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EFFICIENT MULTIGRID BASED SOLVERS FOR ISOGEOMETRIC ANALYSIS

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Abstract. Introduced in [1], Isogeometric Analysis (IgA) has become widely accepted in academia and industry. However, the system matrices arising from discretizations based on Isogeometric Analysis (IgA) tend to become ill-conditioned for higher values of the approximation order p, and hence, the efficient solution of linear systems of equations is a challenging task. In this paper, we propose the use of p-multigrid based methods for the efficient solution of IgA based discretizations. The overall idea is to construct a multigrid hierarchy based on different values of p, where the coarse grid correction is obtained at level $p = 1$. Consequently, the residual equation is solved at level $p = 1$, so that the coarse grid correction can be obtained using established solution techniques for low-order Lagrange finite elements. Numerical results are presented for a two-dimensional benchmarkproblem to compare the performance of p-multigrid used as a solver and as a preconditioner for a Krylov subspace method. It follows from a Local Fourier Analysis, that the coarse grid correction and the smoothing procedure complement each other quite well. Moreover, the obtained convergence rates indicate that p-multigrid methods have the potential to efficiently solve IgA discretizations.

1 INTRODUCTION

Introduced in [1], Isogeometric Analysis (IgA) has become widely accepted in academia and industry. Over the years, IgA has been applied in fluid mechanics [2], shape optimization [3] and structural mechanics [4]. The use of B-splines and Non-Uniform Rational B-splines (NURBS) within IgA enables a highly accurate representation of geometries. Another advantage of IgA over standard Finite Elements is the C^{p-1} continuity of the B-spline basis functions, leading to numerical approximations with high continuity.

However, solving the resulting linear systems remains a challenging task. Due to the growing support of the basis functions, the bandwidth of the resulting system matrix after discretization becomes wider for increasing p, making the use of a (sparse) direct solver more and more expensive.

The use of (standard) iterative solvers is not straightforward, since the system matrices arising in IgA can become ill-conditioned: The condition number of the Poisson operator is bounded by a constant times h^{-2}, where h is the mesh width, but, in contrast to standard Finite Elements, scales exponentially with the order of the approximation p, cf. [5]. The performance of (standard) iterative solvers thus decreases fast for higher values of p.

Over the past years, the use of different types of solvers have been investigated. In [6], the use of direct solvers has been investigated when either C^0 or C^{p-1} continuous B-spline basis functions were used for discretization. Preconditioners based on solvers for the Sylvester equation have been studied in [7]. Geometric multigrid methods were first studied in [5] for scalar second-order elliptic problems. In [8], a spectral analysis of the two-grid method was performed by determining the reduction factors of the coarse grid correction and smoother. Recently, a geometric multigrid method has been presented in [9] which exhibits convergence rates independent of h and p.

In this paper we propose an alternative solution strategy that is based on p-multigrid techniques, where the resulting p-multigrid method is used both as a solver and as a preconditioner in a Krylov subspace iterative method. The approach makes use of a hierarchy of B-spline based discretizations of different approximations orders. This is in contrast to (geometric) h-multigrid methods, in which a hierarchy of coarser and finer meshes is constructed. The coarse grid correction in the suggested p-multigrid method is determined at level $p = 1$, enabling the use of (established) solution techniques developed for low-order Lagrange finite elements.

The remainder of this paper is organised as follows. The considered model problem and spline spaces used for discretization are presented in Section 2. In Section 3, the p-multigrid method is described in more detail. Numerical results, obtained on a non-trivial two-dimensional geometry, are presented in Section 4. Furthermore, results are compared with h-multigrid methods. Conclusions are finally drawn in Section 5.

2 MODEL PROBLEM AND DISCRETIZATION

As our model problem we consider the Poisson equation:

$$-\Delta u = f \qquad \text{on } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$, $f \in L^2(\Omega)$ and $u = 0$ on the boundary $\partial\Omega$. Let $\mathcal{V} = H_0^1(\Omega)$ denote the space of functions in $H^1(\Omega)$ that vanish on the boundary $\partial\Omega$, where $H^1(\Omega)$ denotes the first order Sobolev space. Multiplying equation (1) with an arbitrary function $v \in \mathcal{V}$, integrating over Ω and applying integration by parts leads to the variational form:
Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in \mathcal{V}. \tag{2}$$

For the spatial discretization, tensor product B-spline basis functions of order p are considered based on an open uniform knot vector and knot span size (which is associated with the mesh width) h.

