

Series 07 Aerospace Materials 10

Finite Difference Methods for Stress Analysis of Adhesive Bonded Joints

The Design of a MATLAB Adhesive Toolbox

M.J.L. van Tooren/Z.C. Roza



Delft University Press



726600

Finite Difference Methods for Stress Analysis of Adhesive Bonded Joints

The Design of a MATLAB Adhesive Toolbox

Bibliotheek TU Delft



C 3021908

2392
320
1

Series 07: Aerospace Materials 10

Finite Difference Methods for Stress Analysis of Adhesive Bonded Joints

The Design of a MATLAB Adhesive Toolbox

M.J.L. van Tooren / Z.C. Roza



Delft University Press / 1998

Published and distributed by:

Delft University Press
Mekelweg 4
2628 CD Delft
The Netherlands
Telephone +31 (0)15 278 32 54
Fax +31 (0)15 278 16 61
e-mail: DUP@DUP.TUDelft.NL

by order of:

Faculty of Aerospace Engineering
Delft University of Technology
Kluyverweg 1
P.O. Box 5058
2600 GB Delft
The Netherlands
Telephone +31 (0)15 278 14 55
Fax +31 (0)15 278 18 22
e-mail: Secretariaat@LR.TUDelft.NL
website: <http://www.lr.tudelft.nl/>

Cover: Aerospace Design Studio, 66.5 x 45.5 cm, by:
Fer Hakkaart, Dullenbakkersteeg 3, 2312 HP Leiden, The Netherlands
Tel. +31 (0)71 512 67 25

90-407-1717-6

Copyright © 1998 by Faculty of Aerospace Engineering

All rights reserved.

No part of the material protected by this copyright notice may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage and retrieval system, without written permission from the publisher: Delft University Press.

Printed in The Netherlands

Contents

Preface

List of symbols

Summary

1	Introduction	1
2	Shear stress calculations for joints loaded in tension or compression	
2.1	Single lap joints	2
2.1.1	Derivation of governing differential equations	2
2.1.2	Analytical solution: Volkersen Method	4
2.1.3	Numerical solution: Finite Difference Method	5
2.2	Tapered lap joints	7
2.2.1	Derivation of governing differential equations	7
2.2.2	Solution by the Finite Difference Method	7
2.3	Stacked joints	10
2.3.1	Derivation of governing differential equations	10
2.3.2	Solution by the Finite Difference Method	11
3	Shear stress calculations for joints loaded in shear	
3.1	A plate bonded to an undeformable surface	14
3.2	Single lap joints	18
3.2.1	Derivation of governing differential equations	18
3.2.2	Analytical solution: Volkersen Method	20
3.2.3	Numerical solution: Finite Difference Method	21

4	Shear and peel stress calculations for joints loaded in tension or compression	
4.1	Definitions of boundary conditions	22
4.2	Single lap joints	25
4.2.1	Derivation of governing differential equations	25
4.2.2	Numerical solution 1: Two high order difference equations	27
4.2.3	Numerical solution 2: Six first order difference equations	29
4.3	Tapered lap joints	33
4.3.1	Derivation of governing differential equations	33
4.3.2	Numerical solution: Six first order difference equations	34
5	Application : The design of the SIMONA under crown fitting	38
6	Conclusions and recommendations	41
	Literature	42
Appendix A:	MATLAB Adhesive Toolbox Users Guide	43
Appendix B:	Adhesive Toolbox program listings	50

List of symbols

γ, γ_a	Adhesive shear deformation
τ, τ_a	Adhesive shear stress
μ_a	Poisson ratio adhesive layer
σ_i	Tensile stress in adherent layer
μ_i	Poisson ratio adherent layer
τp_i	Adherent layer shear stress
γp_i	Adherent layer shear deformation
σ_y	Peel stress of the adhesive layer
ε_y	Adhesive strain
d	Adhesive layer thickness
d_{ct}	Diameter carbon tube
E_a	Adhesive Youngs modulus
E_i	Youngs modulus adherent layer
F_{lim}	Limit load
F_{ult}	Ultimate load
G_a	Adhesive shear modulus
G_{p_i}, G_{p_l}	Adherent shear modulus
K_i	Moment reduction factor
$K_{i_{hs}}$	Hart-Smith reduction factor
$K_{i_{red}}$	Extra reduction factor
$K_{i_{tap}}$	Extra reduction factor for tapered joints
l	Length of the joint
$L_{non-tap}$	Length of the non-tapered area
M_i	Applied joint edge moment
P	Applied tension or compression load to a adherent layer
P_{tot}	Total applied tension or compression load per unit width
q_i	Applied shear stress to the adherent layer
t_{af}	Thickness aluminium fitting
t_{ct}	Thickness carbon tube
t_p, t_i	Adherent thickness
T_{ratio}	Joint taper ratio
u, U	Displacement in x-direction
v, V	Displacement in y-direction
V_i	Applied joint edge shear force

Summary

The purpose of this report is to present finite difference models (FDM) for adhesively bonded joints and the implementation in a computerized toolbox. Based on a literature study, linear elastic finite difference models are derived. These are programmed in a MATLAB Toolbox.

The first model in the adhesive toolbox is the single lap joint loaded in tension or compression. The model does not take into account the peel stresses and the solution therefore can be directly compared to the Volkersen analytical solution. The results from the FDM models show good agreement to the Volkersen solution.

The second model in the adhesive toolbox is the tapered lap joint loaded in tension or compression. Also this model does not take into account the peel stresses. The results found from the FDM model show clearly, the positive effect of joint taper.

The third model in the adhesive toolbox is the stacked joint loaded in tension that can be found in laminated structures. The peel stress is not taken into account in the model. From the results it is remarkable to see how the effect of layer ending transmits through the thickness of the laminate.

The fourth model in the adhesive toolbox is a deformable plate bonded to an undeformable surface with in plane shear loads. The differential equations are analytically solved.

The fifth model in the adhesive toolbox is a single lap joint loaded with in plane shear forces. This model is analytically and numerically (FDM) solved. The analytical solution is similar to that of the Volkersen solution for single lap joints loaded in tension or compression. There is a good agreement between the numerical and analytical solution.

The sixth model in the adhesive toolbox is again the single lap joint loaded in tension or compression. This model takes into account peel stresses as well shear stresses.

The seventh and last model in the adhesive toolbox the tapered lap joint loaded in tension or compression. This model is the most complicated model in the toolbox so far, it takes into account both shear and peel stresses. To find the joint edge loads, the Hart-Smith K-factor plus an extra reduction factor is used. This extra reduction factor is a function of the geometric parameters of the tapered joint.

It can be concluded that computerized finite difference models are very useful design tools for stress analysis of adhesively bonded joints.

1

Introduction

Adhesive bonding has a long tradition in civil and military aviation. Besides riveting it is today the major jointing technique. Due to the increasing use of modern materials, like fibre reinforced plastics and fibre metal laminates, its application is still growing. Although adhesive bonded joints are frequently used in aircraft structures, there is no comprehensive information on mechanical analysis of thin joints available. The basic formula for the mechanical analysis of adhesively bonded joints have been derived by Volkersen, Golland and Reissner, Hart-Smith. They give analytical design formula but the formula are restricted to simple lap joints.

An other approach is to solve the problem with a numerical method. There are two relevant numerical methods namely Finite Element Method (FEM) and Finite Difference Method (FDM).

The application of FEM is frequently mentioned and investigated but very less is found on the use of FDM for analysis of adhesively bonded joints.

Recent studies in the SIMONA project group at the Delft University of Technology, faculty of Aerospace Engineering required the development of an engineering design tool for the design of the mirror support structure of the SIMONA Research Simulator.

The purpose of this report is to present the analysis of a number of different adhesively bonded joints using of the finite difference method and its implementation into a computerized design toolbox. The designed software has to be user friendly and must have to run on a average computer. Therefore MATLAB 4.2 is used as the program language. MATLAB 4.2 has excellent graphics, build in calculation tools and is very user friendly. Based on literature studies, differential equations with boundary conditions are derived for different joint types. These equations are transformed into finite difference equations. Several verification simulations are done after implementation of the finite difference equations in MATLAB. All computer calculations are done on a 486DX33Mhz with 4Mb and a 486DX66Mhz with 12 Mb.

The contents of this report is as follows. Chapter 2 starts with the calculation of the shear stress distribution in tension loaded joints. This is done for normal lap joints, tapered lap joints and stacked joints. The shear stress analysis of shear loaded joints is discussed in chapter 3. In chapter 4 finite difference models are developed for normal and tapered lap joints with combined shear and peel stress. Chapter 5 describes the application of the derived design tools in the design of the SIMONA under crown fitting. Chapter 6 ends this report with conclusions and recommendations on the FDM to improve the models for better and more realistic stress analysis.

2

Shear stress analysis of adhesively bonded joints loaded in tension or compression.

In this chapter the shear stress distribution in adhesively bonded joints loaded in tension is analyzed. No peel stresses are considered. This model is suitable to describe the stress distribution in tube - to - end - fitting joints and multi plated joints, where bending of the adherents is prevented or negligible small. Therefore it is assumed that the adherents don't bend. The adhesive is exclusively loaded with shear stress. Part 2.1 starts with the derivation of equations for single lap joints which are analytically and numerical solved. Expanding the equations for tapered lap joints is the topic of part 2.2. This chapter ends with shear stress calculations in multi plated stacked joints (2.3).

In this stage the adherents are assumed to be linear elastic isotropic materials. Further more it is assumed that the adherents are in plane stress and the joint has unit thickness in z -direction.

2.1 Single lap joints

2.1.1 Derivation of governing differential equations

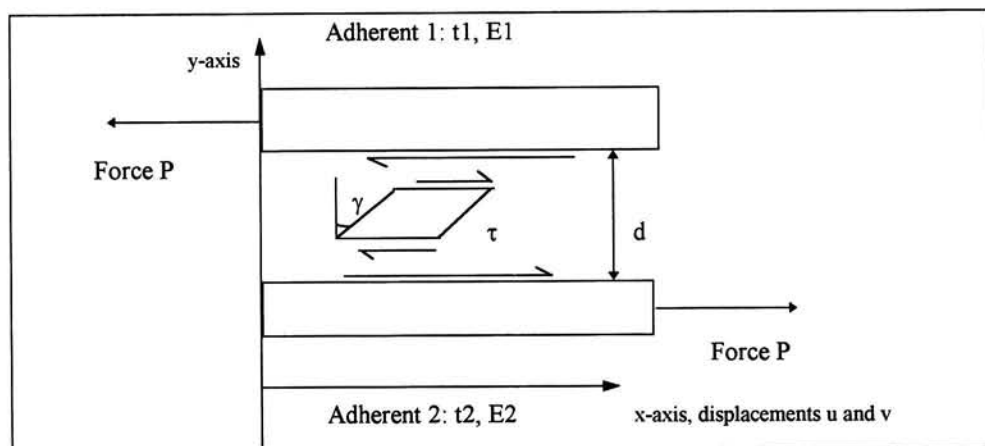


figure 2.1: sign conventions of a single lap joint

The basic equations are derived from the equilibrium of an infinitesimal small segment of adherent one and two.

$$\frac{d^2\sigma_1}{dx^2}t_1 - \tau = 0 \quad (2.1)$$

$$\frac{d^2\sigma_2}{dx^2}t_2 + \tau = 0 \quad (2.2)$$

Expressions 2.3 and 2.4 give the relations between the stresses and the adherent displacements, u and v , respective adherent 1 and 2.

$$\sigma_1 = E_1 \frac{du}{dx} \quad (2.3)$$

$$\sigma_2 = E_2 \frac{dv}{dx} \quad (2.4)$$

The shear strain and stress in the adhesive can be related to the adherent displacements, u and v with the following expressions

$$\gamma = \frac{u - v}{d} \quad (2.5)$$

$$\tau = G_a \gamma \quad (2.6)$$

Differentiation of 2.3 and 2.4 and substitution into 2.1 and 2.2 together with 2.5 and 2.6 results in

$$\frac{d^2u}{dx^2} - \frac{G_a}{d} \frac{1}{E_1 t_1} (u - v) = 0 \quad (2.7)$$

$$\frac{d^2v}{dx^2} + \frac{G_a}{d} \frac{1}{E_2 t_2} (u - v) = 0 \quad (2.8)$$

The boundary conditions belonging to these equations are the following

Boundary conditions adherent 1:

$$\frac{du}{dx} = \frac{P}{E_1 t_1} \quad \text{at } x = 0; \quad \frac{du}{dx} = 0 \quad \text{at } x = l.$$

Boundary conditions of adherent 2:

$$\frac{dv}{dx} = 0 \quad \text{at } x = 0; \quad \frac{dv}{dx} = \frac{P}{E_2 t_2} \quad \text{at } x = l$$

2.1.2. Analytical solution: Volkersen Method

Already in 1938 Volkerson found the analytical solution for equations 2.7 and 2.8 (see reference 7).

$$\tau(x) = \frac{G_a P}{d} \frac{1}{\lambda \sinh(\lambda \cdot l)} \left\{ \frac{\cosh(\lambda \cdot x)}{E_2 t_2} + \frac{\cosh(\lambda \cdot (l - x))}{t_1 E_1} \right\} \quad (2.9)$$

$$\lambda = \sqrt{\frac{G_a}{d} \left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right)} \quad (2.10)$$

This final solution is programmed in MATLAB m-file Volkers.m (See appendix A and B). Figure 2.1 shows the results of a symmetrical and an asymmetrical single lap joint.

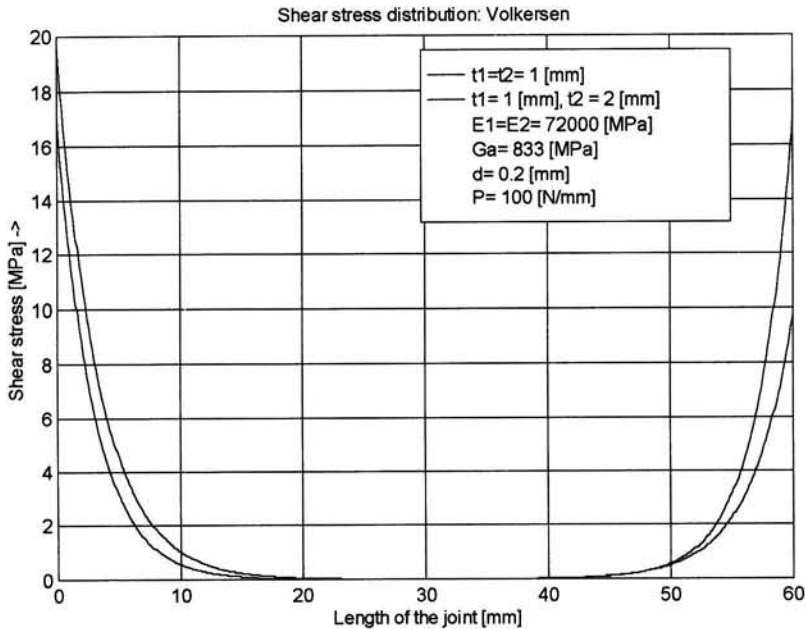
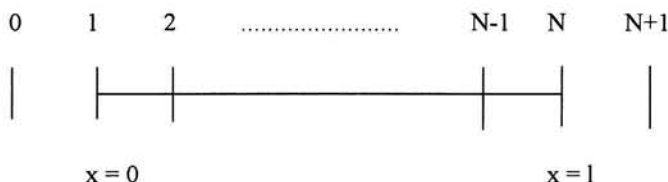


figure 2.1: volkersons shear stress distribution

2.1.3 Numerical solution: Finite Difference Method

To solve the equations (2.7) and (2.8) with its boundary conditions, the interval 0 - 1 is divided into N-1 equal parts.



For each part a finite difference equation is derived with the following expression.

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} = \frac{d^2 u}{dx^2} \Big|_{x_i} + O(h^2) \quad (2.11)$$

This leads to a set of $2N-2$ linear algebraic equations with $2N+2$ unknown variables of the next form.

$$U_{j-1} + (-2 - (\Delta x^2) \frac{Ga}{d} \frac{1}{E_1 t_1}) U_j + U_{j+1} + ((\Delta x^2) \frac{Ga}{d} \frac{1}{E_1 t_1}) V_j = 0 \quad (2.12)$$

for $j = 1$ to N

$$V_{j-1} + (-2 - (\Delta x^2) \frac{Ga}{d} \frac{1}{E_2 t_2}) V_j + V_{j+1} + ((\Delta x^2) \frac{Ga}{d} \frac{1}{E_2 t_2}) U_j = 0 \quad (2.13)$$

for $j = 1$ to N

Substitution of the next boundary conditions in these two equations reduces the number of unknown variables to $2N-2$.

$$U_1 - U_0 = \Delta x \frac{P}{E_1 t_1} \quad \text{and} \quad V_{N+1} - V_N = \Delta x \frac{P}{E_2 t_2}$$

$$U_{N+1} - U_N = 0 \quad \text{and} \quad V_1 - V_0 = 0$$

The set of equations can be written in the following matrix expression.

$$A \cdot \bar{U} = B$$

The matrix is implemented in the MATLAB m-file normjts.m (See appendix A and B). A standard matlab routine will solve the equations by Gaussian elimination. Subtracting of the displacements gives the shear stress in the adhesive. The calculations are done for the same parameters as in 2.1.1 and displayed in figure (2.2). Comparison of the two methods shows the similarity of the results.

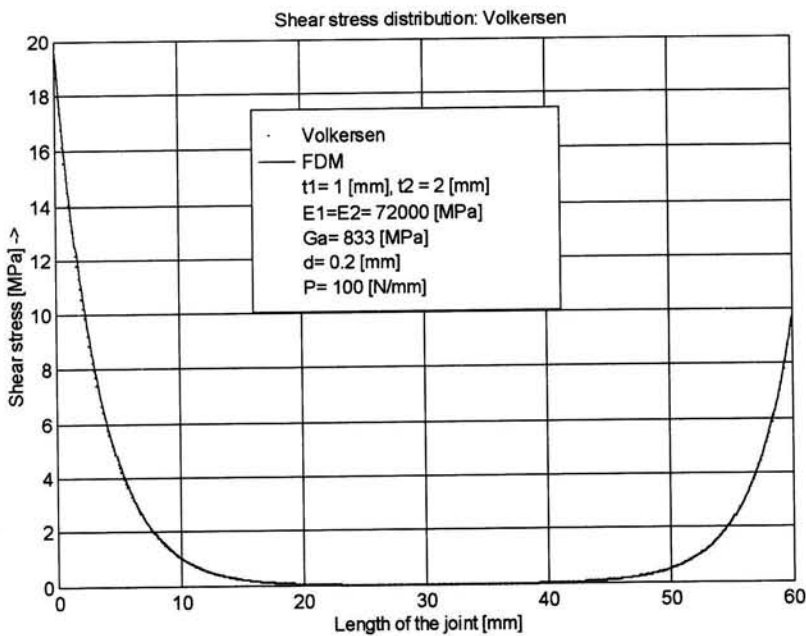


figure 2.2: — F.D.M, Volkersen Method

2.2 Tapered lap joints

2.2.1 Derivation of governing differential equations

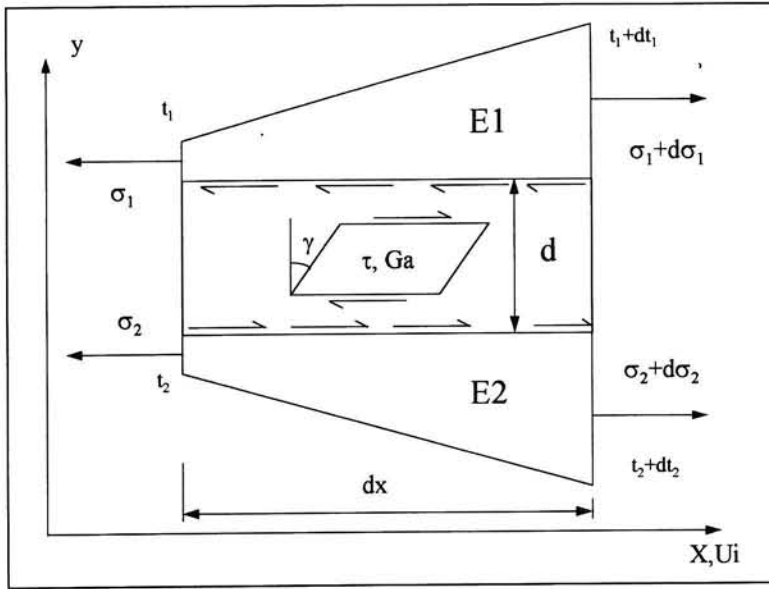


figure 2.4: sign conventions for tapered lap joints

The basic equations are derived from the equilibrium of an infinitesimal small segment of the adherents 1 and 2. In these equations t_1 and t_2 are functions of x .

$$\frac{d^2 u_1}{dx^2} t_1 - \frac{Ga}{E_1} \frac{1}{d} (u_1 - u_2) + \frac{du_1}{dx} \frac{dt_1}{dx} = 0 \quad (2.14)$$

$$\frac{d^2 u_2}{dx^2} t_2 + \frac{Ga}{E_2} \frac{1}{d} (u_1 - u_2) + \frac{du_2}{dx} \frac{dt_2}{dx} = 0 \quad (2.15)$$

2.2.2 Solution by the finite difference method

For each point a finite difference equation is derived with formula 2.11, 2.14 and 2.15 and the next expression.

$$\frac{u_{i+1} - u_i}{\Delta x} = \left. \frac{du}{dx} \right|_{x_i} + O(h) \quad (2.16)$$

Substitution of these equations into 2.14 and 2.15 leads to a set of $2N-2$ linear algebraic equations with $2N+2$ unknown variables.

$$U_{1,j-1} \left\{ \frac{t_{1,j}}{\Delta x^2} \right\} + \left\{ -\frac{1}{\Delta x} \left(\frac{dt_1}{dx} \right)_j - 2 \left(\frac{t_{1,j}}{\Delta x^2} \right) - \frac{Ga}{d} \frac{1}{E_1} \right\} U_{1,j} + \left\{ \frac{1}{\Delta x} \left(\frac{dt_1}{dx} \right)_j + \frac{t_{1,j}}{\Delta x^2} \right\} U_{1,j+1} + \dots$$

$$\dots + \left\{ \frac{Ga}{d} \frac{1}{E_1} \right\} U_{2,j} = 0 \quad \text{for } j = 1 \text{ to } N \quad (2.17)$$

$$U_{2,j-1} \left\{ \frac{t_{2,j}}{\Delta x^2} \right\} + \left\{ -\frac{1}{\Delta x} \left(\frac{dt_2}{dx} \right)_j - 2 \left(\frac{t_{2,j}}{\Delta x^2} \right) - \frac{Ga}{d} \frac{1}{E_2} \right\} U_{2,j} + \left\{ \frac{1}{\Delta x} \left(\frac{dt_2}{dx} \right)_j + \frac{t_{2,j}}{\Delta x^2} \right\} U_{2,j+1} + \dots$$

$$\dots + \left\{ \frac{Ga}{d} \frac{1}{E_2} \right\} U_{1,j} = 0 \quad \text{for } j = 1 \text{ to } N \quad (2.18)$$

Substitution of the following boundary conditions reduces the number of unknown variables to $2N-2$ and the equations can be solved.

