Impurity necklaces in the two-dimensional electron gas

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(Received 10 January 1995)

Impurity necklaces are scattering sites randomly distributed on a line normal to the electron transport direction in wide wires of the two-dimensional electron gas. The level density and correlation function of the transmission matrix eigenvalues of these systems are expressed exactly as a function of a short-range impurity scattering potential. The nonuniversal dependence of the averages and the fluctuations of the transport properties on microscopic parameters can be explained by the absence of long-range level repulsion.

I. INTRODUCTION

The study of transport phenomena in disordered mesoscopic systems[1] is often focused on "universal" properties, which do not depend on details of the sample geometry or the nature of the disorder. Most prominent are the universal conductance fluctuations (UCF) relative to the ensemble averaged value.[2–4] Recently random matrix theory[5] (RMT) of transport has been invoked to explain the universality.[6–8] This approach is based on Landauer's formula, which for zero magnetic field and nonmagnetic systems reads

$$G = \frac{2e^2}{h}\mathrm{Tr}\ \mathbf{T} = \frac{2e^2}{h}\sum_n T_n, \tag{1}$$

where T_n are the eigenvalues of the transmission matrix $\mathbf{T}$. The distribution function of the transmission matrix eigenvalues ("levels") of sufficiently random systems are interpreted in terms of classical fictitious charges in one dimension, which repel each other with the two-dimensional Coulomb interaction.[9] The universality of the conductance fluctuations is explained by the spectral rigidity caused by the level repulsion, which reflects the negligible weight of highly symmetrical matrices in the maximum entropy ensemble average. However, RMT is not a theory of everything.[10] The single-particle potential, which confines the charges, must be obtained by other means. RMT is based on the isotropy assumption that incoming electrons in a specified mode are homogeneously distributed over all outgoing modes, which appears reasonable only in the limit where the length of the sample is much larger than its width and the electron mean free path. Even in this long-wire limit, Beenakker and Rejaei[11] identified an attractive level interaction on top of the logarithmic Coulomb repulsion, which causes a small but disconcerting deviation between the UCF's obtained by RMT (Ref. 12) and microscopic calculations.[2–4] Universal features in the eigenvalue density are completely lost in the presence of tunnel barriers.[13,14] Conductance fluctuations in metallic point contacts have been measured to be much smaller than the universal predictions.[15] Since RMT is increasingly applied to other problems,[16,17] it is important to learn more about the regime of applicability and the accuracy of RMT by deliberate research of nonuniversal transport phenomena in disordered systems.

The present paper is devoted to the theory of transport through a disordered region that is much shorter (or thinner) than the average distance between scatterers and much wider than the Fermi wavelength, i.e., without appreciable effects of size quantization, in an otherwise ballistic constriction or wire. Such a system can be interpreted as the opposite to the long-wire, quasi-one-dimensional or diffusive limit, which is therefore said to be in the *short channel* limit. Physically this model represents, e.g., a rough metallic heterointerface as relevant for the perpendicular giant magnetoresistance in metallic magnetic multilayers,[18,19] or a line of scatterers intentionally introduced into a wire of the two-dimensional electron gas. The latter system, i.e., a disordered line in the two-dimensional electron gas (see Fig. 1), will be called an *impurity necklace*. We have shown recently that it is possible to separate the transmitted amplitude into a specular and diffuse contribution.[20] The specular part is transmitted without change of momentum, and also the diffuse contribution is not scattered isotropically into all outgoing states. RMT has been used to describe trans-

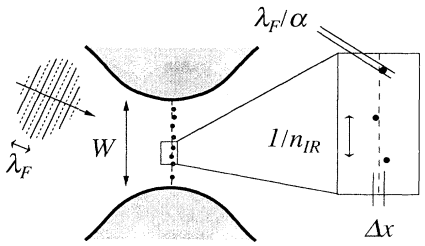


FIG. 1. Schematic view of an impurity necklace in the two-dimensional electron gas. λ_F is the Fermi wavelength, W the channel width, λ_F/α the range of the scattering potentials, Δx the spreading of the impurity positions in the transport direction, and n_{IR} the line density of scatterers.

port in ballistic cavities under the condition that the drain contact does not "see" the source directly.[16] We notice that in the short channel limit an analogous condition is violated. Therefore, for necklaces and interfaces the isotropy assumption must not be made. Furthermore, the number of "pearls" on the necklace is necessarily smaller than the number of channels, which means that the transmission matrix elements are not distributed independently, thus violating the maximum entropy assumption. RMT is therefore not applicable.

In the following, a microscopic theory is presented for the level density

$$P(T) = \left\langle \sum_n \delta(T - T_n) \right\rangle \tag{2}$$

and the correlation function

$$P(T,T') = \left\langle \sum_n \delta(T - T_n) \sum_m \delta(T' - T_m) \right\rangle, \tag{3}$$

where the ensemble average over the randomness is denoted by $\langle \cdots \rangle$ Beenakker[12,21] emphasized that these functions allow direct calculation of various physical properties and its variances, such as shot-noise power[22] and the transport properties of Normal/Superconducting (N/S) and $S/N/S$ Josephson junctions.[23] In general, for a property a described by a linear statistic $a(T)$

$$\langle a \rangle = \left\langle \sum_n a(T_n) \right\rangle = \int dT\ a(T)\ P(T) \tag{4}$$

and $\mathrm{Var}(a) = \langle a^2 \rangle - \langle a \rangle^2$ with

$$\langle a^2 \rangle = \int dT dT'\ a(T)\ a(T')\ P(T,T'). \tag{5}$$

Nazarov[13] derived a Green's function expression for $P(T)$. Its semiclassical approximation, which is valid in the diffusive and weak scattering limit, agreed with the bimodal distribution obtained before by a scaling equation.[24] $P(T,T')$ has been computed by RMT,[8] and by a Fokker-Planck equation parametrized by the mean free path.[11] The author is not aware of fully microscopic calculations of these distribution functions for realistic systems, which is not so surprising, since the diagrammatic theory of the conductance fluctuations is already quite involved.[2–4] Only due to the simplifications that are possible in the present limit does the algebra become manageable.[25] With minor modifications the present results are also applicable to other transport problems, e.g., the transparency of a thin diffusing medium to light.[26] The nonuniversal properties found in the present study include a strong dependence on dimensionality. Here we focus on a two-dimensional system. A brief account on some results in three dimensions and the relevance for the giant magnetoresistance effect in metallic multilayers is given in Ref. 27.

The paper is organized as follows. In Sec. II, the transmission matrix is derived from the Schrödinger equation.[20] These results are used in Sec. III to obtain general expressions for the level density and correlation functions. In the weak scattering limit, these functions can be written explicitly in terms of the microscopic scattering parameters, which can be evaluated easily, as shown in Sec. IV. The paper concludes with a discussion in Sec. V.

