique used is based on the assumption that the ship-fluid system - and therefore the fluid reactive forces - are linear. All (experimental) investigations (see e.g. refs. [20, 23, 26, 31]) indicate that this basic linearity assumption is a good working approximation for small to moderate displacements of real ship forms. Especially in ref. [30] it was shown experimentally that this assumption holds true for displacements with an order of magnitude as occurring during the contact between ship and fender.
- Fluid reactive forces from viscous origin have been neglected. The

largest influence can be expected in the sway mode of motion. An estimate of these forces can be made using the empirical formula:

$$F_{2,\text{viscous}} = -\frac{1}{2}\rho\, C_D L D\, \dot{x}_2 |\dot{x}_2| \ ,$$

where $F_{2,\text{ viscous}}$ = fluid reactive force on the ship from viscous origin in the sway mode of motion,

C_D = drag coefficient.

Inclusion of this force in the mathematical model did not change the results significantly.

Since the fluid is assumed to be inviscid (and incompressible) and to move irrotationally, the mathematical model is only suitable to describe transient ship motions and does certainly not apply to (nearly) steady motions. For, in that case the viscous effects are no longer small with respect to the 'potential part' of the mathematical model and they may change the picture entirely.

For remarks on the influence of the effects of the strip theory, the neglect of viscosity and the end effects on the hydrodynamic coefficients is referred to ref. [30].

- The fluid reactive forces are taken into account by way of the hydrodynamic coefficients, which were determined theoretically and experimentally (see ref. [30]). On closer examination of the relevant quantities of the berthing ship problem, comparison of experimental results with results as calculated on the one side by means of the two-dimensional theory and on the other hand by means of the theory adapted to the three-dimensional situation shows that the accuracy of the hydrodynamic coefficients - especially in the lower frequency range - may be an important factor. Fairly large differences in the hydrodynamic coefficients for the lower (circular) frequencies may occur without the result of a significant change in the (maximum values of the) fender forces, the lengths of time of the contact between ship and fender, the points of time at which the fender forces reach their maxima, the translations of the ship's centre of gravity G and the angles of rotation of the ship's longitudinal axis of symmetry.

- Certain approximations are involved in the numerical calculation of the i.r.f.'s and the set of two integro-differential eqs. ($41^{a,b}$). The accuracy of the numerical calculation of the i.r.f.'s is discussed in Appendix V. Farther, to arrive at an accuracy as great as possible the calculation of the eqs. ($41^{a,b}$) was carried out with a time step Δt = 0.01 s (see further Section III.3.b).

III.4: EXTENSION OF THE MATHEMATICAL MODEL TO OTHER SITUATIONS

In this Section III the approach of analysing the berthing of a ship with the aid of the 'i.r.f.'-technique was applied to a (schematized) ship berthing to an open structure equipped with one single undamped, (non-)linear fender.
Naturally, the same approach can also be used for an open structure equipped with two or more - damped or undamped - (non-)linear fenders. The case of a ship berthing to a solid structure equipped with fenders - e.g. a quay parallel to the ship - can be treated in precisely the same way as described in this resport by using the 'i.r.f.'-technique, provided that the relevant i.r.f.'s, and therefore the hydrodynamic coefficients, are known.
Farther, the 'i.r.f.'-technique as applied enables the inclusion of other arbitrarily in time varying forces; in this context it can be thought of forces exerted upon the ship by wind, waves, current, tugs and mooring lines.

SECTION IV: CONCLUSIONS

On account of the theoretical and experimental research of the berthing ship problem as presented in this report the following conclusions can be drawn.

The fact that the hydrodynamic coefficients are dependent on the circular frequency necessitates a time-domain approach for the analysis of the berthing ship problem, in which the fluid reactive forces are described by way of the hydrodynamic coefficients and the remaining forces are taken into account over their entire time history. The 'i.r.f.'-technique used satisfies these requirements.

The two important assumptions of fluid idealization and linearity of the ship-fluid system are very well acceptable for the quantitative analysis of transient motions of shiplike bodies.

The formulation of the mathematical model presented is sufficiently accurate to be a valuable basis for the qualitative and quantitative description of the typical behaviour of a ship berthing to a structure equipped with fenders as well as for the determination of the response of the fenders themselves.
The fenders may be damped or undamped, linear or non-linear; combinations of these options may occur.
In addition to the fender forces other external forces upon the ship such as forces exerted by wind, waves, current, tugs and mooring lines can be incorporated in the mathematical model as well.
The calculated results show a (very) satisfactory agreement with values obtained from measurements on (small) scale models. The viscosity of the fluid does not influence significantly the relevant quantities which play a part in the berthing phenomenon.

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APPENDICES

APPENDIX I: THE STABILITY OF THE LINEAR SHIP-FLUID SYSTEM

The stability of the linear ship-fluid system can be defined as follows (see ref. [A1]).
Suppose that the (linear) system is subjected to two different arbitrary pulses (i.e. input signals) and thereupon, in both cases, is let to take its own course. Then the (linear) system is stable if the difference of the two results (i.e. output signals) converges for $t \to \infty$, and asymptotically stable if the above-mentioned difference tends to zero for $t \to \infty$.
According to eq. (6^b) it applies for the linear ship-fluid system in the time domain:

$$(6^b) \qquad \dot{x}_j(t) = \sum_{i=1}^{6} \int_{-\infty}^{t} f_i(\tau)\, k_{ij}(t - \tau)\, d\tau , \qquad j = 1, 2, .., 6.$$

At the time $t = 0$ an arbitrary pulse in the i-direction is exerted upon the ship:

$$f_k(t) = \alpha_{ik}\, \zeta_i\, \delta_i(t),$$

where α_{ik} = Kronecker alpha, $\alpha_{ik} = 1$ for $k = i$, $\alpha_{ik} = 0$ for $k \neq i$,

ζ_i = (arbitrary) coefficient specifying the magnitude of the pulse in the i-direction,

$\delta_i(t)$ = unit pulse (i.e. Dirac or delta function) in the i-direction.

Substitution of $f_k(t)$ into eq. (6^b) yields:

$$\text{case I,} \qquad \dot{x}_j(t)\Big|_I = \zeta_i \int_{-\infty}^{t} \delta_i(\tau)\, k_{ij}(t - \tau)\, d\tau = \zeta_i\, k_{ij}(t).$$

Then, at the time $t = t_1$ ($t_1 \geq 0$) an other arbitrary pulse, also in the i-direction, is exerted upon the ship:

$$f_k(t) = \alpha_{ik}\,\eta_i\,\delta_i(t - t_1),$$

where η_i = (arbitrary) coefficient specifying the magnitude of the pulse in the i-direction.

Substitution of this expression for $f_k(t)$ into eq. (6^b) gives:

case II, $$\dot{x}_j(t)\Big|_{II} = \eta_i \int_{-\infty}^{t} \delta_i(\tau - t_1)\,k_{ij}(t-\tau)\,d\tau = \eta_i\,k_{ij}(t-t_1).$$

The difference of these two results (i.e. output signals) can be written as:

$$\dot{x}_j(t)\Big|_{II} - \dot{x}_j(t)\Big|_{I} = \eta_i\,k_{ij}(t - t_1) - \zeta_i\,k_{ij}(t).$$

In case $t \to \infty$ the limit of this expression then yields:

$$\lim_{t\to\infty} \{\dot{x}_j(t)\Big|_{II} - \dot{x}_j(t)\Big|_{I}\} = \eta_i \lim_{t\to\infty} k_{ij}(t - t_1) - \zeta_i \lim_{t\to\infty} k_{ij}(t),$$

which can be reduced to:

$$\lim_{t\to\infty} \{\dot{x}_j(t)\Big|_{II} - \dot{x}_j(t)\Big|_{I}\} = (\eta_i - \zeta_i)\,k_{ij}(\infty),$$

since $\lim_{t\to\infty} k_{ij}(t - t_1) = k_{ij}(\infty)$.

The linear ship-fluid system now is stable in any case, if this limit does converge, i.e. if $k_{ij}(\infty)$ = constant:

- if $k_{ij}(\infty) = 0$, the linear ship-fluid system is asymptotically stable; the modes of motion (heave, roll and pitch) have a restoring force;
- if $k_{ij}(\infty)$ = constant $\neq 0$, the linear ship-fluid system is stable, the modes of motion (surge, sway and yaw) have no restoring force.

APPENDIX II: THE BEHAVIOUR OF $k_{ii}(t)$ FOR $i = 1, 2, 6$ AS $t \to \infty$

In order to give a physical interpretation as well as an explanation of the behaviour of the impulse response function as t tends to infinity for ship motions without restoring force, it is started from a specific case - this for the sake of simplicity - with the following features: the ship moves merely in the horizontal plane at zero forward speed, the water is calm (no waves, no current) and has unrestricted horizontal dimensions, the ship motions (surge, sway and yaw) are uncoupled.

In case of uncoupled motions it can be written for the linear ship-fluid system in the time domain (see eq. (19)):

$$\dot{x}_i(t) = \int_{-\infty}^{t} f_i(\tau)\, k_{ii}(t - \tau)\, d\tau, \qquad i = 1, 2, 6.$$

Upon the ship now such a force (moment) is exerted, that it translates (rotates) with a constant velocity, viz.:

$$\dot{x}_i = v_{oi}\, U(t) \ ,$$

where v_{oi} = constant velocity of the ship in the i-direction,

$$U(t) = \text{unit step function} = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2} & \text{for } t = 0 \\ 1 & \text{for } t > 0\ . \end{cases}$$

For $t > 0$ it then holds good:

$$v_{oi} = \int_{0}^{t} f_i(\tau)\, k_{ii}(t - \tau)\, d\tau.$$

Taking the Laplace transform of this expression one obtains:

$$\frac{v_{oi}}{s} = \mathcal{L}\{f_i(t)\}\ \mathcal{L}\{k_{ii}(t)\}\ , \qquad \mathrm{Re}[s] > \mathrm{Re}[s_1],$$

where $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$ = Laplace transform of the function f(t),

s = complex variable,

s_1 = certain complex number,

Re[..] = real part of ... ;

in doing so it is supposed that $\mathcal{L}\{f_i(t)\}$ and $\mathcal{L}\{k_{ii}(t)\}$ exist and that at least one of these Laplace transforms converges absolutely. If $\mathcal{L}\{f(t)\}$ does exist and f(t) has a limit for $t \to \infty$ then it holds (see ref. [42]):

$$\lim_{s \downarrow 0} s\, \mathcal{L}\{f(t)\} = \lim_{t \to \infty} f(t).$$

Making use of this lemma one obtains:

$$\lim_{t \to \infty} f_i(t) = f_i(\infty) = \lim_{s \downarrow 0} s\, \mathcal{L}\{f_i(t)\},$$

or:

$$f_i(\infty) = \lim_{s \downarrow 0} \frac{v_{oi}}{\mathcal{L}\{k_{ii}(t)\}} = \lim_{s \downarrow 0} \frac{v_{oi}}{\int_0^\infty k_{ii}(t)\, e^{-st}\, dt} =$$

$$= \frac{v_{oi}}{\int_0^\infty k_{ii}(t)\, dt}.$$

Under the circumstances mentioned above a steady state will come into being for great values of t, since it is supposed that the fluid is inviscid and incompressible and moves irrotationally. In view of linearity only small velocities are considered. Waves which are generated in the beginning, have already travelled away from the ship for great values of t. Separation of flow and vortex shedding, which do occur in reality and produce a resistance that is proportional to certain (positive) power of the velocity, have to be neglected in this linearized approach. Concluding it can be stated that the resulting force (moment) upon the ship for great values of t must equal zero, so:

$$f_i(\infty) = 0 = \frac{v_{oi}}{\int_0^\infty k_{ii}(t)\, dt} \quad ,$$

from which it follows that:

$$\int_0^\infty k_{ii}(t)\, dt \longrightarrow \infty \quad ,$$

in other words, this integral does not converge absolutely.
Now there are two further possibilities, viz.: $k_{ii}(\infty) > 0$ and $k_{ii}(\infty) = 0$. If $k_{ii}(\infty) > 0$, the ship - after getting a pulse at $t = 0$ - will keep a final velocity greater than zero; if $k_{ii}(\infty) = 0$, the ship slows down until its velocity equals zero. This can be explained as follows:
let

$$f_i(t) = \delta_i(t),$$

where $\delta_i(t)$ = unit pulse (i.e. Dirac or delta function) in the i-direction,
then it holds good that:

$$\dot{x}_i(t) = \int_0^\infty \delta_i(\tau)\, k_{ii}(t - \tau)\, d\tau = k_{ii}(t),$$

and consequently:

$$\dot{x}_i(\infty) = k_{ii}(\infty).$$

As the ship at constant (rotational) velocity does not encounter any resistance, and as it 'forgets' its original (rotational) velocity with respect to the water, the ship generally will keep - after a pulse - in the long run a constant (rotational) velocity; this implies that $k_{ii}(\infty)$ = constant $\neq 0$ for i = 1, 2, 6.

APPENDIX III: COMPLEMENTARY REMARKS ON THE HYDRODYNAMIC COEFFICIENTS $a_{ii}(\omega)$ AND $b_{ii}(\omega)$ FOR $i = 1, 2, 6$

This appendix has to be considered as a supplement upon casu quo an extension of ref. [30].

In ref. [30] the hydrodynamic coefficients of a ship (model) were determined in case of swaying and yawing at zero speed of advance. The water was calm (no waves, no current) and shallow and had relatively large horizontal dimensions; the plane of the bottom was horizontal. Farther the ship (model) had been schematized to a rigid prismatic body with a rectangular cross-section and a symmetrical mass distribution, so that any coupling between sway and yaw motion did not exist.

The main dimensions of the schematized ship (model) were:

length (on the water-line)	L	2.438 m
beam	B	0.375 m
draught	D	0.150 m
block coefficient		1.000 .

The hydrodynamic coefficients were determined for two water depths, viz. $h = 0.200$ m and $h = 0.175$ m.

Farther, in the following is:

ρ = specific mass density of fluid = 1000 kg m^{-3},

g = acceleration due to gravity = 9.81 m s^{-2}.

A-III.1: ADDED MASS FOR SWAYING MOTION AT ZERO CIRCULAR FREQUENCY

In case $\omega = 0$ the values of the sway added mass ($a_{22}(\omega)$) from ref. [30] can be compared with those from ref. [A2]. Since in both references use is made of the (two-dimensional) strip theory, the results are given per unit length; this is indicated by adding an accent ' to the hydrodynamic coefficient in question.

The values are presented in dimensionless form in the table below. The agreement between the results from the two respective references is considered to be satisfactory.