The one-dimensional B-spline basis functions are defined recursively by the Cox-de Boor formula [1]:

$$\phi_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{else} \end{cases} \tag{3}$$

and for $p > 0$:

$$\phi_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}\phi_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}}\phi_{i+1,p-1}(\xi). \tag{4}$$

Let the tensor product spline space of order p and mesh width h be denoted by $\mathcal{V}_{h,p}$:

$$\mathcal{V}_{h,p} := \left\{ u_{h,p} = \sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{c}_{(i,j)}\phi_{i,p}(x)\phi_{j,p}(y) \;\mid\; \mathbf{c} \in \mathbb{R}^{N^2} \right\}, \tag{5}$$

where N denotes the number of basis functions in a single direction and $\mathbf{c}$ the coefficient vector corresponding to $u_{h,p}$. The Galerkin formulation of the variational problem (2) is then obtained using these spline spaces, which leads to:

Find $u_{h,p} \in \mathcal{V}_{h,p}$ such that

$$(\nabla u_{h,p}, \nabla v_{h,p}) = (f, v_{h,p}) \qquad \forall v_{h,p} \in \mathcal{V}_{h,p}. \tag{6}$$

Adopting the B-spline basis functions, the discretized problem becomes, written as a linear system:

$$\mathbf{A}_{h,p}\mathbf{u}_{h,p} = \mathbf{f}_{h,p}. \tag{7}$$

Here, $\mathbf{A}_{h,p}$ is the stiffness matrix corresponding to the discretization with B-spline basis functions of order p and mesh width h.

3 P-MULTIGRID

A p-multigrid algorithm is formulated to solve equation (7). Starting from the space $\mathcal{V}_{h,1}$, refinement in p is applied, which leads to a sequence of spline spaces $\mathcal{V}_{h,1}, \ldots, \mathcal{V}_{h,p}$. In this paper, basis functions are considered which are C^{p-1} continuous, implying that the spline spaces are not nested. Based on a random initial guess $\mathbf{u}^0_{h,p}$, a single iteration of the two-grid correction scheme consists of the following steps:

1. Apply a fixed number of ν_1 presmoothing steps :

$$\mathbf{u}^{(0,m+1)}_{h,p} = \mathbf{u}^{(0,m)}_{h,p} + \mathcal{S}(\mathbf{f}_{h,p} - \mathbf{A}_{h,p}\mathbf{u}^{(0,m)}_{h,p}), \quad m = 0, \ldots, \nu_1 - 1 \tag{8}$$

   where $\mathcal{S}$ is a smoother (i.e. Jacobi, Gauss-Seidel) for the fine grid problem.

2. Determine the residual at level p and project it onto the space $\mathcal{V}_{h,p-1}$ using the restriction operator I_p^{p-1} (to be defined):

$$\mathbf{r}_{h,p-1} = I_p^{p-1}\left(\mathbf{f}_{h,p} - \mathbf{A}_{h,p}\mathbf{u}^{(0,\nu_1)}_{h,p}\right). \tag{9}$$

3. Determine the coarse grid correction by solving the residual equation at level $p-1$:

$$\mathbf{A}_{h,p-1}\mathbf{e}_{h,p-1} = \mathbf{r}_{h,p-1}. \tag{10}$$

4. Project the correction $\mathbf{e}_{h,p-1}$ onto the space $\mathcal{V}_{h,p}$ using the prolongation operator I_{p-1}^{p} (to be defined) and update $\mathbf{u}_{h,p}^{(0,\nu_1)}$:

$$\mathbf{u}_{h,p}^{(0,\nu_1)} := \mathbf{u}_{h,p}^{(0,\nu_1)} + I_{p-1}^{p}(\mathbf{e}_{h,p-1}). \tag{11}$$

5. Apply a fixed number of ν_2 postsmoothing steps to obtain $\mathbf{u}_{h,p}^{(0,\nu_1+\nu_2)} := \mathbf{u}_{h,p}^{1}$.

The two-grid correction scheme is applied recursively until level $p = 1$ has been reached, which results in a V-cycle. Alternatively, different schemes can be applied leading, for example, to a W-cycle, see Figure 1.

