Convexification of the Quantum Network Utility Maximization Problem

Kar, Sounak; Wehner, Stephanie

DOI
10.1109/TQE.2024.3523889

Publication date
2025

Document Version
Final published version

Published in
IEEE Transactions on Quantum Engineering

Citation (APA)
Kar, S., & Wehner, S. (2025). Convexification of the Quantum Network Utility Maximization Problem. *IEEE Transactions on Quantum Engineering* , *6*, Article 4100314. https://doi.org/10.1109/TQE.2024.3523889

Received 2 September 2024; revised 4 December 2024; accepted 26 December 2024; date of publication 30 December 2024; date of current version 26 February 2025.

Digital Object Identifier 10.1109/TQE.2024.3523889

Convexification of the Quantum Network Utility Maximization Problem

SOUNAK KAR AND STEPHANIE WEHNER
QuTech, Kavli Institute of Nanoscience, Quantum Computer Science, Department of Software Technologies, Faculty Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, 2628, CJ Delft, The Netherlands

Corresponding authors: Sounak Kar; Stephanie Wehner (e-mail: s.kar-1@tudelft.nl; s.d.c.wehner@tudelft.nl).

This work was supported in part by the Dutch Research Council (NWO) Vici Program under Grant VI.C.222.029.

ABSTRACT Network utility maximization (NUM) addresses the problem of allocating resources fairly within a network and explores the ways to achieve optimal allocation in real-world networks. Although extensively studied in classical networks, NUM is an emerging area of research in the context of quantum networks. In this work, we consider the quantum network utility maximization (QNUM) problem in a static setting, where a user's utility takes into account the assigned quantum quality (fidelity) via a generic entanglement measure, as well as the corresponding rate of entanglement generation. Under certain assumptions, we demonstrate that the QNUM problem can be formulated as an optimization problem with the rate allocation vector as the only decision variable. Using a change-of-variable technique known in the field of geometric programming, we then establish sufficient conditions under which this formulation can be reduced to a convex problem: a class of optimization problems that can be solved efficiently and with certainty even in high dimensions. We further show that this technique preserves convexity, enabling us to formulate convex QNUM problems in networks where some routes have certain entanglement measures that do not readily admit convex formulation while others do. This allows us to compute the optimal resource allocation in networks where heterogeneous applications run over different routes.

INDEX TERMS Convex optimization, network utility maximization (NUM), quantum networks.

NOTATION

$[n]$	$\{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$
l	Number of links in the network.
r	Number of routes in the network.
d_j	Positive constant characterizing the physical attributes of the jth link; see (3)
w_j	Werner parameter of the generated pairs in the jth link.
μ_j	Corresponding entanglement generation capacity of the jth link, $\mu_j := d_j(1 - w_j)$
x_i	Rate allocated to the ith route.
y_i	$\ln(x_i)$
a_{ji}	Binary variable taking value 1 iff the ith route passes through the jth link.
A	Link–route incidence matrix, i.e., $((A))_{ji} = a_{ji}$
A_j	jth row of A, i.e., $(a_{j1}, a_{j2}, \ldots, a_{jr})$
u_i	End-to-end Werner parameter of the ith route, i.e., $u_i = \prod_{j\in[l]} w_j^{a_{ji}}$
$u_i(\vec{y})$	$\prod_{j\in[l]}(1 - \langle A_j, e^{\vec{y}}\rangle/d_j)^{a_{ji}}, \vec{y}\in\mathbb{R}^r$
f_i	Entanglement measure for the ith route, $f_i : [0, 1] \to [0, b_i]$, nondecreasing and twice differentiable on $\{z : f_i(z) > 0\}\setminus\{1\}$, and $f_i(0) = 0$
$c^{(i)}$	$\sup\{z : f_i(z) = 0\}$
T	$\{\vec{y}\in\mathbb{R}^r : \langle A_j, e^{\vec{y}}\rangle < d_j \; \forall j\}$
S_i	$\{\vec{y}\in T : u_i(\vec{y}) > c^{(i)}\}$
F_i	$\ln f_i$, $F_i : (c^{(i)}, 1] \to \mathbb{R}$
$c_1^{(i)}$	Unique inflection point of F_i; F_i is concave in $(c^{(i)}, c_1^{(i)}]$ and convex in $(c_1^{(i)}, 1)$

The Link Index $j \in [l]$ *and the Route Index* $i \in [r]$

I. INTRODUCTION

Quantum networks are envisaged to facilitate a variety of applications, including quantum key distribution (QKD) [1], [2], enhanced sensing [3], [4], and blind quantum computation [5], [6]. Unlike classical networks, where an application's quality of service (QoS) typically depends on the available transmission rate, the QoS of a quantum network application relies on the quality of entanglement and the rate at which it is distributed between the sender and the receiver. Furthermore, the QoS metric varies according to the underlying application, and the dependence of the QoS metric on the quality of entanglement can sometimes be captured via a suitable entanglement measure [7].

To support diverse applications and multiple users, a network must plan and distribute the resources for communication accordingly. Two central concepts of resource distribution in networks are efficiency and fairness [8], which have been the focus of network utility maximization (NUM) [9], [10] in classical networks. Since NUM is essentially a resource distribution problem, it borrows the mathematical framework of fairness from welfare economics [11], which explores the notion of *equitable* resource allocation among contenders. The core idea is to formulate a social welfare metric that takes into account the well-being of individuals. Mathematically, this entails encoding individual well-being via suitable utility functions and aggregating them into a single social metric. Subsequently, the social metric is maximized over possible resource allocations to find the optimal resource distribution. Interestingly, there are social welfare metrics for which the optimal allocation is not necessarily Pareto-efficient, i.e., starting from the optimal allocation, it is possible to increase an individual's utility without affecting others [12].

In classical NUM, the transmission rate is usually the (communication) resource allocated across routes, defined as paths on the network graph. To do so, the utility function of a route is formulated according to the QoS metric of the underlying application, such as delay, jitter, or throughput. Individual utilities are then aggregated into the (social) network utility function, which is optimized with respect to rate allocation. Often, the individual utilities are concave functions of the rate allocation, enabling a central entity with the knowledge of individual utilities to compute the optimal rate allocation by solving a convex optimization problem. Since the global optimum can already be found efficiently and with certainty for convex problems, classical NUM literature explores other aspects such as decentralized implementations and their stability, i.e., the convergence of such implementations to the optimal allocation vector [9], [10], [13].

In contrast to classical NUM, the quantum network utility maximization (QNUM) problem [7] aims to maximize the aggregate utility of the network over achievable entanglement quality and generation rate. This is because the utility of a route in this case involves both the generation rate and the quality, where the latter is encoded via an entanglement measure [14] (including the secret key rate) that depends on the underlying application. Moreover, the rate and quality of entanglement generation are related—the relation being governed by the entanglement generation scheme and the physical attributes of the quantum communication links. It was shown in [7] that the aggregate utility function is not necessarily convex in the rate and quality of entanglement generation, meaning that there is no known theoretical guarantee for finding a globally optimal allocation for the QNUM problem.

In light of the above, this article aims to find conditions under which the QNUM problem leads to a convex formulation. Our first observation is that the QNUM problem can be reformulated as an optimization problem with the entanglement generation rate as the sole decision variable in certain networks, specifically when entangled links are generated using the single-click protocol [15], which was experimentally demonstrated in [16]. Borrowing a change-of-variable technique from geometric programming [17], we then provide sufficient conditions for the QNUM problem to allow for convex reformulation.[1] Using the fact that the reformulation preserves convexity, we show that the QNUM problem can be transformed into a convex problem in the presence of certain route entanglement measures, some of which do not immediately lead to convex formulations while the rest do. Our result has the implication that the optimal resource allocation for such QNUM problems can be computed efficiently and with certainty, even in large networks.

The rest of this article is organized as follows. We first discuss the related work in Section II and subsequently provide the relevant background on the state description of the quantum communication links and convex optimization in Section III. We then formally describe the assumptions made and state our main results in Section IV. In Section V, we show that certain entanglement measures satisfy the conditions laid out in Section IV, while the results are applied to an example network in Section VI. We present the proofs of the results in Section VII. Finally, Section VIII concludes this article.

II. RELATED WORK

QNUM is an emerging area of research introduced in [7]. In this work, the authors define the QNUM problem by characterizing the utility of individual routes based on the rate and fidelity of the entanglement allocated to these routes, with fidelity encoded through specific measures of entanglement. The study demonstrates that, depending on the choice of entanglement measure, the QNUM problem may not be a convex optimization problem. Lee et al. [18] adopt a different approach, where the utility of a route is primarily determined by the execution rate of the corresponding task and its associated computational requirements. The network utility is then defined as the maximum sum of the route utilities, subject to feasible rate allocations. The article further explores the achievable network utility for various example networks in the context of distributed quantum computing. In this work, we primarily adopt the framework of [7]. We slightly generalize this setup by allowing the routes to have different entanglement measures to describe respective utilities. Our focus is to show that the QNUM problem can be transformed into a convex problem, at least for the measures of entanglement considered in [7] when link-level entanglements are generated using the single-click protocol [16]. Notably, Vardoyan and Wehner [7] also consider this generation scheme to define the rate–fidelity tradeoff in their analysis.

[1] A solution of the reformulation has a (known) one-to-one correspondence with a solution of the original problem.

A related but different problem where quantum network utility is considered in the context of network planning is given in [19]. Here, the authors optimize the repeater placements in a network to maximize the network utility. In contrast, our work assumes a fixed topology and predefined routes, focusing solely on distributing link-level resources to these routes. In addition to QNUM, other forms of network resource sharing have also been explored in the quantum network literature. For example, Gauthier et al. [20], [21] focus on sharing the resources of an entanglement generation switch at the center of a network. A crucial difference between these studies and our work is that we are concerned with sharing link-level resources in a general topology.

III. PRELIMINARIES

In this work, we consider an entanglement distribution network with a setup identical to [7]. The end nodes on this network represent users, and they are connected via the repeater nodes. Two nodes are adjacent if and only if there is a direct (or elementary) quantum communication link between them. Furthermore, a route is a path between two users, i.e., a sequence of adjacent nodes linking the corresponding end nodes.

Before delving into the details of the QNUM problem, we first introduce two key concepts central to our analysis. First, we describe the link-level entanglement generation scheme, which defines the rate–fidelity tradeoff in the QNUM problem, i.e., determines the feasible region of the underlying optimization. Next, we present the state description of the links. Given that QNUM is an optimization problem—and optimization problems are generally challenging to solve[2]—we provide a brief overview of convex optimization, a class of problems that can be efficiently solved, even in large dimensions.

A. LINK GENERATION AND STATE DESCRIPTION

We assume that entanglements in the elementary links are generated using the single-click protocol [15], experimentally demonstrated in [16]. In this scheme, the generated state has the following form:

$$\rho = (1-\alpha)|\Psi^+\rangle\langle\Psi^+| + \alpha|\uparrow\uparrow\rangle\langle\uparrow\uparrow| \tag{1}$$

where α is the bright-state population and $|\Psi^+\rangle$ is a Bell-state orthogonal to the bright state $|\uparrow\uparrow\rangle$. Each generation attempt succeeds with probability

$$p_{\text{elem}} = 2\kappa\eta\alpha$$

where $\kappa \in (0, 1)$ is a constant that accounts for the inefficiencies other than photon loss in the fiber. Furthermore, η denotes the transmissivity between one end of the link and its midpoint, where the heralding station is located. For a link of length L km, its transmissivity is given by $\eta = 10^{-0.02L}$.

