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UPPER AND LOWER BOUNDS FOR THE OPTIMAL CONSTANT IN THE EXTENDED SOBOLEV INEQUALITY. DERIVATION AND NUMERICAL RESULTS

SH. M. NASIBOV AND E. J. M. VELING*

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Abstract. We prove and give numerical results for two lower bounds and eleven upper bounds to the optimal constant $k_0 = k_0(n, \alpha)$ in the inequality

$$\|u\|_{2n/(n-2\alpha)} \leq k_0 \|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}, \quad u \in H^1(\mathbb{R}^n),$$

for $n = 1$, $0 < \alpha \leq 1/2$, and $n \geq 2$, $0 < \alpha < 1$.

This constant k_0 is the reciprocal of the infimum $\lambda_{n,\alpha}$ for $u \in H^1(\mathbb{R}^n)$ of the functional

$$\Lambda_{n,\alpha} = \frac{\|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}}{\|u\|_{2n/(n-2\alpha)}}, \quad u \in H^1(\mathbb{R}^n),$$

where for $n = 1$, $0 < \alpha \leq 1/2$, and for $n \geq 2$, $0 < \alpha < 1$.

The lowest point in the point spectrum of the Schrödinger operator $\tau = -\Delta + q$ on $\mathbb{R}^n$ with the real-valued potential q can be expressed in $\lambda_{n,\alpha}$ for all $q_- = \max(0, -q) \in L^p(\mathbb{R}^n)$, for $n = 1$, $1 \leq p < \infty$, and $n \geq 2$, $n/2 < p < \infty$, and the norm $\|q_-\|_p$.

1. Introduction

Here, we present the derivations and the results of some numerical evaluations for the optimal constant $k_0 = k_0(n, \alpha)$ in the estimate

$$\|u\|_{2n/(n-2\alpha)} \leq k_0 \|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}, \quad u \in H^1(\mathbb{R}^n), \quad (1)$$

for $n = 1$, $0 < \alpha \leq 1/2$, and $n \geq 2$, $0 < \alpha < 1$.

For $n = 1$, k_0 is known explicitly (see [1], [2], [3] and [4, Lemma 2.1, (2.4)])

$$k_0(1, \alpha) = 2^\alpha \alpha^{\alpha/2} (1 - \alpha)^{-(1-\alpha)/2} (1 - 2\alpha)^{(1-2\alpha)/2} B\left(\frac{1}{2}, \frac{1}{2\alpha}\right)^{-\alpha}, \quad (2)$$

for $0 < \alpha < 1/2$, and $k_0(1, 1/2) = 1$,

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where $B(p, q)$ is the Beta Function

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad \Re p > 0, \quad \Re q > 0. \quad (3)$$

For $n \geq 2$, a number of authors has dealt with estimates for $k_0(n, \alpha)$ for some specific values or in a general sense: [5], [6], [7], [8], [9], [10], [11], [4], [12], [13], [14], [15].

The value k_0 equals the reciprocal value of the infimum $\lambda_{n,\alpha}$ of the functional $\Lambda_{n,\alpha}$:

$$\lambda_{n,\alpha} = \inf_{u \in H^1(\mathbb{R}^n)} \Lambda_{n,\alpha}, \quad \text{with} \quad (4)$$

$$\Lambda_{n,\alpha} = \frac{\|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}}{\|u\|_{2n/(n-2\alpha)}}, \quad u \in H^1(\mathbb{R}^n), \quad (5)$$

where $0 < \alpha \leq 1/2$ if $n = 1$, and $0 < \alpha < 1$ if $n \geq 2$.

One of the motivations to study this functional comes from the fact that the lowest point in the point spectrum of the Schrödinger operator can be expressed by the infimum $\lambda_{n,\alpha}$ of this functional $\Lambda_{n,\alpha}$. So, for the Schrödinger operator $\tau = -\Delta + q$ on $\mathbb{R}^n$ with the real-valued potential q such that $q = q_+ - q_-$, where

$$q_+ = \max(0, q) \in L_{loc}^2(\mathbb{R}^n), \quad (6)$$

$$\begin{aligned} q_- = \max(0, -q) \in L^p(\mathbb{R}^n), \quad n = 1: \quad 1 \leq p < \infty, \\ n \geq 2: \quad n/2 < p < \infty. \end{aligned} \quad (7)$$

the lowest point in the point spectrum for all such q expressed as

$$l(n, \alpha) = \inf_{q_- \in L^p(\mathbb{R}^n)} \inf_{u \in H^1(\mathbb{R}^n)} \frac{\|\nabla u\|_2^2 + \int_{\mathbb{R}^n} q |u|^2 dx}{\|u\|_2^2} \|q_-\|_p^{-1/(1-\alpha)}, \quad (8)$$

with $\alpha = n/(2p)$,

will be

$$l(n, \alpha) = -(1-\alpha)\alpha^{\alpha/(1-\alpha)}\lambda_{n,\alpha}^{-2/(1-\alpha)}, \quad \begin{aligned} 0 < \alpha \leq 1/2 \text{ if } n = 1, \\ 0 < \alpha < 1 \text{ if } n \geq 2, \end{aligned} \quad (9)$$

see among others [10], [4].

The corresponding Euler equation belonging to the infimum $\lambda_{n,\alpha}$ of the functional $\Lambda_{n,\alpha}(u)$ reads

$$-\alpha \frac{\Delta u}{\|\nabla u\|_2^2} + (1-\alpha) \frac{u}{\|u\|_2^2} - \frac{u|u|^p}{\|u\|_{p+2}^{p+2}} = 0, \quad (10)$$

$$\text{with } \rho = \frac{4\alpha}{(n-2\alpha)}, \quad \alpha = \frac{\rho n}{2(\rho+2)},$$

which can be scaled in the form (see [10], [4])

$$\begin{aligned} -\frac{d^2}{dr^2}u - \frac{(n-1)}{r}\frac{d}{dr}u - u|u|^p + u &= 0, \quad r = |x| > 0, \\ \frac{d}{dr}u(0) &= 0, \quad \lim_{r \rightarrow \infty} u(r) = 0. \end{aligned} \quad (11)$$

We have used a scaling such that

$$\alpha \|u\|_2^2 = (1 - \alpha) \|\nabla u\|_2^2 = \alpha(1 - \alpha) \|u\|_{p+2}^{p+2}, \quad (12)$$

which is always possible by scaling the function and the argument. And the infimum $\lambda_{n,\alpha}$ will then be found as (with $\bar{u}_{n,\alpha}$ the unique positive (see [16]) solution of (11))

$$\frac{1}{k_0(n, \alpha)} = \lambda_{n,\alpha} = \alpha^{\alpha/2} (1 - \alpha)^{(n(1-\alpha)-2\alpha)/(2n)} [\|\bar{u}_{n,\alpha}\|_2^2]^{\alpha/n} = \chi(\alpha) \left(\frac{\|\bar{u}_{n,\alpha}\|_2^2}{1 - \alpha} \right)^{\alpha/n},$$

for $0 < \alpha < 1$, $n \geq 2$, (13)

$$\text{with } \chi(\alpha) = \sqrt{\alpha^\alpha (1 - \alpha)^{1-\alpha}}. \quad (14)$$

The values $k_0(n, \alpha)$ for $\alpha = 1$ is covered by the special form of the Sobolev embedding

$$\|w\|_t \leq \frac{1}{C_T(n, s)} \|\nabla w\|_s, \quad t = sn/(n-s), \quad 1 \leq s < n, \quad w \in H^{1,s}(\mathbb{R}^n), \quad (15)$$

where $C_T(n, s)$ is the optimal constant and

$$\begin{aligned} H^{1,s}(\mathbb{R}^n) &= \text{completion of } \{w \mid w \in C^1(\mathbb{R}^n), \|u\|_{1,s}^s = \|u\|_s^s + \|\nabla u\|_s^s < \infty\} \\ &\text{with respect to the norm } \|\cdot\|_{1,s}. \end{aligned} \quad (16)$$

If we take $\alpha = 1$ and $s = 2$ in (1), we have $k_0(n, 1) = 1/\lambda_{n,1} = 1/C_T(n, 2)$, $n \geq 3$. Since $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$, it follows that $\lambda_{2,1} = C_T(2, 2) = 0$, and so $k_0(2, 1)$ is not defined. The numbers $C_T(n, s)$ are known explicitly by the work of [17] and [18], see also [19]

$$C_T(n, s) = n^{1/s} \left(\frac{n-s}{s-1} \right)^{(s-1)/s} \left[\sigma_n B\left(\frac{n}{s}, n+1 - \frac{n}{s}\right) \right]^{1/n}, \quad 1 < s < n, \quad (17)$$

$$C_T(n, 1) = n\omega_n^{1/n}, \quad n \geq 2, \quad (18)$$

where σ_n the surface area of the unit ball in $\mathbb{R}^n$, ω_n the volume of the unit ball in $\mathbb{R}^n$

$$\omega_n = \pi^{n/2} / \Gamma(1 + n/2), \quad (19)$$

$$\sigma_n = n\omega_n = 2\pi^{n/2} / \Gamma(n/2), \quad (20)$$

$$B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b), \quad a, b > 0, \quad (21)$$

and there is equality in (15) for functions of the form

$$w_{n,s}(x_1, \dots, x_n) = \left\{ a + b|x|^{s/(s-1)} \right\}^{1-n/s}, \quad a, b > 0, \quad 1 < s < n. \quad (22)$$

From now on, we concentrate on the optimal constant $k_0(n, \alpha)$. Firstly, we list a number of estimates, two lower bounds and eleven different upper bounds for $k_0(n, \alpha)$ with references if published. Thereafter, we proof the estimates also for the published bounds.

