HEAT TRANSFER IN TURBULENT TUBE FLOW

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOGESCHOOL TE DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS DR. O. BOTTEMA, HOOGLERAAR IN DE AFDELING DER ALGEMENE WETENSCHAPPEN, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP WOENSDAG 14 NOVEMBER 1956 DES NAMIDDAGS TE 4 UUR DOOR

HENDRIKUS LUDOVICUS BECKERS

Natuurkundig ingenieur

GEBOREN TE MAASTRICHT

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MARTINUS NIJHOFF

1956

Dit proefschrift is goedgekeurd door de promotor

Prof. Dr. J. A. PRINS

AAN DE NAGEDACHTENIS VAN A. CUSTERS

AAN MIJN OUDERS EN MIJN VROUW

STELLINGEN

I

De berekening van de tengevolge van inwendige wrijvingswarmte ontstane temperatuurverdeling in een laminaire stroming door een buis is bij een willekeurige warmteweerstand van de buiswand nauwelijks gecompliceerder dan bij de door Brinkman beschouwde waarden nul en oneindig.

Brinkman, H. C., Appl. Sci. Res. **A2**, (1950), 120.

II

De door Dörr gegeven oplossingsmethode van de bij de theorie der draagvlakken in loodrechte cascade-opstelling voorkomende singuliere integraalvergelijking

$$f(x) = \frac{1}{2h}\int_{-1}^{+1} g(y) \operatorname{cotgh}\left\{\frac{\pi(x-y)}{h}\right\} dy$$

is nodeloos omslachtig.

Dörr, J., Ingenieur Archiv, Erstes Heft, (1951), 66.

III

Het door Minard en Johnson berekende verband tussen de snelheden van de continue faze en gedispergeerde faze bij tegenstroom-extractie in een sproeikolom berust op een principieel onjuiste fysische benadering.

Minard, G. W. and Johnson, A. I., Chem. Engng Progr. **48**, (1952), 62.

IV

De voorstelling van Krebs, dat het polymere bestanddeel van vloeibare zwavel uit ringvormige macromoleculen zou bestaan in plaats van uit ketenvormige, is onwaarschijnlijk.

Krebs, H., Z. anorg. Chem. **272**, (1953), 288.

V

Het verdient aanbeveling voor een antiferromagnetisch kristal de term Curie- of Néelpunt te reserveren voor het overgangspunt bij afwezigheid van een uitwendig magneetveld.

VI

Het uitspreken en schrijven van de operator van Laplace als delta verdient geen aanbeveling.

VII

Op pagina 37 van Science News **34** (1954) blijkt de auteur niet op de hoogte te zijn van de lange voorgeschiedenis van het door hem behandelde verschijnsel.

Leidenfrost, De aquae communis qualitatibus, Duisburg 1758.
Boutigny (d'Evreux), M. P. H., Etudes sur le corps à l'état spheroïdal, Paris 1883.

VIII

In „Orgelbouwkunde", door A. P. Oosterhof en Mr. A. Bouman (Amsterdam 1947), is het volgende onjuist:

a) de verklaring van de werking van een zwelkast (pag. 147).
b) het schema van een zogenaamd unit-orgel (pag. 260).

IX

In tegenstelling tot de projectielen der luchtdoelartillerie behoeft bij raketten die een zogenaamde hondebaan doorlopen, het zonder meer opvoeren van de snelheidsverhouding raket-doel nog geen verhoging der gevechtswaarde tengevolge te hebben.

X

Het zou de maatschappij ten goede komen, wanneer algemener aanvaard werd, dat ook een partiële opleiding aan een Technische Hogeschool, eindigend met het volbrengen van bepaalde propaedeutische of kandidaats-examens, voor een aantal beroepen een voldoende voorbereiding vormt.

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H. L. BECKERS

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CHAPTER I. INTRODUCTION

§ 1. *Statement of the problem.* There are two types of flow: that without eddying, called laminar flow, and the eddying type, called turbulent flow. The former type occurs in slow movement, or, to put it more accurately, when the Reynolds number is less than 2300; the latter type occurs above that value. The Reynolds number is defined as $Re = 2r_0u_b/\nu$, r_0 *) being the radius of the tube, u_b the mean velocity and ν the kinematic viscosity. ν is a measure of the rate at which velocity differences in the fluid smooth out; this phenomenon is called internal friction. Another material constant a is in the same way a measure of the equalization rate of temperature differences, the so-called phenomenon of heat conductivity. In the case of turbulent motion these two constants are increased by amounts A_m and A_q, respectively, because the eddies, too, promote exchange of momentum and energy between different points of the flowing medium. These quantities are not constants, however; they depend on the flow condition, which makes the calculation of their effect more difficult.

For the velocity distribution and heat transfer in the case of laminar flow the differential equations have not only been drawn up; they have also been solved for the most important cases, often by the mathematical method of eigenfunctions [14] [15] [20].

In the case of turbulent flow the variable quantities A_m and A_q occur in the differential equations. These are not known a priori, but they can be found from the velocity distribution along the cross-section of the tube, for which a largely empirically derived law is known. Some attempts have already been made to calculate the heat transfer on these assumptions. In this thesis we shall try to do so systematically with the aid of eigenfunctions and eigenvalues calculated numerically. The first eigenfunction and the constants belonging to it can be worked out with sufficient accuracy on a normal computer; for a more precise calculation of the constants belonging to the second eigenfunction (and the following ones) an electronic computer would in general be indispensable.

We state our problem as follows: A liquid or gas having an initial temperature T_i flows through a smooth cylindrical tube. Starting from a given section $z = 0$ the fluid cools down owing to

*) For list of symbols see p. 44.

heat passing through the wall to the environment of temperature T_0. We presuppose that the flow is hydrodynamically fully developed and that the radial heat current from the inside of the wall to the environment obeys Newton's law:

$$-\lambda\left(\frac{\partial T}{\partial r}\right)_{r=r_0}=\alpha_0\,(T_{r=r_0}-T_0). \tag{1}$$

Though our theory applies to all values of α_0, in our numerical calculations we have as yet taken the case $\alpha_0=\infty$ only.

In a laminar flow we can then derive the following differential equation for the temperature distribution:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda r\frac{\partial T}{\partial r}\right)=\varrho c u\frac{\partial T}{\partial z}\,. \tag{2}$$

In deriving this differential equation use has been made of the fact that in a volume element the longitudinal supply of heat by convection must be equal to the lateral heat discharge by conduction. (It can be demonstrated that except for extremely small values of *Pr* the longitudinal supply by conduction is negligible compared with the supply by convection.)

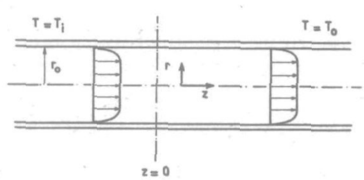

Fig. 1. Turbulent flow from hot to cold section in cylindrical tube.

If we assume that the material constants are independent of velocity and temperature, we may write (2) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(a r\frac{\partial T}{\partial r}\right)=u\frac{\partial T}{\partial z}\,. \tag{3}$$

In principle this equation is also valid for turbulent flow. The expression for the radial heat discharge, however, becomes rather more complicated, as not only the heat conduction but also the "turbulent diffusion" carries off the heat radially. This turbulent diffusion is a result of turbulent mixing and can to a certain extent be regarded as a macroscopic analogue of the chaotic movement of molecules, which results in molecular conduction.

For molecular conduction in the radial direction the classical formula

$$\frac{q_m}{\varrho c} = - a \frac{\partial T}{\partial r} \tag{4}$$

is valid. Analogously, for "turbulent diffusion" we may write

$$\frac{q_t}{\varrho c} = - A_q \frac{\partial T}{\partial r}. \tag{5}$$

A_q is here the thermal eddy diffusivity. We shall see later on how this coefficient depends upon position.

The total radial heat discharge is equal to the sum of turbulent and molecular diffusion:

$$\frac{q}{\varrho c} = - (a + A_q) \frac{\partial T}{\partial r}. \tag{6}$$

The differential equation for the thermal balance in a turbulent flow then becomes

$$\frac{1}{r} \frac{\partial}{\partial r} \left[(a + A_q) r \frac{\partial T}{\partial r} \right] = u \frac{\partial T}{\partial z}. \tag{7}$$

Here again the longitudinal heat transfer by conduction and turbulent diffusion is negligible in comparison to convection.

The solution of (7) requires that the velocity and eddy diffusivity be known as functions of the radius r. These functions are slightly more complicated in a turbulent than in a laminar flow, where A_q is zero and u is a quadratic function of the radius r.

The principal difficulties in the case of turbulence are:

1) The velocity and eddy diffusivities of momentum and heat have never yet been calculated theoretically; nor is it to be expected that this will be done in the near future; they have therefore to be taken from measurement [5] [13].

2) For velocity and eddy diffusivity of momentum empirical formulae have already been devised [5] [6] [19]. Accordingly, we shall take these as our starting point, and from them we shall derive formulae suited to our problem. To describe the velocity distribution

and eddy diffusivities as functions of r it has, however, proved necessary to divide the section of the tube into three parts:

a) The turbulent core extending from the centre of the tube to rather near the wall. In this region the flow is completely turbulent, which means that the molecular diffusion of momentum is much smaller than the turbulent diffusion. At high *Pr*-values the molecular heat diffusion can then be ignored. At lower *Pr*- and *Re*-values, however, the molecular heat diffusion near the boundary of the turbulent core is not quite small enough to be ignored.

b) The laminar boundary layer, adjoining the wall. The flow here is so nearly laminar that the turbulent diffusion is negligible in relation to the molecular diffusion.

c) Between the laminar boundary layer and the turbulent core there is an area of transition, called the transition layer, where the flow changes from laminar to turbulent. In this region molecular and turbulent diffusion are of the same order and both therefore have to be taken into account.

§ 2. *Relation between velocity distribution and eddy diffusivities of momentum and heat.* For the sum of radial heat discharge by molecular conduction and turbulent diffusion we found

$$\frac{q}{\varrho c} = -\,(a + A_q)\,\frac{\partial T}{\partial r}\,.$$

Likewise, for the radial momentum transfer in a turbulent flow we may write

$$\frac{\tau}{\varrho} = -\,(\nu + A_m)\,\frac{\partial u}{\partial r}\,. \tag{8}$$

$-\,A_m \partial u/\partial r$ represents the „turbulent momentum diffusion" resulting from turbulent mixing and $-\,\nu \partial u/\partial r$ the molecular momentum diffusion; A_m is the eddy diffusivity of momentum.

There is a close similarity between turbulent diffusion of heat and that of momentum, as in both cases some material property (heat or momentum) is transmitted by mixing of the material. The amount of material transported from one layer of the fluid to the other determines the amount of heat or momentum transmitted. From this it would look as if A_q should be equal to A_m, but opinions on this point differ a good deal [8]). In the literature on the subject the values 1 and 1.6 are those most frequently found, even 2 being mentioned.

Since, in our calculations this relation only occurs in combination with other dimensionless quantities, to which a series of values is given in the calculations, we may leave the exact value of $A = A_q/A_m$ unsettled till the end. The results found can, if necessary, be worked out later for the value of A that proves to be the most correct one.

With the aid of the hypothesis mentioned A_q can be calculated, provided A_m is known. Indeed, for the radial momentum transmission we found (8), so that

$$A_m = -\nu - \frac{\tau/\varrho}{\partial u/\partial r}. \tag{9}$$

In the case of hydrodynamically fully developed flow, the shearing stress τ can be directly calculated. The work performed by pressure is then completely converted into frictional work, so that per unit of length we have

$$2\pi \mathrm{d}(r\tau)\,\mathrm{d}z = -\,2\pi r\,\mathrm{d}r\,\mathrm{d}p \tag{10}$$

or

$$\tau = -\frac{r}{2}\frac{\mathrm{d}p}{\mathrm{d}z}.$$

By introducing

$$\tau_0 = -\,\tfrac{1}{2} r_0 \frac{\mathrm{d}p}{\mathrm{d}z}, \tag{11}$$

the shearing stress at the wall, we find for τ

$$\tau = \tau_0\, r/r_0. \tag{12}$$

By substitution of (12) formula (9) reduces to

$$A_m = -\nu - \frac{\tau_0 r/r_0}{\varrho\partial u/\partial r}. \tag{13}$$

Making use of (13) we then find for A_q

$$A_q = -A\left(\nu + \frac{\tau_0 r/r_0}{\varrho\partial u/\partial r}\right). \tag{14}$$

In the following section we shall consider more closely the velocity as a function of the radius. When this function is known, A_q also is known as a function of the radius.

§ 3. *Velocity distribution and eddy diffusivities.*

a) In the laminar boundary layer. In this layer it is difficult to determine the flow velocity experimentally, as its thickness is in general slight. It is safe to assume, however, that the theoretical approximation of the velocity is reasonably correct. For, in this area the eddy diffusivity is virtually zero and r/r_0 is practically equal to 1, so that (13) is transformed into

$$\nu = -\frac{\tau_0}{\varrho} \Big/ \frac{\partial u}{\partial r}. \tag{15}$$

Upon integration it follows that the velocity is

$$u = \frac{\tau_0}{\varrho} \frac{r_0}{\nu} \left(1 - \frac{r}{r_0}\right); \tag{16}$$

τ_0/ϱ has the dimension of the square of a velocity, for which reason the "frictional velocity" $u^* = \sqrt{\tau_0/\varrho}$ may be introduced. After division by u^* we find from (16)

$$\frac{u}{u^*} = \frac{u^* r_0}{\nu} (1 - \xi), \tag{17}$$

ξ being equal to r/r_0. $u^* r_0/\nu$ is dimensionless, having the form of a Reynolds modulus. We therefore use the symbol Re' for this group, so that

$$u/u^* = Re'(1 - \xi) \equiv Re'\eta. \tag{18}$$

$\eta \equiv 1 - \xi$ represents the reduced distance to the wall of the tube.

The value of Re' for a given flow can readily be derived from the "usual" Reynolds number, $Re = 2r_0 u_b/\nu$, as follows: With the aid of the measured friction factor C_w $(= \tau_0/\frac{1}{2}\varrho u_b^2)$ of the tube we find via (11)

$$\frac{2\tau_0}{r_0} = -\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{c_w \varrho u_b^2}{r_0}, \tag{19}$$

or

$$Re' = Re\sqrt{\frac{c_w}{8}}. \tag{20}$$

At a given velocity distribution the relation Re'/Re can also be found by calculating the amount of fluid passing through the tube at that distribution. We applied this calculation to provide the usual Re-number corresponding to chosen Re'-values.

The thickness of the laminar boundary layer we can find from the experimental velocity distribution. As long as (18) applies, turbulent diffusion can be ignored and the flow is still laminar; when the velocity diverges from it, we can assume that the transition layer has been reached (see fig. 2). From measurements it is found that the boundary layer extends from the wall to $\eta = 5/Re'$ [5] [6] [8] [19].