In addition, we assume that the elementary links generated as (1) are further converted to Werner states of same fidelity. This can be done by applying transformations uniformly at random from a set of operations that involve identical rotations on each qubit [22], [23]. This leads to the following state description for these links:

$$\rho_w = w|\Psi^+\rangle\langle\Psi^+| + (1-w)\mathbb{I}_4/4. \tag{2}$$

Equating fidelities from (1) and (2) gives

$$1-\alpha = \frac{1+3w}{4} \Rightarrow \alpha = \frac{3(1-w)}{4}.$$

The fidelity of a link can range between 0.25 and 1. That is, we allow w to take values in its theoretically possible range [0,1]. However, we will see later that the QNUM problem may impose more stringent requirements on the link fidelities. The state description (2) also leads to a convenient description for end-to-end entanglements on the routes, which are created by swapping the elementary links on a given route (see Assumption A1 in Section IV-A).

Observe that if a link attempts entanglement generation every T units of time, the resulting entanglement generation rate is given by

$$\frac{p_{\text{elem}}}{T} = \underbrace{\frac{3\kappa\eta}{2T}}_{=:d}(1-w). \tag{3}$$

Equation (3) describes the effective rate–fidelity tradeoff in elementary link generation.

B. CONVEX OPTIMIZATION

Optimization problems are ubiquitous in quantitative scientific disciplines. While general optimization problems are hard to solve, convex optimization problems can be solved efficiently and the corresponding tools are part of any standard numerical software [24]. Here, we briefly provide their definitions, which are part of standard textbooks; readers interested in a more comprehensive introduction to the subject may refer to [25] and [26].

Definition III.1 (General Optimization Problems): An optimization problem in general has the following form:

$$\min_{\vec{z}\in\mathbb{U}} g(\vec{z}) \tag{4}$$

where $g : \text{dom}(g) \to \mathbb{R}$ is called the objective function and $\mathbb{U} \subset \text{dom}(g) \subset \mathbb{R}^m$ is called the feasible set of the problem. If there exists a $\vec{z}_* \in \mathbb{U}$ such that

$$g(\vec{z}_*) = \min_{\vec{z}\in\mathbb{U}} g(\vec{z})$$

$\vec{z}_*$ is called a global minimum of (4).

Unless the objective function g and the feasible set $\mathbb{U}$ have special structures, there is no guaranteed way of finding a global minimum. However, for convex optimization problems, one can efficiently find a global minimum. We now describe the notion of convex sets and convex functions, which are essential for defining such problems.

[2]Here, *solving* refers to finding a global optimum.

Definition III.2 (Convex Sets): A set $\mathbb{U} \subset \mathbb{R}^m$ is convex if for any $\vec{z}_1, \vec{z}_2 \in \mathbb{U}$ and $\theta \in (0, 1)$

$$\theta \vec{z}_1 + (1-\theta)\vec{z}_2 \in \mathbb{U}.$$

That is, the line segment connecting any two pints in $\mathbb{U}$ lies completely in $\mathbb{U}$.

Definition III.3 (Convex Functions): A function $g : \text{dom}(g) \to \mathbb{R}$ is convex if $\text{dom}(g)$ is a convex set and for any $\vec{z}_1, \vec{z}_2 \in \text{dom}(g)$ and $\theta \in (0, 1)$

$$g(\theta \vec{z}_1 + (1-\theta)\vec{z}_2) \le \theta g(\vec{z}_1) + (1-\theta) g(\vec{z}_2)$$

i.e., the line segment connecting any two points on the graph of g lies pointwise above the graph.

For differentiable and twice-differentiable functions on open convex sets, there are convenient characterizations of convexity. We use the latter in this article.

Definition III.4 (First-Order Characterization of Convex Functions): A differentiable function $g : \text{dom}(g) \to \mathbb{R}$ on the open convex set $\text{dom}(g)$ is convex if and only if

$$g(\vec{z}_2) \ge g(\vec{z}_1) + g'(\vec{z}_1)^T(\vec{z}_2 - \vec{z}_1)$$

for all $\vec{z}_1, \vec{z}_2 \in \text{dom}(g)$, where

$$g'(\vec{z}_1) := \left(\frac{\partial g}{\partial z_1}(\vec{z}_1), \frac{\partial g}{\partial z_2}(\vec{z}_1), \ldots, \frac{\partial g}{\partial z_m}(\vec{z}_1)\right).$$

That is, the tangent hyperplane at any point lies below the graph.

Definition III.5 (Second-Order Characterization of Convex Functions): A twice-differentiable function $g : \text{dom}(g) \to \mathbb{R}$ on the open convex set $\text{dom}(g)$ is convex if and only if the Hessian $D^2 g$ of g is positive semidefinite at every $\vec{z} \in \text{dom}(g)$, where

$$D^2 g(\vec{z}) := \begin{bmatrix} \frac{\partial^2 g}{\partial z_1^2}(\vec{z}) & \frac{\partial^2 g}{\partial z_1 \partial z_2}(\vec{z}) & \ldots & \frac{\partial^2 g}{\partial z_1 \partial z_m}(\vec{z}) \\ \frac{\partial^2 g}{\partial z_2 \partial z_1}(\vec{z}) & \frac{\partial^2 g}{\partial z_2^2}(\vec{z}) & \ldots & \frac{\partial^2 g}{\partial z_2 \partial z_m}(\vec{z}) \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial^2 g}{\partial z_m \partial z_1}(\vec{z}) & \frac{\partial^2 g}{\partial z_m \partial z_2}(\vec{z}) & \ldots & \frac{\partial^2 g}{\partial z_m^2}(\vec{z}) \end{bmatrix}.$$

We can now define a convex optimization problem.

Definition III.6 (Convex Optimization Problems): An optimization problem of the form (4) is called a convex optimization (or minimization) problem if the feasible set $\mathbb{U}$ is a convex set and the objective function g is convex.

So far we have described optimization problems in the so-called abstract form. The feasible region may also be defined in terms of inequalities and equalities involving functions called constraints. If the constraints are expressed as follows, a convex optimization problem is said to be in standard form

$$\begin{aligned} &\min g(\vec{z}) \\ \text{s.t.} \quad & g_k(\vec{z}) \le 0, \quad k \in [n_1] \\ & h_k(\vec{z}) = 0, \quad k \in [n_2] \end{aligned} \tag{5}$$

where g_k's are convex and h_k's are affine. Furthermore, $[n] := \{1, 2, \ldots, n\}$.

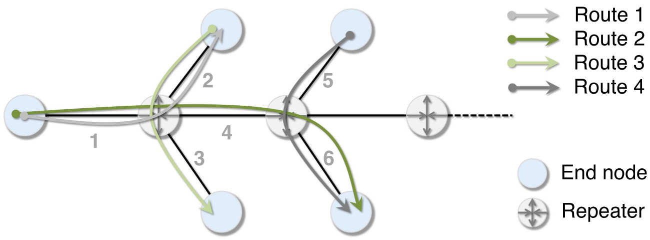


FIGURE 1. **Entanglement distribution network with numbered links. Link j's Werner parameter is w_j, $j \in [6]$. The end-to-end (e2e) Werner parameters of the routes are products of corresponding link-level w_js. For example, routes 1 and 2 have e2e Werner parameters $w_1 w_2$ and $w_1 w_4 w_6$, respectively. The positive rate allocations (x_1, x_2, x_3, x_4) must satisfy six constraints, one for each link (6). For example, $x_4 \le d_5(1 - w_5)$ (link 5) and $x_2 + x_4 \le d_6(1 - w_6)$ (link 6). The utility of a route is the product of the allocated rate and a measure of e2e entanglement, e.g., route 2 has utility $x_2 f_2(w_1 w_4 w_6)$.**

A remarkable feature of convex functions is that any local minimum is also a global minimum. This helps us compute a global minimum of a (twice) differentiable convex function via efficient algorithms such as variants of gradient descent or Newton's method, among others. These algorithms are implemented in most modern numerical software.

IV. ASSUMPTIONS AND MAIN RESULTS

A. SETUP AND ASSUMPTIONS

As mentioned in Section III, we consider an entanglement distribution network similar to [7]; see Fig. 1 for an example. Our main objective is to determine a globally optimal allocation among the routes, with a mathematical guarantee, before the network begins its operation. This is because we assume that the routes are served concurrently for the entire duration of network operation. To achieve this main goal, we also make a few simplifications in the setup that are described in the individual assumptions.

- A1 *Entanglement swapping:* Entanglements between the end nodes are generated in two steps: first at the link level, i.e., between two adjacent nodes. Next, entanglement swaps are performed at the repeater nodes along the route, producing end-to-end entanglement [27]. Similar to [7], we make the simplifying assumption that the route fidelities take the best possible value given the corresponding link fidelities. This is equivalent to considering that link-level entanglements are generated simultaneously and swapped immediately, such that they do not decohere in memory during storage. Since swapping two Werner states results in a Werner state with a Werner parameter equal to the product of the parameters of the input states [28], this leads to a straightforward expression for the resulting end-to-end entanglement.
- A2 *Static network:* Analogous to classical NUM, our goal is to distribute communication resources, i.e., entanglement generation rate and quality, between routes connecting end users. As is the case with classical NUM in its basic form, the routes and the applications are fixed. This implies that the utility

of a route changes only when its share of resources is modified. For our analysis, we focus *only* on the subnetwork $\mathcal{G}$ consisting of relevant routes and the corresponding links. We assume that $\mathcal{G}$ has r routes and l links.

A3 *Rate and fidelity of entanglement generation:* Recall that the rate–fidelity tradeoff of the elementary link generation is governed by (3), which is inspired by the single-click protocol. Accordingly, if we fix the fidelity of link $j \in [l]$ by setting its Werner parameter to w_j, the corresponding maximum entanglement generation rate will be $\mu_j := d_j(1 - w_j)$, where $d_j := 3\kappa_j\eta_j/2T_j$. Here, μ_j can be interpreted as the capacity of link j when it produces Werner states with parameter w_j.

A4 *Arbitrary but fixed quality of entanglement:* The Werner parameter of a link-level entanglement w_j, $j \in [l]$ can be chosen arbitrarily for optimizing the network utility but remains fixed once chosen. In other words, we perform a one-shot analysis, where the values of w_j's are set in advance and cannot be adjusted dynamically during the operation of the network, as the routes are served concurrently. This also implies that the contributions of the jth link toward the end-to-end Werner parameters are the same across the routes passing through that link. Observe from (3) that increasing the value of the Werner parameter reduces the entanglement generation rate.