Adherent 1.

$$U_1 - U_0 = \Delta x \frac{P}{E_1 t_1(0)} \quad \text{at } x = 0;$$

$$U_{N+1} - U_N = 0$$

$$\text{at } x = 1;$$

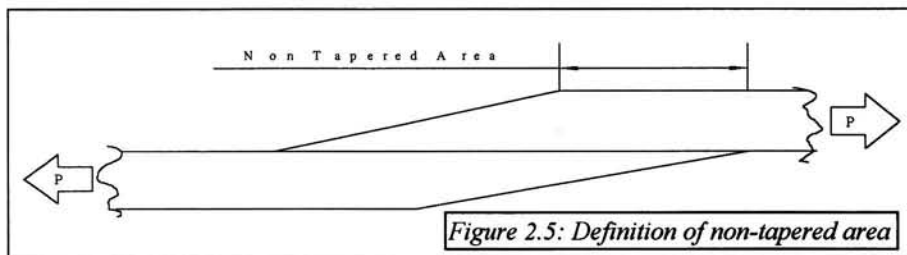
Adherent 2.

$$V_{N+1} - V_N = \Delta x \frac{P}{E_2 t_2(1)} \quad \text{at } x = 1$$

$$V_1 - V_0 = 0$$

$$\text{at } x = 0$$

Together with a geometric modeling routine the equations are entered in the MATLAB m-file `tapjntss.m` (See appendix A and B). The equations are solved by Gaussian elimination. The positive effect of joint taper can be seen in figure (2.6a), several shear stress distributions are given for increasing taper. Taper reduces the shear stress maximum at the end of the joint. The model can also be used for analysis of tapered joints with tapered areas smaller than the overlap length. This is called a non - tapered area. Calculation results are displayed in figure (2.6b).



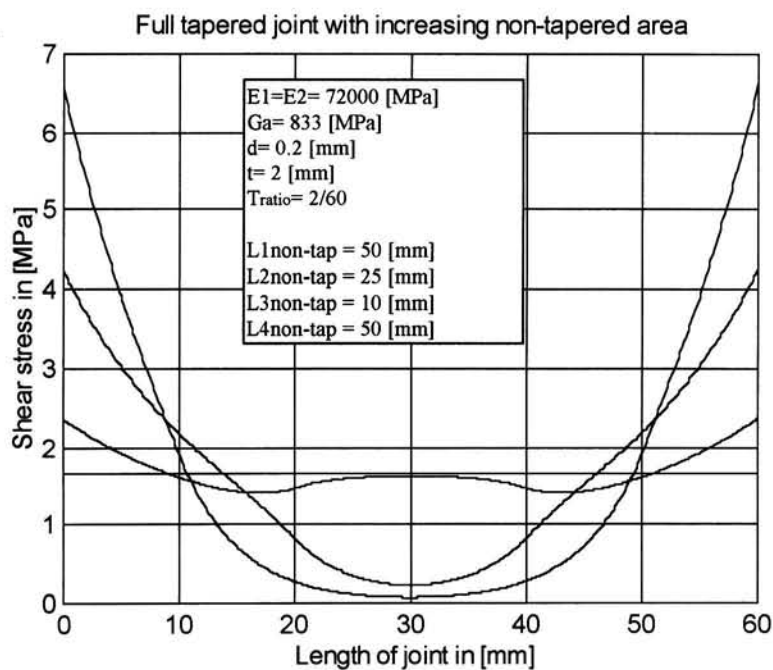
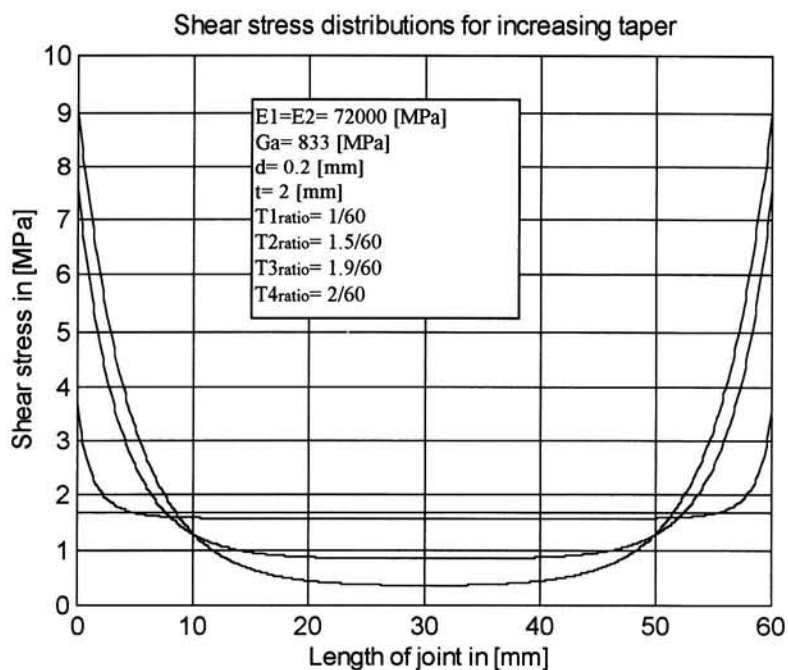


Figure 2.6a and 2.6b: Shear stress distributions in tapered lap joints.

2.3 Stacked lap joints

In this paragraph multi layer adhesively bonded joints will be analyzed which can be found in composite and laminated structures. The general geometric of these joints is given in figure 2.7.

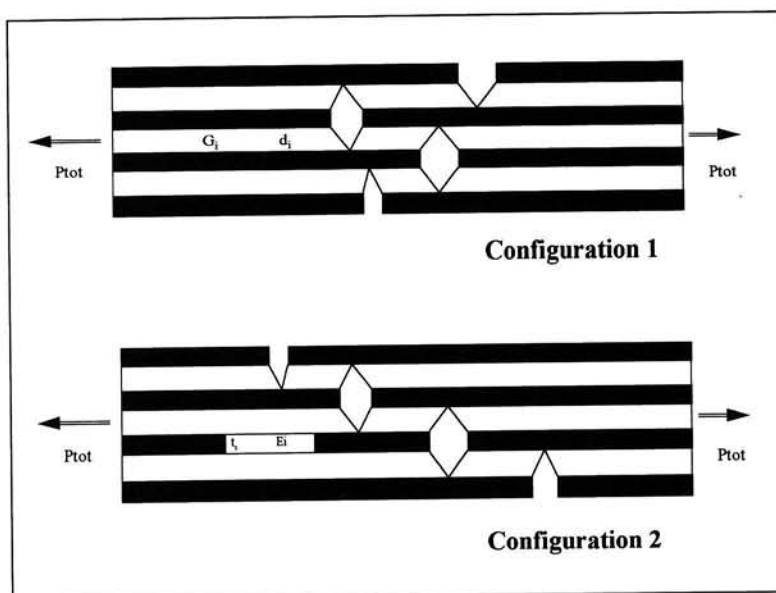


figure 2.7: stacked joint configurations

To solve the problem two assumptions have to be made with respect to the boundary conditions. The first assumption is found by applying the principle of the Saint Venant. Which says that local disturbances remain bounded to the direct area of the disturbance. Therefore it is reasonable to assume that all plates are stiffly attached to each other at some distance from the actual bond.

Secondly it is assumed that the peel stresses in the adhesive between the facing plate ends are negligible compared to the shear stresses.

2.3.1 Derivation of the governing differential equations

For the upper plates the equilibrium of an infinitesimal small segment yields

$$\frac{d^2 u_{1,j}}{dx^2} - \frac{G_1}{d_1 E_1 t_1} (u_{1,j} - u_{2,j}) = 0 \quad \text{for } j = 1 \text{ and } 2 \quad (2.19)$$

For the lower (N th) plates the next expression can be derived

$$\frac{d^2 u_{N,j}}{dx} - \frac{G_{N-1}}{d_{N-1} E_N t_N} (u_{N-1,j} - u_{N,j}) = 0 \quad \text{for } j = 1 \text{ and } 2 \quad (2.20a)$$

In the intermediated (i th) layers the next expression follows from the equilibrium equations for an infinitesimal small segment of each layer.

$$\frac{d^2 u_{i,j}}{dx} - \frac{G_{i-1}}{d_{i-1} E_i t_i} (u_{i-1,j} - u_{i,j}) - \frac{G_i}{d_i E_i t_i} (u_{i,j} - u_{i+1,j}); \quad \text{for } j = 1 \text{ and } 2 \quad (2.20b)$$

The boundary conditions for this problem at 'infinity' are found using assumption 1, which allow for the assumption that the displacements u are equal for all plates at a limited distance from the last plate ending. The distance can be found with the program or guessed educatedly. For the loaded sides of each layer the next expression yields

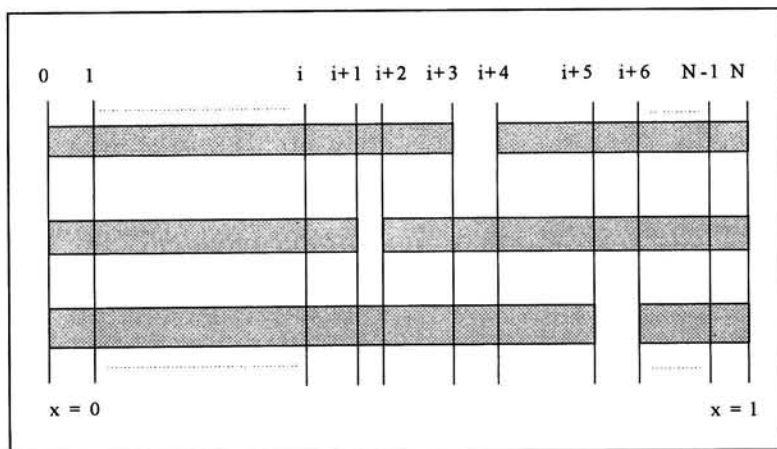
$$\|P_i\| = \frac{E_i t_i P_{\text{tot}}}{(E_1 t_1 + E_2 t_2 + \dots + E_{N-1} t_{N-1} + E_N t_N)} \quad \text{for } i = 1 \text{ to } N \quad (2.21)$$

Assumption 2 for the unloaded sides of each layer yields

$$\|P_i\| = 0 \quad \text{for } i = 1 \text{ to } N \quad (2.22)$$

2.3.2 Solution by the finite difference method

With the formulas (2.11) and (2.16), the differential equations are written in difference equations. To do this the plates must be divided in properly chosen intervals.



With the use of a geometric modeling sequence the plates are partitioned and modeled as above. This together with the difference equations and boundary conditions are programmed in MATLAB m-file `stackjoin.m` (See appendix A and B).

Figure 2.9 shows the results for a joint of type number 1.

Used parameters: $E_1 = E_2 = E_3 = 72000$ [Mpa]
 $t_1 = 1$ [mm], $t_2 = 2$ [mm], $t_3 = 1$ [mm].
 $G_1 = G_2 = 833$ [Mpa]
 $d_1 = d_2 = 0.2$ [mm]
 Applied total load $P = 100$ [Mpa]

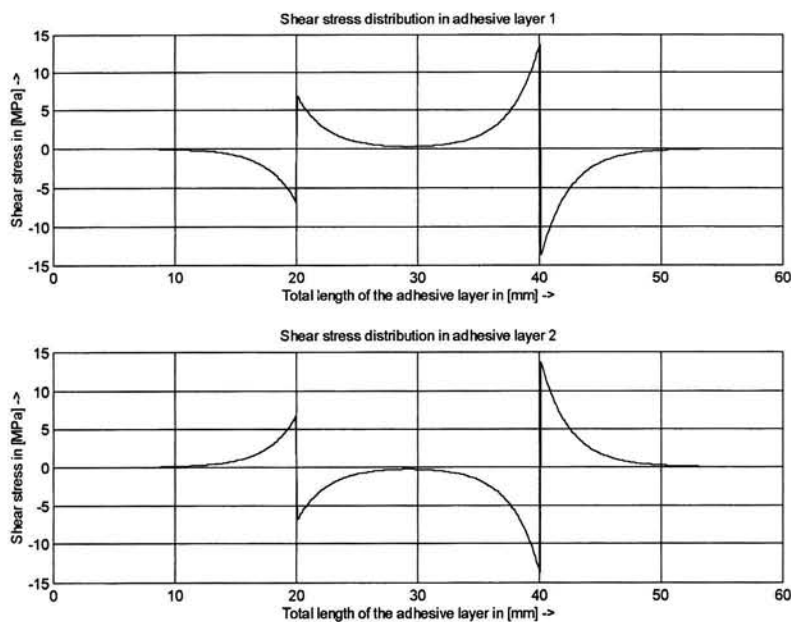


figure 2.9: Shear stress distributions in stacked joint of configuration number 1 with three adherent layers.

Figure 2.10 at the next page shows the result for a joint of type 2. It is remarkable to see how the effect of layer ending transmits through the thickness of the laminate.

Used parameters:

All adhesive layers have similar $G = 833$ [Mpa] and thickness $d = 0.2$ [mm].
 All adherent layers have similar $E = 72000$ [Mpa].
 Thickness of adherent layers: $t_1 = t_3 = 3$ [mm], $t_2 = t_4 = 2$ [mm], $t_5 = 1$ [mm].
 Applied total load $P = 100$ [Mpa]

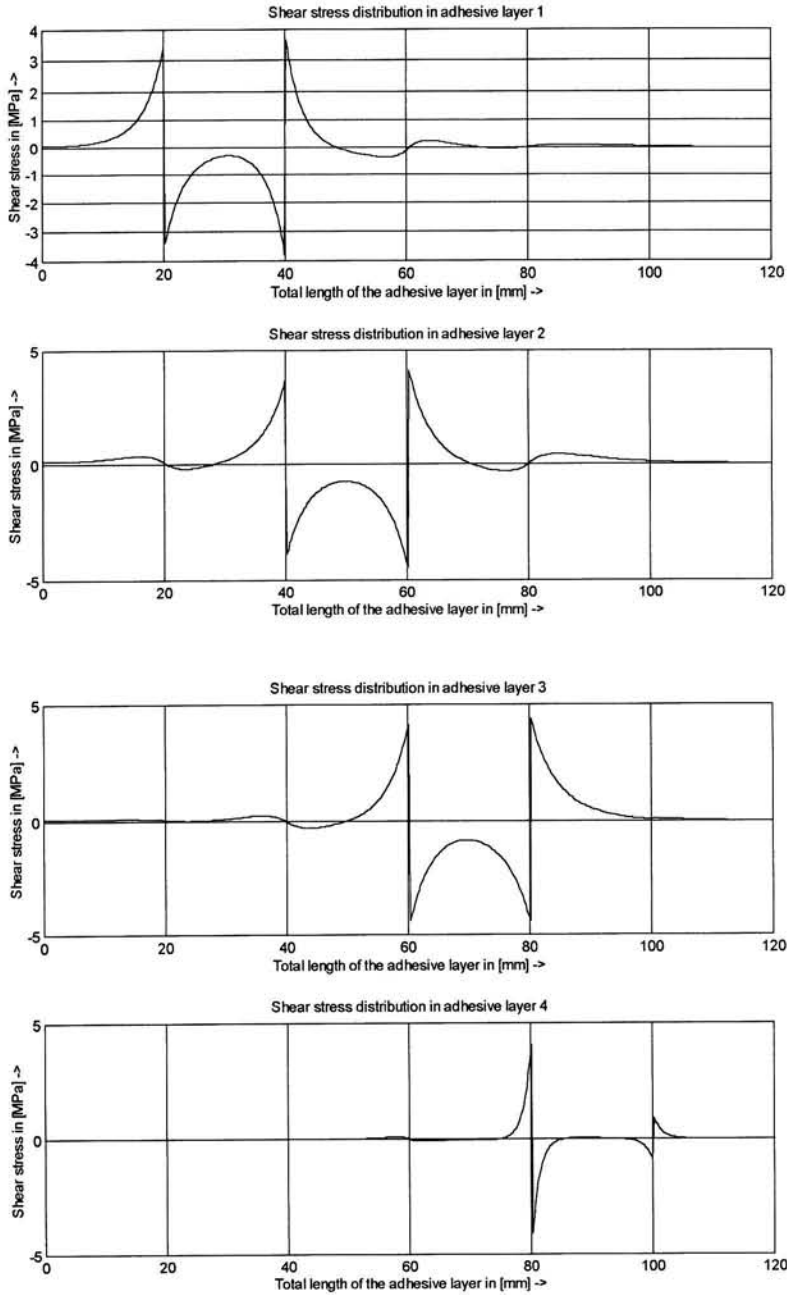


figure 2.10: Shear stress distributions in stacked joint of configuration number 2 with five adherent layers.

3

Shear stress analysis of joints with in plane shear loads

In this chapter the development of some theory and tools for plates loaded with constant shear at the edges is described. Besides joints loaded in tension or compression there are many joints in aircraft structures which are loaded with shear forces. At some distance from the place where the loads are introduced into the plate, it is assumed that the shear stress is constant over the width of the plate. With this assumption the problem reduces to a one-dimensional problem. Part 3.1 starts with a deformable plate loaded with shear stress, bonded to an undeformable surface. For two boundary conditions an analytical and a numerical solution are derived. The theory of 3.1 is expanded into two deformable plates in part 3.2. Both an analytical and a finite difference solution are given.

3.1 A plate bonded to an undeformable surface

In this part a plate loaded in shear adhesively bonded to an undeformable surface is studied.

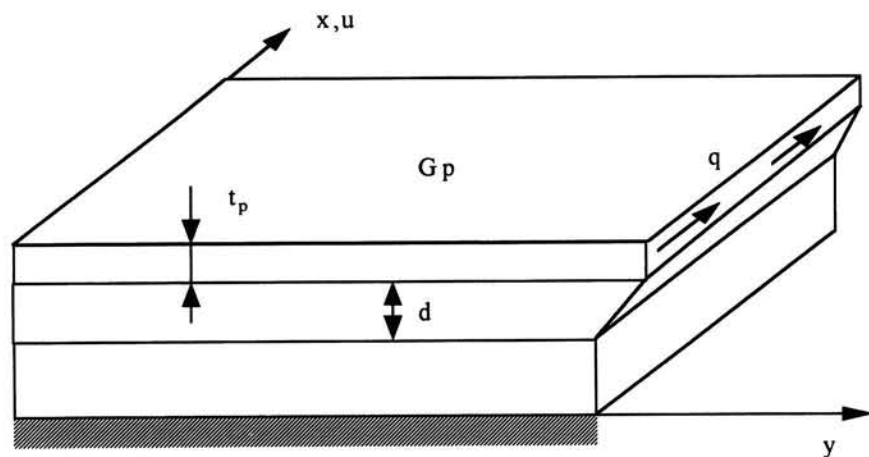


Figure 3.1: A plate bonded to a undeformable surface

In deriving the governing equations only shear deformations and stresses need to be considerate. Because of the assumption that q is constant the problem is reduced to a one-dimensional problem. Only the force equilibrium in x -direction has to be analyzed.

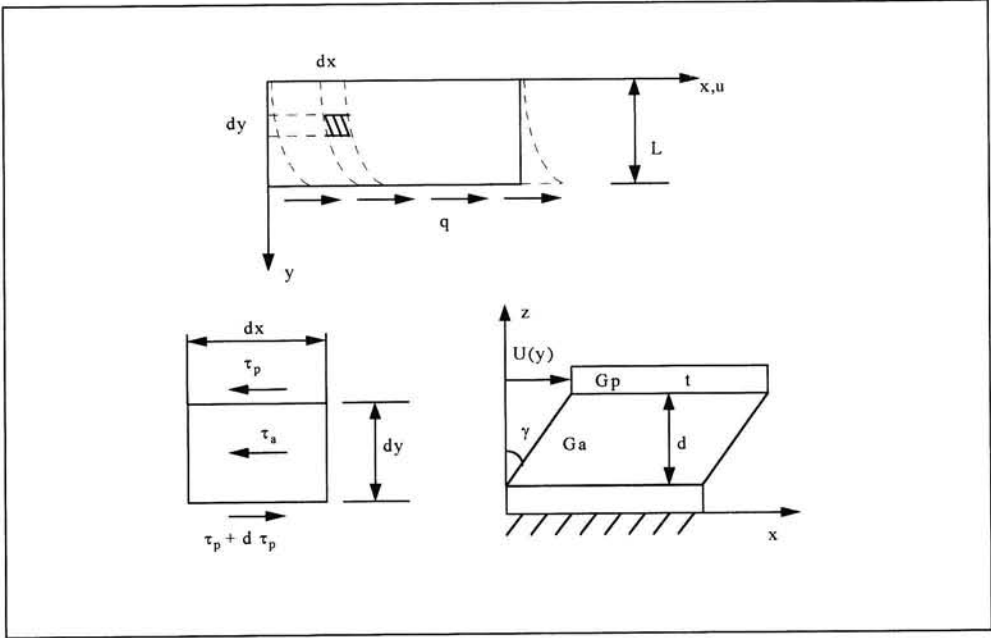


figure 3.2: Sign conventions and loads

Equilibrium of an infinitesimal small segment in x-direction yields

$$\frac{d\tau_p}{dy} dy \cdot t_p = \tau_a dy \quad (3.1)$$

The following expressions relate the shear stress to the shear deformation.

$$\tau_p = G_p \cdot \gamma_p \quad (3.2)$$

$$\tau_a = G_a \cdot \gamma_a \quad (3.3)$$

The compatibility equations are given by

$$\gamma_a(y) = \frac{u(y)}{d} \quad (3.4)$$

$$\gamma_p = \frac{du}{dy} \quad (3.5)$$

Differentiating of 3.2 and substitution with 3.3 to 3.5 into 3.1 gives

$$\frac{d^2u}{dy^2} - \left(\frac{Ga}{Gp \cdot t_p \cdot d} \right) \cdot u = 0 \quad (3.6)$$

The general solution of this second order differential equation is found in Van Duijn (Ref 13)

$$u = C_1 \cdot e^{y\sqrt{a}} + C_2 \cdot e^{-y\sqrt{a}} \quad \text{with } a = \frac{Ga}{Gp \cdot t_p \cdot d} \quad (3.7)$$

Constants C_1 and C_2 are calculated from the boundary conditions

$$\frac{du}{dy} = \frac{q}{Gp} \quad \text{at } y = l \quad (3.8)$$

When the other plate end is fixed, the second boundary condition becomes

$$u(0) = 0 \quad \text{at } x = 0 \quad (3.9)$$

When the other plate end is free and no loads are applied there, the second boundary condition is.

$$\frac{du}{dy} = 0 \quad (3.10)$$

The final solution belonging to boundary conditions 3.8 and 3.9 is.

$$u = \frac{q}{Gp \cdot \sqrt{a}} \cdot \frac{1}{(e^{l\sqrt{a}} + e^{-l\sqrt{a}})} \cdot (e^{y\sqrt{a}} - e^{-y\sqrt{a}}) \quad (3.11)$$

For boundary conditions 3.8 and 3.10 the final solution is given by

$$u = \frac{q}{Gp \cdot \sqrt{a}} \cdot \frac{1}{(e^{l\sqrt{a}} - e^{-l\sqrt{a}})} \cdot (e^{y\sqrt{a}} + e^{-y\sqrt{a}}) \quad (3.12)$$

Both solutions are programmed in the MATLAB m-file shlojnt1.m . Figure 3.3 shows both solutions for the same parameters. Striking is the fact that both solutions are similar. Because only the very first part of the adhesive carries the load, the other end ($x = 0$) is almost shear free. This is only true for long plates, for small joint length a difference will be shown.