$\frac{a'_{22}(0)}{\rho BD}$ ↘	ref. [A2]	ref. [30]	
	2-D pot. theory	2-D pot. theory	2-D long-w. appr.
$\frac{h}{D} = 1.333$	3.968	3.955	2.992
$\frac{h}{D} = 1.167$	6.850	7.087	5.972

$\frac{B}{D} = 2.50$

A-III.2: ADDED MASS FOR SWAYING MOTION AT HIGH CIRCULAR FREQUENCIES

Concerning the (special) problem of determining the sway added mass at high circular frequencies work has been done with regard to its relevance to ship vibrations. Reference is made to ref. [A3] where an excellent review is given of published data. In ref. [A4] experimentally sway added masses are determined for $\omega \rightarrow \infty$ by means of an electric analogon, taking into account the influence of a restricted water depth; the results apply to the case of zero forward speed on water with unrestricted horizontal dimensions; the sway added masses are given per unit length for ships with rectangular cross-sections. On basis of the data from ref. [A4] the following (dimensionless) added mass for the swaying motion at high circular frequencies is predicted:

$$\frac{a_{22}(\omega)}{\rho LBD} \simeq 0.35 \quad , \quad \omega \rightarrow \infty.$$

For comparison the following table with data from ref. [30] is provided:

$\frac{h}{D}$	$\frac{a_{22}(\infty)}{\rho LBD}$
1.333	0.356
1.167	0.377
1.000	0.433

The agreement of these data with those from ref. [A4] is satisfactory.

A-III.3: ESTIMATION OF HYDRODYNAMIC DAMPING FORCE COEFFICIENTS FOR HORIZONTAL MOTIONS AT HIGH CIRCULAR FREQUENCIES

Generally it holds good that the hydrodynamic coefficient of the damping force approaches asymptotically to zero with increasing circular frequencies.

To investigate the asymptotic behaviour of the damping coefficient for $\omega \to \infty$ a two-dimensional approach may be used. The relation between the damping coefficient and the amplitude of the radiated waves at infinity in case of a ship on water with unrestricted (horizontal) dimensions at zero forward speed is given in ref. [A5] to be (per unit length):

$$b'_{ii}(\omega) = \frac{\rho g^2}{\omega^3} R_i^2(\omega),$$

where $R_i(\omega)$ = ratio of the amplitude of the radiated waves at infinity to the amplitude of the (ship) motion in the i-direction = wave making coefficient for (ship) motion in the i-direction.

For shallow water the bottom has to be horizontal.

When the ship is approximated by a vertical barrier extending to the (sea) bottom (thus ignoring the keel clearance, which is permissible if the wave length is small compared with the draught of the ship) the behaviour of the damping coefficients for horizontal motions at high circular frequencies can be determined from refs. [A6] and [A7]. For the surge and sway modes of motion the ship then may be regarded as a piston type wave maker. According to refs. [A6] and [A7], respectively, the wave making coefficient has the form:

$$R_i(\omega) = \frac{2\sinh^2(m_0h)}{m_0h + \sinh(m_0h)\cosh(m_0h)} = \frac{2\{\cosh(2m_0h) - 1\}}{2m_0h + \sinh(2m_0h)}, \quad i = 1, 2,$$

where m_0 = usual wave number = positive root of $\omega^2 = gm_0 \tanh(m_0h)$. At high circular frequencies it holds good that $m_0 = \omega^2/g$. For the surge and sway modes of motion the wave making coefficient approaches to a constant value when $\omega \to \infty$:

$$R_i(\omega) = 2, \qquad \omega \to \infty, \qquad i = 1,2.$$

Consequently, for high circular frequencies the surge and sway damping force coefficients can be approximated by (per unit length):

$$b'_{ii}(\omega) = \frac{4\rho g^2}{\omega^3}, \qquad \omega \to \infty, \qquad i = 1,2.$$

On account of the fact that:

$$b_{ii}(\omega) = L\, b'_{ii}(\omega) \quad ,$$

the sway damping force coefficient becomes for high circular frequencies:

$$b_{22}(\omega) = \frac{4\rho g^2 L}{\omega^3}, \qquad \omega \to \infty \quad .$$

According to ref. [30] it holds good for the yaw damping moment coefficient $b_{66}(\omega)$ that at high circular frequencies:

$$b_{66}(\omega) = \frac{1}{12} L^2\, b_{22}(\omega), \qquad \omega \to \infty.$$

In general form the hydrodynamic damping force coefficients for horizontal motions at high circular frequencies now can be represented by:

$$b_{ii}(\omega) = \frac{q_i}{\omega^3} \quad , \qquad \omega \to \infty, \qquad i = 1,\ 2,\ 6,$$

where q_i = constant in approximative expression for $b_{ii}(\infty)$ in case $\omega \to \infty$;

q_i is dependent on the mode of motion, but independent of the water depth.

If the hydrodynamic damping force coefficients for horizontal motions are known from $\omega = 0$ to $\omega = \omega_1$ - where ω_1 represents a certain circular frequency -, then it is possible to choose the values of the constant q_i such that the high frequency approximations correspond with the known parts of the damping curves.

According to ref. [30] the sway damping force coefficient in case of zero keel clearance is:

$$b_{22}(\omega)\Big|_{h=D} = 2\rho L \frac{\omega}{m_0^2} \frac{2\sinh^2(m_0 h)}{m_0 h + \sinh(m_0 h)\cosh(m_0 h)} \quad .$$

For high circular frequencies this expression changes into:

$$b_{22}(\omega)\Big|_{h=D} = \frac{4\rho g^2 L}{\omega^3} \quad , \qquad \omega \to \infty \quad ,$$

which is identical with the corresponding expression for $b_{22}(\omega)$ derived above.

For high circular frequencies the sway damping force coefficient in dimensionless form is:

$$\frac{b_{22}}{\rho LBD}\sqrt{\frac{B}{g}} = \frac{4g^2}{BD}\sqrt{\frac{B}{g}}\frac{1}{\omega^3} \quad , \qquad \omega \to \infty \ .$$

In ref. [30] values for $b_{22}(\rho LBD)^{-1}(B/g)^{1/2}$ are presented as function of $\omega(B/g)^{1/2}$ with h/D as parameter. From these it appears that:

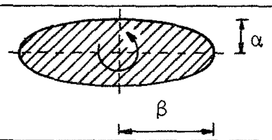
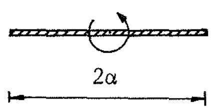
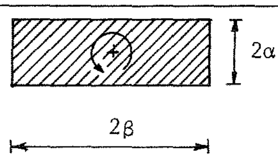
- in case of large values of $\omega(B/g)^{1/2}$ (say $\omega(B/g)^{1/2} > 3.9$) $b_{22}(\rho LBD)^{-1}(B/g)^{1/2}$ becomes independent of h/D, and
- for $\omega(B/g)^{1/2} > 2.8$, independent of h/D,

$$\frac{b_{22}}{\rho LBD}\sqrt{\frac{B}{g}} = \frac{4g^2}{BD}\sqrt{\frac{B}{g}}\frac{1}{\omega^3}, \qquad \omega \to \infty \qquad ,$$

makes a good approximation for the (dimensionless) sway damping force coefficient.

A-III.4: HYDRODYNAMIC COEFFICIENTS FOR YAWING MOTION

From the experimental results as presented in ref. [30] it appears that the dimensionless added mass-moment of inertia for the yawing motion $a_{66}(\frac{1}{12} L^2 \rho LBD)^{-1}$ is not very dependent on the circular frequency, at least for the frequency range considered in the experiments (i.e. for low values of $\omega(B/g)^{1/2}$). In other words, the influence of the free-surface of the fluid seems to play a minor part. Also the influence of the (dimensionless) water depth h/D is not very important. One thing and another seems to justify an approximation of the added mass-moment of inertia for the yawing motion - certainly for the lower frequency range - by means of the relevant value for 'infinite' water. The concept of 'infinite' water has to be understood as follows: the fluid domain has unrestricted horizontal dimensions, there is neither influence of a bottom, nor of a free-surface (no waves). This implies that $a_{66}(\omega)$ = constant and $b_{66}(\omega) = 0$.
In the following table expressions are given for the added mass-moment of inertia per unit length (= a'_{66}) in case of three elementary forms.

		a'_{66}	reference
I		$\frac{1}{8}\pi\rho(\alpha^2-\beta^2)^2$	see ref. [A8]
II		$\frac{1}{8}\pi\rho\alpha^4$	see ref. [A8]
III		coeff. $*\ \pi\rho\beta^4$ coeff. $= f(\frac{\alpha}{\beta})$	see ref. [A9]

Application of these respective expressions for a'_{66} to the schematized ship (model) as used in ref. [30] yields successively:

	I, II, III applied to schematized ship, see ref [30]					
	α	β	coeff.	a'_{66} kg m	$a_{66} = a'_{66} * D$ kg m^2	$\frac{a_{66}}{\frac{1}{12}L^2\rho LBD}$
I	$\frac{1}{2}$ B	$\frac{1}{2}$ L	—	826.74	123.93	1.823
II	$\frac{1}{2}$ L	—	—	867.40	130.02	1.912
III	$\frac{1}{2}$ B	$\frac{1}{2}$ L	$\frac{\alpha}{\beta} = \frac{B}{L} = 0.154$ coeff. $\simeq 0.149$	1033.94	154.99	2.280

In case of a ship harmonically oscillating on shallow water the influence of the (three-dimensional) end effects (i.e. the circulation of water around bow and stern) decreases with increasing circular frequency. In behalf of the determination of $a_{66}(\omega)$ and $b_{66}(\omega)$ then use can be made of the (two-dimensional) strip theory (see ref. [30]):

$$\left.\begin{aligned} a_{66}(\omega) &= \frac{1}{12} L^2 a_{22}(\omega) \\ b_{66}(\omega) &= \frac{1}{12} L^2 b_{22}(\omega) \end{aligned}\right\} \text{ for the higher circular frequencies.}$$

APPENDIX IV: TWO MORE DIRECT METHODS TO DERIVE EQ. (35^a)

A-VI.1: METHOD IN WHICH USE IS MADE OF BOTH FOURIER AND LAPLACE TRANSFORMS

For the case of uncoupled ship motions the f.r.f. or harmonic transfer function of the linear ship-fluid system reads (see eq. (31)):

(31) $$K_{ii}(i\omega) = \frac{1}{\{m_{ii} + a_{ii}(\omega)\}i\omega + b_{ii}(\omega)} .$$

Since - according to eqs. ($33^{a,b}$) - $a_{ii}(0)$ = constant $\neq 0$ and $b_{ii}(0) = 0$ $K_{ii}(i\omega)$ contains a singularity (i.e. a pole) for $\omega = 0$. By means of eqs. ($33^{a,b}$) it can be derived that:

$$K_{ii}(i\omega)\Big|_{\omega\downarrow 0} = \lim_{\omega\to 0} \frac{1}{\{m_{ii} + a_{ii}(\omega)\}i\omega + b_{ii}(\omega)} =$$

$$= \lim_{\omega\to 0} \frac{1}{\{m_{ii} + a_{ii}(0) + a_{ii}^{(2)}\omega^2 + ..\}i\omega + b_{ii}^{(2)}\omega^2 + ..} \simeq$$

$$\simeq \frac{1}{m_{ii} + a_{ii}(0)} \frac{1}{i\omega} ;$$

in other words, in the neighbourhood of the pole $\omega = 0$ $K_{ii}(i\omega)$ behaves as:

$$K_{ii}(i\omega)\Big|_{\omega\downarrow 0} = \frac{1}{m_{ii} + a_{ii}(0)} \frac{1}{i\omega} .$$

If there should be further any poles, these probably lie in the left half-plane; the presence of hydrodynamic damping points that way. From a physical point of view this pole for $\omega = 0$ means that in case of a translation (rotation) in the horizontal plane with constant (rotational) velocity - i.e. $\omega = 0$ - no force (moment) is required. According to eq. (21^a) the f.r.f. $K_{ii}(i\omega)$ is the Fourier transform of the i.r.f. $k_{ii}(t)$. Now $K_{ii}(i\omega)$ has a pole for $\omega = 0$, and this indicates that $k_{ii}(t)$ is not absolutely integrable. For the determination of $K_{ii}(i\omega)$ therefore no use can be made of an ordinary Fourier transform and it has to be passed on to the Laplace transform technique.

N.B. The Fourier transform has to be considered as a special case of the Laplace transform.

Suppose that the pole in $K_{ii}(i\omega)$ can be isolated and that $K_{ii}(i\omega)$ can be written as:

(AIV-1) $$K_{ii}(i\omega) = K_{ii}^{(p)}(i\omega) + K_{ii}^{(r)}(i\omega),$$

where $K_{ii}^{(p)}(i\omega) = \dfrac{1}{m_{ii} + a_{ii}(0)} \dfrac{1}{i\omega}$ = part of $K_{ii}(i\omega)$ containing the pole for $\omega = 0$,

$K_{ii}^{(r)}(i\omega)$ = part of $K_{ii}(i\omega)$ without poles.

As $K_{ii}^{(r)}(i\omega)$ does not contain poles, this function can be treated with ordinary Fourier transform techniques.

$K_{ii}^{(p)}(i\omega)$ is purely imaginary, consequently $Re[K_{ii}(i\omega)] = Re[K_{ii}^{(r)}(i\omega)]$, so that $Re[K_{ii}^{(r)}(i\omega)]$ is not influenced by isolating the pole.

Because of the pole for $\omega = 0$ $k_{ii}(t)$ cannot be determined as being the inverse Fourier transform of $K_{ii}(i\omega)$; however, the determination of $k_{ii}(t)$ as the inverse Laplace transform of the transfer function is possible. By replacing $i\omega$ in eq. (AIV-1) with $s = \lambda + i\omega$, where $\lambda = Re[s] > 0$ and $\omega = Im[s]$, $K_{ii}(i\omega)$ can be transformed into:

(AIV-2) $$K_{ii}(s) = \frac{1}{m_{ii} + a_{ii}(0)} \frac{1}{s} + K_{ii}^{(r)}(s) \quad ,$$

where $K_{ii}(s)$ = Laplace transform of $k_{ii}(t)$ = transfer function for the i-direction in response to a force excitation in the i-direction,

$K_{ii}^{(r)}(s)$ = part of $K_{ii}(s)$ without poles,

$\text{Im}[\ldots]$ = imaginary part of ...

Taking the inverse Laplace transform of $K_{ii}(s)$ one obtains:

$$k_{ii}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_{ii}(s)\, e^{st}\, ds =$$

$$= \frac{1}{m_{ii} + a_{ii}(0)}\, U(t) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_{ii}^{(r)}(s)\, e^{st}\, ds,$$

where c = real constant.