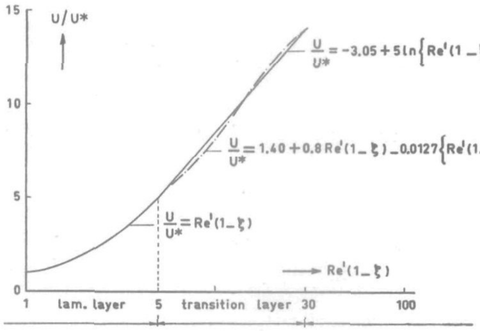

Fig. 2. Reduced velocity distribution of turbulent flow in the laminar boundary layer and the transition layer.
–·– Approximation by a polynomial.

b) The transition layer. In this region the velocity distribution can be approximated by [5] [6] [8] [19]

$$u/u^* = \mathrm{C(onstant)} + 5 \ln (Re'\eta). \qquad (21)$$

From the condition that for $\eta = 5/Re'$ (the boundary between the transition layer and the laminar boundary layer) the values found from (18) and (21) must be equal, it follows that the value of C is $-$ 3.05, so that (21) is transformed into

$$u/u^* = -\,3.05 + 5 \ln (Re'\eta). \qquad (22)$$

From (22) we find for A_m

$$A_m/\nu = -\,1 + 0.2\, Re'\xi(1 - \xi) \qquad (23)$$

and for A_q

$$\frac{A_q}{a} = A\,\frac{\nu}{a}\,[-\,1 + 0.2\,Re'\xi(1-\xi)]. \tag{24}$$

In this equation ν/a is Prandtl's number Pr. In analogy to this number we shall represent the dimensionless group $A.Pr$ by Pr'. When $A = 1$, Pr' is equal to Pr. Thus, the formula for A_q becomes

$$A_q/a = -\,Pr' + 0.2\,Pr'Re'\xi(1-\xi). \tag{25}$$

c) The turbulent core. For this region Reichardt [13]) has drawn up the so-called "middle law" for the velocity distribution:

$$\frac{u}{u^*} = \frac{u_m}{u^*} + 2.5 \ln\left(\frac{1-\xi^2}{1+2\xi^2}\right). \tag{26}$$

The familiar logarithmic formula [5]) [6]) for the velocity distribution is

$$u/u^* = 5.5 + 2.5 \ln\,(Re'\eta). \tag{27}$$

The "middle law" is therefore in the nature of an extension of (27). From (27) it follows that the maximum velocity for $\xi = 0$ is

$$u_m/u^* = 5.5 + 2.5 \ln Re'. \tag{28}$$

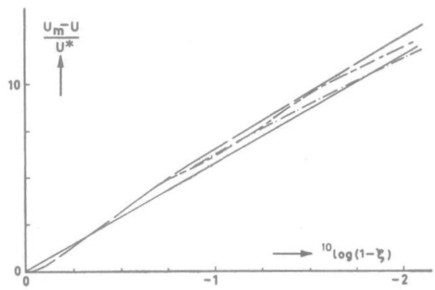

Fig. 3. Reduced velocity distribution of turbulent flow in the turbulent core
—— —— Reichardt's "middle law" (26).
— - — - — Approximation by a polynomial (65).
— · — · — Approximation by formula (88).
———— Logarithmic velocity distribution (27).

We shall use this u_m/u^* value also in the distribution according to (26).

For the boundary between the transition layer and the turbulent core, where the velocities found by these formulae must be equal, it follows from (22) and (27) that

$$- 3.05 + 5 \ln (Re'\eta) = 5.5 + 2.5 \ln (Re'\eta), \tag{29}$$

so that the boundary is at $1 - \xi = \eta = 30/Re'$. We shall adhere to this formula also when using (26).

The condition that at the boundary of the transition layer and the turbulent core the velocities found from (22) and (26) must be equal, cannot be precisely complied with, owing to the adaption of u_m/u^* according to (28); closeness of compliance depends in some degree on the Re'-number.

From (26), with (13) in which we can ignore ν in comparison to A_m, we find

$$\frac{A_m}{\nu} = \frac{Re'}{15} (1 - \xi^2)(1 + 2\xi^2) \tag{30}$$

and (see fig. 4)

$$\frac{A_q}{a} = \frac{Pr'Re'}{15} (1 - \xi^2)(1 + 2\xi^2). \tag{31}$$

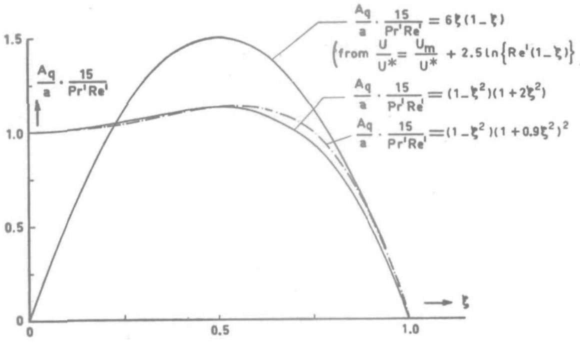

Fig. 4. Variation across the tube (with ξ) of reduced thermal eddy diffusivity in the turbulent core according to some velocity distributions considered.

To recapitulate: In the laminar region, extending to $\eta = 5/Re'$, (18) is valid for the velocity. In the transition layer the formulae for the velocity and the eddy diffusivity of heat are respectively (22) and (25), and in the turbulent core (26) and (31).

It should again be noted that we assume the eddy diffusivities, local velocity and material values occurring to be independent of the local temperature.

Chapter II. investigation of the differential equation

§ 4. *The general solution.* The given differential equation has the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r(a+A_q)\frac{\partial T}{\partial r}\right\} = u\frac{\partial T}{\partial z}. \tag{32}$$

It is our object to determine the respective general solutions T_1, T_2 and T_3 of this differential equation for the turbulent, the transition and the laminar regions, making use of the corresponding A_q and velocity distribution. Next, we determine the constants occurring in the general solutions in such a manner that the three solutions are continuous at the boundaries of the areas, the initial and boundary conditions of the problem also being satisfied.

Before deriving the form of the general solution we reduce (32) by introducing the variables

$$\xi = \frac{r}{r_0}, \; \zeta' = A\frac{z}{r_0}, \quad \vartheta = \frac{T - T_0}{T_i - T_0}$$

and the quantities $f(\xi) = (a + A_q)/a$, $\varphi = u/u_m$ and $P\acute{e}_m = Re_m Pr'$, Re_m being equal to $u_m r_0/\nu$. We reduce the velocity u by dividing it by u_m, so that the reduced velocity throughout the section is less than or equal to 1, which is of some advantage in the numerical calculation.

After reduction (32) becomes

$$\frac{1}{\xi}\frac{\partial}{\partial \xi}\left[\xi f(\xi)\frac{\partial \vartheta}{\partial \xi}\right] = P\acute{e}_m\,\varphi(\xi)\frac{\partial \vartheta}{\partial \zeta'}. \tag{33}$$

By supposing ϑ to be the product of a function of ξ and a function of ζ': $\vartheta = E(\xi)\cdot H(\zeta')$, (33), when divided by ϑ, is transformed into

$$\frac{1}{\varphi(\xi)}\cdot\frac{1}{\xi E(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\frac{\mathrm{d}E}{\mathrm{d}\xi}\right] = P\acute{e}_m\frac{1}{H(\zeta')}\frac{\mathrm{d}H(\zeta')}{\mathrm{d}\zeta'}. \tag{34}$$

The left-hand term is a function of ξ only and the right-hand one a function of ζ'. As (34) must be valid for any $\zeta' > 0$ and $0 \leqslant \xi \leqslant 1$, the left-hand and right-hand terms of this equation should be equal to the same constant in order to obtain a solution of E and H. This method is called "separation of variables". The value of this constant is negative, as $\mathrm{d}H/\mathrm{d}\zeta'$ is negative; we can therefore take this constant to be equal to $-\beta_k^2$, so that

$$\frac{1}{\varphi(\xi)}\,\frac{1}{\xi E(\xi)}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\,\frac{\mathrm{d}E}{\mathrm{d}\xi}\right] = P\acute{e}_m\,\frac{1}{H(\zeta')}\,\frac{\mathrm{d}H(\zeta')}{\mathrm{d}\zeta'} = -\,\beta_k^2$$

is valid.

Upon integration the solution $\exp(-\,\beta_k^2\zeta'/P\acute{e}_m)$ for H follows; for E_k we find the differential equation

$$\frac{1}{\xi}\,\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\,\frac{\mathrm{d}E_k}{\mathrm{d}\xi}\right] + \beta_k^2\varphi(\xi)\,E_k = 0. \tag{35}$$

Every value of β_k^2 has a corresponding function $H_k(\zeta')$ and a differential equation in the form of (35). The values of β_k^2 are determined by the boundary conditions. By satisfying these we obtain a series of e i g e n v a l u e s β_1, β_2, β_3 etc. with the respective corresponding e i g e n f u n c t i o n s E_1, E_2, E_3 etc. .

The general solution is then a linear combination of the following form [9] [15] [20] [21]:

$$\vartheta = \sum_0^\infty{}_k\, C_k E_k\, \mathrm{e}^{-\beta_k{}^2\,\zeta'/P\acute{e}_m}. \tag{36}$$

It should be noted that for expanding the E_k series it is necessary on account of convergence to determine the poles of the differential equation. This will be dealt with more fully in § 7.

If the above conditions have been complied with, (35) produces for E_k two independent series Y_k and Z_k; E_k is in general a linear combination of these two series.

The complete general solution assumes the form

$$\vartheta = \sum_0^\infty{}_k\,[A_kY_k(\xi) + B_kZ_k(\xi)]\exp\left(-\,\beta_k^2\zeta'/P\acute{e}_m\right). \tag{37}$$

A_k, B_k and β_k^2 are constants that have still to be determined.

§ 5. *Adaptation of the solution to the initial and boundary conditions.* For the turbulent core, the transition layer and the laminar boundary layer we can represent the respective general solutions by

$$\vartheta_I \;= \sum_0^\infty {}^k \, [A_{Ik} Y_{Ik} + B_{Ik} Z_{Ik}] \exp\,(-\,\beta^2_{Ik}\,\zeta'/Pé_m), \tag{38}$$

$$\vartheta_{II} \;= \sum_0^\infty {}^k \, [A_{IIk} Y_{IIk} + B_{IIk} Z_{IIk}] \exp\,(-\,\beta^2_{IIk}\,\zeta'/Pé_m), \tag{39}$$

$$\vartheta_{III} = \sum_0^\infty {}^k \, [A_{IIIk} Y_{IIIk} + B_{IIIk} Z_{IIIk}] \exp\,(-\,\beta^2_{IIIk}\,\zeta'/Pé_m). \tag{40}$$

To determine the value of the constants in these equations we shall consider the conditions that the solutions ϑ_I, ϑ_{II} and ϑ_{III} have to satisfy:

a) $\partial\vartheta_I/\partial\xi = 0$, for $\xi = 0$, which can be directly derived from considerations of symmetry.

b) $\partial\vartheta_{III}/\partial\xi = -\,Nu_0\,\vartheta_{III}$ for $\xi = 1$, Nu_0 being equal to $\alpha_0 r_0/\lambda$. For, the current of heat at the inside perpendicular to the wall, $q = -\,\lambda\,(\partial T/\partial r)_{r=r_0}$, must be equal to the current of heat as given by Newton's law: $q = \alpha_0(T_{r=r_0} - T_0)$.

c) In order that the calculations shall have physical significance the solutions obtained must pass over into each other at the boundaries. This means that at the limit of the turbulent region and the transition layer (let $\xi = \xi_I$) $\vartheta_I = \vartheta_{II}$ and at the boundary of the transition layer and the laminar layer (let $\xi = \xi_{II}$) ϑ_{II} must be equal to ϑ_{III}.

d) An obvious condition would be that the values derived from the temperature distributions at the boundaries should be equal. In general, however, this requirement cannot be satisfied simultaneously with the initial condition. For then the eigenfunctions would not remain orthogonal across the section of the tube (see (50)). The condition that at the boundaries the radial flow of heat shall be equal does prove to be capable of being satisfied. From a physical point of view this is certainly to be preferred, so that finally we choose the conditions

$$f_I(\xi)\,\frac{\partial\vartheta_I}{\partial\xi} = f_{II}(\xi)\,\frac{\partial\vartheta_{II}}{\partial\xi} \qquad \text{for} \quad \xi = \xi_I \quad \text{and}$$

$$f_{II}(\xi)\,\frac{\partial\vartheta_{II}}{\partial\xi} = f_{III}(\xi)\,\frac{\partial\vartheta_{III}}{\partial\xi} \qquad \text{for} \quad \xi = \xi_{II} \tag{41}$$

(the radial flow of heat being $-\,\varrho c(a + A_q)\partial T/\partial r$).

In general at the boundaries the reduced eddy diffusivities are not equal. When they are so, (41) means that the derivates are also equal.

e) The last condition that has to be fulfilled is the initial one $\vartheta = 1$ for $\zeta' = 0$ and $0 \leqslant \xi \leqslant 1$.

If we consider conditions *a—d* more closely, we see that with each of these conditions a linear combination of exponentials must be equal to another linear combination of exponentials. For instance, for ξ_I, ϑ_I and ϑ_{II} must be equal, that is to say

$$\sum_0^\infty {}_k E_{I_k}(\xi_I) \exp(-\beta_{Ik}^2 \zeta'/Pé_m) = \sum_0^\infty {}_k E_{II_k}(\xi_I) \exp(-\beta_{IIk}^2 \zeta'/Pé_m). \quad (42)$$

Since, for $\xi = \xi_I$, $E_{I_k}(\xi_I)$ and $E_{II_k}(\xi_I)$ are constant, there must exist a β_{II_k} for every β_{I_k}, in such a manner that for every value of ζ' an exponential in the left-hand member is equal to an exponential in the right-hand member. If this were not the case, only the trivial solution $\vartheta_I = \vartheta_{II} = \vartheta_{III} = 0$ would exist. Therefore, (42) will be fulfilled if

$$\beta_{I_k} = \beta_{II_k} \quad \text{and} \quad E_{I_k}(\xi_I) = E_{II_k}(\xi_I). \quad (43)$$

The other conditions can also be worked out in this way, so that we find one series of eigenvalues β_1, β_2, β_3, etc. valid for all the three regions, in other words

$$\beta_{I_k} = \beta_{II_k} = \beta_{III_k} = \beta_k. \quad (44)$$

Conditions *a–e* are transformed into

a) $\dfrac{dE_{I_k}}{d\xi} = 0$ for $\xi = 0$,

b) $E_{I_k}(\xi) = E_{II_k}(\xi)$ and $f_I(\xi)\dfrac{dE_{I_k}}{d\xi} = f_{II}(\xi)\dfrac{dE_{II_k}}{d\xi}$ for $\xi = \xi_I$,

c) $E_{II_k}(\xi) = E_{III_k}(\xi)$ and $f_{II}(\xi)\dfrac{dE_{II_k}}{d\xi} = f_{III}(\xi)\dfrac{dE_{III_k}}{d\xi}$ for $\xi = \xi_{II}$,

d) $\dfrac{dE_{III_k}}{d\xi} = -Nu_0' E_{III_k}$ for $\xi = 1$,

e) $\vartheta = 1$ for $\zeta' = 0$ and $0 \leqslant \xi \leqslant 1$.

