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Real Zeros in Finite Models of the Completed Riemann Zeta Function

A Conditional de Branges Approach

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Layman's summary

The Riemann Hypothesis is one of the biggest unsolved problems in mathematics about prime numbers, such as $2, 3, 5, 7, 11, \ldots$. Prime numbers do not seem to occur in a simple pattern, but their distribution is not completely random. The mathematician Riemann found that prime numbers are connected to the zeros of a special function, namely the Riemann zeta function. The better we understand these zeros, the better we understand the structure behind the distribution of the prime numbers.

Rather than studying the zeta function directly, we look at a simpler finite model derived from it. We then ask ourselves what kind of mathematical structure could force the zeros of this simpler finite model to lie on one straight real line.

The main idea is to compare these zeros with the resonant frequencies of a vibrating string. When one string vibrates, not every possible motion fits with the fixed endpoints of the string. Only certain vibration patterns fit, and these patterns determine the frequencies we hear. For the finite model, we try to build a similar link: each zero should correspond to one of the resonant frequencies of the model. Since these frequencies are real numbers, this would explain why the zeros are forced onto one real line.

There is one important catch. To interpret the zeros in this way, the finite model has to satisfy some additional assumptions. These assumptions are exactly the properties needed to make the link with resonant frequencies precise. If they could be proved for finite models that get closer and closer to the zeta function, then the same argument could be transferred back to the zeta function, which could imply the Riemann Hypothesis.

Summary

The Riemann Hypothesis (RH) can be written as a question about the zeros of the completed zeta function, which is built from the analytic continuation of the Riemann zeta function. Since this function is hard to analyse directly, we replace it by a simpler truncated model. Nevertheless, this model still keeps a strong connection with the original zeta problem.

After recentering to the critical line, the completed zeta function can be written as a cosine-transform formula with theta kernel $\Phi(u)$. This formula is made finite in two ways: the integral is cut off at a finite point $a > 0$, and the kernel Φ is replaced by the partial kernel Φ_N, containing only the first N terms. This yields

$$F_{a,N}(z) := \int_0^a \Phi_N(u) \cos(zu) \, \mathrm{d}u.$$

As $a \to \infty$ and $N \to \infty$, the functions $F_{a,N}$ converge locally uniformly to the recentered completed zeta function.

The central question is then what additional structure can be used to study the zeros of $F_{a,N}$. The approach is to construct a Hilbert space of entire functions starting from $F_{a,N}$. To do this, we first combine $F_{a,N}$ with its derivative and define

$$E_{a,N,\tau} := F_{a,N} + \mathrm{i}\tau F'_{a,N}, \qquad \tau > 0.$$

We assume that this function satisfies the so-called Hermite–Biehler condition. This is the condition that makes the Hilbert-space construction possible. The resulting Hilbert space is denoted by $\mathcal{H}(E_{a,N,\tau})$ and is called a de Branges space.

In this space, we study the multiplication operator given by multiplication by the variable z: a function G is sent to the function $z \mapsto zG(z)$. This formula alone does not determine the operator, because the domain is part of the definition. Indeed, not every $G \in \mathcal{H}(E_{a,N,\tau})$ has the property that the function $z \mapsto zG(z)$ again belongs to $\mathcal{H}(E_{a,N,\tau})$. The domain must therefore be chosen carefully.

To use this operator for the zero problem, we want to bring it into spectral theory. This is where self-adjoint extensions enter the argument. In finite dimensions, the spectrum of a matrix is the set of its eigenvalues, and Hermitian matrices have only real eigenvalues. Self-adjoint operators are the Hilbert-space analogue of this: their spectrum is real. Under the assumptions stated later, one of the self-adjoint extensions of the multiplication operator has the zeros of $F_{a,N}$ as its spectral points.

For $F_{a,N}$, the Hermite–Biehler condition is already a very strong assumption: it implies that $F_{a,N}$ has only real zeros. Therefore the construction through $E_{a,N,\tau}$, the de Branges space $\mathcal{H}(E_{a,N,\tau})$, and the multiplication operator should not be seen merely as another way to obtain real zeros. Its role is different: it shows how these zeros fit into the de Branges-space construction, where they appear as spectral points of a self-adjoint extension.

Moreover, if the Hermite–Biehler condition could be proved for a sequence F_{a_j,N_j} with $a_j \to \infty$ and $N_j \to \infty$, then the local uniform convergence to the recentered completed zeta function would rule out non-real zeros of the limiting function. Indeed, any non-real zero of the recentered completed zeta function would force nearby non-real zeros of F_{a_j,N_j} for large j, which contradicts the finite real-zero property. Hence such a result would imply the Riemann Hypothesis.

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Contents

Notation and conventions

Throughout the report,

$$\mathbb{N}_{\geq 1} := \{1, 2, 3, \ldots\}.$$

All inner products are linear in the first variable and conjugate-linear in the second.

For an entire function $E : \mathbb{C} \to \mathbb{C}$, define its reflected function by

$$E^{\#}(z) := \overline{E(\overline{z})}.$$

The upper half-plane is denoted by

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}.$$

The Fourier transform is taken with the convention

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(u) e^{-2\pi \mathrm{i} u \xi} \, \mathrm{d}u.$$

In the table below, functions are listed as objects. When needed, their defining formula is written using map notation.

RH	Riemann Hypothesis,
ζ	Riemann zeta function,
ξ	completed Riemann zeta function,
Ξ_{dBN}	$z \mapsto \frac{1}{8} \xi \left(\frac{1}{2} + \frac{\mathrm{i}z}{2} \right)$,
$\theta,\ \psi$	theta function and theta tail,
ϕ_n	the n-th theta-kernel term,
$\Phi,\ \Phi_N$	theta kernel and partial theta kernel,
$F_{a,N}$	$z \mapsto \int_0^a \Phi_N(u) \cos(zu) \, \mathrm{d}u$ truncated theta-kernel model,
$f_{a,N}$	$u \mapsto \frac{1}{2} \Phi_N(\lvert u \rvert) \mathbf{1}_{[-a,a]}(u)$,
$E_{a,N,\tau}$	$z \mapsto F_{a,N}(z) + \mathrm{i}\tau F'_{a,N}(z)$,
$\mathcal{HB}$	Hermite–Biehler class,
$\mathcal{H}(E)$	de Branges space generated by E,
$K_E(w, z)$	reproducing kernel of $\mathcal{H}(E)$,
S_E	multiplication operator in $\mathcal{H}(E)$,
S_β	self-adjoint extension of S_E,
$\mathrm{Dom}(T)$	domain of an operator T,
$\sigma(T)$	spectrum of an operator T,
$\sigma_p(T)$	point spectrum of an operator T,
$\mathrm{Zeros}(F)$	zero set of a function F.

1 Introduction

In the year 1737, one of the first connections between the prime numbers and the zeta function was found by Euler through the Euler product. The zeta function ζ is first defined by a series which only converges for $\mathrm{Re}(s) > 1$. More than a hundred years later, in 1859, Riemann studied $\zeta(s)$ as a complex function and extended it by analytic continuation. From this continuation one defines the completed zeta function, denoted by $\xi(s)$. The non-trivial zeros of $\xi(s)$ are the relevant zeros for the Riemann Hypothesis. These lie in the critical strip

$$0 < \mathrm{Re}(s) < 1.$$

Riemann conjectured that in this critical strip, all these non-trivial zeros lie on one particular line:

$$\mathrm{Re}(s) = \frac{1}{2}.$$

The completed zeta function is symmetric with respect to this line, and after recentering $\xi(s)$ to the critical line, the RH translates into a question about real zeros of an entire function:

$$\Xi_{\mathrm{dBN}}(z) = \frac{1}{8}\xi\left(\frac{1}{2} + \frac{\mathrm{i}z}{2}\right).$$

This function can be written as a cosine transform with theta kernel $\Phi(u)$:

$$\Xi_{\mathrm{dBN}}(z) = \int_0^\infty \Phi(u)\cos(zu)\ \mathrm{d}u.$$

In Section 2, the kernel is written as the series $\Phi(u) = \sum_{n=1}^\infty \phi_n(u)$, and $\Phi_N(u) = \sum_{n=1}^N \phi_n(u)$ denotes the partial theta kernel obtained by keeping only the first N terms. The full integral is still difficult to study directly, so we replace it by finite approximations. For $a > 0$ and $N \in \mathbb{N}_{\geq 1}$, define

$$F_{a,N}(z) = \int_0^a \Phi_N(u)\cos(zu)\ \mathrm{d}u.$$

Thus N specifies how many terms of the theta kernel are retained, while a specifies where the integral is cut off. The functions $F_{a,N}$ are therefore truncated theta-kernel models of the recentered completed zeta function.

These truncated models remain connected to the recentered completed zeta function. As $a \to \infty$ and $N \to \infty$, the functions $F_{a,N}$ converge locally uniformly to Ξ_{dBN}. This convergence allows one to compare zeros on bounded regions: if such a region has no zero of the limiting function on its boundary, then for all sufficiently large a and N, the function $F_{a,N}$ has the same number of zeros in that region as Ξ_{dBN}, counted with multiplicity. Therefore, if the approximating functions had only real zeros, a non-real zero of the limiting function would be impossible. This is the zero-counting comparison used later in the report.

The truncated theta-kernel model also has a Fourier interpretation. Since the integral is taken over a finite interval $[0, a]$ and z appears as the frequency variable, the functions $F_{a,N}$ can be viewed as finite Fourier-type models. Fourier analysis is closely tied to Hilbert spaces, because inner products and orthogonality provide a way to compare frequency components. This suggests the following question: can one construct a Hilbert space of entire functions from $F_{a,N}$ in which the frequency variable z can be used to study the zeros of $F_{a,N}$?

To construct such a Hilbert space, we first combine $F_{a,N}$ with its derivative and define $E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}$, where $\tau > 0$. The key assumption will be that $E_{a,N,\tau}$ satisfies the Hermite–Biehler condition, meaning that $|E^{\#}_{a,N,\tau}(z)| < |E_{a,N,\tau}(z)|$ for $z \in \mathbb{C}_+$, and that $E_{a,N,\tau}$

has no real zeros. This condition is defined in Section 4. It is the condition that makes it possible to construct a Hilbert space of entire functions, denoted by $\mathcal{H}(E_{a,N,\tau})$; this space is called a de Branges space.

Inside this space we study the multiplication operator given by multiplication by the variable z: a function G is sent to the function $z \mapsto zG(z)$. This formula alone does not determine the final operator, because the domain is part of the definition. Indeed, not every $G \in \mathcal{H}(E_{a,N,\tau})$ has the property that the function $z \mapsto zG(z)$ again belongs to $\mathcal{H}(E_{a,N,\tau})$. The domain must therefore be chosen carefully. In Section 6 we formulate the assumptions under which the corresponding multiplication operator has self-adjoint extensions. In Section 7, under these regularity assumptions and after excluding the relevant exceptional obstruction, the relevant extension is related to the zeros of $F_{a,N}$.

This matters because self-adjoint operators have real spectrum. In finite dimensions this corresponds to the fact that Hermitian matrices have real eigenvalues. In the construction used here, the zeros of $F_{a,N}$ are interpreted as spectral points of a self-adjoint extension.

For the function $E_{a,N,\tau}$ defined from $F_{a,N}$, the assumption $E_{a,N,\tau} \in \mathcal{HB}$ already implies that $F_{a,N}$ has only real zeros. Thus the operator-theoretic construction does not remove this assumption and should not be read as an independent proof of the real-zero property. The construction adds a spectral interpretation: in the de Branges space $\mathcal{H}(E_{a,N,\tau})$, multiplication by z is studied as an operator, and for the relevant self-adjoint extension the zeros of $F_{a,N}$ appear as spectral points.

The difficult point is proving the Hermite–Biehler condition for the truncated theta-kernel models. In this report this condition is kept as an assumption. If it could be proved for a sequence F_{a_j,N_j} with $a_j \to \infty$ and $N_j \to \infty$, then every function in that sequence would have only real zeros. By the local-uniform convergence and zero-counting comparison described above, this would rule out non-real zeros of the recentered completed zeta function. Such a result would imply the Riemann Hypothesis.

The aim of this report is to work out this conditional mechanism: the Hermite–Biehler assumption gives real zeros for the truncated theta-kernel models, the de Branges construction interprets these zeros spectrally, and the zero-transfer argument passes the real-zero conclusion to the recentered completed zeta function.

2 The completed Riemann zeta function

We start from the Riemann zeta function in the half-plane $\mathrm{Re}(s) > 1$ and construct the completed zeta function ξ. The main ingredient is the theta transformation law, which follows from the Poisson summation formula and relates the values of the theta function at x and $1/x$. This symmetry allows the small-x part of a Mellin integral to be rewritten in terms of large x.

This leads to the analytic continuation of ξ and to the functional equation $\xi(s) = \xi(1-s)$. After recentering around the critical line, we derive the cosine-transform formula that will be truncated in Section 3.

Definition 2.1 (Riemann zeta function)**.** Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. The Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Proposition 2.2. *The series defining $\zeta(s)$ is absolutely convergent for every $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1$.

Proof. Write $s = \sigma + \mathrm{i}t$, where $\sigma = \mathrm{Re}(s) > 1$ and $t \in \mathbb{R}$. For every $n \in \mathbb{N}_{\geq 1}$,

$$\left|\frac{1}{n^s}\right| = \left|e^{-s\log n}\right| = \left|e^{-\sigma \log n} e^{-\mathrm{i}t\log n}\right| = n^{-\sigma}.$$

Therefore $\sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$, which is a convergent p-series since $\sigma > 1$. Hence the defining series of $\zeta(s)$ is absolutely convergent. □

Example 2.3 (The value at $s = 2$)**.** A famous example is the value of the zeta function at $s = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The series converges, since it is a p-series with $p = 2 > 1$.