2. Lower bounds

2.1. Lower bound 1

$$k_0 > \underline{k}_0(\alpha) = \left[\frac{\alpha^\alpha}{\pi^\alpha e^\alpha (1-\alpha)^\alpha \left[\ln \left(\frac{1}{1-\alpha} \right) \right]^\alpha} \right]^{1/2}, \quad n = 2, \quad 0 < \alpha < 1. \quad (23)$$

2.2. Lower bound 2

$$k_0 > \underline{\underline{k}}_0(n, \alpha) = \left[\frac{1}{n^n} \left(\frac{2}{\pi} \right)^{2\alpha} (n-2\alpha)^{n-2\alpha} \right]^{1/4}, \quad n \geq 2, \quad 0 < \alpha < 1. \quad (24)$$

3. Upper bounds

3.1. Upper bound 1

$$k_0 < \overline{k}_0(n, \alpha) = \frac{1}{\chi(\alpha)} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\alpha/n} k_B \left(\frac{2n}{n+2\alpha} \right), \quad (25)$$

for $n \geq 2$, $0 < \alpha < 1$,

with $\chi(\alpha)$ defined in (14), σ_n defined in (20),

with $B(p, q)$ defined in (3),

$$\text{and with } k_B(p) = \left[\left(\frac{p}{2\pi} \right)^{1/p} \left(\frac{p'}{2\pi} \right)^{-1/p'} \right]^{n/2}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (26)$$

See [10, Theorem 1], [12, Proposition 1] and [15, Theorem 1]. Remark that

$$n = 2, \quad B \left(1, \frac{1-\alpha}{\alpha} \right) = \frac{\alpha}{1-\alpha}.$$

3.2. Upper bound 2

$$k_0 < \overline{\overline{k_0}}(n, \alpha) = \frac{1}{\chi(\alpha)} \left[k_B \left(\frac{n}{n-2\alpha} \right) k_B^2 \left(\frac{2n}{n+2\alpha} \right) \|G(x)\|_{n/(n-2\alpha)} \right]^{1/2}, \quad (27)$$

for $n \geq 2$, $0 < \alpha < 1$,

with $\chi(\alpha)$ defined in (14), $k_B(p)$ defined in (26),

$$\text{and with } G(x) = \frac{K_{(n-2)/2}(|x|)}{|x|^{(n-2)/2}}, \quad K_\nu \text{ is the modified Bessel function.} \quad (28)$$

See [10, Theorem 2] and [15].

Remark that for $n = 2$, $\alpha = 1/2$

$$\|G(x)\|_2 = \left(2\pi \int_0^\infty K_0^2(r) r dr \right)^{1/2} = \pi^{1/2},$$

and for $n = 3$, and general α

$$\|G(x)\|_{3/(3-2\alpha)} = \sqrt{\frac{\pi}{2}} (4\pi)^{(3-2\alpha)/3} \left(\frac{3-2\alpha}{3} \right)^{2-2\alpha} \left[\Gamma \left(\frac{6-6\alpha}{3-2\alpha} \right) \right]^{(3-2\alpha)/3},$$

$$\text{because } K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \exp(-x).$$

3.3. Upper bound 3

$$k_0 < \overline{\overline{\overline{k_0}}}(n, \alpha) = \frac{1}{\chi(\alpha)} \frac{1}{\sqrt{(1-\alpha)}} k_B \left(\frac{n}{n-\alpha} \right) k_B \left(\frac{2n}{n+2\alpha} \right) \times \|G(x)\|_{n/(n-\alpha)}, \quad \text{for } n \geq 2, \quad 0 < \alpha < 1, \quad (29)$$

with $\chi(\alpha)$ defined in (14), with $k_B(p)$ defined in (26),

and with $G(x)$ defined in (28).

3.4. Upper bound 4

$$k_0 < \overline{\overline{k_{D,1}}}(n, \alpha) = A(n, \alpha)^\gamma, \quad n \geq 2, \quad 0 < \alpha < 1, \quad (30)$$

$$\text{with } A(n, \alpha) = \left[\frac{2\alpha(n-\alpha)}{\pi n(n-2\alpha)^2} \right]^{\theta/2} \left[1 - \frac{n\alpha}{2(n-\alpha)} \right]^{(n-2\alpha)/(2n)} \quad (31)$$

$$\times \left[\frac{\Gamma\left(\frac{n}{\alpha} - 1\right)}{\Gamma\left(\frac{n}{\alpha} - 1 - \frac{n}{2}\right)} \right]^{\theta/n},$$

and with $\theta = \frac{\alpha(n-2\alpha)}{2n-2\alpha-\alpha n}$, $\gamma = \frac{2n-2\alpha-\alpha n}{n-2\alpha}$. (32)

3.5. Upper bound 5

$$k_0 < \overline{k_{D,2}}(n, \alpha) = A(n, \alpha)^\alpha \overline{k_0}(n, \alpha)^{1-\theta}, \quad n \geq 2, \quad 0 < \alpha < 1, \quad (33)$$

with $A(n, \alpha)$ defined in (31), $\overline{k_0}(n, \alpha)$ defined in (25),

and with $\theta = \frac{\alpha(n-2\alpha)}{2n-2\alpha-\alpha n}$, defined in (32).

Compare [4, Theorem 1.7 (1.30)].

3.6. Upper bound 6

$$k_0 < \overline{k_{D,3}}(n, \alpha) = A(n, \alpha)^\alpha \overline{\overline{k_0}}(n, \alpha)^{1-\theta}, \quad n \geq 2, \quad 0 < \alpha < 1, \quad (34)$$

with $A(n, \alpha)$ defined in (31), θ defined in (32),

and with $\overline{\overline{k_0}}(n, \alpha)$ defined in (27).

Compare [4, Theorem 1.7 (1.30)].

3.7. Upper bound 7

$$k_0 < \overline{k_{I,1}}(n, \alpha) = 1/k_{V,1}(n, \alpha), \quad n \geq 3, \quad 1/2 < \alpha < 1, \quad (35)$$

$$\text{with } k_{V,1}(n, \alpha) = \overline{k_0} \left(n, \frac{1}{2} \right)^{-\alpha_1} k_T(n)^{-(1-\alpha_1)}, \quad \alpha_1 = 2(1-\alpha), \quad (36)$$

with $\overline{k_0}(n, \alpha)$ defined in (25),

$$\text{and with } k_T(n) = \frac{1}{C_T(n, 2)} = \frac{1}{\sqrt{\pi n(n-2)}} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \right]^{1/n}, \quad (37)$$

where $C_T(n, 2)$ is defined in (17).

See [4, Theorem 1.7, (1.30), $\theta' = 1/2$, $\theta'' = 1$, with the restriction $n \geq 3$].

3.8. Upper bound 8

$$k_0 < \overline{k_{I,2}}(n, \alpha) = 1/k_{V,2}(n, \alpha), \quad n \geq 3, \quad \alpha_V < \alpha < 1, \quad (38)$$

$$\text{with } k_{V,2}(n, \alpha) = \overline{k_0}(n, \alpha_V)^{-\alpha_2} k_T(n)^{-(1-\alpha_2)}, \quad (39)$$

with $\overline{k_0}(n, \alpha)$ defined in (25), $k_T(n)$ defined in (37),

$$\text{and with } \alpha_2 = \frac{1-\alpha}{1-\alpha_V}, \quad (40)$$

where α_V follows from

$$\alpha_V = \alpha_V(n) = \frac{n}{2p_V}, \text{ where } p_V \text{ is the solution of} \quad (41)$$

$$\ln\left(\frac{n-p}{p-1}\right) + \frac{n-p}{p(p-1)} + \psi(p) - \psi(n+1-p) = 0, \quad (42)$$

$$\psi(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}, \quad x > 0, \quad 1 < p < n, \quad n \geq 2.$$

See [4, Theorem 1.7 (1.30), $\theta' = \theta_N (= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geq 3$]. See Section 5.3 for numerical values of $\alpha_V(n)$, $n = 2, \dots, 10$.

3.9. Upper bound 9

$$k_0 < \overline{k_{I,3}}(n, \alpha) = 1/k_{V,3}(n, \alpha), \quad n \geq 3, \quad \alpha_V < \alpha < 1, \quad (43)$$

with α_V defined in (41),

$$\text{with } k_{V,3}(n, \alpha) = k_{L,V}(n, \alpha_V)^{\alpha_2} k_T(n)^{-(1-\alpha_2)}, \quad \alpha_2 \text{ defined in (40),} \quad (44)$$

$$\text{with } k_{L,V}(n, \alpha) = [\alpha C_T(n, 2\alpha)]^\alpha, \quad (45)$$

with $C_T(n, s)$ defined in (17), that is

$$C_T(n, s) = n^{1/s} \left(\frac{n-s}{s-1}\right)^{(s-1)/s} \left[\sigma_n B\left(\frac{n}{s}, n+1-\frac{n}{s}\right)\right]^{1/n}, \quad 1 < s < n,$$

and with $k_T(n)$ defined in (37), $k_T(n) = 1/C_T(n, 2)$.

Compare [4, Theorem 1.7 (1.30) and (1.32), $\theta' = \theta_N (= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geq 3$].