Conditions *a*, *b*, *c* and *d* produce six homogeneous linear equations in the constants A_{I_k}, A_{II_k}, A_{III_k}, B_{I_k}, B_{II_k} and B_{III_k}. If a non-zero solution is to exist for this series of constants, the determinant of these six equations must be zero. As this determinant is only a function of β_k^2, it gives us by iteration the series of required eigenvalues (see § 10).

The six equations only give us the ratios between the required constants; in other words, the constants calculated from them all contain an arbitrary common factor γ_k. The solutions obtained can then be represented by

$$\vartheta_I = \sum_0^\infty {}^k \gamma_k \, \Xi_{Ik} \exp\left(-\beta_k^2 \zeta'/P\acute{e}_m\right), \tag{45}$$

$$\vartheta_{II} = \sum_0^\infty {}^k \gamma_k \, \Xi_{IIk} \exp\left(-\beta_k^2 \zeta'/P\acute{e}_m\right), \tag{46}$$

$$\vartheta_{III} = \sum_0^\infty {}^k \gamma_k \, \Xi_{IIIk} \exp\left(-\beta_k^2 \zeta'/P\acute{e}_m\right). \tag{47}$$

The expansion of Ξ_{Ik} is chosen in such a way that $\Xi_{Ik} = 1$ for $\xi = 0$.

As in the calculation for a laminar flow, we utilize the orthogonality of the eigenfunctions Ξ_k to determine γ_k from the initial condition e), which has still to be fulfilled. Filling in the initial values $\vartheta_I = \vartheta_{II} = \vartheta_{III} = 1$ and $\zeta' = 0$ in (45), (46) and (47), multiplying the two members of each equation by $\xi\varphi_I\Xi_{Ik}$, $\xi\varphi_{II}\Xi_{IIk}$ and $\xi\varphi_{III}\Xi_{IIIk}$ respectively, we find, after integration over the interval $0 \leqslant \xi \leqslant 1$,

$$\int_0^1 \xi\varphi \, \Xi_k \, \mathrm{d}\xi = \gamma_k \int_0^1 \xi\varphi \Xi_k \Xi_k \, \mathrm{d}\xi. \tag{48}$$

Integration of (35) gives

$$\int_0^1 \xi\varphi\Xi_k \, \mathrm{d}\xi = \frac{-1}{\beta_k^2} \left(\frac{\mathrm{d}\Xi_{IIIk}}{\mathrm{d}\xi}\right)_{\xi=1}. \tag{49}$$

Likewise

$$\int_0^1 \xi\varphi \, \Xi_k \Xi_l \, \mathrm{d}\xi = \left\{ \begin{array}{l} = 0 \quad \text{for } k \neq l \\[1ex] = \xi f_I(\xi) \left[\dfrac{\mathrm{d}\Xi_{Ik}}{\mathrm{d}(\beta_k^2)} \dfrac{\mathrm{d}\Xi_{Ik}}{\mathrm{d}\xi} - \Xi_{Ik} \dfrac{\mathrm{d}^2\Xi_{Ik}}{\mathrm{d}\xi \mathrm{d}(\beta_k^2)}\right]_0^{\xi_I} \\[2ex] + \xi f_{II}(\xi) \left[\dfrac{\mathrm{d}\Xi_{IIk}}{\mathrm{d}(\beta_k^2)} \dfrac{\mathrm{d}\Xi_{IIk}}{\mathrm{d}\xi} - \Xi_{IIk} \dfrac{\mathrm{d}^2\Xi_{IIk}}{\mathrm{d}\xi \, \mathrm{d}(\beta_k^2)}\right]_{\xi_I}^{\xi_{II}} \\[2ex] + \xi f_{III}(\xi) \left[\dfrac{\mathrm{d}\Xi_{IIIk}}{\mathrm{d}(\beta_k^2)} \dfrac{\mathrm{d}\Xi_{IIIk}}{\mathrm{d}\xi} - \Xi_{IIIk} \dfrac{\mathrm{d}^2\Xi_{IIIk}}{\mathrm{d}\xi \, \mathrm{d}(\beta_k^2)}\right]_{\xi_{II}}^{1} \\[1ex] \qquad \text{for } k = l \end{array} \right\}. \tag{50}$$

Equation (48) gives

$$\gamma_k = \frac{\int_0^1 \xi\varphi\,\Xi_k\,\mathrm{d}\xi}{\int_0^1 \xi\varphi\,\Xi_k\Xi_k\,\mathrm{d}\xi}, \tag{51}$$

so that the general solution of ϑ becomes

$$\vartheta_I = \sum_0^\infty {}_k \frac{\int_0^1 \xi\varphi\Xi_k\,\mathrm{d}\xi}{\int_0^1 \xi\varphi\Xi_k\Xi_k\,\mathrm{d}\xi} \Xi_{Ik}(\xi) \exp\left(-\beta_k^2\zeta'/P\acute{e}_m\right) \text{ for } 0 \leqslant \xi \leqslant \xi_I, \tag{52}$$

$$\vartheta_{II} = \sum_0^\infty {}_k \frac{\int_0^1 \xi\varphi\Xi_k\,\mathrm{d}\xi}{\int_0^1 \xi\varphi\Xi_k\Xi_k\,\mathrm{d}\xi} \Xi_{IIk}(\xi) \exp\left(-\beta_k^2\zeta'/P\acute{e}_m\right) \text{ for } \xi_I \leqslant \xi \leqslant \xi_{II}, \tag{53}$$

$$\vartheta_{III} = \sum_0^\infty {}_k \frac{\int_0^1 \xi\varphi\Xi_k\,\mathrm{d}\xi}{\int_0^1 \xi\varphi\Xi_k\Xi_k\,\mathrm{d}\xi} \Xi_{IIIk}(\xi) \exp\left(-\beta_k^2\zeta'/P\acute{e}_m\right) \text{ for } \xi_{II} \leqslant \xi \leqslant 1. \tag{54}$$

§ 6. *Expressions for some physically important quantities.* In actual practice much use is made of the „cup mixing mean temperature" defined by

$$\vartheta_m = \frac{\int_0^1 \xi\varphi\vartheta\,\mathrm{d}\xi}{\int_0^1 \xi\varphi\,\mathrm{d}\xi} \tag{55}$$

and of the coefficient of "total" heat transfer α defined by

$$q = \alpha\vartheta_m, \tag{56}$$

in which q represents the flow of heat from the fluid towards the wall. The corresponding "total" Nusselt modulus is

$$Nu = \alpha r_0/\lambda = -\left(\frac{\partial\vartheta}{\partial\xi}\right)_{\xi=1} \Big/ \vartheta_m. \tag{57}$$

The integral $\int_0^1 \xi\varphi\vartheta\,\mathrm{d}\xi$ is already known. $\int_0^1 \xi\varphi\,\mathrm{d}\xi$ represents the reduced amount of fluid passing through the tube and follows from the equation

$$u_m \int_0^{r_0} 2\pi r \varphi \, \mathrm{d}r = \pi r_0^2 u_b,$$

which becomes upon reduction

$$\int_0^1 \xi \varphi \, \mathrm{d}\xi = \tfrac{1}{2} u_b / u_m = \tfrac{1}{4} Re/Re' (5.5 + 2.5 \ln Re'). \tag{58}$$

For ϑ_m and Nu we therefore find:

$$\vartheta_m = \sum_0^\infty {}_k \left\{ \frac{-\gamma_k \left(\dfrac{\mathrm{d}\Xi_{III k}}{\mathrm{d}\xi} \right)_{\xi=1}}{\beta_k^2 \int_0^1 \xi \varphi \, \mathrm{d}\xi} \right\} \exp\left(-\beta_k^2 \zeta'/P\acute{e}_m\right) \tag{59}$$

and

$$Nu = \frac{- \sum_0^\infty {}_k \left[\gamma_k \left(\dfrac{\mathrm{d}\Xi_{III k}}{\mathrm{d}\xi} \right)_{\xi=1} \right] \exp\left(- \beta_k^2 \zeta'/P\acute{e}_m\right)}{- \sum_0^\infty {}_k \left\{ \dfrac{\gamma_k \left(\dfrac{\mathrm{d}\Xi_{III k}}{\mathrm{d}\xi} \right)_{\xi=1}}{\beta_k^2 \int_0^1 \xi \varphi \, \mathrm{d}\xi} \right\} \exp\left(- \beta_k^2 \zeta'/P\acute{e}_m\right)}. \tag{60}$$

For high ζ' values in both series only the first term is of importance, so that

$$Nu_\infty = \beta_0^2 \int_0^1 \xi \varphi \, \mathrm{d}\xi. \tag{61}$$

When ζ' approximates zero, Nu becomes infinite. For values above a certain $\zeta' = \zeta_i'$ Nu will not differ by more than 5% from the final value Nu_∞. The tract between 0 and ζ_i' is called the thermal entrance region. In this region Nu decreases from an infinitely large value for $\zeta' = 0$ to a value of 1.05 Nu_∞ for $\zeta' = \zeta_i'$. In the calculations we made, only the first two terms in the series expansion of (60) are of any significance; the following terms may be ignored. We can then write for Nu

$$\frac{Nu}{Nu_\infty} = \frac{1 + \sigma \exp\left[(\beta_0^2 - \beta_1^2)\, \zeta'/P\acute{e}_m\right]}{1 + \sigma \dfrac{\beta_0^2}{\beta_1^2} \exp\left[(\beta_0^2 - \beta_1^2)\, \zeta'/P\acute{e}_m\right]} \tag{62}$$

with
$$\sigma = \gamma_1 \left(\frac{\mathrm{d}\Xi_{III1}}{\mathrm{d}\xi}\right)_{\xi=1} \Big/ \gamma_0 \left(\frac{\mathrm{d}\Xi_{III0}}{\mathrm{d}\xi}\right)_{\xi=1}. \tag{63}$$

To obtain an idea of the influence of the transition layer and laminar boundary layer upon the Nusselt number as compared with the influence of the core, we also calculated the fictitious "total" Nusselt number, ignoring the transition layer and the laminar boundary layer. This means that we suppose that $\vartheta = 0$ for $\xi = \xi_I$. For the limiting value of this total Nusselt number ($Nu_{t\infty}$) we find
$$Nu_{t\infty} = \beta_{t\infty}^2 \int_0^{\xi_I} \xi\varphi \, \mathrm{d}\xi. \tag{64}$$

These formulae will be used in the final representation of results in chapter IV.

Chapter III. Expansions in series

§ 7. *Approximation to the velocity distribution and eddy diffusivity of heat.* In § 8 we shall expand E_k in powers of ξ. For this purpose we must approximate the velocity and eddy diffusivities by a polynomial in ξ. We shall again consider the three regions in succession:

a) The turbulent core. In this region it is found that the velocity can be approximated by the polynomial
$$\varphi = \frac{u}{u_m} = \frac{5.5 + 2.5 \ln Re' - 7.1\, \xi^2 - 6.5\, \xi^{26}}{5.5 + 2.5 \ln Re'}. \tag{65}$$

For low Re'-numbers this representation fits very well (see fig. 3). For $f(\xi)$ we might in principle be able to use (31), derived from Reichardt's middle law. The poles of the differential equation (35) are, however, given by the roots of the equation $\xi f(\xi) = 0$. Looking apart from the poles ± 1 outside the turbulent core, the singular points following from this equation are: $\xi = 0$ and $\xi = \pm \frac{1}{2} i \sqrt{2}$. A solution expanded from $\xi = 0$ in a power series of ξ would only converge within the area $|\xi| < \frac{1}{2}\sqrt{2}$. To get an analytical continuation we should therefore have to expand from several points. To avoid this we constructed an approximation to $f(\xi)$ avoiding poles. This is (see fig. 4)
$$f(\xi) = \frac{Pr'Re'}{15} (1 - \xi^2)\,(1 + 0.9\, \xi^2)^2. \tag{66}$$

With this formula the singular points are $\xi = 0$ and $\xi = \pm i/\sqrt{0.9}$, so that for the interval $0 \leqslant \xi \leqslant 1$ the series expanded from $\xi = 0$ converges. Furthermore we see in fig. 4 that for small values of ξ (66) approximates $f(\xi)$ according to the middle law (26) and for larger values $f(\xi)$ according to the logarithmic formula (27).

b) The transition layer. The approximation (see fig. 2) found for φ is

$$\varphi = \frac{u}{u_m} = \frac{1.40 + 0.8\, Re'\eta - 0.0127\, (Re'\eta)^2}{5.5 + 2.5 \ln Re'}. \tag{67}$$

From (25) we find for $f(\xi)$

$$f(\xi) = (a + Aq)/a = 1 - Pr' + 0.2\, Pr'Re'\xi(1 - \xi). \tag{68}$$

The singular points are $\xi = 0$ and two points that are in the laminar layer for $Re' > 5Pr'$. Hence, if $Re' > 5Pr'$, (68) can be utilized provided we expand the series from the boundary of the turbulent core and the transition layer. In this case the singular points are outside the interval $1 - 30/Re' < \xi < 1 - 5/Re'$.

In practice the condition that Re' must be larger than $5Pr'$ is mostly fulfilled. For instance, for $Re = 10^4$, $Re' \approx 300$, so that Pr' only has to be smaller than 75.

c) The laminar boundary layer. In this region no difficulties arise. Here we have

$$f(\xi) = 1 \text{ and } \varphi = Re'\eta/5.5 + 2.5 \ln Re'. \tag{69}$$

§ 8. *Numerical solution by expansion of E_k in powers of ξ.* By means of the polynomials introduced in the preceding section for φ and $f(\xi)$ it is possible to expand the eigenfunctions Ξ_{I_k}, Ξ_{II_k} and Ξ_{III_k} in powers of ξ. We shall again take the three regions successively.

a) The turbulent core. When the approximations have been introduced, the differential equation for Ξ_{I_k} becomes

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi(1 - \xi^2)\,(1 + 0.9\xi^2)^2 \frac{\mathrm{d}E_{I_k}}{\mathrm{d}\xi}\right] +$$

$$\frac{15\beta_k^2}{Re'Pr'}\left[\frac{5.5 + 2.5 \ln Re' - 7.1\xi^2 - 6.5\xi^{26}}{5.5 + 2.5 \ln Re'}\right] E_{I_k} = 0. \tag{70}$$

This equation is satisfied by the two independent solutions

$$Y_{I_k} = a_0 + a_2\xi^2 + a_4\xi^4 + \ldots. \quad (71)$$

and

$$Z_{I_k} = a_1\xi + a_3\xi^3 + a_5\xi^5 + \ldots., \quad (72)$$

the coefficients a_n being polynomials of β_k^2. Owing to the conditions $\mathrm{d}\Xi_{I_k}/\mathrm{d}\xi = 0$ for $\xi = 0$, Z_{I_k} has to be rejected, so that for Ξ_{I_k} we find the solution $\Xi_{I_k} = A_{I_k} \cdot Y_{I_k}$, a_0 and A_{I_k} being taken equal to 1. The other constants A_{II_k}, A_{III_k}, B_{I_k}, B_{II_k} and B_{III_k} are then determined by the six conditions already mentioned, except for the factor γ_k.

To calculate the characteristic quantities of § 6, Ξ_{I_k} and $\mathrm{d}\Xi_{I_k}/\mathrm{d}\xi$ must be determined for $\xi = 0$ and $\xi = 1 - 30/Re'$, the boundary of the turbulent core.