Before continuing the analytic continuation argument, we need to introduce three standard tools: the Mellin transform, the Schwartz space, and the Poisson summation formula.

Definition 2.4 (Mellin transform)**.** Let $f : (0,\infty) \to \mathbb{C}$ be locally integrable. For $s \in \mathbb{C}$, the Mellin transform of f is defined by

$$\mathcal{M}f(s) := \int_0^{\infty} f(x) x^{s-1}\, \mathrm{d}x,$$

whenever this improper integral is absolutely convergent, that is, whenever $\int_0^{\infty} |f(x)x^{s-1}|\, \mathrm{d}x < \infty$. Here $x^{s-1} = e^{(s-1)\log x}$, with the real logarithm on $(0,\infty)$.

Definition 2.5 (Schwartz space)**.** The Schwartz space on $\mathbb{R}$ is

$$\mathcal{S}(\mathbb{R}) := \left\{ f \in C^{\infty}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \left|t^m f^{(k)}(t)\right| < \infty \text{ for all } m, k \in \mathbb{N}_0 \right\}.$$

Remark 2.6. This means that f is smooth and that f, together with all its derivatives, decays faster than any polynomial in $|t|$ as $|t| \to \infty$. We use this space because the Poisson summation formula holds for functions in $\mathcal{S}(\mathbb{R})$. In addition, $\mathcal{S}(\mathbb{R})$ is well suited to Fourier analysis: the Fourier transform is well-defined on $\mathcal{S}(\mathbb{R})$ and maps $\mathcal{S}(\mathbb{R})$ onto itself. In particular, the Gaussian $f_x(t) = e^{-\pi x t^2}$ belongs to $\mathcal{S}(\mathbb{R})$ for every $x > 0$.

We use the following form of the Poisson summation formula; see [2, Section 12.3, Theorem 12.9].

Theorem 2.7 (Poisson summation formula)**.** *Let $f \in \mathcal{S}(\mathbb{R})$ and define its Fourier transform by*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx.$$

Then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n).$$

Proposition 2.8 (Theta transformation law)**.** *For $x > 0$, define $\theta(x) := \sum_{n \in \mathbb{Z}} e^{-\pi x n^2}$. Then*

$$\theta(x) = x^{-1/2} \theta\left(\frac{1}{x}\right).$$

Proof. Apply the Poisson summation formula from Theorem 2.7 to $f_x(t) := e^{-\pi x t^2}$. With our Fourier convention, one has $\widehat{f_x}(\xi) = x^{-1/2} e^{-\pi \xi^2 / x}$. Hence

$$\theta(x) = \sum_{n \in \mathbb{Z}} f_x(n) = \sum_{n \in \mathbb{Z}} \widehat{f_x}(n) = x^{-1/2} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 / x} = x^{-1/2} \theta(1/x).$$

□

The theta transformation law from Proposition 2.8 is the key step in the analytic continuation of the zeta function. It rewrites the theta function at x in terms of the theta function at $1/x$. This will allow us to rewrite the part of a Mellin integral over $(0, 1)$ as an integral over $(1, \infty)$. Before using this, we first introduce the Gamma function. This function is a special Mellin transform: it is the Mellin transform of e^{-t}.

Definition 2.9 (Gamma function)**.** For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, the Gamma function is defined by

$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} \, dt.$$

Proposition 2.10 (Meromorphic continuation of Γ)**.** *The function Γ is holomorphic on $\mathrm{Re}(s) > 0$ and extends meromorphically to $\mathbb{C}$. Its only poles are simple poles at $0, -1, -2, \ldots$.*

Proof. For $\sigma = \mathrm{Re}(s) > 0$, the integral converges near 0 like $t^{\sigma - 1}$ and near ∞ because of the exponential factor e^{-t}. Hence Γ is holomorphic on $\mathrm{Re}(s) > 0$. Integration by parts gives $\Gamma(s+1) = s\Gamma(s)$. Thus, if $N \in \mathbb{N}$ is chosen such that $\mathrm{Re}(s + N) > 0$, define

$$\Gamma(s) := \frac{\Gamma(s+N)}{s(s+1)\cdots(s+N-1)}.$$

This gives a meromorphic continuation to $\mathbb{C}$, with possible poles only at $0, -1, -2, \ldots$. It remains to show that these singularities are not removable. Fix $m \in \mathbb{N}_0$ and choose $N = m + 1$. Near $s = -m$ we have

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)\cdots(s+m)}.$$

The numerator is holomorphic near $s = -m$ and has value $\Gamma(1) = 1$ at $s = -m$. In the denominator, exactly one factor vanishes at $s = -m$, namely $s + m$; all other factors are non-zero there. Hence the denominator has a simple zero at $s = -m$, while the numerator is non-zero. Therefore Γ has a simple pole at $s = -m$.

□

Definition 2.11 (Positive theta part)**.** For $x > 0$, set

$$\psi(x) := \sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

Using the theta function from Proposition 2.8, we have

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x} = 1 + 2\psi(x).$$

Combining this with the theta transformation law from Proposition 2.8 gives

$$1 + 2\psi(x) = \theta(x) = x^{-1/2}\theta(1/x) = x^{-1/2}(1 + 2\psi(1/x)).$$

Thus

$$\psi(x) = \frac{1}{2}(x^{-1/2} - 1) + x^{-1/2}\psi(1/x).$$

The connection with $\zeta(s)$ appears after multiplying $\psi(x)$ by $x^{s/2-1}$ and integrating over $(0, \infty)$. Termwise integration then produces the powers n^{-s} that occur in the Dirichlet series for $\zeta(s)$.

Proposition 2.12 (Mellin representation)**.** *For* $\mathrm{Re}(s) > 1$,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^{\infty} \psi(x) x^{s/2-1} \, \mathrm{d}x.$$

Proof. Write $s = \sigma + it$ with $\sigma > 1$. We first justify termwise integration. Since

$$\sum_{n=1}^{\infty} \int_0^{\infty} \left| e^{-\pi n^2 x} x^{s/2-1} \right| \, \mathrm{d}x = \Gamma\left(\frac{\sigma}{2}\right) \pi^{-\sigma/2} \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < \infty,$$

Fubini's theorem allows us to interchange summation and integration. Hence

$$\begin{aligned}
\int_0^{\infty} \psi(x) x^{s/2-1} \, \mathrm{d}x &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 x} x^{s/2-1} \, \mathrm{d}x \\
&= \sum_{n=1}^{\infty} (\pi n^2)^{-s/2} \int_0^{\infty} e^{-y} y^{s/2-1} \, \mathrm{d}y \\
&= \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} (\pi n^2)^{-s/2} \\
&= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).
\end{aligned}$$

In the second equality we used the substitution $y = \pi n^2 x$, and in the third equality we used Definition 2.9. □

Proposition 2.13 (Symmetrized Mellin formula)**.** *For* $\mathrm{Re}(s) > 1$,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \psi(x) \left(x^{s/2-1} + x^{-(s+1)/2} \right) \mathrm{d}x.$$

The right-hand side gives the meromorphic continuation of the left-hand side to $\mathbb{C}$.

Proof. Start from Proposition 2.12 and split the Mellin integral at $x = 1$. On $(0, 1)$ we use the identity derived after Definition 2.11:

$$\psi(x) = \frac{1}{2}(x^{-1/2} - 1) + x^{-1/2}\psi(1/x).$$

Then

$$\int_0^1 \psi(x)x^{s/2-1}\,\mathrm{d}x = \frac{1}{2}\int_0^1 (x^{-1/2} - 1)x^{s/2-1}\,\mathrm{d}x + \int_0^1 x^{-1/2}\psi(1/x)x^{s/2-1}\,\mathrm{d}x.$$

The first term equals

$$\frac{1}{2}\left(\int_0^1 x^{(s-1)/2-1}\,\mathrm{d}x - \int_0^1 x^{s/2-1}\,\mathrm{d}x\right) = \frac{1}{2}\left(\frac{2}{s-1} - \frac{2}{s}\right) = \frac{1}{s(s-1)}.$$

In the remaining integral, substitute $u = 1/x$. Since $\mathrm{d}x = -u^{-2}\,\mathrm{d}u$, we get

$$\int_0^1 x^{-1/2}\psi(1/x)x^{s/2-1}\,\mathrm{d}x = \int_1^\infty \psi(u)u^{-(s+1)/2}\,\mathrm{d}u.$$

Combining this with the integral over $(1, \infty)$ gives the formula.

Finally, for $x \geq 1$ we have

$$0 \leq \psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} \leq \sum_{n=1}^\infty e^{-\pi n x} = \frac{e^{-\pi x}}{1 - e^{-\pi x}},$$

so $\psi(x)$ decays exponentially as $x \to \infty$. This decay dominates the powers of x locally uniformly for s in compact subsets of $\mathbb{C}$, and therefore the integral over $(1, \infty)$ is entire in s. □

Definition 2.14 (Completed zeta function on $\mathrm{Re}(s) > 1$)**.** For $\mathrm{Re}(s) > 1$, define

$$\xi_0(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

This is well defined, since $\zeta(s)$ is defined there by Definition 2.1 and $\Gamma(s/2)$ is defined by Definition 2.9.

Theorem 2.15 (Analytic continuation and functional equation)**.** *The function ξ_0 from Definition 2.14 extends uniquely to an entire function on $\mathbb{C}$. This extension is denoted by ξ and is called the completed Riemann zeta function. For every $s \in \mathbb{C}$, it satisfies*

$$\xi(s) = \xi(1-s).$$

Moreover, for every $s \in \mathbb{C}$,

$$\xi(s) = \frac{1}{2} + \frac{1}{2}s(s-1)\int_1^\infty \psi(x)\left(x^{s/2-1} + x^{-(s+1)/2}\right)\mathrm{d}x.$$

Proof. By Proposition 2.13, the factor $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$, with only the displayed possible poles at $s = 0$ and $s = 1$. The integral in the statement is entire by the same proposition. Multiplication by $\frac{1}{2}s(s-1)$ removes these two displayed poles, so the formula in the statement defines an entire extension of ξ_0. This extension is unique by the identity theorem. Replacing s by $1-s$ interchanges the two powers of x, and $(1-s)((1-s)-1) = s(s-1)$. Hence the formula is unchanged, so $\xi(s) = \xi(1-s)$. □

Remark 2.16 (Critical strip, critical line and recentering). The Euler product representation shows that $\zeta(s)$ has no zeros for $\mathrm{Re}(s) > 1$. We also use, without proof, the classical fact that $\zeta(s)$ has no zeros on the line $\mathrm{Re}(s) = 1$; see [6, Chapter 4, Section 4.2] and [9, Chapter III, Sections 3.2–3.3]. Since the pole of ζ at $s = 1$ is cancelled in ξ and Theorem 2.15 gives $\xi(1) = 1/2$, there are no zeros of ξ on $\mathrm{Re}(s) = 1$. If $\mathrm{Re}(s) = 0$, then $\mathrm{Re}(1 - s) = 1$, so the functional equation also excludes zeros on $\mathrm{Re}(s) = 0$.

The zeros of ζ outside the critical strip are the trivial zeros at the negative even integers. In the completed function, these trivial zeros cancel the poles of the Gamma factor. Thus the zeros of ξ correspond to the non-trivial zeros of ζ, namely the zeros in

$$0 < \mathrm{Re}(s) < 1.$$

The line

$$\mathrm{Re}(s) = \frac{1}{2}$$

is called the critical line. The functional equation sends a zero at s to a zero at $1-s$. Moreover, the integral formula in Theorem 2.15 gives

$$\xi(\bar{s}) = \overline{\xi(s)}.$$

Together, these symmetries single out the critical line as the central symmetry line of the strip.

To study zeros relative to this line, write $s = 1/2 + \mathrm{i}w$. Then s lies on the critical line exactly when $w \in \mathbb{R}$. Therefore the Riemann Hypothesis can be reformulated as the statement that all zeros of the recentered function

$$w \mapsto \xi\left(\frac{1}{2} + \mathrm{i}w\right)$$

are real. The cosine-transform representation introduced next is useful because it is written directly in this recentered variable.

Definition 2.17 (Theta kernel)**.** For $n \in \mathbb{N}_{\geq 1}$ and $u \geq 0$, define

$$\phi_n(u) := \left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}, \qquad \Phi(u) := \sum_{n=1}^{\infty} \phi_n(u).$$

The normalization below is the one used in the de Bruijn–Newman setting; see [5] and [7].