At level $p = 1$, the residual equation (10) is solved approximately by means of a Conjugate Gradient (CG) solver. As a stopping criterium for the CG method, a reduction of the relative residual with a factor 10^{-8} is used. In our experience, only a limited improvement of the p-multigrid convergence was shown if a solution obtained with a very low tolerance ($\epsilon = 10^{-14}$) was used on the coarsest level. The stiffness matrices, needed at each level for the smoothing procedure, can be obtained by rediscretization. The solution $\mathbf{u}_{h,p}^{1}$ is used as an initial guess for the next cycle.

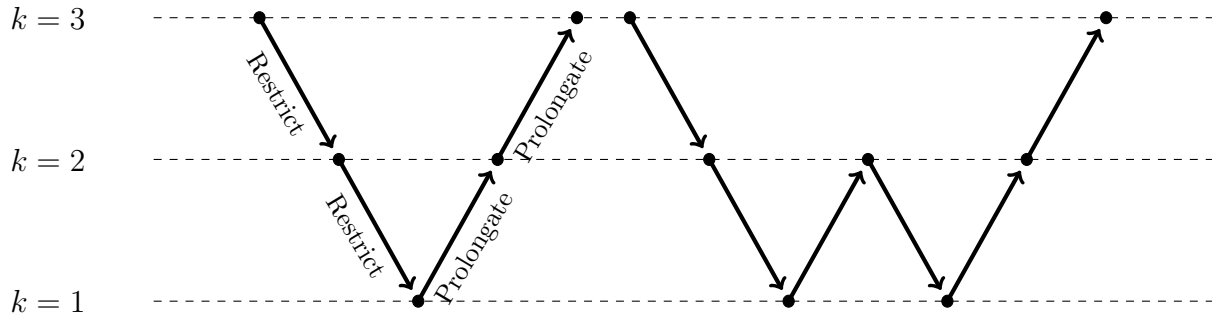

Figure 1: Description of a V-cycle and W-cycle.

Prolongation/Restriction

The correction $\mathbf{e}_{h,k-1}$ at level $k-1$ is prolongated by projecting it onto the space $\mathcal{V}_{h,k}$. The prolongation operator $I_{k-1}^{k} : \mathcal{V}_{h,k-1} \to \mathcal{V}_{h,k}$ is defined such that

$$I_{k-1}^{k}(v) = v \qquad \forall v \in \mathcal{V}_{h,k-1}. \tag{12}$$

This prolongation operator has been used before in literature [10, 11, 12].

The variational form of equation (12) is given by:

$$\left(I_{k-1}^{k}(v), w\right)_{h,k} = (v, w)_{h,k} \qquad \forall v \in \mathcal{V}_{h,k-1},\ \forall w \in \mathcal{V}_{h,k}. \tag{13}$$

It is sufficient to test the equation for all higher-order basis functions $\phi_{m,k} := \phi_{m,k}(x, y)$. Since $v \in \mathcal{V}_{h,k-1}$, we can write

$$v = \sum_{i=1}^{\#\mathcal{V}_{h,k-1}} \mathbf{v}_{i,k-1}\phi_{i,k-1} \tag{14}$$

for some coefficient vector $\mathbf{v}$. Using $w = \phi_{m,k}$, $I_{k-1}^{k}(v) = v$ and equation (14), we can write consequently:

$$\left(\sum_{n=1}^{\#\mathcal{V}_{h,k-1}} \mathbf{v}_{n,k-1}\phi_{n,k-1}, \phi_{m,k}\right)_k = \left(\sum_{m=1}^{\#\mathcal{V}_{h,k}} \mathbf{v}_{m,k}\phi_{m,k}, \phi_{m,k}\right)_k \quad \forall v \in V_{h,k-1},\ \forall \phi_{m,k} \in \mathcal{V}_{h,k}. \tag{15}$$

The equation above is equivalent to

$$\mathbf{P}_k^{k-1}\mathbf{v}_{k-1} = \mathbf{M}_k\mathbf{v}_k, \tag{16}$$

where

$$(\mathbf{P}_k^{k-1})_{(i,j)} := \int_\Omega \phi_{i,k-1}\phi_{j,k}\,\mathrm{d}\Omega \qquad\qquad (\mathbf{M}_k)_{(i,j)} := \int_\Omega \phi_{i,k}\phi_{j,k}\,\mathrm{d}\Omega. \tag{17}$$