$K_{ii}^{(r)}(s)$ does not contain poles in the right half-plane. Therefore the integral in the right-hand member of the above expression can be written as an inverse Fourier transform (to that end s is replaced by $i\omega$):

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)}\, U(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{ii}^{(r)}(i\omega)\, e^{i\omega t}\, d\omega \quad .$$

$$K_{ii}^{(r)}(i\omega) = \text{Re}\left[K_{ii}^{(r)}(i\omega)\right] + i\, \text{Im}\left[K_{ii}^{(r)}(i\omega)\right] = \text{Re}\left[K_{ii}(i\omega)\right] +$$

$$+ i\, \text{Im}\left[K_{ii}^{(r)}(i\omega)\right] \quad ,$$

substitution of which into the expression for $k_{ii}(t)$ yields:

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} U(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \{Re\left[K_{ii}(i\omega)\right] \cos(\omega t) +$$

$$- Im\left[K_{ii}^{(r)}(i\omega)\right] \sin(\omega t)\}d\omega +$$

$$+ \frac{i}{2\pi} \int_{-\infty}^{\infty} \{Re \left[K_{ii}(i\omega)\right] \sin(\omega t) + Im\left[K_{ii}^{(r)}(i\omega)\right] \cos(\omega t)\} d\omega .$$

$k_{ii}(t)$ is a real function of t; as a result it applies

$$\int_{-\infty}^{\infty} Re\left[K_{ii}(i\omega)\right]\sin(\omega t)\, d\omega = - \int_{-\infty}^{\infty} Im\left[K_{ii}^{(r)}(i\omega)\right] \cos(\omega t)\, d\omega$$

for all t,

so that

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} U(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} Re\left[K_{ii}(i\omega)\right] \cos(\omega t)\, d\omega +$$

$$- \frac{1}{2} \int_{-\infty}^{\infty} Im\left[K_{ii}^{(r)}(i\omega)\right] \sin(\omega t)\, d\omega .$$

Since $k_{ii}(t) = 0$ for $t < 0$ (see eq. (7^b)), the second term in the right-hand member of the above expression is an even function of t, and the third term is an odd function of t, it must hold good that:

$$\int_{-\infty}^{\infty} Re\left[K_{ii}(i\omega)\right] \cos(\omega t)\, d\omega = - \int_{-\infty}^{\infty} Im\left[K_{ii}^{(r)}(i\omega)\right] \sin(\omega t)\, d\omega ,$$

by means of which $k_{ii}(t)$ can be written as:

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} U(t) + \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t)\, d\omega \ .$$

Generally it holds that $\mathrm{Re}\left[K_{ii}(i\omega)\right]$ is an even function of ω (see eqs. (23) and (25^b)), so that $k_{ii}(t)$ finally becomes:

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} U(t) + \frac{2}{\pi} \int_{0}^{\infty} \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t)\, d\omega.$$

Using eq. (32^a) $k_{ii}(t)$ then can be written as:

$$(35^a) \qquad k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} + \frac{2}{} \int_{0}^{\infty} \frac{b_{ii}(\omega)\cos(\omega t)}{\{m_{ii} + a_{ii}(\omega)\}^2 \omega^2 + b_{ii}^2(\omega)}\, d\omega$$

for $t > 0$,

$$(22) \qquad k_{ii}(t) = 0 \quad \text{for} \quad t \leqq 0 \ .$$

A-IV.2: FORMAL METHOD IN WHICH USE IS MADE OF LAPLACE TRANSFORMS

As indicated in the foregoing, it generally holds good that the transfer function of a linear system is identical to the Laplace transform of the i.r.f. (see also refs. [37, 38]).
The harmonic transfer function or f.r.f. of the linear ship-fluid system in case of uncoupled ship motions can be written as (see eq. (31)):

$$(\text{AIV-3}) \qquad K_{ii}(i\omega) = \frac{1}{\{m_{ii} + a_{ii}(i\omega)\}\, i\omega + b_{ii}(i\omega)} \ ;$$

in section A-IV.1 it has been shown that $K_{ii}(i\omega)$ contains a pole for $\omega = 0$. If in this expression for $K_{ii}(i\omega)$ $i\omega$ is replaced by $s = \lambda + i\omega$, it can be passed on to the Laplace notation. The transfer function $K_{ii}(s)$ then takes the form:

(AIV-4) $$K_{ii}(s) = \frac{1}{\{m_{ii} + a_{ii}(s)\}s + b_{ii}(s)} \quad ,$$

and the i.r.f. $k_{ii}(t)$ - through the inverse Laplace transform of $K_{ii}(s)$ -:

(AIV-5) $$k_{ii}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_{ii}(s)\, e^{st}\, ds =$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{\{m_{ii} + a_{ii}(s)\}s + b_{ii}(s)}\, ds \quad .$$

The integrand of this integral has a pole for $s = 0$. This means that in eq. (AIV-5) $c = 0$ and that the path of integration is formed by a semi-circle around the origin - situated in the right half-plane and with radius $r \downarrow 0$ - , plus the respective positive and negative imaginary axis connected to that.
Now $k_{ii}(t)$ can be written as the sum of three integrals:

(AIV-6) $$k_{ii}(t) = \frac{1}{2\pi i} \left(I_{I} + I_{II} + I_{III}\right) \quad ,$$

where $$I_{I} = \lim_{r\downarrow 0} \int_{-i\infty}^{-ir} K_{ii}(s)\, e^{st}\, ds \quad ,$$

$$I_{II} = \lim_{r\downarrow 0} \int_{r\supset} K_{ii}(s)\, e^{st}\, ds \ ,$$

$$I_{III} = \lim_{r\downarrow 0} \int_{ir}^{i\infty} K_{ii}(s)\, e^{st}\, ds \quad ,$$

with $K_{ii}(s)$ according to eq. (AIV-4).
In the integral I_{II} s can be represented by $s = re^{i\phi}$, where $r = |s|$ and $\phi = \arg(s)$; then $ds = ir\, e^{i\phi} d\phi$, so that I_{II} changes into:

$$I_{II} = \lim_{r \downarrow 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{tre^{i\phi}}}{\{m_{ii} + a_{ii}(re^{i\phi})\}re^{i\phi} + b_{ii}(re^{i\phi})}\, ire^{i\phi}\, d\phi =$$

$$= i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\lim\limits_{r \downarrow 0} e^{tre^{i\phi}}}{\lim\limits_{r \downarrow 0} a_{ii}(re^{i\phi}) + m_{ii} + \lim\limits_{r \downarrow 0} \dfrac{b_{ii}(re^{i\phi})}{re^{i\phi}}}\, d\phi \;.$$

On the analogy of eqs. $(33^{a,b})$ it can be written for small values of r:

$$\left.\begin{aligned} a_{ii}(s) &= a_{ii}(re^{i\phi}) = a_{ii}(0) + a_{ii}^{(2)} r^2 e^{2i\phi} + \text{terms of higher order in } r \\ b_{ii}(s) &= b_{ii}(re^{i\phi}) = \qquad\qquad b_{ii}^{(2)} r^2 e^{2i\phi} + \text{terms of higher order in } r \end{aligned}\right\} \text{for } r \to 0,$$

from which it follows:

$$\lim_{r \downarrow 0} a_{ii}(re^{i\phi}) = a_{ii}(0) \;, \quad \lim_{r \downarrow 0} \frac{b_{ii}(re^{i\phi})}{re^{i\phi}} = \lim_{r \downarrow 0} b_{ii}^{(2)}\, re^{i\phi} = 0 \;.$$

Besides it holds good that:

$$\lim_{r \downarrow 0} e^{tre^{i\phi}} = 1 \;.$$

Substitution of one thing and another into the expression for I_{II} yields:

$$I_{II} = i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{m_{ii} + a_{ii}(0)}\, d\phi = \frac{\pi i}{m_{ii} + a_{ii}(0)} \;.$$

In the integral I_{III} it applies $s = i\omega$, $ds = id\omega$ and $\omega > 0$, so that I_{III} takes the form:

$$I_{III} = \lim_{r \downarrow 0} \int_r^\infty \frac{ie^{i\omega t}}{\{m_{ii} + a_{ii}(i\omega)\}i\omega + b_{ii}(i\omega)} d\omega \quad .$$

In the integral I_I it applies $s = -i\omega$, $ds = -id\omega$ and $\omega > 0$, so that I_I takes the form:

$$I_I = - \lim_{r \downarrow 0} \int_\infty^r \frac{ie^{-i\omega t}}{-\{m_{ii} + a_{ii}(-i\omega)\}i\omega + b_{ii}(-i\omega)} d\omega =$$

$$= - \lim_{r \downarrow 0} \int_r^\infty \frac{ie^{-i\omega t}}{\{m_{ii} + a_{ii}(-i\omega)\}i\omega - b_{ii}(-i\omega)} d\omega \quad ;$$

$a_{ii}(i\omega)$ and $b_{ii}(i\omega)$ are even functions of ω: $a_{ii}(i\omega) = a_{ii}(-i\omega)$, $b_{ii}(i\omega) = b_{ii}(-i\omega)$, so that I_I becomes:

$$I_I = - \lim_{r \downarrow 0} \int_r^\infty \frac{i\, e^{-i\omega t}}{\{m_{ii} + a_{ii}(i\omega)\}i\omega - b_{ii}(i\omega)} d\omega \quad .$$

For $(I_I + I_{III})$ then it can be derived:

$$I_I + I_{III} = \lim_{r \downarrow 0} 2i \int_r^\infty \frac{\omega\{m_{ii} + a_{ii}(i\omega)\} \sin(\omega t) + b_{ii}(i\omega) \cos(\omega t)}{\{m_{ii} + a_{ii}(i\omega)\}^2\omega^2 + b_{ii}^2(i\omega)} d\omega .$$

From eq. (AIV-3) it follows:

$$\frac{b_{ii}(i\omega)}{\{m_{ii} + a_{ii}(i\omega)\}^2\omega^2 + b_{ii}^2(i\omega)} = \mathrm{Re}\left[K_{ii}(i\omega)\right] \quad ,$$

$$\frac{\omega\{m_{ii} + a_{ii}(i\omega)\}}{\{m_{ii} + a_{ii}(i\omega)\}^2\omega^2 + b_{ii}^2(i\omega)} = - \mathrm{Im}\left[K_{ii}(i\omega)\right] \quad .$$

Substitution of the results obtained so far into the expression for $k_{ii}(t)$ eq. (AIV-6) yields:

$$k_{ii}(t) = \frac{1}{2}\frac{1}{m_{ii} + a_{ii}(0)} + \frac{1}{\pi} \lim_{r \downarrow 0} \int_r^{\infty} \{\mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) +$$

$$- \mathrm{Im}\left[K_{ii}(i\omega)\right] \sin(\omega t)\} \, d\omega =$$

$$= \frac{1}{2}\frac{1}{m_{ii} + a_{ii}(0)} + \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) \, d\omega +$$

$$- \frac{1}{\pi} \int_0^{\infty} \mathrm{Im}\left[K_{ii}(i\omega)\right] \sin(\omega t) \, d\omega \; .$$

As $k_{ii}(t) = 0$ for $t < 0$ (see eq. (7^b)), the first two terms in the right-hand member of the above expression are even functions of t, and the third term is an odd function of t, it must hold good that:

$$- \frac{1}{\pi} \int_0^{\infty} \mathrm{Im}\left[K_{ii}(i\omega)\right] \sin(\omega t) \, d\omega = \frac{1}{2}\frac{1}{m_{ii} + a_{ii}(0)} +$$

$$+ \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) \, d\omega \; ,$$

substitution of which into the expression for $k_{ii}(t)$ finally gives:

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} + \frac{2}{\pi} \int_0^{\infty} \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) \, d\omega \; .$$

Making use of eq. (32^a) $k_{ii}(t)$ then can be written:

$$(35^a) \qquad k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} + \frac{2}{\pi} \int_0^\infty \frac{b_{ii}(\omega)\ \cos(\omega t)}{\{m_{ii} + a_{ii}(\omega)\}^2 \omega^2 + b_{ii}^2(\omega)}\, d\omega$$

for $t > 0$,

$$(22) \qquad k_{ii}(t) = 0 \quad \text{for} \quad t \leqq 0 \ .$$

APPENDIX V: NUMERICAL CALCULATION OF EQ. (35^a): $k_{ii}(t)$ FOR $i = 2, 6$

Eq. (35^a) reads as follows:

$$(35^a) \qquad k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} + \frac{2}{\pi} \int_0^\infty \frac{b_{ii}(\omega)\cos(\omega t)}{\{m_{ii} + a_{ii}(\omega)\}^2 \omega^2 + b_{ii}^2(\omega)} \, d\omega \, , \qquad t > 0 \, .$$

With

$$k_{ii}(\infty) = \frac{1}{m_{ii} + a_{ii}(0)} = \text{constant} \neq 0$$

and eq. (32^a)

$$(32^a) \qquad \mathrm{Re}\left[K_{ii}(i\omega)\right] = K_{ii}^{(c)}(\omega) = \frac{b_{ii}(\omega)}{\{m_{ii} + a_{ii}(\omega)\}^2 \omega^2 + b_{ii}^2(\omega)}$$

eq. (35^a) takes the form:

$$(\text{AV-1}) \qquad k_{ii}(t) = k_{ii}(\infty) + \frac{2}{\pi} \int_0^\infty \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) \, d\omega \, , \qquad t > 0 \, .$$

Suppose now that the hydrodynamic coefficients $a_{ii}(\omega)$ and $b_{ii}(\omega)$ are known (by calculations and/or measurements) from $\omega = 0$ to $\omega = \omega_M$.

Eq. (AV-1) then can be written as:

$$(\text{AV-2}) \qquad k_{ii}(t) = k_{ii}(\infty) + \frac{2}{\pi} \int_0^{\omega_M} \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) \, d\omega \, +$$

$$+ \frac{2}{\pi} \int_{\omega_M}^\infty \mathrm{Re}\left[K_{ii}(i\omega)\right] \cos(\omega t) \, d\omega \, .$$

Since along the closed interval $[0, \omega_M]$ $a_{ii}(\omega)$ and $b_{ii}(\omega)$ - and consequently $Re[K_{ii}(i\omega)]$ - are known in discretized form, the first integral in eq. (AV-2) can be solved numerically. The solution of this integral can be represented by:

$$\text{(AV-3)} \qquad \int_0^{\omega_M} Re\left[K_{ii}(i\omega)\right] \cos(\omega t)\, d\omega = I_M(t) + R_M(t) \quad ,$$

where $I_M(t)$ = result of numerical integration along the closed interval $[0, \omega_M]$,

$R_M(t)$ = discretization error (i.e. process error) in consequence of the numerical process of integration along $[0, \omega_M]$.