3.10. Upper bound 10

$$k_0 < \overline{k_{L,V}}(n, \alpha) = [\alpha_V C_T(n, 2\alpha_V)]^{-\alpha}, \quad (46)$$

$$\begin{aligned}
n &\geq 2, \quad 0 < \alpha \leq \alpha_V, \\
k_0 &< \overline{k_{L,V}}(n, \alpha) = 1/k_{L,V}(n, \alpha) = [\alpha C_T(n, 2\alpha)]^{-\alpha}, \\
n &\geq 2, \quad \alpha_V \leq \alpha < 1,
\end{aligned} \tag{47}$$

with α_V defined in (41), $C_T(n, s)$ defined in (17).

See [4, Theorem 1.7, (1.32)].

3.11. Upper bound 11

$$k_0 < \overline{k_B}(n, \alpha) = k_T(n)^\alpha, \quad n \geq 3, \quad 0 < \alpha < 1, \tag{48}$$

with $k_T(n)$ defined in (37).

See [4, Theorem 1.7 (1.33), $\theta' = 0$, $\theta'' = 1$, with the restriction $n \geq 3$].

4. Proofs

4.1. Lower bounds

We take as trial function in (5) the function

$$u_{n,\alpha} = a \exp(-br^\mu), \quad a, b, \mu > 0. \tag{49}$$

We need the following general integral (see [20, (5.9.1)])

$$\int_0^\infty \exp(-mr^\mu) r^{\nu-1} dr = \frac{1}{\mu} \left(\frac{1}{m} \right)^{\nu/\mu} \Gamma\left(\frac{\nu}{\mu}\right). \tag{50}$$

For this trial function the following three integrals become ($\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$, the surface area of the unit ball in $\mathbb{R}^n$, see (20))

$$\int_{\mathbb{R}^n} u_{n,\alpha}^2(x) dx = \sigma_n \int_0^\infty a^2 e^{-2br^\mu} r^{n-1} dr = \sigma_n a^2 \frac{1}{\mu} \left(\frac{1}{2b} \right)^{n/\mu} \Gamma\left(\frac{n}{\mu}\right), \tag{51}$$

$$\begin{aligned}
\int_{\mathbb{R}^n} (\nabla u_{n,\alpha}(x))^2 dx &= \sigma_n \int_0^\infty a^2 b^2 \mu^2 r^{2(\mu-1)} e^{-2br^\mu} r^{n-1} dr \\
&= \sigma_n a^2 \frac{\mu}{4} \left(\frac{1}{2b} \right)^{(n-2)/\mu} \Gamma\left(2 + \frac{n-2}{\mu}\right),
\end{aligned} \tag{52}$$

$$\begin{aligned}
\int_{\mathbb{R}^n} u_{n,\alpha}^{\rho+2}(x) dx &= \sigma_n \int_0^\infty a^{\rho+2} e^{-(\rho+2)br^\mu} r^{n-1} dr \\
&= \sigma_n a^{\rho+2} \frac{1}{\mu} \left(\frac{1}{(\rho+2)b} \right)^{n/\mu} \Gamma\left(\frac{n}{\mu}\right).
\end{aligned} \tag{53}$$

4.2. Lower bound 1

For $n = 2$, and general μ the three integrals (51), (52) and (53) become

$$\int_{\mathbb{R}^2} u_{2,\alpha}^2(x) dx = 2\pi \int_0^\infty a^2 e^{-2br^\mu} r dr = \sigma_2 a^2 \frac{1}{\mu} \left(\frac{1}{2b} \right)^{2/\mu} \Gamma\left(\frac{2}{\mu}\right), \quad (54)$$

$$\int_{\mathbb{R}^2} (\nabla u_{2,\alpha}(x))^2 dx = 2\pi \int_0^\infty a^2 b^2 \mu^2 r^{2(\mu-1)} e^{-2br^\mu} r dr = \sigma_2 a^2 \frac{\mu}{4} \Gamma(2), \quad (55)$$

$$\begin{aligned} \int_{\mathbb{R}^2} u_{2,\alpha}^{\rho+2}(x) dx &= 2\pi \int_0^\infty a^{\rho+2} e^{-(\rho+2)br^\mu} r dr \\ &= \sigma_2 a^{\rho+2} \frac{1}{\mu} \left(\frac{1}{(\rho+2)b} \right)^{2/\mu} \Gamma\left(\frac{2}{\mu}\right). \end{aligned} \quad (56)$$

Let a, b be variable and μ fixed, we use the two scaling relations (12)

$$\alpha \sigma_2 a^2 \frac{1}{\mu} \left(\frac{1}{2b} \right)^{2/\mu} \Gamma\left(\frac{2}{\mu}\right) = (1-\alpha) \sigma_2 a^2 \frac{\mu}{4} \Gamma(2), \quad (57)$$

$$\sigma_2 a^2 \frac{1}{\mu} \left(\frac{1}{2b} \right)^{2/\mu} \Gamma\left(\frac{2}{\mu}\right) = (1-\alpha) \sigma_2 a^{\rho+2} \frac{1}{\mu} \left(\frac{1}{(\rho+2)b} \right)^{2/\mu} \Gamma\left(\frac{2}{\mu}\right). \quad (58)$$

This gives for the optimal values for $(a, b) = (a_0, b_0)$

$$a^\rho = a_0^\rho = \left(\frac{\rho+2}{2} \right)^{\frac{\mu+2}{\mu}}, \quad b^{2/\mu} = b_0^{2/\mu} = \frac{2\rho \Gamma\left(\frac{2}{\mu}\right)}{\mu^2 2^{2/\mu}}.$$

$$\begin{aligned} k_0(2, \alpha) &= \frac{1}{\chi(\alpha)} \left(\frac{1-\alpha}{\|\bar{u}_{2,\alpha}\|_2^2} \right)^{\alpha/2} \\ &> \underline{k}_0(\alpha) = \frac{1}{\chi(\alpha)} \left\{ \frac{(1-\alpha)2\rho}{2\pi \left[\mu^{\rho/2} \left(\frac{\rho}{2} + 1 \right)^{1+2/\mu} \right]^{2/\rho}} \right\}^{\alpha/2}. \end{aligned} \quad (59)$$

Consider now μ as variable to minimize $\underline{k}_0(\alpha)$ by maximizing the denominator

$$\begin{aligned} \max_{0 < \mu < \infty} \left[\mu^{\rho/2} \left(\frac{\rho}{2} + 1 \right)^{1+2/\mu} \right] &= \left[\frac{2e \ln(1+\rho/2)}{\rho/2} \right]^{\rho/2} (1+\rho/2), \\ \text{for } \mu_0 &= \frac{2 \ln(1+\rho/2)}{\rho/2}. \end{aligned}$$

This gives for (59)

$$\underline{k}_0(\alpha) = \frac{1}{\chi(\alpha)} \left\{ \frac{2(1-\alpha)(\rho/2)^2}{2\pi e \ln(1+\rho/2)(1+\rho/2)^{2/\rho}} \right\}^{\alpha/2}$$

$$= \left[\frac{\alpha^\alpha}{\pi^\alpha e^\alpha (1-\alpha)^\alpha \left[\ln\left(\frac{1}{1-\alpha}\right) \right]^\alpha} \right]^{1/2}, \quad (60)$$

which equals (23).

4.3. Lower bound 2

For general n and $\mu = 2$ the three integrals (51), (52) and (53) become

$$\int_{\mathbb{R}^n} u_{n,\alpha}^2(x) dx = \sigma_n \int_0^\infty a^2 \exp(-2br^2) r^{n-1} dr = \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b} \right)^{n/2} \Gamma\left(\frac{n}{2}\right), \quad (61)$$

$$\begin{aligned} \int_{\mathbb{R}^n} (\nabla u_{n,\alpha}(x))^2 dx &= \sigma_n \int_0^\infty a^2 b^2 4r^2 \exp(-2br^2) r^{n-1} dr \\ &= \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b} \right)^{(n-2)/2} \Gamma\left(1 + \frac{n}{2}\right), \end{aligned} \quad (62)$$

$$\begin{aligned} \int_{\mathbb{R}^n} u_{n,\alpha}^{\rho+2}(x) dx &= \sigma_n \int_0^\infty a^2 \exp(-(\rho+2)r^2) r^{n-1} dr \\ &= \sigma_n a^{\rho+2} \frac{1}{2} \left(\frac{1}{(\rho+2)b} \right)^{n/2} \Gamma\left(\frac{n}{2}\right). \end{aligned} \quad (63)$$

Using the two scaling relations (12)

$$\alpha \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b} \right)^{n/2} \Gamma\left(\frac{n}{2}\right) = (1-\alpha) \sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b} \right)^{(n-2)/2} \Gamma\left(1 + \frac{n}{2}\right), \quad (64)$$

$$\sigma_n a^2 \frac{1}{2} \left(\frac{1}{2b} \right)^{n/2} \Gamma\left(\frac{n}{2}\right) = (1-\alpha) \sigma_n a^{\rho+2} \frac{1}{2} \left(\frac{1}{(\rho+2)b} \right)^{n/2} \Gamma\left(\frac{n}{2}\right), \quad (65)$$

we get $(a, b) = (a_0, b_0)$

$$a^\rho = a_0^\rho = \frac{1}{1-\alpha} \left(\frac{n}{n-2\alpha} \right)^{n/2}, \quad b = b_0 = \frac{\alpha}{n(1-\alpha)},$$

where we use all the time the relation $\rho = \frac{4\alpha}{n-2\alpha}$. Using (61) and (13) we find lower bound 2 (24)

$$\underline{k}_0(n, \alpha) = \left[\frac{1}{n^n} \left(\frac{2}{\pi} \right)^{2\alpha} (n-2\alpha)^{n-2\alpha} \right]^{1/4}, \quad n \geq 2, \quad 0 < \alpha < 1. \quad (66)$$

4.4. Upper bounds

We introduce the standard notations

$$r = \frac{2n}{n-2\alpha}, \quad \rho = r-2 = \frac{4\alpha}{n-2\alpha}, \quad (67)$$

and so

$$\alpha = \frac{\rho n}{2(\rho+2)} = \frac{n}{2} \left(\frac{r-2}{r} \right). \quad (68)$$

For the proof of upper bound 1 we need a less well-known inequality which we present here as Lemma.