The vector of coefficients $\mathbf{v}_k$ is therefore given by

$$\mathbf{v}_k = (\mathbf{M}_k)^{-1}\mathbf{P}_k^{k-1}\mathbf{v}_{k-1}. \tag{18}$$

Hence, the prolongation operator to determine $\mathbf{v_k}$ from $\mathbf{v_{k-1}}$ is defined by

$$I_{k-1}^{k} := (\mathbf{M}_k)^{-1}\mathbf{P}_k^{k-1}. \tag{19}$$

The restriction operator $I_k^{k-1} : \mathcal{V}_{h,k} \to \mathcal{V}_{h,k-1}$ is defined as the Hilbert adjoint of the prolongation operator, which implies [12]:

$$(I_k^{k-1}(w), v)_{k-1} = (w, I_{k-1}^{k}(v))_k = (w, v)_k \qquad \forall v \in \mathcal{V}_{h,k-1}, \forall w \in \mathcal{V}_{h,k}. \tag{20}$$

The last equality follows from the definition of the prolongation operator. Combining the equations leads to

$$\begin{aligned}(I_k^{k-1}(\phi_{m,k}), \phi_{n,k-1})_{k-1} &= \sum_{l=1}^{\#\mathcal{V}_{k-1}} \mathbf{R}(l, m)(\phi_{l,k-1}, \phi_{n,k-1})_{k-1}, \\ &= (\phi_{m,k}, \phi_{n,k-1})_k. \end{aligned} \tag{21}$$

Equation (21) has to hold $\forall m = 1, 2, \ldots, \#\mathcal{V}_k$ and $\forall n = 1, 2, \ldots, \#\mathcal{V}_{k-1}$.

This leads to the following matrix equation:

$$\mathbf{R}\mathbf{M}_{k-1} = \mathbf{P}_k^{k-1}. \tag{22}$$

Hence, the restriction matrix is given by

$$\mathbf{R} = (\mathbf{M}_{k-1})^{-1}\mathbf{P}_k^{k-1}. \tag{23}$$

Consequently, we have

$$I_k^{k-1} := (\mathbf{M}_{k-1})^{-1}\mathbf{P}_k^{k-1}. \tag{24}$$

The need to solve a linear system involving the consistent mass matrix is circumvented by applying direct row-sum mass lumping for both the prolongation and restriction operator, which is possibly due to the partition of unity property and non-negativity of the B-spline basis functions.

4 NUMERICAL RESULTS

To assess the quality of the p-multigrid method, a variety of numerical experiments have been performed. Results have been obtained for different approximation orders p, mesh widths h and smoothing steps ν, where p-multigrid is used both as a solver and as a preconditioner within a Conjugate Gradient method. Furthermore, different types of multigrid cycles are considered and results are compared with a (standard) h-multigrid method. Finally, the reduction factors of a single smoothing step and a coarse grid correction are determined to obtain more insight in the interaction between both components of the p-multigrid method.

The following model problem is considered: Let $\Omega \subset \mathbb{R}^2$ be the quarter annulus in the first quadrant of the Cartesian coordinate system with inner radius 1 and outer radius 2. The right hand side f is chosen such that the analytical solution is given by:

$$u(x,y) = (x^2 + y^2 - 3\sqrt{x^2+y^2} + 2) \cdot \sin\left(2\arctan\left(\frac{y}{x}\right)\right). \tag{25}$$

This model problem can be found in [5], where it is used to assess the quality of h-multigrid methods. The initial guess $\mathbf{u}_{h,p}^{(0)}$ is chosen to be a random vector, with entries taken from a uniform distribution on $[-1, 1]$. Prolongation and restriction are performed with the lumped versions of the operators presented in Section 3. Boundary conditions are imposed by using Nitsche's method. Defining $\mathbf{r}_{h,p}^{(0)}$ and $\mathbf{r}_{h,p}^{(i)}$ as, respectively, the residual based on the initial guess and the residual after the i^{th} multigrid cycle, the following stopping criterium is adopted:

$$\frac{||\mathbf{r}_{h,p}^{(i)}||}{||\mathbf{r}_{h,p}^{(0)}||} < 10^{-8}.$$

In all numerical experiments, the same number of pre- and postsmoothing steps has been applied ($\nu = \nu_1 = \nu_2$). For both presmoothing and postsmoothing, the Gauss-Seidel method is adopted.

