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Marynets, K.; Pantova, D.H.

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Fractional Boundary Value Problems with Parameter-Dependent and Asymptotic Conditions

Kateryna Marynets * and Dona Pantova

Delft Institute of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628CD Delft, The Netherlands; d.h.pantova@tudelft.nl
* Correspondence: k.marynets@tudelft.nl

Abstract

We study a nonlinear fractional differential equation, defined on a finite and infinite interval. In the finite interval setting, we attach initial conditions and parameter-dependent boundary conditions to the problem. We apply a dichotomy approach, coupled with the numerical-analytic method, to analyze the problem and to construct a sequence of approximations. Additionally, we study the existence of bounded solutions in the case when the fractional differential equation is defined on the half-axis and is subject to asymptotic conditions. Our theoretical results are applied to the Arctic gyre equation in the fractional setting on a finite interval.

Keywords: fractional differential equations; parameter-dependent boundary conditions; constructive approximations; asymptotic conditions; Arctic gyre

1. Introduction

In recent years, fractional initial boundary value problems (FIBVPs) have attracted attention in the field of mathematical modeling due to the ability of fractional operators to capture nonlocal and memory effects. Such effects are relevant in many physical processes, thus making fractional differential equations (FDEs) a suitable modeling tool (see, for instance, [1–5] and the references therein). Examples of applications of fractional calculus include viscoelasticity, wave propagation through inhomogeneous media, porous media flow, and anomalous diffusion. In viscoelasticity, fractional models have been employed to describe the behavior of polymers and elastomers (see discussions in [6,7]), to analyze the vibrations of materials with viscoelastic damping [8], and to characterize the mechanical response of magneto-sensitive rubbers [9]. In the context of wave propagation, fractional derivatives are used to model power-law attenuation phenomena, such as those observed in acoustic wave transmission through inhomogeneous media [10] and in the attenuation of seismic waves traveling through the Earth's interior [11]. Applications in porous media include modeling groundwater flow [12], the dynamics of unsaturated aquifers [13], nonlinear infiltration processes [14], and viscoplastic soil compression [15]. Furthermore, fractional calculus has been instrumental in advancing the understanding of anomalous diffusion, with applications ranging from modeling random walks on fractal structures [16], to describing diffusion in complex systems [17], hydrodynamic transport through fracture networks [18], and anomalous transport processes in porous and fractured media [19]. However, the nonlocal nature of fractional calculus operators poses some challenges. Exact solutions to FDEs, especially those arising in modeling and engineering applications, are

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rarely obtainable, in particular when they contain nonlinearities. Classical analytical techniques often have limited applicability or require substantial modification for the fractional setting. Moreover, the computational complexity of numerically solving FDEs is generally higher than that of their integer-order counterparts. This has prompted the development of approximation methods that avoid discretization and can be used in the analysis of FBVPs. One such technique, applicable to FIBVPs, is the numerical–analytic technique. It has been extensively applied to the investigation of boundary value problems (BVPs) for ordinary differential equations (see [20–22]), and has subsequently been adapted for the study of BVPs of fractional order, subject to various types of boundary conditions [23–26]. It combines the construction of closed-form approximations of the solution of the FIBVP with the numerical solution of a system of algebraic equations.

FBVPs, defined on domains of unknown length arise when both non-local effects and moving interfaces are present. Some examples include the fractional Stefan problem, which models phase changes in materials with memory [27], fractional models of American pricing options [28,29], and fluid dynamics in porous media [30,31]. These problems present significant challenges due to the coupling between the unknown boundary position and the solution itself, which requires the development of approaches capable of effectively handling these interactions. The numerical-analytic method has previously been applied to first-order ODEs with non-fixed right boundaries under two-point conditions [20], but, to our knowledge, has not yet been extended to FDEs with unknown domain length. This paper aims to generalize the method to FDEs of order $p \in (1,2]$ with three-point boundary conditions. The novelty in the problem setting consists of the fractional derivative and the three-point condition. Moreover, the interval-splitting method applied here, which extends the applicability of the numerical-analytic method and improves its convergence, has not previously been applied to an FDE with non-fixed boundary. A key advantage of the method is that it can accommodate various types of boundary conditions and does not require complete knowledge of the initial data. Additionally, it avoids discretization, reducing the problem to solving an algebraic equation, which simplifies implementation. It is worth noting that there exist other techniques for constructing closed-form approximations to FIBVPs, subject to different types of boundary constraints. Two well-known approaches are the Variational Iteration Method (VIM) which uses correction functionals and variational theory [32], and the Homotopy Perturbation Method (HPM) which constructs a homotopy with an embedding parameter to build an approximating series [33]. Unlike these semi-analytical approaches, the numerical-analytic method not only constructs approximate solutions but also serves as a framework for rigorous analysis of FBVPs, allowing for the derivation of existence, uniqueness, and solvability results.