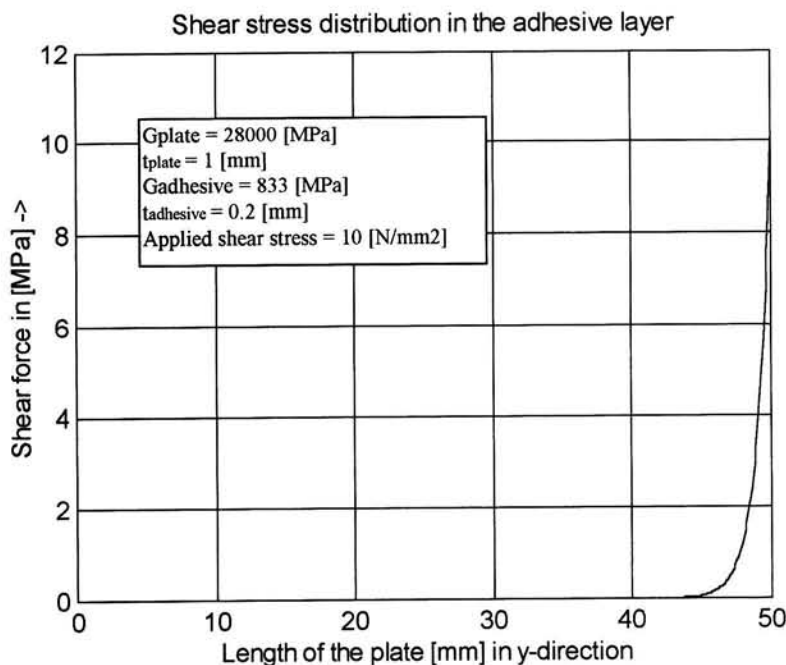


figure 3.3: Comparison between a fixed end and a free end of a joint loaded with shear

3.2 Single lap joints

In this section the analysis of an adhesively bonded joint between two deformable plates loaded with uniform shear stress is described. The theory of this part is an extension of part 3.1. It is shown that the analytical solution and numerical solution have a form similar to the solution for a lap joint loaded in tension (2.1).

3.2.1 Derivation of governing differential equations

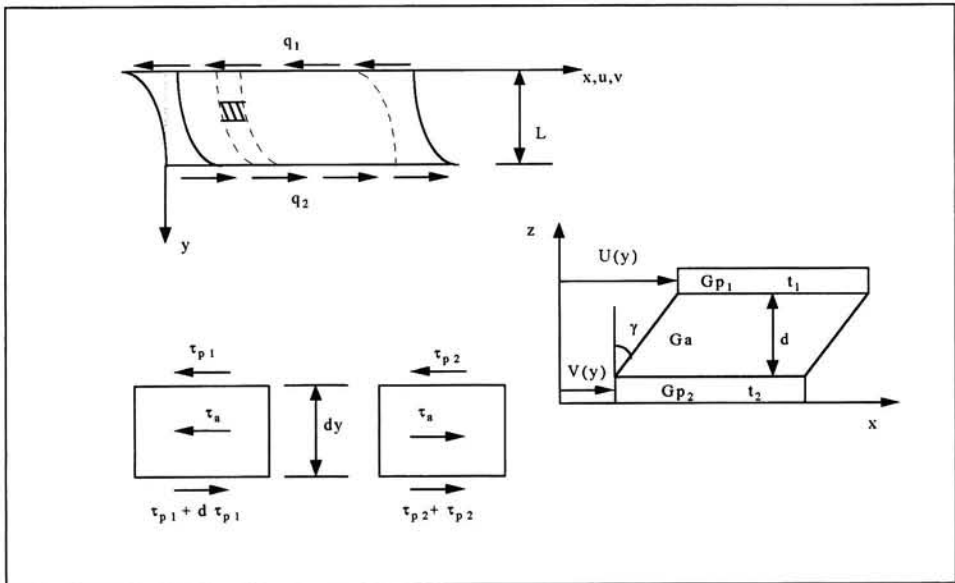


figure 3.4: Sign conventions and loads for two plates loaded with shear.

Equilibrium equations can be derived for small strips of both plates

Adherent 1

$$\frac{d\tau_{p1}}{dy} dy \cdot t_{p1} = \tau_a dy \quad (3.13)$$

Adherent 2

$$\frac{d\tau_{p2}}{dy} dy \cdot t_{p2} = -\tau_a dy \quad (3.14)$$

Relating shear deformation to shear stress leads to the next expressions.

$$\tau_{p1} = Gp_1 \cdot \gamma_{p1} \quad (3.15)$$

$$\tau_{p2} = Gp_2 \cdot \gamma_{p2} \quad (3.16)$$

$$\tau_a = Ga \cdot \gamma_a \quad (3.17)$$

To solve the set of equations, the following compatibility equations are needed.

$$\gamma_{p1} = \frac{du}{dy} \quad (3.18)$$

$$\gamma_{p2} = \frac{dv}{dy} \quad (3.19)$$

$$\gamma_a = \frac{u - v}{d} \quad (3.20)$$

Combination of equations 3.13 to 3.20 leads to the following two second order differential equations

$$\frac{d^2u}{dy^2} - \left(\frac{Ga}{Gp_1 \cdot t_{p1} \cdot d} \right) \cdot (u - v) = 0 \quad (3.21)$$

$$\frac{d^2v}{dy^2} + \left(\frac{Ga}{Gp_2 \cdot t_{p2} \cdot d} \right) \cdot (u - v) = 0 \quad (3.22)$$

The boundary conditions belonging to this problem are

Adherent 1:

$$\frac{du}{dy} = 0 \quad \text{at } x = 0 \quad \frac{du}{dy} = \frac{q_1}{Gp_1} = \frac{V}{Gp_1 \cdot t_1} \quad \text{at } x = l$$

Adherent 2:

$$\frac{dv}{dy} = \frac{q_2}{Gp_2} = \frac{V}{Gp_2 \cdot t_2} \quad \text{at } x = 0 \quad \frac{dv}{dy} = 0 \quad \text{at } x = l$$

3.2.2 Analytical solution: Volkersen method

Looking to equations 3.21 and 3.22 there is a clear resemblance between equations 2.7 and 2.8 of part 2.1.1., also the boundary conditions have the same structure. The only differences are that E is replaced by G and P is replaced by V. Because both sets are of the same structure and have similar boundary conditions, the solution is of the same structure. This means that the Volkersen solution holds also for 3.21 and 3.22. Only E need to be replaced by G and P need to be replaced by V.

$$\tau(x) = \frac{G_a V}{d} \frac{1}{\lambda \sinh(\lambda \cdot l)} \left\{ \frac{\cosh(\lambda \cdot x)}{G_2 t_2} + \frac{\cosh(\lambda \cdot (l - x))}{G_1 t_1} \right\} \quad (3.23)$$

$$\lambda = \sqrt{\frac{G_a}{d} \left(\frac{1}{G_1 t_1} + \frac{1}{G_2 t_2} \right)} \quad (3.24)$$

proof

subtract equations 3.21 and 3.22

$$\frac{d^2 u}{dy^2} - \left(\frac{Ga}{G_{p1} \cdot t_{p1} \cdot d} \right) \cdot (u - v) - \frac{d^2 v}{dy^2} - \left(\frac{Ga}{G_{p2} \cdot t_{p2} \cdot d} \right) \cdot (u - v) = 0 \quad (3.25)$$

$$\tau(x) = Ga \frac{u - v}{d} = \frac{G_a V}{d} \frac{1}{\lambda \sinh(\lambda \cdot l)} \left\{ \frac{\cosh(\lambda \cdot x)}{G_2 t_2} + \frac{\cosh(\lambda \cdot (l - x))}{t_1 G_1} \right\} \quad (3.26)$$

Differentiating 3.26 twice leads to

$$\frac{d^2 u}{dx^2} - \frac{d^2 v}{dx^2} = \frac{V \cdot \lambda}{\sinh(\lambda \cdot l)} \left\{ \frac{\cosh(\lambda \cdot x)}{G_2 t_2} + \frac{\cosh(\lambda \cdot (l - x))}{t_1 G_1} \right\} \quad (3.27)$$

Enter 3.26 and 3.27 in equation 3.25 yields

$$0 \equiv 0$$

Differentiating equation 3.26 once and entering the boundary conditions proves that equation 3.23 is the solution belonging to the given boundary conditions. The difference in sign for the shear stress is caused by the different sign convention used by Volkersen.

3.2.3 Numerical solution: Finite difference method

Since equations 3.21 and 3.22 have a structure similar to equations 2.7 and 2.8 the FDM solution can be derived in a similar fashion. This is therefore not repeated here.

Both the analytical and numerical solution are programmed in MATLAB. For details see `svolkers.m` and `shlojnt2.m` in appendix A. By comparing solution 3.23 with the FDM solution it is clear that the adjusted Volkersen formula 3.23 is indeed the correct analytical solution of equations 3.21 and 3.22. This is displayed in figure 3.5.

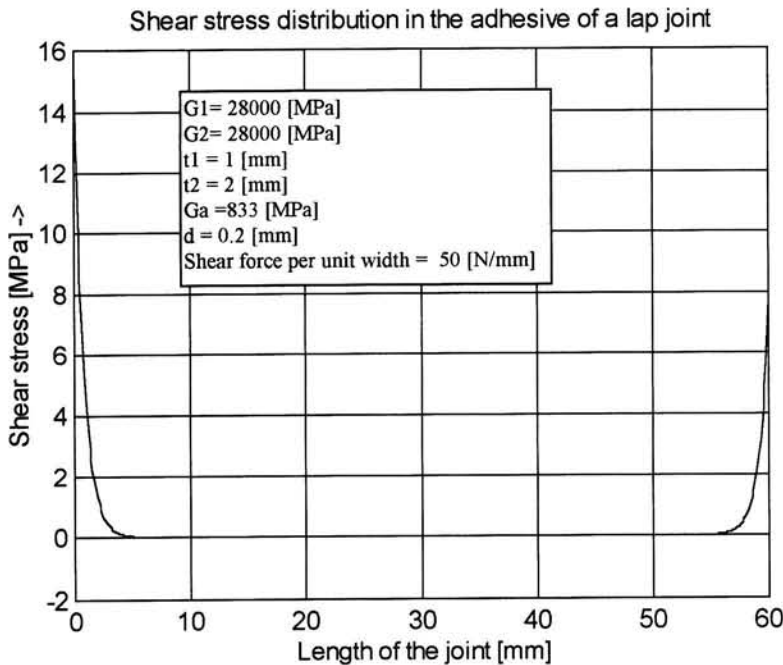


figure 3.5: Comparison between Volkersen for shear loads and Numerical Solution for joints loaded with shear.

4

Shear and peel stress analysis of adhesively bonded joints loaded in tension or compression.

In many airplane structures plates are adhesively bonded and loaded with tensile forces. Because of their flexibility, besides shear stresses also peel stresses will occur in these joints.

This chapter discusses how to model these effects in normal and tapered joints (4.2 and 4.3).

Part 4.1 gives some brief theory about edge moments and edge shear forces.

4.1 Definitions of boundary conditions

One of the major problems in analyzing peel and shear stress is the calculation of loads applied to the adherent/ adhesive sandwich. Here only major remarks are presented in order to enable understanding of the following parts. Thorough examination of this particular problem is beyond the scope of this report.

Tensile loads applied to a lap joint cause a bending moment in both adherents. This is caused by an offset between the lines of action of the tensile loads. To satisfy moment and force equilibrium also shear forces will be introduced.

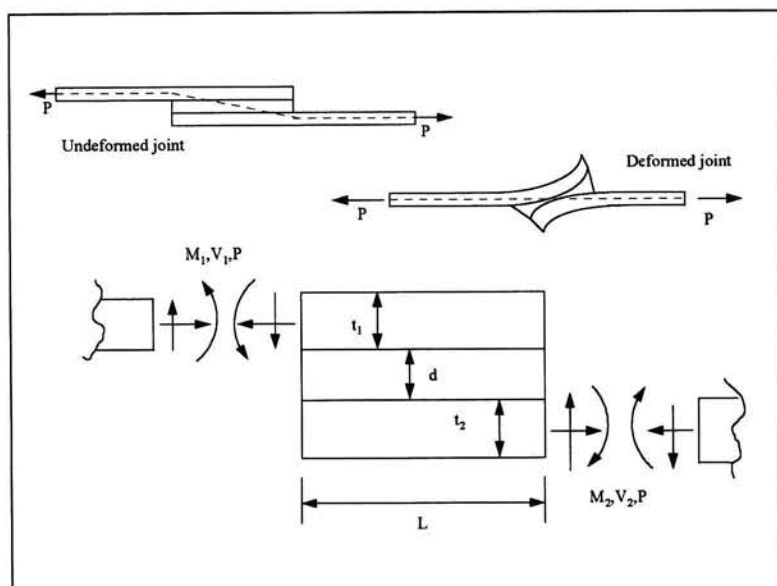


figure 4.1: load sign conventions and joint deformations

The maximum moments and shear forces that can occur are derived from the equilibrium equations for the undeformed joint

$$M_{1_{\max}} = -\frac{P \cdot (t_1 + d)}{2} \quad (4.1)$$

$$M_{2_{\max}} = -\frac{P \cdot (t_2 + d)}{2} \quad (4.2)$$

and $V_1 = V_2 = 0$

In practice the joint adherents will deform (bent) under the applied loads. The effect of this is that the lines of action come closer to each other and the end moments will decrease. The way this effect is generally modeled, is by introducing a K-factor ($0 < K < 1$). This K-factor is a function of geometry, stiffness of the adherents and applied force.

$$M_1^* = K_1 \cdot M_{1_{\max}} \quad 0 < K_1 < 1 \quad (4.3)$$

$$M_2^* = K_2 \cdot M_{2_{\max}} \quad 0 < K_2 < 1 \quad (4.4)$$

With the reduced moments, the following equation for the shear forces is found

$$V_1 = V_2 = \frac{M_2^* - M_1^* - P \cdot \left(\frac{t_1 + t_2}{2} + d \right)}{l} \quad (4.5)$$

The problem that remains now is the estimation of the K-factor. This report is not meant to make a thorough analysis in order to find such a realistic K-factor. Many people have done this before. In Van Ingen and Vlot (Ref 1) the K-factor of Hart-Smith for normal joints is recommended. Therefore this factor will be used in the calculation programs of this report.

$$K_{i_{hs}} = \frac{1}{1 + \lambda \frac{1}{2} + \frac{1}{6} \left(\lambda \frac{1}{2} \right)^2} \quad \text{with} \quad \lambda = \sqrt{\frac{12 \cdot P (1 - \mu_i^2)}{E_i t_i^3}} \quad (4.6)$$

From figure 4.1 it can be seen that the edge moments will be larger with increasing stiffness of the adherents. In other words the K-factor will increase with the adherent stiffness.

The Hart-Smith K-factor only holds for normal lap joints. No formula is given for fully or partially tapered adherents.

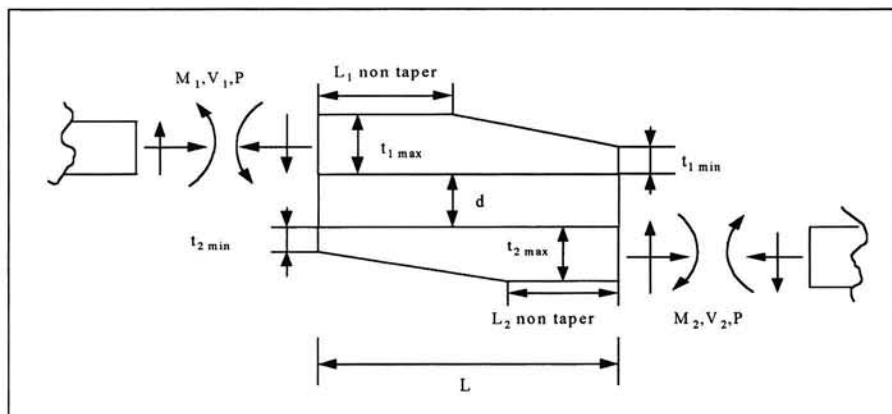


figure 4.2: *load sign conventions and joint deformations*

Due to their geometric tapered joints bent easier than normal lap joints. This indicates that smaller bending moments are required in the deformed position. The flexibility of the adherents depends on the rate of taper and the non-tapered length. It is suggested here that this can be taken into account by adding an extra reduction factor to the Hart-Smith K-factor.

$$K_{i\text{tap}} = K_{i\text{hs}} \cdot K_{i\text{red}} \quad 0 < K_{i\text{red}} < 1 \quad (4.7)$$

$K_{i\text{red}}$ must be at least a function of l , $l_{\text{non-tapered}}$, t_{max} , t_{min} . Using dimension analysis and some engineering 'feeling'

$$K_{i\text{red}} = \frac{1}{1 + \frac{1}{4} \left\{ \left(\frac{1 - l_{\text{nontapered}}}{l} \right) \cdot \left(\frac{t_{\text{max}} - t_{\text{min}}}{t_{\text{max}}} \right) \right\}} \quad (4.8)$$

If full taper is used then the maximum reduction is 0.8. When no taper is used the reduction is 1. The usefulness of this factor has to be further investigated. No proof will be shown here. In the matlab-programs quantitative three options are given for the definitions of the end loads.

- * enter specified end loads
- * using Hart-Smith K-factor
- * using Hart-Smith K-factor plus extra reduction factor.

For the specification of the required end loads it is important that the sign conventions for equilibrium specifications are carefully followed.

4.2 Single lap joints

In this section the mathematical development of a universal analysis of the shear and peel stress distribution in an adhesively bonded joint is discussed. The adherents are in a state of plane strain. They bend under the influence of the edge bending moments.

In part 4.1 the necessary boundary load conditions are derived by the method of Hart-Smith. Part 4.2.1 begins with the derivation of the general governing equations. From the equations of part 4.2.1 two high order differential equations can be obtained. In part 4.2.2. it is proven that those equations cannot be solved by FDM. Therefore in 4.2.3. the problem is solved by the derivation of six coupled first order differential equations which are then solved with the finite difference method.

4.2.1 Derivation of governing differential equations

In deriving the governing equations, the edge loads are chosen such that the equilibrium of the bonded joint is satisfied. The following elementary part of an adherent /adhesive sandwich subjected to a general loading system per unit width is taken to derive the equations.