Theorem 2.18 (Theta-kernel cosine-transform formula)**.** *With the de Bruijn–Newman normalization*

$$\Xi_{\mathrm{dBN}}(z) := \frac{1}{8}\xi\left(\frac{1}{2} + \frac{\mathrm{i}z}{2}\right),$$

one has

$$\Xi_{\mathrm{dBN}}(z) = \int_0^{\infty} \Phi(u)\cos(zu)\ \mathrm{d}u.$$

Equivalently, with $\Xi_{\mathrm{line}}(t) := \xi(1/2 + \mathrm{i}t)$*,*

$$\Xi_{\mathrm{line}}(t) = 8\int_0^{\infty} \Phi(u)\cos(2tu)\ \mathrm{d}u.$$

Proof. Set $s = 1/2+\mathrm{i}w$ in the integral formula from Theorem 2.15. Then $s(s-1) = -(w^2+1/4)$, and

$$x^{s/2-1} = x^{-3/4}e^{\mathrm{i}(w/2)\log x}, \qquad x^{-(s+1)/2} = x^{-3/4}e^{-\mathrm{i}(w/2)\log x}.$$

Thus

$$x^{s/2-1} + x^{-(s+1)/2} = 2x^{-3/4} \cos\left(\frac{w}{2}\log x\right).$$

Substituting these identities into Theorem 2.15 gives

$$\xi\left(\frac{1}{2} + \mathrm{i}w\right) = \frac{1}{2} - \left(w^2 + \frac{1}{4}\right) \int_1^\infty \psi(x) x^{-3/4} \cos\left(\frac{w}{2}\log x\right)\, \mathrm{d}x.$$

Now put $x = e^{4u}$. Since $\mathrm{d}x = 4e^{4u}\,\mathrm{d}u$ and $x^{-3/4}\,\mathrm{d}x = 4e^{u}\,\mathrm{d}u$, with $h(u) := e^u \psi(e^{4u})$ we obtain

$$\xi\left(\frac{1}{2} + \mathrm{i}w\right) = \frac{1}{2} - 4\left(w^2 + \frac{1}{4}\right) \int_0^\infty h(u) \cos(2wu)\, \mathrm{d}u.$$

Using

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} - 1\right) \cos(2wu) = -4\left(w^2 + \frac{1}{4}\right) \cos(2wu),$$

we can transfer the polynomial factor in w onto the cosine. Indeed, integrating by parts twice gives

$$\xi\left(\frac{1}{2} + \mathrm{i}w\right) = \frac{1}{2} + h'(0) + \int_0^\infty (h''(u) - h(u)) \cos(2wu)\, \mathrm{d}u.$$

The boundary terms at ∞ vanish because h, h' and h'' decay faster than any exponential as $u \to \infty$; this follows from the factor $e^{-\pi n^2 e^{4u}}$ in the series for $\psi(e^{4u})$. At $u = 0$, the sine boundary term vanishes since $\sin(0) = 0$, while the cosine boundary term gives $h'(0)$.

It remains to compute $h'(0)$. The series for $\theta(x)$ and its derivative series converge locally uniformly on $(0, \infty)$, so

$$\theta'(x) = \sum_{n\in\mathbb{Z}} (-\pi n^2) e^{-\pi n^2 x}.$$

Differentiating the theta transformation law from Proposition 2.8 gives

$$\theta'(x) = -\frac{1}{2} x^{-3/2} \theta(1/x) - x^{-5/2} \theta'(1/x).$$

At $x = 1$, this gives $\theta'(1) = -\theta(1)/4$. Since $\theta = 1 + 2\psi$, we have $\psi = (\theta - 1)/2$ and $\psi' = \theta'/2$. Hence

$$h'(0) = \psi(1) + 4\psi'(1) = \frac{\theta(1) - 1}{2} + 2\theta'(1) = -\frac{1}{2}.$$

Thus the constant $1/2$ cancels with $h'(0)$.

Finally, termwise differentiation of $h(u) = \sum_{n=1}^\infty e^u e^{-\pi n^2 e^{4u}}$ is allowed twice on compact u-intervals, because the differentiated series converge locally uniformly. For $h_n(u) := e^u e^{-\pi n^2 e^{4u}}$ and $A_n(u) := \pi n^2 e^{4u}$, one has $h_n'(u) = h_n(u)(1 - 4A_n(u))$ and

$$h_n''(u) - h_n(u) = 8\left(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}\right) e^{-\pi n^2 e^{4u}}.$$

Summing over n and using Definition 2.17, we obtain $h''(u) - h(u) = 8\Phi(u)$. Therefore

$$\xi\left(\frac{1}{2} + \mathrm{i}w\right) = 8 \int_0^\infty \Phi(u) \cos(2wu)\, \mathrm{d}u.$$

Taking $z = 2w$ gives the de Bruijn–Newman normalization, and taking $w = t$ gives the line convention. □

This section shows that the Riemann Hypothesis can be reformulated as the statement that all zeros of the recentered completed zeta function Ξ_{dBN} are real. The final formula writes Ξ_{dBN} as a cosine transform with theta kernel Φ. In the next section, we introduce the truncated theta-kernel models $F_{a,N}$ by truncating this integral and replacing Φ by its partial kernel Φ_N.

3 Truncated theta-kernel models

The theta-kernel cosine-transform formula from Theorem 2.18 gives

$$\Xi_{\text{dBN}}(z) = \int_0^\infty \Phi(u)\cos(zu)\ \mathrm{d}u.$$

This full integral formula is difficult to study directly, so we truncate it in two ways: the infinite kernel Φ is replaced by a finite partial kernel, and the integral is cut off at a finite point. Thus, for $a > 0$ and $N \in \mathbb{N}_{\geq 1}$, define

$$\Phi_N(u) := \sum_{n=1}^{N} \phi_n(u), \qquad F_{a,N}(z) := \int_0^a \Phi_N(u)\cos(zu)\ \mathrm{d}u.$$

Here N specifies how many kernel terms are kept, while a specifies where the integral is cut off. The functions $F_{a,N}$ are the truncated theta-kernel models. Since Φ_N is a finite sum of continuous functions on $[0, a]$, the integrand $u \mapsto \Phi_N(u)\cos(zu)$ is continuous on $[0, a]$ for each fixed $z \in \mathbb{C}$. Thus the integral defining $F_{a,N}(z)$ is finite for every $z \in \mathbb{C}$, so $F_{a,N}$ is defined on all of $\mathbb{C}$.

The truncated theta-kernel models still approximate Ξ_{dBN}. Below we prove that, along every sequence with $a \to \infty$ and $N \to \infty$, the functions $F_{a,N}$ converge locally uniformly to Ξ_{dBN}. This local uniform convergence will later be used to compare zero counts between $F_{a,N}$ and Ξ_{dBN}.

Proposition 3.1 (Integrability of the theta kernel)**.** *For every $R \geq 0$,*

$$\int_0^\infty e^{Ru} \sum_{n=1}^{\infty} |\phi_n(u)|\ \mathrm{d}u < \infty.$$

In particular, $e^{Ru}\Phi(u)$ is absolutely integrable on $[0, \infty)$.

Proof. There exists a constant $C > 0$ such that, for all $n \in \mathbb{N}_{\geq 1}$ and $u \geq 0$,

$$|\phi_n(u)| \leq C(n^4 + n^2)e^{9u}e^{-\pi n^2 e^{4u}}.$$

Indeed, this follows directly from the definition of ϕ_n by bounding the constants $2\pi^2$ and 3π. For all $n \geq 1$ and $u \geq 0$, we have

$$n^2 e^{4u} \geq \frac{1}{2}n^2 + \frac{1}{2}e^{4u}.$$

Hence

$$e^{-\pi n^2 e^{4u}} \leq e^{-\pi n^2/2}e^{-\pi e^{4u}/2}.$$

Therefore

$$e^{Ru}|\phi_n(u)| \leq C(n^4 + n^2)e^{(R+9)u}e^{-\pi n^2/2}e^{-\pi e^{4u}/2}.$$

Summing from $n = 1$ to ∞ gives

$$e^{Ru}\sum_{n=1}^{\infty} |\phi_n(u)| \leq Ce^{(R+9)u}e^{-\pi e^{4u}/2}\sum_{n=1}^{\infty}(n^4 + n^2)e^{-\pi n^2/2}.$$

The series

$$\sum_{n=1}^{\infty}(n^4 + n^2)e^{-\pi n^2/2}$$

converges since the exponential term dominates, so it can be absorbed into the constant. Thus

$$e^{Ru}\sum_{n=1}^{\infty}|\phi_n(u)| \le C_R e^{(R+9)u}e^{-\pi e^{4u}/2}.$$

The right-hand side is integrable on $[0,\infty)$, since $e^{-\pi e^{4u}/2}$ decays faster than any exponential. □

The functions $F_{a,N}$ also have a compactly supported angular Fourier representation. Define

$$f_{a,N}(u) := \frac{1}{2}\Phi_N(|u|)\mathbf{1}_{[-a,a]}(u).$$

Then $f_{a,N}$ is real, even, and compactly supported, and

$$F_{a,N}(z) = \int_{-a}^{a} f_{a,N}(u)e^{\mathrm{i}zu}\,\mathrm{d}u.$$

Indeed, writing $e^{\mathrm{i}zu} = \cos(zu) + \mathrm{i}\sin(zu)$, the sine term is odd and integrates to zero. With our Fourier convention, $F_{a,N}(z) = \widehat{f}_{a,N}(-z/(2\pi))$, so z plays the role of an angular frequency.

This representation gives the basic analytic properties of $F_{a,N}$. Since the integrand is entire in z and the interval of integration is finite, differentiation under the integral sign shows that $F_{a,N}$ is entire. It is also even, because the cosine is even in z, and it takes real values on the real line, because $F_{a,N}(x) \in \mathbb{R}$ for real x.

Definition 3.2 (Cofinal sequence)**.** A sequence (a_j, N_j) is called cofinal if $a_j \to \infty$ and $N_j \to \infty$.

Proposition 3.3 (Local uniform convergence)**.** *For every cofinal sequence (a_j, N_j),*

$$F_{a_j,N_j} \longrightarrow \Xi_{\mathrm{dBN}}$$

locally uniformly on $\mathbb{C}$.

Proof. Let $K \subset \mathbb{C}$ be compact and choose $R \ge 0$ such that $|\operatorname{Im} z| \le R$ for all $z \in K$. By Theorem 2.18,

$$F_{a_j,N_j}(z) - \Xi_{\mathrm{dBN}}(z) = \int_0^{\infty}\Big(\Phi_{N_j}(u)\mathbf{1}_{[0,a_j]}(u) - \Phi(u)\Big)\cos(zu)\,\mathrm{d}u.$$

For $z \in K$,

$$\Big|\Big(\Phi_{N_j}(u)\mathbf{1}_{[0,a_j]}(u) - \Phi(u)\Big)\cos(zu)\Big| \le e^{Ru}\sum_{n=1}^{\infty}|\phi_n(u)|.$$

The right-hand side is integrable by Proposition 3.1. Since $\Phi_{N_j}(u)\mathbf{1}_{[0,a_j]}(u) \to \Phi(u)$ pointwise and

$$\Big|\Phi_{N_j}(u)\mathbf{1}_{[0,a_j]}(u) - \Phi(u)\Big| \le \sum_{n=1}^{\infty}|\phi_n(u)|,$$

dominated convergence gives

$$\int_0^{\infty} e^{Ru}\Big|\Phi_{N_j}(u)\mathbf{1}_{[0,a_j]}(u) - \Phi(u)\Big|\,\mathrm{d}u \to 0.$$

Hence

$$\sup_{z\in K}|F_{a_j,N_j}(z) - \Xi_{\mathrm{dBN}}(z)| \to 0,$$

so the convergence is locally uniform on $\mathbb{C}$.

□

To turn local uniform convergence into a statement about zeros, we use the following form of Rouché's theorem.

Theorem 3.4 (Rouché's theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain with boundary $\partial\Omega$, and let f and g be holomorphic on a neighbourhood of $\overline{\Omega}$. If*

$$|g(z)| < |f(z)|, \qquad z \in \partial\Omega,$$

then f and $f + g$ have the same zero count in Ω, counted with multiplicity.

For a proof, see [1, Chapter 4, Section 5.2].

Proposition 3.5 (Zero transfer)**.** *Let (a_j, N_j) be a cofinal sequence. If every F_{a_j,N_j} has only real zeros, then Ξ_{dBN} has only real zeros.*

Proof. By Proposition 3.3, $F_{a_j,N_j} \to \Xi_{\mathrm{dBN}}$ locally uniformly. Suppose that Ξ_{dBN} has a non-real zero z_0. Since $\Xi_{\mathrm{dBN}} \not\equiv 0$, its zeros are isolated, so we can choose $r > 0$ such that $\overline{D(z_0, r)} \cap \mathbb{R} = \varnothing$ and Ξ_{dBN} has no zeros on $\partial D(z_0, r)$. Hence $\min_{\partial D(z_0,r)} |\Xi_{\mathrm{dBN}}| > 0$, so local uniform convergence gives, for all sufficiently large j,

$$|F_{a_j,N_j} - \Xi_{\mathrm{dBN}}| < |\Xi_{\mathrm{dBN}}| \qquad \text{on } \partial D(z_0, r).$$

Applying Theorem 3.4 with $f = \Xi_{\mathrm{dBN}}$ and $g = F_{a_j,N_j} - \Xi_{\mathrm{dBN}}$, we find that F_{a_j,N_j} and Ξ_{dBN} have the same zero count in $D(z_0, r)$, counted with multiplicity. Thus F_{a_j,N_j} has a zero in $D(z_0, r)$ for all sufficiently large j. This zero is non-real, contradicting the assumption that F_{a_j,N_j} has only real zeros. □

The next section builds the de Branges Hilbert-space framework for a conditional real-zero argument for $F_{a,N}$.

4 Hermite–Biehler functions and de Branges spaces

In the previous section the truncated theta-kernel models were written in angular Fourier form:

$$F_{a,N}(z) = \int_{-a}^{a} f_{a,N}(u)e^{\mathrm{i}zu}\, \mathrm{d}u.$$

Fourier representations are closely tied to Hilbert spaces: inner products and orthogonality provide a way to compare different frequency components. This section builds the Hilbert-space framework needed for the later de Branges construction. We first introduce Hilbert spaces and Hardy spaces, then define the Hermite–Biehler class and de Branges spaces. For the truncated theta-kernel model, the relevant function will be

$$E_{a,N,\tau} := F_{a,N} + \mathrm{i}\tau F'_{a,N}, \qquad \tau > 0,$$

whose real part is $F_{a,N}$. The condition $E_{a,N,\tau} \in \mathcal{HB}$ will be the central assumption in the conditional argument.