II. TRANSPORT THROUGH A DISORDERED INTERFACE

In the following, the statistical distributions of the transmission matrix eigenvalues are calculated directly from the Schrödinger equation, using many results from Ref. 20. However, in contrast to Ref. 20, focus is here on the two-dimensional geometry, noting that the three-dimensional situation can be treated as well.[27] On the other hand, technical complications due to the presence of a potential step or barrier at the interface and of evanescent states are disregarded. The model consists of (see Fig. 1) short-range impurities on a line such that the spreading Δx is smaller than the average distance $1/n_{IR}$ where n_{IR} is the line density of scatterers.

In the effective mass approximation, the single-electron states at the Fermi energy E_F are solutions of the Schrödinger equation,

$$\left[-\frac{\hbar^2 \nabla^2}{2m^*} + V(\vec{r}) \right] \psi(\vec{r}) = E_F \psi(\vec{r}), \tag{6}$$

where V is the scattering potential. The wave function can be expanded in transverse plane waves

$$\psi(x,y) = \sum_{k_\parallel} c_{k_\parallel}(x) \frac{e^{ik_\parallel y}}{\sqrt{W}}, \tag{7}$$

where W is the width of the wire. Impurity scattering gives rise to mixing between different transverse modes. Using the orthogonality of the transverse wave functions, a one-dimensional equation is obtained:

$$\frac{d^2}{dx^2} c_{k_\parallel}(x) + k_\perp^2(x) c_{k_\parallel}(x) = \sum_{k'_\parallel} V_{k_\parallel, k'_\parallel}(x) c_{k'_\parallel}(x), \tag{8}$$

$$V_{k_\parallel, k'_\parallel}(x) \equiv \frac{2m^*}{W\hbar^2} \int dy\ V(\vec{r}) e^{-i(k_\parallel - k'_\parallel)y}. \tag{9}$$

The scattering potentials are modeled by δ-impurity scatters in the plane $x = 0$:

$$V(\vec{r}) = \sum_\alpha \gamma_\alpha \delta(x) \delta(y - y_\alpha), \tag{10}$$

where y_α is the transverse position of a scattering center and $\gamma_\alpha = \pm\gamma$ the scattering potential. Inserting Eq. (10) into Eq. (8)

$$\frac{d^2}{dx^2} c_{k_\parallel}(x) + k_\perp^2 c_{k_\parallel}(x) = \sum_{k'_\parallel} \tilde{\Gamma}_{k_\parallel, k'_\parallel} \delta(x) c_{k'_\parallel}(x). \tag{11}$$

Here the transverse part of the δ function has been integrated out, and the matrix

$$\tilde{\Gamma}_{k_\parallel,k'_\parallel} \equiv \frac{2m^*}{W\hbar^2}\sum_\alpha \gamma_\alpha e^{-i(k_\parallel - k'_\parallel)y_\alpha} \tag{12}$$

has been introduced. Integrating over x

$$\left.\frac{dc_{k_\parallel}(x)}{dx}\right|_{x=0^+} - \left.\frac{dc_{k_\parallel}(x)}{dx}\right|_{x=0^-} = \sum_{k'_\parallel}\tilde{\Gamma}_{k_\parallel,k'_\parallel}c_{k'_\parallel}(0). \tag{13}$$

Equations (11) and (13) are obeyed by

$$c_{k_\parallel}(x) = \sqrt{\frac{1}{\hbar|k_\perp|}} \times \begin{cases} \delta_{k_\parallel,k'_\parallel}e^{ik_\perp x} + r_{k_\parallel,k'_\parallel}e^{-ik_\perp x}, & x<0 \\ t_{k_\parallel,k'_\parallel}e^{ik_\perp x}, & x>0. \end{cases} \tag{14}$$

We can interpret Eq. (14) in terms of particles incident with parallel momentum $k'_\parallel$. The incoming wave gives rise to an infinite set of modes, propagating for real $k_\perp$, and evanescent (exponentially localized) for imaginary $k_\perp$, the perpendicular wave vector being defined in terms of the Fermi wave vector k_F as $k_\perp^2 = k_F^2 - k_\parallel^2$. The propagating states are normalized to carry unit flux. The transmission (reflection) probabilities are obtained as $|t_{k_\parallel,k'_\parallel}|^2$ ($|r_{k_\parallel,k'_\parallel}|^2$). The conductance is given by the summation over all propagating modes in both indices. The continuity of the wave function at the interfaces implies that

$$r_{k_\parallel,k'_\parallel} = t_{k_\parallel,k'_\parallel} - \delta_{k_\parallel,k'_\parallel}, \tag{15}$$

whereas from current conservation for propagation modes

$$\sum_{k''_\parallel}^{\text{prop.}}[t_{k_\parallel,k''_\parallel}t^\dagger_{k''_\parallel,k'_\parallel} + r_{k_\parallel,k''_\parallel}r^\dagger_{k''_\parallel,k'_\parallel}] = \delta_{k_\parallel,k'_\parallel}. \tag{16}$$

Combining Eqs. (13), (14), and (15) and disregarding evanescent waves,[28] we find that the transmission coefficients are solutions of the equation

$$\sum_{k''_\parallel}^{\text{prop.}}\left[\delta_{k_\parallel,k''_\parallel} + i\tilde{\Gamma}_{k_\parallel,k''_\parallel}\frac{1}{2\sqrt{k_\perp k''_\perp}}\right]t_{k''_\parallel,k'_\parallel} = \delta_{k_\parallel,k'_\parallel}. \tag{17}$$

This can be written in matrix notation as $(\mathbf{I}+i\mathbf{\Gamma})\cdot\mathbf{t} = \mathbf{I}$, where

$$\Gamma_{k_\parallel,k'_\parallel} \equiv \frac{m^*}{W\hbar^2}\sum_\alpha \gamma_\alpha e^{-i(k_\parallel - k'_\parallel)y_\alpha}\frac{1}{\sqrt{k_\perp k'_\perp}}. \tag{18}$$

Note that since we omitted evanescent states, $\mathbf{\Gamma}$ is Hermitian. Expanding $\mathbf{t}$ in a power series in $\mathbf{\Gamma}$, we obtain

$$\mathbf{t} = \sum_{N=0}^\infty (-i\mathbf{\Gamma})^N. \tag{19}$$

The following Ward identity can be derived directly as[29]

$$\mathbf{T} = \mathbf{t}\mathbf{t}^\dagger = \sum_{N=0}^\infty\sum_{M=0}^\infty (-i)^{N-M}\mathbf{\Gamma}^{N+M} = \frac{1}{2}(\mathbf{t}+\mathbf{t}^\dagger). \tag{20}$$

This relation is more than an optical theorem,[20,30] since it holds also for the nondiagonal elements, which is important for the manipulations below. The configurational average can be calculated most conveniently by Green's function methods.[21] The following, general expression is given for future reference:

$$\begin{aligned}\langle T_{k_\parallel,k'_\parallel}\rangle &= \delta_{k_\parallel,k'_\parallel}\mathrm{Re}\langle t_{k_\parallel,k_\parallel}\rangle \\ &= \delta_{k_\parallel,k'_\parallel}\mathrm{Re}\,\frac{1}{1+i\frac{m^*}{\hbar^2 k_\perp}\Sigma},\end{aligned} \tag{21}$$

where Σ is the irreducible self-energy at the Fermi energy and Re denotes the real part.