A5 *Utility of a route:* We denote the allocated rate of route i by x_i and the end-to-end Werner parameter by u_i. To simplify the formulation, we also introduce the (binary) link–route incidence matrix A, where $a_{ji} := ((A))_{ji} = 1$ iff the ith route passes through the jth link. Note that we must have 1) $\sum_{i\in[r]} a_{ji}x_i \le \mu_j$, i.e., the total rate allocated to the incident routes cannot exceed a link's maximum entanglement generation rate and 2) $u_i = \prod_{j\in[l]} w_j^{a_{ji}}$, i.e., the end-to-end Werner parameter is the product of link-level Werner parameters [28]. To reflect the suitability of a Werner state for executing the underlying application for the ith route, we use an entanglement measure (including secret key rate) f_i, where

$$f_i : [0, 1] \to [0, b_i]$$
$$u_i \mapsto f_i(u_i).$$

We assume that f_i is nondecreasing and twice differentiable on $\{z : f_i(z) > 0\}\setminus\{1\}$, and $f_i(0) = 0$ for $i \in [r]$. The utility of a route is assumed to have the form $x_i f_i(u_i)$. Finally, the network utility is formulated as the product of the route utilities. The product form of individual and network utilities ensures that a specified level of network utility is achieved only when each route is allocated both rate and fidelity adequately. Note that it is possible to have different forms for route and network utility functions than ours.

Equipped with the assumptions, we are ready to describe the QNUM problem introduced in [7]. The formulation is from the perspective of a central entity with global knowledge of the network topology, routes, and individual utilities. The entity then calculates the *optimal* allocation for the concurrent execution of applications prior to the network becoming operational. A complete list of parameters describing the network and the auxiliary variables is given in Nomenclature.

B. QNUM PROBLEM

We denote the rate allocation vector for the routes by $\vec{x} = (x_1, x_2, \dots, x_r)$ and the Werner parameter vector for the links by $\vec{w} = (w_1, w_2, \dots, w_l)$. The QNUM problem in its canonical form can then be written as

$$\begin{aligned} \max_{\vec{x},\vec{w}} \quad & \prod_{i=1}^{r} x_i f_i\Big(\prod_{j=1}^{l} w_j^{a_{ji}}\Big) \\ \text{s.t.} \quad & \vec{0} \prec \vec{x}, \\ & \vec{0} \prec \vec{w} \preceq \vec{1} \text{ (fidelity bounds)} \\ & \langle A_j, \vec{x}\rangle \le \mu_j = d_j(1-w_j)\ \forall j \in [l] \text{ (rate constraints).} \end{aligned} \tag{6}$$

Here, $\preceq$ (respectively, $\prec$) denotes elementwise (respectively, strict) inequality and $\langle A_j, \vec{x}\rangle$ denotes the dot product of A_j and $\vec{x}$. Furthermore, the inequalities in $\vec{0} \prec \vec{x}$ and $\vec{0} \prec \vec{w}$ are strict as the objective function is nonnegative and equals zero if any element of $\vec{x}$ or $\vec{w}$ is zero. Note that for any feasible $(\vec{x}, w_j)$, if the last inequality in (6) is strict, i.e., $\langle A_j, \vec{x}\rangle < d_j(1-w_j)$, it is possible to increase w_j further to make it an equality. This is because, with all other variables held constant, an increase in w_j results in a higher or equal value of the objective function, as f_is are nondecreasing by assumption. Thus, if there exists a solution to (6), there will be another solution satisfying

$$\langle A_j, \vec{x}\rangle = d_j(1-w_j) \Rightarrow w_j = 1 - \frac{\langle A_j, \vec{x}\rangle}{d_j} \quad \forall j \in [l] \tag{7}$$

for which the objective function attains a higher or equal value. Therefore, it is sufficient to focus only on solutions satisfying (7), which allows us to eliminate $\vec{w}$ from the set of optimization variables. Also, instead of maximizing (6), we minimize the negative logarithm of the objective function, which leads to the following formulation:

$$\begin{aligned} \min_{\vec{x}} \quad & -\sum_{i=1}^{r}\Bigg(\ln x_i + \ln\Bigg(f_i\Bigg(\prod_{j=1}^{l}\Big(1 - \frac{\langle A_j, \vec{x}\rangle}{d_j}\Big)^{a_{ji}}\Bigg)\Bigg)\Bigg) \\ \text{s.t.} \quad & \vec{0} \prec \vec{x} \\ & 0 < \frac{\langle A_j, \vec{x}\rangle}{d_j} < 1, \quad j \in [l] \\ & c^{(i)} < \prod_{j=1}^{l}\Big(1 - \frac{\langle A_j, \vec{x}\rangle}{d_j}\Big)^{a_{ji}}, \quad i \in [r]. \end{aligned} \tag{8}$$

Here, $c^{(i)} := \sup\{z : f_i(z) = 0\}$.

We now comment on the constraints of formulation (8). The constraint $0 \prec \vec{x}$, together with (7), modifies the constraint $\vec{0} \prec \vec{w} \preceq \vec{1}$ in (6) to $0 < w_j < 1 \; \forall j \in [l]$, as reflected in the second constraint. The last constraint guarantees that the logarithms in the objective function have positive arguments.

In [7, Appendix B], it was shown that the objective function of the QNUM problem is not necessarily convex in rate and quality allocations. To convert (8) into a convex problem, we perform the following change of variable well known in the field of geometric programming [17]:

$$\vec{x} = e^{\vec{y}} := (e^{y_1}, e^{y_2}, \ldots, e^{y_r}). \tag{9}$$

We will show that this leads to a convex formulation under certain conditions. Substituting (9) into (8) yields

$$\begin{aligned}
\min_{\vec{y} \in \mathbb{R}^r} \; & -\sum_{i=1}^{r} \left(y_i + \ln \left(f_i \left(\prod_{j=1}^{l} \left(1 - \frac{\langle A_j, e^{\vec{y}} \rangle}{d_j} \right)^{a_{ji}} \right) \right) \right) \\
\text{s.t.} \; & \frac{\langle A_j, e^{\vec{y}} \rangle}{d_j} < 1, \quad j \in [l] \\
& c^{(i)} < \prod_{j=1}^{l} \left(1 - \frac{\langle A_j, e^{\vec{y}} \rangle}{d_j} \right)^{a_{ji}}, \quad i \in [r].
\end{aligned} \tag{10}$$

Here, the implicit constraint $\vec{y} \in \mathbb{R}^r$ is imposed to ensure $-\vec{\infty} \prec \vec{y}$, i.e., $0 \prec \vec{x}$ as required in (8). Note that this also ensures $\langle A_j, e^{\vec{y}} \rangle > 0$, allowing us to drop this condition from the first constraint. For brevity, we conveniently reuse the notations for w_j and u_i

$$w_j(\vec{y}) := 1 - \frac{\langle A_j, e^{\vec{y}} \rangle}{d_j}, \quad u_i(\vec{y}) := \prod_{j=1}^{l} \left(w_j(\vec{y}) \right)^{a_{ji}}. \tag{11}$$

In order for (10) to be a convex optimization problem, we need its objective function and the feasible region to be convex. To describe the feasible region, we define

$$T_j := \{\vec{y} \in \mathbb{R}^r \colon \langle A_j, e^{\vec{y}} \rangle < d_j\}, \quad j \in [l], \; T := \bigcap_{j \in [l]} T_j$$

$$S_i := \{\vec{y} \in T \colon u_i(\vec{y}) > c^{(i)}\}, \quad i \in [r], \; S := \bigcap_{i \in [r]} S_i. \tag{12}$$

Observe that S is the feasible region for problem (10). In the next section, we show that S is a convex set. We also provide sufficient conditions for the objective function to be convex.

C. RESULTS

While the feasible region of the transformed problem (10) is always convex, the objective function is not in general. We take the following approach to derive the conditions for it to be convex: we consider the contributions from each route to the objective function (i.e., $-y_i - \ln f_i(u_i(\vec{y}))$, $i \in [r]$) and look for conditions for them to be convex individually. Since a sum of convex functions is convex, the objective function is convex if all individual conditions are satisfied. The individual conditions are provided as (Cond. 1) and (Cond. 2) in (13) and (14), respectively. We also show that the change of variable (9) preserves convexity, i.e., if the contribution of a route to the objective function in (8) is convex, so is the corresponding contribution in the reformulation (10). This allows us to apply technique (9) to convexify the contribution from a route without affecting the behavior of already convex contributions from other routes.

Theorem 1: The transformed QNUM problem (10) is feasible, and the set of feasible vectors S is a convex set.

Proof idea: The problem is feasible since it is possible to satisfy constraints in (10) by allocating sufficiently small rates to each route, i.e., by taking $\vec{y} \preceq -M\vec{1}$ for sufficiently large $M > 0$. Convexity of S follows from convexity of each S_i and T_j. See Section VII for a complete proof. □

To establish the convexity of the objective function on S, we first note that $-y_i$ is a convex function of $\vec{y}$. Since a sum of convex functions is convex, we only look for a sufficient condition for $-\ln(f_i(u_i(\vec{y})))$ to be convex, i.e., for $\ln(f_i(u_i(\vec{y})))$ to be concave on S_i, defined in (12). The following proposition, which provides a sufficient condition for $u_i(\vec{y})$ to be concave on S_i, is a stepping stone toward that goal.

Proposition 1: For $i \in [r]$, let $H^{(i)}(\vec{y})$ denote the Hessian of $u_i(\vec{y})$. If

$$\text{(Cond. 1)} \quad c^{(i)} := \sup\{z : f_i(z) = 0\} \geq 1/2 \tag{13}$$

$H^{(i)}(\vec{y})$ is negative semidefinite on S_i.

Proof idea: We compute the Hessian and show that its eigenvalues are nonpositive on S_i if $c^{(i)} \geq 1/2$. See Section VII for details. □

Our main result is the following:

Theorem 2: Let Cond. 1 (13) hold and f_i be twice differentiable on $(c^{(i)}, 1)$. Assume that $F_i(u) := \ln f_i(u)$, $u \in (c^{(i)}, 1]$, has a unique inflection point $c_1^{(i)} \geq c^{(i)}$ satisfying $F_i''(u) \leq 0 \; \forall u \in (c^{(i)}, c_1^{(i)}]$ and $F_i''(u) > 0 \; \forall u \in (c_1^{(i)}, 1)$. Furthermore, let

$$\text{(Cond. 2)} \quad v_i(u) := \frac{u F_i''(u)}{u F_i''(u) + F_i'(u)} + \frac{1}{u} \leq 2 \quad \forall u \in (c_1^{(i)}, 1). \tag{14}$$

Then, $F_i(u_i(\vec{y}))$ is concave on S_i.

Proof idea: Since u_i is concave on S_i and F_i is concave and increasing on $(c^{(i)}, c_1^{(i)}]$, $F_i(u_i(\vec{y}))$ is also concave on $\{\vec{y} \in T \colon c^{(i)} < u_i(\vec{y}) \leq c_1^{(i)}\}$. On $\{\vec{y} \in T \colon c_1^{(i)} < u_i(\vec{y}) < 1\}$, we show that the eigenvalues of its Hessian are nonpositive if Cond. 1 (13) and Cond. 2 (14) are satisfied. See Section VII for details. □

We now show that the change of variable (9) preserves convexity. To that end, we note that the contribution from the ith route to the objective function in formulation (8) is

$$h_x(\vec{x}) := -\ln x_i - G(x)$$

where

$$G(x) := F_i\left(\prod_{j=1}^{l}\left(1-\frac{\langle A_j,\vec{x}\rangle}{d_j}\right)^{a_{ji}}\right). \tag{15}$$

The corresponding contribution in formulation (10) is

$$h(\vec{y}) := -y_i - F_i(u_i(\vec{y})).$$

The following proposition formalizes our argument.