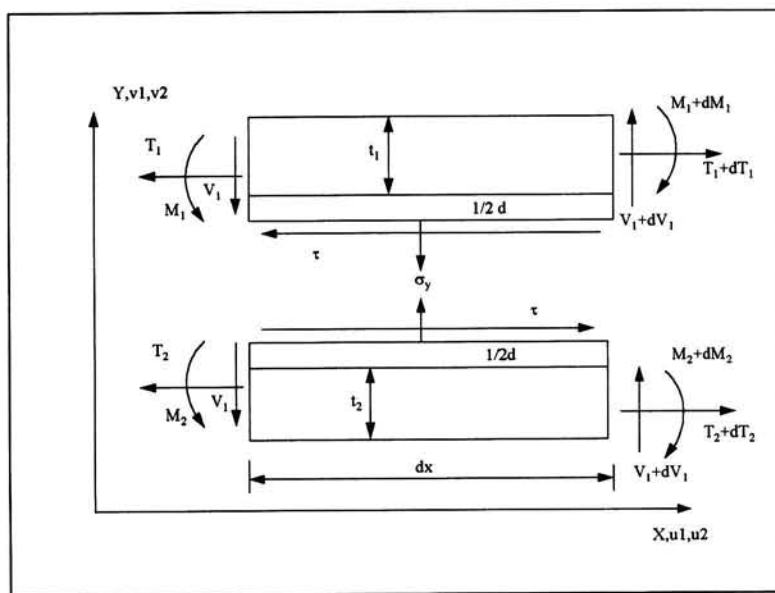


Figure 4.3: *geometry of a single lap joint*

For each adherent combined with a half adhesive layer, the next equilibrium equations hold

Adherent 1.

$$\frac{dT_1}{dx} = \tau$$

$$\frac{dV_1}{dx} = \sigma_y$$

$$\frac{dM_1}{dx} = V_1 - \tau \left(\frac{d + t_1}{2} \right)$$

Adherent 2.

$$\frac{dT_2}{dx} = -\tau \quad (4.9)$$

$$\frac{dV_2}{dx} = -\sigma_y \quad (4.10)$$

$$\frac{dM_2}{dx} = V_2 - \tau \left(\frac{d + t_2}{2} \right) \quad (4.11)$$

The vertical displacements of the adherents are calculated from simple engineering bending theory, Timoshenko 1991 (Ref 14). With the assumption of bending in plane strain the vertical displacements can be written as

Adherent 1.

$$\frac{d^2 V_1}{dx^2} = \frac{-M_1}{12 \cdot (1 - \mu_1^2)} \cdot E_1 t_1^3$$

Adherent 2.

$$\frac{d^2 V_2}{dx^2} = \frac{-M_2}{12 \cdot (1 - \mu_2^2)} \cdot E_2 t_2^3 \quad (4.12)$$

The horizontal displacements of the adherents are caused by tensile and moment loads. Besides plane strain bending, the plate is assumed to be in plane stress tension. Therefore the horizontal displacements of the adherents at the interface of adhesive / adherent become

$$\frac{du_1}{dx} = \frac{1}{E_1 \cdot t_1} \left[T_1 - \frac{6 \cdot (1 - \mu_1^2)}{t_1} \cdot M_1 \right] \quad (4.13)$$

$$\frac{du_2}{dx} = \frac{1}{E_2 \cdot t_2} \left[T_2 + \frac{6 \cdot (1 - \mu_2^2)}{t_2} \cdot M_2 \right] \quad (4.14)$$

Assuming plain strain for relating stresses in the adhesive to strain, the next expressions are produced

$$\sigma_y = \frac{E_a \cdot \varepsilon_y}{(1 - \mu_a^2)} \quad (4.15)$$

$$\tau = \frac{E_a \cdot \gamma}{2 \cdot (1 + \mu_a)} \quad (4.16)$$

To solve the differential equations two compatibility equations are required. These are found by relating the adhesive strains to the adherents displacements as follows

$$\varepsilon_y = \frac{(v_1 - v_2)}{d} \quad (4.17)$$

$$\gamma = \frac{(u_1 - u_2)}{d} \quad (4.18)$$

Together with the boundary conditions the system can now be solved.

4.2.2 Numerical solution 1: 2 high order difference equations

The shear and peel stress distributions can be described with two coupled differential equations of the third and fourth order respectively. These equations are derived from the equations given in 4.2.1

$$\frac{d^3 \tau}{dx^3} - K1 \frac{d\tau}{dx} = -K2 \cdot \sigma_y \quad (4.19)$$

where

$$K1 = f(Ga, d, \mu_1, \mu_2, t_1, t_2, E_1, E_2)$$

$$K2 = f(Ga, d, \mu_1, \mu_2, t_1, t_2, E_1, E_2)$$

and

$$\frac{d^4 \sigma_y}{dx^4} + K3 \cdot \sigma_y = K4 \cdot \frac{d\tau}{dx} \quad (4.20)$$

where

$$K3 = f(Ga, d, \mu_1, \mu_2, t_1, t_2, E_1, E_2)$$

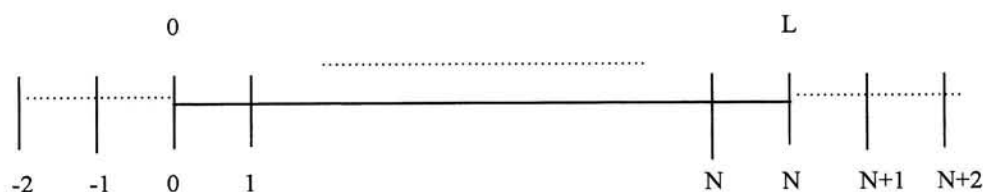
$$K4 = f(Ga, d, \mu_1, \mu_2, t_1, t_2, E_1, E_2)$$

Transforming equations 4.19 and 4.20 into difference equations requires a third order and fourth order difference derivative. They are obtained using Taylor expansions (See Kan 1993). The resulting derivatives are

$$\frac{d^4 \sigma}{dx^4} \Big|_i = \frac{\sigma_{i-2} - 4\sigma_{i-1} + 6\sigma_i - 4\sigma_{i+1} + \sigma_{i+2}}{\Delta x^4} + O(\Delta x^2) \quad (4.21)$$

$$\frac{d^3 \tau}{dx^3} \Big|_i = \frac{\tau_{i+2} - 2\tau_{i+1} + 2\tau_{i-1} - \tau_{i-2}}{2\Delta x^3} + O(\Delta x^2) \quad (4.22)$$

The joint is divided into equal parts with $N+5$ iteration points.



Transforming of the third order differential equation 4.19 leads to a set of $N-1$ equations with $N+3$ unknown variables. Only 3 boundary conditions are given for this equation. For the analytical solution these would be sufficient (See Crocombe 1989). Solving the equation by FDM, one extra boundary condition is needed. The extra boundary conditions is found from a first order approximation

$$\frac{\tau_0 - \tau_1}{\Delta x} = \frac{d\tau}{dx} \Big|_0 + O(\Delta x) \quad (4.23)$$

Transforming of the fourth order differential equation 4.20 produces a set of N difference equations with $N+4$ unknown variables. For this equation the tensile stresses at the four joint edges are known, which are added to the set of equations

$$\overline{\overline{A}} \cdot \overline{\overline{x}} = \overline{\overline{b}} \quad (4.24)$$

The resulting set of equations could not be properly solved, caused by the badly conditioned A -matrix.

4.2.3 Numerical solution 2: six first order difference equations

To overcome the problem caused by using two high order equations, another approach is tried. It is possible to derive six first order differential equations to describe the problem. This method has the disadvantage of a larger number of equations resulting in an increased calculation time. It appears that for accurate results a high number of steps is required.

Starting point of the analysis is formed by the equilibrium equations of an adherent/ adhesive sandwich cutted at a location x .

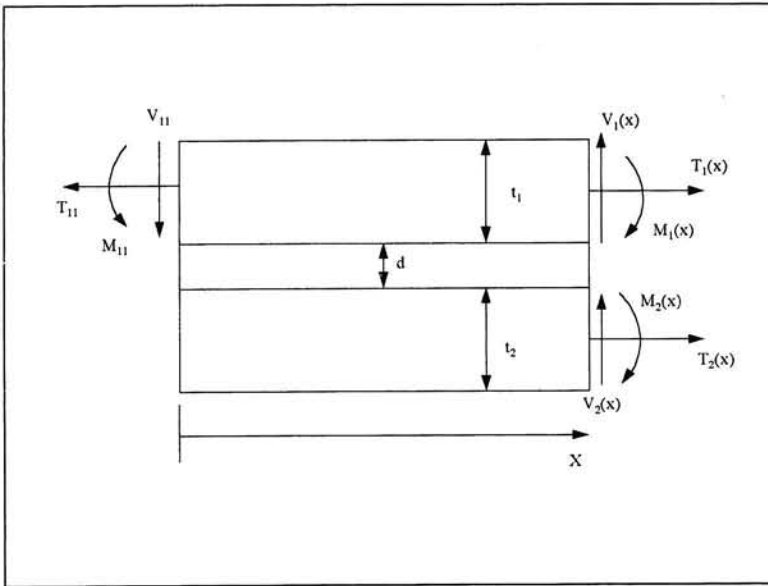


figure 4.4: equilibrium of a cutted adhesive/ adherent sandwich

With the equilibrium equations, $T_2(x)$ and $M_2(x)$ can be expressed as a function of $T_1(x)$ and $M_1(x)$

$$T_2(x) = T_{11} - T_1(x) \quad (4.25)$$

$$M_2(x) = (T_{11} - T_1(x)) \cdot \left(d + \frac{t_1 + t_2}{2} \right) - M_1(x) + M_{11} + V_{11} \cdot x \quad (4.26)$$

Differentiating the compatibility equations once resp twice yields

$$\frac{dy}{dx} = \frac{1}{d} \cdot \left(\frac{du_1}{dx} - \frac{du_2}{dx} \right) \quad (4.27)$$

$$\frac{d^2 \varepsilon_y}{dx^2} = \frac{1}{d} \cdot \left(\frac{d^2 v_1}{dx^2} - \frac{d^2 v_2}{dx^2} \right) \quad (4.28)$$

Together with the deformation formula of the adherents, formula 4.25 and 4.26, these equations are expressed in $T_1(x)$ and $M_1(x)$ only. By adding equilibrium equations 4.6 to 4.8 of adherent 1 a complete set of five equations is produced, four first order and one second order equation.

$$\frac{dT_1}{dx} = \gamma \cdot \frac{E_a}{2 \cdot (1 + \mu_a)} \quad (4.29)$$

$$\frac{dV_1}{dx} = \varepsilon_y \cdot \frac{E_a}{(1 - \mu_a^2)} \quad (4.30)$$

$$\frac{dM_1}{dx} = V_1 - \gamma \cdot \frac{E_a \cdot (d + t_1)}{4 \cdot (1 + \mu_a)} \quad (4.31)$$

$$\begin{aligned} \frac{dy}{dx} = \frac{1}{d \cdot E_2 \cdot t_2} \cdot \left[T_1(x) \cdot \left\{ 1 + \frac{E_2 \cdot t_2}{E_1 \cdot t_1} + \frac{6 \cdot (1 - \mu_2^2)}{t_2} \cdot hl \right\} + 6 \cdot M_1(x) \cdot \left\{ \frac{(1 - \mu_2^2)}{t_2} - \frac{E_2 \cdot t_2 \cdot (1 - \mu_1^2)}{E_1 \cdot t_1^2} \right\} \dots \right. \\ \left. \dots - \frac{6 \cdot (1 - \mu_2^2) \cdot V_{11} \cdot x}{t_2} - T_{11} - \frac{6 \cdot (1 - \mu_2^2) \cdot (M_{11} + T_{11} \cdot hl)}{t_2} \right] \\ \text{with } hl = \frac{t_1 + t_2}{2} + d \end{aligned} \quad (4.32)$$

The second order equation 4.28 is transferred into two first order differential equations

$$\frac{dK}{dx} = \frac{1}{d} \cdot \left[\frac{T_{11} \cdot h1 + M_{11} + V_{11} \cdot x}{D2} - T_1(x) \cdot \frac{h1}{D2} - M_1(x) \cdot \left(\frac{1}{D1} + \frac{1}{D2} \right) \right] \quad (4.33)$$

$$\frac{d\varepsilon_y}{dx} = K \quad (4.34)$$

$$\text{with } D1 = \frac{E_1 \cdot t_1^3}{12 \cdot (1 - \mu_1^2)} \quad \text{and} \quad D2 = \frac{E_2 \cdot t_2^3}{12 \cdot (1 - \mu_2^2)}$$

Using a first order difference derivative, the six differential equations are transformed into a system of 6N algebraic difference equations with 6N+6 unknown variables.

$$\frac{y_{i+1} - y_i}{\Delta x} = \left. \frac{dy}{dx} \right|_i + O(\Delta x) \quad (4.35)$$

There are six boundary loads given which are used as the boundary conditions for this problem. This results in the next system which is programmed in MATLAB m-file normjntp.m (See appendix A and B)

$$\overline{\overline{A}} \cdot \overline{\overline{x}} = \overline{\overline{b}} \quad (4.36)$$

Figure 4.5 shows an example of a shear and peel stress distribution for a asymmetrical single joint with an applied load of 100 [MPa] , using the Hart-Smith K-factor.

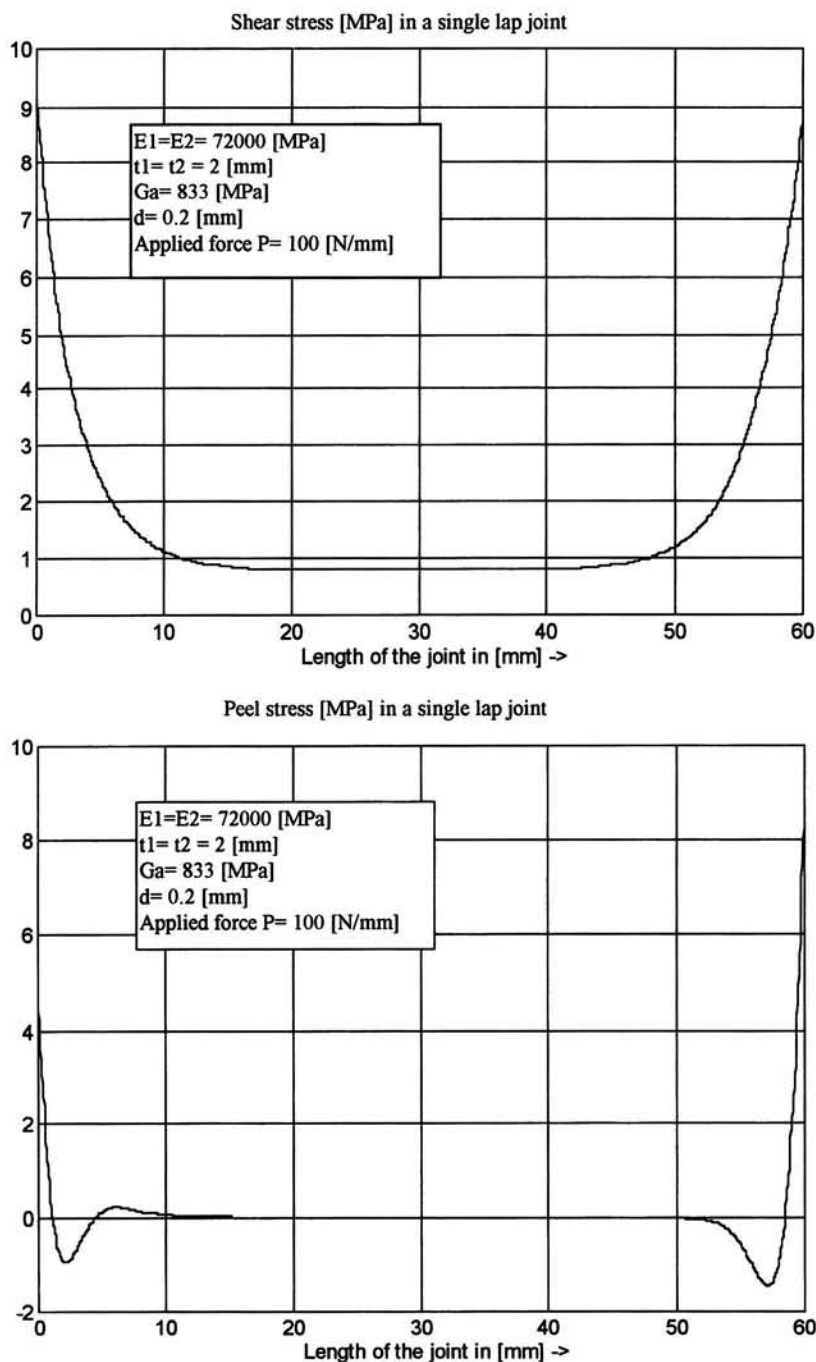


Figure 4.5: Shear and peel stress distribution in a single lap joint ($N = 2000$ steps)

4.3 Tapered lap joints

In this section shear and peel stress distributions in adhesively bonded tapered joints are analyzed. The theory discussed in this part is an extension, with the same assumptions, of the theory of section 4.2. The major difference with the model mentioned in 4.2 is that the geometric parameters are no longer constants but are now a function of x .

4.3.1. Derivation of governing differential equations

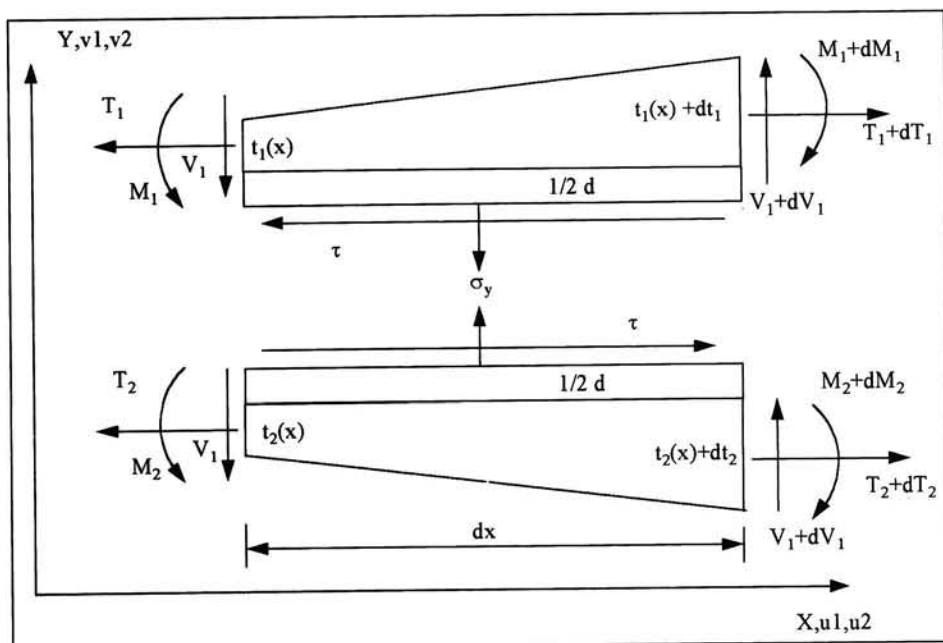


figure 4.6: equilibrium of a tapered lap joint

The force equilibrium equations are the same as those in 4.2. The taper adds an extra term to the moment equation

$$\frac{dM_1}{dx} = V_1 - \tau \left(\frac{d + t_1(x)}{2} \right) - T_1(x) \cdot \frac{1}{2} \cdot \frac{dt_1(x)}{dx}$$

$$\frac{dM_2}{dx} = V_2 - \tau \left(\frac{d + t_2(x)}{2} \right) + T_2(x) \cdot \frac{1}{2} \cdot \frac{dt_2(x)}{dx} \quad (4.37)$$

Also the formula 4.12 for the deformation remains the same. Only here the thickness is a function of x .

4.3.1. Numerical solution: six difference equations

To derive the six governing first order differential equations, moment and force equilibrium of a cutted adhesive/ adherend sandwich is considered.

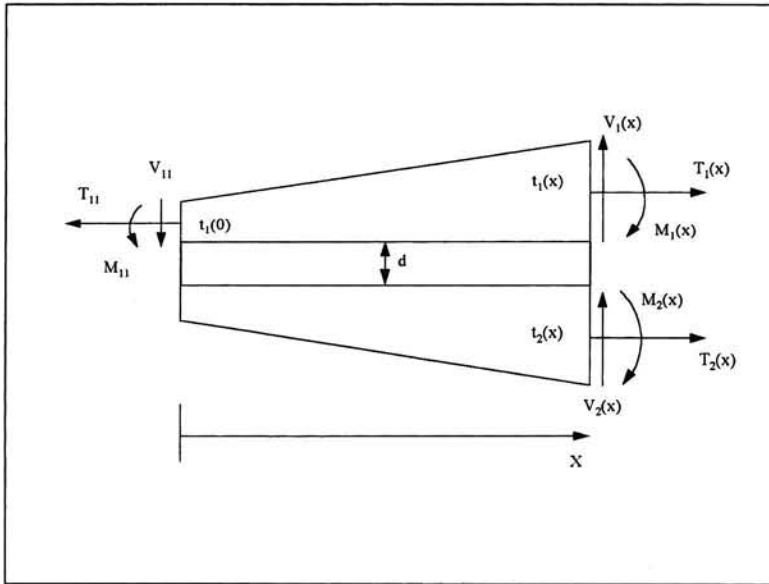


figure 4.7: equilibrium of a cutted tapered joint

From the equilibrium equations $T_2(x)$ and $M_2(x)$ can be expressed as functions of $T_1(x)$ and $M_1(x)$

$$T_2(x) = T_{11} - T_1(x) \quad (4.38)$$

$$M_2(x) = T_{11} \cdot \left(d + \frac{t_1(0) + t_2(x)}{2} \right) - T_1(x) \cdot \left(d + \frac{t_1(x) + t_2(x)}{2} \right) - M_1(x) + M_{11} + V_{11} \cdot x \quad (4.39)$$

Using the compatibility equations 4.24 and 4.25 the follow six governing equations are found

$$\frac{dT_1}{dx} = \gamma \cdot \frac{E_a}{2 \cdot (1 + \mu_a)} \quad (4.40)$$

$$\frac{dV_1}{dx} = \epsilon_y \cdot \frac{E_a}{(1 - \mu_a^2)} \quad (4.41)$$

$$\frac{dM_1}{dx} = V_1 - \gamma \cdot \frac{E_a \cdot (d + t_1(x))}{4 \cdot (1 + \mu_a)} - T_1(x) \cdot \frac{1}{2} \cdot \frac{dt_1(x)}{dx} \quad (4.42)$$

$$\begin{aligned} \frac{d\gamma}{dx} = \frac{1}{d \cdot E_2 \cdot t_2} \cdot \left[T_1(x) \cdot \left\{ 1 + \frac{E_2(x) \cdot t_2(x)}{E_1(x) \cdot t_1(x)} + \frac{6 \cdot (1 - \mu_2^2)}{t_2(x)} \cdot h1(x) \right\} + 6 \cdot M_1(x) \cdot \left\{ \frac{(1 - \mu_2^2)}{t_2(x)} - \dots \right. \right. \\ \left. \left. \dots \frac{E_2(x) \cdot t_2(x) \cdot (1 - \mu_1^2)}{E_1(x) \cdot t_1(x)^2} \right\} - \frac{6 \cdot (1 - \mu_2^2) \cdot V_{11} \cdot x}{t_2(x)} - T_{11} - \frac{6 \cdot (1 - \mu_2^2) \cdot (M_{11} + T_{11} \cdot h2(x))}{t_2(x)} \right] \end{aligned}$$

$$\text{with } h1(x) = \frac{t_1(x) + t_2(x)}{2} + d \quad \text{and} \quad h2(x) = \frac{t_1(0) + t_2(x)}{2} + d \quad (4.43)$$

The second order equation is written as two first order differential equations

$$\frac{dK}{dx} = \frac{1}{d} \cdot \left[\frac{T_{11} \cdot h2(x) + M_{11} + V_{11} \cdot x}{D2(x)} - T_1(x) \cdot \frac{h1(x)}{D2(x)} - M_1(x) \cdot \left(\frac{1}{D1(x)} + \frac{1}{D2(x)} \right) \right] \quad (4.44)$$

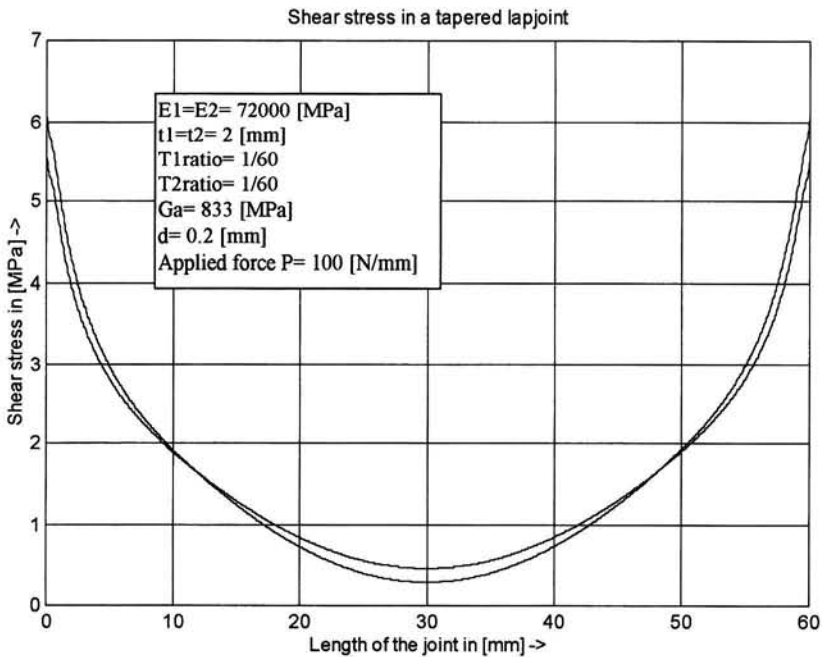
$$\frac{d\epsilon_y}{dx} = K \quad (4.45)$$

$$\text{with } D1(x) = \frac{E_1(x) \cdot t_1(x)^3}{12 \cdot (1 - \mu_1^2)} \quad \text{and} \quad D2(x) = \frac{E_2(x) \cdot t_2(x)^3}{12 \cdot (1 - \mu_2^2)}$$

Equations 4.40 to 4.45 are transformed into six difference equations. Together with the geometric generator of `tapjntss.m` (See appendix A and B), they are programmed in the matlab file `tapjntps.m` (see appendix A and B).