In addition to studying FIBVPs, restricted to finite intervals, one may be interested in the existence and behavior of solutions to FDEs, where the fractional derivatives are defined on the half or whole real axis. BVPs for integer order differential equations on infinite domains arise naturally from the modeling of various physical processes [34–36]. There are also some results for nonlinear FBVPs on infinite intervals [37–40] with fractional boundary conditions.

In the present paper, we address an FIBVP of the Caputo type on a semi-finite domain. In particular, we study a nonlinear FDE of the general form

$$ {}_0^C D_t^p u(t) = f(t, u(t)), \quad t \in [t_0, \infty), \quad p \in (1,2], $$

defined on the half-axis, and subject to asymptotic conditions of the form

$$ \lim_{t\to\infty} u(t) = \phi_0, \quad \lim_{t\to\infty} \{e^t u'(t)\} = 0. $$

We use the fixed-point theory to give some conditions for the existence of bounded solutions. Furthermore, we consider the case when the FDE is restricted to a finite interval of unknown length λ, and coupled with initial and parameter-dependent boundary conditions:

$$u(0) = \psi, \quad u'(0) = \chi, \quad Au(0) + Bu(\lambda) + Cu'(\lambda) = d.$$

We use an interval-splitting method and the numerical-analytic technique to analyze the problem, and to construct a sequence of functions that converges to its exact solution. Finally, our method is applied to the Arctic gyre equation in the fractional setting to illustrate the validity of our results.

Note that the choice of the Caputo operator in the differential equation of our interest is motivated by its ability to incorporate boundary conditions of integer order, which are typically of physical relevance. However, the method presented in this work is not limited to a specific type of fractional operator and can be extended to a broader class of operators. Notable examples include the two-scale fractal derivative, which is particularly effective in modeling multiscale phenomena [41]; He's fractional derivative, designed for systems exhibiting weak memory effects [42]; and the Atangana-Baleanu derivative, which is capable of capturing both short-term dynamics and long-range memory behaviors [43].

The present paper consists of seven sections. Section 2 deals with the existence of bounded solutions to FDEs with asymptotic conditions. In Section 3, we consider an FDE defined on an interval of unknown finite length, subject to initial and three-point boundary conditions. We describe the interval-splitting method used to reduce the original problem to "model-type" problems and the derivation of the approximating sequence. We show the uniform convergence of the sequence to the unique solution of the "model-type" problems and their relation to the original FBVP. In Section 4, the solvability of the FBVP is analyzed. In Section 5, the numerical-analytic method is applied to the Arctic gyre equation in the fractional setting. In Section 6, we discuss the presented method and its advantages, and outline directions for future work. Section 7 gives a summary of the present work.

2. Bounded Solutions of FDEs with Asymptotic Conditions

In this section, we give the problem setting for an FDE on a semi-infinite domain, subject to asymptotic conditions. We study the existence of bounded solutions to the FDE, satisfying the given conditions.
Consider a FDE of the form

$$^{C}D^{p}_{-}u(t) = f(t, u(t)),\ t \geq t_0 \tag{1}$$

with the asymptotic conditions

$$\lim_{t\to\infty} u(t) = \phi_0, \quad \lim_{t\to\infty}\{e^t u'(t)\} = 0, \tag{2}$$

where $p \in (1,2]$, and $^{C}D^{p}_{-}$ denotes the Caputo fractional derivative on the half-axis (see [3], 2.4.48).