Proposition 2: If $h_x(\vec{x})$ is convex, so is $h(\vec{y})$, i.e., the change of variable in (9) preserves convexity.

Proof: Since $-\ln x_i$ is a convex function of $\vec{x}$, we essentially show that if $G(\vec{x})$ is concave, so is $F_i(u_i(\vec{y}))$. Note that $F_i(u_i(\vec{y})) = G(e^{\vec{y}})$. Let us denote the Hessian of $G(\vec{y})$ and $F_i(u_i(\vec{y}))$ by $D^2G(\vec{y})$ and $D^2F(\vec{y})$, respectively. Then

$$D^2F(\vec{y}) = E(\vec{y})D^2G(e^{\vec{y}})E(\vec{y}) + E(\vec{y})\nabla G(e^{\vec{y}})$$

where

$$E(\vec{y}) := \operatorname{diag}(e^{y_1}, e^{y_2}, \ldots, e^{y_r})$$

$$\nabla G(\vec{y}) := \operatorname{diag}\left(\frac{\partial G}{\partial y_1}(\vec{y}), \frac{\partial G}{\partial y_2}(\vec{y}), \ldots, \frac{\partial G}{\partial y_r}(\vec{y})\right).$$

Since F_i is nondecreasing and $0<1-\langle A_j,\vec{x}\rangle/d_j<1$ on our domain of interest, observe from (15) that each $\partial G/\partial y_k$ is nonpositive for $k \in [r]$. Thus, $E(\vec{y})\nabla G(e^{\vec{y}})$ is negative semidefinite. Also, by assumption, $h_x(\vec{x})$ is convex, i.e., $G(\vec{x})$ is concave, which implies that D^2G is negative semidefinite. Hence, $E(\vec{y})D^2G(e^{\vec{y}})E(\vec{y})$ is negative semidefinite as well. This proves that D^2F is negative semidefinite, i.e., $F_i(u_i(\vec{y}))$ is concave, as required. □

We have thus established that if each utility function f_i either satisfies Cond. 1 and Cond. 2 or its contribution (15) to the objective function in formulation (8) is already convex, formulation (10) is a convex optimization problem. We now test these criteria on certain entanglement measures.

V. EXAMPLE ENTANGLEMENT MEASURES

We first show that the entanglement measures considered in [7] that did not *readily* admit convex formulations satisfy Cond. 1 (13) and Cond. 2 (14) and thus can be transformed into a convex problem via formulation (10). We then provide an example where the entanglement measure does not satisfy Cond. 1 (13) but satisfies the hypothesis of Proposition 2, i.e., convexity of the contribution toward the objective function is preserved by the change of variable (9) for routes using this measure of entanglement. We end with an example where the entanglement measure satisfies none of the aforementioned conditions but admits a convex formulation once we impose a cutoff on the end-to-end Werner parameters. The findings of this section are summarized in Table 1.

TABLE 1. Convex Reformulation Approaches for Different Measures of Entanglement

Measure of entanglement	Convex reformulation possible?	Reason
Secret key fraction (16)	Yes, via change of variable (9)	Satisfies cond. 1 (13) and 2 (14)
Distillable entanglement (17)	Yes, via change of variable (9)	Satisfies cond. 1 (13) and 2 (14)
Negativity (18)	Yes	Already convex, (9) preserves convexity (Prop. 2)
Fidelity of teleportation (19)	With additional constraint	By imposing route Werner parameters $\geq 1/2$; see Sect. V-D

A. SECRET KEY FRACTION

We first consider the secret key fraction [29], which has the following form for Werner states with Werner parameter w:

$$f_{\mathrm{sk}}(w) = \max\left(0,\, 1+(1+w)\log_2\frac{1+w}{2}+(1-w)\log_2\frac{1-w}{2}\right). \tag{16}$$

Since, $f_{\mathrm{sk}}(1/2) = 0$, Cond. 1 (13) is satisfied. In particular

$$c^{\mathrm{sk}} := \sup\{w : f_{\mathrm{sk}}(w) = 0\} \approx 0.779944.$$

For $w > c^{\mathrm{sk}}$, we define $F_{\mathrm{sk}} := \ln(f_{\mathrm{sk}})$. Also

$$f'_{\mathrm{sk}}(w) = \log_2\left(\frac{1+w}{1-w}\right) \qquad f''_{\mathrm{sk}}(w) = \frac{2\log_2 e}{1-w^2}$$

$$F'_{\mathrm{sk}}(w) = \frac{f'_{\mathrm{sk}}(w)}{f_{\mathrm{sk}}(w)} \qquad F''_{\mathrm{sk}}(w) = \frac{f''_{\mathrm{sk}}(w)f_{\mathrm{sk}}(w)-(f'_{\mathrm{sk}}(w))^2}{(f_{\mathrm{sk}}(w))^2}.$$

In Fig. 2(b), we see that F_{sk} has a unique inflection point $c_1^{\mathrm{sk}} \approx 0.968418$. Furthermore, Cond. 2 (14) is seen to be true in Fig. 2(c), where we plot $g_{\mathrm{sk}}(u) = 2-uF''_{\mathrm{sk}}(u)/(uF''_{\mathrm{sk}}(u)+F'_{\mathrm{sk}}(u))-1/u$. Thus, by Theorem 2, the contribution of a route to the objective function in formulation (10) is convex if f_{sk} is used as its measure of entanglement.

B. DISTILLABLE ENTANGLEMENT

Following [7], we consider a lower bound to distillable entanglement. For a Werner state with Werner parameter w, the lower bound can be expressed as

$$f_{\mathrm{de}}(w) = \max\left(0,\, 1+\frac{1+3w}{4}\log_2\left(\frac{1+3w}{4}\right) + \frac{3(1-w)}{4}\log_2\left(\frac{1-w}{4}\right)\right). \tag{17}$$

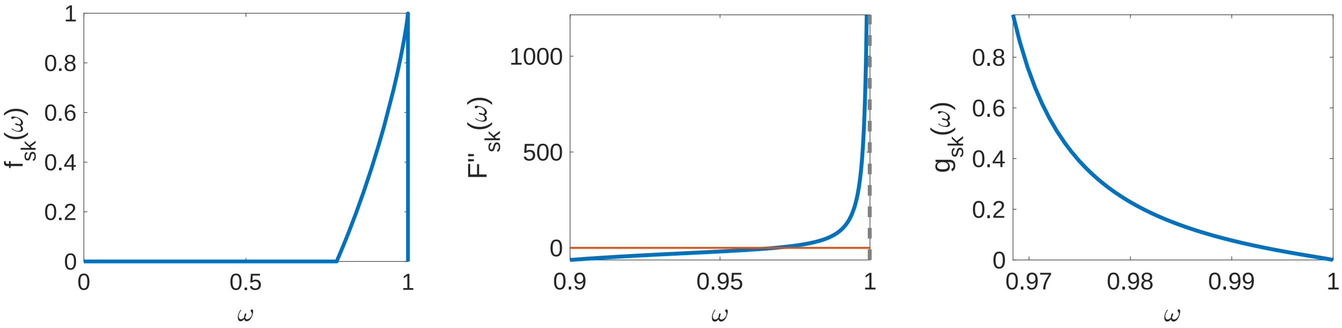
Since $f_{\mathrm{de}}(1/2) = 0$, Cond. 1 (13) is met. Indeed

$$c^{\mathrm{de}} := \sup\{w : f_{\mathrm{de}}(w) = 0\} \approx 0.747613.$$

For $w > c^{\mathrm{de}}$, we define $F_{\mathrm{de}} := \ln(f_{\mathrm{de}})$. Furthermore

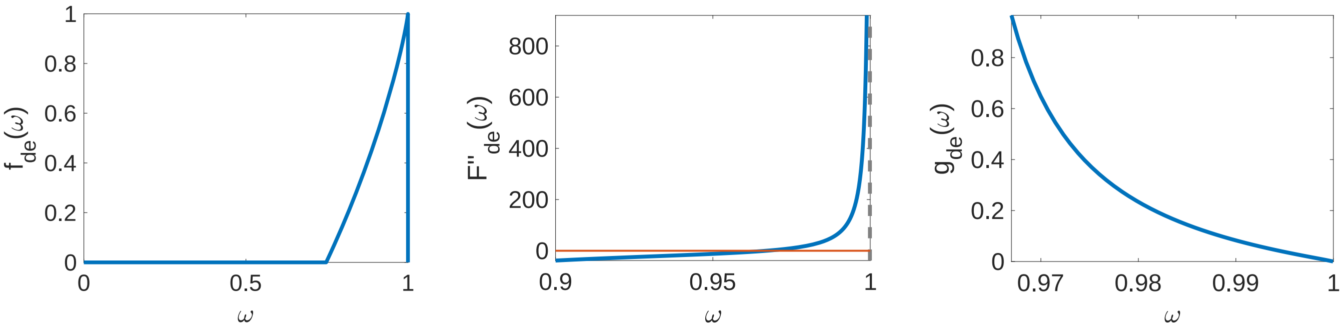
$$f'_{\mathrm{de}}(w) = \frac{3}{4}\log_2\left(\frac{1+3w}{1-w}\right) \qquad f''_{\mathrm{de}}(w) = \frac{3\log_2 e}{-3w^2+2w+1}$$

$$F'_{\mathrm{de}}(w) = \frac{f'_{\mathrm{de}}(w)}{f_{\mathrm{de}}(w)} \qquad F''_{\mathrm{de}}(w) = \frac{f''_{\mathrm{de}}(w)f_{\mathrm{de}}(w)-(f'_{\mathrm{de}}(w))^2}{(f_{\mathrm{de}}(w))^2}.$$



(a) Entanglement measure f_{sk} (b) Second derivative of $\ln f_{\text{sk}}$ (c) Condition (14)

FIGURE 2. **(a) Secret key fraction satisfies $\sup\{w : f_{\text{sk}}(w) = 0\} \approx 0.779944 \geq 1/2$. (b) Unique inflection point of its logarithm F_{sk} is approximately 0.968418. (c) For $w > 0.968418$, we plot $g_{\text{sk}}(u) = 2 - uF''_{\text{sk}}(u)/(uF''_{\text{sk}}(u) + F'_{\text{sk}}(u)) - 1/u$ showing that Cond. 2 (14) is satisfied. .**



(a) Entanglement measure f_{de} (b) Second derivative of $\ln f_{\text{de}}$ (c) Condition (14)

FIGURE 3. **(a) Lower bound to distillable entanglement (17) satisfies $\sup\{w : f_{\text{de}}(w) = 0\} \approx 0.747613 \geq 1/2$. (b) The unique inflection point of $F_{\text{de}} := \ln(f_{\text{de}})$ is approximately 0.966984, beyond which Cond. 2 (14) is shown to be satisfied by plotting $g_{\text{de}}(u) = 2 - uF''_{\text{de}}(u)/(uF''_{\text{de}}(u) + F'_{\text{de}}(u)) - 1/u$ in (c).**

In Fig. 3(b), we see that F_{de} has a unique inflection point $c_1^{\text{de}} \approx 0.966984$. In Fig. 3(c), we plot $g_{\text{de}}(u) = 2 - uF''_{\text{de}}(u)/(uF''_{\text{de}}(u) + F'_{\text{de}}(u)) - 1/u$, to show that Cond. 2 (14) is met. Thus, by Theorem 2, the contribution of a route to the objective function in formulation (10) is convex if f_{de} is used as its measure of entanglement.