Definition 4.1 (Hilbert space)**.** A Hilbert space is a vector space $\mathcal{H}$ over $\mathbb{R}$ or $\mathbb{C}$ with an inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ such that the norm

$$\|f\|_{\mathcal{H}} := \sqrt{\langle f, f\rangle_{\mathcal{H}}}$$

makes $\mathcal{H}$ complete. Completeness means that every Cauchy sequence in $\mathcal{H}$ converges to an element of $\mathcal{H}$.

Remark 4.2. The relevant Hilbert spaces are complex Hilbert spaces, since Hardy spaces and de Branges spaces consist of holomorphic or entire functions.

To define the de Branges space, two ingredients are needed first: the Hardy space $H^2(\mathbb{C}_+)$ and the Hermite–Biehler class. The upper half-plane appears here because its boundary is the real line. In the previous section the real variable z was interpreted as a frequency variable, so the real line is the frequency axis used here. The Hardy space allows us to study holomorphic functions above this axis, while still measuring their boundary values with an L^2 norm on $\mathbb{R}$.

Definition 4.3 (Hardy space $H^2(\mathbb{C}_+)$)**.** Let

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

be the upper half-plane. The Hardy space $H^2(\mathbb{C}_+)$ consists of all holomorphic functions $G : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|G\|^2_{H^2(\mathbb{C}_+)} := \sup_{y>0} \int_{\mathbb{R}} |G(x + \mathrm{i}y)|^2 \, \mathrm{d}x < \infty.$$

Remark 4.4. This norm gives uniform L^2 control of the horizontal slices $G(\cdot + \mathrm{i}y)$ as $y > 0$ varies; see [4, Chapter 1]. The boundary value theorem below turns this upper half-plane control into an $L^2(\mathbb{R})$ boundary norm.

Theorem 4.5 (Boundary values for $H^2(\mathbb{C}_+)$)**.** *If $G \in H^2(\mathbb{C}_+)$, then there exists a boundary function $G^* \in L^2(\mathbb{R})$ such that*

$$G(\cdot + \mathrm{i}y) \to G^* \qquad \textit{in } L^2(\mathbb{R}) \textit{ as } y \downarrow 0.$$

Moreover,

$$\|G\|_{H^2(\mathbb{C}_+)} = \|G^*\|_{L^2(\mathbb{R})}.$$

Thus $H^2(\mathbb{C}_+)$ can be identified isometrically with a closed subspace of $L^2(\mathbb{R})$.

Proof. This is the boundary value theorem for Hardy spaces on the upper half-plane; see [4, Chapter 1]. □

Remark 4.6. Hardy spaces are useful here because they combine holomorphic functions on the upper half-plane with an L^2 boundary norm on the real line. This fits the Fourier point of view from Section 3: the real line is the frequency axis for the angular Fourier representation of $F_{a,N}$. Thus Hardy spaces provide the analytic link between the truncated theta-kernel model and the de Branges Hilbert-space construction introduced below.

Definition 4.7 (Conjugate reflection)**.** For an entire function E, define

$$E^{\#}(z) := \overline{E(\overline{z})}.$$

If $E^{\#} = E$, then E is called real entire.

The functions $F_{a,N}$ from Section 3 are real entire in this sense.

Definition 4.8 (Hermite–Biehler class)**.** Following de Branges [4, Chapter 2], an entire function E belongs to the Hermite–Biehler class $\mathcal{HB}$ if E has no real zeros and

$$\left|E^{\#}(z)\right| < |E(z)|, \qquad \mathrm{Im}(z) > 0.$$

Remark 4.9. A first attempt would be to use $F_{a,N}$ itself as the Hermite–Biehler function. However, this does not work. Since $F_{a,N}$ is real entire, we have $F^{\#}_{a,N} = F_{a,N}$. But the Hermite–Biehler inequality would become

$$|F_{a,N}(z)| < |F_{a,N}(z)|, \qquad \mathrm{Im}(z) > 0,$$

which is impossible. Thus a real entire function alone cannot be a Hermite–Biehler function, and an imaginary part has to be added to $F_{a,N}$.

Definition 4.10 (The function $E_{a,N,\tau}$)**.** For $a > 0$, $N \in \mathbb{N}_{\geq 1}$ and $\tau > 0$, define

$$E_{a,N,\tau}(z) := F_{a,N}(z) + \mathrm{i}\tau F'_{a,N}(z).$$

Since $F_{a,N}$ is real entire, we have $F^{\#}_{a,N} = F_{a,N}$. Differentiating this identity gives $(F'_{a,N})^{\#} = F'_{a,N}$, so $F'_{a,N}$ is also real entire. Hence

$$E^{\#}_{a,N,\tau}(z) = F_{a,N}(z) - \mathrm{i}\tau F'_{a,N}(z).$$

Remark 4.11. The derivative is added as an imaginary part because the Hermite–Biehler inequality for

$$F_{a,N}(z) + \mathrm{i}\tau F'_{a,N}(z)$$

can be rewritten, away from the zeros of $F_{a,N}$, as a sign condition on the logarithmic derivative

$$\frac{F'_{a,N}}{F_{a,N}}.$$

This quotient is closely related to zero-counting: in the argument principle, the logarithmic derivative is integrated around a contour to count zeros, including multiplicities. In the present argument we do not use it directly to count zeros; rather, it gives a local algebraic form of the Hermite–Biehler inequality for the truncated theta-kernel model, away from the zeros of $F_{a,N}$.

Assumption 4.12 (Assumption on $E_{a,N,\tau}$)**.** For the function $E_{a,N,\tau}$, assume that

$$E_{a,N,\tau} \in \mathcal{HB}.$$

By the definition of $\mathcal{HB}$, this means two things. First, $E_{a,N,\tau}$ has no real zeros. Second, it satisfies the Hermite–Biehler inequality

$$\left|F_{a,N}(z) - \mathrm{i}\tau F'_{a,N}(z)\right| < \left|F_{a,N}(z) + \mathrm{i}\tau F'_{a,N}(z)\right|, \qquad z \in \mathbb{C}_+.$$

The next proposition makes the Hermite–Biehler inequality more concrete. It is not an additional assumption, but only an algebraic rewriting of the inequality away from the zeros of $F_{a,N}$. The full assumption remains $E_{a,N,\tau} \in \mathcal{HB}$.

Proposition 4.13 (Logarithmic derivative form)**.** *For $z \in \mathbb{C}_+$ with $F_{a,N}(z) \neq 0$, the Hermite–Biehler inequality*

$$\left|E^{\#}_{a,N,\tau}(z)\right| < |E_{a,N,\tau}(z)|$$

is equivalent to

$$\operatorname{Im} \frac{F'_{a,N}(z)}{F_{a,N}(z)} < 0.$$

Proof. Write

$$m(z) := \frac{F'_{a,N}(z)}{F_{a,N}(z)}.$$

Since $F_{a,N}(z) \neq 0$, we may divide the Hermite–Biehler inequality by $|F_{a,N}(z)|$. Using

$$E_{a,N,\tau} = F_{a,N}(1 + \mathrm{i}\tau m), \qquad E^{\#}_{a,N,\tau} = F_{a,N}(1 - \mathrm{i}\tau m),$$

the inequality becomes

$$|1 - \mathrm{i}\tau m(z)| < |1 + \mathrm{i}\tau m(z)|.$$

Write $m(z) = \alpha + \mathrm{i}\beta$, with $\alpha, \beta \in \mathbb{R}$. Then

$$|1 - \mathrm{i}\tau m|^2 = (1 + \tau\beta)^2 + \tau^2\alpha^2, \qquad |1 + \mathrm{i}\tau m|^2 = (1 - \tau\beta)^2 + \tau^2\alpha^2.$$

Hence the inequality is equivalent to

$$(1 + \tau\beta)^2 < (1 - \tau\beta)^2,$$

which is equivalent to $\beta < 0$. Since $\beta = \operatorname{Im} m(z)$, this proves the claim. □

Remark 4.14 (Strength of the assumption)*.* Proposition 4.13 does not replace Assumption 4.12. It only rewrites the strict inequality at points where $F_{a,N}$ is non-zero. The full Hermite–Biehler assumption is stronger, because it also includes the condition that $E_{a,N,\tau}$ has no real zeros.

Indeed, if $F_{a,N}$ had a zero in $\mathbb{C}_+$, then at that zero the two sides of the Hermite–Biehler inequality would have the same modulus, which is impossible. Since $F_{a,N}$ is real entire, non-real zeros occur in conjugate pairs. Hence the Hermite–Biehler inequality excludes all non-real zeros of $F_{a,N}$.

The no-real-zero condition for $E_{a,N,\tau}$ also has a consequence on the real line. Since $F_{a,N}$ and $F'_{a,N}$ are real-valued on $\mathbb{R}$, a real zero of $E_{a,N,\tau}$ is exactly a point $x \in \mathbb{R}$ where

$$F_{a,N}(x) = F'_{a,N}(x) = 0.$$

Thus Assumption 4.12 also excludes multiple real zeros of $F_{a,N}$. In particular, the assumption already forces $F_{a,N}$ to have only real, simple zeros.

This is a very strong assumption. If it could be proved along a cofinal sequence of truncated theta-kernel models, then the finite real-zero property needed in Proposition 3.5 would already follow. Together with the zero-transfer argument from Section 3, this would imply the Riemann Hypothesis. Therefore any conclusion about the Riemann Hypothesis based only on Assumption 4.12 would be circular unless the assumption is proved independently.

From now on, whenever we use the de Branges space $\mathcal{H}(E_{a,N,\tau})$, we assume the full condition

$$E_{a,N,\tau} \in \mathcal{HB}$$

from Assumption 4.12, not only the logarithmic-derivative inequality from Proposition 4.13.

Definition 4.15 (de Branges space)**.** Let $E \in \mathcal{HB}$. Following de Branges [4, Chapter 2], the de Branges space $\mathcal{H}(E)$ consists of all entire functions G such that

$$\frac{G}{E} \in H^2(\mathbb{C}_+), \qquad \frac{G^\#}{E} \in H^2(\mathbb{C}_+).$$

Its norm is

$$\|G\|^2_{\mathcal{H}(E)} := \int_{\mathbb{R}} \left| \frac{G(x)}{E(x)} \right|^2 \, \mathrm{d}x.$$

Equivalently, the inner product is

$$\langle G, H \rangle_{\mathcal{H}(E)} := \int_{\mathbb{R}} \frac{G(x)\overline{H(x)}}{|E(x)|^2} \, \mathrm{d}x.$$

Remark 4.16. The conditions $G/E \in H^2(\mathbb{C}_+)$ and $G^\#/E \in H^2(\mathbb{C}_+)$ say that both G and its reflection are controlled in the upper half-plane after normalization by E. The norm is an L^2 norm on the real line with weight $1/|E|^2$, so E determines the weight, norm, and inner product of the de Branges space. The next proposition shows that this norm is also complete.

Proposition 4.17 (Hilbert-space structure)**.** *If $E \in \mathcal{HB}$, then $\mathcal{H}(E)$ is a Hilbert space with the inner product above.*

Proof. The inner product is well-defined because a Hermite–Biehler function has no real zeros. It is positive, since

$$\|G\|^2_{\mathcal{H}(E)} = \int_{\mathbb{R}} \frac{|G(x)|^2}{|E(x)|^2} \, \mathrm{d}x \geq 0.$$

If this norm is zero, then $G/E = 0$ almost everywhere on $\mathbb{R}$. Since E has no real zeros, it follows that $G = 0$ almost everywhere on $\mathbb{R}$. The function G is continuous on $\mathbb{R}$, so it vanishes on all of $\mathbb{R}$, and therefore $G \equiv 0$ by the identity theorem.

The remaining point is completeness. This is a theorem in de Branges theory: for $E \in \mathcal{HB}$, the space defined above is complete with respect to the norm $\|G\|_{\mathcal{H}(E)} = \|G/E\|_{L^2(\mathbb{R})}$. We use this result here rather than reproducing its proof; see [4, Chapter 2, Theorem 21]. □

Remark 4.18 (Connection with the truncated theta-kernel model). For the truncated theta-kernel model, the relevant space is $\mathcal{H}(E_{a,N,\tau})$, where $E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}$. Assuming $E_{a,N,\tau} \in \mathcal{HB}$, the function $F_{a,N}$ is the real part of a Hermite–Biehler function. This brings the truncated theta-kernel model into the de Branges setting. The next section explains how multiplication by the variable z acts on functions in this space, and why this is useful for studying the zeros of $F_{a,N}$.

5 Reproducing kernels and the multiplication operator

In the previous section we constructed, under the assumption $E_{a,N,\tau} \in \mathcal{HB}$, the de Branges space

$$\mathcal{H}(E_{a,N,\tau}), \qquad E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N},$$

associated with the truncated theta-kernel model. Two objects in this space will be used repeatedly. First, kernel functions translate point evaluation into Hilbert-space language: instead of only looking at the value of an entire function G at a point w, this value can be recovered from an inner product. Second, we will study multiplication by the variable z on functions in this space.