First, we state the following lemma, which gives the relationship between the fractional integral and the Caputo fractional derivative on the half-axis.

Lemma 1. *Let $u(t) \in C^n(\mathbb{R}^+)$ and $p \in (n-1, n)$. Then,*

$$(I^{p}_{-}\,{}^{C}D^{p}_{-}u)(t) = u(t) + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{k!} \lim_{\zeta\to\infty} u^{(k)}(\zeta)(\zeta - t)^k. \tag{3}$$

The next lemma shows the equivalence between Problems (1) and (2) and the corresponding integral equation.

Lemma 2. *Let there exist positive constants k and $c > 1$, such that*

$$|f(t,u(t))| \leq ke^{-ct}. \tag{4}$$

Then, Problems (1) and (2) are equivalent to the integral equation

$$u(t) = \phi_0 + \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} f(s,u(s))ds. \tag{5}$$

Lemmas 1 and 2 can be proved easily, thus, their proofs are left to the reader.

Let us now prove conditions that assure existence of a unique solution of the integral Equation (5), which also satisfies the asymptotic constraints (2).

Theorem 1. *Assume that there exists a function $a : [t_0, \infty] \to \mathbb{R}^+$, such that*

$$\int_{t_0}^{\infty} s^{p-1} a(s)ds < \infty, \tag{6}$$

$$|f(t,u) - f(t,v)| \leq a(t)|u-v|, \;\; t \geq t_0, \;\; u,v, \in \mathbb{R}, \tag{7}$$

and the condition in (4) holds. Then, for all $\phi_0 \in \mathbb{R}$, integral Equation (5) has a unique continuous solution $u : [t_0, \infty] \to \mathbb{R}$ satisfying $\lim_{t\to\infty} u(t) = \phi_0$.

Proof. Since the improper integral of the positive function $a(s)$ is finite, i.e., (6), for any positive constant $c > 0$, there exists a sufficiently large T_0, such that the integral from T_0 is bounded by c. Thus, we can choose $c = \Gamma(p)$ and find a $T_0 \geq t_0$, such that

$$\int_{T_0}^{\infty} s^{p-1} a(s)ds < \Gamma(p).$$

On the Banach space X of continuous and bounded functions $u : [T_0, \infty) \to \mathbb{R}$, endowed with the norm $||u(t)|| := \sup_{t \geq T_0} |u(t)|$, define the operator

$$[F(u)](t) := \phi_0 + \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} f(s,u(s))ds \tag{8}$$

for $t \geq T_0$. Then, for $u \in X$, we have

$$\begin{aligned} |[F(u)](t)| &\leq |\phi_0| + \left| \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} f(s,u(s))ds \right| \\ &\leq |\phi_0| + \int_t^{\infty} |I_-^q f(s,u(s))|ds < \infty, \end{aligned}$$

that is, $F : X \to X$. Now, let $u, v \in X$ and consider

$$||[F(u)] - [F(v)]|| \leq \sup_{t \geq T_0} \frac{1}{\Gamma(p)} \int_t^{\infty} (s-t)^{p-1} |f(s,u) - f(s,v)|ds$$

$$\leq \sup_{t \geq T_0} \frac{1}{\Gamma(p)} \int_t^{\infty} s^{p-1} a(s)|u-v|ds \leq ||u-v|| \sup_{t \geq T_0} \frac{1}{\Gamma(p)} \int_t^{\infty} s^{p-1} a(s)ds$$

$$\leq ||u-v|| \frac{1}{\Gamma(p)} \int_{T_0}^{\infty} s^{p-1} a(s)ds < ||u-v||,$$

which implies that the operator F is a contraction on X, hence it has a unique fixed point in X by the contraction principle, and this fixed point is the unique solution to Equation (5) on $t \in [T_0, \infty)$ [44].