If we represent the terms $a_{2n}\,\xi^{2n}$ in the expansion (71) by T_{2n}, we have $\Xi_{I_k} = \sum_0^\infty{}^n\, T_{2n}$ and

$$\frac{\mathrm{d}\Xi_{I_k}}{\mathrm{d}\xi} = \sum_0^\infty{}^n \frac{2n}{\xi} T_{2n}. \quad (73)$$

With the aid of the differential equation (70) we find for T_{2n} the recurrent relation

$$T_{2n} = -0.8\xi^2\,\frac{2n-2}{2n}\,T_{2n-2} + 0.99\,\xi^4\,\frac{2n-4}{2n}\,T_{2n-4} + 0.81\xi^6\,\frac{2n-6}{2n}\,T_{2n-6}$$

$$-\frac{1}{4n^2}\left(\frac{\beta_k^2}{Pr'}\right)(\xi^2 q_0 T_{2n-2} + \xi^4 q_1\, T_{2n-4} + \xi^{28}\, q_2\, T_{2n-28}), \quad (74)$$

in which q_0, q_1 and q_2 are functions of Re' only.

By applying the recurrent relation (74) in the series (71) we can determine Ξ_{I_k} and $\mathrm{d}\Xi_{I_k}/\mathrm{d}\xi$ for $\xi = \xi_I$, expressed as a power series in β_k^2/Pr'.

The method of calculation described was applied for $Re' = 311.5$ ($Re = 9.68 \times 10^3$) and $Re' = 2370$ ($Re = 9.81 \times 10^4$). When this was done numerically, the following points were found to be of importance:

1. Pr' only occurs in the combination β_k^2/Pr'. Thus if we expand in powers of β_k^2/Pr', there is no need to calculate the series for different Pr' values.

2. Considering the magnitude of the eigenvalues in laminar flow we may expect that for the first two eigenvalues $15\beta_k^2/Re'Pr'$

becomes smaller than, say, 10 (see (70)). We therefore determine Ξ_{Ik} and $\mathrm{d}\Xi_{Ik}/\mathrm{d}\xi$ for $\xi = \xi_I$ as functions of $t^2 = 10^{-n}\beta_k^2/Pr'$, n being given by the condition that 10^n is of the same order as Re'.

3. The convergence of the series represented by (73) is poor, becoming worse as Re' becomes greater. For the values of Re' mentioned we require about 30 terms before the term $-\beta_k^2/4n^2Pr'$ (. . . .) can be ignored in the recurrent relation. For the subsequent terms in the series the recurrent relation may be simplified to

$$T_{2n} = -0.8\,\xi_I^2\frac{2n-2}{2n}T_{2n-2} + 0.99\,\xi_I^4\frac{2n-4}{2n}T_{2n-4} + + 0.81\,\xi_I^6\frac{2n-6}{2n}T_{2n-6}. \tag{75}$$

This relation is comparatively simple, so that the rest of the series can be given in an analytical form.

Suppose, for instance, that after T_{2N+4} the term mentioned can be ignored in (74); Ξ_{Ik} can then be formulated as $\sum\limits_0^{N+2}{}^n T_{2n} + \sum\limits_{N+3}^{\infty}{}^n T_{2n}$. Inserted in the differential equation (70) we then get

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(\sum_0^{N+2}{}^nT_{2n}\Big)\right]+\beta_k^2\varphi(\xi)\sum_0^{N+2}{}^nT_{2n}+\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_{N+3}^{\infty}{}^nT_{2n}\right)\right]=0$$

or

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sum_{N+3}^{\infty}{}^nT_{2n}\right)\right]=-\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi f(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(\sum_0^{N+2}{}^nT_{2n}\Big)\right]-\beta_k^2\varphi(\xi)\sum_0^{N+2}{}^nT_{2n}. \tag{76}$$

Because T_{2n} satisfies the recurrent relation (74), the right-hand member can be written as a linear combination of T_{2N}, T_{2N+2} and T_{2N+4}, by which the remainder $\sum\limits_{N+3}^{\infty}{}^n T_{2n}$ is determined by integration of this combination as a function of ξ.

4. In order to examine the convergence more closely we shall work out Ξ_{Ik} as an example. As we have seen, Ξ_{Ik} is obtained by addition of the terms T_{2n}; we hence write

$$\begin{aligned} T_0 &= 1,\\ T_2 &= m_{22}t^2,\\ T_4 &= m_{24}t^2 + m_{44}t^4,\\ T_6 &= m_{26}t^2 + m_{46}t^4 + m_{66}t^6,\\ &\dots\dots\dots\dots\dots \end{aligned}$$

Upon addition Ξ_{Ik} becomes a series in t^n, or

$$\Xi_{Ik} = \sum_0^\infty {}^n T_{2n} = 1 + r_2 t^2 + r_4 t^4 + r_6 t^6 + \dots . \tag{77}$$

As we already stated, the convergence of the series $r_n = \sum_0^\infty {}^p m_{n,\, n+2p}$ is poor. It should be observed, however, that the sequence $r_2, r_4, r_6, r_8, \dots.$ does converge well. Here two or three terms may sometimes be sufficient for the first eigenvalue.

b) The transition layer. As we saw in the preceding paragraph, we must expand E_{IIk} from $\xi = 1 - 30/Re'$. Upon introduction of the new variable $\omega = \xi + 30/Re' - 1$, the differential equation changes into

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left[(v_0 + v_1\omega + v_2\omega^2 + v_3\omega^3)\frac{\mathrm{d}E_{IIk}}{\mathrm{d}\omega}\right] + \\ + \beta_k^2(w_0 + w_1\omega + w_2\omega^2 + w_3\omega^3)E_{IIk} = 0, \tag{78}$$

where v_n and w_n are functions of Re' and Pr'.

This differential equation is satisfied by two series of the form

$$Y_{IIk} = 1 + a_2\omega^2 + a_4\omega^4 + \dots . \tag{79}$$

and

$$Z_{IIk} = \omega + b_2\omega^2 + b_3\omega^3 + \dots . \tag{80}$$

In the same way as for Ξ_{Ik}, Y_{IIk} and Z_{IIk} and their derivatives are determined as functions of $t^2 = 10^{-n}\beta_k^2/Pr'$ for the values $\omega = 0$ and $\omega = 25/Re'$, the boundaries of the transition layer. The coefficients a_n and b_n are here functions of Pr' and Re', so that these have to be determined for every Pr' value. Convergence is very good, and in the series obtained

$$Y_{IIk} = 1 + l_2 t^2 + l_4 t^4 + \dots . \tag{81}$$

$$Z_{IIk} = m_0 + m_2 t^2 + m_4 t^4 + \dots . \tag{82}$$

two or three terms are sufficient. For higher Re' values the first term alone is sufficient.

c) The laminar boundary layer. After substitution of $f(\xi)$ and $\varphi(\xi)$ the differential equation is

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}E_{IIIk}}{\mathrm{d}\xi}\right) + \frac{\beta_k^2 Re'(1-\xi)}{5.5 + 2.5\ln Re'}E_{IIIk} = 0. \tag{83}$$

Here again, upon expansion from $\eta = 1 - \xi = 0$, two independent series

$$Y_{IIIk} = 1 + a_2\eta^2 + \dots . \tag{84}$$

$$Z_{IIIk} = \eta + b_2\eta^2 + \dots . \tag{85}$$

occur. They converge well. In the calculations we carried out we only considered the cases in which the thermal resistance of the wall is zero, so that $\vartheta_{III} = 0$ for $\eta = 0$. This means that only Z_{IIIk} has to be retained. In the series

$$Z_{IIIk} = z_0 + z_2 t^2 + \dots .$$

for higher Re'-values the first term alone is sufficient.

§ 9. *Numerical solution by expansion in powers of β_k^2.* Regarding the calculation in § 8 we observed that the convergence of Ξ_{Ik}, upon expansion in a power series of ξ, is poor, so that this expansion cannot be applied to high Re'-values. On the other hand, the convergence of the power series of Ξ_{Ik} in t^2 (77) was found to be good.

It also followed from this calculation that for higher Re'-values of the eigenfunctions in the transition layer and in the laminar boundary layer, except for Y_{IIk}, the first term of the series in β_k^2 is sufficient. With not too low Re'-values this approximation may be qualified as very good for the first eigenvalue and good for the second.

Utilizing these observations we apply a different method for calculating Ξ_{Ik} and its derivative, using for higher Re'-values only the first term of the series in the transition layer and the laminar boundary layer which we can calculate analytically. For lower values of Re', however, we must use the first two terms of the power series of Y_{IIk} in β_k^2 to get a good approximation to the second eigenvalue.

a) The turbulent core. As we saw above, we can derive for Ξ_{Ik} the series

$$\Xi_{Ik} = e_0 + e_2(\xi)\beta_k^2 + e_4(\xi)\beta_k^4 + \dots , \tag{86}$$

$e_n(\xi)$ being a function of ξ.

If we insert this series into the differential equation (70), we get the following relation between e_n and e_{n-2}:

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi f_I(\xi)\frac{\mathrm{d}e_n}{\mathrm{d}\xi}\right) + \varphi(\xi)\, e_{n-2} = 0. \tag{87}$$

This means that every subsequent e_n-function can be determined from the preceding e_{n-2} function by integration.

Since, when $\xi = 0$, Ξ_{Ik} is equal to 1, and hence independent of β_k^2, $e_0 = 1$. Every following e_n-function can therefore be determined in principle. These integrations are, however, difficult to carry out by ordinary means, so we have to calculate the integrals numerically.

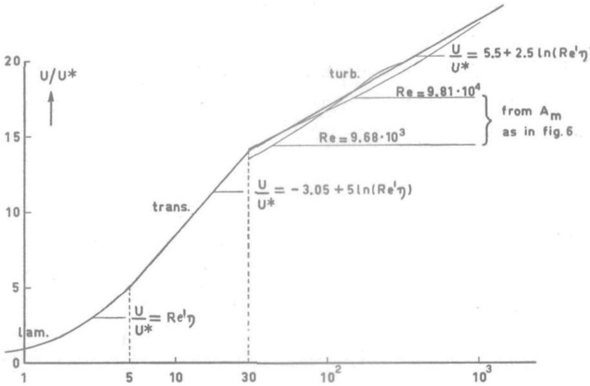

Fig. 5. Approximation to the velocity distribution in the turbulent core by (88), for $Re = 9.68 \times 10^3$ and 9.81×10^4.

One of the first advantages of numerical integration is that the velocity does not have to be approximated by a polynomial. We therefore use the velocity which follows from (66) given for A_q:

$$\varphi = \frac{-\,7.5}{u_m/u^*} \int \frac{\mathrm{d}(\xi^2)}{(1-\xi^2)(1+0.9\xi^2)^2} =$$

$$\frac{\dfrac{u_m}{u^*} + \dfrac{15}{7.22} \ln \left(\dfrac{1-\xi^2}{1+0.9\xi^2} \right) - \dfrac{13.5\xi^2}{3.8(1+0.9\xi^2)}}{u_m/u^*} \tag{88}$$

(see figs. 3, 5 and 6).

For smaller values of ξ the velocity distribution (88) is a good approximation to the middle law. From figs. 5 and 6 it follows that for greater values of ξ the velocity distribution (88) diverges somewhat from the logarithmic distribution (27). The spread of the measurements, however, is such that this seems permissible [5] [13] [19].

In applying the calculation described in § 8 we have ignored the molecular diffusion of heat. For lower Pr'-values this is no longer permissible. In these cases instead of formula (66) for $f_I(\xi)$ we must use

$$\frac{a + A_q}{a} = 1 + \frac{Pr'Re'}{15}(1 - \xi^2)\,(1 + 0.9\xi^2)^2 = 1 + f_I(\xi).$$

Since, when this formula is introduced, Pr' and Re' cannot be combined with β_k^2, we should have to carry out the integration again for every Pr' and Re'. We can, however, utilize the fact that $f_I(\xi) = A_q/a$ is almost constant, to make a correction for ignoring the molecular diffusion.

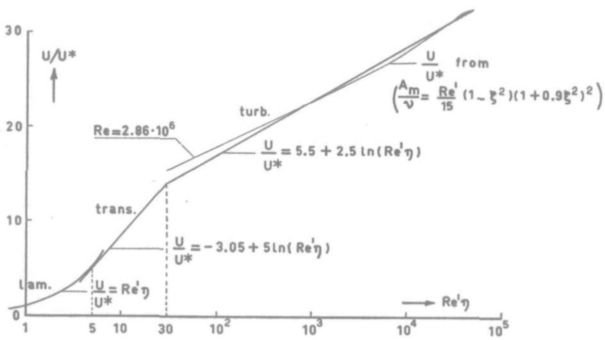

Fig. 6. Approximation to the velocity distribution in the turbulent core by (88), for $Re = 2.86 \times 10^6$.

Instead of $f_I(\xi)$ we introduce $f_I'(\xi) = [1 + 1/\bar{f}_I(\xi)]\,f_I(\xi)$, $\bar{f}_I(\xi)$ being the mean value of $f_I(\xi)$ over the turbulent region. $f_I(\xi)/\bar{f}_I(\xi)$ is virtually equal to 1 and therefore represents the term for molecular diffusion (a/a).