Definition 5.1 (Continuous linear functional)**.** Let $\mathcal{H}$ be a complex Hilbert space. A linear functional on $\mathcal{H}$ is a linear map

$$L : \mathcal{H} \to \mathbb{C}.$$

It is called continuous if there exists a constant $C > 0$ such that

$$|L(G)| \le C \, \|G\|_{\mathcal{H}} \qquad (G \in \mathcal{H}).$$

Theorem 5.2 (Riesz representation theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space, with inner product linear in the first variable. If $L : \mathcal{H} \to \mathbb{C}$ is a continuous linear functional, then there exists a unique vector $h_L \in \mathcal{H}$ such that*

$$L(G) = \langle G, h_L \rangle_{\mathcal{H}} \qquad (G \in \mathcal{H}).$$

Proof. This is the Riesz representation theorem for Hilbert spaces; see [3, Chapter I, Section 3, Theorem 3.4]. □

Definition 5.3 (Reproducing kernel Hilbert space)**.** Let $\mathcal{H}$ be a Hilbert space of complex-valued functions on a set Ω. We say that $\mathcal{H}$ is a reproducing kernel Hilbert space if, for every $w \in \Omega$, the evaluation map

$$\mathrm{ev}_w : \mathcal{H} \to \mathbb{C}, \qquad \mathrm{ev}_w(G) := G(w),$$

is a continuous linear functional.

The evaluation map is now of the same type as the linear functionals in the Riesz representation theorem: it can be represented by an inner product with a unique vector in the Hilbert space.

Proposition 5.4 (Existence of kernel functions)**.** *Let $\mathcal{H}$ be a reproducing kernel Hilbert space. Then for every $w \in \Omega$ there exists a unique function $K_w \in \mathcal{H}$ such that*

$$G(w) = \langle G, K_w \rangle_{\mathcal{H}} \qquad (G \in \mathcal{H}).$$

The function K_w is called the reproducing kernel at w.

Proof. Fix $w \in \Omega$. Since $\mathcal{H}$ is a reproducing kernel Hilbert space, the map ev_w is a continuous linear functional. By the Riesz representation theorem, there exists a unique vector $K_w \in \mathcal{H}$ such that

$$\mathrm{ev}_w(G) = \langle G, K_w \rangle_{\mathcal{H}} \qquad (G \in \mathcal{H}).$$

Since $\mathrm{ev}_w(G) = G(w)$, this gives the reproducing identity. □

Before writing down the de Branges kernel, we first recall the general Hilbert space idea behind reproducing kernels.

Remark 5.5 (Kernel interpretation). The kernel function K_w is the vector in the Hilbert space which represents evaluation at the point w. Instead of only looking at the pointwise value of an entire function at w, this value can be recovered from an inner product with K_w. Thus point evaluation becomes part of the Hilbert-space structure.

For a general reproducing kernel Hilbert space, we denoted the representing vector by K_w. In a de Branges space it is useful to write the same object as a function of two variables:

$$K_E(w,z) := K_w(z).$$

Theorem 5.6 (Reproducing kernel of a de Branges space). *Let $E \in \mathcal{HB}$. Then $\mathcal{H}(E)$ is a reproducing kernel Hilbert space. Its reproducing kernel is*

$$K_E(w,z) = \frac{\overline{E(w)}E(z) - \overline{E^\#(w)}E^\#(z)}{2\pi \mathrm{i}(\overline{w}-z)}.$$

Thus, for every $G \in \mathcal{H}(E)$,

$$G(w) = \langle G, K_E(w,\cdot)\rangle_{\mathcal{H}(E)}\,.$$

At points where $z = \overline{w}$, the formula is understood by analytic continuation.

Proof. We use the de Branges reproducing-kernel formula without proof; see [4, Chapter 2, Theorem 19] or compare it with [8, Chapter 3, Theorem 5]. □

Definition 5.7 (Kernel for the truncated theta-kernel model). For the truncated theta-kernel model, assuming $E_{a,N,\tau} \in \mathcal{HB}$, define

$$K_{a,N,\tau}(w,z) := K_{E_{a,N,\tau}}(w,z).$$

Thus

$$K_{a,N,\tau}(w,z) = \frac{\overline{E_{a,N,\tau}(w)}E_{a,N,\tau}(z) - \overline{E^\#_{a,N,\tau}(w)}E^\#_{a,N,\tau}(z)}{2\pi \mathrm{i}(\overline{w}-z)}.$$

Remark 5.8 (Meaning for the kernel of the truncated theta-kernel model). The kernel $K_{a,N,\tau}$ is the reproducing kernel of the de Branges space $\mathcal{H}(E_{a,N,\tau})$. Thus, for every function $G \in \mathcal{H}(E_{a,N,\tau})$ and every $w \in \mathbb{C}$,

$$G(w) = \langle G, K_{a,N,\tau}(w,\cdot)\rangle_{\mathcal{H}(E_{a,N,\tau})}\,.$$

Using the de Branges inner product, this means

$$G(w) = \int_{\mathbb{R}} \frac{G(x)\overline{K_{a,N,\tau}(w,x)}}{|E_{a,N,\tau}(x)|^2}\,\mathrm{d}x.$$

So the kernel translates evaluation at a point into an inner product in $\mathcal{H}(E_{a,N,\tau})$.

After introducing point evaluation, the second object is multiplication by z as this operation was suggested by the Fourier variable. To make this precise, we first introduce the basic notion of what an operator is.

Definition 5.9 (Operator). Let $\mathcal{H}$ be a complex Hilbert space. An operator S on $\mathcal{H}$ is a linear map

$$S : \mathrm{Dom}(S) \subseteq \mathcal{H} \to \mathcal{H},$$

where $\mathrm{Dom}(S)$ is a linear subspace of $\mathcal{H}$ called the domain of S.

Definition 5.10 (Densely defined operator)**.** An operator S on $\mathcal{H}$ is called densely defined if

$$\overline{\mathrm{Dom}(S)} = \mathcal{H}.$$

In the de Branges spaces used later, this means that every function in the space can be approximated in norm by functions from $\mathrm{Dom}(S)$.

Definition 5.11 (Closed operator)**.** An operator $S : \mathrm{Dom}(S) \subseteq \mathcal{H} \to \mathcal{H}$ is called closed if, whenever $G_n \in \mathrm{Dom}(S)$, $G_n \to G$ in $\mathcal{H}$ and $SG_n \to H$ in $\mathcal{H}$, then

$$G \in \mathrm{Dom}(S), \qquad SG = H.$$

Remark 5.12. Closedness says that the operator behaves well under limits: if $G_n \in \mathrm{Dom}(S)$, $G_n \to G$ and $SG_n \to H$, then $G \in \mathrm{Dom}(S)$ and $SG = H$. This will be useful when we discuss adjoints and self-adjoint extensions.

Definition 5.13 (Adjoint, symmetric and self-adjoint operators)**.** Let S be a densely defined operator on a Hilbert space $\mathcal{H}$. An element $V \in \mathcal{H}$ belongs to $\mathrm{Dom}(S^*)$ if there exists an element $W \in \mathcal{H}$ such that

$$\langle SG, V\rangle_{\mathcal{H}} = \langle G, W\rangle_{\mathcal{H}} \qquad \text{for all } G \in \mathrm{Dom}(S).$$

In that case W is unique, and we define $S^*V := W$. The operator S is called symmetric if it is densely defined and

$$\langle SG, H\rangle_{\mathcal{H}} = \langle G, SH\rangle_{\mathcal{H}} \qquad (G, H \in \mathrm{Dom}(S)).$$

The operator S is called self-adjoint if

$$\mathrm{Dom}(S) = \mathrm{Dom}(S^*)$$

and the two operators agree on this common domain. In this case we write $S = S^*$.

Remark 5.14 (Symmetric versus self-adjoint). For a symmetric operator, the identity $\langle SG, H\rangle_{\mathcal{H}} = \langle G, SH\rangle_{\mathcal{H}}$ holds for all $G, H \in \mathrm{Dom}(S)$. This does not yet mean that S is self-adjoint, because the adjoint may have a larger domain. For a self-adjoint operator, both the formula and the domain agree. This distinction is essential here: the multiplication operator will satisfy the symmetric identity on its specified domain, while real spectral conclusions require a self-adjoint operator.

Definition 5.15 (Multiplication operator)**.** Let $E \in \mathcal{HB}$. The multiplication operator S_E on $\mathcal{H}(E)$ is defined by

$$(S_E G)(z) := zG(z),$$

with domain

$$\mathrm{Dom}(S_E) := \{G \in \mathcal{H}(E) : zG(z) \in \mathcal{H}(E)\}.$$

Remark 5.16. The formula $G(z) \mapsto zG(z)$ does not automatically define an operator on all of $\mathcal{H}(E)$. Even if G has finite de Branges norm, the function $zG(z)$ may fail to have finite de Branges norm. Therefore the domain is part of the definition of the operator. Later we will see this concretely: the adjoint can have a larger domain than the original multiplication operator.

Proposition 5.17 (Basic properties of the multiplication operator)**.** *Let $E \in \mathcal{HB}$. Then S_E is linear and closed. If $\mathrm{Dom}(S_E)$ is dense in $\mathcal{H}(E)$, then S_E is symmetric.*

Proof. First we prove linearity. Let $G, H \in \mathrm{Dom}(S_E)$ and $\alpha, \beta \in \mathbb{C}$. Since $\mathcal{H}(E)$ is a vector space, $\alpha G + \beta H \in \mathcal{H}(E)$. Moreover, since $G, H \in \mathrm{Dom}(S_E)$, we have $zG, zH \in \mathcal{H}(E)$. Hence

$$z(\alpha G + \beta H) = \alpha z G + \beta z H \in \mathcal{H}(E).$$

Thus $\alpha G + \beta H \in \mathrm{Dom}(S_E)$ and

$$S_E(\alpha G + \beta H) = \alpha S_E G + \beta S_E H.$$

Next we prove closedness. Let $G_n \in \mathrm{Dom}(S_E)$ and suppose that $G_n \to G$ and $S_E G_n \to H$ in $\mathcal{H}(E)$. By the reproducing identity from Theorem 5.6, convergence in $\mathcal{H}(E)$ implies pointwise convergence. Indeed, for each $z \in \mathbb{C}$,

$$G_n(z) = \langle G_n, K_E(z, \cdot)\rangle_{\mathcal{H}(E)} \longrightarrow \langle G, K_E(z, \cdot)\rangle_{\mathcal{H}(E)} = G(z),$$

and similarly $(S_E G_n)(z) \to H(z)$. Since $(S_E G_n)(z) = zG_n(z)$, taking pointwise limits gives $H(z) = zG(z)$ for every $z \in \mathbb{C}$. Because $H \in \mathcal{H}(E)$, this means that $zG(z) \in \mathcal{H}(E)$. Hence $G \in \mathrm{Dom}(S_E)$ and $S_E G = H$. Therefore S_E is closed.

Finally, assume that $\mathrm{Dom}(S_E)$ is dense in $\mathcal{H}(E)$. For $G, H \in \mathrm{Dom}(S_E)$, using the definition of the de Branges inner product gives

$$\langle S_E G, H\rangle_{\mathcal{H}(E)} = \int_{\mathbb{R}} \frac{xG(x)\overline{H(x)}}{|E(x)|^2}\, \mathrm{d}x.$$

Since $x \in \mathbb{R}$, we have $\overline{xH(x)} = x\overline{H(x)}$. Hence

$$\int_{\mathbb{R}} \frac{xG(x)\overline{H(x)}}{|E(x)|^2}\, \mathrm{d}x = \int_{\mathbb{R}} \frac{G(x)\overline{xH(x)}}{|E(x)|^2}\, \mathrm{d}x = \langle G, S_E H\rangle_{\mathcal{H}(E)}.$$

Together with the density assumption, this shows that S_E is symmetric. □

Remark 5.18 (Multiplication by the frequency variable). The variable z is the frequency variable in the angular Fourier representation of $F_{a,N}$. Multiplication by z therefore matches the usual Fourier intuition. If $A(z) = \int_{\mathbb{R}} f(u)e^{\mathrm{i}zu}\, \mathrm{d}u$ with f smooth and compactly supported, then integration by parts gives formally

$$zA(z) = \frac{1}{\mathrm{i}} \int_{\mathbb{R}} f(u) \frac{\mathrm{d}}{\mathrm{d}u} e^{\mathrm{i}zu}\, \mathrm{d}u = -\frac{1}{\mathrm{i}} \int_{\mathbb{R}} f'(u) e^{\mathrm{i}zu}\, \mathrm{d}u.$$

So multiplication by z on the frequency side corresponds formally to differentiation in the u-variable. This is only a motivation; the function $f_{a,N}$ is compactly supported but need not be smooth.

The reason for choosing S_E is that multiplication by z uses the same variable that appeared in the angular Fourier representation of $F_{a,N}$. This is also the variable in which the zeros of $F_{a,N}$ are located. We want to relate these zeros directly to real numbers coming from a self-adjoint operator. Other operators could of course be studied, but the de Branges construction used here is built around multiplication by the independent variable z. For example, multiplication by z^2 would give different information: real squared values do not force the original values to be real.

For the truncated theta-kernel model, the construction is

$$F_{a,N} \longrightarrow E_{a,N,\tau} \longrightarrow \mathcal{H}(E_{a,N,\tau}) \longrightarrow S_{E_{a,N,\tau}}.$$

Thus the operator is part of the de Branges construction, and not an unrelated object added afterwards.

The multiplication operator has now been introduced. We have shown that it is linear and closed, and that it is symmetric whenever its domain is dense. We have not shown that it is self-adjoint. The next section studies self-adjoint extensions of this operator.

6 Deficiency indices and self-adjoint extensions

In the previous section we introduced the multiplication operator

$$(S_E G)(z) := zG(z), \qquad \mathrm{Dom}(S_E) := \{G \in \mathcal{H}(E) : zG(z) \in \mathcal{H}(E)\}.$$

We showed that S_E is linear and closed, and that it is symmetric whenever its domain is dense. The remaining issue is the domain itself: it may still be too small for self-adjointness. We therefore ask whether S_E admits a self-adjoint extension.

To study this question, one uses deficiency spaces. These spaces compare the domain of a symmetric operator with the domain of its adjoint. Von Neumann's extension theorem then gives a criterion for the existence of a self-adjoint extension.