If $T_0 = t_0$, we are done. If $t_0 < T_0$, $u(T_0)$, and $u'(T_0)$ are determined and the solution of (1) can be extended from $[T_0, \infty)$ to $[t_0, \infty)$. This is because, since $a(t)$ is positive and (6) holds, $a(t)$ is bounded for $t \geq t_0$, which implies that $f(t, u(t))$ is Lipschitz continuous in u. Hence (1) has a unique solution on $[T_0, \infty)$, which can be continued to $[t_0, \infty)$. Moreover, condition (4) on $f(t, u(t))$ prevents the blow up of solutions in finite time. Therefore, the solutions on $[T_0, \infty)$ can be extended to $[t_0, \infty)$. $\square$

3. Analysis of the Parameter-Dependent FIBVP on a Finite Interval

In this section, we consider an FDE on a finite interval whose length is denoted by the unknown parameter λ, and can in principle be extended indefinitely, that is, we can let $\lambda \to \infty$. We attach parameter-dependent boundary conditions to the FDE and use an interval-splitting method to redefine the original problem as a system of "model" type problems on smaller domains. The numerical-analytic technique is applied to construct approximations to the solution of each problem. We establish a connection between the solutions to the "model" problems and the original FIBVP, and give necessary and sufficient conditions for the existence of solutions (for details of the technique, see [20]).

3.1. Problem Setting and Interval Splitting

We consider the FIBVP for the FDE of the form:

$$ {}_{0}^{C}D_{t}^{p} u(t) = f(t, u(t)), \quad t \in J := [0, \lambda] \tag{9} $$

for some $p \in (1, 2]$, subjected to the parameter-dependent boundary conditions

$$ Au(0) + Bu(\lambda) + Cu'(\lambda) = d \tag{10} $$

and the initial conditions

$$ u(0) = \alpha_0, \;\; u'(0) = \chi_0. \tag{11} $$

Here, ${}_{0}^{C}D_{t}^{p}$ is the Caputo derivative with lower limit at 0 (see [3], 2.4.15), $u : J \to D \subset \mathbb{R} \in C^2(J, \mathbb{R})$, $f : G \to \mathbb{R}$, $G = J \times D$, and D is a closed and bounded domain. The constants in the boundary and initial conditions $A, B, C, d, \alpha_0, \chi_0 \in \mathbb{R}$ are given scalars, and the end point of the interval J is an unknown parameter $\lambda \in \mathbb{R}$.

We aim to find a solution $u : J \to D$ of the FDE (9), which satisfies the parameter-dependent boundary conditions (10), and the given initial conditions (11) in the space $C^2(J, \mathbb{R})$.

For this purpose, we will construct a sequence of approximate solutions, and as it will be seen in Theorem 2, the convergence of this sequence is contingent upon the function $f(t, u(t))$ satisfying a Lipschitz condition on J. If this fails to hold, the uniform convergence of the sequence cannot be guaranteed. To deal with this difficulty, we will use a dichotomy-type approach, similar to [26], but for a more general setting. The interval-splitting method can also be applied to speed up the convergence of the iterative sequence, and in the case when the interval of definition of the FBVP (9)–(11) needs to be extended.

Let us decompose the interval $J = [0, \lambda]$ into N subintervals. Without loss of generality, let each subinterval have length λ/N, and denote them by $J_j = [\lambda_{j-1}, \lambda_j] :=$

$\left[(j-1)\lambda/N, j\lambda/N\right]$ for $j = 1, ..., N$. We denote the solution on each J_j by $u_j(t)$, and the values of $u_j(t)$ and $u_j'(t)$ at the end point of the subintervals in the following way:

$$u_j(\lambda_j) = u_{j+1}(\lambda_j) = \alpha_j, \quad u_j'(\lambda_j) = u_{j+1}'(\lambda_j) = \chi_j, \quad j = 1, ..., N-1. \tag{12}$$

The solution $u(t)$ to FBVP (9)–(11) on the whole interval $t \in [0, \lambda]$ is defined by $u_j(t)$ piece-wise on each J_j. Thus, the boundary conditions (12) are chosen in such a way that they ensure that $u(t)$ is a smooth function. Here, α_j and χ_j are unknown parameters to be calculated. Note that the boundary condition at the value of $u_N'(\lambda_N = \lambda)$ is given by $\chi_N = C^{-1}(d - A\alpha_0 - B\alpha_N)$.