Upon division by

$$\frac{Pr'Re'}{15}\left(1 + \frac{1}{\bar{f}_I(\xi)}\right)$$

the differential equation for Ξ_{I_k} becomes

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\xi(1-\xi^2)(1+0.9\xi^2)^2\frac{\mathrm{d}\Xi_{I_k}}{\mathrm{d}\xi}\right] + \frac{15\,\beta_k^2}{Pr'Re'\left(1+\dfrac{1}{\bar{f}_I(\xi)}\right)}\varphi(\xi)\Xi_{I_k} = 0 \quad (89)$$

or, upon insertion of the new variable $\varkappa = \xi^2$,

$$\frac{\mathrm{d}}{\mathrm{d}\varkappa}\left[(1-\varkappa)(1+0.9\varkappa)^2\frac{\mathrm{d}\Xi_{Ik}}{\mathrm{d}\varkappa}\right]+\frac{15\beta_k^2}{4Pr'Re'}\left(\frac{1}{1+\dfrac{1}{\bar{f}_I(\xi)}}\right)\varphi(\xi)\Xi_{Ik}=0. \quad (90)$$

If we consider (88), we see that it has the form

$$\varphi = 1 - \frac{\chi(\varkappa)}{u_m/u^*}. \quad (91)$$

From (89) it further appears that Re' and Pr' only occur in the combination

$$t^2 = \frac{15\beta_k^2}{Pr'Re'}\left(\frac{1}{1+\dfrac{1}{\bar{f}_I(\xi)}}\right),$$

for which reason we expand Ξ_{Ik} in a power series of t^2, so that

$$\Xi_{Ik} = \bar{e}_0 + \bar{e}_2 t^2 + \bar{e}_4 t^4 + \ldots. \quad (92)$$

For $\bar{e}_n$ the recurrent relation

$$\frac{\mathrm{d}}{\mathrm{d}\varkappa}\left[(1-\varkappa)\,(1+0.9\varkappa)^2\frac{d\bar{e}_n}{\mathrm{d}\varkappa}\right]+\left[1-\frac{\chi(\varkappa)}{u_m/u^*}\right]\bar{e}_{n-2}=0 \quad (93)$$

is valid. The quantity $(u_m/u^*)^{-1}$ can be maintained as parameter in the numerical integrations. Expansion in this manner makes it possible to carry out the integrations only once over the interval $0 \leqslant \varkappa < 1$, after which the coefficients $\bar{e}_n$ are known as functions of ξ and $(u_m/u^*)^{-1}$. Hereafter they are independent of the values of Pr' and Re'. In carrying out the integrations the interval $0 \leqslant \varkappa < 1$ is divided into the sub-intervals

$$\varkappa = 0, 0.1, 0.2, \ldots. 0.9, 0.91, 0.92, \ldots. 0.99, 0.991, \ldots.$$

This division was chosen on account of the logarithmic behaviour of the function χ, which makes this function infinite for $\varkappa = 1$. In our calculations we went no further than $\varkappa = 0.999$. For, if the boundary of the turbulent core is at $\xi_I = \sqrt{0.999}$, Re must be about 3.5×10^6. Thus the procedure is as follows: Take $\bar{e}_0 = 1$. From (93) it follows that for the boundary of the n^{th} sub-interval it is true that

$$\left[(1-\varkappa)(1+0.9\varkappa)^2\left(\frac{d\bar{e}_2}{d\varkappa}\right)\right]_{\varkappa=\varkappa_n}-\left[(1-\varkappa)(1+0.9\varkappa)^2\left(\frac{\mathrm{d}\bar{e}_2}{\mathrm{d}\varkappa}\right)\right]_{\varkappa=0}=$$

$$-\int\limits_0^{\varkappa_n}\left(1-\frac{\chi(\varkappa)}{u_m/u^*}\right)\bar{e}_0\mathrm{d}\varkappa \tag{94}$$

$$=-\sum_0^{n-1}{}^m\left\{\left[\left(1-\frac{\chi(\varkappa)}{u_m/u^*}\right)\bar{e}_0\right]_{\varkappa=\varkappa_m}+\left[\left(1-\frac{\chi(\varkappa)}{u_m/u^*}\right)\bar{e}_0\right]_{\varkappa=\varkappa_{m+1}}\right\}\left(\frac{\varkappa_{m+1}-\varkappa_m}{2}\right). \tag{95}$$

In this manner the derivative $\mathrm{d}\bar{e}_2/\mathrm{d}\varkappa$ is determined for the boundary of each interval, after which $\bar{e}_2$ is found by numerical integration. The functions $\bar{e}_4$, $\bar{e}_6$, etc. can also be calculated in this way for the boundaries of the sub-divisions.

If necessary, we can find by interpolation between these intervals the functions $\bar{e}_n$ and $\mathrm{d}\bar{e}_n/\mathrm{d}\varkappa$ for $\xi = \xi_I$ at the given Re'-value.

b) The transition layer. For $\omega = 0$ the values of Y_{IIk}, Z_{IIk} and their derivatives can be directly calculated from the series (79) and (80):

$$Y_{IIk}=1,\ \frac{\mathrm{d}Y_{IIk}}{\mathrm{d}\xi}=0,\ Z_{IIk}=0 \text{ and } \frac{\mathrm{d}Z_{IIk}}{\mathrm{d}\xi}=1. \tag{96}$$

As, in the expansion of $\mathrm{d}Y_{IIk}/\mathrm{d}\xi$, Z_{IIk} and $\mathrm{d}Z_{IIk}/\mathrm{d}\xi$ the first term is sufficient for $\omega = 25/Re'$, it may be derived from differential equation (78) that

$$\left(\frac{\mathrm{d}Y_{IIk}}{\mathrm{d}\xi}\right)_{\xi=5/Re'}=\frac{-1}{[\xi f_{II}(\xi)]_{\omega=25/Re'}}\int\limits_{\omega=0}^{\omega=25/Re'}\beta_k^2\xi\varphi(\xi)\,\mathrm{d}\xi, \tag{97}$$

$$\frac{\mathrm{d}Z_{IIk}}{\mathrm{d}\xi}=\frac{[\xi f_{II}(\xi)]_{\omega=0}}{[\xi f_{II}(\xi)]_{\omega=25/Re'}} \tag{98}$$

and

$$Z_{IIk}=[\xi f_{II}(\xi)]_{\omega=0}\int\limits_{\omega=0}^{\omega=25/Re'}\frac{1}{\xi f_{II}(\xi)}\,\mathrm{d}\xi. \tag{99}$$

If we use the second term too, we can represent Y_{IIk} for $\omega = 25/Re'$ by

$$Y_{IIk}=1-\int\limits_{\omega=0}^{\omega=25/Re'}\left[\frac{1}{\xi f_{II}(\xi)}\int\limits_{\omega=0}^{\omega}\beta_k^2\xi\varphi(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\xi. \tag{100}$$

Because again the condition that $\varphi(\xi)$ is a polynomial of ξ is not needed, we can here use the logarithmic formula (22) for $\varphi(\xi)$ to calculate the functions Y_{IIk}, Z_{IIk}, $\mathrm{d}Y_{IIk}/\mathrm{d}\xi$ and $\mathrm{d}Z_{IIk}/\mathrm{d}\xi$.

c) The laminar boundary layer. The first term in the expansion of the series of Z_{IIIk} for $1 - \xi = 5/Re'$ is found by integration of the differential equation (83)

$$Z_{IIIk} = -\ln(1 - 5/Re') \tag{101}$$

and

$$\frac{\mathrm{d}Z_{IIIk}}{\mathrm{d}\xi} = -1/(1 - 5/Re'). \tag{102}$$

Chapter IV. results

§ 10. *Calculations.*

a) Calculation of β_k^2 for $Re = 9.81 \times 10^4$ ($Re' = 2370$) and $Pr' = 1$. It is not practicable to give all the series and calculations carried out. As an example we shall examine for one case the calculation of β_0^2 and β_1^2, the case mentioned in the heading: $Re' = 2370$ and $Pr' = 1$. The numerically calculated series Ξ_{Ik} in powers of $t^2 = 10^{-3}\,\beta_k^2/Pr'$ for $\xi_I = 1 - 30/Re'$ is:

$$\Xi_{Ik} = 1 - 2.073\,4t^2 + 0.902\,16t^4 - 0.166\,34t^6 + 0.017\,15t^8 - \\ - 0.001\,143t^{10} + 0.000\,058t^{12} - 0.000\,001\,4t^{14} + \ldots\ldots \tag{103}$$

For the derivative of Ξ_{Ik} with respect to ξ for $\xi_I = 1 - 30/Re'$ we find the series

$$\frac{\mathrm{d}\Xi_{Ik}}{\mathrm{d}\xi} = -30.970\,1t^2 + 21.196\,2t^4 - 5.077\,4t^6 + 0.631\,3t^8 - \\ - 0.049\,37t^{10} + 0.002\,573t^{12} - 0.000\,101t^{14}. \tag{104}$$

Furthermore, for the transition layer we must know the functions

$$Y_{IIk},\ Z_{IIk},\ \mathrm{d}Y_{IIk}/\mathrm{d}\xi \quad \text{and} \quad \mathrm{d}Z_{IIk}/\mathrm{d}\xi$$

for

$$\xi_I = 1 - 30/Re' \text{ and } \xi_{II} = 1 - 5/Re'.$$

For $\xi_I = 1 - 30/Re'$, or in other words for $\omega = 0$, it follows from the given formula (96) that

$$Y_{IIk} = 1,\ Z_{IIk} = 0,\ \mathrm{d}Y_{IIk}/\mathrm{d}\xi = 0 \text{ and } \mathrm{d}Z_{IIk}/\mathrm{d}\xi = 1.$$

With the aid of the formulae (97), (98), (99) and (100) we then find for $\xi_{II} = 1 - 5/Re'$

$$Y_{IIk} = 1 - 0.012\,264t^2,\ Z_{IIk} = 0.022\,240,$$
$$\mathrm{d}Y_{IIk}/\mathrm{d}\xi = -\,4.530\,7t^2,\ \mathrm{d}Z_{IIk}/\mathrm{d}\xi = 5.873\,85.$$

As we henceforward assume Nu_0 to be infinite, we need only know in the laminar region the function Z_{IIIk} and its derivative for $\xi = 1$ and $\xi = 1 - 5/Re'$.

From (84) it follows that for $\xi = 1$ we have $Z_{IIIk} = 0$ and $\mathrm{d}Z_{IIIk}/\mathrm{d}\xi = -\,1$. If we again only take the first term in the series $Z_{IIIk} = z_0 + z_2 t^2 + \ldots.$ for $\xi_{II} = 1 - 5/Re'$, we find with the aid of formulae (101) and (102)

$$Z_{IIIk} = 0.002\,111\,9 \text{ and } \mathrm{d}Z_{IIIk}/\mathrm{d}\xi = -\,1.002\,114.$$

The conditions, already mentioned in § 5, which the calculated functions now still have to satisfy, are

$$a)\ \Xi_{Ik}(\xi_I) = \frac{A_{IIk}}{\gamma_k} Y_{IIk}(\xi_I) + \frac{B_{IIk}}{\gamma_k} Z_{IIk}(\xi_I), \tag{105}$$

$$b)\ f_I'(\xi_I)\left(\frac{\mathrm{d}\Xi_{Ik}}{\mathrm{d}\xi}\right)_{\xi=\xi_I} = f_{II}(\xi_I)\left[\frac{A_{IIk}}{\gamma_k}\left(\frac{\mathrm{d}Y_{IIk}}{\mathrm{d}\xi}\right)_{\xi=\xi_I} + \frac{B_{IIk}}{\gamma_k}\left(\frac{\mathrm{d}Z_{IIk}}{\mathrm{d}\xi}\right)_{\xi=\xi_I}\right], \tag{106}$$

$$c)\ \frac{A_{IIk}}{\gamma_k} Y_{IIk}(\xi_{II}) + \frac{B_{IIk}}{\gamma_k} Z_{IIk}(\xi_{II}) = \frac{B_{IIIk}}{\gamma_k} Z_{IIIk}(\xi_{II}), \tag{107}$$

$$d)\ f_{II}(\xi_{II})\left[\frac{A_{IIk}}{\gamma_k}\left(\frac{\mathrm{d}Y_{IIk}}{\mathrm{d}\xi}\right)_{\xi=\xi_{II}} + \frac{B_{IIk}}{\gamma_k}\left(\frac{\mathrm{d}Z_{IIk}}{\mathrm{d}\xi}\right)_{\xi=\xi_{II}}\right] = \frac{B_{IIIk}}{\gamma_k}\left(\frac{\mathrm{d}Z_{IIIk}}{\mathrm{d}\xi}\right)_{\xi=\xi_{II}} \tag{108}$$

If we substitute in (105) and (106) the values found for $Y_{IIk}(\xi_I)$, $Z_{IIk}(\xi_I)$, $(\mathrm{d}Y_{IIk}/\mathrm{d}\xi)_{\xi=\xi_I}$ and $(\mathrm{d}Z_{IIk}/\mathrm{d}\xi)_{\xi=\xi_I}$, we find for A_{IIk}/γ_k and B_{IIk}/γ_k

$$\frac{A_{IIk}}{\gamma_k} = \Xi_{Ik}(\xi_I),\quad \frac{B_{IIk}}{\gamma_k} = \frac{f_I'(\xi_I)}{f_{II}(\xi_I)}\left(\frac{\mathrm{d}\Xi_{Ik}}{\mathrm{d}\xi}\right)_{\xi=\xi_I}$$

respectively.

After inserting A_{IIk}/γ_k and B_{IIk}/γ_k in (107) and (108) and after elimination of B_{IIIk}/γ_k we get the eigenvalue equation

$$1 - 4.646\,5t^2 + 2.693\,45t^4 - 0.604\,27t^6 + 0.072\,78\,t^8 - \\ -\,0.005\,582t^{10} + 0.000\,295t^{12} - 0.000\,011\,0t^{14} = 0. \tag{109}$$

TABLE I

Pr'		$Re = 9.68 \times 10^3$	$Re = 3.74 \times 10^4$	$Re = 9.81 \times 10^4$	$Re = 2.86 \times 10^6$
0.1	β_0^2	15.10	29.207	53.346	653.5
	β_1^2	85.7	198.6	405.7	6.9×10^3
1.0	β_0^2	43.70	119.8	249.28	3846
	β_1^2	485	1432.2	3238	6.28×10^4
10	β_0^2	105.4	319.48	701.9	1.2707×10^4
	β_1^2	4127	1.2544×10^4	2.900×10^4	5.798×10^5
100	β_0^2	—	451.71	1008.0	1.9358×10^4
	β_1^2	—	1.2085×10^5	2.8065×10^5	5.629×10^6

TABLE II

Pr'		$Re = 9.68 \times 10^3$		$Re = 9.81 \times 10^4$	
		according to § 8	according to § 9	according to § 8	according to § 9
1.0	β_0^2	43.26	43.70	256.39	249.28
	β_1^2	457.1	485	2698	3238
10	β_0^2	106.3	105.4	729.4	701.9
	β_1^2	4029	4127	2.348×10^4	2.900×10^4

By iteration it follows from this that β_0^2 and β_1^2 are

$$\beta_0^2 = 0.249\,28 \times 10^3 \text{ and } \beta_1^2 = 3.238 \times 10^3.$$

The first two eigenvalues now being known (table I), as well as the corresponding eigenfunctions, the relevant thermal quantities can be calculated as shown in § 6.

b) In tables III and IV for several Re' values at some Pr'-values, the first two eigenfunctions are given for various values of ξ. In addition, these eigenfunctions are also given as a function of ξ in the figures 7, 8, 9 and 10.

As was to be expected, the first eigenfunction flattens out as a function of ξ at higher Pr'-values. From this it follows that at least for the case of a homogeneous entrance temperature the subsequent eigenfunctions play a smaller and smaller part at increasing Pr' values. With the aid of the given eigenfunctions Ξ_0 and Ξ_1 and other given quantities it is now also possible to calculate the temperature distribution and heat transfer for non-homogeneous entrance temperatures, provided these are symmetrical with respect to ξ.

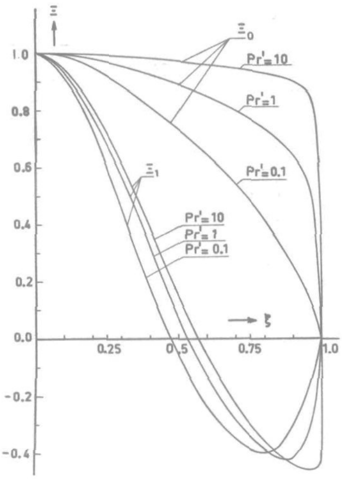

Fig. 7. Variation of first two eigenfunctions Ξ_0 and Ξ_1 with ξ for various Pr'.
$Re = 9.68 \times 10^3$,
$Re' = 311.5$.