C. NEGATIVITY

In [7], it was already shown that negativity allows for convex formulation via (6). However, this example is relevant to QNUM problems in networks where one route's entanglement measure is defined by negativity, and another route's entanglement measure requires the variable transformation in (9) to achieve a convex formulation.

For a Werner state with Werner parameter w, negativity can be expressed as

$$f_{\text{neg}}(w) = \max\left(0, \frac{3w - 1}{4}\right). \tag{18}$$

In [7, Appendix A], it was shown that the function

$$J(\vec{x}, \vec{w}) := -\ln x_i - \ln\left(f_{\text{neg}}\left(\prod_{j=1}^{l} w_j^{a_{ji}}\right)\right)$$

is convex on $S' := \{(\vec{x}, \vec{w}) : \vec{0} \prec \vec{x}, \vec{0} \prec \vec{w}, \prod_j w_j > 1/3\}$. This implies that $J(\vec{x}, \vec{w})$ is also convex on the convex subdomain $S'' := \{(\vec{x}, \vec{w}) \in S' : w_j \leq 1 - \langle A_j, \vec{x}\rangle / d_j \ \forall j\}$. Since extended-value extensions of convex functions are convex

$$\tilde{J}(\vec{x}, \vec{w}) := \begin{cases} J(\vec{x}, \vec{w}), & (\vec{x}, \vec{w}) \in S'' \\ \infty, & \text{otherwise} \end{cases}$$

is convex as well. Therefore

$$\begin{aligned} \underline{J}(\vec{x}) &= \inf_{\vec{w} \in \mathbb{R}^l} \tilde{J}(\vec{x}, \vec{w}) \\ &= -\ln x_i - \ln\left(f_{\text{neg}}\left(\prod_{j=1}^{l}\left(1 - \frac{\langle A_j, \vec{x}\rangle}{d_j}\right)^{a_{ji}}\right)\right) \end{aligned}$$

is convex by [25, Sect. 3.2.5]. Note that we have used the fact that $-\ln(f_{\text{neg}})$ is a decreasing function in the last step. Thus, f_{neg} satisfies the hypothesis of Proposition 2 and $\underline{J}(e^{\vec{y}})$ is convex in $\vec{y}$. That is, a route with $f_i = f_{\text{neg}}$ allows for a convex formulation via (10) even after performing the change of variable (9).

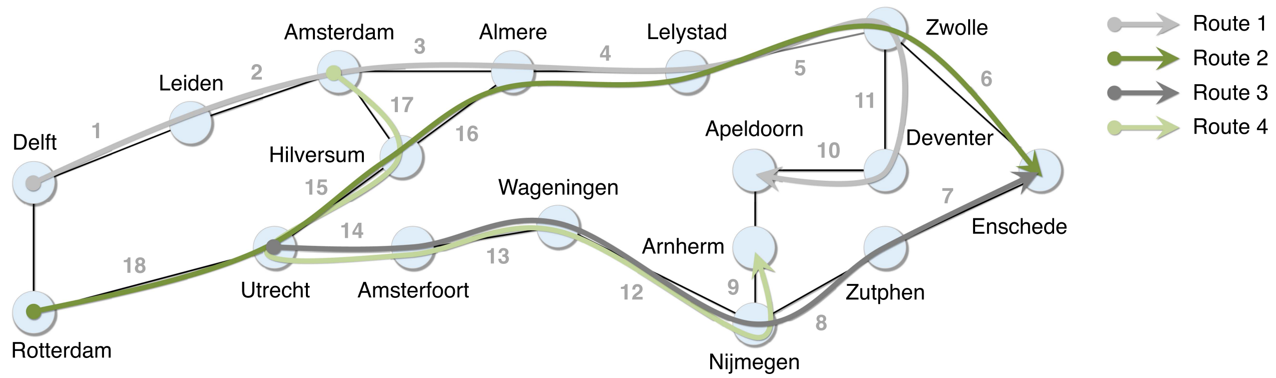


FIGURE 4. A subgraph of the SURFnet topology from [30], figure not to scale. Users run QKD on routes 1 and 2 and teleport via routes 3 and 4 (see Table 2), which involves 18 links annotated here. The length of a link determines its transmissivity (see Table 3). Consistent with current hardware capabilities, entanglement generation is assumed to be attempted every $T_j = 10^{-3}$ s and the nonfiber induced inefficiencies coefficient is $\kappa_j = 0.1$ for each link $j \in [18]$. Optimal allocation is provided in Table IV.

D. FIDELITY OF TELEPORTATION

The fidelity of teleportation with a Werner state with Werner parameter w as a shared entanglement resource is given by

$$f_{\rm F}(w) = \frac{1+w}{2}, \quad 0 \le w \le 1. \tag{19}$$

Observe that $f_{\rm F}(0) = 1/2$, i.e., it does not satisfy assumption A5, which requires $f_i(0) = 0$. However, the corresponding QNUM formulation (10) is similar and the contribution from the ith route is

$$-y_i - \ln\left(f_{\rm F}\left(\prod_{j=1}^{l}\left(1-\frac{\langle A_j, e^{\vec{y}}\rangle}{d_j}\right)^{a_{ji}}\right)\right)$$
$$\text{s.t.} \quad 0 \le \frac{\langle A_j, e^{\vec{y}}\rangle}{d_j} < 1, \quad j \in [l]. \tag{20}$$

The objective function in (20) is not convex in general, but if we restrict the end-to-end Werner parameter to (1/2, 1], it becomes convex. That is, we require

$$0 < \frac{\langle A_j, e^{\vec{y}}\rangle}{d_j} < 1, \quad j \in [l]$$
$$1/2 < \prod_{j=1}^{l}\left(1-\frac{\langle A_j, e^{\vec{y}}\rangle}{d_j}\right)^{a_{ji}}. \tag{21}$$

The region corresponding to (21) is convex by Theorem 1 and the objective function in (20) is convex on this domain due to the following: 1) we can write the objective function as $-y_i - \ln((1+u_i(\vec{y}))/2)$; 2) $u_i(\vec{y})$ is concave on $\{\vec{y}: u_i(\vec{y}) > 1/2\}$ by Proposition 1 (see the proof in Section VII) and hence $(1+u_i(\vec{y}))/2$ is concave; 3) logarithm is concave and increasing implying that $\ln(1+u_i(\vec{y}))/2$ is concave; and 4) $-y_i$ is convex in $\vec{y}$.

VI. NUMERICAL EXAMPLE

In this section, we work out an example on a subgraph [30] derived from the network topology of SURFnet, the national research network of the Netherlands. We assume that each node in the network is also equipped to serve as a repeater and can support multiple routes. We then consider the scenario if certain pairs of nodes were to perform QKD or teleportation between themselves on this real-world fiber network. The corresponding routes and the relevant link IDs are shown in Fig. 4. We also describe the routes with corresponding applications in Table 2.

TABLE 2. Routes With Corresponding Links and Applications. See Also Fig. 4

Route ID (i)	End nodes	Links	Application
1	(Delft, Apeldoorn)	(1, 2, 3, 4, 5, 11, 10)	QKD
2	(Rotterdam, Enschede)	(18, 15, 16, 4, 5, 6)	QKD
3	(Utrecht, Enschede)	(14, 13, 12, 8, 7)	Teleportation
4	(Amsterdam, Arnherm)	(17, 15, 14, 13, 12, 9)	Teleportation

For routes performing QKD (respectively, teleportation), we assume that the relevant measure of entanglement is the secret key fraction (respectively, fidelity of teleportation), i.e., $f_i = f_{\rm sk},\ i \in [2]$ and $f_i = f_{\rm F},\ i \in \{3, 4\}$; see (16) and (19). As explained in Section V-D, we also impose additional constraints that the end-to-end Werner parameters of routes 3 and 4 are more than 1/2 so that the resulting QNUM problem is convex. To cast the QNUM problem in this network to formulation (10), we now need the constants d_js for the relevant links.

Recall from assumption A3 that $d_j = 3\kappa_j\eta_j/2T_j$, where κ_j, η_j, and T_j denote the constant reflecting inefficiencies other than photon loss in the fiber, the transmissivity, and the frequency of the entanglement generation attempt for the jth link, respectively. Similar to an example in [7], we assume that $\kappa_j = 0.1$ and $T_j = 10^{-3}$ s for all 18 links. The transmissivities can be calculated as $\eta_j = 10^{-0.02L_j}$, where L_j is the length of the jth link in kilometers. The final values of the constants d_js are provided in Table 3.

We now solve the QNUM problem via formulation (10) using MATLAB's `fmincon`. The convex nature of the problem ensures that the output y_i's are indeed globally optimal.

TABLE 3. Relevant Links From Fig. 4 and Derived d_j

Link ID (j)	Link	Length (km)	d_j
1	(Delft, Leiden)	30.6	89.84
2	(Leiden, Amsterdam)	60.4	53.79
3	(Amsterdam, Almere)	38.9	77.47
4	(Almere, Lelystad)	44.2	69.44
5	(Lelystad, Zwolle)	47.7	65.12
6	(Zwolle, Enschede)	78.7	40.76
7	(Zutphen, Enschede)	60	54.17
8	(Nijmegen, Zutphen)	58.1	56.25
9	(Nijmegen, Arnhem)	25.7	99.02
10	(Apeldoorn, Deventer)	24.4	100.98
11	(Deventer, Zwolle)	44.7	68.75
12	(Wageningen, Nijmegen)	66.3	49.35
13	(Amersfoort, Wageningen)	62.5	52.40
14	(Utrecht, Amersfoort)	33.8	84.63
15	(Utrecht, Hilversum)	36.7	80.54
16	(Hilversum, Almere)	35.4	82.41
17	(Amsterdam, Hilversum)	30.2	90.52
18	(Rotterdam, Utrecht)	70	46.82

TABLE 4. Optimal Rate and Fidelity Allocations

Route ID (i)	y_i	Rate (x_i) in pairs/sec.	Werner param. (u_i)	Fidelity
1	-0.1877	0.8289	0.9029	0.9272
2	-0.3475	0.7065	0.8677	0.9008
3	1.6633	5.2769	0.5000	0.6250
4	1.2690	3.5573	0.5362	0.6522

The optimal rate allocations and link-level Werner parameters are then computed as $x_i = e^{y_i}$ and using (11), respectively. This prescribes the optimal setting for the link-level generation rates and the fidelities. The resulting end-to-end Werner parameters and fidelities for the optimal allocation are provided in Table 4.