LEMMA 1. See [21] and [13, Lemma 1]. For $u \in L^2(\mathbb{R}^n)$, $|x|u \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $0 < \alpha < 1$,

$$\|u\|_{\frac{2n}{n+2\alpha}} \leq \frac{1}{\chi(\alpha)} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\alpha/n} \| |x|u \|_2^\alpha \|u\|_2^{1-\alpha}. \quad (69)$$

Equality will be reached for functions

$$u(x) = \frac{A}{\left(B + C|x|^2 \right)^{\frac{n+2\alpha}{4\alpha}}}, \quad \text{with } A, B, C \text{ arbitrary.}$$

Proof. We start with the inequality

$$\int_{\mathbb{R}^n} f^s g^t dx \leq \left(\int_{\mathbb{R}^n} f dx \right)^s \left(\int_{\mathbb{R}^n} g dx \right)^t, \quad s+t=1, \quad (70)$$

and we make the choices

$$s = p/2, \quad t = 1 - p/2. \quad f^s = \left(|u|^2 (a + b|x|^2) \right)^{p/2}, \quad g^t = \left(a + b|x|^2 \right)^{-p/2}.$$

This makes for (70)

$$\int_{\mathbb{R}^n} |u|^p dx \leq \left(\int_{\mathbb{R}^n} \left(|u|^2 (a + b|x|^2) \right) dx \right)^{p/2} \left(\int_{\mathbb{R}^n} \left(a + b|x|^2 \right)^{-\frac{p/2}{1-p/2}} dx \right)^{(1-p/2)},$$

or for $p = (\rho+2)/(\rho+1) = 2n/(n+2\alpha)$ and so $\rho = 4\alpha/(n-2\alpha)$

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^p dx &= \|u\|_{\frac{\rho+2}{\rho+1}}^{\frac{\rho+2}{\rho+1}} \leq \left(\int_{\mathbb{R}^n} \left(|u|^2 (a + b|x|^2) \right) dx \right)^{\frac{\rho+2}{2(\rho+1)}} \\ &\quad \times \left(\int_{\mathbb{R}^n} \left(a + b|x|^2 \right)^{-\frac{\rho+2}{\rho}} dx \right)^{\frac{\rho}{2(\rho+1)}}. \end{aligned} \quad (71)$$

We define

$$I_0 = \left(\int_{\mathbb{R}^n} \left(a + b|x|^2 \right)^{-\frac{\rho+2}{\rho}} dx \right).$$

In a standard way this integral can be calculated as

$$I_0 = a^{-\frac{(4-(n-2)\rho)}{2\rho}} b^{-\frac{n}{2}} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{\rho+2}{\rho} - \frac{n}{2} \right) \right].$$

We make now the choice

$$b = \|u\|_2^2 / \| |x| u \|_2^2,$$

such that (71) transforms into

$$\begin{aligned} \|u\|_{\frac{\rho+2}{\rho+1}}^2 &\leq \left(\int_{\mathbb{R}^n} \left(|u|^2 \left(a + b |x|^2 \right) \right) dx \right) \\ &\quad \times \left(a^{-\frac{(4-(n-2)\rho)}{2\rho}} b^{-\frac{n}{2}} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{\rho+2}{\rho} - \frac{n}{2} \right) \right] \right)^{\frac{\rho}{(\rho+2)}}, \end{aligned}$$

or

$$\|u\|_{\frac{\rho+2}{\rho+1}}^2 \leq (a+1) a^{-(1-\alpha)} \|u\|_2^{2-n\frac{2\alpha}{n}} \| |x| u \|_2^{2(-\frac{n}{2})\frac{2\alpha}{n}} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\frac{2\alpha}{n}}.$$

We still have the free parameter a . We minimize the function $h(a) = (a+1) a^{-(1-\alpha)}$. By standard means this minimum will be found for $a_0 = (1-\alpha)/\alpha$ and $h(a_0) = \alpha^{-\alpha} (1-\alpha)^{-1+\alpha} = \chi^{-2}(\alpha)$, by (14). Finally, we arrive at

$$\|u\|_{\frac{\rho+2}{\rho+1}} = \|u\|_{\frac{2n}{n+2\alpha}} \leq \frac{1}{\chi(\alpha)} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\frac{\alpha}{n}} \|u\|_2^{1-\alpha} \| |x| u \|_2^\alpha.$$

Equality in (70) will be reached if $f = Cg$, C arbitrary, so

$$\left(|u|^2 \left(a + b |x|^2 \right) \right) = C \left(a + b |x|^2 \right)^{-\frac{p/2}{1-p/2}}, \quad a, b \text{ arbitrary},$$

or

$$u(x) = C \left(a + b |x|^2 \right)^{-\frac{\rho+1}{\rho}} = \frac{C}{\left(A + B |x|^2 \right)^{\frac{n+2\alpha}{4\alpha}}}, \quad a, A, b, B \text{ arbitrary.} \quad \square$$

LEMMA 2. See [4, Theorem 1.7, Case i), formula (1.30)]. For $0 < \alpha < 1$, $n \geq 2$ there holds the logconvexity of $k_0(n, \alpha)$

$$\begin{aligned} k_0(n, \alpha) &< (k_0(n, \alpha'))^\theta (k_0(n, \alpha''))^{1-\theta}, \quad 0 < \theta < 1, \\ \text{with } \alpha &= \theta \alpha' + (1-\theta) \alpha'', \quad \alpha' \neq \alpha''. \end{aligned} \quad (72)$$

Proof. By the Hölder inequality

$$\|v\|_r < \|v\|_{r'}^\theta \|v\|_{r''}^{1-\theta}, \quad 0 < \theta < 1, \quad 1/r = \theta/r' + (1-\theta)/r'', \quad r' \neq r'', \quad (73)$$

which inequality is strict, since $r' \neq r''$. For the choice $r = 2n/(n-2\alpha)$, the condition for application of (73) implies $\alpha = \theta \alpha' + (1-\theta) \alpha''$, and so

$$\Lambda_{N,\alpha}(v) = \frac{\|\nabla v\|_2^\alpha \|v\|_2^{1-\alpha}}{\|v\|_r} > \left(\frac{\|\nabla v\|_2^{\alpha'} \|v\|_2^{1-\alpha'}}{\|v\|_{r'}} \right)^\theta \left(\frac{\|\nabla v\|_2^{\alpha''} \|v\|_2^{1-\alpha''}}{\|v\|_{r''}} \right)^{1-\theta}$$

$$= \Lambda_{N,\alpha'}^\theta(v) \Lambda_{N,\alpha''}^{1-\theta}(v), \quad (74)$$

and this implies the assertion of Lemma 2, since (see (4))

$$\frac{1}{k_0(n, \alpha)} = \lambda_{n, \alpha} = \inf_{u \in H^1(\mathbb{R}^n)} \Lambda_{n, \alpha}. \quad \square$$

4.5. Upper bound 1

See the proof in [12, Proposition 1] or [15, Theorem 1]. For completeness we sketch the proof. We use the following sharp form of the Hausdorff-Young inequality due to Babenko (see [22, Section II. Babenko's inequality])

$$\begin{aligned} \|u\|_{\frac{2n}{n-2\alpha}} &\leq k_b \left(\frac{2n}{n+2\alpha} \right) \|\hat{u}\|_{\frac{2n}{n+2\alpha}}, \\ \text{with } \hat{u} &= \left(\frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} \exp(-i\langle x, \xi \rangle) u(x) dx. \end{aligned} \quad (75)$$

Application of Lemma 1 (69) for the Fourier Transform of u , the function $\hat{u}$, gives (combined with (75))

$$\begin{aligned} \|u\|_{\frac{2n}{n-2\alpha}} &\leq k_b \left(\frac{2n}{n+2\alpha} \right) \|\hat{u}\|_{\frac{2n}{n+2\alpha}} \\ &\leq k_b \left(\frac{2n}{n+2\alpha} \right) \frac{1}{\chi(\alpha)} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\alpha/n} \|\xi\|_2^\alpha \|\hat{u}\|_2^{1-\alpha}. \end{aligned}$$

Due to the Parseval-Steklov relations for Fourier transforms $\|\hat{u}\|_2 = \|u\|_2$ and $\|\xi\|_2 \|\hat{u}\|_2 = \|\nabla u\|_2$, we arrive at formula (25), the first upper bound, so

$$\overline{k}_0(n, \alpha) = k_b \left(\frac{2n}{n+2\alpha} \right) \frac{1}{\chi(\alpha)} \left[\frac{\sigma_n}{2} B \left(\frac{n}{2}, \frac{n(1-\alpha)}{2\alpha} \right) \right]^{\alpha/n}. \quad (76)$$

4.6. Upper bound 2

See the proof in [15, Theorem 1]. For completeness we sketch the proof. We apply the Beckner-Young's Inequality, see [22, Section III. Young's inequality], for $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$,

$$\begin{aligned} \|f * g\|_r &\leq (A_p A_q A_{r'})^n \|f\|_p \|g\|_q, \quad 1 \leq p, q, r < \infty, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \\ \text{where } A_p &= \left[p^{1/p} / p'^{(1/p')} \right]^{1/2}, \quad \text{with } \frac{1}{p} + \frac{1}{p'} = 1. \end{aligned} \quad (77)$$

Note that $k_b(p) = (2\pi)^{(-1/p+1/p')n/2} A_p^n$.