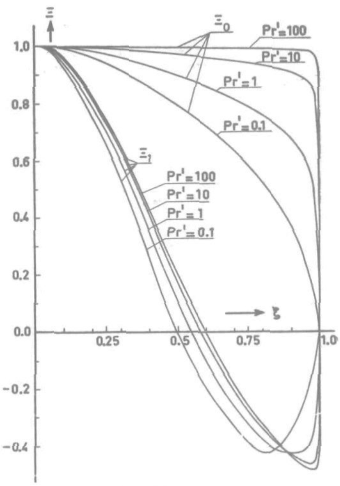

Fig. 8. The same as fig. 7.
$Re = 3.74 \times 10^4$.
$Re' = 1000$.

TABLE III

$Re = 9.68 \times 10^3$

Pr'	0.1		1.0		10	
ξ	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_1	Ξ_1
0.0000	1.0000	1.000	1.0000	1.000	1.0000	1.000
0.2236	0.9409	0.685	0.9756	0.744	0.9938	0.772
0.5000	0.7399	—0.058	0.8891	0.080	0.9715	0.154
0.7071	0.5308	—0.364	0.7918	—.0295	0.9454	—0.237
0.8062	0.4070	—0.400	0.7304	—0.398	0.9284	—0.366
0.8944	0.2617	—0.303	0.6541	—0.418	0.9067	—0.435
1.0000	0.0000	0.000	0.0000	0.000	0.0000	0.000

$Re = 3.74 \times 10^4$

Pr'	0.1		1.0		10		100	
ξ	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_0	Ξ_1
0.0000	1.0000	1.000	1.0000	1.000	1.0000	1.000	1.0000	1.000
0.2236	0.9545	0.709	0.9785	0.757	0.9942	0.782	0.9992	0.789
0.5000	0.7966	—0.007	0.9017	0.112	0.9732	+0.182	0.9961	0.204
0.7071	0.6264	—0.344	0.8144	—0.271	0.9486	—0.211	0.9925	—0.193
0.8062	0.5226	—0.425	0.7587	—0.394	0.9325	—0.354	0.9901	—0.339
0.8944	0.3974	—0.350	0.6890	—0.423	0.9118	—0.432	0.9871	—0.429
1.0000	0.0000	0.000	0.0000	0.000	0.0000	0.000	0.0000	0.000

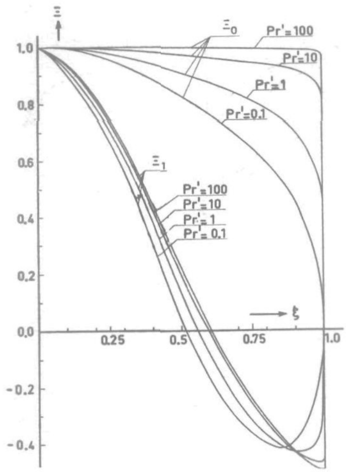

Fig. 9. The same as fig. 7.
$Re = 9.81 \times 10^4$,
$Re' = 2370$.

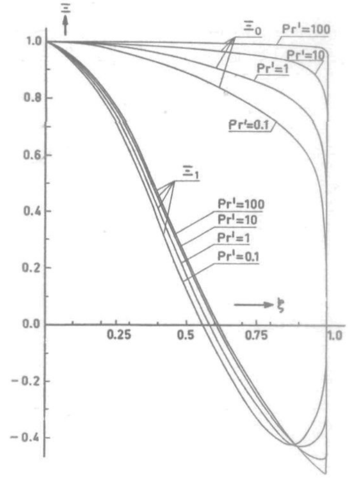

Fig. 10. The same as fig. 7.
$Re = 2.86 \times 10^6$,
$Re' = 50000$.

TABLE IV

$Re = 9.81 \times 10^4$								
Pr'	0.1		1.0		10		100	
ξ	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_0	Ξ_1
0.0000	1.0000	1.000	1.0000	1.000	1.0000	1.000	1.0000	1.000
0.2236	0.9619	0.727	0.9809	0.765	0.9946	0.787	0.9992	0.793
0.5000	0.8277	0.032	0.9125	0.134	0.9748	0.195	0.9963	0.216
0.7071	0.6807	—0.324	0.8342	—0.254	0.9515	—0.202	0.9929	—0.183
0.8062	0.5894	—0.407	0.7838	—0.377	0.9361	—0.342	0.9906	—0.329
0.8944	0.4777	—0.395	0.7203	—0.430	0.9163	—0.429	0.9877	—0.427
1.0000	0.0000	0.000	0.0000	0.000	0.0000	0.000	0.0000	0.000
$Re = 2.86 \times 10^6$								
Pr'	0.1		1.0		10		100	
ξ	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_0	Ξ_1	Ξ_0	Ξ_1
0.0000	1.0000	1.000	1.0000	1.000	1.0000	1.000	1.0000	1.000
0.2236	0.9762	0.762	0.9859	0.781	0.9953	0.797	0.9993	0.802
0.5000	0.8911	0.123	0.9349	0.178	0.9782	0.225	0.9967	0.242
0.7071	0.7937	—0.264	0.8754	—0.218	0.9580	—0.173	0.9935	—0.157
0.8062	0.7312	—0.386	0.8366	—0.357	0.9444	—0.326	0.9914	—0.313
0.8944	0.6529	—0.420	0.7870	—0.426	0.9270	—0.423	0.9887	—0.417
1.0000	0.0000	0.000	0.0000	0.000	0.0000	0.000	0.0000	0.000

TABLE V

Pr'		$Re = 9.68 \times 10^3$	$Re = 3.74 \times 10^4$	$Re = 9.81 \times 10^4$	$Re = 2.86 \times 10^6$
0.1	γ_0	1.485	1.391	1.341	1.229
	γ_1	—0.658	—0.554	—0.513	—0.363
1.0	γ_0	1.256	1.203	1.179	1.137
	γ_1	—0.366	—0.308	—0.272	—0.204
10	γ_0	1.079	1.059	1.052	1.045
	γ_1	—0.104	—0.082	—0.075	—0.066
100	γ_0		1.009	1.008	1.008
	γ_1		—0.010	—0.010	—0.010

TABLE VI

Pr'		$Re = 9.68 \times 10^3$	$Re = 3.74 \times 10^4$	$Re = 9.81 \times 10^4$	$Re = 2.86 \times 10^6$
0.1	C_0	0.852	0.896	0.923	0.958
	C_1	0.083	0.048	0.040	0.023
1.0	C_0	0.926	0.982	0.984	0.990
	C_1	0.020	0.015	0.012	0.008
10	C_0	0.998	0.999	0.999	0.999
	C_1	0.002	0.001	0.001	0.001

c) The coefficients C_k in the expansion of the series

$$\vartheta_m = C_0 \exp\left(-\beta_0^2 \zeta'/P\acute{e}_m\right) + C_1 \exp\left(-\beta_1^2 \zeta'/P\acute{e}_m\right) + \ldots . \tag{110}$$

are, according to (59), given by

$$C_k = -\gamma_k \left(\frac{\mathrm{d}\Xi_{IIIk}}{\mathrm{d}\xi}\right)_{\xi=1} \bigg/ \beta_k^2 \int_0^1 \xi \varphi(\xi)\, \mathrm{d}\xi. \tag{111}$$

By substituting expression (51) for γ_k it follows that the coefficients C_k are positive. The coefficients C_k must in addition fulfill the initial condition

$$C_0 + C_1 + \ldots . = 1 \quad \text{for } \zeta' = 0. \tag{112}$$

For a given Re'-value the sum of $C_2 + C_3 + \ldots .$ rapidly decreases as Pr' increases, so that for high Pr'-values $C_0 + C_1$ becomes practically equal to the limiting value 1. The numerical value of $\int_0^1 \xi\varphi(\xi)\,\mathrm{d}\xi$, following from the numerical calculations, is some per cent higher than the value obtained by integrating the velocity distribution according to the formulae (18), (22) and (88) (see table VIII).

TABLE VII

Pr'		$Re=9.68\times10^3$	$Re=3.74\times10^4$	$Re=9.81\times10^4$	$Re=2.86\times10^6$
0.1	$(d\Xi_0/d\xi)_{\xi=1}$	—3.517	—7.7805	—16.042	—241.7
	$(d\Xi_1/d\xi)_{\xi=1}$	4.41	7.062	13.79	203.0
1.0	$(d\Xi_0/d\xi)_{\xi=1}$	—13.07	—40.461	—90.856	—1589.3
	$(d\Xi_1/d\xi)_{\xi=1}$	10.7	29.12	62.12	1100
10	$(d\Xi_0/d\xi)_{\xi=1}$	—39.576	—124.589	—291.15	—5764.5
	$(d\Xi_1/d\xi)_{\xi=1}$	28.1	70.12	161.25	3370
100	$(d\Xi_0/d\xi)_{\xi=1}$		—184.18	—436.29	—9124.0
	$(d\Xi_1/d\xi)_{\xi=1}$		89.67	222.64	4952

TABLE VIII

Pr'		$Re=9.68\times10^3$	$Re=3.74\times10^4$	$Re=9.81\times10^4$	$Re=2.86\times10^6$
0.1	σ	0.556	0.362	0.329	0.248
1.0	σ	0.239	0.184	0.158	0.124
10	σ	0.068	0.044	0.040	0.037
100	σ		0.005	0,005	0.005
$\int_0^1 \xi\varphi \, d\xi$ according to (18), (22) and (88)		0.3914	0.4103	0.4152	0.4393
$\int_0^1 \xi\varphi \, d\xi$ numerically		0.4059	0.4135	0.4368	0.4745
$P\acute{e}_m/Pr' = Re_m$		6184	22769	59080	1.6275×10^6

Since all calculations are based upon the numerical value of $\int_0^1 \xi\varphi(\xi)\, d\xi$, we have used this value too for calculating C_k and Nu. In table VI at several Re'-values, the values of C_0 and C_1 are given for various Pr'-numbers.

d) Coefficient of heat transfer outside the thermal entrance region. The calculation described in § 8 has only been applied for $Re' = 311.5$ and $Re' = 2370$. For the latter value the convergence was already poor. The calculation described in § 9 was applied for $Re' = 311.5$, 1000, 2370 and 5×10^4, the Pr'-values being 0.1, 1.0, 10, 100. The results given in figs. 15 and 17 for other Pr'-values were found by interpolation from the calculations.

Let us first consider the limiting value Nu_∞ for the total Nusselt modulus. In fig. 11 the calculated values are plotted against Re, with Pr' as the parameter. From this graph it appears that $\log(Nu_\infty)$

for high Re-values increases almost linearly as a function of log (Re). For low Re-values the turbulent diffusion is less; in consequence, the flow becomes laminar, as it were, with a turbulent velocity

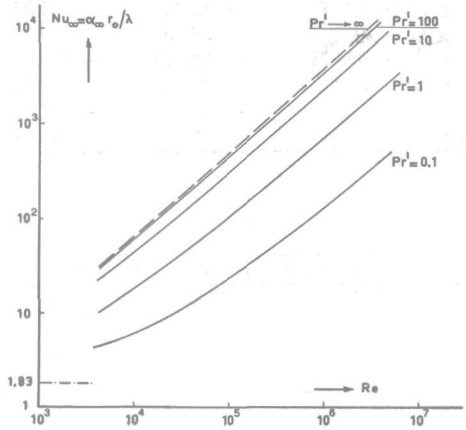

Fig. 11. Limiting Nusselt numbers against Re for various Pr', as calculated. — · — ·· — Universal value for laminar flow (1.83).

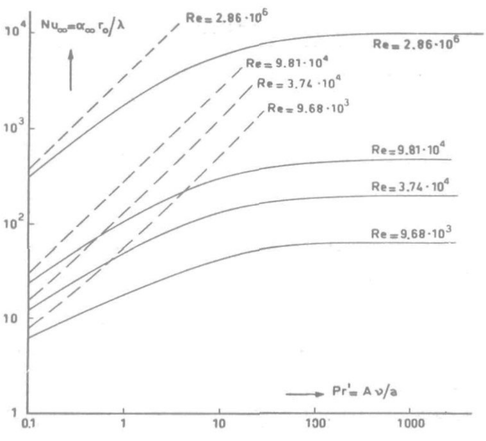

Fig. 12. Limiting Nusselt numbers against Pr' for various Re, as calculated. ----- Fictitious limiting Nusselt numbers, as calculated by neglecting the transition layer and laminar boundary layer.

distribution, so that the Nu_∞-number will approximate the value of Nu_∞ for laminar flow: $Nu_\infty = 1.83$. As the coefficient of turbulent diffusion is proportional to $Pr'Re'$, at lower Pr'-values this Nu_∞-value will already be reached at higher Re-values.

From fig. 12, giving the Nu_∞-values plotted as a function of Pr', with Re as the parameter, it appears that Nu_∞ reaches a constant value if Pr' becomes very high. This constant value can be found by the following calculation.

At higher Pr'-values the turbulent diffusion in the turbulent core and in the transition layer becomes so great that the temperature across the entire section as far as the laminar boundary layer is smoothed out. In the laminar boundary layer the heat is only exchanged by molecular diffusion. At high Pr'-values this becomes less, so that this layer comes to offer the greatest resistance to the discharge of heat.

In this layer the general solution given by formula (40) is valid for the temperature. At the boundary of the laminar layer and the core in which a homogeneous reduced temperature ϑ_t prevails we have the following conditions:

1) temperature ϑ_t is equal to that for $\xi = \xi_I$ obtained from the general solution (40) for the laminar layer; therefore

$$\vartheta_t = \sum_0^\infty{}^k \gamma_k B_{IIIk} (Z_{IIIk})_{\xi=\xi_I} \quad \exp\left(- \beta_k^2 \zeta' / P\acute{e}_m\right); \tag{113}$$

2) the heat discharged at this boundary via the laminar layer is equal to the decrease in heat content of the core. At the boundary between the core and the laminar boundary layer it is true that

$$- \pi r_0^2 \xi_I^2 \, u_b \varrho c \frac{\partial T}{\partial z} = - \lambda \left(\frac{\partial T}{\partial r} \right)_{r=r_0 \xi_I} 2\pi r_0 \, \xi_I. \tag{114}$$

After reduction and application of (113), equation (114) changes into

$$- \frac{Pr'Re}{P\acute{e}_m} \beta_k^2 \, \xi_I Z_{IIIk} = 4 \left(\frac{\mathrm{d}Z_{IIIk}}{\mathrm{d}\xi} \right)_{\xi=\xi_I}. \tag{115}$$

As Z_{IIIk} and $(\mathrm{d}Z_{IIIk}/\mathrm{d}\xi)\xi = \xi_I$ can be represented by the first terms in their series expansion to β_k^2, we find from (115) that

$$Nu_\infty = \beta_0^2 \int_0^1 \xi \varphi(\xi) \, \mathrm{d}\xi = - \frac{4\, P\acute{e}_m}{RePr'\xi_I} \left(\frac{\mathrm{d}Z_{III0}}{\mathrm{d}\xi} \right)_{\xi=\xi_I} \frac{\int_0^1 \xi\varphi(\xi)\,\mathrm{d}\xi}{(Z_{III0})_{\xi=\xi_I}} \tag{116}$$

(see table IX). In fig. 12 we see that for $Pr' > 50$ the final value is already fairly closely approximated. An approximate calculation

TABLE IX

		$Nu_\infty = \alpha_\infty r_0/\lambda$		
Pr'	$Re = 9.68 \times 10^3$	$Re = 3.74 \times 10^4$	$Re = 9.81 \times 10^4$	$Re = 2.86 \times 10^6$
0.1	6.13	12.08	23.30	310.1
1.0	17.74	49.55	108.89	1825
10	42.78	132.10	306.61	6029
100		186.78	440.3	9186
∞	63.32	201.5	475.5	10000

was also carried out for high Pr'-values by including only the first term in the expansion of the series for $(\Xi_{I0})_{\xi_I}$ and $(\mathrm{d}\Xi_{I0}/\mathrm{d}\xi)_{\xi=\xi_I}$ to β_0^2. This was very satisfactory when Pr' was higher than 5. As this calculation is only of importance in ascertaining the influence of the other terms in the expansion of the series, we shall not go any further into it.