Definition 6.1 (Deficiency spaces and deficiency indices)**.** Let S be a densely defined symmetric operator on a complex Hilbert space $\mathcal{H}$. The deficiency spaces of S are

$$\mathcal{N}_+ := \ker(S^* - \mathrm{i}I), \qquad \mathcal{N}_- := \ker(S^* + \mathrm{i}I).$$

Equivalently,

$$\mathcal{N}_+ = \{G \in \mathrm{Dom}(S^*) : S^*G = \mathrm{i}G\}, \qquad \mathcal{N}_- = \{G \in \mathrm{Dom}(S^*) : S^*G = -\mathrm{i}G\}.$$

Their dimensions

$$n_+ := \dim \mathcal{N}_+, \qquad n_- := \dim \mathcal{N}_-$$

are called the deficiency indices of S.

Remark 6.2 (Meaning of deficiency indices). The deficiency spaces measure how many extra elements appear in the adjoint domain $\mathrm{Dom}(S^*)$ at the non-real values i and $-$i. To see why this matters, suppose that S is self-adjoint and let $G \in \mathcal{N}_+$. Then $S^*G = \mathrm{i}G$, and since $S = S^*$, we get $SG = \mathrm{i}G$. Hence

$$\langle SG, G\rangle_{\mathcal{H}} = \mathrm{i}\,\|G\|_{\mathcal{H}}^2.$$

On the other hand, since S is self-adjoint, it is symmetric. Therefore

$$\langle SG, G\rangle_{\mathcal{H}} = \langle G, SG\rangle_{\mathcal{H}} = \overline{\langle SG, G\rangle_{\mathcal{H}}}.$$

So $\langle SG, G\rangle_{\mathcal{H}}$ is real. Hence $\mathrm{i}\,\|G\|_{\mathcal{H}}^2$ must be real. This is only possible if $\|G\|_{\mathcal{H}} = 0$, so $G = 0$. Thus $\mathcal{N}_+ = \{0\}$. The same argument gives $\mathcal{N}_- = \{0\}$. Hence a self-adjoint operator has deficiency indices $(0, 0)$. If the deficiency spaces are non-zero, then the adjoint has extra directions which are not present in the original domain of S. These directions describe how the domain can be enlarged when constructing self-adjoint extensions.

For the multiplication operator in a de Branges space, the extra directions in the adjoint domain can be described very concretely as they are given by reproducing kernels.

Proposition 6.3 (Kernel vectors in the adjoint domain)**.** *Let $E \in \mathcal{HB}$, and assume that the multiplication operator S_E on $\mathcal{H}(E)$ is densely defined. For every $w \in \mathbb{C}$,*

$$K_E(w, \cdot) \in \mathrm{Dom}(S_E^*), \qquad S_E^* K_E(w, \cdot) = \overline{w}\, K_E(w, \cdot).$$

Proof. Let $G \in \mathrm{Dom}(S_E)$. By the reproducing identity,

$$\langle S_E G, K_E(w, \cdot)\rangle_{\mathcal{H}(E)} = (S_E G)(w) = wG(w) = w\,\langle G, K_E(w, \cdot)\rangle_{\mathcal{H}(E)}.$$

Since the inner product is conjugate-linear in the second variable,

$$w \langle G, K_E(w,\cdot)\rangle_{\mathcal{H}(E)} = \langle G, \overline{w}\, K_E(w,\cdot)\rangle_{\mathcal{H}(E)} .$$

Hence $K_E(w,\cdot) \in \mathrm{Dom}(S_E^*)$ and

$$S_E^* K_E(w,\cdot) = \overline{w}\, K_E(w,\cdot).$$

□

Corollary 6.4 (Kernel vectors outside the original domain)**.** *Let $w \in \mathbb{C} \setminus \mathbb{R}$, and assume that S_E is densely defined and symmetric. If $K_E(w,\cdot) \not\equiv 0$, then*

$$K_E(w,\cdot) \in \mathrm{Dom}(S_E^*), \qquad K_E(w,\cdot) \notin \mathrm{Dom}(S_E).$$

Consequently,

$$\mathrm{Dom}(S_E) \neq \mathrm{Dom}(S_E^*),$$

so S_E is not self-adjoint.

Proof. The first inclusion follows from Proposition 6.3. Suppose that $K_E(w,\cdot) \in \mathrm{Dom}(S_E)$. Since S_E is symmetric, every element of $\mathrm{Dom}(S_E)$ also belongs to $\mathrm{Dom}(S_E^*)$, and the two operators agree on $\mathrm{Dom}(S_E)$. Hence

$$S_E K_E(w,\cdot) = S_E^* K_E(w,\cdot) = \overline{w}\, K_E(w,\cdot).$$

By the definition of S_E, this means that for every $z \in \mathbb{C}$,

$$z K_E(w,z) = \overline{w}\, K_E(w,z).$$

Equivalently,

$$(z - \overline{w}) K_E(w,z) = 0, \qquad z \in \mathbb{C}.$$

This identity has to hold for every $z \in \mathbb{C}$. Hence, for every $z \neq \overline{w}$, the factor $z - \overline{w}$ is non-zero, and therefore $K_E(w,z) = 0$. Thus $K_E(w,\cdot)$ vanishes on $\mathbb{C} \setminus \{\overline{w}\}$. Since $K_E(w,\cdot)$ is entire, the identity theorem gives $K_E(w,\cdot) \equiv 0$, contradicting the assumption. Therefore $K_E(w,\cdot) \notin \mathrm{Dom}(S_E)$. □

Remark 6.5 (Deficiency directions). For the multiplication operator in a de Branges space, the extra directions in the adjoint domain are described by reproducing kernels. By Proposition 6.3, we have

$$S_E^* K_E(-\mathrm{i},\cdot) = \mathrm{i} K_E(-\mathrm{i},\cdot), \qquad S_E^* K_E(\mathrm{i},\cdot) = -\mathrm{i} K_E(\mathrm{i},\cdot).$$

Therefore, by Definition 6.1,

$$K_E(-\mathrm{i},\cdot) \in \ker(S_E^* - \mathrm{i}I), \qquad K_E(\mathrm{i},\cdot) \in \ker(S_E^* + \mathrm{i}I).$$

These kernels already appeared as the vectors representing point evaluation. Here they also show how the adjoint domain can be larger than the original domain of S_E.

The two deficiency spaces now have to be paired in order to obtain a self-adjoint extension. This pairing is done by a unitary map.

Definition 6.6 (Unitary map)**.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A linear map

$$U : \mathcal{H}_1 \to \mathcal{H}_2$$

is called unitary if it is bijective and preserves the inner product:

$$\langle Uf, Ug \rangle_{\mathcal{H}_2} = \langle f, g \rangle_{\mathcal{H}_1} \qquad (f, g \in \mathcal{H}_1).$$

Equivalently, U is a bijective isometry:

$$\|Uf\|_{\mathcal{H}_2} = \|f\|_{\mathcal{H}_1}.$$

Theorem 6.7 (von Neumann extension theorem)**.** *Let S be a closed densely defined symmetric operator with deficiency spaces $\mathcal{N}_+$ and $\mathcal{N}_-$. Then S has a self-adjoint extension if and only if*

$$n_+ = n_-.$$

When this equality holds, each unitary map

$$U : \mathcal{N}_+ \to \mathcal{N}_-$$

determines one self-adjoint extension S_U, and every self-adjoint extension arises in this way. More precisely, S_U has domain

$$\mathrm{Dom}(S_U) = \mathrm{Dom}(S) \dotplus \{h_+ + Uh_+ : h_+ \in \mathcal{N}_+\},$$

and acts on this domain by

$$S_U(G + h_+ + Uh_+) = SG + \mathrm{i}h_+ - \mathrm{i}Uh_+, \qquad G \in \mathrm{Dom}(S),\ h_+ \in \mathcal{N}_+.$$

Here $\dotplus$ denotes a direct sum of subspaces.

Proof. We use von Neumann's extension theorem without proof; see [3, Chapter X, Section 2, Theorem 2.20]. □

Assumption 6.8 (Regular de Branges operator setting)**.** For the de Branges spaces used below, we assume that the multiplication operator S_E is densely defined. By Proposition 6.3 and Definition 6.1, the kernel functions satisfy

$$K_E(-\mathrm{i}, \cdot) \in \ker(S_E^* - \mathrm{i}I), \qquad K_E(\mathrm{i}, \cdot) \in \ker(S_E^* + \mathrm{i}I).$$

We assume that these inclusions are equalities after taking spans:

$$\ker(S_E^* - \mathrm{i}I) = \mathrm{span}\{K_E(-\mathrm{i}, \cdot)\}, \qquad \ker(S_E^* + \mathrm{i}I) = \mathrm{span}\{K_E(\mathrm{i}, \cdot)\}.$$

We also assume that $K_E(-\mathrm{i}, \cdot)$ and $K_E(\mathrm{i}, \cdot)$ are not identically zero.

Proposition 6.9 (Deficiency indices in the regular de Branges setting)**.** *Let $E \in \mathcal{HB}$ and assume that Assumption 6.8 holds. Then the multiplication operator S_E has deficiency indices $n_+ = n_- = 1$.*

Proof. By Assumption 6.8,

$$\mathcal{N}_+ = \mathrm{span}\{K_E(-\mathrm{i}, \cdot)\}, \qquad \mathcal{N}_- = \mathrm{span}\{K_E(\mathrm{i}, \cdot)\}.$$

Since the two kernel functions are assumed to be non-zero, both spaces are one-dimensional. Hence

$$n_+ = \dim \mathcal{N}_+ = 1, \qquad n_- = \dim \mathcal{N}_- = 1.$$

□

Remark 6.10 (Regularity assumptions for the truncated theta-kernel model). For the truncated theta-kernel model we use

$$E = E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}.$$

If $E_{a,N,\tau} \in \mathcal{HB}$, then $\mathcal{H}(E_{a,N,\tau})$ is a de Branges space. For the self-adjoint-extension argument we also need extra operator properties of S_E. We therefore assume the regular de Branges operator setting from Assumption 6.8. In particular, we assume that $\mathrm{Dom}(S_E)$ is dense and that

$$\mathcal{N}_+ = \mathrm{span}\{K_E(-\mathrm{i}, \cdot)\}, \qquad \mathcal{N}_- = \mathrm{span}\{K_E(\mathrm{i}, \cdot)\}.$$

The density assumption is motivated by the Fourier-side representation discussed in Remark 5.18. There, multiplication by the frequency variable z corresponds formally to differentiating in the u-variable. For smooth compactly supported Fourier data this differentiation stays within the same $L^2(-a, a)$-type setting, and such test functions are dense in $L^2(-a, a)$. This gives a reason to expect the multiplication domain to contain many elements.

This argument is only heuristic. The de Branges norm is not simply the $L^2(-a, a)$-norm, and the actual function $f_{a,N}$ has both the absolute value at $u = 0$ and the cutoff at $\pm a$. Thus the density of $\mathrm{Dom}(S_E)$, as well as the description of the deficiency spaces by $K_E(-\mathrm{i}, \cdot)$ and $K_E(\mathrm{i}, \cdot)$, is kept as a separate assumption.

Once this regular operator setting is assumed, the two deficiency spaces are one-dimensional. A unitary map between them is then determined by one complex number of modulus one, which gives the one-phase family of self-adjoint extensions.

Corollary 6.11 (Self-adjoint extensions of the truncated theta-kernel multiplication operator)**.** *Assume $E_{a,N,\tau} \in \mathcal{HB}$ and assume that the regular de Branges operator setting from Assumption 6.8 holds for $E_{a,N,\tau}$. Then the self-adjoint extensions of $S_{E_{a,N,\tau}}$ are parametrized by one phase*

$$e^{\mathrm{i}\theta}, \qquad \theta \in [0, 2\pi).$$

Proof. By Proposition 5.17, the operator $S_{E_{a,N,\tau}}$ is closed. Moreover, by Assumption 6.8, its domain is dense, so Proposition 5.17 also shows that $S_{E_{a,N,\tau}}$ is symmetric. By Proposition 6.9, its deficiency indices are

$$n_+ = n_- = 1.$$

Therefore Theorem 6.7 applies. Choose normalized generators $e_+ \in \mathcal{N}_+$ and $e_- \in \mathcal{N}_-$. Since both deficiency spaces are one-dimensional, a unitary map $U : \mathcal{N}_+ \to \mathcal{N}_-$ is completely determined by the image of e_+. Because U preserves norms, Ue_+ must be a unit vector in the one-dimensional space $\mathcal{N}_-$. Hence there exists a phase $e^{\mathrm{i}\theta}$, with $\theta \in [0, 2\pi)$, such that

$$Ue_+ = e^{\mathrm{i}\theta} e_-.$$

Conversely, each choice of such a phase defines a unitary map $U : \mathcal{N}_+ \to \mathcal{N}_-$. By Theorem 6.7, each such unitary map determines a self-adjoint extension. □

For the truncated theta-kernel model, the construction has now reached

$$F_{a,N} \longrightarrow E_{a,N,\tau} \longrightarrow \mathcal{H}(E_{a,N,\tau}) \longrightarrow S_{E_{a,N,\tau}} \longrightarrow \text{self-adjoint extensions.}$$

The extensions are parametrized by one phase. The next section introduces the spectrum of these extensions and explains how it is connected to zeros of real entire functions.

7 Spectrum and zeros of the truncated theta-kernel model

In the previous section, we obtained self-adjoint extensions of the multiplication operator S_E under the regular de Branges operator setting. We now study the spectra of these extensions. The de Branges spectral theorem describes these spectra through zeros of real entire functions. For the truncated theta-kernel model, the choice $\beta = \pi/2$ gives the spectral function $-F_{a,N}$. Thus this section explains how the zeros of $F_{a,N}$ obtain a spectral interpretation through the self-adjoint extension $S_{\pi/2}$.