We apply this inequality (77) for the solution of (11) $\bar{u}_{n,\alpha}(r)$ written as $\psi_0(x)$, $x \in \mathbb{R}^n$, in convolution form. ψ_0 satisfies

$$\Delta \psi_0 - \psi_0 = -\psi_0^{\rho+1}. \quad (78)$$

By application of the Fourier Transform on the equation

$$\Delta \psi_{0,\delta} - \psi_{0,\delta} = \delta, \quad x \in \mathbb{R}^n,$$

with δ the Dirac delta function, we find for the Fourier Transform $\widehat{\psi_{0,\delta}}$

$$\widehat{\psi_{0,\delta}} = -\left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{(1+\xi^2)}, \quad \text{because } \widehat{\delta} = \left(\frac{1}{2\pi}\right)^{n/2},$$

which gives for $\psi_{0,\delta}$

$$\psi_{0,\delta} = -\left(\frac{1}{2\pi}\right)^{n/2} G(x), \quad \text{with } G(x) = \frac{K_{(n-2)/2}(|x|)}{|x|^{\frac{n-2}{2}}},$$

see [23, Chapter 8, p. 289]. And so we find for ψ_0 the integral equation

$$\psi_0 = -\left(\frac{1}{2\pi}\right)^{n/2} G * (-\psi_0^{\rho+1}) = \left(\frac{1}{2\pi}\right)^{n/2} G * \psi_0^{\rho+1}. \quad (79)$$

Now, we apply (77) with $f = G$, $g = \psi_0^{\rho+1}$, $r = \rho + 2$, $p = (\rho + 2)/2$, $q = (\rho + 2)/(\rho + 1)$, so $r' = q$, and we have

$$\begin{aligned} \|\psi_0\|_{\rho+2} &= \left(\frac{1}{2\pi}\right)^{n/2} \|G * \psi_0^{\rho+1}\|_{\rho+2} \\ &\leq \left(\frac{1}{2\pi}\right)^{n/2} \left(A_{(\rho+2)/2} A_{(\rho+2)/(\rho+1)}^2\right)^n \|G\|_{(\rho+2)/2} \|\psi_0^{\rho+1}\|_{(\rho+2)/(\rho+1)} \\ &= k_b \left(\frac{\rho+2}{2}\right) k_b^2 \left(\frac{\rho+2}{\rho+1}\right) \|G\|_{(\rho+2)/2} \|\psi_0\|_{\rho+2}^{\rho+1}. \end{aligned} \quad (80)$$

From (80) we get

$$\|\psi_0\|_{\rho+2}^{\rho+2} \geq \left[k_b \left(\frac{\rho+2}{2}\right) k_b^2 \left(\frac{\rho+2}{\rho+1}\right) \|G\|_{(\rho+2)/2}\right]^{-\left(\frac{\rho+2}{\rho}\right)}. \quad (81)$$

By (12) this becomes

$$\|\psi_0\|_2^2 \geq (1-\alpha) \left[k_b \left(\frac{\rho+2}{2}\right) k_b^2 \left(\frac{\rho+2}{\rho+1}\right) \|G\|_{(\rho+2)/2}\right]^{-\left(\frac{\rho+2}{\rho}\right)},$$

and by (13) we have

$$\chi(\alpha) \left(\frac{\|\bar{u}_{n,\alpha}\|_2^2}{1-\alpha}\right)^{\alpha/n} = \frac{1}{k_0(n,\alpha)}.$$

Since $\|\bar{u}_{n,\alpha}\|_2^2 = \|\psi_0\|_2^2$ (by definition) and $\alpha/n = \rho/(2(\rho+2))$

$$k_0(n, \alpha) \leq \frac{1}{\chi(\alpha)} \left[k_b \left(\frac{\rho+2}{2} \right) k_b^2 \left(\frac{\rho+2}{\rho+1} \right) \|G\|_{(\rho+2)/2} \right]^{1/2}.$$

This equals the announced upper bound 2 (27), because $(\rho+2)/2 = n/(n-2\alpha)$ and $(\rho+2)/(\rho+1) = 2n/(n+2\alpha)$:

$$\begin{aligned} k_0(n, \alpha) &\leq \frac{1}{\chi(\alpha)} \left[k_B \left(\frac{n}{n-2\alpha} \right) k_B^2 \left(\frac{2n}{n+2\alpha} \right) \|G(x)\|_{n/(n-2\alpha)} \right]^{1/2} \\ &= \bar{k}_0(n, \alpha). \end{aligned} \quad (82)$$

4.7. Upper bound 3

We follow the same strategy as for the upper bound 2. We apply (77) with $f = G$, $g = \psi_0^{\rho+1}$, $p = 2(\rho+2)/(\rho+4)$, $q = (\rho+2)/(\rho+1)$, $r = 2$, so $r' = 2$, and we have

$$\begin{aligned} \|\psi_0\|_2 &= \left(\frac{1}{2\pi} \right)^{n/2} \|G * \psi_0^{\rho+1}\|_2 \\ &\leq \left(\frac{1}{2\pi} \right)^{n/2} (A_{2(\rho+2)/(\rho+4)} A_{(\rho+2)/(\rho+1)})^n \times \|G\|_{2(\rho+2)/(\rho+4)} \|\psi_0^{\rho+1}\|_{(\rho+2)/(\rho+1)} \\ &= k_b \left(\frac{2(\rho+2)}{\rho+4} \right) k_b \left(\frac{\rho+2}{\rho+1} \right) \|G\|_{2(\rho+2)/(\rho+4)} \|\psi_0\|_{\rho+2}^{\rho+1}. \end{aligned} \quad (83)$$

By (12) this becomes

$$(1-\alpha) \|\psi_0\|_{\rho+2}^{\rho+2} \leq \left[k_b \left(\frac{2(\rho+2)}{\rho+4} \right) k_b \left(\frac{\rho+2}{\rho+1} \right) \|G\|_{2(\rho+2)/(\rho+4)} \right]^2 \|\psi_0\|_{\rho+2}^{2(\rho+1)}.$$

This can be rewritten as

$$\|\psi_0\|_{\rho+2}^\rho \geq (1-\alpha) \left[k_b \left(\frac{2(\rho+2)}{\rho+4} \right) k_b \left(\frac{\rho+2}{\rho+1} \right) \|G\|_{2(\rho+2)/(\rho+4)} \right]^{-2}, \quad (84)$$

and by (13) we have

$$\chi(\alpha) \left(\frac{\|\bar{u}_{n,\alpha}\|_2^2}{1-\alpha} \right)^{\alpha/n} = \chi(\alpha) \left(\|\bar{u}_{n,\alpha}\|_{\rho+2}^{\rho+2} \right)^{\alpha/n} = \frac{1}{k_0(n, \alpha)}.$$

Since $\|\bar{u}_{n,\alpha}\|_{\rho+2}^{\rho+2} = \|\psi_0\|_{\rho+2}^{\rho+2}$ (by definition) and $\alpha/n = \rho/(2(\rho+2))$ there follows

$$k_0(n, \alpha) \leq \frac{1}{\chi(\alpha)} \frac{1}{\sqrt{(1-\alpha)}} \left[k_b \left(\frac{2(\rho+2)}{\rho+4} \right) k_b \left(\frac{\rho+2}{\rho+1} \right) \|G\|_{2(\rho+2)/(\rho+4)} \right].$$

This equals the announced upper bound 3 (29), because $2(\rho+2)/(\rho+4) = n/(n-\alpha)$ and $(\rho+2)/(\rho+1) = 2n/(n+2\alpha)$:

$$k_0(n, \alpha) \leq \frac{1}{\chi(\alpha)} \frac{1}{\sqrt{(1-\alpha)}} \left[k_b \left(\frac{n}{(n-\alpha)} \right) k_b \left(\frac{2n}{(n+2\alpha)} \right) \|G\|_{n/(n-\alpha)} \right] \\ = \overline{\overline{\overline{k_0}}}(n, \alpha). \quad (85)$$

4.8. Upper bound 4

We start with the inequality

$$\|u\|_{2p} \leq A \|\nabla u\|_2^\theta \|u\|_{p+1}^{1-\theta}, \quad u \in L^{p+1}(\mathbb{R}^n), \nabla u \in L^2(\mathbb{R}^n), |u|^{2p} \in L^1(\mathbb{R}^n), \quad (86)$$

for $n = 2$, $p > 1$, and for $n \geq 3$, $1 < p \leq n/(n-2)$,

$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)}, \quad (87)$$

with the optimal constant

$$A = \left(\frac{y(p-1)^2}{2\pi n} \right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y} \right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{n}{2})} \right)^{\frac{\theta}{n}}, \quad y = \frac{p+1}{p-1}, \quad (88)$$

see [24, Theorem 1].