Good results are obtained with more than 1000 points. From figure 4.8 it follows that the effect of taper is less than in the case with only shear stress. This is caused by the end loads.

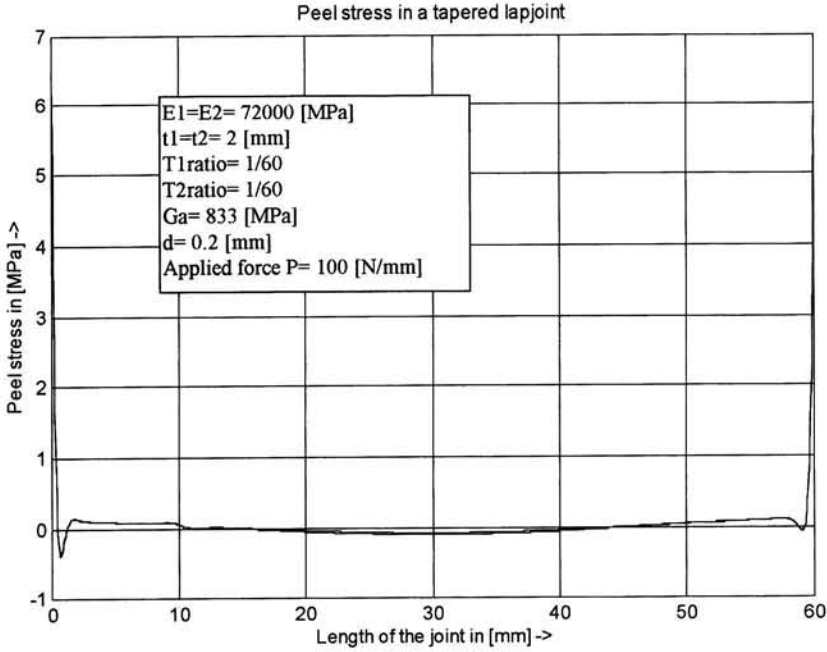
Through application of the extra reduction factor both peel and shear stress magnitudes are smaller than for a Hart-Smith K-factor only. The validity of this reduction factor however needs to be proved analytically and in practice.



Shear stress max: 6.8 MPa for Hart-Smith K-factor only.

Shear stress max: 5.5 MPa for Hart-Smith plus extra reduction factor

figure 4.8a: Shear stress distribution in a tapered lap joint ($N = 1000$ steps)
for Hart-Smith K-factor and for extra reduction factor.



Peel stress max: 6.2 MPa for Hart-Smith K-factor only.

Peel stress max: 5.6 MPa for Hart-Smith plus extra reduction factor

*figure 4.8b: Peel stress distribution in a tapered lap joint ($N = 1000$ steps)
for Hart-Smith K-factor and for extra reduction factor.*

5

The design of the SIMONA under crown fitting

This chapter is an example of how to use the adhesive design toolbox in a practical design task. The example used here is the design of the SIMONA Research Simulator under crown fitting. The simulator consist of three major parts, hydraulic motion system, composite shuttle and display system. To hold and connect the the mirror of the visual display system to the shuttle, the mirror is connected to an under and an upper crown structure. These mirror support structures consist of carbon tubes which are adhesively bonded to aluminum fittings at both ends. In all cases the under crown fittings carry the highest loads.

The aim is now to design a fitting without taper, but with low shear stresses to prevent creep.

Because the problem deals with bonding tubes, the peel stresses are negligible small. So the theory of part 2.1 can be used. The given parameters are

Loads:

$$F_{lim} = 15327 \text{ [N]}$$

$$\text{Safety factor} = 2.7$$

$$F_{ult} = 2.7 * 15327 = 41383 \text{ [N]}$$

Aluminum fitting:

$$E1 = 72000 \text{ [MPa]}$$

Adhesive: 3MDP460 (epoxy)

$$G_a = 833.33 \text{ [Mpa]}$$

$$\tau_{max} = 31 \text{ [N/mm}^2\text{]}$$

Carbon tube:

$$E2 = 114000 \text{ [Mpa]}$$

$$t_{ct} = 3.65 \text{ [mm]}$$

$$d_{ct} = 40 \text{ [mm]}$$

The parameters to design are

$$\text{fitting thickness: } t_{af}$$

$$\text{fitting length: } l, \quad 35 < l < 65 \text{ [mm]}$$

$$\text{adhesive thickness: } d, \quad 0.1 < d < 0.6 \text{ [mm]}$$

Initial values taken are $l = 35$ and $d = 0.1$. Choose the following options

- * Both adherents the same stiffness: $t_{af} = (114/72) * 3.65 = 5.78 \text{ [mm]}$
- * Less stiffness for the fitting: $t_{af} = 4.78 \text{ [mm]}$
- * More stiffness for the fitting: $t_{af} = 6.78 \text{ [mm]}$

$$\text{Applied load is estimated as } P = (F_{ult} / (\pi * d_{ct})) = 330 \text{ [Mpa]}$$

Using the analytical solution of Volkersen the next shear stress distributions follow.

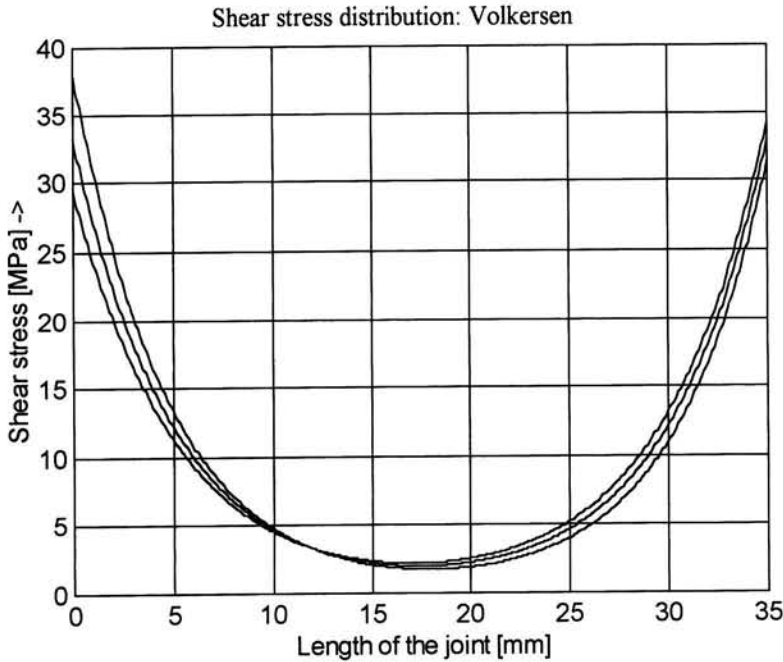


figure 6.1: the effect of fitting thickness on the peak stresses

$$t_{af} = 4.78 \text{ [mm]}, \tau_{max} = 38 \text{ [N/mm}^2\text{]}$$

$$t_{af} = 6.78 \text{ [mm]}, \tau_{max} = 34 \text{ [N/mm}^2\text{]}$$

$$t_{af} = 5.78 \text{ [mm]}, \tau_{max} = 33 \text{ [N/mm}^2\text{]}$$

Figure 6.1 shows that choosing equal stiffness, that is $t_{af} = 5.78$ [mm], for both adherents give the lowest maximum shear stress in the adhesive.

For a joint length of $l = 35$ [mm] the shear stress are calculated with $d = 0.1, 0.2, \dots, 0.5, 0.6$ [mm]. The same is done for $l = 65$ [mm]. Figure 6.2 display the shear stress distributions. With a joint length of 35 [mm] and an adhesive thickness of 0.2 [mm] the maximum shear stress stays below τ_{max} . But to prevent creep in the adhesive layer, the shear stress must be much lower. Eventually there is a length of 65 [mm] chosen and an adhesive thickness of 0.6 [mm].

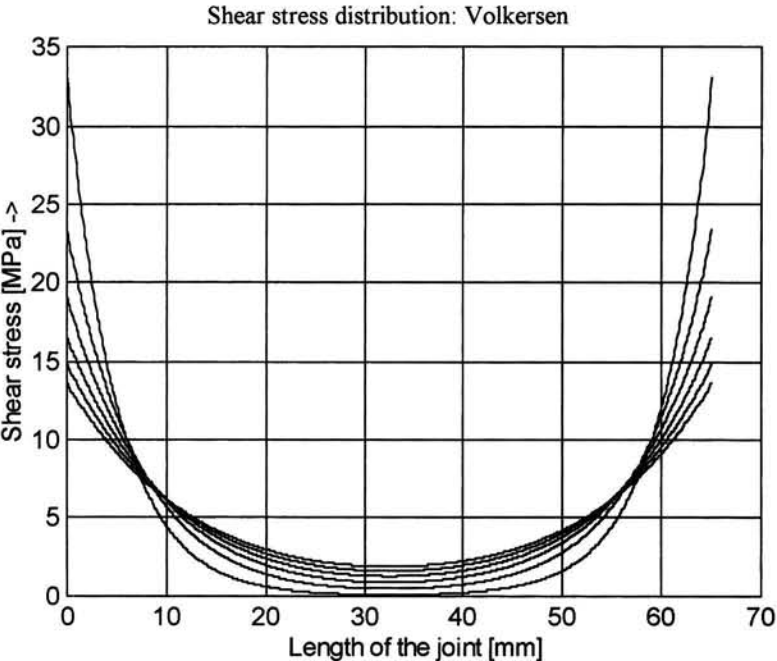
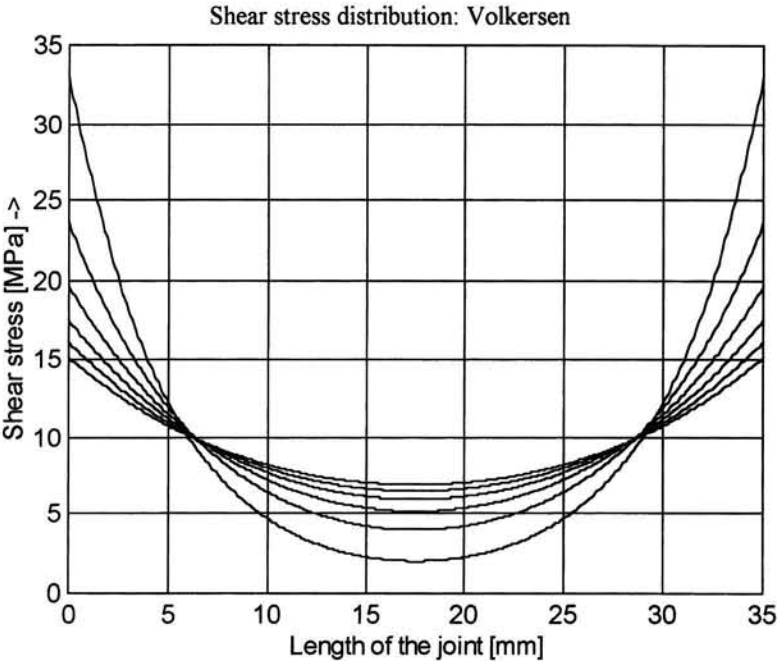


figure 6.2: the effect of adhesive thickness for a joint with length 35 and 65 [mm]

6

Conclusions and recommendations

Conclusions

Finite difference methods (FDM) and models are very powerful tools for linear elastic analysis for adhesively bonded joints. They are easy to program in a computer language like MATLAB 4.2. The design MATLAB Adhesive Toolbox runs on all today's 486 computers with at least 4Mb.

To get an optimal performance of the toolbox, at least a 486DX2 with 12Mb or higher (Pentium) is required.

Although the derived models and tools are concerning linear elastic joints without spew-fillet, they give very good results for low loads. Because the design tools are build for general dissimilar adherents a very wide range of joint configurations can be analyzed. Just by changing the input parameters or material properties the tool calculates new adhesive stresses and the operator can compare these result with previous calculations. Both numerical or graphical.

This makes FDM and in particular the Adhesive Toolbox excellent engineering design tools for adhesive stress analysis.

Recommendations

To improve the adhesive toolbox and get more realistic results. It is recommended to do some more research on and expand the existing models with the following topics:

1. A thorough analysis needs to be made on realistic joint edge loads in a deformed adherent/ adhesive sandwich.
2. In high loaded joints besides elastic deformation also plastic deformation of the adhesive will occur. It is therefore recommended to expand the elastic adhesive model to a elastic-plastic model. One could use Hart-Smith or Crocombees model.
3. All adhesive joint have in practice spew-fillets added deliberately or not. These fillets have severe effects on the stress magnitudes in the adhesive. To get more realistic stress analysis these fillets must be modeled and added to the existing models.

Adhesive theories

1. *J.W. van Ingen / A. Vlot*, 'Stress Analysis of Adhesively Bonded Single Lap Joints' Delft: April 1993. Report LR 740.
2. *K. Ikegami and T. Sugibayashi*, 'A method of estimating strength of adhesive bonded joints of metals', int J. adhesion, vol 21, 1987. pp 211 - 227
3. *F.Erdogan and M. Ratwani*, 'Stress distribution in bonded joints', Lehigh University Bethlehem Pennsylvania: January 25, 1971.
4. *D.A. Bigwood and Crocombe*, 'Elastic analysis and engineering design formula for bonded joints', int J. adhesion, vol 9, October 1989. pp 229 - 242.
5. *D.A. Bigwood and Crocombe*, 'Non-linear adhesive bonded joint analysis', int J. adhesion, vol 10, January 1990. pp 31 - 41.
6. *J.H. Williams*, 'Stresses in adhesive between dissimilar adherends', int J. adhesion, vol 7, July 1975. pp 97 - 107.
7. *F. Delale, F. Erdogan and M.N. Aydinoglu*, 'Stresses in adhesively bonded joints: A Closed-Form Solution', J. Composites Materials Vol. 15, May 1981. pp 249 - 271.
8. *I.U. Ojalvo and H.L. Eidinoff*, 'Bond thickness effects upon stresses in single lap adhesive joints', AIAA Journal Vol 16, 1973. pp 204 - 211.
9. *M.Goland and E. Reissner*, 'The stress in cemented joints', Journal of applied mechanics, March 1944. PP A17-A27.

Mathematics and Numerical Mathematics

10. *W.E. Boyce and R.C. DiPrima*, 'Elementary differential equations and boundary value problems', 4th edition, John Wiley & Sons inc: 1986.
11. *J. van Kan*, 'Numerieke wiskunde voor technici', 2e editie, Delftse Uitgevers Maatschappij. Delft: 1991.
12. *A. Segal en J. van Kan*, 'Numerieke methoden voor partiele differentiaalvergelijkingen'. Delftse Uitgevers Maatschappij. Delft: 1993.
13. *C.J. van Duijn*, 'Analyse van differentiaalvergelijkingen'. Delftse Uitgevers Maatschappij. Delft: 1994.

Miscellaneous

14. *S.P. Timoshenko and D.H. Young*, 'Theory of structures', 2th edition, McGraw Hill 1965.
15. *Gere and Timoshenko*, 'Mechanics of Materials', 3th edition, Chapman Hall 1991.
16. *Kermit Sigmon*, 'MATLAB PRIMER', 3th edition. Department of Mathematics, University of Florida, 1993.

Appendix A

MATLAB Adhesive Toolbox Users Guide

MATLAB Adhesive Toolbox Users Guide

The MATLAB Adhesive Toolbox consists of twelve so called m-files

- Volkers
- Svolkers
- Normjnts
- Normjntp
- Tapjntss
- Tapjntps
- Stack
- Stackcal
- Shlojnt1
- Shlojnt2
- Hartsm
- Hartsm2

Just by typing the name of the m-file after the MATLAB prompt will start the program. For people who are not familiar with MATLAB, type 'help' or 'help help' to get all information about the program. Also very cheap MATLAB Primers are available at most universities. The Primers discusses all the basic MATLAB commands.

Installing the Toolbox and computer requirements

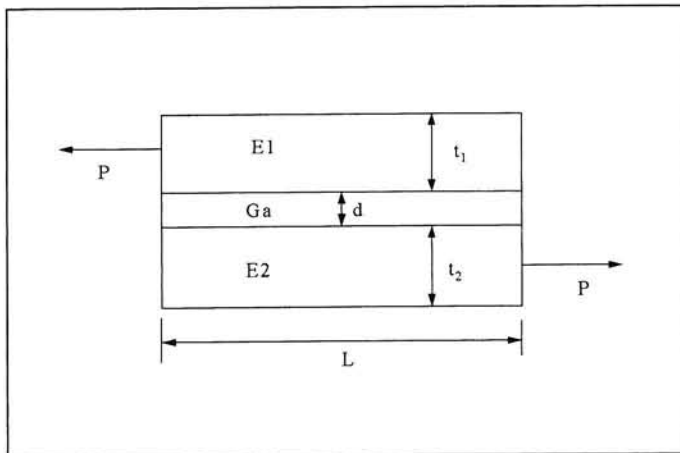
- * Copy the file: adhesive to C:/ matlab/ toolbox.
- * Define a new search path in the master start up m-file 'MatlabBC' by adding the next line: `'C:\MATLAB\toolbox\adhesive',...`
- * Computer requirements: at least a 486DX with 4Mb. For optimal working of the program a 486DX2 with 12Mb or higher is recommended.

volkers.m*Purpose:*

Calculation of the volkersen shear stress distribution in a normal lap joint loaded with tensile forces.

Synopsis:

volkers

Description:

P = tensile force per unit width

L = length of the joint

E1 = E modules of plate 1

t1 = thickness of plate 1

t2 = thickness of plate 2

E2 = E modules of plate 2

d = thickness of the adhesive

Ga = shear modules of the adhesive

references:

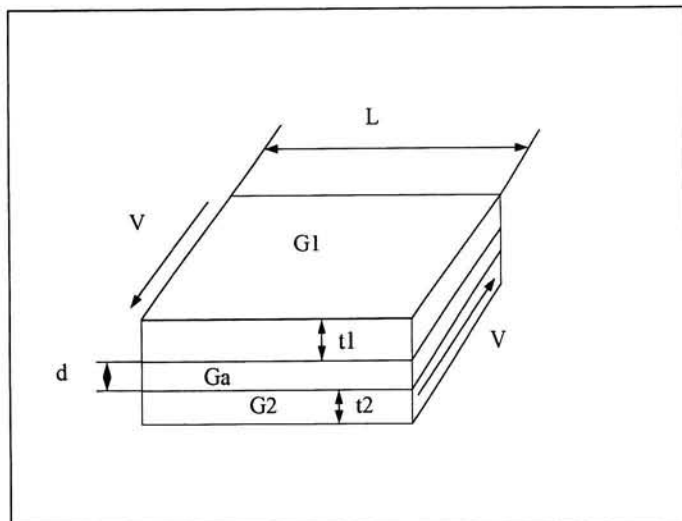
Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

svolkers.m*Purpose:*

Calculation of the volkersen shear stress distribution in a normal lap joint loaded with shear forces.

Synopsis:

svolkers

*Description:*

V = shear force per unit width

L = length of the joint

E1 = E modules of plate 1

t1 = thickness of plate 1

t2 = thickness of plate 2

E2 = E modules of plate 2

d = thickness of the adhesive

Ga = shear modules of the adhesive

references:

Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

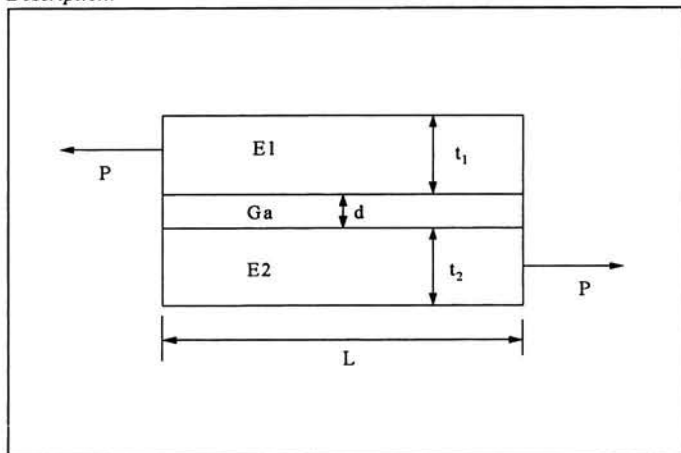
normjnts.m*Purpose:*

Finite difference calculation of the shear stress distribution in a normal lap joint loaded with tensile forces.