Definition 7.1 (Resolvent and spectrum)**.** Let $T : \mathrm{Dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a closed densely defined operator on a Hilbert space $\mathcal{H}$. The resolvent set $\rho(T)$ consists of all $\lambda \in \mathbb{C}$ such that

$$T - \lambda I : \mathrm{Dom}(T) \to \mathcal{H}$$

is bijective and has a bounded inverse. The spectrum of T is

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

Definition 7.2 (Eigenvalues and eigenvectors)**.** A number $\lambda \in \mathbb{C}$ is called an eigenvalue of T if there exists a non-zero element $G \in \mathrm{Dom}(T)$ such that

$$TG = \lambda G.$$

The element G is called an eigenvector. The set of all eigenvalues is called the point spectrum and is denoted by $\sigma_p(T)$.

Remark 7.3. For a finite-dimensional matrix, the spectrum is just the set of eigenvalues. For operators on infinite-dimensional Hilbert spaces, the spectrum can be larger, because not every spectral point has to come from an eigenvector. Still, the main idea is the same: the spectrum is the set of values naturally attached to the operator. The eigenvalues are the most concrete part of this set.

Remark 7.4. For a self-adjoint operator the spectrum is real (see [3, Chapter X, Section 2, Corollary 2.9]):

$$T = T^* \qquad \Longrightarrow \qquad \sigma(T) \subseteq \mathbb{R}.$$

Note that if the zeros of $F_{a,N}$ can be identified with the spectrum of a self-adjoint extension, then those zeros must be real.

Before stating the de Branges spectral theorem, we introduce the notation used in its statement. We first recall what it means for a system in a Hilbert space to be orthogonal and complete. Then we split a function E into its real and imaginary parts, and define the spectral functions whose zeros will describe the spectra of the self-adjoint extensions.

Definition 7.5 (Orthogonal and complete systems)**.** Let $\mathcal{H}$ be a Hilbert space. A family $(e_\lambda)_{\lambda\in\Lambda}$ is called orthogonal if

$$\langle e_\lambda, e_\mu \rangle_{\mathcal{H}} = 0 \qquad (\lambda \neq \mu).$$

It is called orthonormal if additionally $\|e_\lambda\|_{\mathcal{H}} = 1$ for every $\lambda \in \Lambda$. The family is called complete if

$$\overline{\mathrm{span}}\{e_\lambda : \lambda \in \Lambda\} = \mathcal{H}.$$

A complete orthonormal system is called an orthonormal basis.

Remark 7.6. Completeness means that the family is large enough to approximate every element of the Hilbert space. In the de Branges spectral theorem, the complete systems will consist of reproducing kernels.

By Corollary 6.11, the self-adjoint extensions are parametrized by a von Neumann phase $e^{i\theta}$, with θ modulo 2π. In the de Branges notation below we instead use $\beta \in [0, \pi)$. This is not a different family: replacing β by $\beta + \pi$ only changes s_β by a minus sign, and hence gives the same zero set and the same extension. We therefore do not need an explicit formula relating β and θ.

Definition 7.7 (Real and imaginary parts of an entire function)**.** Let E be an entire function and define

$$A := \frac{E + E^{\#}}{2}, \qquad B := \frac{E - E^{\#}}{2i}.$$

Then A and B are real entire functions, and $E = A + iB$, $E^{\#} = A - iB$. For the truncated theta-kernel model $E_{a,N,\tau} = F_{a,N} + i\tau F'_{a,N}$, the functions $F_{a,N}$ and $F'_{a,N}$ are real entire. Hence

$$E^{\#}_{a,N,\tau} = F_{a,N} - i\tau F'_{a,N}, \qquad A = F_{a,N}, \quad B = \tau F'_{a,N}.$$

Definition 7.8 (Spectral functions and spectral sets)**.** Let $E = A + iB$ be as above. For $\beta \in [0, \pi)$, define

$$s_\beta(z) := B(z) \cos \beta - A(z) \sin \beta.$$

Since A and B are real entire, s_β is also real entire. We denote its zero set by

$$\Lambda_\beta := \{z \in \mathbb{C} : s_\beta(z) = 0\}.$$

Remark 7.9. The parameter β selects one real linear combination of the two real entire functions A and B. In the regular de Branges theorem below, the zeros of s_β are the spectral points of the corresponding self-adjoint extension S_β.

Theorem 7.10 (de Branges spectral theorem, regular case)**.** *Let $E \in \mathcal{HB}$, write $E = A + iB$, and assume that the regular de Branges operator setting from Assumption 6.8 holds for S_E. Fix $\beta \in [0, \pi)$, let s_β and Λ_β be as in Definition 7.8, and let S_β be the corresponding self-adjoint extension. Assume moreover that the exceptional de Branges obstruction is absent for this value of β, in the following sense: s_β does not define a non-zero element of $\mathcal{H}(E)$ orthogonal to all kernels $K_E(\lambda, \cdot)$ with $\lambda \in \Lambda_\beta$. Then*

$$\Lambda_\beta \subseteq \mathbb{R}, \qquad \sigma(S_\beta) = \Lambda_\beta.$$

Moreover, the kernels $K_E(\lambda, \cdot)$, $\lambda \in \Lambda_\beta$, form a complete orthogonal system in $\mathcal{H}(E)$.

Proof sketch following de Branges. We use a condensed form of several results from de Branges' theory. The phase-level orthogonality and the possible exceptional function come from de Branges' theorem on orthogonal sets in $\mathcal{H}(E)$; see [4, Chapter 2, Theorem 22]. The operator-theoretic identification with the corresponding self-adjoint extension uses the regular de Branges theory of multiplication by the independent variable; see [4, Chapter 2, Theorems 27–29]. We spell out the mechanism in the notation of this report, but we do not reprove the full de Branges spectral theory.

We first explain why the spectral function has the form $s_\beta = B \cos \beta - A \sin \beta$. For real t, the functions A and B are real-valued and $E(t) = A(t) + iB(t)$. Since E has no real zeros, we may write $E(t) = |E(t)|e^{i\varphi(t)}$ for a phase function φ. Then $A(t) = |E(t)| \cos \varphi(t)$ and $B(t) = |E(t)| \sin \varphi(t)$, so

$$s_\beta(t) = B(t) \cos \beta - A(t) \sin \beta = |E(t)| \sin(\varphi(t) - \beta).$$

Hence $s_\beta(t) = 0$ if and only if $\varphi(t) = \beta \pmod{\pi}$. Thus, on the real line, the zeros of s_β are precisely the phase-level points with phase β modulo π.

We also see directly that these zeros are real. Since

$$e^{-\mathrm{i}\beta}E - e^{\mathrm{i}\beta}E^{\#} = 2\mathrm{i}(B\cos\beta - A\sin\beta) = 2\mathrm{i}s_\beta,$$

a zero of s_β in $\mathbb{C}_+$ would imply $|E(z)| = |E^{\#}(z)|$, contradicting the Hermite–Biehler inequality $|E^{\#}(z)| < |E(z)|$ in $\mathbb{C}_+$. Since s_β is real entire, non-real zeros occur in conjugate pairs, so zeros in $\mathbb{C}_-$ are excluded as well. Therefore $\Lambda_\beta \subseteq \mathbb{R}$.

Now let $\lambda, \mu \in \Lambda_\beta$ with $\lambda \neq \mu$. By the de Branges kernel formula from Theorem 5.6, for real λ and μ,

$$K_E(\lambda,\mu) = \frac{A(\lambda)B(\mu) - B(\lambda)A(\mu)}{\pi(\lambda-\mu)}.$$

Because $s_\beta(\lambda) = s_\beta(\mu) = 0$, the two non-zero real vectors $(A(\lambda), B(\lambda))$ and $(A(\mu), B(\mu))$ lie on the same line through the origin in $\mathbb{R}^2$. Hence $A(\lambda)B(\mu) - B(\lambda)A(\mu) = 0$, and therefore $K_E(\lambda,\mu) = 0$. By the reproducing property,

$$\langle K_E(\lambda,\cdot), K_E(\mu,\cdot)\rangle_{\mathcal{H}(E)} = K_E(\lambda,\mu),$$

so the kernels corresponding to distinct points of Λ_β are orthogonal.

The completeness is the non-elementary part. De Branges' phase theorem says that the only possible functions in $\mathcal{H}(E)$ orthogonal to all kernels from this phase-level set are constant multiples of $e^{-\mathrm{i}\beta}E - e^{\mathrm{i}\beta}E^{\#}$; see [4, Theorem 22]. In our notation,

$$e^{-\mathrm{i}\beta}E - e^{\mathrm{i}\beta}E^{\#} = 2\mathrm{i}s_\beta.$$

Thus the only possible obstruction to completeness is the exceptional function represented, in our notation, by s_β. By the additional assumption in the theorem, this obstruction is absent: s_β does not give a non-zero vector in $\mathcal{H}(E)$ orthogonal to all these kernels. Hence the kernels $K_E(\lambda,\cdot)$, $\lambda \in \Lambda_\beta$, form a complete orthogonal system in $\mathcal{H}(E)$.

It remains to relate this kernel system to the operator S_β. For real λ, Proposition 6.3 gives

$$S_E^* K_E(\lambda,\cdot) = \lambda K_E(\lambda,\cdot).$$

Thus $K_E(\lambda,\cdot)$ is an eigenvector of the adjoint operator. It becomes an eigenvector of the self-adjoint extension S_β precisely when it lies in $\mathrm{Dom}(S_\beta)$.

The regular de Branges operator theory describes this domain condition in terms of the same phase parameter; see [4, Chapter 2, Theorems 27–29]. In the notation used here, the condition is

$$e^{-\mathrm{i}\beta}E(\lambda) - e^{\mathrm{i}\beta}E^{\#}(\lambda) = 0.$$

Using $e^{-\mathrm{i}\beta}E - e^{\mathrm{i}\beta}E^{\#} = 2\mathrm{i}s_\beta$, this is equivalent to $s_\beta(\lambda) = 0$, or $\lambda \in \Lambda_\beta$. Hence

$$K_E(\lambda,\cdot) \in \mathrm{Dom}(S_\beta) \qquad \Longleftrightarrow \qquad \lambda \in \Lambda_\beta.$$

For such λ,

$$S_\beta K_E(\lambda,\cdot) = S_E^* K_E(\lambda,\cdot) = \lambda K_E(\lambda,\cdot).$$

Thus every $\lambda \in \Lambda_\beta$ is an eigenvalue of S_β, and hence

$$\Lambda_\beta \subseteq \sigma_p(S_\beta) \subseteq \sigma(S_\beta).$$

For the reverse inclusion, we again use the operator-theoretic part of the regular de Branges theory: it identifies the resolvent of the self-adjoint extension by the same phase condition. Equivalently, if a point is not in Λ_β, then it is not a spectral point of S_β; see [4, Chapter 2, Theorems 27–29]. Therefore

$$\sigma(S_\beta) \subseteq \Lambda_\beta.$$

Combining both inclusions gives $\sigma(S_\beta) = \Lambda_\beta$. This proves the spectral identification and the complete orthogonal kernel system. □

The theorem gives a complete orthogonal system of kernel functions. To obtain an orthonormal basis, we normalize these kernels.

Definition 7.11 (Normalized kernel basis)**.** For $\lambda \in \Lambda_\beta$, define

$$e_\lambda(z) := \frac{K_E(\lambda, z)}{\|K_E(\lambda, \cdot)\|_{\mathcal{H}(E)}}.$$

Then $\{e_\lambda : \lambda \in \Lambda_\beta\}$ is the normalized kernel system associated with the extension S_β.

After normalization, this system becomes an orthonormal basis. This is similar to a Fourier expansion: instead of expanding a function in sines and cosines, we expand it in normalized reproducing kernels. The coefficients are again given by inner products.

Corollary 7.12 (Expansion in normalized kernels)**.** *In the regular de Branges setting of Theorem 7.10, every $G \in \mathcal{H}(E)$ has the norm-convergent expansion*

$$G = \sum_{\lambda \in \Lambda_\beta} \langle G, e_\lambda \rangle_{\mathcal{H}(E)} \, e_\lambda,$$

and

$$\|G\|^2_{\mathcal{H}(E)} = \sum_{\lambda \in \Lambda_\beta} \left| \langle G, e_\lambda \rangle_{\mathcal{H}(E)} \right|^2.$$

Proof. By Theorem 7.10, the kernels $K_E(\lambda, \cdot)$, $\lambda \in \Lambda_\beta$, form a complete orthogonal system. After normalization, the functions e_λ form an orthonormal basis of $\mathcal{H}(E)$. The expansion and the norm identity are therefore the Hilbert-space formulas for an orthonormal basis; see [2, Theorem 7.4 and Corollary 7.6]. □

We now apply the theorem to the truncated theta-kernel model. For the extension corresponding to $\beta = \pi/2$, the spectral function reduces to $F_{a,N}$, up to a sign.