Although for high circular frequencies $a_{ii}(\omega)$ might be equated to $a_{ii}(\infty)$ = constant and an estimate of $b_{ii}(\omega)$ is available (see Appendix III), an analytical solution of the second integral in eq. (AV -2) is probably not possible because of the complicated form of its integrand: only a rough estimate can be made. Therefore $I_M(t)$ has to be used as approximation for the integral in eq. (AV-1). On account of the behaviour of $a_{ii}(\omega)$ and $b_{ii}(\omega)$ one obtains by majorating the second integral in eq. (AV-2):

$$\text{(AV-4)} \qquad \left| \int_{\omega_M}^{\infty} Re\left[K_{ii}(i\omega)\right] \cos(\omega t)\, d\omega \right| < \frac{b_{ii}(\omega_M)}{\{m_{ii} + a_{ii}(\omega_M)\}^2 \omega_M^2} \cdot$$

$$\cdot \int_{-\frac{\pi}{2t}}^{+\frac{\pi}{2t}} \cos(\omega t)\, d\omega = \frac{b_{ii}(\omega_M)}{\{m_{ii} + a_{ii}(\omega_M)\}^2 \omega_M^2} \frac{2}{t} \quad ;$$

this expression can be regarded as an estimate for the error which arises by truncating the (numerical) process of integration (i.e. the truncation error). To provide that the result of the numerical calculation of the first integral in eq. (AV-2) - i.e. $I_M(t)$ - represents a sufficiently accurate and reliable solution for the integral in eq. (AV-1), it can be stated that the following condition must be fulfilled:

$$\text{(AV-5)} \quad \frac{2}{\pi}\left[R_M(t) + \frac{b_{ii}(\omega_M)}{\{m_{ii} + a_{ii}(\omega_M)\}^2 \omega_M^2} \frac{2}{t}\right] << \{k_{ii}(\infty) + \frac{2}{\pi} I_M(t)\} \quad .$$

For convenience' sake in the following $\text{Re}\left[K_{ii}(i\omega)\right]$ is represented by the general form:

$$f(\omega) = \text{Re}\left[K_{ii}(i\omega)\right] \quad .$$

Suppose that $a_{ii}(\omega)$ and $b_{ii}(\omega)$ - and therefore $f(\omega)$ - are given in a discretizised form at the abscissas

$$\omega = \omega_0, \omega_1, \ldots\ldots\ldots, \omega_M \qquad ,$$

with successive intervals

$$(\Delta\omega)_n = \omega_n - \omega_{n-1} = \omega_{n+1} - \omega_n, \qquad n = 1, 3, 5, \ldots, M-1 \ ,$$

where n = subscript used as running index representing a real positive odd integer,

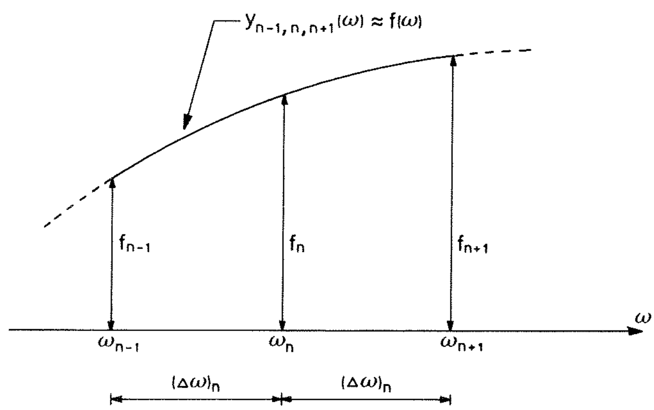
M = subscript representing a real positive even integer.

Then eq. (AV-3) can be written as:

$$I_M(t) + R_M(t) = \int_0^{\omega_M} f(\omega) \cos(\omega t)\, d\omega =$$

$$= \sum_{n=1,3,\ldots}^{M-1} \int_{\omega_{n-1}}^{\omega_{n+1}} f(\omega)\cos(\omega t)\,d\omega \quad .$$

Let for $\omega = \omega_{n-1}, \omega_n, \omega_{n+1}$ the values of $f(\omega)$ be represented by f_{n-1}, f_n and f_{n+1}, respectively.



Applying the Lagrange three point interpolation formula for equally spaced abscissas on the closed interval $[\omega_{n-1}, \omega_{n+1}]$, $f(\omega)$ can be approximated by (see ref. [A10]):

$$f(\omega)\Big|_{n-1,\ n,\ n+1} \simeq y_{n-1,\ n,\ n+1}(\omega) = a_n\omega^2 + b_n\omega + c_n \quad ,$$

where
$$a_n = \frac{f_{n-1} - 2\,f_n + f_{n+1}}{2\{(\Delta\omega)_n\}^2} \quad ,$$

$$b_n = -\,2\,a_n\,\omega_n - \frac{f_{n-1} - f_{n+1}}{2(\Delta\omega)_n} \quad ,$$

$$c_n = a_n\omega_n^2 + \omega_n\frac{f_{n-1} - f_{n+1}}{2(\Delta\omega)_n} + f_n \quad .$$

The discretization error on the interval $[\omega_{n-1}, \omega_{n+1}]$ as a consequence of this approximation for $f(\omega)$ is (see ref. [A10]):

$$E_{n-1,\, n,\, n+1}(\omega) = (\omega - \omega_{n-1})(\omega - \omega_n)(\omega - \omega_{n+1}) \frac{1}{3!} \frac{d^3 f(\xi)}{d\omega^3} \quad ,$$

with $\quad \omega_{n-1} \leqq \omega \leqq \omega_{n+1} \quad , \qquad \omega_{n-1} \leqq \xi \leqq \omega_{n+1} \quad .$

On the interval $[\omega_{n-1}, \omega_{n+1}]$ $f(\omega)$ then can be written as:

$$f(\omega)\Big|_{n-1,\, n,\, n+1} = y_{n-1,\, n,\, n+1}(\omega) + E_{n-1,\, n,\, n+1}(\omega) \quad ,$$

from which it follows:

$$\int_{\omega_{n-1}}^{\omega_{n+1}} f(\omega) \cos(\omega t)\, d\omega = \int_{\omega_{n-1}}^{\omega_{n+1}} y_{n-1,\, n,\, n+1}(\omega) \cos(\omega t)\, d\omega \, +$$

$$+ \int_{\omega_{n-1}}^{\omega_{n+1}} E_{n-1,\, n,\, n+1}(\omega) \cos(\omega t)\, d\omega \quad .$$

$$\int_{\omega_{n-1}}^{\omega_{n+1}} y_{n-1,\, n,\, n+1}(\omega) \cos(\omega t)\, d\omega = \int_{\omega_{n-1}}^{\omega_{n+1}} (a_n \omega^2 + b_n \omega + c_n) \cos(\omega t)\, d\omega =$$

$$= \frac{1}{t^2} \{(2a_n \omega_{n+1} + b_n) \cos(\omega_{n+1} t) - (2a_n \omega_{n-1} + b_n) \cos(\omega_{n-1} t)\} \, +$$

$$+ \frac{1}{t} \{(f_{n+1} - a_n \frac{2}{t^2}) \sin(\omega_{n+1} t) - (f_{n-1} - a_n \frac{2}{t^2}) \sin(\omega_{n-1} t)\} \quad .$$

$$\int_{\omega_{n-1}}^{\omega_{n+1}} E_{n-1,\ n,\ n+1}(\omega)\ \cos(\omega t)\ d\omega =$$

$$= \int_{\omega_{n-1}}^{\omega_{n+1}} (\omega - \omega_{n-1})(\omega - \omega_n)(\omega - \omega_{n+1})\ \frac{1}{3!}\ \frac{d^3 f(\xi)}{d\omega^3}\ \cos(\omega t)\ d\omega \quad ;$$

let $\left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.}$ be the maximum value of $\frac{d^3 f(\xi)}{d\omega^3}$ on $\left[\omega_{n-1},\ \omega_{n+1}\right]$, and suppose that $\left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max}$ is distributed uniformly on this interval;

then it holds good:

$$\int_{\omega_{n-1}}^{\omega_{n+1}} E_{n-1,\ n,\ n+1}(\omega)\ \cos(\omega t)\ d\omega \leqq$$

$$\leqq \frac{1}{3!}\ \left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.} \int_{\omega_{n-1}}^{\omega_{n+1}} (\omega - \omega_{n-1})(\omega - \omega_n)(\omega - \omega_{n+1})\ \cos(\omega t)\ d\omega \ ,$$

with

$$\frac{1}{3!}\ \left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.} \int_{\omega_{n-1}}^{\omega_{n+1}} (\omega - \omega_{n-1})(\omega - \omega_n)(\omega - \omega_{n+1})\ \cos(\omega t)\ d\omega =$$

$$= \frac{1}{t^2}\ \{\cos(\omega_{n+1} t) - \cos(\omega_{n-1} t)\}\left[2\{(\Delta\omega)_n\}^2 - \frac{6}{t^2}\right]\ \frac{1}{3!}\ \left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.} +$$

$$- 3(\Delta\omega)_n\ \frac{2}{t^3}\ \{\sin(\omega_{n+1} t) + \sin(\omega_{n-1} t)\}\ \frac{1}{3!}\ \left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.} \quad .$$

For $I_M(t)$ and $R_M(t)$ now it can be written:

(AV-6) $$I_M(t) = \sum_{n=1,3,..}^{M-1} \int_{\omega_{n-1}}^{\omega_{n+1}} y_{n-1,\ n,\ n+1} \cos(\omega t)\, d\omega \ ,$$

(AV-7) $$R_M(t) \leqq \sum_{n=1,3,..}^{M-1} \left| \frac{1}{3!} \left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.} \int_{\omega_{n-1}}^{\omega_{n+1}} (\omega - \omega_{n-1}) \cdot (\omega - \omega_n)(\omega - \omega_{n+1}) \cos(\omega t)\, d\omega \right| \ ,$$

respectively. With respect to the evaluation of $\left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.}$ the following has to be remarked.

In the numerical calculation $\left.\frac{d^3 f(\xi)}{d\omega^3}\right|_{max.}$ was taken to be the maximum out of the values of $\frac{d^3 f(\omega)}{d\omega^3}$ at the three respective abscissas $\omega = \omega_{n-1}, \omega_n, \omega_{n+1}$;

$\frac{d^3 f(\omega_n)}{d\omega^3}$ was calculated numerically by means of a five point formula for equally spaced abscissas (see ref. [A10]).

Since in $f(\omega) = Re[K_{ii}(i\omega)]$ - with $Re[K_{ii}(i\omega)]$ according to eq. (32^a) - $a_{ii}(0)$ = constant $\neq 0$ and $b_{ii}(0) = 0$, $f(0)$ takes an indeterminate form. With eqs. (33^a) and (33^b) $f(0)$ can be approximated by:

$$f(0) = Re[K_{ii}(0)] \simeq \lim_{\omega \to 0} \frac{b_{ii}^{(2)}\omega^2 + \dots}{\{m_{ii} + a_{ii}(0) + a_{ii}^{(2)}\omega^2 + ..\}^2\omega^2 + \{b_{ii}^{(2)}\omega^2 + ..\}^2} =$$

$$= \frac{b_{ii}^{(2)}}{\{m_{ii} + a_{ii}(0)\}^2} \ .$$

Making use of the expressions derived above, now - for $t > 0^+$ - the numerical calculation can be carried out: $k_{ii}(t)$ is calculated from

$$k_{ii}(t) = \frac{1}{m_{ii} + a_{ii}(0)} + \frac{2}{\pi} I_M(t) \quad ,$$

with $I_M(t)$ according to eq. (AV-6), while the truncation error and the discretization error, except for the factor $\frac{2}{\pi}$, are estimated by means of the expressions (AV-4) and (AV-7), respectively. This calculation was done starting from values for the hydrodynamic coefficients $a_{ii}(\omega)$ and $b_{ii}(\omega)$ given at the following circular frequencies: for $0 \leqq \omega < 25\,s^{-1}$ with frequency step=$0.1\ s^{-1}$, for $25 \leqq \omega < 50\ s^{-1}$ with frequency step = $0.5\ s^{-1}$, for $50 \leqq \omega < 80\ s^{-1}$ with frequency step = $1.0\ s^{-1}$. In doing so condition (AV-5), which warrants the accuracy of the numerical calculation, was fulfilled amply: in all cases considered the value of the left-hand member in condition (AV-5) remained - for all t - smaller than 0.01% of the value of the right-hand member.
By systematic variation of ω_M it was found that the influence of the hydrodynamic coefficients at higher circular frequencies on the i.r.f. $k_{ii}(t)$ indeed is of minor importance; when, for instance, $\omega_M = 30.0\ s^{-1}$, the ratio of the left-hand member to the right-hand member in condition (AV-5) is - for all t - still smaller than 0.01.

It is obvious that the numerical procedure described above is not suited to calculate the value of $k_{ii}(t)$ at the time $t = 0^+$. At $t = 0^+$ for the numerical integration a Simpson routine (see ref. [A10]) was applied along the same range of circular frequencies as used in the case with $t > 0^+$. It could be shown that the estimated value of the discretization error (see ref. [A10]) remained smaller than 0.01% of the values as calculated for $k_{ii}(0^+)$. It was not possible to give an estimate for the truncation error. However, by means of systematic variation of ω_M it was found that as long as $\omega_M \geqq 30.0\ s^{-1}$ - in all cases considered - the combined influence of the truncation of the numerical process of integration and the discretization error is within 1% of the values calculated for $k_{ii}(0^+)$.

The i.r.f. $k_{ii}(t)$ was calculated numerically at intervals of time amounting to $\Delta t = 0.01$ s. In all cases considered the upper limit of the calculation was taken as $t = 7.50$ s; this time range was regarded to be sufficiently long, since it was found that for $t > 7.50$ s $k_{ii}(t) \approx k_{ii}(\infty)$.

Supplementary note

The value of $k_{ii}(0^+)$ $(i = 2, 6)$ as calculated above can be checked c.q. approximated in a rather simple way.
Suppose that the uncoupled motions of the schematized ship can be described by the equation(s) of motion:

$$m^*_{ii} \ddot{x}_i(t) = f_i(t) \quad , \qquad i = 1, 2, \ldots, 6,$$

where $m^*_{ii} = m_{ii}$ + added mass(-moment of inertia), representing the 'mass effect'.
It is assumed that within a very short length of time the (hydrodynamic) damping of the ship-fluid system may be neglected. For $t < 0$ there is a state of rest. At the time $t = 0$ the ship is subjected to a unit pulse in the i-direction:

$$f_i(t) = \zeta_i \; \delta_i(t) \quad , \qquad \zeta_i = 1 .$$

The equation(s) of motion then become(s):

$$m^*_{ii} \ddot{x}_i(t) = \delta_i(t) \quad .$$

Taking the Laplace transform of this expression one obtains:

$$m^*_{ii} \left[s \, L\{\dot{x}_i(t)\} - \dot{x}_i(0) \right] = 1 \quad .$$

Since $\dot{x}_i(0) = 0$ this yields:

$$L\{\dot{x}_i(t)\} = \frac{1}{m^*_{ii} s} \quad .$$

According to eq. (19) it holds good for uncoupled ship motions:

$$(19) \qquad \dot{x}_i(t) = \int_{-\infty}^{t} f_i(\tau)\, k_{ii}(t - \tau)\, d\tau \,, \qquad i = 1, 2, \ldots, 6.$$

With $f_i(t) = \zeta_i\, \delta_i(t)$, $\zeta_i = 1$, it can be derived by taking the Laplace transform of this expression (see also Appendix II):

$$L\{\dot{x}_i(t)\} = L\{k_{ii}(t)\} \quad .$$

Therefore it can be written:

$$L\{k_{ii}(t)\} = \frac{1}{m^*_{ii} s} \quad ,$$

from which it follows by taking the inverse Laplace transform:

$$k_{ii}(t) = \frac{1}{m^*_{ii}}\, U(t) \quad .$$

As small lengths of time correspond with high (circular) frequencies, it applies:

$$m^*_{ii} = m_{ii} + a_{ii}(\infty) \quad ,$$

so that

$$k_{ii}(0^+) = \frac{1}{m_{ii} + a_{ii}(\infty)} \quad .$$

This expression can be used to check the values of $k_{ii}(0^+)$ as calculated above for the cases i = 2 and i = 6.