No bending of the adherends.

Synopsis: normjnts

Description:



P = tensile force per unit width

L = length of the joint

E1 = E modules of plate 1

t1 = thickness of plate 1

t2 = thickness of plate 2

E2 = E modules of plate 2

d = thickness of the adhesive

Ga = shear modules of the adhesive

references:

Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

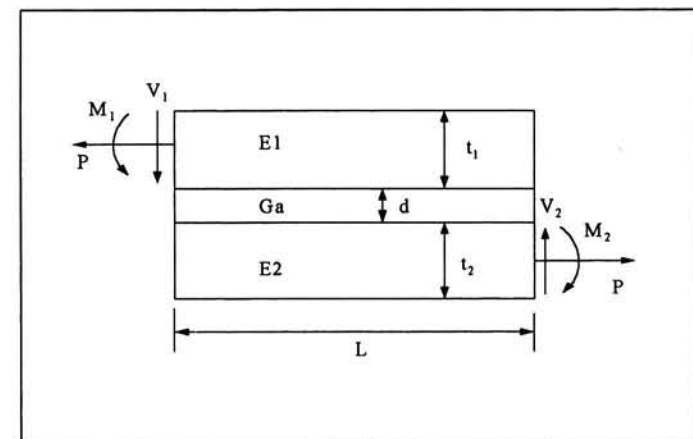
normjntp.m*Purpose:*

Finite difference calculation of the shear stress distribution in a normal lap joint loaded with tensile forces.

Bending of the adherends is allowed.

Synopsis: normjntp

Description: uses hartsm.m



P = tensile force per unit width

E1 = E modules of plate 1

L = length of the joint

t1 = thickness of plate 1

t2 = thickness of plate 2

E2 = E modules of plate 2

Ga = shear modules of the adhesive

d = thickness of the adhesive

M1 = boundary moment left

M2 = boundary moment right

V1 = boundary shear force left

V2 = boundary shear force right

boundary conditions options:

1 = Enter your own boundary loads with the above defined sign convention

2 = Calculation of moments and shear force by use of the Hart-Smith K-factor

references: Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

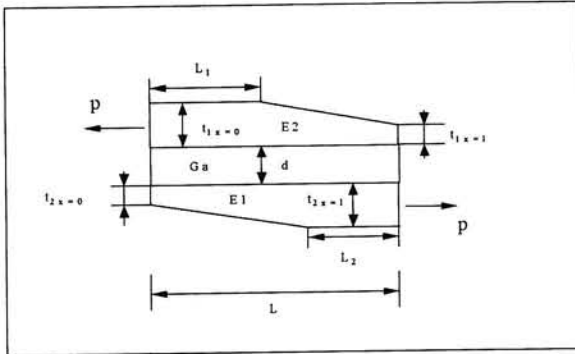
tapjntss.m

Purpose:

Finite difference calculation of the shear stress distribution in a tapered lap joint loaded with tensile forces. No bending of the adherends.

Synopsis: tapjntss

Description:



P = tensile force per unit width

L = length of the joint

E1 = E modules of plate 1

t1 = thickness of plate 1

E2 = E modules of plate 2

t2 = thickness of plate 2

Ga = shear modules of the adhesive

d = thickness of the adhesive

references:

Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

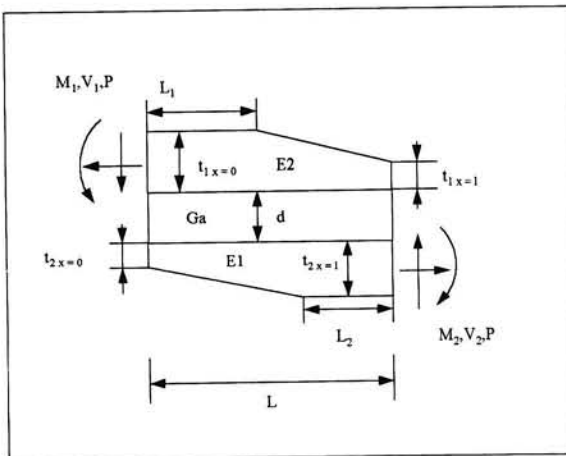
tapjntps.m

Purpose:

Finite difference calculation of the shear stress distribution in a tapered lap joint loaded with tensile forces. Bending of the adherends is allowed.

Synopsis: tapjntps

Description: uses hartsm2.m



P = tensile force per unit width

L = length of the joint

E1 = E modules of plate 1

t1 = thickness of plate 1

E2 = E modules of plate 2

t2 = thickness of plate 2

d = thickness of the adhesive

Ga = shear modules of the adhesive

M1 = boundary moment left

M2 = boundary moment right

V1 = boundary shear force left

V2 = boundary shear force right

Boundary conditions options:

1 = Enter your own boundary loads with the above defined sign convention.

2 = Calculation of moments and shear force by use of the Hart-Smith K-factor.

3 = Calculation of moments and shear force by use of the Hart-Smith K-factor + an extra reduction factor caused by the taper.

references: Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

stack.m*Purpose:*

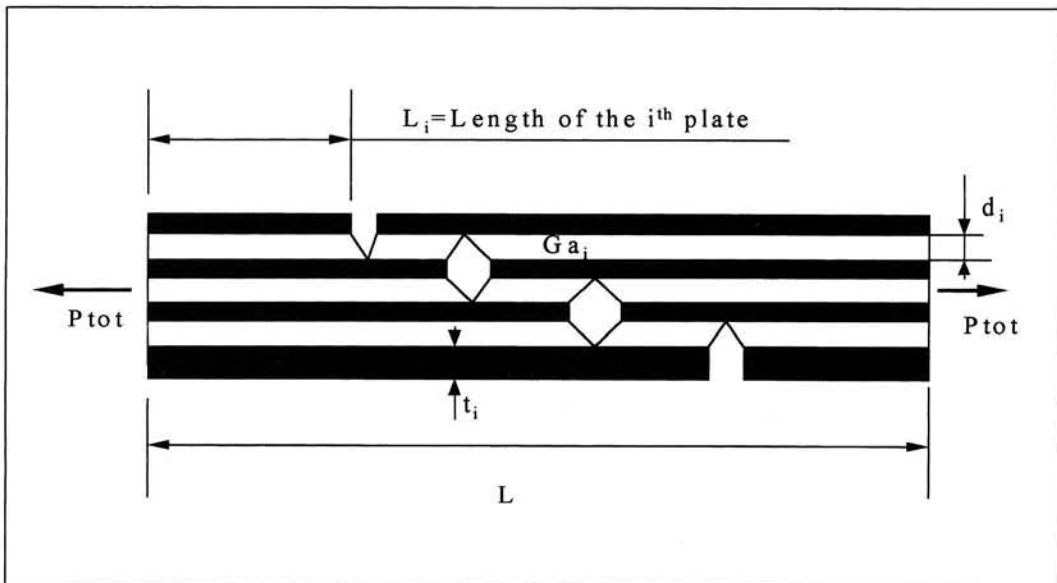
Finite difference calculation of the shear stress distribution in the adhesive layers of a laminated stacked joint loaded with tensile forces. Bending of the adherends is allowed.

Synopsis:

stack

Description:

uses stackcal.m



P = total tensile force per unit width

L = total length of the joint

$G a_i$ = shear modulus of the i^{th} adhesive layer

E_i = E modulus of the i^{th} plate

t_i = thickness of the i^{th} plate

d_i = thickness of the i^{th} adhesive layer

references:

Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

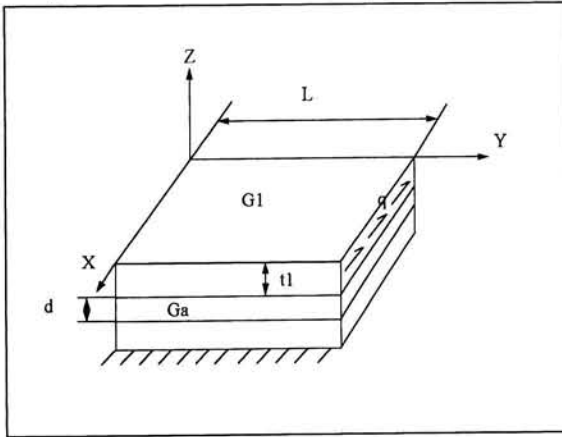
shlojnt1.m

Purpose:

Calculation of the analytical solution for a deformable plate bonded to a undeformable surface loaded with shear stress.

Synopsis: shlojnt1

Description:



q = constant shear stress

L = length of the joint

$G1$ = shear modulus of plate 1

$t1$ = thickness of plate 1

Ga = shear modulus of the adhesive

d = thickness of the adhesive

boundary conditions at $y = 0$:

1 = a fixed end

2 = a free end with no loads

references:

Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

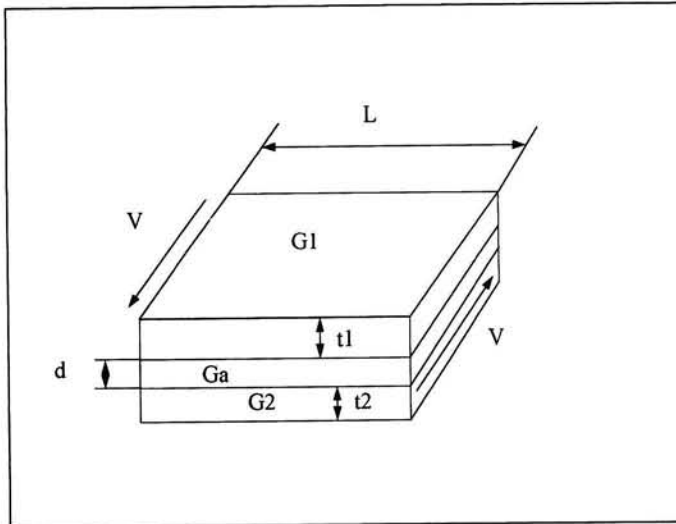
shlojnt2.m

Purpose:

Finite difference calculation of the shear stress distribution in a normal lap joint loaded with shear forces.

Synopsis: shlojnt2

Description:



V = shear force per unit width

L = length of the joint

$G1$ = shear modulus of plate 1

$t1$ = thickness of plate 1

$G2$ = shear modulus of plate 2

$t2$ = thickness of plate 2

Ga = shear modulus of the adhesive

d = thickness of the adhesive

references:

Roza Z.C., 'The design of a MATLAB Adhesive Toolbox', TU- Delft: Jan 1996.

Appendix B

Adhesive Toolbox Program Listings

HARTSM.M

```
%*****
%
%   This m-file calculates the end moment loads for a lap
%   joint using the Hart-Smith K-factor.
%
%*****

M1m=-(P1*(t1+d))/2;
M2m=(P1*(t2+d))/2;
lab1=((12*P1*(1-mu1^2))/(E1*t1^3))^0.5;
lab2=((12*P1*(1-mu2^2))/(E2*t2^3))^0.5;

Kh1=1/(1+(lab1*1/2)+(1/6)*(lab1*1/2)^2);
Kh2=1/(1+(lab2*1/2)+(1/6)*(lab2*1/2)^2);

M1=Kh1*M1m;
M2=Kh2*M2m;
```

```
%*****
```

```
%
```

```
%      This m-file calculates the end moment loads for a lap  
%      joint using the Hart-Smith K-factor.
```

```
%
```

```
%*****
```

```
M1m=-(P1*(t1(1)+d))/2;
```

```
M2m=(P1*(t22+d))/2;
```

```
lab1=((12*P1*(1-mu1^2))/(E1*t1(1)^3))^0.5;
```

```
lab2=((12*P1*(1-mu2^2))/(E2*t22^3))^0.5;
```

```
Kh1=1/(1+(lab1*l/2)+(1/6)*(lab1*l/2)^2);
```

```
Kh2=1/(1+(lab2*l/2)+(1/6)*(lab2*l/2)^2);
```

```
M1=Kh1*M1m;
```

```
M2=Kh2*M2m;
```

```
Kr1=1/(1+0.25*((l-l1)/l)*(t1(1)-t11)/t1(1));
```

```
Kr2=1/(1+0.25*((l-l2)/l)*(t22-t2(1))/t22);
```

```

%*****
%
% This program calculates the shear stress and peel stress
% in a single bonded lapjoint. The used method is FDM.
%
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****

disp('*****');
disp('This program calculates the shear stress and peel stress');
disp('in a single lapjoint by means of FDM');
disp('*****');

clear;

%***** inputs *****;

l=input('Length of the joint in [mm]: ');
n=input('Total number of iteration steps ( > 1000): ');

d=input('Thickness in [mm] of the adhesive layer: ');
ma=input('poisson ratio of the adhesive: ');
Ea=input('E modulus in [MPa] of the adhesive: ');

E1=input('E modulus in [MPa] of plate number 1: ');
mu1=input('Poisson ratio of plate number 1: ');
t1=input('Thickness in [mm] of plate number 1: ');

E2=input('E modulus in [MPa] of plate number 2: ');
mu2=input('Poisson ratio of plate number 2: ');
t2=input('Thickness in [mm] of plate number 2: ');

P1=input('Force per unit width [N/mm]: ');

disp(' ');
disp('*****');
disp('Enter your own end moments/shear force (=1) or the program');

disp('calculates moments/shear forces by Hart-Smith (=2) K-factor');
dum=input(' What is your choose ?: ');

```

```

if dum == 1
    M1=input('Moment per unit width [N/mm/mm] left: ');
    V1=input('Shear force per unit width [N/mm] left: ');
    M2=input('Moment per unit width [N/mm/mm] right: ');
    V2=input('Shear force per unit width [N/mm] right: ');
else
    hartsm;
    V1=(M2-M1-P*((t1+t2)/2+d))/l;
    V2=V1;
end;

disp(' The calculated shear force (V) for the given moment:');
disp(V1);
disp(' ');
disp('I am calculating ');

%*****;
% Caculation of the A matrix;
%*****;

dx=l/n;                                % step size;

Gal=-dx*Ea/(2*(1+ma));

D1=(E1*t1^3)/(12*(1-mu1^2));    D2=(E2*t2^3)/(12*(1-mu2^2));

h1=(t1+t2)/2 +d;

Ca=-dx*Ea/(1-ma^2);                C=dx*Ea*(t1+d)/(4*(1+ma));

%*****
A=sparse(6*n,6*n);

for i=2:n-1

    A(i,i-1)=-1;                    *** Equations for P;
    A(i,i)=1;
    A(i,3*(n-1)+i)=Gal;

    A(n+i,n+i-2)=-1;                *** Equations for V;
    A(n+i,n+i-1)=1;
    A(n+i,5*n-1+i)=Ca;

```

```

A(2*n+i,2*n-3+i)=-1;
A(2*n+i,2*n-2+i)=1;
A(2*n+i,n-2+i)=-dx;
A(2*n+i,3*n-3+i)=C;

```

```

*** Equations for M;

```

```

end;

```

```

%*****

```

```

A(1,1)=1;          A(1,3*(n-1)+1)=Ga1;
A(n,n-1)=-1;       A(n,4*n-3)=Ga1;

```

```

A(n+1,n)=1;        A(n+1,5*n)=Ca;
A(2*n,2*n-2)=-1;   A(2*n,6*n-1)=Ca;

```

```

A(2*n+1,2*n-1)=1;   A(2*n+1,3*n-2)=C;
A(3*n,3*n-3)=-1;    A(3*n,4*n-3)=C;      A(3*n,2*n-2)=-dx;

```

```

%*****

```

```

C2=(1+(E2*t2)/(E1*t1)+(6*(1-mu2^2)*h1)/t2)/(E2*t2*d);
C3=6*((1-mu2^2)/t2-E2*t2*(1-mu1^2)/(E1*t1*t1))/(E2*t2*d);

```

```

for i=2:n-1

```

```

A(3*n+i,3*n-3+i)=-1;
A(3*n+i,3*n+i-2)=1;
A(3*n+i,i-1)=-C2*dx;
A(3*n+i,2*n-3+i)=-C3*dx;

```

```

*** Equations for Gamma;

```

```

A(4*n+i,4*n-2+i)=-1;
A(4*n+i,4*n-2+i+1)=1;
A(4*n+i,2*n-3+i)=(dx/d)*(1/D1+1/D2);
A(4*n+i,i-1)=dx*h1/(d*D2);

```

```

*** Equations for K;

```

```

A(5*n+i,5*n-1+i)=-1;
A(5*n+i,5*n+i)=1;
A(5*n+i,4*n-2+i)=-dx;

```

```

*** Equations for Epsilon

```

```

end;

```

```

%*****

```

```

A(3*n+1,3*n-2)=-1;   A(3*n+1,3*n-1)=1;
A(4*n,4*n-3)=-1;    A(4*n,4*n-2)=1;
A(4*n,n-1)=-dx*C2;  A(4*n,3*n-3)=-dx*C3;

```

```

A(4*n+1,4*n-1)=-1;
A(5*n,5*n-2)=-1;
A(5*n,3*n-3)=(dx/d)*(1/D1+1/D2);
A(4*n+1,4*n)=1;
A(5*n,5*n-1)=1;
A(5*n,n-1)=dx*h1/(d*D2);

A(5*n+1,5*n)=-1;
A(6*n,6*n-1)=-1;
A(5*n+1,5*n+1)=1;
A(6*n,6*n)=1;
A(5*n+1,4*n-1)=-dx;
A(6*n,5*n-2)=-dx;

%*****;
% Caculation of the B matrix;
%*****;

B=sparse(6*n,1);

B(1,1)=P1;
B(n+1,1)=V1;
B(2*n+1,1)=M1+dx*V1;

B(3*n+1,1)=(C2*P1+C3*M1)*dx+(-P1-(6*(1-mu2^2)*(M1+P1*h1))/t2)*(dx/(E2*t2*
2*d));

B(4*n,1)=(-(6*(1-mu2^2)*V1*(1-dx))/t2-P1-(6*(1-mu2^2)*(M1+P1*h1))/t2)*(dx
2*d));

B(4*n+1,1)=-((dx/d)*(M1/D1+M1/D2)+dx*h1*P1/(d*D2))+(dx/(d*D2))*(M1+P1*h1)

B(5*n,1)=(dx/(d*D2))*(M1+P1*h1+V1*(1-dx));

for i=2:n-1

    B(3*n+i,1)=(-(6*(1-mu2^2)*V1*(i-1)*dx)/t2-P1-(6*(1-mu2^2)*(M1+P1*
2)*dx/(E2*t2*d));
    B(4*n+i,1)=(dx/(d*D2))*(M1+P1*h1+V1*(i-1)*dx);
end;

%*****;
% Calculation of U vector;
%*****;

U=A\B;

%***** Results *****

x=0:dx:1;
U=full(U);

disp('*****');

```

```

disp('Your results are displayed in figure 1 and 2');
disp('*****');
disp('Maximum shear stress at x=0 and x=1 :');
disp(U(3*n-2,1)*Ga1/dx);disp(U(4*n-2,1)*Ga1/dx);
disp('Maximum peel stress at x=0 and x=1 :');
disp(U(5*n,1)*(-Ca/dx));disp(U(6*n,1)*(-Ca/dx));

figure(1);
plot(x,U(5*n:6*n,1)*(-Ca/dx));
xlabel('Length of the joint in [mm] ->');ylabel('Peel stress in [MPa] ->');
title('Peel stress in a normal lapjoint');grid;
figure(2);
plot(x,U(3*n-2:4*n-2,1)*(Ga1/dx));
xlabel('Length of the joint in [mm] ->');ylabel('Shear stress in [MPa] ->');
title('Shear stress in a normal lapjoint');grid;

%***** Checking the results *****

Tau=trapz(x,U(3*n-2:4*n-2,1)*(Ga1/dx));
Dwk=trapz(x,U(5*n:6*n,1)*(-Ca/dx));

disp('abs(P) found by integrating Tau:');
disp(Tau);
disp('abs(Shear force V) found by integrating Sigma:');
disp(Dwk);

```

```

%*****;
%Finite difference method for solving lap joints;
%This program calculates the shear stress in a lapjoint;
%
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****;

disp('*****');
disp('This program calculates the shear stress in the adhesive');
disp('of a lap joint under tensile force by the FDM');
disp('*****');

clear;

%***** inputs *****;

l=input('Length of the joint in [mm]: ');

d=input('Thickness in [mm] of the adhesive layer: ');
Ga=input('Shear modulus in [MPa] of the adhesive: ');

n=input('Total number of iteration steps: ');

E1=input('E modulus in [MPa] of plate number 1: ');
t1=input('Thickness in [mm] of plate number 1: ');

E2=input('E modulus in [MPa] of plate number 2: ');
t2=input('Thickness in [mm] of plate number 2: ');

P=input('Tensile force per unit width [N/mm]: ');

%*****;
% Berekening van de A matrix;
%*****;

dx=1/(n-1); % step size numerical method in [mm];