p-Multigrid

Table 1 denotes the number of cycles needed to reach a converged solution with a p-multigrid solver for different values of ν and p. A fixed mesh size of 2^{-4} is chosen in each direction. The coarse grid operators are obtained by rediscretization (R) and applying the Galerkin Condition (G). Results indicate that for a fixed value of p and h, the number of cycles depends inversly on the number of smoothing steps ν. Doubling the number of smoothing steps, results in a decrease of the number of cycles with a factor of approximately two. A dependence on the approximation order p is also clearly observed: Increasing the order p increases the number of cycles needed to converge.

The V_R-cycle and W_R-cycle methods show for all configurations the same convergence behaviour. Application of the Galerkin Condition leads for all configurations to almost the same number of V-cycles, except for the case $p = 4$ and $\nu = 1$. For this combination, the p-multigrid method diverges, which is denoted with (x). Therefore, all further results presented in this paper are obtained by rediscretization.

Table 1: Poisson problem on quarter annulus: ν-dependence of the number of multigrid cycles

p :	2	3	4	p :	2	3	4	p :	2	3	4
$V_R(\cdot,\cdot)$				$V_G(\cdot,\cdot)$				$W_R(\cdot,\cdot)$			
$\nu = 1$	33	71	231	$\nu = 1$	35	72	x	$\nu = 1$	33	71	231
$\nu = 2$	17	36	116	$\nu = 2$	18	36	116	$\nu = 2$	17	36	116
$\nu = 4$	9	18	58	$\nu = 4$	9	18	58	$\nu = 4$	9	18	58
$\nu = 8$	5	9	29	$\nu = 8$	5	9	29	$\nu = 8$	5	9	29

The dependence of the number of cycles on the mesh width h is shown in Table 2. For constant p and a constant number of smoothing steps ($\nu = 4$), the number of cycles is independent of the mesh width. Again, the p-dependence can clearly be observed. Furthermore, the different cycle types show similar behaviour as illustrated in Table 1.

Table 2: Poisson problem on quarter annulus: h-independence of the number of multigrid cycles

p :	2	3	4	p :	2	3	4
$V(4,4)$				$W(4,4)$			
$h = 2^{-3}$	9	23	86	$h = 2^{-3}$	9	23	86
$h = 2^{-4}$	9	18	58	$h = 2^{-4}$	9	18	58
$h = 2^{-5}$	9	19	61	$h = 2^{-5}$	9	19	61
$h = 2^{-6}$	9	19	59	$h = 2^{-6}$	9	19	59

Using the p-multigrid as a preconditioner for a Krylov subspace method leads to similar observations. Table 3 shows the number of Conjugate Gradient iterations needed for convergence, which is independent of the mesh width for constant p. Here, the CG

method is preconditioned with a single multigrid cycle in each iteration. In contrast to the previous numerical experiments, forward Gauss-Seidel is used in the first part of the cycle, while backward Gauss-Seidel is adopted in the second part. The use of preconditioned CG solver mitigates the p-dependence, but does not solve this problem completely. Similar conclusions were drawn in [8] for h-multigrid methods.

Table 3: Poisson problem on quarter annulus: h-independence of the number of CG iterations

$p:$	2	3	4	$p:$	2	3	4
$V(1,1)$				$W(1,1)$			
$h = 2^{-3}$	16	29	80	$h = 2^{-3}$	16	29	74
$h = 2^{-4}$	17	30	68	$h = 2^{-4}$	17	30	61
$h = 2^{-5}$	18	30	69	$h = 2^{-5}$	18	31	64
$h = 2^{-6}$	19	30	68	$h = 2^{-6}$	19	31	65

Comparison between p- and h-multigrid

In what follows, we compare p-multigrid based methods with (standard) h-multigrid based methods. Results are obtained when h-multigrid is used either as a solver or as a preconditioner for a Conjugate Gradient method.

Table 4 denotes the results obtained with h-multigrid as a stand-alone solver. As with p-multigrid, we choose the number of levels to coincide with the approximation order p. Note that the number of iterations is equal for all values of ν and p, implying that h-multigrid and p-multigrid are equally effective.