With this, we split the FBVP (9)–(11) into N "model-type" problems, which read

$$\begin{aligned} {}_{\lambda_{j-1}}^{\ \ C}D_t^p u_j(t) =& f(t, u_j(t)) - \frac{1}{\Gamma(n-p)} \sum_{i=0}^{j-1} \int_{\lambda_j}^{\lambda_{j+1}} (t-s)^{n-p-1} u_i^{(n)}(s) ds \\ :=& f_j(t, u_1(t), ..., u_j(t)) = f_j(t, u(t)), \quad t \in J_j, \end{aligned} \tag{13}$$

$$\begin{aligned} u_j(\lambda_{j-1}) =& \alpha_{j-1}, \quad u_j(\lambda_j) = \alpha_j, \\ u_j'(\lambda_{j-1}) =& \chi_{j-1}, \quad u_j'(\lambda_j) = \chi_j, \quad j = 1, ..., N. \end{aligned} \tag{14}$$

The functions $u_j : J_j \to D_j$ are continuous on D_j, and the domains D_j are such that $\cup_{j=1}^N D_j = D$.

Remark 1. *Due to the nonlocality of the Caputo fractional derivative, after defining a new BVP (13), (14) on each subinterval, the right-hand side function had to be adjusted accordingly, since now the Caputo derivative is taken with lower limit λ_{j-1} on each J_j.*

In the following subsection, we give the form of the sequence of successive approximations to the solution of FBVP (13), (14) and a theorem on the convergence of the sequence. The proof of Theorem 2 follows the lines of Theorem 1 in [26], so we omit it.

3.2. Successive Approximations

Let us consider FBVP (13), (14) for a fixed $j \in \{1, ..., N\}$. We connect it with a sequence $\{u_j^m\}$, $m \in \mathbb{Z}_0^+$, for $t \in J_j$, $u_j^0(t; \alpha, \chi, \lambda) \in D_j$, given by

$$\begin{aligned} u_j^0(t; \alpha, \chi, \lambda) =& \alpha_{j-1} + \chi_{j-1}(t - \lambda_{j-1}) + \left(\frac{t - \lambda_{j-1}}{\lambda_1}\right)^p \left(\alpha_j - \alpha_{j-1} - \chi_{j-1}\frac{\lambda}{N}\right), \\ u_j^m(t; \alpha, \chi, \lambda) =& u_j^0(t; \alpha, \chi, \lambda) + \frac{1}{\Gamma(p)} \left[\int_{\lambda_{j-1}}^t (t-s)^{p-1} f_j(s, u^{m-1}(s; \alpha, \chi, \lambda)) ds \right. \\ & \left. - \left(\frac{t - \lambda_{j-1}}{\lambda_1}\right)^p \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} f_j(s, u^{m-1}(s; \alpha, \chi, \lambda)) ds \right]. \end{aligned} \tag{15}$$

Here, λ_1 denotes the length of each subinterval. The sequence above is derived by integrating the modified equation given in (13) and enforcing the boundary conditions (14).

Note that the approximating function $u^m(t; \alpha, \chi, \lambda)$ on the whole interval $t \in J$ is piece-wise given by

$$u^m(t; \alpha, \chi, \lambda) = u_j^m(t; \alpha, \chi, \lambda), \quad t \in J_j. \tag{16}$$

Assume that for the BVP (13), (14), the following conditions are satisfied:

(i) The function $f_j(t, u(t))$ is bounded:

$$|f_j(t, u(t))| \leq M_j, \tag{17}$$

for all $t \in J_j$, $u_j \in D_j$ and some non-negative integer M_j.

(ii) The function $f_j(t, u(t))$ is Lipschitz continuous in $u_j(t)$, i.e.

$$|f_j(t, u_1(t), ..., u_j^1(t)) - f_j(t, u_1(t), ..., u_j^2(t))| \leq K_j |u_j^1(t) - u_j^2(t)| \tag{18}$$

for all $t \in J_j$, $u_j^1, u_j^2 \in D_j$, and a non-negative Lipschitz constant K_j.