```



```
C1=Ga/(d*E1*t1);
C2=Ga/(d*E2*t2);
```

```
%***** second to n-1 th equation *****;
```

```
A=sparse(2*n,2*n);
```

```
for i=2:1:n-1;
A(i,i-1)=1;
A(i,i)=-2-C1*dx^2;
A(i,i+1)=1;
A(i,n+i)=(dx^2)*C1;
A(n+i,i)=(dx^2)*C2;
A(n+i,n+i-1)=1;
A(n+i,n+i)=-2-(dx^2)*C2;
A(n+i,n+i+1)=1;
end;
```

```
% ***** Firts and last equations of the plate*****;
```

```
A(1,1)=-1-(dx^2)*C1;      A(1,2)=1;      A(1,n+1)=(dx^2)*C1;
A(n,n-1)=1;      A(n,n)=-1-(dx^2)*C1;      A(n,2*n)=(dx^2)*C1;
A(n+1,n+1)=-1-(dx^2)*C2;      A(n+1,n+2)=1;      A(n+1,1)=(dx^2)*C2;
A(2*n,n)=(dx^2)*C2;      A(2*n,2*n-1)=1;      A(2*n,2*n)=-1-(dx^2)*C2;
```

```
%*****;
% Calculation of the B matrix;
%*****;
```

```
B=sparse(2*n,1);
```

```
B(1)=-P*dx/(t1*E1);      B(2*n)=P*dx/(t2*E2);
```

```
%*****;
% Calculation of displacement vector u by Gaussian elimination;
%*****;
```

```
U1=A\B;
```

```

%*****;
%      Calculations of the shear stress in the adhesive;
%*****;

for k=1:1:n;
Tau(k)=(Ga/d)*(U1(k)-U1(n+k));
end;

%*****;
%      Displaying and plotting results
%*****;

disp(' ');
disp('Maximum shear stress left:');
disp(Tau(1));
disp('Maximum shear stress right:');
disp(Tau(n));

x=0:dx:1;

plot(x,Tau);

xlabel('Length of the joint [mm]');ylabel('Shear stress [MPa] ->');
title('Shear stress distribution in the adhesive of a lap joint');
grid;

testf=trapz(x,Tau);
disp('Applied tensile force found by integrating Tau');
disp(testf);

```

```

%*****
% This program calculates the shear stress in the adhesive
% layer of a deformable plate bonded to a undeformable plate
% under shear loading. The used method is a analytical solution
% of the governing equations. This is done for two different
% boundary conditons.
%
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratry
%*****

disp('*****');
disp('This program calculates the shear stress in the adhesive ');
disp('layer of a deformable plate bonded to a undeformable plate');
disp('under shear loading. The used method is a analytical solution');
disp('of the governing equations. This is done for two different');
disp('boundary conditons. ');
disp('*****');

clear;

%***** inputs *****;

l=input('Length of the deformable plate in [mm]: ');
Gp=input('Shear modulus in [MPa] of the deformable plate : ');
tp=input('Thickness in [mm] of the deformable plate : ');

Ga=input('Shear modulus in [MPa] of the adhesive: ');
d=input('Thickness in [mm] of the adhesive layer: ');

n=input('Total number of iteration steps: ');

q=input('The applied shear stress [N/mm^2]: ');

disp('*****');
disp(' Select one of the two boundry conditons:');
disp(' (1): one fixed end at y = 0');
disp(' (2): a free end at y = 0');
bound=input('Enter your choose : ');

%***** Calculations of displacements *****

dy=1/n;           % step size;

```

```

if bound==1

    a=Ga/(Gp*d*tp);
    C1=q/((a^0.5)*Gp*(exp(1*a^0.5)+exp(-1*a^0.5)));
    y=0:dy:1;
    U=zeros(n+1);

    U=C1*(exp(y*a^0.5)-exp(-y*a^0.5));

else

    a=Ga/(Gp*d*tp);
    C1=q/((a^0.5)*Gp*(exp(1*a^0.5)-exp(-1*a^0.5)));
    y=0:dy:1;
    U=zeros(n+1);

    U=C1*(exp(y*a^0.5)+exp(-y*a^0.5));

end;

Tau=(Ga/d)*U;
disp('*****');
disp('The shear stress at y = 0 is:');
disp(Tau(1));
disp('The shear stress at y = 1 is:');
disp(Tau(n+1));
testf=trapz(y,Tau);
disp('Applied shear stress found by integrating Tau');
disp(testf);
plot(y,Tau);grid;
xlabel('Length of the plate [mm] in y-direction');
ylabel('Shear force in [MPa] ->');
title('Shear stress distribution in the adhesive layer');

```

```

%*****
% This program calculates the shear stress in the adhesive
% layer of two bonded deformable plates under shear loading.
% The used method is the finite difference method.
%
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****

```

```

disp('*****');
disp('This program calculates the shear stress in the adhesive ');
disp('layer of two bonded deformable plates under shear loading ');
disp('The used method is the finite difference method');
disp('*****');

```

```
clear;
```

```
%***** inputs *****;
```

```
l=input('Length of the joint in [mm]: ');
```

```
Ga=input('Shear modulus in [MPa] of the adhesive: ');
```

```
d=input('Thickness in [mm] of the adhesive layer: ');
```

```
G1=input('Shear modulus in [MPa] of plate number 1: ');
```

```
t1=input('Thickness in [mm] of plate number 1: ');
```

```
G2=input('Shear modulus in [MPa] of plate number 2: ');
```

```
t2=input('Thickness in [mm] of plate number 2: ');
```

```
n=input('Total number of iteration steps: ');
```

```
V=input('Shear force per unit width [N/mm]: ');
```

```

%*****;
% Calculation of the A matrix;
%*****;

```

```
dx=l/(n-1); % step size numerical method in [mm];
```

```
C1=Ga/(d*G1*t1);
```

```
C2=Ga/(d*G2*t2);
```

```

%***** second to n-1 th equation *****;

A=sparse(2*n,2*n);

for i=2:1:n-1;
A(i,i-1)=1;
A(i,i)=-2-C1*dx^2;
A(i,i+1)=1;
A(i,n+i)=(dx^2)*C1;
A(n+i,i)=(dx^2)*C2;
A(n+i,n+i-1)=1;
A(n+i,n+i)=-2-(dx^2)*C2;
A(n+i,n+i+1)=1;
end;

% ***** Firts and last equations of the plate*****;

A(1,1)=-1-(dx^2)*C1;    A(1,2)=1;    A(1,n+1)=(dx^2)*C1;

A(n,n-1)=1;    A(n,n)=-1-(dx^2)*C1;    A(n,2*n)=(dx^2)*C1;

A(n+1,n+1)=-1-(dx^2)*C2;    A(n+1,n+2)=1;    A(n+1,1)=(dx^2)*C2;

A(2*n,n)=(dx^2)*C2;    A(2*n,2*n-1)=1;    A(2*n,2*n)=-1-(dx^2)*C2;

%*****;
% Calculation of the B matrix;
%*****;

B=sparse(2*n,1);

B(1)=-V*dx/(t1*G1);    B(2*n)=V*dx/(t2*G2);

%*****;
% Calculation of displacement vector u by Gaussian elimination;
%*****;

U1=A\B;

%*****;
%      Calculations of the shear stress in the adhesive;
%*****;

```

```

for k=1:1:n;
Tau(k)=(Ga/d)*(U1(k)-U1(n+k));
end;

%*****;
%      Displaying and plotting results
%*****;

disp('*****');
disp('Maximum shear stress left:');
disp(Tau(1));
disp('Maximum shear stress right:');
disp(Tau(n));

x=0:dx:l;

plot(x,Tau);

xlabel('Length of the joint [mm]');ylabel('Shear stress [MPa] ->');
title('Shear stress distribution in the adhesive of a lap joint');
grid;

testf=trapz(x,Tau);
disp('Applied shear force found by integrating Tau');
disp(testf);

```

```

%*****
%
%      Stack.m
%      This program calculates the shear stresses in a
%      multi plated stackered joint. The used method is
%      the " finite difference method ".
%
%      Copyright (c) 1995-96, Z.C. Roza
%      TU-Delft: faculty of aerospace engineering
%      Structures and materials Laboratory
%*****

clear

disp('*****');
disp(' ');
disp(' This program calculates the shear stresses in multi plated');
disp(' stackered joints by use of the finite difference method. ');
disp(' ');
disp('*****');
disp(' ');

%***** Input parameters *****;

PT=input('The applied force per unit width [n/mm]: ');
LT=input('Total lenght of bonded laminated plates: ');
N=input('Total number of iteration steps: ');
NP=input('Total number of adherend layers: ');

for i=1:NP
    disp(' Enter parameters of plate number:'),disp(i);
    L(i)=input('Lenght of the plate: ');
    E(i)=input('E modulus in [MPa] of the plate: ');
    t(i)=input('Thickness in [mm] of the plate : ');
    disp(' ');
end;

for i=1:NP-1
    disp(' Enter parametrs of adhesive layer number:'),disp(i);
    d(i)=input('Thickness in [mm] of the adhesive layer : ');
    Ga(i)=input('Shear modulus in [MPa]of the adhesive : ');
    disp(' ');
end;

```



```
clc
```

```
%***** parameter checking sequence *****
```

```
disp('*****')
```

```
disp(' ');
```

```
disp(' Entered parameters of adherend layers from top to bottom.')
```

```
disp(' L(i):   E(i):   t(i):');
```

```
disp(' ');
```

```
for i=1:NP
```

```
    disp([L(i),E(i),t(i)]);
```

```
end;
```

```
disp(' ');
```

```
disp(' Entered parameters of adhesive layers from top to bottom.')
```

```
disp(' d(i):   Ga(i):');
```

```
disp(' ');
```

```
for i=1:NP-1
```

```
    disp([d(i),Ga(i)]);
```

```
end;
```

```
disp(' ');
```

```
disp('***** Press any key to continu *****');
```

```
pause;
```

```
YN=input('Are the entered parameters correct [yes= 1 / no= 2]: ');
```

```
disp('I am calculating ');
```

```
if YN == 1
```

```
    stackcal;
```

```
else
```

```
    stack;
```

```
    disp(' ');
```

```
end;
```

```

%*****
%
% This program calculates the shear stresses in multi plated stackered
% joints using finite difference method. The used input file is caled
% stack.m
%
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****

%***** Building step size matrix DX *****

DX=LT/(N-1);

for i=1:NP
    n(i,1)=round(L(i)/DX)+1;
%    n(i,2)=round((LT-L(i))/DX)+1;

%    dx(i,1)=L(i)/(n(i,1)-1);
%    dx(i,2)=(LT-L(i))/n(i,2);
end;

%***** Calculations of the end loads *****;

for i=1:NP
    P(i)=E(i)*PT*t(i)/(E*t');
end;

%*****;
% Calculation of the A and B matrix;
%*****;

dx2=DX^2;
A=sparse(N*NP,N*NP);
B=sparse(N*NP);

%***** Top plate equations *****

C1=Ga(1)/(d(1)*E(1)*t(1));

for i=2:n(1,1)-1
%***** left hand side ****;
```

```

A(i,i-1)=1;
A(i,i)=-2-C1*dx2;
A(i,i+1)=1;
A(i,N+i)=C1*dx2;

```

```
end;
```

```

A(1,1)=-1-dx2*C1;          A(1,2)=1;          A(1,N+1)=C1*dx2;

```

```
B(1)=-DX*P(1)/(E(1)*t(1));
```

```
A(n(1,1),n(1,1)-1)=1;  A(n(1,1),n(1,1))=-1-dx2*C1;
```

```
A(n(1,1),N+n(1,1))=C1*dx2;
```

```
for i= n(1,1)+2:N-1;          %***** right hand side *****;
```

```

A(i,i-1)=1;
A(i,i)=-2-C1*dx2;
A(i,i+1)=1;
A(i,N+i)=C1*dx2;

```

```
end;
```

```
A(n(1,1)+1,n(1,1)+1)=-1-dx2*C1;          A(n(1,1)+1,n(1,1)+2)=1;
```

```
A(n(1,1)+1,N+n(1,1)+1)=dx2*C1;
```

```
A(N,N-1)=1;          A(N,N)=-1-dx2*C1;          A(N,2*N)=dx2*C1;
```

```
B(N)=DX*P(1)/(E(1)*t(1));
```

```
%***** Bottom plate equations *****
```

```
Cnp=Ga(NP-1)/(d(NP-1)*E(NP)*t(NP));
```

```
Ndum=N*(NP-1);
```

```
Ndum1=Ndum+n(NP,1);
```

```
for i=Ndum+2:Ndum1-1;          %**** left hand side ****;
```

```

A(i,i-1)=1;
A(i,i)=-2-Cnp*dx2;
A(i,i+1)=1;
A(i,-N+i)=Cnp*dx2;

```

```
end;
```

```

A(Ndum+1,Ndum+1)=-1-Cnp*dx2;      A(Ndum+1,Ndum+2)=1;
A(Ndum+1,Ndum-N+1)=dx2*Cnp;
B(Ndum+1)=-DX*P(NP)/(E(NP)*t(NP));

A(Ndum1,Ndum1)=-1-dx2*Cnp;      A(Ndum1,Ndum1-1)=1;
A(Ndum1,Ndum1-N)=dx2*Cnp;

for i=Ndum1+2:NP*N-1              %**** right hand side *****,
    A(i,i-1)=1;
    A(i,i)=-2-Cnp*dx2;
    A(i,i+1)=1;
    A(i,-N+i)=Cnp*dx2;
end;

A(Ndum1+1,Ndum1+1)=-1-dx2*Cnp;  A(Ndum1+1,Ndum1+2)=1;
A(Ndum1+1,Ndum1-N+1)=dx2*Cnp;

A(NP*N,NP*N)=-1-dx2*Cnp;        A(NP*N,NP*N-1)=1;
A(NP*N,NP*N-N)=dx2*Cnp;

B(NP*N)=DX*P(NP)/(E(NP)*t(NP));

%***** Intermediate plates *****,

for i=2:NP-1;                    %**** Number of plates **

    T1=Ga(i-1)/(d(i-1)*E(i)*t(i));
    T2=Ga(i)/(d(i)*E(i)*t(i));
    T3=T1+T2;

    Ndum=N*(i-1);
    Ndum1=Ndum+n(i,1);

    for j=Ndum+2:Ndum1-1;        %**** left hand side ****

```

```

A(j,j-1)=1;
A(j,j)=-2-T3*dx2;
A(j,j+1)=1;
A(j,-N+j)=T1*dx2;
A(j,N+j)=T2*dx2;

```

```
end;
```

```

A(Ndum+1,Ndum+1)=-1-T3*dx2;      A(Ndum+1,Ndum+2)=1;
A(Ndum+1,Ndum-N+1)=dx2*T1;      A(Ndum+1,Ndum+N+1)=dx2*T2;
B(Ndum+1)=-DX*P(i)/(E(i)*t(i));
A(Ndum1,Ndum1)=-1-dx2*T3;      A(Ndum1,Ndum1-1)=1;
A(Ndum1,Ndum1-N)=dx2*T1;      A(Ndum1,Ndum1+N)=dx2*T2;

```

```
for j=Ndum1+2:i*N-1;           %**** right hand side ****
```

```

A(j,j-1)=1;
A(j,j)=-2-T3*dx2;
A(j,j+1)=1;
A(j,-N+j)=T1*dx2;
A(j,N+j)=T2*dx2;

```

```
end;
```

```

A(Ndum1+1,Ndum1+1)=-1-T3*dx2;    A(Ndum1+1,Ndum1+2)=1;
A(Ndum1+1,Ndum1-N+1)=dx2*T1;    A(Ndum1+1,Ndum1+N+1)=dx2*T2;

```

```

A(N*i,N*i)=-1-dx2*T3;    A(N*i,N*i-1)=1;
A(N*i,N*i-N)=dx2*T1;    A(N*i,N*i+N)=dx2*T2;
B(N*i)=DX*P(i)/(E(i)*t(i));

```

```
end;
```

```

%*****;
% Caculation of the u vector by Gaussian elimination;

```

```

%*****;

B1=B';
U1=A\B1;

%*****;
% Calculation of the shear stresses in the adhesive layers;
%*****;

Tau=sparse(NP-1,N);

for i=1:NP-1;

    Gad=Ga(i);
    dd=d(i);
    for j=1:N;
        Tau(i,j)=(Gad/dd)*(U1((i-1)*N+j)-U1((i*N)+j));
    end;
end;

%*****;
%      Plotting and plotting the results
%*****;

x=0:DX:LT;

Tau=full(Tau);
subplot(2,1,1);plot(x,Tau(1,:)); grid;
xlabel('Total length of the adhesive layer in [mm] ->');
ylabel('Shear stress in [MPa] ->');
title('Shear stress distribution in a multi plated stackerd joint

subplot(2,1,2);plot(x,Tau(2,:)); grid;
xlabel('Total length of the adhesive layer in [mm] ->');
ylabel('Shear stress in [MPa] ->');
title('Shear stress distribution in a multi plated stackerd joint

disp(' PRESS ANY KEY TO CONTINU');
pause;
end;

%*****
%
%      Intergrating Tau for numerical fault indication

```

```
%
%*****
for j=1:NP-1;
    V(j)=0;
    for i=1:N-1
        V(j)=V(j)+Tau(j,i)*DX;
    end;
end;

disp(' P(i) found by integrating Tau over lenght l:');
disp(V);
```

```
%*****;  
% Volkersens analytical solution for single lapjoints loaded  
% with shear.  
%  
% Copyright (c) 1995-96, Z.C. Roza  
% TU-Delft: faculty of aerospace engineering  
% Structures and materials Laboratory  
%*****;  
  
disp('*****');  
disp('This program calculates the shear stress in a lap joint');  
disp('which is loaded with shear. This done by a modified');  
disp('analytical solution of Volkersen');  
disp('*****');  
  
clear;  
  
%*** Inputs***;  
l=input('Length of the joint in [mm]: ');  
  
d=input('Thickness in [mm] of the adhesive layer: ');  
Ga=input('Shear modulus in [MPa] of the adhesive: ');  
  
G1=input('E modulus in [MPa] of plate number 1: ');  
t1=input('Thickness in [mm] of plate number 1: ');  
  
G2=input('E modulus in [MPa] of plate number 2: ');  
t2=input('Thickness in [mm] of plate number 2: ');  
  
Q=input('Shear force per unit width [N/mm]: ');  
  
%*****;
```



```

% formula of Volkersen;
%*****;

labda=sqrt((Ga/d)*(1/(G1*t1)+1/(G2*t2)));

V1=Ga*Q/(d*labda*sinh(labda*l));

v=[0:0.1:1]';

Tauvolk=V1*(cosh(labda*(v)) / (G2*t2) + cosh(labda*(1-v)) / (G1*t1));

end;

%*****;
%      Displaying and plotting results
%*****;

disp(' ');
disp('Maximum shear stress:');
disp(Tauvolk(1));

x=0:0.1:1;

plot(x,Tauvolk);

xlabel('Length of the joint [mm]');ylabel('Shear stress [MPa] ->');
title('Shear stress distribution: Volkerson');
grid;

```

```

%*****
% This program calculates the shear and peel stresses
% in a tapered lap joint with the finite difference
% method.
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****

clear

%***** Input *****;

disp('*****')
disp('')
disp('This program calculates the shear stress distribution')
disp('and peel stress distribution in the adhesive layer of')
disp('a tapered lap joint.')
disp('')
disp('*****')
disp('')
l=input('Length of the joint in [mm]: ');
n=input('Total number of iteration steps ( > 1000): ');

t1=zeros(n+1,1);          t2=zeros(n+1,1);

d=input('Thickness in [mm] of the adhesive layer: ');
ma=input('poisson ratio of the adhesive: ');
Ea=input('E modulus in [MPa] of the adhesive: ');

E1=input('E modulus in [MPa] of plate number 1: ');
mu1=input('Poisson ratio of plate number 1: ');
t1(1)=input('Thickness in [mm] of plate number 1 at X=0: ');
t1l=input('Thickness in [mm] of plate number 1 at X=1: ');
l1=input('length in [mm] of non tapered area of plate 1: ');

E2=input('E modulus in [MPa] of plate number 2: ');
mu2=input('Poisson ratio of plate number 2: ');
t2(1)=input('Thickness in [mm] of plate number 2 at X=0: ');
t2l=input('Thickness in [mm] of plate number 2 at X=1: ');
l2=input('length of [mm] of non tapered area of plate 2: ');

P1=input('Force per unit width [N/mm]: ');

disp(' ');

```

```

disp('*****');
disp('Enter your own end moments/shear force (=1),the program');

disp('calculates moments/shear forces by Hart-Smith (=2) K-factor');
disp('or the program uses Hart-Smith K-factor plus an extra reduction');
disp('factor for taper (=3).');
dum=input(' What is your choose ? : ');

if dum == 1
    M1=input('Moment per unit width [N/mm/mm] left: ');
    V1=input('Shear force per unit width [N/mm] left: ');
    M2=input('Moment per unit width [N/mm/mm] right: ');
    V2=input('Shear force per unit width [N/mm] right: ');

    elseif dum == 2
        hartsm2;
        V1=(M2-M1-P1*((t1(1)+t22)/2+d))/l;
        V2=V1;

    else
        hartsm2;
        M1=Kr1*M1;
        M2=Kr2*M2;
        V1=(M2-M1-P1*((t1(1)+t22)/2+d))/l;
        V2=V1;

end;

disp(' The calculated shear force (V) for the given moment:');
disp(V1);
disp(' ');
disp('I am calculating ');

%*****
% Calculation of joint taper
%*****

dx=1/n;                                % step size;

if l1 == 0
    dt1x(1)=(t11-t1(1))/l;

```

```

for i=2:n+1
t1(i)=t1(i-1)+dx*dt1x(1);
dt1x(i)=dt1x(1);
end;

else

dtx1=(t11-t1(1))/(1-round(l1/dx)*dx);
dt1x(1)=0;

for i=2:round(l1/dx +1)
t1(i)=t1(1);
dt1x(i)=0;
end;

for k=round(l1/dx +2):n+1
t1(k)=t1(k-1)+dtx1*dx;
dt1x(k)=dtx1;
end;

end;

if l2 == 0

dt2x(1)=(t22-t2(1))/1;

for i=2:n+1
t2(i)=t2(i-1)+dx*dt2x(1);
dt2x(i)=dt2x(1);
end;

else

dtx2=(t22-t2(1))/(1-round(l2/dx)*dx);
dt2x(1)=dtx2;

for r=2:n-round(l2/dx)-1+1
t2(r)=t2(r-1)+dtx2*dx;
dt2x(r)=dtx2;
end;

for q=n-round(l2/dx)+1:n+1
t2(q)=t22;
dt2x(q)=0;
end;

```

end;

```

%*****;
% Caculation of the inertia momemt, moments arms, constants and C(i) ;
%*****;

```

```

D1=zeros(n+1,1);      D2=zeros(n+1,1);
h1=zeros(n+1,1);      h2=zeros(n+1,1);
C=zeros(n+1,1);

```

for i=1:n+1

```

    D1(i)=(E1*t1(i)^3)/(12*(1-mu1^2));
    D2(i)=(E2*t2(i)^3)/(12*(1-mu2^2));
    h1(i)=(t1(i)+t2(i))/2 +d;
    h2(i)=(t1(i)+t2(i))/2 +d;
    C(i)=dx*Ea*(t1(i)+d)/(4*(1+ma));