In fig. 12, besides Nu_∞, the $Nu_{t\infty}$-values are given as a function of Pr' at several Re-values. $Nu_{t\infty}$ approaches Nu_∞ for low Pr'-values, the difference between them rapidly increasing as Pr' becomes larger. At lower Pr'-values molecular diffusion becomes large. The resistance of the boundary layer and the transition layer then rapidly decreases, so that the total Nusselt number comes to be more strongly affected by the resistance of the turbulent core; this is indeed seen in the figure. That the difference between $Nu_{t\infty}$ and Nu_∞ now rapidly becomes larger for increasing Pr'-values is self-evident.

It should be observed that $Nu_{t\infty}$ was calculated by letting Ξ_{I0} be zero for $\xi = \xi_I$. As Ξ_{I0} has been expanded in powers of

$$t^2 = \frac{4\,\beta_k^2}{15\,Re'}\left(\frac{1}{Pr' + Pr'/\bar{f}_I(\xi)}\right),$$

we find from this series the eigenvalues, and hence also $Nu_{t\infty}$, as a function of Pr', so that $Nu_{t\infty}$ is given by

$$Nu_{t\infty} = \text{Constant}\,[Pr' + Pr'/\bar{f}_I(\xi)]. \tag{117}$$

In table X the values of $Nu_{t\infty}/[Pr' + Pr'/\bar{f}_I(\xi)]$ are recorded for four values of Re.

TABLE X

	$Re = 9.68 \times 10^3$	$Re = 3.74 \times 10^4$	$Re = 9.81 \times 10^4$	$Re = 2.86 \times 10^6$
$\beta_{t0}^2 / Pr' + Pr'/\bar{f}(\xi)$	150.3	333.5	637.0	7664
$Pr'/\bar{f}(\xi)$	0.0483	0.0164	0.0071	0.00035
$Nu_{t\infty} / Pr' + Pr'/\bar{f}(\xi)$	53.18	133.10	274.8	3633.5

Before comparing the results obtained for $Re' = 311.5$ and $Re' = 2370$ by the methods of calculation described in §§ 8 and 9, we shall observe the differences that exist between these calculations:

a. For the calculation in § 8 the velocity was approximated by a polynomial. As a rule, however, the influence of the velocity distribution on heat transfer is not very great, so that the influence of this approximation will be but slight.

b. Since the molecular diffusion in the turbulent core is not ignored, the first eigenvalues according to the calculation in § 9 will become higher than those of § 8. This will be particularly evident with low Pr'- and Re-values.

c. In the calculation in § 9 only the first term was used for the series in the transition layer and the laminar boundary layer. The influence of this has already been examined and proved to be negligible.

d. The power series obtained for Ξ_{Ik} in β_k^2 converges rapidly, so that the series can be broken off after a number of terms. In the calculation according to § 8, however, after a number of terms the term β_k^2 (......) in the recurrent relation for this series was ignored in the power series to β_k^2 for Ξ_{Ik}. The influence of this makes itself felt in every term of the power series in β_k^2. For an elucidation of this point (77) is referred to. Especially with poor convergence the resultant divergence will become large, certainly for the second eigenvalue, as the higher powers of β_k^2 then become more important.

In table II we see that the first eigenvalue for $Re' = 311.5$ and $Pr' = 1$ as calculated by § 9 will be greater than the value obtained by § 8. This is in accordance with remark *b*, which apparently has a greater influence here as remark *d*.

For $Re' = 311.5$ and $Pr' = 1$ and for $Re' = 2370$ the factor $(1 + 1/\bar{f}_I(\xi))$ no longer greatly affects the eigenvalues to be compared. The first eigenvalues found according to § 8 are here higher. Both for $Re' = 311.5$ and 2370 the second eigenvalues found according to § 9 are higher than those found by the method of § 8.

Remark *d* above provides the explanation for this point. That the influence on the second eigenvalue is much greater is self-evident. For the first eigenvalue the first terms in the power series to β_k^2 are

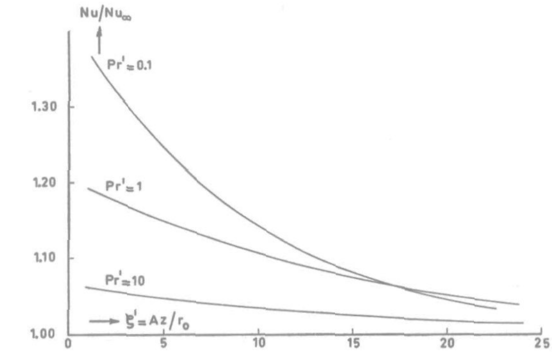

Fig. 13. Variation of local Nusselt number Nu along the tube, divided by the limiting value Nu_∞, for various Pr'. $Re = 9.68 \times 10^3$.

the most important, the higher powers also being of great importance for the higher eigenvalues. It is precisely upon these higher powers that the ignoring of β_k^2 (....) in the recurrent relation has most influence. The conclusion is therefore that the calculation of Ξ_{Ik} according to § 8 is only satisfactory for very low Re'-values.

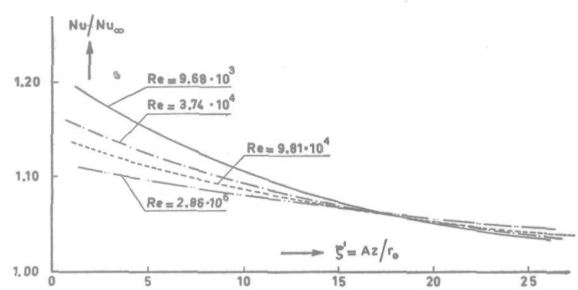

Fig. 14. Variation of local Nusselt number Nu along the tube, divided by the limiting value Nu_∞, for various Re. $Pr' = 1.0$.

e) The thermal entrance region. The thermal entrance lengths are shown in fig. 17 as a function of Re with Pr' as parameter. The shape of the curves in this graph is found to be greatly dependent on the Pr'-values.

At higher Pr'-values the thermal entrance length decreases as Re' increases. This decrease is the more rapid as the Pr'-value is

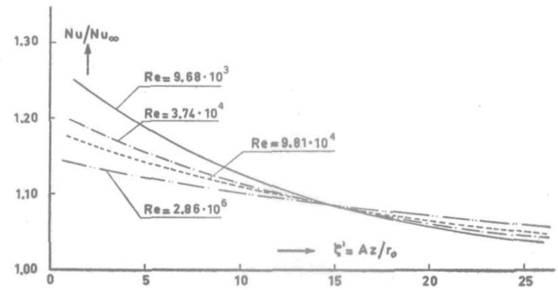

Fig. 15. Idem. $Pr' = 0.5$.

higher. At lower Pr'-values the thermal entrance length increases as Re becomes higher. For Pr'-values in the vicinity of 1 the length changes little with Re.

In figs 13, 14, 15 and 16 the relation of the local total Nusselt number and the limiting value is given as a function of ζ'.

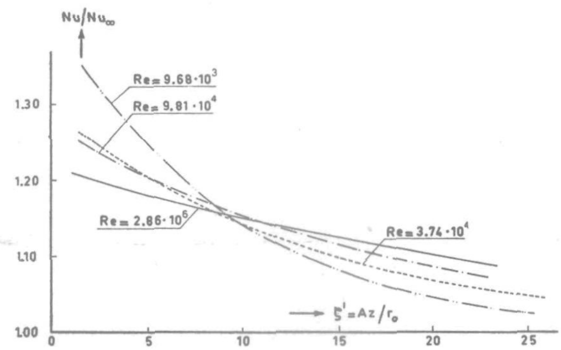

Fig. 16. Idem. $Pr' = 0.1$.

§ 11. *Comparison with other results.* Our theoretical results agree satisfactorily with experimental data which are usually rather roughly summarized by formulae of the type

$$Nu = 0.0135\, Re^{0.8} Pr^{0.3}.$$

A precise comparison has little sense as experimental boundary conditions are usually different from ours and often ill defined. The latter fact also explains the great spread of experimental points.

As regards theoretical results of other workers, a precise comparison is not possible, as they start from simpler assumptions than we. With the exception of Latzko, they oversimplify the differential equation by assuming some simple longitudinal decrease in temperature.

In calculating the heat transfer for turbulent flow Latzko [4]) also has made use of the method of eigenvalues and eigenfunctions, employing the seventh root formula for the velocity:

$$u/u_m = (1 - \xi^2)^{1/7}. \tag{118}$$

In his calculations he does not differentiate between the turbulent core and the boundary layers. By assuming that $Pr' = 1$ and $A = 1$ he finds for $(a + A_q)$

$$a + A_q = \nu + A_m = -\frac{\tau}{\varrho} \Big/ \frac{\mathrm{d}u}{\mathrm{d}r}. \tag{119}$$

This distribution of diffusion differs considerably from that used by us. From his calculations it follows that.

$$Nu_\infty = 0.0192\, Re^{3/4}. \tag{120}$$

For low values of Re this fits in with our calculations for the Pr'-value mentioned. For higher Re-values his results are somewhat lower. From the formula given by Latzko for Nu it follows that the length of the thermal entrance region is

$$z/r_0 = \zeta_i = 0.530\, Re^{1/4}. \tag{121}$$

For lower values of Re these results are substantially lower than the values calculated by us (see fig. 17).

Marris [22]) [23]), in his calculation of Nu_∞, has simplistically supposed that the longitudinal temperature gradient is constant, hence independent of ζ'. Under this assumption the temperature distribution can be found directly by integration.

Deissler [16]) [17]) and several other authors go still further and suppose that the longitudinal heat discharge can be ignored. This means that the radial heat discharge has no effect upon the temperature distribution. The temperature distribution and Nu_∞ can then be calculated from the formula for radial heat discharge:

$$\frac{q}{\varrho c} = -\,(a + A_q)\frac{\partial T}{\partial r}\,.$$

Our values unexpectedly correspond rather well with Marris's results. For higher Pr'-values the values calculated by Deissler for Nu_∞ are somewhat higher. This is due to the introduction of a coefficient of turbulent diffusion in the laminar boundary layer. Especially at high Pr'-values, where the laminar boundary layer produces the greatest resistance, this makes itself felt in the results.

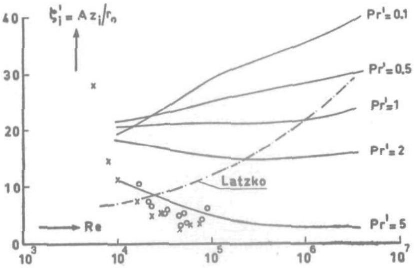

Fig. 17. Variation of thermal entrance length with Re for various Pr'. The reduced thermal entrance length ζ_i' is defined as the value of $\zeta' = Az/r_0$ for which $Nu = 1.05\ Nu_\infty$.

o Experimental points [24]) for $Pr \approx 7$,

× Experimental points [24]) $60 < Pr < 500$,

— · — For Latzko's curve see text.

An increase in the diffusion by introducing a coefficient of turbulent diffusion will of course greatly affect the calculated resistance of this laminar boundary layer. It is found that the ad hoc introduction of this coefficient of turbulent diffusion, based more or less on considerations of dimension, makes Deissler's values for high Pr'-values agree better with experimental results. For our calculation with eigenfunctions this would mean that with high Pr'-values in the differential equation (83) for the laminar boundary layer it is not permissible to suppose that $f_{III}(\xi)$ is equal to 1.

For calculating Nu in the thermal entrance region Deissler assumes that there is an expanding thermal boundary layer. The thickness of this boundary layer as a function of the longitudinal coordinate is fixed by the condition that the total amount of heat discharged must be equal to the decrease in heat content by the formation of the thermal boundary layer.

J. P. Hartnett [24]) measured the length of the thermal entrance region with several high Pr'-values. Our calculated lengths prove to be in good agreement with these measurements (see fig. 17). It should finally be observed that these experimental values and also the theoretical results of Deissler (except for $Pr' = 0.73$) refer to the case when the heat discharged by the wall is constant (and hence not to a constant wall temperature as we have assumed).

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LIST OF SYMBOLS

a	thermal diffusivity of the fluid ($\lambda/\varrho c$)
c	specific heat per unit of mass (for gases c_p)
λ	thermal conductivity of the fluid
ν	kinematic viscosity
ϱ	specific mass
A	ratio A_q/A_m
A_m	eddy diffusivity of momentum
A_q	eddy diffusivity of heat
C_w	friction factor ($\tau_0/\frac{1}{2}\varrho u_b^2$)
E_k	eigenfunction in ξ
$f(\xi)$	reduced coefficient of heat diffusion $[(a + A_q)/a]$
Nu_0	Nusselt number for tube wall ($\alpha_0 r_0/\lambda$)
Nu	total Nusselt number ($\alpha r_0/\lambda$)
$Pé_m$	Péclet number ($Re_m Pr'$)
Pr	Prandtl number (ν/a)
Pr'	adapted Prandtl number ($A\nu/a$)
$\mathrm{d}p/\mathrm{d}z$	pressure gradient in tube
q_m	radial heat flow density due to molecular conduction ($-\lambda\partial T/\partial r$)
q_t	radial heat flow density due to turbulent diffusion
r	radial distance to the axis of the tube
r_0	internal radius of tube
Re	Reynolds modulus ($2r_0 u_b/\nu$)
Re'	Reynolds modulus ($r_0 u^*/\nu$)
Re_m	Reynolds modulus ($r_0 u_m/\nu$)
T	temperature of the fluid
T_i	initial or entrance temperature
T_m	cup mixing mean temperature
T_0	ambient temperature
u^*	frictional velocity ($\sqrt{\tau_0/\varrho}$)
u_m	maximum velocity
u_b	mean velocity ($u_b = \int_0^{r_0} 2\pi r u \, \mathrm{d}r/\pi r_0^2$),
u	local velocity
y	distance from the wall
z	longitudinal coordinate
α_0	coefficient of heat transfer as defined by $q = \alpha_0(T_{wall} - T_0)$
α	coefficient of total heat transfer from fluid to environment $[q = \alpha(T_m - T_0)]$
β_k	eigenvalue corresponding to E_k
ζ'	reduced longitudinal coordinate (Az/r_0)

ζ_i' reduced thermal entrance length
η reduced distance from the wall (y/r_0)
ϑ reduced temperature $[(T - T_0)/(T_i - T_0)]$
ϑ_m reduced cup mixing mean temperature
ξ reduced variable radius (r/r_0)
τ shearing stress
τ_0 shearing stress at the wall
φ reduced velocity (u/u_m)
ω reduced distance in the transition layer to the boundary of the turbulent core

SUMMARY

The differential equation for the temperature distribution in a laminar flow through a smooth cylindrical tube can be solved on certain assumptions by separating the variables; the result is a temperature distribution given by a linear combination of eigenfunctions. For a fully developed turbulent flow the same differential equation applies in principle with the following differences: *a*) the velocity distribution is only known from measurement; consequently only empirical formulae are available for the calculations; *b*) the thermal diffusivity a is increased by an amount A_q due to turbulent mixing, and the value of this thermal eddy diffusivity depends upon the velocity distribution. The cross-section of the tube is divided into three concentric parts, according to the magnitude of the frictional eddy diffusivity A_m: the turbulent core, where the molecular momentum diffusion is negligible in relation to the eddy diffusivity of momentum; the laminar boundary layer, where the molecular momentum diffusion prevails; and the transition layer, where eddy diffusivity and molecular diffusion of momentum are of the same order. Separately for each of these regions formulae have been drawn up for the velocity and momentum eddy diffusivity A_m. Given the hypothesis that $A_q/A_m = A$ is a constant, A_q follows from A_m. In this thesis for each of these three regions a general solution is given for the differential equation valid in that area, by separating the variables. The constants in these solutions have been chosen in such a way as to make the temperatures and heat currents continuous at the boundaries between the three regions. For a constant wall temperature the first two eigenfunctions and eigenvalues are calculated for several Re and Pr values. Furthermore, the heat transfer and temperature distribution for a homogeneous entrance temperature are given. The calculated eigenfunctions can also be used for calculating the heat transfer and heat distribution for a non-homogeneous entrance temperature, provided the latter is symmetrical with respect to the axis of the tube.