Next, we apply the Cauchy-Schwarz's Inequality in the form

$$\|u\|_{p+1} \leq \|u\|_{2p}^\eta \|u\|_2^{1-\eta}, \quad \text{for } \eta = \frac{p}{p+1}, \quad (89)$$

and insert this inequality in the right-hand side of (86) to obtain

$$\|u\|_{2p} \leq A \|\nabla u\|_2^\theta \|u\|_{2p}^{\eta(1-\theta)} \|u\|_2^{(1-\eta)(1-\theta)},$$

or

$$\|u\|_{2p}^{1-\eta(1-\theta)} \leq A \|\nabla u\|_2^\theta \|u\|_2^{(1-\eta)(1-\theta)},$$

or

$$\|u\|_{2p} \leq A^{\frac{1}{1-\eta(1-\theta)}} \|\nabla u\|_2^{\frac{\theta}{1-\eta(1-\theta)}} \|u\|_2^{\frac{(1-\eta)(1-\theta)}{1-\eta(1-\theta)}}. \quad (90)$$

For the choice of $p = n/(n-2\alpha)$ as in (1) we find after some calculations, using (87)

$$\theta = \frac{\alpha(n-2\alpha)}{2n-2\alpha-\alpha n}, \quad \frac{\theta}{1-\eta(1-\theta)} = \alpha, \quad (91) \\ \frac{(1-\eta)(1-\theta)}{1-\eta(1-\theta)} = 1-\alpha, \quad y = \frac{n-\alpha}{\alpha},$$

and

$$\frac{1}{1-\eta(1-\theta)} = \frac{2n-2\alpha-\alpha n}{n-2\alpha} \equiv \gamma. \quad (92)$$

Using the identities (91) and (92) we arrive at

$$\|u\|_{2n/(n-2\alpha)} \leq A^\gamma \|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}, \quad (93)$$

which is inequality (1) and where A^γ equals, using $y = n/\alpha - 1$, $p - 1 = 2\alpha/(n - 2\alpha)$

$$A^\gamma = \left(\frac{2\alpha(n-\alpha)}{\pi n(n-2\alpha)^2} \right)^{\frac{\alpha}{2}} \left(1 - \frac{n\alpha}{2(n-\alpha)} \right)^{(2n-2\alpha-\alpha n)/(2n)} \times \left(\frac{\Gamma(\frac{n}{\alpha}-1)}{\Gamma(\frac{n}{\alpha}-1-\frac{n}{2})} \right)^{\frac{\alpha}{n}}, \quad (94)$$

so we found the announced upper bound 4 (30)

$$\overline{k_{D,1}}(n, \alpha) = A^\gamma, \quad \text{with } A = A(n, \alpha) \text{ defined in (31)}. \quad (95)$$

4.9. Upper bound 5

We observe that there holds trivially

$$k_0(n, \alpha) = k_0(n, \alpha)^\theta k_0(n, \alpha)^{1-\theta}. \quad (96)$$

Make now the choice $\theta = \alpha(n - 2\alpha)/(2n - 2\alpha - \alpha n)$ see (32), then

$$k_0(n, \alpha)^\theta < \overline{k_{D,1}}(n, \alpha)^\theta = (A(n, \alpha)^\gamma)^\theta = A(n, \alpha)^\alpha, \quad (97)$$

since $\gamma\theta = \alpha$ (see (92)) and further

$$k_0(n, \alpha)^{1-\theta} < \overline{k_0}(n, \alpha)^{1-\theta}. \quad (98)$$

Insertation of (97) and (98) into (96) gives upper bound 5:

$$k_0 < \overline{k_{D,2}}(n, \alpha) = A(n, \alpha)^\alpha \overline{k_0}(n, \alpha)^{1-\theta}, \quad n \geq 2, \quad 0 < \alpha < 1. \quad (99)$$

4.10. Upper bound 6

There holds trivially

$$k_0(n, \alpha) = k_0(n, \alpha)^\theta k_0(n, \alpha)^{1-\theta}. \quad (100)$$

Make now the choice $\theta = \alpha(n - 2\alpha)/(2n - 2\alpha - \alpha n)$ see (32), then

$$k_0(n, \alpha)^\theta < \overline{k_{D,1}}(n, \alpha)^\theta = (A(n, \alpha)^\gamma)^\theta = A(n, \alpha)^\alpha, \quad (101)$$

since $\gamma\theta = \alpha$ (see (92)) and further

$$k_0(n, \alpha)^{1-\theta} < \overline{\overline{k_0}}(n, \alpha)^{1-\theta}. \quad (102)$$

Insertation of (101) and (102) into (100) gives upper bound 6:

$$k_0 < \overline{k_{D,3}}(n, \alpha) = A(n, \alpha)^\alpha \overline{\overline{k_0}}(n, \alpha)^{1-\theta}, \quad n \geq 2, \quad 0 < \alpha < 1. \quad (103)$$

By the way, it is clear that in this way more upper bounds can be constructed.

4.11. Upper bound 7

This inequality is an application of [4, Theorem 1.7, (1.30), $\theta' = 1/2$, $\theta'' = 1$, with the restriction $n \geq 3$], as follows. Apply Lemma 2 with the choices $\alpha' = 1/2$, $\alpha'' = 1$ and $\theta = 2(1 - \alpha)$. See the results for the case $\alpha = 1$ in the Introduction, equation (15). Application of (72) for $n \geq 3$:

$$\begin{aligned} k_0(n, \alpha) &< \overline{k_0}\left(n, \frac{1}{2}\right)^{2(1-\alpha)} k_0(n, 1)^{2\alpha-1} = \overline{k_0}\left(n, \frac{1}{2}\right)^{2(1-\alpha)} (C_T(n, 2))^{-2\alpha+1} \\ &= \overline{k_0}\left(n, \frac{1}{2}\right)^{2(1-\alpha)} (k_T(n))^{2\alpha-1}, \quad n \geq 3, \quad 1/2 < \alpha < 1. \end{aligned} \quad (104)$$

The last restriction comes from the requirement that $\theta < 1$. We made the choice to bound $k_0\left(n, \frac{1}{2}\right)$ by $\overline{k_0}\left(n, \frac{1}{2}\right)$. Equation (104) represents the announced upper bound 7

$$\overline{k_{I,1}}(n, \alpha) = \overline{k_0}\left(n, \frac{1}{2}\right)^{\alpha_1} k_T(n)^{(1-\alpha_1)}, \alpha_1 = 2(1 - \alpha), n \geq 3, 1/2 < \alpha < 1. \quad (105)$$

4.12. Upper bound 8

This inequality is an application of [4, Theorem 1.7, (1.30), $\theta' = \theta_N (= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geq 3$], as follows. Apply Lemma 2 with the choices $\alpha' = \alpha_V$, $\alpha'' = 1$ and $\theta = \alpha_2 = (1 - \alpha)/(1 - \alpha_V)$. See the results for the case $\alpha = 1$ in the Introduction, equation (15). Application of (72) for $n \geq 3$ and for $\alpha_V < \alpha < 1$:

$$\begin{aligned} k_0(n, \alpha) &< \overline{k_0}(n, \alpha_V)^{\alpha_2} k_0(n, 1)^{1-\alpha_2} = \overline{k_0}(n, \alpha_V)^{\alpha_2} (C_T(n, 2))^{-(1-\alpha_2)} \\ &= \overline{k_0}(n, \alpha_V)^{\alpha_2} (k_T(n))^{(1-\alpha_2)}, \quad n \geq 3, \quad \alpha_V < \alpha < 1. \end{aligned} \quad (106)$$

We again made the choice to bound $k_0(n, \alpha_V)$ by $\overline{k_0}(n, \alpha_V)$. The value α_V can be chosen freely and has been chosen here as the argument value for the optimum of the expression $\alpha C_T(n, 2\alpha)$, see further at the proof for upper bound 10. Equation (106) represents the announced upper bound 8

$$\overline{k_{I,2}}(n, \alpha) = \overline{k_0}(n, \alpha_V)^{\alpha_2} k_T(n)^{(1-\alpha_2)}, \alpha_2 = (1 - \alpha)/(1 - \alpha_V), n \geq 3, \alpha_V < \alpha < 1. \quad (107)$$

4.13. Upper bound 9

This inequality is an application of [4, Theorem 1.7, (1.30), $\theta' = \theta_N (= \alpha_V)$, $\theta'' = 1$, with the restriction $n \geq 3$], as follows. Apply Lemma 2 with the choices $\alpha' = \alpha_V$, $\alpha'' = 1$ and $\theta = \alpha_2 = (1 - \alpha)/(1 - \alpha_V)$. See the results for the case $\alpha = 1$ in the Introduction, equation (15). Application of (72) for $n \geq 3$ and for $\alpha_V < \alpha < 1$:

$$\begin{aligned} k_0(n, \alpha) &< \overline{k_{L,V}}(n, \alpha_V)^{\alpha_2} k_0(n, 1)^{1-\alpha_2} = (\alpha_V C_T(n, 2\alpha_V))^{-\alpha_V \alpha_2} (C_T(n, 2))^{-(1-\alpha_2)} \\ &= (\alpha_V C_T(n, 2\alpha_V))^{-\alpha_V \alpha_2} (k_T(n))^{(1-\alpha_2)}, \quad n \geq 3, \quad \alpha_V < \alpha < 1. \end{aligned} \quad (108)$$