III. DISTRIBUTION FUNCTIONS OF TRANSMISSION MATRIX EIGENVALUES

We are interested in the probability distribution of the transmission matrices. The δ function in Eq. (2) can be expressed as a Fourier integral

$$P(T) = \int\frac{dq}{2\pi}e^{-iqT}\langle\mathrm{Tr}\,e^{iq\mathbf{T}}\rangle. \tag{22}$$

In order to evaluate the expectation value of the exponent, we need expressions for $\langle\mathbf{T}^n\rangle$. To this end it is noted that the electron-electron propagator

$$\begin{aligned}\mathbf{t}\mathbf{t} &= \sum_{N=0}^\infty\sum_{M=0}^\infty(-i\mathbf{\Gamma})^{N+M} \\ &= \sum_{N=0}^\infty(1+N)(-i\mathbf{\Gamma})^N = \left(1+\gamma\frac{\partial}{\partial\gamma}\right)\mathbf{t},\end{aligned} \tag{23}$$

can be expressed as a derivative with respect to the coupling constant to the impurities. Equation (23) is a second useful Ward identity. Using these relations and its Hermitian conjugate repeatedly we find

$$\mathbf{T}^{n+1} = \sum_{N=0}^\infty f_n(N)(-i\mathbf{\Gamma})^{2N} = f_n\left(\eta\frac{\partial}{\partial\eta}\right)\mathbf{T}(\eta)\Big|_{\eta=\gamma^2}, \tag{24}$$

where $f_n(N) = \binom{N+n}{n}$ is a polynomial of N. $\mathbf{T}(\eta)$ denotes the transmission matrix as a function of the squared scattering potential, which is set to the physical value $\eta = \gamma^2$ after differentiation. Assuming that the order of differentiation and configurational averaging may be exchanged, $P(T)$ can be expressed in terms of the series expansion

$$\begin{aligned}P(T) = \sum_{k_\parallel}\int\frac{dq}{2\pi}e^{-iqT}\Bigg(1 + \sum_{n=1}^\infty\frac{(iq)^n}{n!} \\ \times f_{n-1}\left(\eta\frac{\partial}{\partial\eta}\right)\langle T_{k_\parallel,k_\parallel}(\eta)\rangle\Bigg).\end{aligned} \tag{25}$$

Referring to Eq. (21), we see that $P(T)$ depends only on the self-energy. For calculations Eq. (25) is cumbersome. By using a (slightly modified) Stieltjes transform in the variable η and separating a possibly not square integrable component, the following more convenient expression can be derived:

$$P(T) = (g_0 - \langle g(\infty)\rangle)\delta(T) + \langle g(\infty)\rangle\delta(T-1) + \frac{1}{\pi}\frac{1}{T(1-T)}\mathrm{Im}\left[\left\langle g\left(\frac{\eta T}{T-1} - i0^+\right) - g(\infty)\right\rangle\right], \tag{26}$$

where $g_0 = \mathrm{Tr}\,\mathbf{I}$ is the number of channels, $g(\eta) = \mathrm{Tr}\,\mathbf{T}(\eta)$, and 0^+ is a positive infinitesimal. We see that the complete distribution function can be obtained by an infinitesimal analytical continuation of the squared coupling constant into the complex plane.

The correlation function Eq. (3) can be expressed as

$$P(T,T') = \int\frac{dq}{2\pi}\int\frac{dq'}{2\pi}e^{-iqT}e^{-iq'T'}\langle \mathrm{Tr}\,e^{iq\mathbf{T}}\,\mathrm{Tr}\,e^{iq'\mathbf{T}}\rangle \tag{27}$$

which, as mentioned above, provides information about the fluctuations of the transport properties due to individual deviations of a typical impurity configuration compared with the ensemble average. We can proceed as before and obtain

$$\langle \mathrm{Tr}\,e^{iq\mathbf{T}}\,\mathrm{Tr}\,e^{iq'\mathbf{T}}\rangle = \lim_{\eta\to\eta'}\sum_{k_\parallel}\sum_{k'_\parallel}\left\langle\left[1+\sum_{n=1}^{\infty}\frac{(iq)^n}{n!}f_{n-1}\left(\eta\frac{\partial}{\partial\eta}\right)T_{k_\parallel,k_\parallel}(\eta)\right]\left[1+\sum_{n'=1}^{\infty}\frac{(iq')^{n'}}{(n')!}f_{n'-1}\left(\eta'\frac{\partial}{\partial\eta'}\right)T_{k'_\parallel,k'_\parallel}(\eta')\right]\right\rangle. \tag{28}$$

The product of the traces of the transition matrix elements reads

$$g(\eta)g(\eta') = \sum_{k_\parallel}\sum_{k'_\parallel}T_{k_\parallel,k_\parallel}(\eta)T_{k'_\parallel,k'_\parallel}(\eta') = \frac{1}{2}\mathrm{Re}[\mathrm{Tr}\,\mathbf{t}(\eta)\,\mathrm{Tr}\,\mathbf{t}(\eta') + \mathrm{Tr}\,\mathbf{t}(\eta)\,\mathrm{Tr}\,\mathbf{t}^\dagger(\eta')], \tag{29}$$

which cannot be reduced any further. The vertex correction of the configurational average of Eq. (29) contains all information about the fluctuations.