Table 4: Poisson problem on quarter annulus: ν-dependence of the number of multigrid cycles

$p:$	2	3	4	$p:$	2	3	4
$V(\cdot,\cdot)$				$W(\cdot,\cdot)$			
$\nu = 1$	33	71	231	$\nu = 1$	33	71	231
$\nu = 2$	17	36	116	$\nu = 2$	17	36	116
$\nu = 4$	9	18	58	$\nu = 4$	9	18	58
$\nu = 8$	5	9	29	$\nu = 8$	5	9	29

Using h-multigrid as a preconditioner for a Conjugate Gradient method leads to the results presented in Table 5. Note that the number of iterations of the CG method is independent of the refinement level, but depends on the approximation order p. Table 3 denotes the results obtained with p-multigrid. For $p = 2$ and $p = 3$, the number of iterations needed for the CG method to converge, is comparable when either p-multigrid or h-multigrid is adopted as a preconditioner. However, for $p = 4$, the difference between both types of multigrid becomes clearly visible.

Table 5: Poisson problem on quarter annulus: h-independence of the number of CG iterations

	$p:$	2	3	4		$p:$	2	3	4
$V(1,1)$					$W(1,1)$				
$h = 2^{-3}$		15	25	59	$h = 2^{-3}$		15	25	56
$h = 2^{-4}$		14	25	56	$h = 2^{-4}$		15	24	54
$h = 2^{-5}$		15	24	52	$h = 2^{-5}$		15	24	49
$h = 2^{-6}$		16	24	49	$h = 2^{-6}$		16	24	47

Local spectral analysis

To obtain a deeper insight in the effect of a single smoothing step and a coarse grid correction for p-multigrid, a local spectral analysis [13] is carried out for different values of p. To do so, the homogeneous problem $\Delta u = 0$ is considered with homogeneous Dirichlet boundary conditions and, hence, $u = 0$ as its exact solution. For an initial guess $\mathbf{u}^0_{h,p}$, the error reduction factors

$$r^S(\mathbf{u}^0_{h,p}) = \frac{||S(\mathbf{u}^0_{h,p})||}{||\mathbf{u}^0_{h,p}||}, \qquad r^{CGC}(\mathbf{u}^0_{h,p}) = \frac{||CGC(\mathbf{u}^0_{h,p})||}{||\mathbf{u}^0_{h,p}||},$$

are computed. Here, $S(\cdot)$ and $CGC(\cdot)$ denote a single smoothing step with Gauss-Seidel and a coarse grid correction, respectively. As initial guess, the generalized eigenvectors are chosen which satisfy

$$\mathbf{A}_{h,p}\mathbf{v}_i = \lambda_i \mathbf{M}_{h,p}\mathbf{v}_i, \ i = 1, \ldots, N,$$

where $\mathbf{M}_{h,p}$ denotes the consistent mass matrix. A similar analysis has been performed in [8].

Figure 2 denotes the reduction factors of the generalized eigenvectors $\mathbf{v}_i$, which are determined for different approximation orders ($p = 2, 3$ and 4). The figures on the left side show the results for the Poisson equation on the unit square, whereas those on the right side are obtained on the quarter annulus geometry. In general, the coarse grid correction reduces the lower part of the spectrum, whereas the smoother reduces the rest of the spectrum. Hence, as with h-multigrid, the coarse grid correction and smoother are complementary to each other. However, the obtained reduction factors for the smoother increase for higher values of p, especially on the quarter annulus, indicating that the smoother has difficulties.

5 CONCLUSIONS

In this paper, we presented p-multigrid based methods to solve linear systems arising in IgA. Numerical experiments show that the number of cycles/iterations is independent of the mesh width h, but (strongly) depends on the approximation order. p-multigrid showed to be competitive to h-multigrid when used as a stand-alone solver, but less efficient as a preconditioner within a Conjugate Gradient method. The coarse grid correction and

the smoothing procedure in p-multigrid complement each other quite well, although the smoother has more difficulties for the model problem on the quarter annulus.

This study should be considered as a first step for the application of p-multigrid methods in the efficient solution of linear systems arising in IgA. In forthcoming works, the use of p-multigrid methods, enhanced with a more effective smoother, will be further investigated. Furthermore, coarsening in both h and p will be considered to reduce the size of the linear system on the coarsest level.