Here, we bounded $k_0(n, \alpha_V)$ by $\overline{k_{L,V}}(n, \alpha_V)$, i.e. the upper bound 10 (46). The value α_V can be chosen freely and has been chosen here as the argument value for the optimum of the expression $\alpha C_T(n, 2\alpha)$, see further at the proof for upper bound 10. Equation (108) represents the announced upper bound 9

$$\begin{aligned} \overline{k_{L,3}}(n, \alpha) &= (\alpha_V C_T(n, 2\alpha_V))^{-\alpha_V \alpha_2} k_T(n)^{(1-\alpha_2)}, \\ \alpha_2 &= (1-\alpha)/(1-\alpha_V), \quad n \geq 3, \quad \alpha_V < \alpha < 1. \end{aligned} \quad (109)$$

4.14. Upper bound 10

Firstly, we prove

$$k_0(n, \alpha) < (\alpha C_T(n, 2\alpha))^{-\alpha}, \quad n \geq 2, \quad 1/2 < \alpha < 1. \quad (110)$$

This result has been given in [4, Theorem 1.7, (1.31)] and was inspired by [6, (1.5)], by making the transformation $w = u^{1/\alpha}$ for $v > 0$ in (15) as follows

$$\begin{aligned} C_T(n, s) &\leq \frac{\|\nabla w\|_s}{\|w\|_t} = \frac{\|\nabla u^{1/\alpha}\|_s}{\|u^{1/\alpha}\|_t} = \frac{1/\alpha \|u^{(1-\alpha)/\alpha} \nabla u\|_s}{\|u^{1/\alpha}\|_t} \quad [t = sn/(n-s)] \\ &= \frac{1}{\alpha} \frac{\left(\int (\nabla u)^s u^{s(1-\alpha)/\alpha} dx \right)^{1/s}}{\left(\int u^{t/\alpha} dx \right)^{1/t}} \quad \begin{array}{l} \text{[apply Hölder inequality,} \\ 1/P + 1/Q = 1] \end{array} \\ &\leq \frac{1}{\alpha} \frac{\left(\int (\nabla u)^{sP} dx \right)^{1/(sP)} \left(\int u^{Qs(1-\alpha)/\alpha} dx \right)^{1/(sQ)}}{\left(\int u^{t/\alpha} dx \right)^{1/t}} \quad \begin{array}{l} \text{[take } P = 2/s, \\ Q = 2/(2-s)] \end{array} \\ &= \frac{1}{\alpha} \frac{\left(\int (\nabla u)^2 dx \right)^{1/2} \left(\int u^{Qs(1-\alpha)/\alpha} dx \right)^{(2-s)/(2s)}}{\left(\int u^{t/\alpha} dx \right)^{1/t}} \quad \begin{array}{l} \text{[take } s = 2\alpha, \text{ and} \\ r = t/\alpha = 2n/(n-2\alpha)] \end{array} \\ &= \frac{1}{\alpha} \frac{\|\nabla u\|_2 \|u\|_2^{(1-\alpha)/\alpha}}{\|u\|_r^{1/\alpha}} = \frac{1}{\alpha} (\Lambda_{n,\alpha}(u))^{1/\alpha}, \end{aligned} \quad (111)$$

for the choice $s = 2\alpha$. We have to restrict α to the interval $1/2 \leq \alpha \leq 1$ to give $C_T(n, 2\alpha)$ a meaning. Again, the inequality is strict since $w = \overline{u}_{n,\alpha}^\alpha$ does not equal a function $w_{n,s}$ (see (22)), with $s = 2\alpha$. So (111) implies

$$\lambda_{n,\alpha} = \inf_{u \in H^1(\mathbb{R}^n)} \Lambda_{n,\alpha}(u) > (\alpha C_T(n, 2\alpha))^\alpha,$$

and this equivalent with

$$k_0(n, \alpha) = 1/\lambda_{n,\alpha} < (\alpha C_T(n, 2\alpha))^{-\alpha}, \quad n \geq 2, \quad 1/2 < \alpha < 1.$$

Application of Lemma 2 with $\alpha'' = 0$, $\theta = \alpha/\alpha'$, and $k_0(n, 0) = 1$ gives

$$k_0(n, \alpha) < \left((\alpha' C_T(n, 2\alpha'))^{-\alpha'} \right)^{\alpha/\alpha'} = (\alpha' C_T(n, 2\alpha'))^{-\alpha}.$$

Since α' can still be chosen freely, we can improve this inequality by maximizing the $(\alpha' C_T(n, 2\alpha'))$. In a standard way we find that there is a unique value $\alpha_V \in (1/2, 1)$ which optimizes this expression, see [4, Proof Theorem 1.7, (1.32)] for details. Finally we find the announced upper bound 10

$$k_0 < \overline{k_{L,V}}(n, \alpha) = [\alpha_V C_T(n, 2\alpha_V)]^{-\alpha}, \quad n \geq 2, \quad 0 < \alpha \leq \alpha_V, \quad (112)$$

$$k_0 < \overline{k_{L,V}}(n, \alpha) = 1/k_{L,V}(n, \alpha) = [\alpha C_T(n, 2\alpha)]^{-\alpha}, \quad n \geq 2, \quad \alpha_V \leq \alpha < 1, \quad (113)$$

where the value for α_V follows from

$$\alpha_V = \alpha_V(n) = \frac{n}{2p_V}, \quad \text{where } p_V \text{ is the solution of} \quad (114)$$

$$\ln\left(\frac{n-p}{p-1}\right) + \frac{n-p}{p(p-1)} + \psi(p) - \psi(n+1-p) = 0, \quad (115)$$

$$\psi(x) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}, \quad x > 0, \quad 1 < p < n, \quad n \geq 2.$$

In both expressions (112) and (113) the second argument in C_T is larger than 1, as required. The value α_V has also been used in the upper bounds 8 and 9.

4.15. Upper bound 11

This inequality is a combination of the Hölder inequality (73)

$$\|u\|_r < \|u\|_{r'}^\theta \|u\|_{r''}^{1-\theta}, \quad 0 < \theta < 1, \quad 1/r = \theta/r' + (1-\theta)/r'', \quad r' \neq r'', \quad (116)$$

and the Sobolev embedding (15)

$$\|u\|_t \leq \frac{1}{C_T(n, 2)} \|\nabla u\|_2, \quad t = 2n/(n-2), \quad n \geq 3. \quad (117)$$

For the choice $r = 2n/(n-2\alpha)$, $\theta = \alpha$, $r'' = 2$ in (116), we find $r' = 2n/(n-2)$, which is just the value applicable for the Sobolev embedding (117). These two estimates combined gives

$$\|u\|_{2n/(n-2\alpha)} < \left(\frac{1}{C_T(n, 2)}\right)^\alpha \|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha} = k_T(n)^\alpha \|\nabla u\|_2^\alpha \|u\|_2^{1-\alpha}, \quad n \geq 3. \quad (118)$$

So, we found the announced upper bound 11

$$k_0 < \overline{k_B}(n, \alpha) = k_T(n)^\alpha, \quad n \geq 3, \quad 0 < \alpha < 1. \quad (119)$$

5. Numerical evaluations lower and upper bounds

In order to assess the quality of the estimates we have calculated the numbers $\lambda_{n,\alpha}$ for $n = 2, 3, 4, 5, 10$ and $\alpha = 0.05 + (i-1)0.005$, $i = 1, 2, 3, \dots, 176$ up till $\theta = 0.925$. The method is the same as used in the paper [4]. This method to find $\lambda_{n,\alpha}$ consists of a

shooting technique to find that value $\bar{u}(0) = u_0$ such that $\bar{u}(r)$ is a positive solution of (11) with $\lim_{r \rightarrow \infty} \bar{u}(r) = 0$. Therefore, we transformed the interval $r \in (0, \infty)$ into $s = r/(1+r) \in (0, 1)$. The transformed differential equation becomes, with $w(s) = u(r)$, $0 < s < 1$,

$$(1-s)^4 \frac{d^2}{ds^2} w + \left\{ \left(\frac{(n-1)}{s} - 2 \right) (1-s)^3 \right\} \frac{d}{ds} w - w|w|^{(n+2\alpha)/(n-2\alpha)-1} - w = 0,$$

$$w(0) = v_0, \quad \frac{d}{ds} w(0) = 0. \quad (120)$$

The aim now is to find a value v_0 such that for $w(0) = v_0$, $\frac{d}{ds} w(0) = 0$, we find $w(1) = 0$. We solved the transformed differential equation (120) by means of a numerical integration method (Runge-Kutta of the fourth order) with a self-adapting step-size routine such that a prescribed maximal relative error (ε_{rel}) in each component ($w(s)$, $\frac{d}{ds} w(s)$) has been satisfied. We made the choice $\varepsilon_{rel} = 10^{-15}$. For every value of v_0 the numerical integrator will find some point $s = s(v_0) \in (0, 1)$ where either $w(s) < 0$, or $\frac{d}{ds} w(s) > 0$. At that point s the integration will be stopped. This integrator is coupled to a numerical zero-finding routine (see ([25])), which can also be applied for finding a discontinuity. The function f for which such a discontinuity has to be found is specified by if $w(s(v_0)) < 0$, $f(v_0) = -(1-s(v_0))$ else (that means thus $\frac{d}{ds} w(s(v_0)) > 0$) $f(v_0) = (1-s(v_0))$. The sought value v_0 has been found if this numerical routine has come up with two values v_0 and v_0^1 such that $|v_0 - v_0^1| < r_p |v_0| + a_p$, (with $r_p = a_p = 10^{-15}$ relative and absolute precisions, respectively) and $|f(v_0)| \leq |f(v_0^1)|$, while $sign(f(v_0)) = -sign(f(v_0^1))$. During the integration processes the norms in (12) will be calculated. As a check upon this procedure the following expressions

$$\|\bar{u}_{n,\alpha}\|_2^2/(1-\alpha), \quad \|\nabla \bar{u}_{n,\alpha}\|_2^2/\alpha, \quad \|\bar{u}_{n,\alpha}\|_{2n/(n-2\alpha)}^{2n/(n-2\alpha)}, \quad (121)$$

are compared. They should be all equal, see (12). The eigenvalue $\lambda_{n,\alpha}$ is found then by (13).