A simple application is the (normalized) shot-noise power of the conductivity[22]

$$P_{\mathrm{SN}} = \sum_n\langle T_n(1-T_n)\rangle = -\eta\frac{\partial}{\partial\eta}\langle g(\eta)\rangle \tag{30}$$

and its variance

$$\mathrm{Var}P_{\mathrm{SN}} = \lim_{\eta\to\eta'}\eta\frac{\partial}{\partial\eta}\eta'\frac{\partial}{\partial\eta'}\langle g(\eta)g(\eta')\rangle. \tag{31}$$

IV. WEAK SCATTERING AND LOW IMPURITY DENSITY LIMIT

The above results are exact for the given model. In particular, they are valid for two and three dimensions (in the latter case the parallel momentum is a vector) and are not limited to weak scattering or any assumptions about the distribution of the scatterers. Concrete results are obtained rather easily when the scattering potential is weak, but also for strong scattering potentials, provided the density of scatterers is small. The single-site approximation is valid in this limit, where the scattering at a single impurity is treated exactly, but the phase coherence during multiple scattering is disregarded in the self-energy. This approximation is also called (single-site) coherent potential approximation. In the following, analytic results are discussed in this approximation for two-dimensional (2D) systems. Three-dimensional systems show quantitative differences,[27] which will be discussed in more detail in a separate paper.

A. Level densities

The self-energy in the single-site approximation reads in two dimensions

$$\Sigma_S = \frac{n_{IR}\bar{\gamma} - in_{IR}\frac{m^*}{\hbar^2}\frac{\gamma^2}{2\pi}(\pi - 2i\ln|\alpha+\sqrt{\alpha^2-1}|)}{1+(\frac{m^*\gamma}{2\hbar^2})^2(\pi-2i\ln|\alpha+\sqrt{\alpha^2-1}|)^2}, \tag{32}$$

where n_{IR} is the density of scatterers with average scattering strength $\bar{\gamma}$ and α is a cut-off parameter in the wave-vector summations. Since we disregard evanescent states, α will be set to unity in the following. The single-site approximation is valid in the weak scattering limit, where the scattering parameter $\eta_B = (\frac{m^*}{\hbar^2})^2\frac{n_{IR}\gamma^2}{2k_F}$ is sufficiently smaller than unity, but is believed to be applicable more generally at elevated temperatures where inelastic scattering destroys subtle phase coherence effects. In three limiting cases, analytical results can be obtained easily. The first, and rather trivial one, is that of a homogeneous δ-function barrier potential without disorder. This problem is equivalent to the disordered interface when only the first term in the self-energy is taken into account. In this limit

$$\Sigma = n_{IR}\bar{\gamma}. \tag{33}$$

It is straightforward to derive the distribution func-

tion for a barrier in a channel of the two-dimensional electron gas and a dimensionless scattering parameter $\eta_b = (\frac{m^* n_{IR} \bar{\gamma}}{k_F \hbar^2})^2$ to be

$$P_b(T) = g_0 \frac{1}{2} \frac{\eta_b}{(1-T)^{\frac{3}{2}}} \frac{1}{\sqrt{1-T-\eta_b T}}, \tag{34}$$

for $T \leq T^b_{\max} = 1/[1+\eta_b]$ and zero otherwise. $g_0 = \frac{k_F W}{\pi}$ denotes the number of channels in a ballistic wire of width W.

A second simple limit is obtained for weak scattering such that $n_{IR}(\frac{m^* \gamma}{\hbar^2})^2 \ll 1$. In that case the Born approximation is valid:

$$\Sigma_B = -i \frac{m^*}{\hbar^2} \frac{n_{IR} \gamma^2}{2}. \tag{35}$$

In terms of the scattering parameter $\eta_B = (\frac{m^*}{\hbar^2})^2 \frac{n_{IR}\gamma^2}{2k_F}$, the distribution function reads

$$P_B(T) = g_0 \frac{\eta_B^2 T}{(1-T)^2} \frac{1}{\sqrt{(1-T)^2 - \eta_B^2 T^2}} \tag{36}$$

for $T \leq T^B_{\max} = 1/[1+\eta_B]$, and zero otherwise, where g_B is the (dimensionless) conductance. These results differs qualitatively from the distribution of long wires. The number of states with small transmission is strongly suppressed and there are no completely transmitted states.

In the limit of very few strong scatterers, the self-energy is independent of the scattering potential

$$\Sigma_S(\infty) = -i \frac{2 n_{IR} \hbar^2}{m^*}, \tag{37}$$

which gives the simple result that $g_0 - n_{IR} W$ eigenstates are transmitted ($T = 1$) and $N_{IR} = n_{IR} W$ states are reflected. In this limit the shot-noise power vanishes, just like in the case of ballistic point contact.

In the intermediate regime, we can use Eqs. (26) and (32) to obtain

$$P_S(T) = g^S_\infty \delta(T-1) + g_0 \frac{T}{1-T} \frac{\eta_B^2}{1-T(1+\eta_B/\eta_\infty)} \times \frac{1}{\sqrt{(1-T)^2 - (\eta_B T)^2}}, \tag{38}$$

for $T \leq T^S_{\max} = 1/[1+\eta_B+\eta_B/\eta_\infty]$, and zero otherwise, where $\eta_\infty = 2n_{IR}/k_F$ and g^S_∞ is the conductance in the limit of strong scattering potentials,

$$\frac{g^S_\infty}{g_0} = 1 - \frac{\eta_\infty \pi}{2} + \begin{cases} \frac{\eta_\infty^2}{\sqrt{1-\eta_\infty^2}} \ln\left(\frac{1+\sqrt{1-\eta_\infty^2}}{\eta_\infty}\right), & \eta_\infty < 1 \\ \frac{\eta_\infty^2}{\eta_\infty^2 - 1} \arccos\left(\frac{1}{\eta_\infty}\right), & \eta_\infty > 1, \end{cases} \tag{39}$$

which is simply the number of completely transmitted states.

These results (Fig. 2) can be understood as follows. In the weak scattering limit, the incoming and outgoing states are plane waves, which are extended over the complete area of the junction. Even the electrons with $k_\perp = k_F$ will be therefore scattered back with a finite probability, which fixes $T^B_{\max}$. If the strength of the scatterers increases, the states which diagonalize the transmission matrix are not plane-wave states anymore, but will adjust to the scattering potential such that some states are localized at those parts of the interfaces where there are no impurities, thus establishing states that have a transmission probability of unity. In the strong scattering limit, all electrons either hit a scatterer and are completely reflected, or are completely transmitted through the "empty" spaces between the scatterers.