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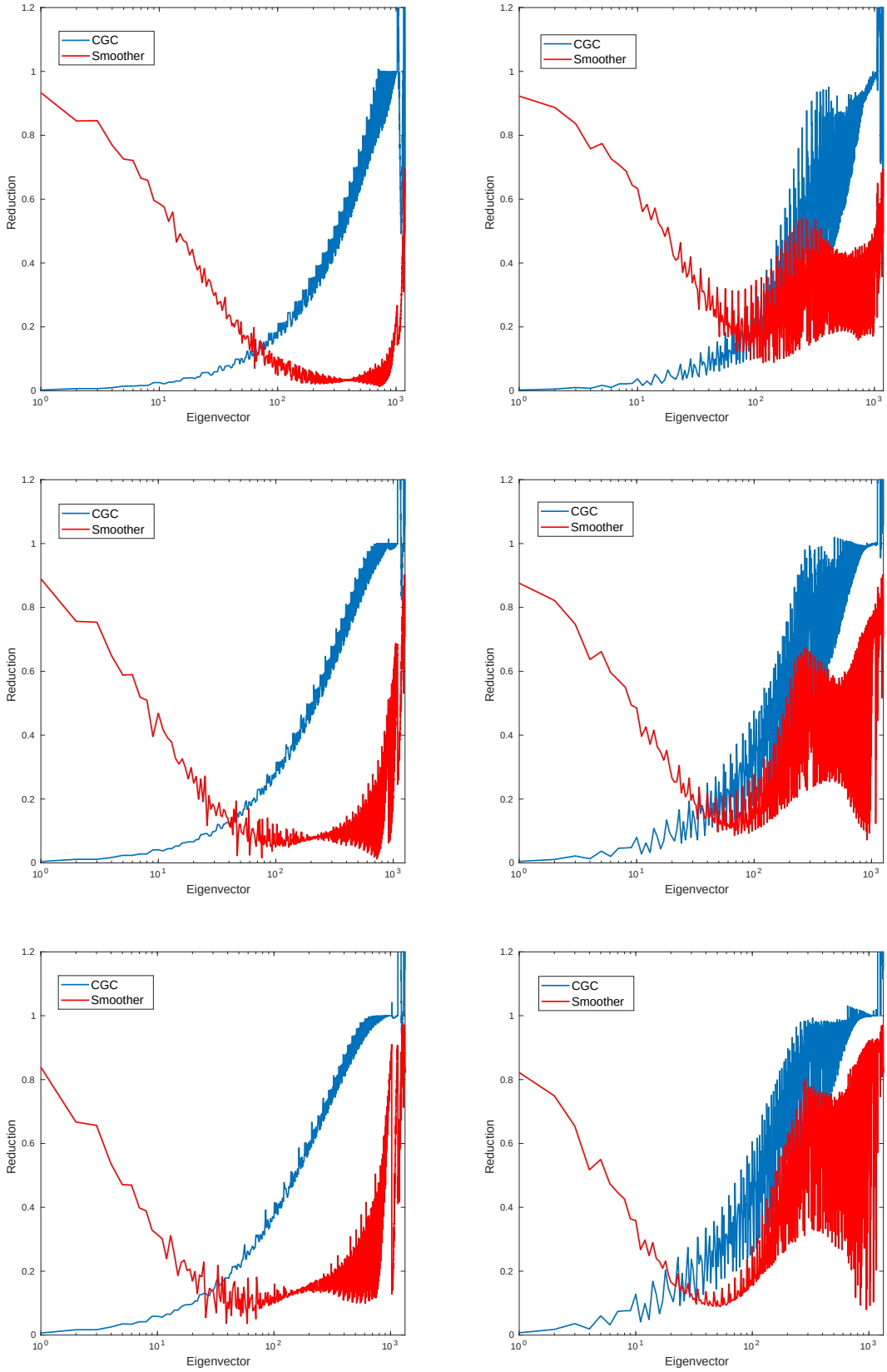

Figure 2: Reduction factors of $(\mathbf{v}_i)$ for $p = 2, 3, 4$ $(\downarrow)$. The figures on the left illustrate results on the unit square, whereas the figures on the right illustrate results on the quarter annulus.