SAMENVATTING EN ENKELE NABESCHOUWINGEN

Samenvatting. De voor de temperatuurverdeling in een laminaire stroming door een gladde cylindrische buis geldende differentiaalvergelijking is, bij toepassing van enkele toelaatbare benaderingen, op te lossen door splitsing der variabelen. De temperatuurverdeling wordt dan gegeven door een lineaire combinatie van eigenfuncties. De in deze combinatie voorkomende coëfficiënten worden zo gekozen, dat aan de gegeven begin- en randvoorwaarden wordt voldaan.

Voor een volledig ontwikkelde turbulente stroming geldt in wezen dezelfde differentiaalvergelijking. De voornaamste verschillen zijn echter:

a) de snelheidsverdeling is alleen bekend uit metingen, waardoor we voor de berekeningen slechts empirische formules ter beschikking hebben.

b) de temperatuurvereffeningscoëfficiënt a wordt vermeerderd met een bedrag A_q, veroorzaakt door de turbulente menging. Deze „turbulente diffusiecoëfficiënt'' voor de warmte is een functie van de snelheidsverdeling.

In verband met de grootte van de turbulente diffusiecoëfficiënt voor de impuls (A_m) kan de doorsnede van de buis verdeeld worden in drie gebieden:

1) de turbulente kern, waar de moleculaire impulsdiffusie te verwaarlozen is t.o.v. de turbulente menging.

Dit betekent niet, dat in dit gebied ook de moleculaire warmtediffusie te verwaarlozen is. De verhouding van A_q en a is namelijk evenredig met het product van Pr' en Re'. Voor lage Pr'-waarden is de verwaarlozing van a t.o.v. A_q voor lage Re'-waarden niet meer gemotiveerd. In dit proefschrift wordt hiermee rekening gehouden door een correctiefactor in the voeren.

2) de laminaire grenslaag, waar de moleculaire diffusie overheerst.

3) de bufferlaag, gelegen tussen de laminaire grenslaag en de turbulente kern. In deze laag zijn turbulente menging en moleculaire

diffusie van de impuls van dezelfde orde, zodat ze beide in rekening moeten worden gebracht.

Voor ieder van deze gebieden zijn afzonderlijke formules opgesteld voor de gemeten snelheidsverdeling en de daaruit volgende turbulente diffusiecoëfficiënt voor de impuls.

Door verder uit te gaan van de hypothese, dat de verhouding $A = A_q/A_m$ constant is, kan men de grootte van A_q uit A_m berekenen.

In dit proefschrift wordt voor ieder van deze drie gebieden door splitsing der variabelen een algemene oplossing gegeven voor de in dat gebied geldende differentiaalvergelijking. De constanten in deze oplossingen worden zo gekozen, dat de temperaturen en warmtestromen aan de randen der gebieden continu zijn.

In het bijzonder wordt voor het geval van een medium, dat met een homogene intreetemperatuur door een buis stroomt en daarbij warmte afstaat aan een omgeving die een constante temperatuur heeft, de warmteoverdracht nagegaan bij verschillende waarden van het getal van Reynolds, *Re*, en het met A vermenigvuldigde getal van Prandtl, Pr'. Opgemerkt zij, dat hierbij de thermische weerstand van de wand gelijk aan nul wordt genomen.

Nabeschouwingen. De berekende grafieken van de eigenfuncties en de totale *Nu*-waarde als functie van *Re* en Pr' behoeven geen nadere uitleg. Alleen figuur 17, waar de inlooplengte gegeven is als functie van *Re* bij verschillende Pr'-waarden zullen we hier nog nader beschouwen.

Voor een laminair door een buis stromend medium geldt, bij een homogene intreetemperatuur, voor de inlooplengte z_i de formule:

$$z_i/r_0 = 0{,}14\ Pr\ Re.$$

De inlooplengte is voor een laminaire stroming dus evenredig met *Pr*. Dit kan men op eenvoudige wijze verklaren. Het in een laminaire stroming op te bouwen temperatuurprofiel (d.w.z. de bij de eerste *e*-macht van de ontwikkeling van de temperatuur naar z/r_0 en r/r_0 behorende functie, die het verloop als functie van de straal geeft) is voor verschillende *Pr*-waarden gelijk. Voor grotere waarden van a, dus voor kleinere *Pr*-waarden, zal de transversale warmteafgifte groter worden, m.a.w. de tweede en volgende *e*-machten zullen sneller te verwaarlozen zijn t.o.v. de eerste *e*-macht. We zouden hierbij eigenlijk beter kunnen spreken van het afbreken van het intreeprofiel tot een bepaald eindprofiel.

Dat de inlooplengte voor een laminaire stroming evenredig is met het getal van Reynolds (d.w.z. met de stroomsnelheid) is zonder meer duidelijk.

Toegepast voor grotere *Re*-waarden, waarvoor de stroming turbulent wordt, levert de genoemde formule een beduidend grotere inlooplengte dan die, welke in werkelijkheid voor een turbulente stroming geldt. Wanneer dus een laminaire stroming door vergroting van *Re* turbulent wordt, zal de inlooplengte in het overgangsgebied snel kleiner worden.

Uit figuur 17 blijkt, dat voor een turbulente stroming de inlooplengte bij grote *Re*-waarden voor grotere *Pr'*-waarden kleiner wordt, dus omgekeerd als bij een laminaire stroming. In het overgangsgebied moeten de voor verschillende waarden van *Pr'* gegeven krommen elkaar dus snijden, hetgeen in figuur 17 inderdaad tot uiting komt.

We zullen nu nagaan waarom de inlooplengte bij een turbulente stroming op de in figuur 17 gegeven wijze afhangt van *Pr'* en *Re*. Beschouwen we daartoe bij een turbulente stroming de eerste eigenfunctie voor de verschillende *Pr'*- en *Re*-waarden (figuren 7, 8, 9 en 10), dan blijkt, dat de doorsnede van de buis ruwweg te verdelen is in een kern, waar de temperatuurgradiënt klein is en een randgebied, waar de temperatuur snel daalt tot de omgevingstemperatuur. De breedte van dit randgebied is bij een bepaalde *Re*-waarde groter voor grotere *a*-waarde, dus voor kleinere *Pr'*-waarde. De oorzaak hiervan moet gezocht worden in de turbulente menging. Voor een grotere waarde van *a* zal bij dezelfde *Re*-waarde de transversale warmteafvoer via de bufferlaag en de laminaire grenslaag groter worden voor een zelfde temperatuurgradiënt. Daarentegen blijft de turbulente diffusie gelijk, waardoor de nivellering van de temperatuur zich niet zo snel naar de wand kan uitbreiden, zodat het genoemde randgebied zich verder naar het midden van de buis kan uitstrekken. Hierdoor zal de temperatuurgradiënt in het randgebied kleiner worden, waardoor de vergroting van de transversale afvoer van de warmte naar de wand verkleind wordt.

Opgemerkt zij nog, dat het „eindprofiel" gelijk is aan het produkt van de eerste eigenfunctie en de coëfficiënt γ_0.

Voor grotere *a*-waarde spelen zich dus de volgende processen af:

1) het genoemde randgebied wordt groter, zodat de hoeveelheid af te voeren warmte voor het verkrijgen van het eindprofiel groter wordt.

2) de warmteafgifte aan de wand is evenredig met het product van a en de temperatuurgradiënt. Voor grotere a-waarde wordt de warmteafgifte aan de wand enerzijds vergroot door vergroting van a, maar anderzijds verkleind door verkleining van de temperatuurgradiënt. Uit tabel VII blijkt, dat het uiteindelijke resultaat is, dat de warmteafgifte groter wordt voor grotere waarde van a. De vergroting wordt echter vooral voor niet te hoge Pr'-waarden aanzienlijk verkleind doordat de temperatuurgradiënt kleiner wordt.

Het gevolg van het in de punten 1) en 2) genoemde is dus, dat voor grote Re-waarden de hoeveelheid af te voeren warmte voor grotere a-waarde meer toeneemt dan de warmteafvoer aan de wand. De inlooplengte wordt daardoor voor grotere a-waarde, dus voor kleinere Pr'-waarde, groter. Voor kleinere Re-waarden zal de af te voeren warmte door het kleiner worden van de turbulente menging – die immers evenredig is met Re' – minder sterk gaan afhangen van de Pr'-waarde. Daardoor zal de warmteafvoer voor grotere a-waarde in verhouding meer toenemen dan de hoeveelheid af te voeren warmte, waardoor bij voldoend kleine Re-waarde de inlooplengte voor grotere a, dus voor kleinere Pr'-waarde, kleiner wordt.

Met het bovenstaande is het verloop van de krommen in figuur 17 verklaard.

De in dit proefschrift uitgevoerde berekeningen betreffen alleen het geval, dat de thermische weerstand van de buiswand nul is ($Nu_0 = \infty$). Om de invloed van een eindige Nu_0-waarde op de inlooplengte en de temperatuurverdeling na te gaan, zullen in de toekomst nog nadere berekeningen moeten worden uitgevoerd. Hierbij kan gebruik gemaakt worden van de reeds bepaalde reeksen. De invloed van Nu_0 op de eerste eigenwaarde en dus op Nu_∞ is echter snel na te gaan. Door ook de eerste oplossing (84) te gebruiken voor het samenstellen van de eigenfunctie in de laminaire grenslaag, kan voldaan worden aan de wet van Newton voor de warmteoverdracht aan de wand van de buis. Door in de verkregen eigenwaardevergelijking niet eerst een bepaalde waarde voor Nu_0 in te vullen, maar de eigenwaarde te kiezen, waarna we de bijbehorende Nu_0 vinden, vermijden we het hele iteratieproces.

Voor de vier gegeven Re-waarden is bij de verschillende Pr'-waarden de invloed van Nu_0 nagegaan.

Om een juister beeld te krijgen van deze invloed, voeren we de grootheid Nu_i in, gedefiniëerd als

$$Nu_i = -\left(\frac{\partial \vartheta}{\partial \xi}\right)_{\xi=1} \Big/ (\vartheta_m - \vartheta_{\xi=1}) \, .$$

$1/Nu_i$ stelt de gereduceerde weerstand van het stromende medium voor. Inderdaad geldt voor de totale Nu de formule:

$$1/Nu_\infty = 1/Nu_0 + 1/Nu_i,$$

m.a.w. de totale gereduceerde weerstand is gelijk aan de som van de gereduceerde wandweerstand en bovengenoemde door het stromende medium veroorzaakte weerstand.

Voor een oneindig grote waarde van Nu_0 is Nu_i gelijk aan de in dit proefschrift berekende waarde van Nu_∞. Op grond van het overeenkomstig gedrag bij een laminaire stroming kan men zich afvragen of bij een eindige waarde van Nu_0 de weerstand $1/Nu_i$ niet gelijk blijft, zodat de bij iedere waarde van Nu_0 behorende waarde van Nu_∞ gevonden kan worden met de formule

$$1/Nu_\infty = 1/Nu_0 + (1/Nu_\infty)_{Nu_0=\infty}.$$

Voor een laminaire stroming begint de invloed van Nu_0 op Nu_i pas merkbaar te worden bij lage Nu_0-waarden, in die zin, dat Nu_i groter wordt. Voor een turbulente stroming is de invloed eveneens pas merkbaar bij kleine Nu_0-waarden en die invloed beperkt zich dan tot een vergroting van enkele procenten. In de praktijk kan dus zeker met de laatstgenoemde formule volstaan worden, temeer omdat voor lagere Nu_0 de wandweerstand steeds groter wordt t.o.v. $1/Nu_i$. De invloed van de afwijking van Nu_i op de totale weerstand wordt daardoor steeds kleiner.

De in dit proefschrift berekende grootheden kunnen ook gebruikt worden voor niet-homogene intreetemperaturen, mits deze symmetrisch zijn t.o.v. de as van de buis. Voor niet-symmetrische temperaturen zal ook de tweede oplossing (72) van de voor het turbulente gebied geldende differentiaalvergelijking gebruikt moeten worden.

Voor het nagaan van de warmteoverdracht voor een volledig ontwikkelde stroming tussen twee platen kunnen op dezelfde wijze als in dit proefschrift berekeningen worden uitgevoerd.

Hendrikus Ludovicus Beckers, geboren 16 frebuari 1931, behaalde in 1949 het einddiploma H.B.S.-B. aan de R.K. H.B.S. te Maastricht. Na een jaar werkzaam te zijn geweest op het Centraal Laboratorium der Staatsmijnen, liet hij zich in september 1950 inschrijven aan de T.H. te Delft. In februari 1955 verkreeg hij het diploma van natuurkundig ingenieur, waarna hij zich verbond aan het Koninklijke/Shell-Laboratorium te Amsterdam. Sinds Mei 1955 vervult hij zijn militaire dienstplicht.

ERRATA

1. Oplossing (72), lees:

$$Z_{I_k} = (\ln \xi + b_o + b_2 \xi^2 + \ldots) Y_{I_k} .$$

2. Formule (79), $a_4 \omega^4$, lees: $a_3 \omega^3$.

3. De uitdrukking voor t^2 op blz. 25, 15, lees: 15/4 .

4. Vergelijkingen (90), (93) en (94), $(1 - \varkappa)(1 + 0.9\varkappa)^2$, lees: $\varkappa(1 - \varkappa)(1 + 0.9\varkappa)^2$.

5. Blz. 51, de zin: "Voor niet-symmetrische temperaturen ..... ", lees: Voor niet-symmetrische temperaturen geldt vergelijking (7) niet meer.