Observe that the secret key fraction ensures that the corresponding routes receive significantly higher fidelity compared to the fidelity of teleportation. In fact, the third route is assigned the minimum fidelity required by the constraint that the end-to-end Werner parameter must be more than $1/2$. In contrast, the fourth route is allocated a slightly higher fidelity as it shares a link with the second route, which runs QKD.

VII. PROOFS OF RESULTS

Proof of Theorem 1: We make use of the fact that sublevel (respectively, superlevel) sets of convex (respectively, concave) functions are convex and the feasible set S can be expressed as an intersection of convex sets. Recall from (12) that

$$T_j := \{\vec{y}\in\mathbb{R}^r\colon \langle A_j, e^{\vec{y}}\rangle < d_j\},\qquad j\in[l],\ \ T := \cap_{j\in[l]} T_j$$
$$S_i := \{\vec{y}\in T\colon u_i(\vec{y}) > c^{(i)}\},\qquad i\in[r],\ \ S := \cap_{i\in[r]} S_i.$$

By convexity of the exponential function and the fact that $a_{ji} \geq 0$, $\langle A_j, e^{\vec{y}}\rangle$ is a convex function of $\vec{y}$. Thus, T_j's, being sublevel sets of convex functions, are convex. Furthermore, T being the intersection of T_js is also convex.

Since $c^{(i)} > 0$, we now consider the following set:

$$\tilde{S}_i := \{\vec{y}\in\mathbb{R}^r : \ln(u_i(\vec{y})) > \ln c^{(i)}\}.$$

We will show that $\tilde{S}_i$ is convex. Observe that

$$\ln(u_i(\vec{y})) = \sum_{j=1}^{l} a_{ji}\ln\left(1 - \frac{\langle A_j, e^{\vec{y}}\rangle}{d_j}\right)$$

where $a_{ji} \geq 0$. Thus, to prove that $\tilde{S}_i$ is convex, it suffices to show that each $\ln(1 - \langle A_j, e^{\vec{y}}\rangle/d_j)$ is concave. To this end, we consider $\vec{y}_1, \vec{y}_2 \in \tilde{S}_i$ and $\vec{y}_t = t\vec{y}_1 + (1-t)\vec{y}_2$, $t\in(0,1)$. By convexity of $\langle A_j, e^{\vec{y}}\rangle$

$$\langle A_j, e^{\vec{y}_t}\rangle \leq t\langle A_j, e^{\vec{y}_1}\rangle + (1-t)\langle A_j, e^{\vec{y}_2}\rangle \tag{22}$$

$$\overset{d_j>0}{\Rightarrow} 1 - \frac{\langle A_j, e^{\vec{y}_t}\rangle}{d_j} \geq t\left(1 - \frac{\langle A_j, e^{\vec{y}_1}\rangle}{d_j}\right) + (1-t)\left(1 - \frac{\langle A_j, e^{\vec{y}_2}\rangle}{d_j}\right) \tag{23}$$

$$\Rightarrow \ln\left(1 - \frac{\langle A_j, e^{\vec{y}_t}\rangle}{d_j}\right) \geq \ln\left(t\left(1 - \frac{\langle A_j, e^{\vec{y}_1}\rangle}{d_j}\right) + (1-t)\left(1 - \frac{\langle A_j, e^{\vec{y}_2}\rangle}{d_j}\right)\right) \tag{24}$$

$$\Rightarrow \ln\left(1 - \frac{\langle A_j, e^{\vec{y}_t}\rangle}{d_j}\right) \geq t\ln\left(1 - \frac{\langle A_j, e^{\vec{y}_1}\rangle}{d_j}\right) + (1-t)\ln\left(1 - \frac{\langle A_j, e^{\vec{y}_2}\rangle}{d_j}\right) \tag{25}$$

where we have used monotonicity and concavity of logarithm in (24) and (25), respectively.

Since S can be rewritten as an intersection of convex sets: $S = (\cap_{i\in[r]}\tilde{S}_i) \cap T$, it must be convex. □

The following lemma is crucial for showing that $u_i(\vec{y})$ is concave on S_i.

Lemma 1: Let $0 < b_\iota < 1$, $n \geq 2$, and $0 < t < 1$. Then

$$\sup_{\prod_{\iota=1}^n b_\iota \geq t}\left(\sum_{\iota=1}^{n-1} {}^{1}\!/\!{}_{b_\iota}\right) = n - 2 + {}^{1}\!/\!{}_{t}. \tag{26}$$

Proof: We again use the change-of-variable technique [17]: $e^{p_\iota} = b_\iota$, $p_\iota < 0$, $\iota\in[n]$. Then

$$\sup_{\prod_{\iota=1}^n b_\iota \geq t,\, 0<b_\iota<1\,\forall\iota}\left(\sum_{\iota=1}^{n-1} {}^{1}\!/\!{}_{b_\iota}\right) = \sup_{\sum_{\iota=1}^n p_\iota \geq \ln t,\, p_\iota<0\,\forall\iota}\left(\sum_{\iota=1}^{n-1} e^{-p_\iota}\right).$$

We prove by induction that

$$\sup_{\sum_{\iota=1}^n p_\iota \geq \ln t,\, p_\iota<0\,\forall\iota}\left(\sum_{\iota=1}^{n-1} e^{-p_\iota}\right) = n - 2 + {}^{1}\!/\!{}_{t}. \tag{27}$$

To show (27) for $n = 2$, we observe that maximizing the function on the left-hand side reduces to minimizing p_1 in the feasible set $\{(p_1, p_2) : p_1 + p_2 \geq \ln t,\ p_1, p_2 < 0\}$ as the function e^{-p_1} is decreasing in p_1. Direct substitution gives

$$\sup_{\sum_{\iota=1}^{2} p_\iota \geq \ln t,\ p_\iota < 0\,\forall\iota} e^{-p_1} = 1/t$$

proving the assertion for $n = 2$. Assuming that the hypothesis holds for $n = k-1$

$$\sup_{\sum_{\iota=1}^{k} p_\iota \geq \ln t,\ p_\iota < 0\,\forall\iota} \left(\sum_{\iota=1}^{k-1} e^{-p_\iota}\right)$$

$$= \sup_{\ln t \leq p < 0}\ \sup_{p_1 = p,\ \sum_{\iota=2}^{k} p_\iota \geq \ln t - p,\ p_\iota < 0\,\forall\iota} \left(e^{-p} + \sum_{\iota=2}^{k-1} e^{-p_\iota}\right) \tag{28}$$

$$= \sup_{\ln t \leq p < 0} \left(e^{-p} + \sup_{\sum_{\iota=2}^{k} p_\iota \geq \ln t - p,\ p_\iota < 0\,\forall\iota} \sum_{\iota=2}^{k-1} e^{-p_\iota}\right) \tag{29}$$

$$= \sup_{\ln t \leq p < 0} \left(e^{-p} + k - 3 + e^{p - \ln t}\right) \tag{30}$$

$$= k - 3 + \sup_{\ln t \leq p < 0} \left(e^{-p} + e^{p - \ln t}\right) \tag{31}$$

$$= k - 2 + 1/t \tag{32}$$

where (32) follows from considering the convex function $e^{-x} + e^{x+\lambda}$ over $x \in [-\lambda, 0)$ and observing that its supremum is attained at $-\lambda$. Thus, the assertion is true for $n \geq 2$. □

To prove Theorem 2, we need a similar result.

Lemma 2: Let $0 < b_\iota < 1$, $0 < \beta < 1$, $n \geq 2$, and $0 < t < 1$. Then

$$\sup_{\prod_{\iota=1}^{n} b_\iota = t} \left(\beta/b_1 + \sum_{\iota=2}^{n} 1/b_\iota\right) = n - 2 + \beta + 1/t. \tag{33}$$

Proof: For $n = 2$, $\beta < 1$ gives

$$\sup_{b_1 b_2 = t,\ 0 < b_\iota < 1} (\beta/b_1 + 1/b_2) = \sup_{t < b_1 < 1} (\beta/b_1 + b_1/t) = \beta + 1/t$$

where the last step follows from the fact that $\beta/x + x/t$ is convex on the set $t < x < 1$, and consequently, its supremum on this range must correspond to its value at one of the boundary points. By inspection, we can see that this corresponds to the boundary point $x = 1$.

The induction step now follows similarly to Lemma 1, where we let

$$e^{p_\iota} = b_\iota, \quad p_\iota < 0, \ \iota \in [n].$$

Assuming that (33) holds for $n = k-1$, we have

$$\sup_{\prod_{\iota=1}^{k} b_\iota = t} \left(\beta/b_1 + \sum_{\iota=2}^{k} 1/b_\iota\right)$$

$$= \sup_{\sum_{\iota=1}^{k} p_\iota = \ln t,\ p_\iota < 0\,\forall\iota} \left(\beta e^{-p_1} + \sum_{\iota=2}^{k} e^{-p_\iota}\right) \tag{34}$$

$$= \sup_{\ln t \leq p < 0}\ \sup_{p_k = p,\ \sum_{\iota=1}^{k-1} p_\iota = \ln t - p,\ p_\iota < 0\,\forall\iota} \Bigg(\beta e^{-p_1} + \sum_{\iota=2}^{k-1} e^{-p_\iota} + e^{-p}\Bigg) \tag{35}$$

$$= \sup_{\ln t \leq p < 0} \Bigg(e^{-p} + \sup_{\sum_{\iota=1}^{k-1} p_\iota = \ln t - p,\ p_\iota < 0\,\forall\iota} \Big(\beta e^{-p_1} + \sum_{\iota=2}^{k-1} e^{-p_\iota}\Big)\Bigg) \tag{36}$$

$$= k - 3 + \beta + \sup_{\ln t \leq p < 0} \left(e^{-p} + e^{p - \ln t}\right) \tag{37}$$

$$= k - 2 + \beta + 1/t \tag{38}$$

where the last step involves computing the maxima of the function $e^{-x} + e^{x+\lambda}$ over $x \in [-\lambda, 0)$ as in Lemma 1. Thus, (34) holds for $n \geq 2$. □

Proof of Proposition 1: Our proof is based on two facts: (i) since $u_i(\vec{y})$ is twice differentiable, its Hessian $H^{(i)}$ is symmetric by [31, eq. (8.12.3)] and has real eigenvalues and (ii) each diagonal element of $H^{(i)}$ dominates the absolute sum of the nondiagonal entries of the corresponding row on S_i. We can thus apply Gershgorin's circle theorem [32], which says that the eigenvalues of a matrix are contained in the circles with centers as diagonal elements and the respective absolute sum of off-diagonal elements as radii. In this case, this would imply that the eigenvalues of $H^{(i)}$ are nonpositive on S_i, which is equivalent to concavity of $u_i(\vec{y})$.