Concerning the values of $a_{22}(\infty)$ and $a_{66}(\infty) = \frac{1}{12} L^2 a_{22}(\infty)$ it is referred to Appendix III.

For comparison the values of $(m_{ii} + a_{ii}(\infty))^{-1}$ as well as the values of $k_{ii}(0^+)$ as calculated for the distinct cases are presented in dimensionless form in the table below.

	$\frac{\rho LBD}{m_{22} + a_{22}(\infty)}$	theor. (2-dim.); $\rho LBD\ k_{22}(0^+)$	theor. with modif. a_{22}, b_{22} (3-dim.); $\rho LBD\ k_{66}(0^+)$
$\frac{h}{D}$ = 1.333	0.738	0.738	0.810
$\frac{h}{D}$ = 1.167	0.726	0.726	0.790
	$\frac{m_{66}}{m_{66} + a_{66}(\infty)}$	theor. (2-dim.); $m_{66}\ k_{66}(0^+)$	theor. with modif. a_{66}, b_{66} (3-dim.); $m_{66}\ k_{66}(0^+)$
$\frac{h}{D}$ = 1.333	0.678	0.678	0.784
$\frac{h}{D}$ =1.167	0.665	0.665	0.782

Generally the agreement between the respective results is considered to be satisfactory. The discrepancy between $(m_{ii} + a_{ii}(\infty))^{-1}$ and $k_{ii}(0^{+})$ as calculated 'three-dimensionally' can be declared from the way in which the hydrodynamic coefficients $a_{ii}(\omega)$ and $b_{ii}(\omega)$ in the lower frequency range were modified with respect to their original two-dimensional values, and from the fact that the two-dimensional values of $a_{ii}(\infty)$ were maintained.

APPENDIX VI: CRITERION FOR THE CONVERGENCE OF THE COMPUTATIONAL SCHEME IN CASE OF A 'CENTRIC IMPACT' TO A LINEAR FENDER

For the case of a berthing operation in which $x_{1f} = - e_0 = 0$ - i.e. a 'centric impact', no rotation - to a linear fender, the motion of the schematized ship is given by eq. (41^a) (see Section III.2):

$$(41^a) \qquad \dot{x}_2(t) = v_A + \int_0^t f_2(\tau)\, k_{22}(t - \tau)\, d\tau \quad ;$$

in this expression $f_2(t)$ is the reaction force of the linear fender, which can be represented by:

$$f_2(t) = - c_0 \Delta x_{2f}(t) \quad , \qquad \Delta x_{2f}(t) \geqq 0 \quad ,$$

where c_0 = spring rate of linear fender.
Substitution of $f_2(t)$ into eq. (41^a) yields:

$$(\text{AVI-1}) \qquad \dot{x}_2(t) = v_A - c_0 \int_0^t \Delta x_{2f}(\tau)\, k_{22}(t - \tau)\, d\tau \quad .$$

During the contact between ship and fender the displacement of the ship's centre of gravity $x_{2G}(t)$ equals the impression of the fender $\Delta x_{2f}(t)$.
For clearness' sake the following simplified notations are used:

$$\dot{x}_2(t) = v(t) \; , \qquad \Delta x_{2f}(t) = \Delta y(t) \; , \qquad k_{22}(t) = k(t) \quad ;$$

eq. (AVI-1) then takes the form:

$$(\text{AVI-2}) \qquad v(t) = v_A - c_0 \int_0^t \Delta y(\tau)\, k(t - \tau)\, d\tau \quad .$$

The calculation of eq. (AVI-2) is carried through according to the iteration procedure as described in Section III.2, using equidistant time steps Δt. It is supposed that the contact between ship and fender has a length of time of $N\Delta t$, where N is a real positive integer.

Let $t = n\Delta t$, $\tau = l\Delta t$,

with $n = 1, 2, 3, \ldots, N$, $l = 1, 2, 3, \ldots, n$;

then $v(t) = v(n\Delta t) = v_n$, $v_0 = v_A$,

$\Delta y(\tau) = \Delta y(l\Delta t) = \Delta y_l$, $\Delta y_0 = 0$, $\Delta y_N = 0$,

$k(t - \tau) = k(n\Delta t - l\Delta t) = k\{(n - l)\Delta t\} = k_{n-l}$.

In these expressions is:

n,l = real positive integer - when used as subscript it represents a number of time steps Δt and indicates that the quantity concerned must be taken at the point of time $t = n\Delta t$, $\tau = l\Delta t$, respectively.

Suppose that the calculation of eq. (AVI-2) has arrived to the point of time $t - \Delta t$; in the iteration procedure for point of time t the m^{th}-approximation of v_n then can be written as:

$$v_n^{(m)} = v_A - c_0 \sum_{l=1}^{n} \frac{1}{2} \Delta t(\Delta y_{l-1}\, k_{n-l+1} + \Delta y_l\, k_{n-l}) =$$

$$= v_A - c_0 \sum_{l=1}^{n-1} \frac{1}{2} \Delta t(\Delta y_{l-1}\, k_{n-l+1} + \Delta y_l\, k_{n-l}) +$$

$$- \frac{1}{2} c_0\, \Delta t(k_1\, \Delta y_{n-1} + k_0 \Delta y_n) \quad ;$$

the superscript (m) indicates that in the iteration procedure the m^{th}-approximation is taken of the quantity concerned.

The approximation used for Δy_n is:

$$\Delta y_n = \Delta y_{n-1} + \frac{1}{2} \Delta t \,\{v_n^{(m-1)} + v_{n-1}\} \quad .$$

Elimination of Δy_n from the last two equations yields:

$$v_n^{(m)} = v_A - c_0 \sum_{l=1}^{n=1} \frac{1}{2} \Delta t(\Delta y_{l-1}\ k_{n-l+1} + \Delta y_l\ k_{n-l}) +$$

$$- \frac{1}{2} c_0\ \Delta t\ \Delta y_{n-1}\ (k_0 + k_1) - \frac{1}{4} c_0\ \Delta t^2\ k_0\ \{v_n^{(m-1)} + v_{n-1}\} \quad .$$

In a completely analogous way it applies for the $(m-1)^{th}$-approximation of v_n:

$$v_n^{(m-1)} = v_A - c_0 \sum_{l=1}^{n-1} \frac{1}{2} \Delta t(\Delta y_{l-1}\ k_{n-l+1} + \Delta y_l\ k_{n-l}) +$$

$$- \frac{1}{2} c_0 \Delta t\ \Delta y_{n-1}(k_0 + k_1) - \frac{1}{4} c_0 \Delta t^2 k_0\ \{v_n^{(m-2)} + v_{n-1}\} \quad .$$

Now it can be written:

$$v_n^{(m)} - v_n^{(m-1)} = - \frac{1}{4} c_0\ \Delta t^2\ k_0\{v_n^{(m-1)} - v_n^{(m-2)}\} \quad .$$

The iteration procedure for the calculation of v_n converges if:

$$\frac{\left| v_n^{(m)} - v_n^{(m-1)} \right|}{\left| v_n^{(m-1)} - v_n^{(m-2)} \right|} < 1 \quad , \qquad \text{i.e. if} \quad \left| - \frac{1}{4} c_0\ \Delta t^2\ k_0 \right| < 0 \quad ;$$

in other words the computational scheme in case of a 'centric impact' to a linear fender is convergent if:

$$\Delta t < 2 \sqrt{\frac{1}{c_0\ k_{22}(0^+)}} \quad .$$

N.B. $\quad k_0 = k_{22}(0^+)$.

APPENDIX VII: SOLUTION OF THE BERTHING SHIP PROBLEM USING A LONG-WAVE APPROXIMATION FOR THE MOTION OF THE WATER

In this Appendix VII a simplified approach is given to solve the problem of a (schematized) ship berthing to a(n open) structure equipped with one fender, on shallow water with unrestricted horizontal dimensions.

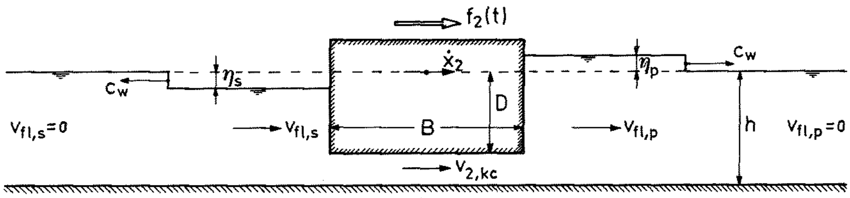
Use is made of a long-wave approximation for the motion of the water, in which - even in case of a small keel clearance - no account is taken of the circulation of water around 'bow' and 'stern' (i.e. the so-called end effects). Therefore, in addition to the fact that this approach of the berthing ship problem is subject to the restrictions of the long-wave theory, it has to be considered as basically two-dimensional.

The berthing ship problem is solved for the same (schematized) ship (model), for the same shallow water depths and for the same berthing lay-out as in the report; likewise the same berthing operations are considered. The assumptions and simplifications concerning the case of the berthing ship, as stated in Section I and recapitulated in Section III.1, do also apply in this Appendix VII. As much as possible the same notations are maintained.

First of all an expression is derived for the i.r.f. for the sway motion. Subsequently this i.r.f., or the allied equation of motion of ship and water combined, is used in determining (expressions for) the relevant quantities which play a part in the berthing ship problem.

A-VII.1: DETERMINATION OF THE I.R.F. $k_{22}(t)$

The schematized ship is supposed to have merely a purely lateral velocity (i.e. in the sway direction), $\dot{x}_2(t)$. Let the heights of the generated long waves on port-side and starboard-side be η_p and η_s, respectively (see fig.); η_p and η_s are assumed to be very small with respect to the water depth: $\eta_p \leqq h$, $\eta_s \leqq h$.



With neglect of friction effects, assuming that the waves propagate without distortion, the velocities of propagation can be represented by $c_w = \sqrt{gh}$. In conformity with the long-wave theory the (horizontal) fluid velocities under the long waves on port-side and starboard-side (i.e. $v_{fl,p}$ and $v_{fl,s}$, respectively) are supposed to be uniformly distributed in the vertical plane and parallel to $\dot{x}_2(t)$; this also holds good at a (very) short distance from the ship's wall. The velocities in the undisturbed fluid region are equal to zero. Farther it is supposed that the velocities in the keel clearance underneath the ship are horizontal and parallel with $\dot{x}_2(t)$, and that they are distributed uniformly; these velocities are indicated by $v_{2,kc}(t)$.

Form the long-wave theory it can be derived that on port-side and starboard-side of the ship the following respective expressions hold good:

$$v_{fl,p}(h + \eta_p) - \eta_p c_w = 0$$

and

$$-\eta_s c_w + v_{fl,s}(h - \eta_s) = 0 \quad .$$

As both $\eta_p \ll h$ and $\eta_s \ll h$, $v_{fl,p}$ and $v_{fl,s}$ can be written as:

(AVII-1^{a,b}) $$v_{fl,p} = \frac{g}{c_w}\,\eta_p \;, \qquad v_{fl,s} = \frac{g}{c_w}\,\eta_s \;.$$

Application of the law of conservation of mass on port-side, c.q. starboard-side of the ship yields:

$$v_{fl,p}(h + \eta_p) = v_{2,kc}(h - D) + \dot{x}_2(D + \eta_p)$$

and

$$v_{fl,s}(h - \eta_s) = v_{2,kc}(h - D) + \dot{x}_2(D - \eta_s) \;.$$

It is assumed that the water depth h and the draught of the ship D are of the same order of magnitude; therefore, both $\eta_p \ll D$ and $\eta_s \ll D$, so that the above expressions become:

(AVII-2^{a,b}) $$\begin{cases} v_{fl,p}h = v_{2,kc}(h - D) + \dot{x}_2 D \;, \\ v_{fl,s}h = v_{2,kc}(h - D) + \dot{x}_2 D \;. \end{cases}$$

Applying the law of conservation of momentum to the mass of water underneath the ship one obtains (per unit length):

$$\rho B(h - D)\,\dot{v}_{2,kc} = -\,\rho g(h - D)(\eta_p + \eta_s) \;,$$

or

(AVII-3) $$\dot{v}_{2,kc} + g\,\frac{\eta_p + \eta_s}{B} = 0 \;.$$

Let the velocity $\dot{x}_2(t)$ be the result of an external force $f_2(t)$ acting upon the schematized ship in the sway direction. The equation of motion of the ship then can be represented by:

$$\rho LBD\ddot{x}_2 = - \rho g \; (D - \eta_s)L(\eta_p + \eta_s) - \frac{1}{2}\rho gL \; (\eta_p + \eta_s)^2 + f_2(t) \; ,$$

or,

$$\text{(AVII-4)} \qquad \rho LBD\ddot{x}_2 = - \rho gDL \; (\eta_p + \eta_s) + f_2(t) \; .$$

Elimination of $v_{fl,p}$ and $v_{fl,s}$ from eqs. (AVII-$1^{a,b}$) and eqs. (AVII-$2^{a,b}$) yields:

$$\frac{g}{c_w}\eta_p h = v_{2,kc} \; (h - D) + \dot{x}_2 D \; , \qquad \frac{g}{c_w}\eta_s h = v_{2,kc}(h - D) + \dot{x}_2 D \; ,$$

from which it follows:

$$\eta_p + \eta_s = \frac{2}{c_w} \{v_{2,kc} \; (h - D) + \dot{x}_2 D\} \; .$$

By substitution of this expression for $\eta_p + \eta_s$ into eq. (AVII-3) and eq. (AVII-4) one obtains:

$$\text{(AVII-5)} \qquad \dot{v}_{2,kc} + \frac{2g}{Bc_w} v_{2,kc} \; (h - D) + \frac{2g}{Bc_w} \dot{x}_2 D = 0$$

and

$$\text{(AVII-6)} \qquad \ddot{x}_2 + \frac{2g}{Bc_w} v_{2,kc} \; (h - D) + \frac{2g}{Bc_w} \dot{x}_2 D = \frac{1}{\rho LBD} f_2(t) \; ,$$

respectively. Elimination of $v_{2,kc}$ from these two expressions then yields:

$$\text{(AVII-7)} \qquad \dddot{x}_2 + \frac{2c_w}{B}\ddot{x}_2 - \frac{1}{\rho LBD}\dot{f}_2(t) - \frac{1}{\rho LBD}\frac{2g(h-D)}{Bc_w} f_2(t) = 0 \;;$$

eq. (AVII-7) has to be considered as the equation of motion in the sway direction of ship and water combined.