Corollary 7.13 (Truncated theta-kernel model case)**.** *Assume that the hypotheses of Theorem 7.10 hold for $E = E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}$ and $\beta = \pi/2$. Then*

$$\sigma(S_{\pi/2}) = \mathrm{Zeros}(F_{a,N}) \subseteq \mathbb{R}.$$

Moreover, with $K_{a,N,\tau} := K_{E_{a,N,\tau}}$, the normalized kernels

$$e_\lambda := \frac{K_{a,N,\tau}(\lambda, \cdot)}{\|K_{a,N,\tau}(\lambda, \cdot)\|_{\mathcal{H}(E_{a,N,\tau})}}, \qquad \lambda \in \mathrm{Zeros}(F_{a,N}),$$

form an orthonormal basis of $\mathcal{H}(E_{a,N,\tau})$. Hence every $G \in \mathcal{H}(E_{a,N,\tau})$ has the norm-convergent expansion

$$G = \sum_{\lambda \in \mathrm{Zeros}(F_{a,N})} \langle G, e_\lambda \rangle_{\mathcal{H}(E_{a,N,\tau})} \, e_\lambda.$$

Proof. For $E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N} = A + \mathrm{i}B$ we have $A = F_{a,N}$ and $B = \tau F'_{a,N}$. Hence

$$s_\beta = \tau F'_{a,N} \cos\beta - F_{a,N} \sin\beta, \qquad s_{\pi/2} = -F_{a,N}.$$

Theorem 7.10 gives

$$\sigma(S_{\pi/2}) = \mathrm{Zeros}(s_{\pi/2}) = \mathrm{Zeros}(F_{a,N}) \subseteq \mathbb{R}.$$

It also gives a complete orthogonal system of kernels indexed by these zeros. After normalization this system becomes the orthonormal basis above, and the expansion follows from Corollary 7.12. □

Remark 7.14. The realness of the zeros of $F_{a,N}$ is already forced by the Hermite–Biehler assumption on $E_{a,N,\tau}$. The additional point of Corollary 7.13 is the operator interpretation: the zeros of $F_{a,N}$ are not only real, but also occur as the spectrum of the self-adjoint extension $S_{\pi/2}$ and as the indices of an orthonormal basis of reproducing kernels.

Remark 7.15 (On the regularity assumption). For $\beta = \pi/2$, the possible obstruction to completeness from Theorem 7.10 is $s_{\pi/2} = -F_{a,N}$. Thus, when Corollary 7.13 assumes that the hypotheses of Theorem 7.10 hold, it also assumes that this obstruction does not occur. However, it is not proved here; it remains part of the conditional framework.

Remark 7.16 (Kernel expansion at the zeros of $F_{a,N}$). The previous corollary also gives a more concrete interpretation of the zeros of $F_{a,N}$. They are the points indexing the kernel basis of $\mathcal{H}(E_{a,N,\tau})$. Indeed, for every $G \in \mathcal{H}(E_{a,N,\tau})$ and $\lambda \in \mathrm{Zeros}(F_{a,N})$, we have

$$G(\lambda) = \langle G, K_{a,N,\tau}(\lambda, \cdot)\rangle_{\mathcal{H}(E_{a,N,\tau})}.$$

The orthonormal expansion first holds in the $\mathcal{H}(E_{a,N,\tau})$ norm. Since point evaluation is continuous in a reproducing kernel Hilbert space, we may evaluate this norm-convergent expansion at any fixed point $z \in \mathbb{C}$. Using

$$\|K_{a,N,\tau}(\lambda, \cdot)\|^2_{\mathcal{H}(E_{a,N,\tau})} = K_{a,N,\tau}(\lambda, \lambda),$$

we obtain, for every fixed $z \in \mathbb{C}$, the pointwise identity

$$G(z) = \sum_{\lambda \in \mathrm{Zeros}(F_{a,N})} G(\lambda) \frac{K_{a,N,\tau}(\lambda, z)}{K_{a,N,\tau}(\lambda, \lambda)}.$$

Thus the zeros of $F_{a,N}$ are not only the spectral points of $S_{\pi/2}$; they are also the points at which functions in the de Branges space are evaluated in this expansion.

Remark 7.17 (Back to the zeta model). We have reached the finite spectral mechanism. Under the Hermite–Biehler and regular de Branges assumptions, the zeros of $F_{a,N}$ are the spectral points of the self-adjoint extension $S_{\pi/2}$, and therefore they are real. Even more, these zeros index an orthonormal basis of reproducing kernels in $\mathcal{H}(E_{a,N,\tau})$.

Together with the local uniform convergence $F_{a,N} \to \Xi_{\mathrm{dBN}}$ along cofinal sequences and the zero-transfer argument from Section 3, this gives the conditional mechanism

$$F_{a,N} \longrightarrow E_{a,N,\tau} \in \mathcal{HB} \longrightarrow \mathcal{H}(E_{a,N,\tau}) \longrightarrow S_{\pi/2} \longrightarrow \mathrm{Zeros}(F_{a,N}) = \sigma(S_{\pi/2}) \subseteq \mathbb{R}.$$

What remains is to prove the Hermite–Biehler property, together with the regular de Branges assumptions, along a suitable cofinal sequence of truncated theta-kernel models.

8 Assumptions, main theorem and conclusion

We now put the different parts of the argument together. After recentering the critical line, the completed zeta function is written as the cosine transform

$$\Xi_{\mathrm{dBN}}(z) = \int_0^\infty \Phi(u)\cos(zu)\ \mathrm{d}u.$$

In this normalization, real values of z correspond to points on the critical line. The truncated theta-kernel models considered above are obtained by keeping only the first N terms of the kernel and by cutting off the integral at $a > 0$:

$$F_{a,N}(z) = \int_0^a \Phi_N(u)\cos(zu)\ \mathrm{d}u.$$

Equivalently, $F_{a,N}$ is the angular Fourier transform of the compactly supported even function

$$f_{a,N}(u) = \frac{1}{2}\Phi_N(|u|)\mathbf{1}_{[-a,a]}(u).$$

Thus z already appears as a frequency variable before the de Branges theory is introduced. This Fourier viewpoint motivates the later use of Hilbert-space methods and multiplication by the variable z.

As $a \to \infty$ and $N \to \infty$ along a cofinal sequence, the functions $F_{a,N}$ converge locally uniformly to Ξ_{dBN}. This keeps the truncated theta-kernel models connected to the completed zeta function. It also gives the zero-transfer step: if every truncated theta-kernel model F_{a_j,N_j} in such a cofinal sequence has only real zeros, then the limiting function Ξ_{dBN} has only real zeros as well.

To connect the Fourier model with de Branges theory, we consider, for $\tau > 0$,

$$E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}.$$

The main assumption is that this function belongs to the Hermite–Biehler class. Under this assumption, $E_{a,N,\tau}$ generates the de Branges space $\mathcal{H}(E_{a,N,\tau})$. In that space we study the multiplication operator given by multiplication by the variable z. Under the regular de Branges operator assumptions, this operator has self-adjoint extensions. For the extension corresponding to $\beta = \pi/2$, the associated spectral function is

$$s_{\pi/2} = -F_{a,N}.$$

This gives the operator interpretation obtained in Section 7:

$$\sigma(S_{\pi/2}) = \mathrm{Zeros}(F_{a,N}) \subseteq \mathbb{R}.$$

The realness of the zeros is not the surprising part by itself, because the Hermite–Biehler assumption on $E_{a,N,\tau}$ already forces $F_{a,N}$ to have only real zeros. The additional point is structural: the zeros of $F_{a,N}$ become the spectrum of a self-adjoint extension and also index an orthonormal basis of reproducing kernels in $\mathcal{H}(E_{a,N,\tau})$.

Thus the conditional mechanism can be summarized as

$$F_{a,N} \longrightarrow E_{a,N,\tau} \in \mathcal{HB} \longrightarrow \mathcal{H}(E_{a,N,\tau}) \longrightarrow S_{\pi/2} = S^*_{\pi/2} \longrightarrow \mathrm{Zeros}(F_{a,N}) = \sigma(S_{\pi/2}) \subseteq \mathbb{R}.$$

Here $S_{\pi/2}$ is the self-adjoint extension corresponding to the phase $\beta = \pi/2$.

The conditional mechanism rests on three inputs. First, we assume the Hermite–Biehler condition

$$E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N} \in \mathcal{HB}.$$

Second, we assume the regular de Branges operator setting from Assumption 6.8: the multiplication operator is densely defined, its deficiency spaces are generated by the kernel functions

$$K_E(-\mathrm{i},\cdot) \quad \text{and} \quad K_E(\mathrm{i},\cdot),$$

and the possible exceptional obstruction in Theorem 7.10 is absent for the phase $\beta = \pi/2$. Third, we choose a cofinal sequence (a_j, N_j), so that the truncated theta-kernel models converge locally uniformly to Ξ_{dBN}.

The main conditional theorem is the following.

Theorem 8.1 (Main conditional theorem). *Assume that there exists a cofinal sequence (a_j, N_j) and parameters $\tau_j > 0$ such that*

$$E_{a_j,N_j,\tau_j} = F_{a_j,N_j} + \mathrm{i}\tau_j F'_{a_j,N_j} \in \mathcal{HB}$$

for every j. Assume also that Assumption 6.8 and the hypotheses of Theorem 7.10 hold for the spaces $\mathcal{H}(E_{a_j,N_j,\tau_j})$ and for the extension corresponding to $\beta = \pi/2$. Then every function F_{a_j,N_j} in this cofinal sequence has only real zeros, and these zeros are the spectral points of the self-adjoint extension $S^{(j)}_{\pi/2}$.

Under these assumptions, Ξ_{dBN} has only real zeros. Equivalently, all non-trivial zeros of ζ lie on the critical line.

Proof. Fix j. Since $E_{a_j,N_j,\tau_j} \in \mathcal{HB}$, Remark 4.14 implies that F_{a_j,N_j} has only real, simple zeros. The spectral result of Theorem 7.10, applied to E_{a_j,N_j,τ_j} and $\beta = \pi/2$, adds the operator interpretation. It gives

$$s^{(j)}_{\pi/2} = -F_{a_j,N_j}$$

and

$$\mathrm{Zeros}(F_{a_j,N_j}) = \sigma(S^{(j)}_{\pi/2}).$$

Thus the finite real-zero statement comes from the Hermite–Biehler assumption, while the de Branges spectral theorem identifies these zeros as the spectral points of the self-adjoint extension $S^{(j)}_{\pi/2}$.

It remains to pass to the limit. By Proposition 3.3,

$$F_{a_j,N_j} \longrightarrow \Xi_{\mathrm{dBN}}$$

locally uniformly on $\mathbb{C}$. Suppose, for contradiction, that Ξ_{dBN} has a non-real zero z_0. Choose a small disk Ω around z_0 such that $\overline{\Omega} \cap \mathbb{R} = \emptyset$ and such that Ξ_{dBN} has no zeros on $\partial\Omega$. By Proposition 3.5, which follows from Rouché's theorem (Theorem 3.4), the functions F_{a_j,N_j} and Ξ_{dBN} have the same number of zeros in Ω, counted with multiplicity, for all sufficiently large j.

But Ω does not meet the real axis, while all zeros of F_{a_j,N_j} are real. This is a contradiction. Therefore Ξ_{dBN} has no non-real zeros.

Finally,

$$\Xi_{\mathrm{dBN}}(z) = \frac{1}{8}\xi\left(\frac{1}{2} + \frac{\mathrm{i}z}{2}\right).$$

Thus real zeros in the z-variable correspond exactly to zeros of ξ on the critical line $\mathrm{Re}(s) = 1/2$. Since the zeros of ξ are the non-trivial zeros of ζ, this gives the stated form of the Riemann Hypothesis. □

This theorem shows clearly where the conditional step lies. The central unproved input is not just a local algebraic inequality, but the full Hermite–Biehler condition

$$E_{a,N,\tau} \in \mathcal{HB}.$$

Away from the zeros of $F_{a,N}$, the inequality part of this condition can be rewritten as

$$\operatorname{Im} \frac{F'_{a,N}(z)}{F_{a,N}(z)} < 0, \qquad z \in \mathbb{C}_+.$$

However, this is only a rewriting at points where $F_{a,N}$ is non-zero. The full Hermite–Biehler assumption also includes the absence of real zeros of $E_{a,N,\tau}$ and the global strict inequality throughout the upper half-plane.

This distinction is important. Proving the Hermite–Biehler condition along a cofinal sequence should not be presented as an obviously easier problem than the Riemann Hypothesis itself. The assumption is very strong: if it were proved along such a sequence, then the finite real-zero property needed for the zero-transfer argument would already follow.

The value of the construction lies in the operator-theoretic interpretation it gives. Under the stated assumptions, the zeros of the truncated theta-kernel models are not just real numbers; they are the spectral points of self-adjoint extensions of multiplication operators in de Branges spaces. For such spectral points λ, the kernels $K_E(\lambda, \cdot)$ form the orthogonal system described by the de Branges spectral theorem. In this way the construction connects Fourier-type truncations, reproducing kernels, and spectral theory.

For future work, one would have to prove the Hermite–Biehler condition for functions of the form

$$E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}$$

along a suitable cofinal sequence. A few isolated examples would not be enough for the limiting argument. What is needed is a sequence with $a_j \to \infty$ and $N_j \to \infty$ for which the Hermite–Biehler condition and the regular de Branges spectral input stay valid. This is the hard part of the approach.

Thus the result is conditional. Starting from the theta-kernel formula for the completed zeta function, we introduced truncated theta-kernel models and studied the de Branges spaces generated by

$$E_{a,N,\tau} = F_{a,N} + \mathrm{i}\tau F'_{a,N}.$$

Under the Hermite–Biehler and regular operator assumptions, and after excluding the relevant exceptional obstruction, the zeros of $F_{a,N}$ can be identified with the spectral points of a self-adjoint extension. The corresponding reproducing kernels form the complete orthogonal system described by the de Branges spectral theorem. If these assumptions could be verified along a cofinal sequence, then local uniform convergence and the Rouché zero-transfer argument would transfer the real-zero conclusion to Ξ_{dBN}, and hence to the Riemann Hypothesis.

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