To establish fact (ii), we observe that the sign of the row-sums of $H^{(i)}$ is the opposite of the sign of sums of the form $n - \sum_{j \neq j',\ j \in [n]} 1/w_j(\vec{y})$ for some $n \in \mathbb{N}$ and $j' \in [n]$. If $0 < w_j(\vec{y}) < 1$ and $\prod_{j \in [n]} w_j(\vec{y}) > 1/2$, such sums are nonnegative as $\sum_{j \neq j',\ j \in [n]} 1/w_j(\vec{y})$ cannot exceed n (see Lemma 1). The details are as follows:

We suppose that the ith route passes through n links where $n \leq l$. Without loss of generality, we can number these links as $1, 2, \ldots, n$. Then, for $k \in [r]$

$$\frac{\partial u_i(\vec{y})}{\partial y_k} = \frac{\partial}{\partial y_k}\Big(\prod_{j=1}^{n} w_j(\vec{y})\Big) = \sum_{j=1}^{n} \underbrace{\frac{\partial w_j(\vec{y})}{\partial y_k} \prod_{j' \in [n]\setminus\{j\}} w_{j'}(\vec{y})}_{v_{jk}^{(i)}(\vec{y})}. \tag{39}$$

From now on, we frequently drop the argument $\vec{y}$ when it is clear from context. Note that $v_{jk}^{(i)} = 0$ iff $a_{jk} = 0$, i.e., the kth route does not pass through the jth link. Specifically

$$v_{jk}^{(i)} = -\frac{a_{jk} e^{y_k}}{d_j} \prod_{j' \in [n]\setminus\{j\}} w_{j'}. \tag{40}$$

Also

$$\sum_{m=1}^{r} \frac{\partial w_{j'}}{\partial y_m} = \sum_{m=1}^{r} \frac{-a_{j'm} e^{y_m}}{d_{j'}} = w_{j'} - 1 \tag{41}$$

$$\frac{\partial}{\partial y_m}\left(-\frac{e^{y_k}}{d_j}\right) = -\frac{e^{y_k}}{d_j} \mathbb{1}_{m=k} \tag{42}$$

where $\mathbb{1}_A$ denotes the indicator function, which takes value 1 on the set A and 0 otherwise. This leads to the following expression for the Hessian:

$$H_{kk}^{(i)}(\vec{y}) = \sum_{j=1}^{n} \frac{\partial v_{jk}^{(i)}}{\partial y_k} = -\sum_{j=1}^{n} \frac{a_{jk} e^{y_k}}{d_j} \prod_{j' \in [n]\setminus\{j\}} w_{j'} \left(1 + \sum_{j'' \in [n]\setminus\{j\}} \frac{\partial w_{j''}}{\partial y_k} \frac{1}{w_{j''}}\right)$$

$$H_{km}^{(i)}(\vec{y}) = H_{mk}^{(i)}(\vec{y}) = -\sum_{j=1}^{n} \frac{a_{jk} e^{y_k}}{d_j} \prod_{j' \in [n]\setminus\{j\}} w_{j'} \left(\sum_{j'' \in [n]\setminus\{j\}} \frac{\partial w_{j''}}{\partial y_m} \frac{1}{w_{j''}}\right), \quad m \in [r] \setminus k.$$

Since $\partial w_j/\partial y_s \le 0 \;\forall j, s$, $H_{km}^{(i)}(\vec{y}) \ge 0$ for $k \ne m$. Therefore

$$H_{kk}^{(i)}(\vec{y}) + \sum_{m \in [r]\setminus k} \left|H_{km}^{(i)}(\vec{y})\right|$$

$$= H_{kk}^{(i)}(\vec{y}) + \sum_{m \in [r]\setminus k} H_{km}^{(i)}(\vec{y}) \tag{43}$$

$$\overset{(42)}{=} \sum_{j=1}^{n} v_{jk}^{(i)} \left(1 + \sum_{j'' \in [n]\setminus\{j\}} \frac{1}{w_{j''}} \left(\frac{\partial w_{j''}}{\partial y_k} + \sum_{m \in [r]\setminus k} \frac{\partial w_{j''}}{\partial y_m}\right)\right) \tag{44}$$

$$= \sum_{j=1}^{n} v_{jk}^{(i)} \left(1 + \sum_{j'' \in [n]\setminus\{j\}} \frac{1}{w_{j''}} \sum_{m=1}^{r} \frac{\partial w_{j''}}{\partial y_m}\right) \tag{45}$$

$$\overset{(43)}{=} \sum_{j=1}^{n} v_{jk}^{(i)} \left(1 + \sum_{j'' \in [n]\setminus\{j\}} \frac{w_{j''} - 1}{w_{j''}}\right) \tag{46}$$

$$= \sum_{j=1}^{n} v_{jk}^{(i)} \left(n - \sum_{j'' \in [n]\setminus\{j\}} 1/w_{j''}\right). \tag{47}$$

Recall that on S_i, $0 < w_{j''} < 1$ for all $j'' \in [n]$ and $\{\prod_{j'' \in [n]} w_{j''} > c^{(i)}\}$. We can thus apply Lemma 1 to see that

$$n - \sum_{j'' \in [n]\setminus\{j\}} 1/w_{j''} \ge n - \sup_{\prod_{j'' \in [n]} w_{j''} \ge c^{(i)}} \left(\sum_{j'' \in [n]\setminus\{j\}} 1/w_{j''}\right)$$

$$= 2 - 1/c^{(i)}$$

which is nonnegative for $c^{(i)} \ge 1/2$. Also, observe from (41) that $v_{jk}^{(i)} \le 0$, implying that the right-hand side of (45) is nonpositive. Therefore, all eigenvalues of $H^{(i)}$ are nonpositive on S_i due to Gershgorin's circle theorem [32]. □

Proof of Theorem 2: Let us denote by D^2F the Hessian of $F_i(u_i(\vec{y})) = \ln f_i(u_i(\vec{y}))$. We will again exploit the symmetry of the Hessian and use Gershgorin's circle theorem [32] to show that the eigenvalues of D^2F are nonpositive on S_i.

To prove symmetry, we observe that since $u_i(\vec{y})$ is twice differentiable and f_i is twice differentiable by assumption, so is $f_i(u_i(\vec{y}))$. Furthermore, $f_i(u_i(\vec{y})) > 0$ on S_i. Thus, $F_i(u_i(\vec{y}))$ is twice differentiable on S_i and D^2F is symmetric by [31, eq. (8.12.3)].

Also, D^2F can be explicitly written as

$$D^2F(\vec{y}) = F_i''(u_i(\vec{y}))(u_i'(\vec{y}))^T u_i'(\vec{y}) + F_i'(u_i(\vec{y}))H^{(i)}(\vec{y}) \tag{48}$$

where

$$u_i'(\vec{y}) := \left(\frac{\partial u_i(\vec{y})}{\partial y_1}, \frac{\partial u_i(\vec{y})}{\partial y_2}, \ldots, \frac{\partial u_i(\vec{y})}{\partial y_r}\right).$$

We consider D^2F separately on the following subsets of S_i: $\{\vec{y} \in T : c^{(i)} < u_i(\vec{y}) \le c_1^{(i)}\}$, $\{\vec{y} \in T : c_1^{(i)} < u_i(\vec{y}) < 1\}$. This is because $F_i''(u) \le 0$ for $u \in (c^{(i)}, c_1^{(i)}]$ and $F_i''(u) > 0$ for $u \in (c_1^{(i)}, 1)$ by assumption. Since $(u_i')^T u_i'$ is positive semidefinite, $F_i' \ge 0$ (as f_i is increasing and $F_i = \ln(f_i)$), $H^{(i)}$ is negative semidefinite on S_i (due to the assumption that Cond. 1 (13) holds), and $F_i'' \le 0$ on the first subset, D^2F is negative semidefinite on this subset as well.

We now consider D^2F on $\{\vec{y} \in T : c_1^{(i)} < u_i(\vec{y}) < 1\}$ (the second subset of S_i), where $F_i''(u_i(\vec{y}) > 0$. Expanding (48)

$$D^2F_{kk}(\vec{y}) = F_i''\left(u_i(\vec{y})\right)\left(\frac{\partial}{\partial y_k} u_i(\vec{y})\right)^2 + F_i'\left(u_i(\vec{y})\right) H_{kk}^{(i)}(\vec{y})$$

$$D^2F_{km}(\vec{y}) = D^2F_{mk}(\vec{y})$$

$$= F_i''(u_i(\vec{y})) \frac{\partial u_i(\vec{y})}{\partial y_m} \frac{\partial u_i(\vec{y})}{\partial y_k} + F_i'\left(u_i(\vec{y})\right) H_{mk}^{(i)}(\vec{y})$$

where $m \in [r] \setminus k$.

Since $\partial u_i/\partial y_s \le 0 \;\forall s$ [see (40) and (42)], $H_{mk}^{(i)}(\vec{y}) \ge 0$, $F_i' \ge 0$, and $F_i'' > 0$ on the second subset, $D^2F_{km}(\vec{y}) \ge 0$ for $m \in [r] \setminus k$. Furthermore

$$\sum_{m=1}^{r} \frac{\partial u_i(\vec{y})}{\partial y_m} = \sum_{j=1}^{n} \sum_{m=1}^{r} \frac{-a_{jm} e^{y_m}}{d_j} \prod_{j' \in [n]\setminus\{j\}} w_{j'} \tag{49}$$

$$= \sum_{j=1}^{n} (w_j - 1) \prod_{j' \in [n]\setminus\{j\}} w_{j'} \tag{50}$$

$$= u_i \left(n - \sum_{j=1}^{n} 1/w_j\right). \tag{51}$$

Therefore

$$D^2F_{kk}(\vec{y}) + \sum_{m \in [r]\setminus k} \left|D^2F_{km}(\vec{y})\right| \tag{52}$$

$$= D^2F_{kk}(\vec{y}) + \sum_{m \in [r]\setminus k} D^2F_{km}(\vec{y}) \tag{53}$$

$$= F_i''(u_i(\vec{y})) \frac{\partial u_i(\vec{y})}{\partial y_k} \sum_{m=1}^{r} \frac{\partial u_i(\vec{y})}{\partial y_m} + F_i'(u_i(\vec{y})) \sum_{m=1}^{r} H_{km}^{(i)} \tag{54}$$

$$\overset{(54)}{=} F_i''(u_i(\vec{y})) \frac{\partial u_i(\vec{y})}{\partial y_k} u_i(\vec{y}) \left(n - \sum_{j'=1}^{n} 1/w_{j'}\right) + F_i'(u_i(\vec{y})) \sum_{m=1}^{r} H_{km}^{(i)} \tag{55}$$

$$\overset{(41),(45)}{=} F_i''(u_i(\vec{y})) \left(\sum_{j=1}^{n} v_{jk}^{(i)}\right) u_i(\vec{y}) \left(n - \sum_{j'=1}^{n} 1/w_{j'}\right)$$

$$+ F_i'\left(u_i(\vec{y})\right) \sum_{j=1}^{n} v_{jk}^{(i)} \left(n - \sum_{j'' \in [n]\setminus\{j\}} 1/w_{j''}\right) \tag{56}$$

$$= \sum_{j=1}^{n} v_{jk}^{(i)} \Bigg(\left(u_i F_i''(u_i) + F_i'(u_i) \left(n - \sum_{j' \in [n]\setminus\{j\}} 1/w_{j'}\right)\right.$$

$$\left. - u_i F_i''(u_i) 1/w_j \right) \tag{57}$$

$$= \underbrace{\left(u_i F_i''(u_i) + F_i'(u_i)\right)}_{>0} \sum_{j=1}^{n} \underbrace{v_{jk}^{(i)}}_{\leq 0} \left(n - \sum_{j' \in [n]\setminus\{j\}} 1/w_{j'} \right.$$

$$\left. - \underbrace{\frac{u_i F_i''(u_i)}{u_i F_i''(u_i) + F_i'(u_i)}}_{=\beta(u_i)} \frac{1}{w_j} \right). \tag{58}$$