```

end;

```

Gal=-dx*Ea/(2*(1+ma));      Ca=-dx*Ea/(1-ma^2);

```

```

%*****;
% Caculation of the A matrix;
%*****;

```

A=sparse(6*n,6*n);

for i=2:n-1

```

    A(i,i-1)=-1;          %** Equations for P;
    A(i,i)=1;
    A(i,3*(n-1)+i)=Gal;

```

```

    A(n+i,n+i-2)=-1;      %** Equations for V;
    A(n+i,n+i-1)=1;
    A(n+i,5*n-1+i)=Ca;

```

```

    A(2*n+i,2*n-3+i)=-1;  %** Equations for M;
    A(2*n+i,2*n-2+i)=1;
    A(2*n+i,n-2+i)=-dx;
    A(2*n+i,3*n-3+i)=C(i);
    A(2*n+i,i-1)=(dx/2)*dt1x(i);

```

```

end;

%*****

A(1,1)=1;                A(1,3*(n-1)+1)=Ga1;
A(n,n-1)=-1;            A(n,4*n-3)=Ga1;

A(n+1,n)=1;             A(n+1,5*n)=Ca;
A(2*n,2*n-2)=-1;       A(2*n,6*n-1)=Ca;

A(2*n+1,2*n-1)=1;       A(2*n+1,3*n-2)=C(1);
A(3*n,3*n-3)=-1;       A(3*n,4*n-3)=C(n);
A(3*n,2*n-2)=-dx;      A(3*n,n-1)=(dx/2)*dt1x(n);

%*****

C2=zeros(n+1,1);        C3=zeros(n+1,1);

for i=1:n

C2(i)=(1+(E2*t2(i))/(E1*t1(i))+(6*(1-mu2^2)*h1(i))/t2(i))/(E2*t2(i)*d);
C3(i)=6*((1-mu2^2)/t2(i)-E2*t2(i)*(1-mu1^2)/(E1*t1(i)*t1(i)))/(E2*t2(i)*d

end;

for i=2:n-1

    A(3*n+i,3*n-3+i)=-1;                *** Equations for Gamma;
    A(3*n+i,3*n+i-2)=1;
    A(3*n+i,i-1)=-C2(i)*dx;
    A(3*n+i,2*n-3+i)=-C3(i)*dx;

    A(4*n+i,4*n-2+i)=-1;                *** Equations for K;
    A(4*n+i,4*n-2+i+1)=1;
    A(4*n+i,2*n-3+i)=(dx/d)*(1/D1(i)+1/D2(i));
    A(4*n+i,i-1)=dx*h1(i)/(d*D2(i));

    A(5*n+i,5*n-1+i)=-1;                *** Equations for Epsilon;
    A(5*n+i,5*n+i)=1;
    A(5*n+i,4*n-2+i)=-dx;

end;

%*****

A(3*n+1,3*n-2)=-1;        A(3*n+1,3*n-1)=1;
A(4*n,4*n-3)=-1;         A(4*n,4*n-2)=1;

```

```

A(4*n,n-1)=-dx*C2(n);    A(4*n,3*n-3)=-dx*C3(n);

A(4*n+1,4*n-1)=-1;          A(4*n+1,4*n)=1;
A(5*n,5*n-2)=-1;          A(5*n,5*n-1)=1;
A(5*n,3*n-3)=(dx/d)*(1/D1(n)+1/D2(n));
A(5*n,n-1)=dx*h1(n)/(d*D2(n));

A(5*n+1,5*n)=-1;          A(5*n+1,5*n+1)=1;          A(5*n+1,4*n-1)=-dx;
A(6*n,6*n-1)=-1;          A(6*n,6*n)=1;          A(6*n,5*n-2)=-dx;

%*****
% Caculation of the B matrix;
%*****

B=sparse(6*n,1);

B(1,1)=P1;
B(n+1,1)=V1;
B(2*n+1,1)=M1+dx*V1-P1*(dx/2)*dt1x(1);

B(3*n+1,1)=(C2(1)*P1+C3(1)*M1)*dx+(-P1-(6*(1-mu2^2)*(M1+P1*h2(1)))/t2(1))
2*t2(1)*d);

B(4*n,1)=(-(6*(1-mu2^2)*V1*(1-dx))/t2(n)-P1-(6*(1-mu2^2)*(M1+P1*h2(n)))/t
dx/(E2*t2(n)*d));

B(4*n+1,1)=-((dx/d)*(M1/D1(1)+M1/D2(1))+dx*h2(1)*P1/(d*D2(1)))+(dx/(d*D2(
1+P1*h2(1)));

B(5*n,1)=(dx/(d*D2(n)))*(M1+P1*h2(n)+V1*(1-dx));

for i=2:n-1

    B(3*n+i,1)=(-(6*(1-mu2^2)*V1*(i-1)*dx)/t2(i)-P1-(6*(1-mu2^2)*(M1+
))) /t2(i))*(dx/(E2*t2(i)*d));
    B(4*n+i,1)=(dx/(d*D2(i)))*(M1+P1*h2(i)+V1*(i-1)*dx);
end;

%*****
% Calculation of U vector;
%*****

U=A\B;

%***** Results *****

```

```

x=0:dx:1;
U=full(U);

disp('*****');
disp('Your results are displayed in figure 1 and 2');
disp('*****');
disp('Maximum shear stress at x=0 and x=1 :');
disp(U(3*n-2,1)*Ga1/dx);disp(U(4*n-2,1)*Ga1/dx);
disp('Maximum peel stress at x=0 and x=1 :');
disp(U(5*n,1)*(-Ca/dx));disp(U(6*n,1)*(-Ca/dx));

figure(1);
plot(x,U(5*n:6*n,1)*(-Ca/dx));
xlabel('Length of the joint in [mm] ->');ylabel('Peel stress in [MPa] ->');
title('Peel stress in a tapered lapjoint');grid;
figure(2);
plot(x,U(3*n-2:4*n-2,1)*(Ga1/dx));
xlabel('Length of the joint in [mm] ->');ylabel('Shear stress in [MPa] ->');
title('Shear stress in a tapered lapjoint');grid;

%***** Checking the results *****

Tau=trapz(x,U(3*n-2:4*n-2,1)*(Ga1/dx));
Dwk=trapz(x,U(5*n:6*n,1)*(-Ca/dx));

disp('abs(P) found by integrating Tau:');
disp(Tau);
disp('abs(Shear force V) found by integrating Sigma:');
disp(Dwk);

```



```

%*****;
% Finite difference method for solving lap joints;
% Dit programma berekent de shear stresses in een lijmnaad;
% m.b.v. de differentie methode;
%
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****;

clear

%***** Input *****;

disp('*****')
disp('')
disp('This program calculates the shear stress distribution')
disp('in the adhesive layer of a lap joint.')
disp('')
disp('*****')
disp('')
l=input('Length in [mm] of the lap joint: ');
d=input('Thickness in [mm] of the adhesive layer : ');
Ga=input('Shear modulus in [MPa]of the adhesive : ');

n=input('Total number of iteration steps: ');
t1=zeros(n,1);          t2=zeros(n,1);

E1=input('E modulus in [MPa] of plate number 1 : ');
t1(1)=input('Thickness in [mm] of plate number 1 at X=0: ');
t11=input('Thickness in [mm] of plate number 1 at X=1: ');
l1=input('length in [mm] of non tapered area of plate 1: ');

E2=input('E modulus in [MPa] of plate number 2 : ');
t2(1)=input('Thickness in [mm] of plate number 2 at X=0 : ');
t22=input('Thickness in [mm] of plate number 2 at X=1 : ');
l2=input('length of [mm] of non tapered area of plate 2: ');

P=input('Force per unit width [MPa/mm]: ');

%***** Calculations of constants *****

dx=1/(n-1);          dx2=dx^2;

%dt1x=(t11-t1(1))/1;    dt2x=(t22-t2(1))/1;

```

```
%***** Calculations of joint taper *****
```

```
%for i=2:n
```

```
    %t1(i)=t1(i-1)+dx*dt1x;
```

```
    %t2(i)=t2(i-1)+dx*dt2x;
```

```
%end;
```

```
if l1 == 0
```

```
    dt1x(1)=(t11-t1(1))/l;
```

```
    for i=2:n
```

```
        t1(i)=t1(i-1)+dx*dt1x(1);
```

```
        dt1x(i)=dt1x(1);
```

```
    end;
```

```
else
```

```
    dtx1=(t11-t1(1))/(1-round(l1/dx)*dx);
```

```
    dt1x(1)=0;
```

```
    for i=2:round(l1/dx +1)
```

```
        t1(i)=t1(1);
```

```
        dt1x(i)=0;
```

```
    end;
```

```
    for k=round(l1/dx +2):n
```

```
        t1(k)=t1(k-1)+dtx1*dx;
```

```
        dt1x(k)=dtx1;
```

```
    end;
```

```
end;
```

```
l == 0
```

```
    dt1x(1)=(t11-t1(1))/l;
```

```
    for i=2:n
```

```
        t1(i)=t1(i-1)+dx*dt1x(1);
```

```
        dt1x(i)=dt1x(1);
```

```
    end;
```

```
else
```

```
    dtx1=(t11-t1(1))/(1-round(l1/dx)*dx);
```

```

dt1x(1)=0;

for i=2:round(l1/dx +1)
t1(i)=t1(1);
dt1x(i)=0;
end;

for k=round(l1/dx +2):n
t1(k)=t1(k-1)+dtx1*dx;
dt1x(k)=dtx1;
end;

end;

if l2 == 0

    dt2x(1)=(t22-t2(1))/l;

    for i=2:n
t2(i)=t2(i-1)+dx*dt2x(1);
dt2x(i)=dt2x(1);
end;

else

    dtx2=(t22-t2(1))/(1-round(l2/dx)*dx);
    dt2x(1)=dtx2;

    for r=2:n-round(l2/dx)-1
t2(r)=t2(r-1)+dtx2*dx;
dt2x(r)=dtx2;
end;

    for q=n-round(l2/dx):n
t2(q)=t22;
dt2x(q)=0;
end;

end;

```

```

%*****;
% Calculation of the A matrix;
%*****;

```

```
C1=Ga/(d*E1);
C2=Ga/(d*E2);
```

```
A=sparse(2*n,2*n);
```

```
for i=2:1:n-1;
A(i,i-1)=t1(i)/(dx2);
A(i,i)=-dt1x(i)/dx -2*t1(i)/dx2 -C1;
A(i,i+1)=dt1x(i)/dx + t1(i)/dx2;
A(i,n+i)=C1;
A(n+i,i)=C2;
A(n+i,n+i-1)=t2(i)/dx2;
A(n+i,n+i)=-dt2x(i)/dx -2*t2(i)/dx2 -C2;
A(n+i,n+i+1)=dt2x(i)/dx + t2(i)/dx2;
end;
```

```
% *****A coefficients *****
```

```
A(1,1)=-dt1x(1)/dx -t1(1)/dx2 -C1;
A(1,2)=dt1x(1)/dx + t1(1)/dx2;
A(1,n+1)=C1;
```

```
A(n,n-1)=t1(n)/dx2;
A(n,n)=-t1(n)/dx2 -C1;
A(n,2*n)=C1;
```

```
A(n+1,n+1)=-dt2x(1)/dx -t2(1)/dx2 -C2;
A(n+1,n+2)=dt2x(1)/dx + t2(1)/dx2;
A(n+1,1)=C2;
```

```
A(2*n,n)=C2;
A(2*n,2*n-1)=t2(n)/dx2;
A(2*n,2*n)=-t2(n)/dx2 -C2;
```

```
% *****;
% Calculation of the B matrix;
% *****;
```

```
B=sparse(2*n,1);
```

```
B(1)=-P/(dx*E1);          B(2*n)=(P*dx/(t2(n)*E2))*(dt2x(n)/dx + t2(n)/dx2)
```

```

%*****;
% Caculation of the u vector by Gaussian elimination;
%*****;

U1=A\B;

%*****;
% Calculation of the shear stresses in the adhesive layer;
%*****;

for k=1:1:n;
Tau(k)=(Ga/d)*(U1(k)-U1(n+k));
end;

Tau(n)=Tau(n-1)+Tau(n-1)-Tau(n-2);

%*****;
%      Plotting and plotting the results
%*****;

disp(' ')
disp('Your results are displayed in figure 1 and 2');
disp(' ')
disp('Shear stress at X=0 : ');disp(Tau(1))
disp('Shear stress at X=1 : ');disp(Tau(n))

x=0:dx:1;

figure(1);
subplot(2,1,1);plot(x,t1);grid;
ylabel('Thickness of plate [mm] ->');xlabel('Length of the plate [mm] ->');
title('Geometry of plate 1');
subplot(2,1,2);plot(x,t2);grid;
ylabel('Thickness of plate [mm] ->');xlabel('Length of the plate [mm] ->');
title('Geometry of plate 2');

figure(2);
plot(x,Tau); grid;
xlabel(' Length of joint in [mm]');ylabel('Shear stress in [MPa]');
title(' Shear stress distribution in the adhesive of a lap joint')

%*****
%
%      Intergrating Tau for numerical fault indication
%

```

%*****

```
V=0;
for i=1:n-1
V=V+Tau(i)*dx;
end;

disp(' P found by integrating Tau over lenght l=');
disp(V);
```

```

%*****;
% Volkersens analytical solution for single lapjoints.;
%
% Copyright (c) 1995-96, Z.C. Roza
% TU-Delft: faculty of aerospace engineering
% Structures and materials Laboratory
%*****;

disp('*****');
disp('This program calculates the shear stress in a lap joint');
disp('by the analytical solution of Volkersen');
disp('*****');

clear;

%*** Inputs***;
l=input('Length of the joint in [mm]: ');

d=input('Thickness in [mm] of the adhesive layer: ');
Ga=input('Shear modulus in [MPa] of the adhesive: ');

E1=input('E modulus in [MPa] of plate number 1: ');
t1=input('Thickness in [mm] of plate number 1: ');

E2=input('E modulus in [MPa] of plate number 2: ');
t2=input('Thickness in [mm] of plate number 2: ');

P=input('Tensile force per unit width [N/mm]: ');

%*****;
% formula of Volkersen;
%*****;

```

```

labda=sqrt((Ga/d)*(1/(E1*t1)+1/(E2*t2)));
V1=Ga*P/(d*labda*sinh(labda*l));
v=[0:0.1:1]';
Tauvolk=V1*(cosh(labda*(v)) / (E2*t2) + cosh(labda*(1-v)) / (E1*t1));
end;

%*****;
%      Displaying and plotting results
%*****;

disp(' ');
disp('Maximum shear stress:');
disp(Tauvolk(1));

x=0:0.1:1;

plot(x,Tauvolk,'w');

xlabel('Length of the joint [mm]');ylabel('Shear stress [MPa] ->');
title('Shear stress distribution: Volkersen');
grid;

```


Series 01: Aerodynamics

01. F. Motallebi, 'Prediction of Mean Flow Data for Adiabatic 2-D Compressible Turbulent Boundary Layers'
1997 / VI + 90 pages / ISBN 90-407-1564-5
02. P.E. Skåre, 'Flow Measurements for an Afterbody in a Vertical Wind Tunnel'
1997 / XIV + 98 pages / ISBN 90-407-1565-3
03. B.W. van Oudheusden, 'Investigation of Large-Amplitude 1-DOF Rotational Galloping'
1998 / IV + 100 pages / ISBN 90-407-1566-1
04. E.M. Houtman / W.J. Bannink / B.H. Timmerman, 'Experimental and Computational Study of a Blunt Cylinder-Flare Model in High Supersonic Flow'
1998 / VIII + 40 pages / ISBN 90-407-1567-X
05. G.J.D. Zondervan, 'A Review of Propeller Modelling Techniques Based on Euler Methods'
1998 / IV + 84 pages / ISBN 90-407-1568-8
06. M.J. Tummers / D.M. Passchier, 'Spectral Analysis of Individual Realization LDA Data'
1998 / VIII + 36 pages / ISBN 90-407-1569-6
07. P.J.J. Moeleker, 'Linear Temporal Stability Analysis'
1998 / VI + 74 pages / ISBN 90-407-1570-X
08. B.W. van Oudheusden, 'Galloping Behaviour of an Aeroelastic Oscillator with Two Degrees of Freedom'
1998 / IV + 128 pages / ISBN 90-407-1571-8
09. R. Mayer, 'Orientation on Quantitative IR-thermography in Wall-shear Stress Measurements'
1998 / XII + 108 pages / ISBN 90-407-1572-6
10. K.J.A. Westin / R.A.W.M. Henkes, 'Prediction of Bypass Transition with Differential Reynolds Stress Models'
1998 / VI + 78 pages / ISBN 90-407-1573-4
11. J.L.M. Nijholt, 'Design of a Michelson Interferometer for Quantitative Refraction Index Profile Measurements'
1998 / 60 pages / ISBN 90-407-1574-2
12. R.A.W.M. Henkes / J.L. van Ingen, 'Overview of Stability and Transition in External Aerodynamics'
1998 / IV + 48 pages / ISBN 90-407-1575-0
13. R.A.W.M. Henkes, 'Overview of Turbulence Models for External Aerodynamics'
1998 / IV + 40 pages / ISBN 90-407-1576-9

Series 02: Flight Mechanics

01. E. Obert, 'A Method for the Determination of the Effect of Propeller Slipstream on a Static Longitudinal Stability and Control of Multi-engined Aircraft'
1997 / IV + 276 pages / ISBN 90-407-1577-7
02. C. Bill / F. van Dalen / A. Rothwell, 'Aircraft Design and Analysis System (ADAS)'
1997 / X + 222 pages / ISBN 90-407-1578-5
03. E. Torenbeek, 'Optimum Cruise Performance of Subsonic Transport Aircraft'
1998 / X + 66 pages / ISBN 90-407-1579-3

Series 03: Control and Simulation

01. J.C. Gibson, 'The Definition, Understanding and Design of Aircraft Handling Qualities'
1997 / X + 162 pages / ISBN 90-407-1580-7
02. E.A. Lomonova, 'A System Look at Electromechanical Actuation for Primary Flight Control'
1997 / XIV + 110 pages / ISBN 90-407-1581-5
03. C.A.A.M. van der Linden, 'DASMAT-Delft University Aircraft Simulation Model and Analysis Tool. A Matlab/Simulink Environment for Flight Dynamics and Control Analysis'
1998 / XII + 220 pages / ISBN 90-407-1582-3
04. S.K. Advani, 'The Kinematic Design of Flight Simulator Motion-Bases'
1998 / XVIII + 244 pages / ISBN 90-407-1671-4
05. J.M. Maciejowski, 'Predictive Control. A Lecture Course Given in the Aerospace Engineering Faculty TU Delft'
1998 / XII + 156 pages / ISBN 90-407-1714-1

Series 05: Aerospace Structures and Computational Mechanics

01. A.J. van Eekelen, 'Review and Selection of Methods for Structural Reliability Analysis'
1997 / XIV + 50 pages / ISBN 90-407-1583-1
02. M.E. Heerschap, 'User's Manual for the Computer Program Cufus. Quick Design Procedure for a CUT-out in a FUSELAGE version 1.0'
1997 / VIII + 144 pages / ISBN 90-407-1584-X
03. C. Wohlever, 'A Preliminary Evaluation of the B2000 Nonlinear Shell Element Q8N.SM'
1998 / IV + 44 pages / ISBN 90-407-1585-8
04. L. Gunawan, 'Imperfections Measurements of a Perfect Shell with Specially Designed Equipment (UNIVIMP)
1998 / VIII + 52 pages / ISBN 90-407-1586-6

Series 07: Aerospace Materials

01. A. Vašek / J. Schijve, 'Residual Strength of Cracked 7075 T6 Al-alloy Sheets under High Loading Rates'
1997 / VI + 70 pages / ISBN 90-407-1587-4
02. I. Kunes, 'FEM Modelling of Elastoplastic Stress and Strain Field in Centre-cracked Plate'
1997 / IV + 32 pages / ISBN 90-407-1588-2
03. K. Verolme, 'The Initial Buckling Behavior of Flat and Curved Fiber Metal Laminate Panels'
1998 / VIII + 60 pages / ISBN 90-407-1589-0
04. P.W.C. Provó Kluit, 'A New Method of Impregnating PEI Sheets for the In-Situ Foaming of Sandwiches'
1998 / IV + 28 pages / ISBN 90-407-1590-4
05. A. Vlot / T. Soerjanto / I. Yeri / J.A. Schelling, 'Residual Thermal Stresses around Bonded Fibre Metal Laminate Repair Patches on an Aircraft Fuselage'
1998 / IV + 24 pages / ISBN 90-407-1591-2
06. A. Vlot, 'High Strain Rate Tests on Fibre Metal Laminates'
1998 / IV + 44 pages / ISBN 90-407-1592-0
07. S. Fawaz, 'Application of the Virtual Crack Closure Technique to Calculate Stress Intensity Factors for Through Cracks with an Oblique Elliptical Crack Front'
1998 / VIII + 56 pages / ISBN 90-407-1593-9
08. J. Schijve, 'Fatigue Specimens for Sheet and Plate Material'
1998 / VI + 18 pages / ISBN 90-407-1594-7
09. J. Schijve, 'The Significance of Fractography for Investigations of Fatigue Crack Growth under Variable-Amplitude Loading'
1998 / IV + 34 pages / ISBN 90-407-1716-8

10. M.J.L. van Tooren / Z.C. Roza, 'Finite Difference Methods for Stress Analysis of Adhesive Bonded Joints. The Design of a MATLAB Adhesive Toolbox'
1998 / VIII + 94 pages / ISBN 90-407-1717-6

Series 08: Astrodynamics and Satellite Systems

01. E. Mooij, 'The Motion of a Vehicle in a Planetary Atmosphere'
1997 / XVI + 156 pages / ISBN 90-407-1595-5
02. G.A. Bartels, 'GPS-Antenna Phase Center Measurements Performed in an Anechoic Chamber'
1997 / X + 70 pages / ISBN 90-407-1596-3
03. E. Mooij, 'Linear Quadratic Regulator Design for an Unpowered, Winged Re-entry Vehicle'
1998 / X + 154 pages / ISBN 90-407-1597-1



3021908

ISBN 904071717-6



9 799040 717177