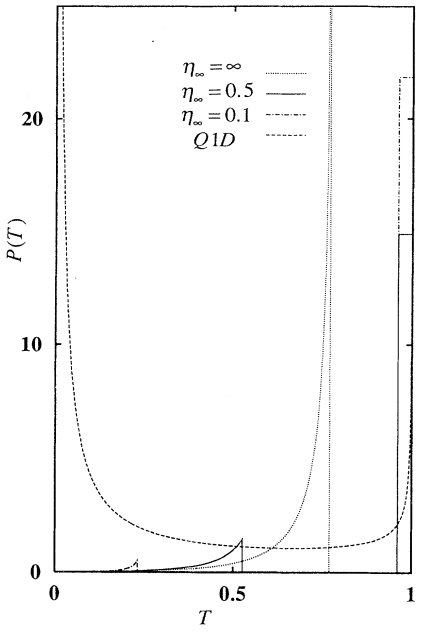


FIG. 2. Level density distribution as calculated in the single-site approximation Eq. (38) for constant $\eta_B = 0.3$ and different values of $\eta_\infty = 2n_{IR}/k_F$. The dashed line labeled $Q1D$ denotes the bimodal distribution of the transmission matrix eigenvalues in long wires (Ref. 22). The blocks on the right represent the integrated strength of the δ functions at $T = 1$.

This can be quantified by the following argument.[31] Since g_0 conducting channels are available with N_{IR} impurities, a unitary transformation can be carried out that locates the nodes of $g_0 - N_{IR}$ states exactly at the impurities. Irrespective of the scattering strength we therefore obtain $g_0 - N_{IR}$ states with transition probability of unity. The reflection probability of states, which have an amplitude on the scatterers, do depend on η. When evanescent states are taken into account, all conducting channels can be constructed in a way to have unit transmission probability, which illustrates the well-known artifact that zero-range potentials do not scatter electrons. It is therefore noted that (i) we should recall that our choice of $\alpha = 1$ does imply that we consider scatterers with a cross section of the order of the Fermi wavelength, (ii) the Born approximation is valid only for high density

of (weak) scatterers even though Eq. (35) appears to be accurate for small n_{IR}, and (iii) the results may depend quite sensitively on a proper treatment of the potential range.

B. Fluctuations

In order to get results for the level correlations we have to approximate Eq. (29). The calculations are more complicated now and the following discussion is restricted to the Born approximation (which has been shown above to be accurate for a high density of weak scatterers). We have to calculate two terms (Fig. 3) representing an electron-electron (Cooperon) and an electron-hole (diffuson) scattering amplitude. Summing the simple ladder and maximally crossed diagrams, we get

$$\langle g(\eta)g(\eta')\rangle - \langle g(\eta)\rangle\langle g(\eta')\rangle$$

$$= 2\left(\frac{m^*}{\hbar^2}\right)^2 \sum_{k_\parallel}\sum_{k'_\parallel} \frac{k_\perp}{(k_\perp+\eta_B k_F)^2}\frac{k'_\perp}{(k'_\perp+\eta'_B k_F)^2} \times \frac{\sigma^2_{\gamma,\gamma'}\Pi_{\eta,\eta'}(k_\parallel,k'_\parallel)}{1-\sigma^2_{\gamma,\gamma'}\Pi_{\eta,\eta'}(k_\parallel,k'_\parallel)^2}, \quad (40)$$

where $\sigma_{\gamma,\gamma'} = n_{IR}\gamma\gamma'/W$ is the simplest irreducible vertex function and

$$\Pi_{\eta,\eta'}(k_\parallel,k'_\parallel) = \left(\frac{m^*}{\hbar^2}\right)^2 \sum_Q \frac{1}{(k_\parallel-Q)_\perp+\eta_B k_F} \times \frac{1}{(k'_\parallel-Q)_\perp+\eta'_B k_F}, \quad (41)$$

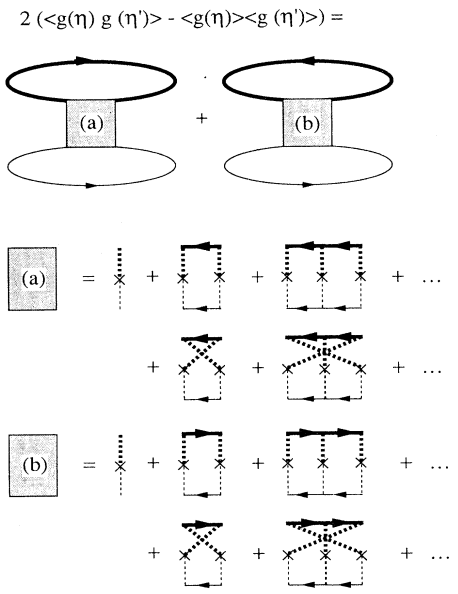


FIG. 3. Lowest order diagrams for $\langle g(\eta)g(\eta')\rangle - \langle g(\eta)\rangle\langle g(\eta')\rangle$.

where $Q_{\max}$ is the largest momentum that leaves the integrand real. It would be rather difficult to calculate the distribution function using Eqs. (28), (29), and (40). Fortunately the second-order approximation to Eq. (40)

$$\langle g(\eta)g(\eta')\rangle - \langle g(\eta)\rangle\langle g(\eta')\rangle$$

$$= \frac{2g_0}{\pi^2}\int_{-1}^{1} dk_\parallel \int_{-1}^{1} dk'_\parallel \frac{k_\perp\eta_B}{(k_\perp+\eta_B)^2}\frac{k'_\perp\eta'_B}{(k'_\perp+\eta'_B)^2} \times \int_0^{Q_{\max}} dQ \frac{1}{(k_\parallel-Q)_\perp+\eta_B}\frac{1}{(k'_\parallel-Q)_\perp+\eta'_B}, \quad (42)$$

where the integration variables are dimensionless, turns out to be an excellent approximation. Disregarding the terms in the denominator in Eq. (40) corresponds to the conventional perturbation treatment in the calculation of conductance fluctuations.[4] The errors in Var(g) are calculated to be $\leq 1\%$ for all $\eta_B \in [0,\infty)$. We see that the variance is proportional to the sample cross section, which reflects the phase space available for scattering of the correlated electron-hole pairs.