5.1. Some numerical results for values for $\alpha = 1/3$, $2/3$ and $n = 2$

Here, we give for $n = 2$ and for particular values of α ($\alpha = 1/3$ and $2/3$) the upper and lower bounds which are applicable. Compare these with [10, $\alpha = 1/3$] and [6, $\alpha = 2/3$].

α	k_0	$\underline{k}_0$	$\underline{\underline{k}}_0$
$n = 2$			
1/3	7.2493833e-001	7.2431703e-001	7.2184608e-001
2/3	6.0129905e-001	5.9737503e-001	5.6854280e-001

Table 1: Functional, $n = 2$, Lower bounds 1 - 2.

α	k_0	$\overline{k_0}$	$\overline{\overline{k_0}}$	$\overline{\overline{\overline{k_0}}}$
$n = 2$				
1/3	7.2493833e-001	7.2978972e-001	7.3987840e-001	7.8567080e-001
2/3	6.0129905e-001	6.4335375e-001	6.1742806e-001	7.2152108e-001

Table 2: Functional, $n = 2$, Upper bounds 1 - 3.

α	$k_{D,1}$	$k_{D,2}$	$k_{D,3}$	$k_{L,V}$
$n = 2$				
1/3	7.3907188e-001	7.3132861e-001	7.3974392e-001	7.7547470e-001
2/3	6.8278406e-001	6.5623746e-001	6.3848696e-001	6.1088706e-001

Table 3: $n = 2$, Upper bounds 4 - 6 and 10.

5.2. Numerical results for $\alpha = 0.05, \dots, 0.925$ ($\Delta = 0.005$) and $n = 2, 3, 4, 5, 10$

In the Supplementary Material to this paper we present tables which give the results of the numerical calculations of the functional $k_0(n, \alpha)$ and the lower and upper bounds, based on the technique described above (see also [4]).

Values "0.0000000e+000" has to be interpreted as "Not Applicable". The lower and upper bounds have been calculated using the software package Matlab™.

5.3. Results for the zeros p_V and $\alpha_V = n/(2p_V)$

The zeros p_V as defined in (42) are given below in the Table 4; $\alpha_V(n) = n/(2p_V)$. The asymptotic expressions are

$$p_V(n) = 2n/3 + 5/18 + O(1/n), \quad n \rightarrow \infty, \tag{122}$$

$$\alpha_V(n) = 3/4 - 5/(16n) + O(1/n^2), \quad n \rightarrow \infty, \tag{123}$$

n	p_V	$p_{V,asympt}$	$p_V - p_{V,asympt}$
		$= 2n/3 + 5/18$	
2	1.6474176e+000	1.6111111e+000	3.6306497e-002
3	2.3044430e+000	2.2777778e+000	2.6665194e-002
4	2.9654018e+000	2.9444444e+000	2.0957401e-002
5	3.6283253e+000	3.6111111e+000	1.7214200e-002
6	4.2923606e+000	4.2777778e+000	1.4582787e-002
7	4.9570820e+000	4.9444444e+000	1.2637555e-002
8	5.6222549e+000	5.6111111e+000	1.1143822e-002
9	6.2877400e+000	6.2777778e+000	9.9621751e-003
10	6.9534493e+000	6.9444444e+000	9.0048448e-003

Table 4: The zeros p_V for $n = 2, \dots, 10$ and their asymptotic approximations.

n	α_V	$\alpha_{V,asympt}$	$\alpha_V - \alpha_{V,asympt}$
		$= 3/4 - 5/(16n)$	
2	6.0701063e-001	5.9375000e-001	1.3260630e-002
3	6.5091652e-001	6.4583333e-001	5.0831867e-003
4	6.7444485e-001	6.7187500e-001	2.5698490e-003
5	6.8902311e-001	6.8750000e-001	1.5231128e-003
6	6.9891612e-001	6.9791667e-001	9.9945530e-004
7	7.0606054e-001	7.0535714e-001	7.0339854e-004
8	7.1145831e-001	7.1093750e-001	5.2081118e-004
9	7.1567845e-001	7.1527778e-001	4.0067485e-004
10	7.1906759e-001	7.1875000e-001	3.1758674e-004

Table 5: The zeros $\alpha_V = n/(2p_V)$ for $n = 2, \dots, 10$ and their asymptotic approximations.

6. Discussion

With respect to the lower bounds it is clear based on the numerical results in the Supplementary Material to this paper (Tables 4-8 and Fig. 3 in "Comparison Functional with Lower bounds for Functional" therein) that the lower bound for $n = 2$, $\underline{k_0}(\alpha)$, is superior to the lower bound $\underline{k_0}(2, \alpha)$.

With respect to the upper bounds the situation is more complicated. For the range of n ($n = 2, 3, 4, 5$ and $n = 10$) and α ($0.05 \leq \alpha \leq 0.925$ with steps $\Delta\alpha = 0.005$) we have examined there are just four upper bounds which are superior, see the Table 6 and the Figures 1, 2, 3, 4 and 5.

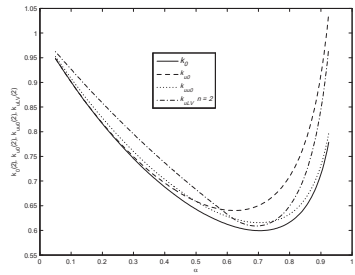


Figure 1: Best bounds for $n = 2$.

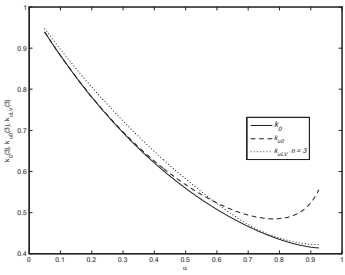


Figure 2: Best bounds for $n = 3$.

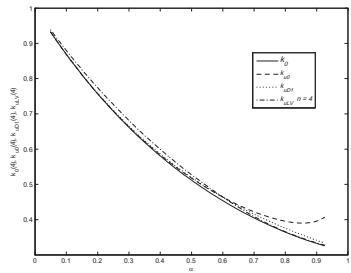


Figure 3: Best bounds for $n = 4$.

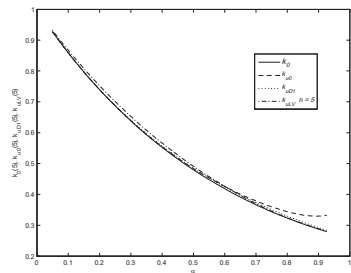


Figure 4: Best bounds for $n = 5$.

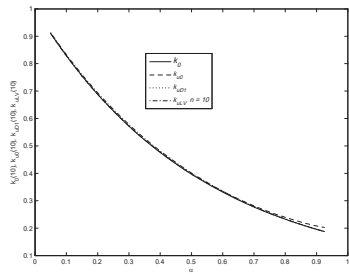


Figure 5: Best bounds for $n = 10$.

n	Range α	Upper bound #	Expression Upper bound
2	(0.050, 0.495)	1	$\overline{k_0}(2, \alpha)$
2	0.500	$1 = 2$	$\overline{k_0}(2, 1/2) = \overline{k_0}(2, 1/2)$
2	[0.505, 0.615)	2	$\overline{k_0}(2, \alpha)$
2	(0.620, 0.745)	10	$\overline{k_{L,V}}(2, \alpha)$
2	(0.750, 0.925)	2	$\overline{k_0}(2, \alpha)$
3	(0.050, 0.590)	1	$\overline{k_0}(3, \alpha)$
3	(0.595, 0.925)	10	$\overline{k_{L,V}}(3, \alpha)$
4	(0.050, 0.590)	1	$\overline{k_0}(4, \alpha)$
4	(0.595, 0.605)	4	$\overline{k_{D,1}}(4, \alpha)$
4	(0.610, 0.925)	10	$\overline{k_{L,V}}(4, \alpha)$
5	(0.050, 0.565)	1	$\overline{k_0}(5, \alpha)$
5	(0.570, 0.630)	4	$\overline{k_{D,1}}(5, \alpha)$
5	(0.635, 0.925)	10	$\overline{k_{L,V}}(5, \alpha)$
10	(0.050, 0.535)	1	$\overline{k_0}(10, \alpha)$
10	(0.540, 0.675)	4	$\overline{k_{D,1}}(10, \alpha)$
10	(0.680, 0.925)	10	$\overline{k_{L,V}}(10, \alpha)$

Table 6: Optimal upper bounds for $n = 2, 3, 4, 5, 10$.

We remark that $\overline{k_0}(2, 1/2) = \overline{\overline{k_0}}(2, 1/2) = 2^1 3^{-3/4} \pi^{-1/4}$, and $\overline{k_0}(3, 3/4) = \overline{\overline{k_0}}(3, 3/4) = 2^{7/4} 3^{-3/2} \pi^{-1/4}$ see [15, equation (12) and (17)].

As can be seen from the figures in the Supplementary Material to this paper, for larger values of n almost all bounds come close to the actual value for $k_0(n, \alpha)$; see the Figures 7, 12, 28, 32, 37, 42, 46 and 51 therein, for $n = 10$.

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