For $t < 0$ there is a state of rest. At the time $t = 0$ the ship is subjected to a unit pulse in the sway direction:

$$f_i(t) = \zeta_i\ \delta_i(t) \;, \qquad \zeta_i = 1, \qquad i = 2.$$

In case of uncoupled motions it can be written for the sway motion of the linear ship-fluid system in the time domain (see eq. (19)):

$$\dot{x}_i(t) = \int_{-\infty}^{t} f_i(\tau)\ k_{ii}(t-\tau)\ d\tau \;, \qquad i = 2.$$

From the last two equations it can be derived:

$$\dot{x}_2(t) = k_{22}(t) \;,$$

with $\dot{x}_2(t) = k_{22}(t) = 0$ for $t \leqq 0$.

Now eqs. (AVII-5) and (AVII-6) can be written as:

$$\dot{v}_{2,kc} + \frac{2g(h-D)}{Bc_w}\ v_{2,kc} + \frac{2gD}{Bc_w}\ k_{22}(t) = 0$$

and

$$\dot{k}_{22}(t) + \frac{2g(h-D)}{Bc_w}\ v_{2,kc} + \frac{2gD}{Bc_w} k_{22}(t) = \frac{1}{\rho LBD}\ \delta_2(t) \;,$$

respectively. Taking the Laplace transforms of these two expressions one obtains:

$$s\ L\{v_{2,kc}(t)\} + \frac{2g(h-D)}{Bc_w}\ L\{v_{2,kc}(t)\} + \frac{2gD}{Bc_w}\ L\{k_{22}(t)\} = 0$$

and

$$s\ L\{k_{22}(t)\} + \frac{2g(h-D)}{Bc_w}\ L\{v_{2,kc}(t)\} + \frac{2gD}{Bc_w}\ L\{k_{22}(t)\} = \frac{1}{\rho LBD}\ ;$$

elimination of $L\{v_{2,kc}(t)\}$ yields:

$$L\{k_{22}(t)\} = \frac{1}{\rho LBD}\ \frac{1}{s}\ \{1 - \frac{2gD}{Bc_w}\ \frac{1}{s + \frac{2c_w}{B}}\}\ .$$

By taking the inverse Laplace transform of this expression one obtains for the i.r.f. for the sway motion:

$$\text{(AVII-8}^a\text{)} \qquad k_{22}(t) = \frac{1}{\rho LBD}\ \{\frac{h-D}{h} + \frac{D}{h}\ e^{-\frac{2c_w}{B}t}\} \qquad \text{for } t > 0\ ,$$

$$\text{(AVII-8}^b\text{)} \qquad k_{22}(t) = 0 \qquad \text{for } t \leqq 0\ .$$

The i.r.f. for the sway motion as calculated from eq. (AVII-$8^{a,b}$), is presented in figs. 4 and 5 as a broken line. Derived analytically from a long-wave approximation for the motion of the water, it can be used for the determination of the relevant quantities playing a part in berthing operations which result in a 'centric impact'.

A-VII.2: DETERMINATION OF BERTHING OPERATIONS

In case of an undamped linear fender - using a long-wave approximation for the motion of the water - analytical expressions can be derived for the relevant quantities figuring in the berthing ship problem. For the situation of a ship berthing to a non-linear fender, generally, the relevant quantities can only be determined by numerical calculations.

For a definition sketch (plan and cross-section) of the berthing lay-out reference is made to fig. 10.

In addition to the assumptions and simplifications concerning the case of the berthing ship, as stated in Section I and recapitulated in Section III.1, the following is supposed to apply.

- The angle of rotation of the ship's longitudinal axis of symmetry around the ox_3-axis during the contact between ship and fender, i.e. $\psi(t)$, remains (very) small.
- The point of contact between ship and fender does not move along the ship's hull during the impression of the fender; i.e. $\overline{AG} = e_0$, when there is contact between ship and fender.

The two above assumptions are affirmed by the results of the experiments and calculations from the report.

- Regarding the motion of the schematized ship in the sway direction, the hydrodynamic effects are taken into account by means of a long-wave approximation for the motion of the water.
- With respect to the rotation of the schematized ship in the horizontal plane, the hydrodynamic effects are only taken into account by means of a constant added mass-moment of inertia for the yawing motion; the (hydrodynamic) damping is neglected. For a justification of this assumption reference is made to Appendix III, Section A-III.4.

A-VII.2.a: CENTRIC IMPACTS: $e_0 = 0$

In case of a centric impact the following relations apply:

$$\psi(t) = 0, \qquad x_{2G}(t) = \Delta x_{2f}(t), \qquad F_{2f}(t) = f(\Delta x_{2f}).$$

For an undamped linear fender the reaction force in the fender then can be written as (see eq. (44^a)):

(AVII-9) $$F_{2f}(t) = \begin{cases} 0 & \text{for} \quad x_{2G}(t) < 0 \;, \\ -c_0 x_{2G}(t) & \text{for} \quad x_{2G}(t) \geqq 0 \;. \end{cases}$$

The purely lateral motion of the schematized ship (i.e. in the sway direction) during its contact with the fender can be represented by eq. (41^a):

(41^a) $$\dot{x}_2(t) = v_A + \int_0^t f_2(\tau)\, k_{22}(t-\tau)\, d\tau \quad ;$$

in this eq. (41^a) $\dot{x}_2(t) = \dot{x}_{2G}(t)$ and $f_2(t) = F_{2f}(t)$, with $F_{2f}(t)$ according to eq. (AVII-9); $k_{22}(t)$ is represented by eq. (AVII-$8^{a,b}$):

$$k_{22}(t) = \alpha_0 + \beta_0 e^{-\gamma_0 t} \qquad \text{for } t > 0 \quad ,$$

$$k_{22}(t) = 0 \qquad \text{for } t \leqq 0 \quad ,$$

where $\alpha_0 = \dfrac{1}{\rho LBD} \dfrac{h - D}{h}$, $\beta_0 = \dfrac{1}{\rho LBD} \dfrac{D}{h}$, $\gamma_0 = \dfrac{2c_w}{B}$.

Substitution of these respective expressions for $\dot{x}_2(t)$, $f_2(t)$ and $k_{22}(t)$ in eq. (41^a) yields:

(AVII-10) $$\dot{x}_{2G}(t) = v_A - c_0 \int_0^t x_{2G}(\tau)\, \{\alpha_0 + \beta_0\, e^{-\gamma_0(t-\tau)}\} d\tau \;.$$

Eq. (AVII-10) can be solved in two ways, which both, naturally, lead to the same result: in the first approach use is made of Laplace transforms, in the second approach the integro-differential equation (AVII-10) is transformed into an ordinary differential equation.

Taking the Laplace transform of eq. (AVII-10) it can be derived, since $x_{2G}(0^+) = 0$:

(AVII-11) $$L\{x_{2G}(t)\} = v_A \frac{s + \gamma_0}{s^3 + \gamma_0 s^2 + (\alpha_0 + \beta_0)c_0 s + \alpha_0\gamma_0 c_0} ,$$

$$\mathrm{Re}[s] > s .$$

As the denominator of the right-hand member of the above expression generally has three (different) roots, this right-hand member can be separated into a sum of partial fractions; $x_{2G}(t)$ then can be determined by taking the inverse Laplace transform of each summand separately.

By differentiating eq. (AVII-10) with respect to t one obtains, using the expression for $k_{22}(t)$ according to eq. (AVII-$8^{a,b}$):

$$\dddot{x}_{2G}(t) = -(\alpha_0 + \beta_0)\, c_0 x_{2G}(t) + \gamma_0\, \{v_A - \dot{x}_{2G}(t)\} - \alpha_0\gamma_0 c_0 \int_0^t x_{2G}(\tau)\, d\tau;$$

differentiation of this expression with respect to t yields:

(AVII-12) $$\dddot{x}_{2G}(t) + \gamma_0 \ddot{x}_{2G}(t) + c_0(\alpha_0 + \beta_0)\, \dot{x}_{2G}(t) + \alpha_0\gamma_0 c_0\, x_{2G}(t) = 0.$$

Eq. (AVII-12) is a linear homogeneous ordinary differential equation of the third order with constant real coefficients.

N.B. As could be expected - since $x_{2G}(0^+) = \ddot{x}_{2G}(0^+) = 0$ and $\dot{x}_{2G}(0^+) = v_A$ -, solution of eq. (AVII-12) by way of the Laplace transform technique leads to an expression for $L\{x_{2G}(t)\}$ which is identical to eq. (AVII-11).

Let tre general solution of eq. (AVII-12) be represented by

(AVII-13) $$x_{2G}(t) = \sum_{m=1}^{3} C_m e^{w_m t} ,$$

where m = subscript representing a real positive integer, $m = 1, 2, 3,$

C_m = constant of integration,

and w_m represents the roots of the characteristic equation:

$$w^3 + \gamma_0 w^2 + (\alpha_0 + \beta_0)\, c_0 w + \alpha_0 \gamma_0 c_0 = 0 \quad .$$

Generally this characteristic equation has three different roots, either three real roots, or one real root plus two rooths which are complex conjugated.

N.B. With $\dot{x}_2(t) = \dot{x}_{2G}(t)$, $f_2(t) = F_{2f}(t)$, eq. (AVII-9) and the respective expressions for α_0, β_0 and γ_0, the equation of motion in the sway direction of ship and water combined, i.e. eq. (AVII-7) yields directly eq. (AVII-12).

The quantity $x_{2G}(t)$ now is determined by solving the linear ordinary differential equation (AVII-12).

Eq. (AVII-12) can be written as:

$$\text{(AVII-14)} \qquad \dddot{x}_{2G}(t) + p_0 \ddot{x}_{2G}(t) + q_0 \dot{x}_{2G}(t) + r_0 x_{2G}(t) = 0 \quad ,$$

where

$$p_0 = \gamma_0 = \frac{2c_w}{B} \quad ,$$

$$q_0 = (\alpha_0 + \beta_0)\, c_0 = \frac{c_0}{\rho LBD} \quad ,$$

$$r_0 = \alpha_0 \gamma_0 c_0 = \frac{2c_w}{B}\, \frac{c_0}{\rho LBD}\, \frac{h - D}{h} \quad .$$

The general solution of eq. (AVII-14) can be written as eq. (AVII-13), where w_m with $m = 1, 2, 3$ represents the roots of the characteristic equation

$$\text{(AVII-15)} \qquad w^3 + p_0 w^2 + q_0 w + r_0 = 0 \quad .$$

The initial conditions of eq. (AVII-14) are:

(AVII-16) $$x_{2G}(0) = 0, \qquad \dot{x}_{2G}(0) = v_A, \qquad \ddot{x}_{2G}(0) = 0.$$

Eq. (AVII-15) is a cubic equation with real coefficients; for the determination of the roots of this equation is referred to ref. [A11].

Let $$w_m = \xi_m - \frac{p_0}{3} = \xi_m - \frac{1}{3}\frac{2c_w}{B} \quad ;$$

ξ_m then satisfies the cubic equation with real coefficients:

$$\xi^3 + a_0\xi + b_0 = 0 \quad ,$$

where $$a_0 = \frac{1}{3}(3q_0 - p_0^2) = \frac{1}{3}\left(3\frac{c_0}{\rho LBD} - \frac{4c_w^2}{B^2}\right) \quad ,$$

$$b_0 = \frac{1}{27}(2p_0^3 - 9p_0q_0 + 27r_0) =$$

$$= \frac{2}{27}\frac{c_w}{B}\left(\frac{8c_w^2}{B^2} - 9\frac{c_0}{\rho LBD} + 27\frac{c_0}{\rho LBD}\frac{h - D}{h}\right) \quad .$$

On account of the values for ρ, g, L, B, D, h and c_0 as used in the report, it holds good that:

$$\frac{b_0^2}{4} + \frac{a_0^3}{27} > 0 \quad ,$$

or, in other words, there is one real root (ξ_1) and there are two roots (ξ_2 and ξ_3) which are complex conjugated; the roots ξ_1, ξ_2 and ξ_3 read as:

$$\xi_1 = A_c + B_c \ , \qquad \xi_2 = -\frac{A_c + B_c}{2} + \frac{A_c - B_c}{2}\ i\sqrt{3} \ ,$$

$$\xi_3 = -\frac{A_c + B_c}{2} - \frac{A_c - B_c}{2}\ i\sqrt{3} \ ,$$

where $$A_c = \sqrt[3]{-\frac{b_0}{2} + \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} \ , \quad B_c = \sqrt[3]{-\frac{b_0}{2} - \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} \ ;$$

A_c and B_c are real constants.
The roots w_1, w_2 and w_3 of eq. (AVII-15) now can be written as:

$$w_1 = P, \qquad w_2 = R + iQ, \qquad w_3 = R - iQ,$$

where
$$P = A_c + B_c - \frac{1}{3}\frac{2c_w}{B} \ ,$$

$$Q = \frac{A_c - B_c}{2}\sqrt{3} \ ,$$

$$R = -\frac{A_c + B_c}{2} - \frac{1}{3}\frac{2c_w}{B} \ .$$

By means of the initial conditions (AVII-16) it can be derived for the constants of integration C_1, C_2 and C_3:

$$C_1 = \frac{-v_A(w_2 + w_3)}{(w_1 - w_2)(w_1 - w_3)} = -v_a \frac{2R}{(P - R)^2 + Q^2} = -2\ v_A\ S \ ,$$

$$C_2 = \frac{v_A(w_1 + w_3)}{(w_2 - w_3)(w_1 - w_2)} = v_A \frac{2QR - i(P^2 - R^2 + Q^2)}{2Q\ \{(P - R)^2 + Q^2\}} = v_A S - i v_A T \ ,$$

and

$$C_3 = \frac{-v_A(w_1 + w_2)}{(w_1 - w_3)(w_2 - w_3)} = v_A \frac{2QR + i(P^2 - R^2 + Q^2)}{2Q\{(P-R)^2 + Q^2\}} = v_A S + iv_A T \quad ,$$

where $S = \frac{R}{(P-R)^2 + Q^2}$, $T = S\,\frac{P^2 - R^2 + Q^2}{2QR}$.