The correlation function calculated using Eqs. (28), (29), and (42) is quite complicated and very singular. We can write down the final result in terms of the "universal" function

$$F(Q,\lambda) = \frac{\lambda\Lambda(Q,\lambda)}{\sqrt{1-\lambda^2}}\left\{\frac{3-2\lambda^2}{1-\lambda^2}+\Lambda(Q,\lambda) \times\left(1-\frac{\lambda}{\sqrt{1-\lambda^2}}\frac{\sqrt{1-\lambda^2}-Q}{\sqrt{1-(Q-\sqrt{1-\lambda^2})^2}}\right)\right\}, \quad (43)$$

where

$$\Lambda(Q,\lambda) = \frac{1}{\sqrt{1-(Q-\sqrt{1-\lambda^2})^2}-\lambda}, \quad (44)$$

such that

$$P(T,T') = \frac{2g_0}{\pi^2}\int_0^{Q_{\max}} dQ\, F\left(Q,\frac{\eta T}{1-T}\right)F\left(Q,\frac{\eta T'}{1-T'}\right), \quad (45)$$

where

$$Q_{\max} = \min\left[1+\sqrt{1-\left(\tfrac{\eta T'}{1-T'}\right)^2}, 1+\sqrt{1-\left(\tfrac{\eta T'}{1-T'}\right)^2}\right].$$

The results are easier to interpret in terms of the function

$$\Xi(\lambda,\lambda') = \int_0^{Q_{\max}} dQ\, \mathrm{P}[\Lambda(Q,\lambda)+\Lambda(-Q,\lambda)] \times \mathrm{P}[\Lambda(Q,\lambda')+\Lambda(-Q,\lambda')], \quad (46)$$

where P indicates the principal value, from which the statistical fluctuations are calculated as

$$\mathrm{Var}(a) = \frac{2g_0}{\pi^2}\int_0^{T^B_{\max}} dT \left[T(1-T)\frac{d}{dT} - 1\right] a(T) \times \int_0^{T^B_{\max}} dT' \left[T'(1-T')\frac{d}{dT'} - 1\right] a(T') \times P_B(T)/g_0 \; P_B(T')/g_0 \; \Xi\left(\frac{\eta T}{1-T}, \frac{\eta T'}{1-T'}\right). \tag{47}$$

All variances scale with the number of modes, i.e., the wire width, and depend on the scattering parameter Another qualitative difference with long channels is the positive correlation of closely spaced eigenvalues in $\Xi(\lambda \to \lambda')$, in contrast to the divergent negative correlation in long channels.[8] The δ-function self-correlation can be separated as

$$\Xi(\lambda,\lambda') = \frac{\pi^2}{2}\frac{\lambda^2}{1-\lambda^2}\delta(\lambda-\lambda') + \mathrm{Re}\int_0^{Q_{\max}} dQ\,[\tilde{\Lambda}(Q,\lambda) + \Lambda(-Q,\lambda)] \times [\tilde{\Lambda}(Q,\lambda') + \Lambda(-Q,\lambda')], \tag{48}$$

where in $\tilde{\Lambda}$ a small imaginary part is introduced into the denominator. The second term on the right-hand side is now positive and smoothly varying at $\lambda = \lambda'$ (or $T = T'$). The results can be understood in terms of phase space arguments (see Fig. 4). The correlations/fluctuations are consequences of the phase space available for scattering to two electron-electron and electron-hole pairs, which scatter simultaneously at the same impurity site. With this scattering a momentum Q is transferred from one pair to the other. A given transmission T is (in the Born approximation) uniquely linked to a certain $k_\perp$ of the scattering state. When $T = T'$ the scattering can make optimal use of the phase space, which means that the correlation is large and positive. Negative correlations are possible close to "resonant" scattering $k_\parallel \to -k_\parallel$, which leaves $k_\perp$ unchanged. Negative correlations occur when only one side of the resonance can be reached by scattering and the other side is blocked, because the companion state cannot accept the momentum. This situation occurs when $1 + \sqrt{1-\lambda^2} = 2\sqrt{1-\lambda'^2}$, or when $\lambda \to 1$ $(T \to T^B_{\max})$.

The original correlation function as defined in Eq. (3) can be recovered from Ξ by partial integration. This does not change the picture much, but in addition to the $\delta(T-T')$ self-correlation in Eq. (48) the first and second derivatives of the δ function contribute too. Introducing a finite broadening parameter by adding a small imaginary part to the integration variable in Eq. (45), $P(T,T')$ is plotted in Fig. 5 close to its diagonal. We can clearly clearly see the presence of $\delta(T-T')$, $\delta'(T-T')$, and $\delta''(T-T')$. If level interaction is still a valid concept at all, this zero-range negative correlation can be interpreted as a very short-range repulsion, which is not effective in suppressing fluctuations. The rest of the interaction is not only weak, but also *attractive*. It should be interesting to find out whether the level attraction identified in Ref. 11 for long wires can be explained by microscopic phase space arguments as well.

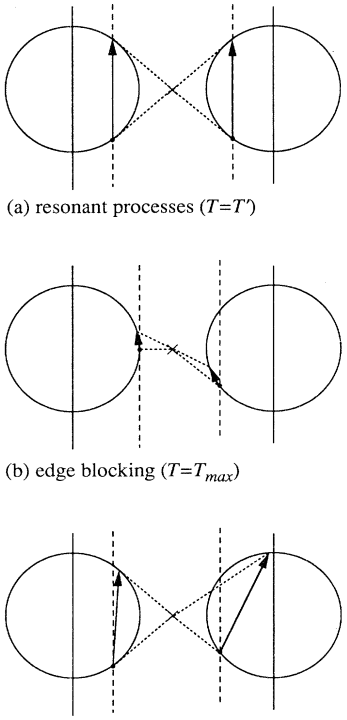


FIG. 4. Singular scattering processes in the correlation function Ξ, Eq. (46).

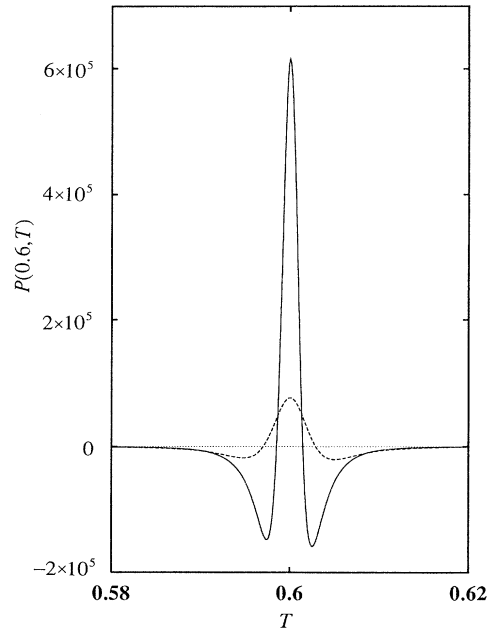


FIG. 5. Correlation function $P(0.6,T)$ close to the diagonal. P has been broadened by adding a small imaginary part (full line: $10^{-3}i$, dashed line: $2\times10^{-3}i$) to the integration variable Q in Eq. (45) to allow for a graphical representation of the singular self-correlation.

C. Transport properties

The physical consequences become evident in Fig. 6, where the averages of three transport properties are plotted for the intermediate regime as a function of the impurity density η_∞ and a fixed $\eta_B = 0.3$. Small η_∞ correspond to strong scattering potentials. Remarkable is the drop of the conductance of the N/S junction below the normal conductance in the Born scattering limit, a direct consequence of the lack of highly transmitting states.