Recall that $\prod_{j'=1}^{n} w_{j'} = u_i$. Also, on the present subset $c_1^{(i)} < u_i(\vec{y}) < 1$ and $F_i''(u_i(\vec{y})) > 0$ by definition, meaning that we have $0 < \beta(u_i) < 1$. Thus, by Lemma 2

$$n - \sum_{j' \in [n]\setminus\{j\}} 1/w_{j'} - \beta(u_i)/w_j$$

$$\geq n - \sup_{\prod_{j'=1}^{n} w_{j'} = u_i} \left(\beta(u_i)/w_j + \sum_{j' \in [n]\setminus\{j\}} 1/w_{j'} \right) \tag{59}$$

$$\overset{(34)}{=} 2 - \beta(u_i) - 1/u_i. \tag{60}$$

Cond. 2 (14) simply says that (60) is nonnegative on $\{\vec{y} \in T : u_i(\vec{y}) > c_1^{(i)}\}$, which via (58) implies that

$$D^2 F_{kk}(\vec{y}) + \sum_{m \in [r]\setminus k} |D^2 F_{km}(\vec{y})| \leq 0.$$

Applying Gershgorin's circle theorem [32], we see that all eigenvalues of D^2F are nonpositive if Cond. 2 (14) is met.

We have thus shown that Cond. 2 (14) is sufficient for D^2F to be negative semidefinite on entire S_i, or equivalently, for $F_i(u_i(\vec{y}))$ to be concave. □

VIII. CONCLUSION

The QNUM problem addresses the issue of efficient and fair distribution of link-level entanglement rate and fidelity among competing routes. In this work, we considered this problem with the objective of finding a globally optimal allocation in a mathematically guaranteed way. To that end, we derived conditions under which the QNUM problem can be formulated as a convex problem. We assumed a static model where links are generated using the single-click protocol, and a central entity, having global knowledge of the network, determines the allocations before the network goes into operation. As a first analysis, our model abstracts away detailed intricacies of the networks and primarily focuses on allocating the link-level resources *optimally*.

We first showed that, in our setup, the QNUM problem can be formulated as an optimization problem solely in terms of rate allocations. We then provided a reformulation and sufficient conditions in terms of the relevant entanglement measures for it to be convex. These conditions were shown to hold for previously considered entanglement measures that did not directly admit a convex QNUM formulation. Furthermore, the reformulation was shown to preserve convexity, i.e., while attempting to convexify the contribution of a route to the objective function, the reformulation did not render convex contributions from other routes nonconvex. We also worked out an example where we derived the optimal rate–fidelity allocations if we were to run QKD and teleportation concurrently on a real-world fiber network. Our findings allow for efficient computation of globally optimal rate and fidelity allocations in an entanglement distribution network supporting diverse applications.

ACKNOWLEDGMENT

The authors thank the anonymous reviewers for their constructive feedback, which greatly improved the quality of this article. The author Sounak Kar also acknowledge Scarlett Gauthier and Kaushik Senthoor for careful reading of an earlier version of this article.

REFERENCES

[1] C. H. Bennett and G. Brassard, "Quantum cryptography: Public key distribution and coin tossing," in *Proc. IEEE Int. Conf. Comput. Syst. Signal Process.*, 1984, pp. 175–179, doi: 10.1016/j.tcs.2014.05.025.

[2] A. K. Ekert, "Quantum cryptography and Bell's theorem," in *Quantum Measurements in Optics.*, Berlin, Germany: Springer, 1992, pp. 413–418, doi: 10.1007/978-1-4615-3386-3_34.

[3] V. Giovannetti, S. Lloyd, and L. Maccone, "Quantum-enhanced measurements: Beating the standard quantum limit," *Science*, vol. 306, no. 5700, pp. 1330–1336, 2004, doi: 10.1126/science.1104149.

[4] R. Jozsa, D. S. Abrams, J. P. Dowling, and C. P. Williams, "Quantum clock synchronization based on shared prior entanglement," *Phys. Rev. Lett.*, vol. 85, no. 9, 2000, Art. no. 2010, doi: 10.1103/PhysRevLett.85.2010.

[5] A. Broadbent, J. Fitzsimons, and E. Kashefi, "Universal blind quantum computation," in *Proc. 50th Annu. IEEE Symp. Found. Comput. Sci.*, 2009, pp. 517–526, doi: 10.1109/FOCS.2009.36.

[6] J. F. Fitzsimons and E. Kashefi, "Unconditionally verifiable blind quantum computation," *Phys. Rev. A*, vol. 96, no. 1, 2017, Art. no. 012303, doi: 10.1103/PhysRevA.96.012303.

[7] G. Vardoyan and S. Wehner, "Quantum network utility maximization," in *Proc. IEEE Int. Conf. Quantum Comput. Eng.*, 2023, pp. 1238–1248, doi: 10.1109/QCE57702.2023.00140.

[8] J. Y. L. Boudec, "Rate adaptation, congestion control and fairness: A tutorial," 22 Nov. 2005. [Online]. Available: https://leboudec.github.io/leboudec/latex/cc/LEB3132.pdf

[9] F. Kelly, "Charging and rate control for elastic traffic," *Eur. Trans. Telecommun.*, vol. 8, no. 1, pp. 33–37, 1997, doi: 10.1002/ett.4460080106.

[10] F. P. Kelly, A. K. Maulloo, and D. K. H. Tan, "Rate control for communication networks: Shadow prices, proportional fairness and stability," *J. Oper. Res. Soc.*, vol. 49, no. 3, pp. 237–252, 1998, doi: 10.1057/palgrave.jors.2600523.

[11] P. O. Johansson, *An Introduction to Modern Welfare Economics*. Cambridge, U.K.: Cambridge Univ. Press, 1991, doi: 10.1017/CBO9780511582417.

[12] J. Rawls, *A Theory of Justice*. Cambridge, MA, USA: Harvard Univ. Press, 1971, doi: 10.2307/j.ctvjf9z6v.

[13] D. P. Palomar and M. Chiang, "A tutorial on decomposition methods for network utility maximization," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 8, pp. 1439–1451, Aug. 2006, doi: 10.1109/JSAC.2006.879350.

[14] M. B. Plenio and S. Virmani, "An introduction to entanglement measures," *Quantum Inf. Comput.*, vol. 7, no. 1, pp. 1–51, 2007, doi: 10.26421/QIC7.1-2-1.

[15] C. Cabrillo, J. I. Cirac, P. Garcia-Fernandez, and P. Zoller, "Creation of entangled states of distant atoms by interference," *Phys. Rev. A*, vol. 59, no. 2, 1999, Art. no. 1025, doi: 10.1103/PhysRevA.59.1025.

[16] P. C. Humphreys et al., "Deterministic delivery of remote entanglement on a quantum network," *Nature*, vol. 558, no. 7709, pp. 268–273, 2018, doi: 10.1038/s41586-018-0200-5.

[17] S. Boyd, S.-J. Kim, L. Vandenberghe, and A. Hassibi, "A tutorial on geometric programming," *Optim. Eng.*, vol. 8, pp. 67–127, 2007, doi: 10.1007/s11081-007-9001-7.

[18] Y. Lee, W. Dai, D. Towsley, and D. Englund, "Quantum network utility: A framework for benchmarking quantum networks," in *Proc. Nat. Acad. Sci.*, vol. 121, no. 17, 2024, Art. no. e2314103121, doi: 10.1073/pnas.2314103121.

[19] S. Pouryousef, H. Shapourian, A. Shabani, and D. Towsley, "Quantum network planning for utility maximization," in *Proc. 1st Workshop Quantum Netw. Distrib. Quantum Comput.*, 2023, pp. 13–18, doi: 10.1145/3610251.361055.

[20] S. Gauthier, G. Vardoyan, and S. Wehner, "An architecture for control of entanglement generation switches in quantum networks," *IEEE Trans. Quantum Eng.*, vol. 4, 2023, Art. no. 4100717, doi: 10.1109/TQE.2023.3320047.

[21] S. Gauthier, T. Vasantam, and G. Vardoyan, "An on-demand resource allocation algorithm for a quantum network hub and its performance analysis," 2024, *arXiv:2405.18066*, doi: 10.48550/arXiv.2405.18066.

[22] W. Dür and H. J. Briegel, "Entanglement purification and quantum error correction," *Rep. Prog. Phys.*, vol. 70, 2007, Art. no. 1381, doi: 10.1088/0034-4885/70/8/R03.

[23] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction," *Phys. Rev. A*, vol. 54, no. 5, 1996, Art. no. 3824, doi: 10.1007/3-540-33133-6_5.

[24] S. Diamond and S. Boyd, "CVXPY: A Python-embedded modeling language for convex optimization," *J. Mach. Learn. Res.*, vol. 17, no. 83, pp. 2909–2913, 2016, doi: 10.5555/2946645.3007036.

[25] S. Boyd and L. Vandenberghe, *Convex Optimization.*, New York, NY, USA: Cambridge Univ. Press, 2004, doi: 10.1017/CBO9780511804441.

[26] Y. Nesterov, *Lectures on Convex Optimization*, vol. 137, Berlin, Germany: Springer, 2018, doi: 10.1007/978-3-319-91578-4.

[27] J.-W. Pan, D. Bouwmeester, H. Weinfurter, and A. Zeilinger, "Experimental entanglement swapping: Entangling photons that never interacted," *Phys. Rev. Lett.*, vol. 80, no. 18, 1998, Art. no. 3891, doi: 10.1103/PhysRevLett.80.3891.

[28] W. J. Munro, K. Azuma, K. Tamaki, and K. Nemoto, "Inside quantum repeaters," *IEEE J. Sel. Top. Quantum Electron.*, vol. 21, no. 3, pp. 78–90, May/Jun., 2015, doi: 10.1109/JSTQE.2015.2392076.

[29] P. W. Shor and J. Preskill, "Simple proof of security of the BB84 quantum key distribution protocol," *Phys. Rev. Lett.*, vol. 85, no. 2, pp. 441–444, 2000, doi: 10.1103/PhysRevLett.85.441.

[30] G. Vardoyan, E. van Milligen, S. Guha, S. Wehner, and D. Towsley, "On the bipartite entanglement capacity of quantum networks," *IEEE Trans. Quantum Eng.*, vol. 5, 2024, Art. no. 4100414, doi: 10.1109/TQE.2024.3366696.

[31] J. Dieudonné, *Foundations of Modern Analysis.*, Redditch, U.K.: Read Books Ltd., 2011.

[32] S. A. Gershgorin, "Über die Abgrenzung der Eigenwerte einer Matrix," *Bull. Acad. Sci.*, vol. 6, pp. 749–754, 1931.