For $x_{2G}(t)$ and $\dot{x}_{2G}(t)$ it then can be derived:

$$x_{2G}(t) = v_A \left[-2\,S\,e^{Pt} + 2\,e^{Rt}\,\{S\cos(Qt) + T\sin(Qt)\}\right]$$

and

$$\dot{x}_{2G}(t) = v_A \left[-\,2PSe^{Pt} + 2R\,e^{Rt}\,\{S\cos(Qt) + T\sin(QT)\} + \right.$$
$$\left. + \,2\,Q\,e^{Rt}\,\{-\,S\sin(QT) + T\cos(Qt)\}\right] \quad ,$$

respectively.

Recapitulating, the analytical solution of the berthing ship problem in case of a centric impact against an undamped linear fender - using a long-wave approximation for the motion of the water - can be represented by:

(AVII-17^a) $$x_{2G}(t) = v_A\,\{-2\,S\,e^{Pt} + 2\,S\,e^{Rt}\cos(Qt) + 2T\,e^{RT}\sin(Qt)\} \quad ,$$

(AVII-17^b) $$\dot{x}_{2G}(t) = v_A\,\{-2\,PS\,e^{Pt} + 2(RS + QT)\,e^{Rt}\cos(Qt) +$$
$$-\,2(QS - RT)\,e^{Rt}\sin(Qt)\} \quad ,$$

(AVII-9) $$F_{2f}(t) = \begin{cases} 0 & \text{for} \quad x_{2G}(t) < 0 \ , \\ -c_o\, x_{2G}(t) & \text{for} \quad x_{2G}(t) \geqq 0 \ , \end{cases}$$

where $$S = \frac{R}{(P-R)^2 + Q^2} \ , \qquad T = S\,\frac{P^2 - R^2 + Q^2}{2QR} \ ,$$

$$P = A_c + B_c - \frac{1}{3}\,\frac{2c_w}{B} \ ,$$

$$Q = \frac{A_c - B_c}{2}\,\sqrt{3} \ ,$$

$$R = -\frac{A_c + B_c}{2} - \frac{1}{3}\,\frac{2c_w}{B} \ ,$$

$$A_c = \sqrt[3]{-\frac{b_0}{2} + \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} \ , \qquad B_c = \sqrt[3]{-\frac{b_0}{2} - \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} \ ,$$

$$a_0 = \frac{1}{3}\,(3q_0 - p_0^2) \ , \qquad b_0 = \frac{1}{27}\,(2p_0^3 - 9p_0q_0 + 27r_0) \ ,$$

$$p_0 = \frac{2c_w}{B} \ , \qquad q_0 = \frac{c_0}{\rho LBD} \ , \qquad r_0 = \frac{2c_w}{B}\,\frac{c_0}{\rho LBD}\,\frac{h-D}{h} \ ,$$

$$c_w = \sqrt{gh} \ .$$

The time histories of the fender forces as calculated from eqs. (AVII-9) and (AVII-17[a]) for the respective spring rates c_0 and water depths h are presented in figs. 14 through 19 as broken lines.

In case of a fender which is damped and/or (non-)linear the real constant coefficients p_0, q_0 and r_0 generally are functions of $x_{2G}(t)$ and $\dot{x}_{2G}(t)$; an analytical solution of the berthing ship problem then is only possible in very special cases. Making use of the i.r.f. for the sway motion, eq. (AVII-$8^{a,b}$), in case of the non-linear fender the time history of the fender force is determined in the same way (i.e. numerically) as in the report; for the respective water depths h and lateral speeds of approach v_A the results are presented in figs. 20 through 23 as broken lines.

A-VII.2.b.: ECCENTRIC IMPACTS: $e_0 \neq 0$

As $\overline{AG} = e_0$ and $\psi(t)$ remains (very) small during the contact between ship and fender, the motion of the (schematized) ship in the x_1-direction can be neglected.

The resulting force and moment, as acting in and about the ship's centre of gravity, are (see eqs. ($39^{a,b}$)):

$$\left.\begin{aligned} &(39^a) \qquad f_2(t) = F_{2f}\cos(\psi) \;, \\ &(39^b) \qquad f_6(t) = -\overline{AG}\,F_{2f}\cos(\psi) \end{aligned}\right\}\quad t \geqq 0 \;;$$

in the case under consideration $\overline{AG} = e_0$.
For an undamped linear fender the reaction force in the fender has the form (see eq. (44^a)):

$$(44^a) \qquad F_{2f}(t) = \begin{cases} 0 & \text{for} \quad \Delta x_{2f}(t) < 0 \;, \\ -c_0\,\Delta x_{2f}(t) & \text{for} \quad \Delta x_{2f}(t) \geqq 0 \;. \end{cases}$$

The equation of motion of ship and water combined for the x_2-direction may be written as (see also eq. (AVII-7)):

(AVII-18) $$\ddot{x}_{2G} + \frac{2c_w}{B}\dot{x}_{2G} - \frac{1}{\rho LBD}\dot{f}_2(t) - \frac{1}{\rho LBD}\frac{2g(h-D)}{Bc_w} f_2(t) = 0$$

The equation of motion describing the rotation of the schematized ship in the horizontal plane reads:

(AVII-19) $$m^{*}_{66}\ \ddot{\psi} = f_6(t) ,$$

where $m^{*}_{66} = m_{66}$ + yaw added mass-moment of inertia;
m^{*}_{66} has to be considered as a constant.
Substitution of eqs. (39^a) and (44^a) into eq. (AVII-18) and subsequent linearization of the terms containing $\psi(t)$ - i.e. $\cos(\psi) \approx 1$, $\sin(\psi) \approx \psi$ - yields as equation of motion for the translation:

(AVII-20) $$\ddot{x}_{2G} + \frac{2c_w}{B}\dot{x}_{2G} + \frac{c_0}{\rho LBD}\frac{d}{dt}\Delta x_{2f} + \frac{2g(h-D)}{Bc_w}\frac{c_0}{\rho LBD}\Delta x_{2f} = 0$$

Likewise, substitution of eqs. (39^b) and (44^a) into eq. (AVII-19) and subsequent linearization of the terms containing $\psi(t)$ yields as equation of motion for the rotation:

(AVII-21) $$m^{*}_{66}\ \ddot{\psi} = e_0\ c_0\ \Delta x_{2f} \quad .$$

The relation between $x_{2G}(t)$ and $\Delta x_{2f}(t)$ reads:

$$x_{2G} = \frac{1}{2} B + \Delta x_{2f} + e_0 \sin(\psi) - \frac{1}{2} B \cos(\psi) \quad .$$

From this expression it can be derived by linearizing the terms containing ψ and/or its derivatives:

(AVII-22)
$$\begin{cases} x_{2G} = \Delta x_{2f} + e_0\psi \,, & \dot{x}_{2G} = \frac{d}{dt}\Delta x_{2f} + e_0\dot{\psi} \,, \\ \ddot{x}_{2G} = \frac{d^2}{dt^2}\Delta x_{2f} + e_0\ddot{\psi} \,, & \dddot{x}_{2G} = \frac{d^3}{dt^3}\Delta x_{2f} + e_0\dddot{\psi} \,. \end{cases}$$

Elimination of x_{2G} from eq. (AVII-20) and eq. (AVII-22) yields:

$$\left[\frac{d^3}{dt^3} + \frac{2c_w}{B}\frac{d^2}{dt^2} + \frac{c_0}{\rho LBD}\frac{d}{dt} + \frac{2g(h-D)}{Bc_w}\frac{c_0}{\rho LBD}\right]\Delta x_{2f} + e_0\dddot{\psi} + \frac{2c_w}{B}e_0\ddot{\psi} = 0;$$

eliminating ψ from this expression by means of eq. (AVII-21) one obtains:

(AVII-23)
$$\left[\frac{d^3}{dt^3} + p_0\frac{d^2}{dt^2} + q_0\frac{d}{dt} + r_0\right]\Delta x_{2f} = 0 \,,$$

where
$$p_0 = \frac{2c_w}{B}\,, \qquad q_0 = c_0\left(\frac{1}{\rho LBD} + \frac{e_0^2}{m^*_{66}}\right)\,,$$

$$r_0 = c_0\,\frac{2c_w}{B}\left(\frac{h-D}{h}\,\frac{1}{\rho LBD} + \frac{e_0^2}{m^*_{66}}\right)\,.$$

Eq. (AVII-23) is a linear homogeneous ordinary differential equation of the third order with constant real coefficients. The initial conditions of eq. (AVII-23) are:

(AVII-24)
$$\Delta x_{2f}(0) = 0, \quad \frac{d}{dt}\Delta x_{2f}(0) = v_A, \quad \frac{d^2}{dt^2}\Delta x_{2f}(0) = 0.$$

Eq. (AVII-23) has the same form as eq. (AVII-14); the same holds for the initial conditions (AVII-24) and (AVII-16) in Section A-VII.2.a.

N.B. Starting from Appendix III, Section A-III.4, for the yaw added mass-moment of inertia a value is chosen which is two times the mass-moment of inertia of the ship around the Gz-axis; this implies that $m^{*}_{66} = 3m_{66}$.

On account of the values for ρ, g, L, B, D, h, m_{66}, e_0 and c_0 as used in the report, it holds good that:

$$\frac{b_0^2}{4} + \frac{a_0^3}{27} > 0 \quad ,$$

where $$a_0 = \frac{1}{3}(3q_0 - p_0^2) = c_0\left(\frac{1}{\rho LBD} + \frac{e_0^2}{m^{*}_{66}}\right) - \frac{4c_w^2}{3B^2} \quad ,$$

$$b_0 = \frac{1}{27}(2p_0^3 - 9p_0q_0 + 27r_0) =$$

$$= \frac{2}{27}\frac{c_w}{B}\left(\frac{8c_w^2}{B^2} - 9\frac{c_0}{\rho LBD} + 27\frac{c_0}{\rho LBD}\frac{h-D}{h} + 18c_0\frac{e_0^2}{m^{*}_{66}}\right) \quad ;$$

consequently the solution of eq. (AVII-23) has the same form as that of eq. (AVII-14). So,

$$\text{(AVII-25)} \qquad \Delta x_{2f}(t) = v_A\left[-2\,S\,e^{Pt} + 2\,e^{Rt}\{S\cos(Qt) + T\sin(Qt)\}\right] \quad ,$$

where $$S = \frac{R}{(P-R)^2 + Q^2} \quad , \qquad T = S\,\frac{P^2 - R^2 + Q^2}{2QR} \quad ,$$

$$P = A_c + B_c - \frac{1}{3}\frac{2c_w}{B} \quad ,$$

$$Q = \frac{A_c - B_c}{2} = \sqrt{3} \quad ,$$

$$R = -\frac{A_c + B_c}{2} - \frac{1}{3}\,\frac{2c_w}{B} \quad ,$$

$$A_c = \sqrt[3]{-\frac{b_0}{2} + \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} \quad , \quad B_c = \sqrt[3]{-\frac{b_0}{2} - \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} \quad ,$$

$$a_0 = \frac{1}{3}\,(3q_0 - p_0^2) \quad , \qquad b_0 = \frac{1}{27}\,(2p_0^3 - 9p_0q_0 + 27r_0) \quad ,$$

$$p_0 = \frac{2c_w}{B} \, , \qquad q_0 = c_0\,\left(\frac{1}{\rho LBD} + \frac{e_0^2}{m_{66}^*}\right) \quad ,$$

$$r_0 = c_0\,\frac{2c_w}{B}\,\left(\frac{h - D}{h}\,\frac{1}{\rho LBD} + \frac{e_0^2}{m_{66}^*}\right) \quad ,$$

$$c_w = \sqrt{gh} \quad ,$$

$m_{66}^* = m_{66}$ + yaw added mass-moment of inertia.

Eq. (AVII-21) can be written as:

$$\dot{\psi} = \frac{e_o c_0}{m_{66}^*}\,\Delta x_{2f} \; ;$$

elimination of Δx_{2f} from this expression by means of eq. (AVII-25) yields:

(AVII-26) $$\dot{\psi}(t) = 2\,v_A\frac{e_0 c_0}{m_{66}^*}\,\{-S\,e^{Pt} + S\,e^{Rt}\,\cos(Qt) + T\,e^{Rt}\,\sin(Qt)\}.$$

The initial conditions of this ordinary differential equation of the second order are:

(AVII-27) $\qquad \psi(0) = 0 \quad , \qquad \dot{\psi}(0) = 0 \quad .$

Now $\psi(t)$ can be determined by direct integration of eq. (AVII-26) with respect to time; the two constants of integration are eliminated by using the initial conditions (AVII-27). In doing so one obtains for $\psi(t)$:

(AVII-28)
$$\psi(t) = 2\, v_A \frac{e_0 c_0}{m^*_{66}} \left[-\frac{S}{P^2} e^{Pt} + \frac{TR^2 + 2RSQ - TQ^2}{(R^2+Q^2)^2} e^{Rt} \sin(Qt) + \right.$$
$$+ \frac{SR^2 - 2TQR - SQ^2}{(R^2+Q^2)^2} e^{Rt} \cos(Qt) +$$
$$+ \left(\frac{S}{P} - \frac{SR - TQ}{R^2 + Q^2}\right) t +$$
$$\left. + \left\{\frac{S}{P^2} - \frac{SR^2 - 2TQR - SQ^2}{(R^2+Q^2)^2}\right\} \right] \quad .$$

According to eq. (AVII-22) it can be written for x_{2G}:

(AVII-29) $\qquad x_{2G} = \Delta x_{2f} + e_0 \psi \quad .$

Since $\Delta x_{2f}(t)$ and $\psi(t)$ are given by eq. (AVII-25) and eq. (AVII-28), respectively, $x_{2G}(t)$ can be determined.