In Fig. 7 the root-mean-square fluctuations of the physical statistics as in Fig. 6 are calculated from Eq. (47). Interesting are the strong nonlinearity of the fluctuations in the shot noise as a function of the scattering strength. The conductance fluctuations in a normal metal system are enhanced up to more than a factor 7 for small η_B when one of the terminals becomes superconducting. This is in contrast to the universal enhancement of the fluctuations in long channels by $3/\sqrt{2}$ (Ref. 12). One should be aware that significant corrections to the results of the weak scattering approximation in Fig. 7 must be expected for large η_B.

V. DISCUSSION

The present results can be directly tested experimentally in wide wires of the high-mobility two-dimensional electron gas,[32] which are selectively doped by focused ion beam implantation.[33] This would also provide a method to systematically explore the emerging level repulsion with increasing channel length. The three-dimensional

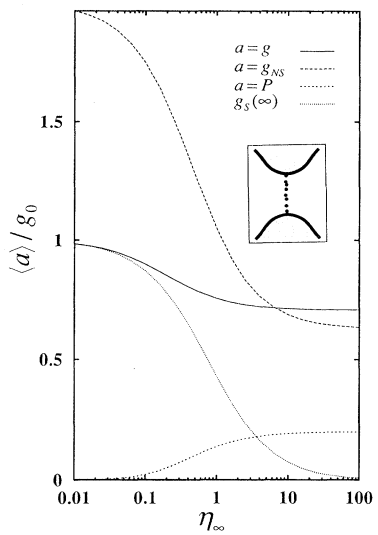


FIG. 6. Average transport properties $\langle a \rangle$ (in units of g_0) of a two-dimensional short disordered channel (inset) as calculated in the single-site approximation of the self-energy as a function of η_∞ and a constant $\eta_B = 0.3$: Normal conductance $a(T) = T$ (full curve), conductance of an N/S junction $a(T) = 2T^2/(2-T)^2$ (long dashes), shot-noise power $a(T) = T(1-T)$ (short dashes). The dotted curve represents the number of completely transmitted states $g_S(\infty)$.

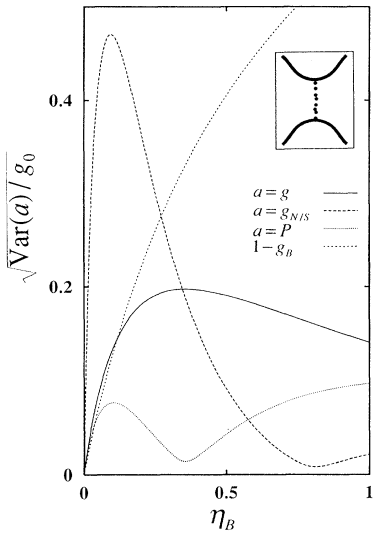


FIG. 7. Fluctuations $\sqrt{\mathrm{Var}(a)}$ of transport properties about the ensemble average (in units of $\sqrt{g_0}$) of a two-dimensional short disordered channel (inset) as calculated in the Born approximation to the self-energy as a function of η_B: Normal conductance (full curve), conductance of an N/S junction (long dashes), shot-noise power (short dashes). The dotted line is the average reflection probability in the Born approximation $1 - g_B$. A similar quantity has been used to estimate the magnitude of the conductance fluctuations in Ref. 15.

version of the present theory describes a metallic point contact,[15] including a short and wide constriction with a thin disordered region in the center, which might be a rough heterointerface or a (sub)monolayer of doping atoms. The conductance fluctuations in ballistic point contacts are caused by remote impurities, which are not treated by the present theory.[34] However, when the main source of scattering comes from the interface it will dominate the fluctuations also.

The above results shed light on a semiclassical approximation for transport in sample with finite system length L (Refs. 35 and 20), which consists of replacing the transmission and reflection coefficients $t_{k_\parallel,k'_\parallel}$, $r_{k_\parallel,k'_\parallel}$ in the concatenation of the scattering matrices by the transmission and reflection probabilities $|t_{k_\parallel,k'_\parallel}|^2$ and $|r_{k_\parallel,k'_\parallel}|^2$, which leads to the recovery of Ohm's law for $L \gg \ell$. This scheme has been shown to be equivalent to shifting all impurities onto a single scattering plane, which may then be treated by the method discussed above. Assuming that the scattering is weak at each point of the sample, we may continue to use the Born approximation, but still let the self-energy become large for large samples. We see that the singularity at $T = 1$ in the bimodal distribution of $P(T)$ for long channels,[24] i.e., the transmitted states, are not recovered in this semiclassical approximation and must therefore be caused by quantum interference. The process is seen to be equivalent to the single barrier vs double barrier tunneling device: the latter supports resonant tunneling states with transmission probability of

unity, in contrast to the single barrier diode. In this semiclassical $-\eta\frac{\partial}{\partial\eta}g \to g$, which means that the Drude result is obtained for the conductance, but that the shot-noise is not suppressed relative to its classical Poisson value. Note however, that the completely transmitting states seem to be not necessary to explain the shot-noise suppression, since de Jong and Beenakker[36] derive the shot-noise suppression semiclassically and conclude that phase coherence it not required for this property. Apparently there are different levels of "semiclassicality."

Theoretically, the present study has to be expanded to include effects of an additional dimension, magnetic fields, potential steps or barriers, and evanescent states. Of particular interest is the double necklace/interface configuration, which should give important additional insights into the level repulsion concept in phase-coherent transport.

The complete absence of universality in the transport through necklaces and interfaces has interesting consequences for material science. Whereas the amplitude of the universal conductance fluctuations obviously does not depend on the microscopic structure of the sample(s) in question, the present results indicate that transport experiments on single necklaces and interfaces do yield additional information on the scattering potentials, which might be a useful technique to characterize these systems.[27]

ACKNOWLEDGMENTS

I acknowledge the hospitality of the NTT Basic Research Laboratories where part of this work has been carried out and helpful discussions with Arne Brataas, Marc de Jong, Seigo Tarucha, and especially Yuli Nazarov. This work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)," which is financially supported by the "Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)."

[1] *Mesoscopic Phenomena in Solids*, edited by B.L. Al'tshuler, P.A. Lee, and R.A. Webb (North-Holland, Amsterdam, 1991).
[2] B.L. Al'tshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)].
[3] P.A. Lee and A.D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).
[4] P.A. Lee, A.D. Stone, and H. Fukuyama, Phys. Rev. B **35**, 1039 (1987).
[5] M.L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, San Diego, 1990).
[6] B.L. Al'tshuler and B.I Shklovskiĭ, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
[7] Y. Imry, Europhys. Lett. **1**, 249 (1986).
[8] A.D. Stone, P.A. Mello, K.A. Muttalib, and J.-L. Pichard, in *Mesoscopic Phenomena in Solids* (Ref. 1), p. 390.
[9] F.J. Dyson, J. Math. Phys. **3**, 140 (1962); **3**, 157 (1962); **3**, 166 (1962).
[10] Limitations of the applicability of RMT in the context of energy level statistics have been discussed by K.B. Efetov, Zh. Eksp. Teor. Fiz. **83**, 833 (1986) [Sov. Phys. JETP **64**, 127 (1986)]; Adv. Phys. **32**, 53 (1983).
[11] C.W.J. Beenakker and B. Rejaei, Phys. Rev. B **49**, 7499 (1994).
[12] C.W.J. Beenakker, Phys. Rev. B **47**, 15 763 (1993).
[13] Yu. V. Nazarov, Phys. Rev. Lett. **73**, 134 (1994).
[14] C.W.J. Beenakker, B. Rejaei, and J.A. Melsen, Phys. Rev. Lett. **72**, 2470 (1994).
[15] P.A.M. Holweg, J.A. Kokkedee, J. Caro, A.H. Verbruggen, S. Radelaar, A.G.M. Jannssen, and P. Wyder, Phys. Rev. Lett. **67**, 2549 (1991).
[16] R.A. Jalabert, J.-L. Pichard, and C.W.J. Beenakker, Europhys. Lett. **27**, 255 (1994); H.U. Baranger and P.A. Mello, Phys. Rev. Lett. **73**, 142 (1994).
[17] C.W.J. Beenakker, Phys. Rev. B **50**, 15 170 (1994).
[18] W.P. Pratt, S.F. Lee, R. Lolee, P.A. Schroeder, and J. Bass, Phys. Rev. Lett. **66**, 3060 (1991); M. Gijs, S.K.J. Lenczowski, and J.B. Giesbers, *ibid.* **70**, 3343 (1993).
[19] G.E.W. Bauer, Phys. Rev. Lett. **69**, 1676 (1992).
[20] A. Brataas and G.E.W. Bauer, Europhys. Lett. **26**, 117 (1994); Phys. Rev. B **49**, 14 684 (1994).
[21] C.W.J. Beenakker, in *Mesoscopic Quantum Physics*, edited by E. Akkermans, G. Montambaux, and J.-L. Pichard (North-Holland, Amsterdam, in press).
[22] M. Büttiker, Phys. Rev. Lett. **65**, 2901 (1990).
[23] C.W.J. Beenakker, Phys. Rev. B **46**, 12 841 (1992).
[24] O.N. Dorokhov, Solid State Commun. **51**, 381 (1984); P.A. Mello, P. Pereyra, and N. Kumar, Ann. Phys. (N.Y.) **181**, 290 (1988).
[25] Microscopic calculations concerning the related, but quite different problem of the distribution function of the conductances can be found in, e.g., B.L. Al'tshuler, V.E. Kravtsov, and I.V. Lerner, in *Mesoscopic Phenomena in Solids* (Ref. 1), p. 449; H.A. Weidenmüller, Physica A **167**, 28 (1990); A.A. Abrikosov, Phys. Scr. **T27**, 148 (1989).
[26] *Analogies in Optics and Microelectronics*, edited by W. van Haeringen and D. Lenstra (Kluwer, Dordrecht, 1990).
[27] G.E.W. Bauer, J. Magn. Magn. Mater. (to be published).
[28] P.H. Bagwell, Phys. Rev. B **41**, 10 354 (1990).
[29] Note that the transmission matrix defined here differs from the matrix of the transmission probabilities for which the same symbol was used in Ref. 20.
[30] Y.B. Levinson, M.I. Lubin, and E.V. Sukhorukov, Phys. Rev. B **45**, 11 936 (1992).
[31] T. Ando (private communication).
[32] S. Tarucha, T. Saku, Y. Tokura, and Y. Hirayama, Phys. Rev. B **47**, 4064 (1993).
[33] S. Tarucha (private communication).
[34] V.I. Kozub, J. Caro, and P.A.M. Holweg, Phys. Rev. B **50**, 15 126 (1994).
[35] B. Shapiro, Phys. Rev. B **35**, 8256 (1987); M. Cahay, M. McLennan, and S. Datta, *ibid.* **37**, 10 125 (1988).
[36] M.J.M. de Jong and C.W.J. Beenakker (unpublished).

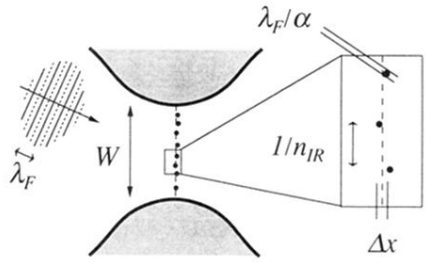
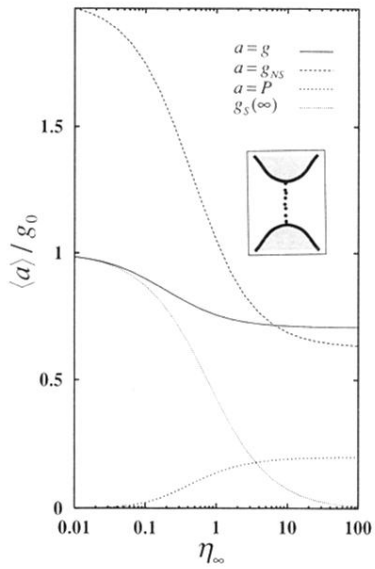


FIG. 1. Schematic view of an impurity necklace in the two-dimensional electron gas. λ_F is the Fermi wavelength, W the channel width, λ_F/α the range of the scattering potentials, Δx the spreading of the impurity positions in the transport direction, and n_{IR} the line density of scatterers.



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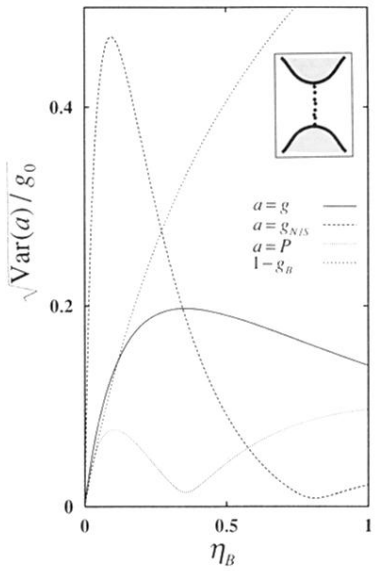

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