Recapitulating, the (analytical) solution of the berthing ship problem in case of an eccentric impact against an undamped linear fender - using a long-wave approximation for the motion of the water - can be represented by:

(AVII-25) $$\Delta x_{2f}(t) = v_A \{- 2\,S\,e^{Pt} + 2\,S\,e^{Rt}\cos(Qt) + 2\,T\,e^{Rt}\sin(Qt)\},$$

(44[a]) $$F_{2f}(t) = \begin{cases} 0 & \text{for } \Delta x_{2f}(t) < 0 \\ -c_0\,\Delta\, x_{2f}(t) & \text{for } \Delta x_{2f}(t) \geqq 0 \end{cases} ,$$

(AVII-28) $$\psi(t) = 2\,v_A \frac{e_0 c_0}{m^*_{66}} \Bigg[- \frac{S}{P^2} e^{Pt} + \frac{TR^2 + 2RSQ - TQ^2}{(R^2 + Q^2)^2} e^{Rt} \sin(Qt) +$$

$$+ \frac{SR^2 - 2TQR - SQ^2}{(R^2 + Q^2)^2} e^{Rt} \cos(Qt) +$$

$$+ (\frac{S}{P} - \frac{SR - TQ}{R^2 + Q^2})\, t +$$

$$+ \{ \frac{S}{P^2} - \frac{SR^2 - 2TQR - SQ^2}{(R^2 + Q^2)^2} \} \Bigg] ,$$

(AVII-29) $$x_{2G}(t) = \Delta x_{2f}(t) + e_0\, \psi(t) ,$$

where $$S = \frac{R}{(P - R)^2 + Q^2} , \qquad T = S\,\frac{P^2 - R^2 + Q^2}{2QR} ,$$

$$P = A_c + B_c - \frac{1}{3}\,\frac{2c_w}{B} ,$$

$$Q = \frac{A_c - B_c}{2}\,\sqrt{3} ,$$

$$R = -\frac{A_c + B_c}{2} - \frac{1}{3}\frac{2c_w}{B} ,$$

$$A_c = \sqrt[3]{-\frac{b_0}{2} + \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} , \quad B_c = \sqrt[3]{-\frac{b_0}{2} - \sqrt{\frac{b_0^2}{4} + \frac{a_0^3}{27}}} ,$$

$$a_0 = \frac{1}{3}(3q_0 - p_0^2) , \quad b_0 = \frac{1}{27}(2p_0^3 - 9p_0q_0 + 27r_0) ,$$

$$p_0 = \frac{2c_w}{B} , \quad q_0 = c_0 \left(\frac{1}{\rho LBD} + \frac{e_0^2}{m_{66}^*}\right) ,$$

$$r_0 = c_0 \frac{2c_w}{B} \left(\frac{h - D}{h}\frac{1}{\rho LBD} + \frac{e_0^2}{m_{66}^*}\right) ,$$

$$c_w = \sqrt{gh} ,$$

$m_{66}^* = m_{66}$ + yaw added mass-moment of inertia.

The time histories of the fender forces, the angles of rotation and the translations of G as calculated from eqs. (AVII-25) and (44^a), eq. (AVII-28) and eq. (AVII-29), for the respective water depths h and values of e_0 are presented in figs. 26 through 33 as broken lines.

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NOMENCLATURE

1. General conventions

- Dots over a quantity mean derivatives with respect to time.
- An accent ' indicates that the quantity concerned must be taken per unit length of the body
- Subscripts:

a	with a harmonically varying variable indicates that the amplitude of that variable is meant;
f	indicates that the quantity concerned must be related to the fender;
fl	indicates that the quantity concerned must be related to the fluid;
i,j,k	are used for a direction or a degree of freedom in a Cartesian co-ordinate system; in general they vary from 1 to 6, unless specified otherwise;
kc	indicates that the quantity concerned must be related to the keel clearance underneath the ship;
p	indicates that the quantity concerned must be related to the fluid region on port-side of the ship;
s	indicates that the quantity concerned must be related to the fluid region on starboard-side of the ship;
G	indicates that the quantity concerned must be related to the ship's centre of gravity G.

- Superscripts:

(c)	is used to indicate a (Fourier) cosine transform;
(s)	is used to indicate a (Fourier) sine transport

- Abbreviations:

f.r.f.	frequency response function;
i.r.f.	impulse response function.

2. Co-ordinate systems

$O\bar{x}_1\bar{x}_2\bar{x}_3$ space fixed right-handed system of Cartesian co-ordinates with origin 0; $O\bar{x}_1\bar{x}_2$ coincides with the water surface at rest; the vertical $O\bar{x}_3$-axis is positive upwards; the forward speed V of the ship coincides with the positive $O\bar{x}_1$-axis.

$ox_1x_2x_3$ right-handed Cartesian co-ordinate system parallel with $O\bar{x}_1\bar{x}_2\bar{x}_3$, but translating with the (constant) ship's speed V; at rest the origin coincides with the ship's centre of gravity G; the longitudinal ox_1-axis is positive in forward direction, the ox_2-axis is positive to port-side, the ox_3-axis is positive upwards.

Gxyz moving right-handed Cartesian co-ordinate system with origin G and fixed with respect to the ship; Gxz coincided with the longitudinal plane of symmetry of the ship; the Gy-axis is positive to port-side, the Gz-axis is positive upwards.

3. List of symbols

Symbols not included in the list below are only used at a specific place and are explained where they occur.

a	distance of G below the plane of the water-line; real constant.
$\hat{a}$	amplitude of harmonically oscillating (pure) sway motion at zero forward speed.
a_n, b_n, c_n	coefficients in expression for $y_{n-1,n,n+1}(\omega)$.
$a_{ii}^{(n)}$	coefficient of term with order n in power series development for $a_{ii}(\omega)$.
a_{jk}	hydrodynamic coefficient of the mass term in the k-equation as a result of motion in the j-direction ($=a_{jk}(\omega)$).
a_{22}	added mass for swaying motion ($=a_{22}(\omega)$); a'_{22} = idem per unit length.

a_{66}	added mass-moment of inertia for yawing motion ($=a_{66}(\omega)$); a'_{66} = idem per unit length.
a_0, b_0	real constant coefficient.
$b_{ii}^{(n)}$	coefficient of term with order n in power series development for $b_{ii}(\omega)$.
b_{jk}	hydrodynamic coefficient of the damping force in the k-equation as a result of motion in the j-direction ($=b_{jk}(\omega)$).
b_{22}	sway damping force coefficient ($=b_{22}(\omega)$).
b_{66}	yaw damping moment coefficient ($=b_{66}(\omega)$).
c	real constant.
c_{jk}	hydrostatic restoring coefficient in the k-equation as a result of a static displacement in the j-direction at zero forward speed.
c_w	velocity of propagation of long wave.
c_0	spring rate of linear fender.
c_1, c_2	respective spring rates of the two linear springs which combined form the non-linear fender.
d_{sc}	initial distance (i.e. at rest) between the two linear spring elements of the non-linear fender.
e_0	(initial) distance of the line of action of the fender to the ship's centre of gravity G before and during the first contact between ship and fender.
f(t)	general expression for a function (of t).
f_n	value of $f(\omega)$ at $\omega = \omega_n$.
f_{ia}	amplitude of a harmonic (force) excitation $f_i(t)$.
$f_i(t)$	forcing function in the i-direction.
g	acceleration due to gravity.
h	water depth at rest (mean water level).
i	$\sqrt{-1}$.
k(t)	simplified notation for $k_{22}(t)$.
$k_j[\ldots]$	kernel for motion in the j-direction.
$k_{ij}(t)$	response for the j-direction to a unit pulse (i.e. Dirac

function at t = 0) in the i-direction = impulse response function (i.r.f.);
k_{ij} = i.r.f. based on the velocity as output signal,
k^{*}_{ij} = i.r.f. based on the displacement/rotation as output signal.

k_{n-1} value of $k(t-\tau)$ at $t = n\Delta t$, $\tau = l\Delta t$.

$k_{22}(t)$ impulse response function for the sway motion.

$k_{66}(t)$ impulse response function for the yaw motion.

l real positive integer; when used as subscript it represents a number of time steps Δt and indicates that the quantity concerned must be taken at the point of time $\tau = l\Delta t$.

m subscript representing a real positive integer.

(m) superscript indicating that (in an iteration procedure) the m^{th}-approximation is taken of the quantity concerned.

m^{*}_{ii} representation of the 'mass effect' of the ship in the equation(s) of motion in case of uncoupled motions (= m_{ii} + added mass(-moment of inertia)).

m_{jk} inertia matrix (i.e. generalized mass) of the ship.

m_0 usual wave number = positive root of $\omega^2 = gm_0 \tanh(m_0 h)$.

m_{11} mass of ship (model) for horizontal (surge) motion.

m_{22} mass of ship (model) for horizontal (sway) motion.

m_{66} mass-moment of inertia of ship (model) around Gz-axis.

n subscript used as running index representing a real positive odd integer; real positive integer -when used as subscript it represents a number of time steps Δt and indicates that the quantity concerned must be taken at the point of time $t = n\Delta t$.

p_i constant in approximative expression for $a_{ii}(\omega)$ in case $\omega \to \infty$.

p_0, q_0, r_0 real constant coefficient.

q_i constant in approximative expression for $b_{ii}(\omega)$ in case $\omega \to \infty$.

r	radius of (semi-)circle around the origin; modulus of s.
s	complex variable.
s_1	certain complex number.
t	time co-ordinate.
t_1	point of time.
$u_j(t)$	response of the ship-fluid system in the j-direction to the set of forcing functions $\{f_i(t)\}$.
$v(t)$	simplified notation for $\dot{x}_2(t)$.
$v_{fl,p}, v_{fl,s}$	(horizontal) fluid velocity on port-side, c.q. starboard-side of the ship.
v_n	value of $v(t)$ at $t = n\Delta t$.
v_A	constant lateral speed of approach towards the berth.
v_{oi}	constant velocity of the ship in the i-direction.
$v_{2,kc}$	horizontal fluid velocity in keel clearance underneath ship.
w_m	root of characteristic equation ($m = 1,2,3$).
x_j	j-th mode of motion of the ship.
x_1	surge motion.
x_2	sway motion.
x_3	heave motion.
x_4	roll motion; Eulerian angle.
x_5	pitch motion; Eulerian angle.
x_6	yaw motion; Eulerian angle.
$x_{1f}(t)$	abscissa of the point of the fender (= constant for all t).
$x_{2f}(t)$	ordinate of the point of the fender.
$x_{1G}(t)$	abscissa of the ship's centre of gravity G at point of time t during the contact between ship and fender.
$x_{2G}(t)$	ordinate of the ship's centre of gravity G at point of time t during the contact between ship and fender.
$y_{n-1,n,n+1}(\omega)$	approximative expression for $f(\omega)$ on the closed interval $[\omega_{n-1},\omega_{n+1}]$, based on the Lagrange three point interpolation formula for equally spaced abscissas.

A	projection of the point of the fender on the ship's longitudinal plane of symmetry.
A_c, B_c	real constant.
B	beam of the ship (model).
C_m	constant of integration (m =1,2,3).
C_D	drag coefficient.
D	draught of ship (model).
E	total amount of energy absorbed by a fender (with linear behaviour).
$E_{n-1,n,n+1}(\omega)$	discretization error on the closed interval $[\omega_{n-1}, \omega_{n+1}]$ as a result of the approximation $y_{n-1,n,n+1}(\omega)$ for $f(\omega)$.
$F_{2f}(t)$	reaction force in fender acting in the x_2-direction.
$F_{2,viscous}$	fluid reactive force on the ship from viscous origin in the sway mode of motion.
G	centre of gravity of the ship (model).
$Im[...]$	imaginary part of ...
$I_M(t)$	result of numerical integration along the closed interval $[0, \omega_M]$.
I_I, I_{II}, I_{III}	symbolic notation of (Laplace) integral.
$K_{ij}(i\omega)$	Fourier transform of $k_{ij}(t)$ = harmonic transfer function for the j-direction in response to a (harmonic force) excitation in the i-direction = frequency response function (f.r.f.).
$K_{ij}^{(c)}(\omega)$	Fourier cosine transform of $k_{ij}(t)$.
$K_{ij}^{(s)}(\omega)$	Fourier sine transform of $k_{ij}(t)$.
$\dot{K}_{ij}^{(c)}(\omega)$	Fourier cosine transform of $\dot{k}_{ij}(t)$.
$\dot{K}_{ij}^{(s)}(\omega)$	Fourier sine transform of $\dot{k}_{ij}(t)$.
$K_{ii}(s)$	Laplace transform of $k_{ii}(t)$ = transfer function for the i-direction in response to a force excitation in the i-direction.
$K_{ii}^{(r)}(s)$	part of $K_{ii}(s)$ without poles.

$K_{ii}^{(p)}(i\omega)$	part of $K_{ii}(i\omega)$ containing a pole for $\omega = 0$.
$K_{ii}^{(r)}(i\omega)$	part of $K_{ii}(i\omega)$ without poles.
L	length of ship (model).
M	subscript representing a real positive even integer.
M_0	reduced or effective mass of the ship (model) for horizontal motion.
N	real positive integer representing the number of time steps of the total durance of the contact between ship and fender; when used as subscript it indicates that the quantity concerned must be taken at the point of time $t = N\Delta t$.
P,Q,R,S,T	real constant.
$Re[...]$	real part of ...
$R_i(\omega)$	wave making coefficient for (ship) motion in the i-direction = ratio of the amplitude of the radiated waves at infinity to the amplitude of the (ship) motion in the i-direction.
$R_M(t)$	discretization error (i.e. process error) in consequence of a numerical process of integration along the closed interval $[0,\omega_M]$.
U(t)	unit step function.
V	constant forward speed of the ship.
X	abscissa of the position of the ship's centre of gravity G in the horizontal plane as measured during the berthing operation by the 'position follower'.
Y	ordinate of the position of the ship's centre of gravity G in the horizontal plane as measured during the berthing operation by the 'position follower'.
$\mathcal{F}\{f(t)\}$	Fourier transform of the function f(t).
$\mathcal{L}\{f(t)\}$	Laplace transform of the function f(t).
α	main dimension of body with elementary form.
$\alpha_0,\beta_0,\gamma_0$	real constant coefficient.
α_{ik}	Kronecker alpha: $\alpha_{ik} = 1$ for $k = i$, $\alpha_{ik} = 0$ for $k \neq i$.

β	main dimension of body with elementary form.
$\delta(t)$	delta function or Dirac function.
$\delta_i(t)$	unit pulse (i.e. Dirac or delta function) in the i-direction.
ζ_i, η_i	(arbitrary) coefficient specifying the magnitude of a pulse in the i-direction.
η_p, η_s	height of long wave generated on port-side, c.q. starboard-side of the ship.
$\theta_j(\omega)$	phase angle of the response in the j-direction to a harmonic (force) excitation in the i-direction.
λ	real part of s.
ξ	certain value of ω on the closed interval $[\omega_{n-1}, \omega_{n+1}]$.
ξ_m	root of cubic equation (m = 1,2,3).
ρ	specific mass density.
τ	integration variable (time).
ϕ	argument of s.
ϕ_i	phase angle of a harmonic (force) excitation $f_i(t)$.
$\psi(t)$	angle of rotation of the ship's longitudinal axis of symmetry around the ox_3-axis at point of time t during the contact between ship and fender.
$\overline{\psi}(t+\Delta t)$	mean value of $\psi(t)$ on the interval of time Δt.
ψ_0	amplitude of harmonically oscillating (pure) yaw motion at zero forward speed.
ω	circular frequency; imaginary part of s.
ω_M, ω_l	certain circular frequency.
Δt	interval of time, time step, time increment.
$\Delta x_{2f}(t)$	impression of the fender.
$(\Delta x_{2f})_{max}$	maximum impression of the linear fender.
$\Delta y(t)$	simplified notation for $\Delta x_{2f}(t)$.
$\Delta y_n, \Delta y_l$	value of Δy at $t = n\Delta t$, $\tau = l\Delta t$, respectively.
$(\Delta\omega)_n$	interval between the successive, equally spaced, abscissas ω_{n-1}, ω_n, and ω_{n+1} (n = 1,3,5,...).