(iii) The set

$$D_{\beta_j} := \{\alpha_{j-1} \in D_j : B(u_j^0(t; \alpha, \chi, \lambda)), \beta_j\Big) \subset D_j \ \ \forall (t, \alpha_j, \chi_j, \lambda) \in J_j \times \Omega_j\} \tag{19}$$

is nonempty, where $\Omega_j = D_j \times X_j \times \Lambda$, $\chi_j \in X_j$, $\lambda \in \Lambda$, and

$$\beta_j = \frac{M_j \lambda_1^p}{2^{2p-1}\Gamma(p+1)}. \tag{20}$$

(iv) The inequality $Q_j < 1$ holds for Q_j, which is defined as

$$Q_j = \frac{K_j \lambda_1^p}{2^{2p-1}\Gamma(p+1)}. \tag{21}$$

The following theorem ensures that if conditions (i)–(iv) hold, for all $j \in \{1, \ldots, N\}$ there exists a limit function $u_j^\infty(t; \alpha, \chi, \lambda) : J_j \times \Omega_j \to D_j$, which is well defined for all artificially introduced parameters $(\alpha_j, \chi_j) \in D_j \times X_j$, and is the unique solution to the FBVP (13), (14) with corresponding index j. Moreover, letting

$$u_\infty(t; \alpha, \chi, \lambda) := u_j^\infty(t; \alpha, \chi, \lambda), \ t \in J_j \tag{22}$$

yields the well-defined smooth function $u_\infty(t; \alpha, \chi, \lambda)$, which satisfies the boundary and initial conditions in the original FIBVP (9)–(11):

$$\begin{aligned} u_\infty(0; \alpha, \chi, \lambda) =& u_1^\infty(0; \alpha, \chi, \lambda) = \alpha_0, \\ (u_\infty)'(0; \alpha, \chi, \lambda) =& (u_1^\infty)'(0; \alpha, \chi, \lambda) = \chi_0, \\ (u_\infty)'(\lambda; \alpha, \chi, \lambda) =& (u_N^\infty)'(\lambda; \alpha, \chi, \lambda) = C^{-1}(d - A\alpha_0 - B\alpha_N). \end{aligned}$$

Theorem 2. *Assume that the FBVP (13), (14) satisfies Conditions (17)–(21). Then, for all fixed $(\alpha_j, \chi_j, \lambda) \in \Omega_j$, it holds:*

1. Functions of the sequence (15) are continuous and satisfy the boundary condition

$$\begin{aligned} u_j^m(\lambda_{j-1}; \alpha, \chi, \lambda) =& \alpha_{j-1}, \quad u_j^m(\lambda_j; \alpha, \chi, \lambda) = \alpha_j, \\ (u_j^m)'(\lambda_{j-1}; \alpha, \chi, \lambda) =& \chi_{j-1}, \quad (u_j^m)'(\lambda_j; \alpha, \chi, \lambda) = \chi_j. \end{aligned}$$

2. The sequence of functions (15) for $t \in J_j$ converges uniformly as $m \to \infty$ to the limit function

$$u_j^\infty(t; \alpha, \chi, \lambda) = \lim_{m \to \infty} u_j^m(t; \alpha, \chi, \lambda). \tag{23}$$

3. The limit function satisfies the boundary conditions

$$u_j^\infty(\lambda_{j-1};\alpha,\chi,\lambda) = \alpha_{j-1}, \quad u_j^\infty(\lambda_j;\alpha,\chi,\lambda) = \alpha_j,$$
$$(u_j^\infty)'(\lambda_{j-1};\alpha,\chi,\lambda) = \chi_{j-1}, \quad (u_j^\infty)'(\lambda_j;\alpha,\chi,\lambda) = \chi_j.$$

4. The limit function (23) is a unique solution to the integral equation

$$u_j(t) = \alpha_{j-1} + \chi_{j-1}(t-\lambda_{j-1}) + \Big(\frac{t-\lambda_{j-1}}{\lambda_1}\Big)^p(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1)$$
$$+\frac{1}{\Gamma(p)}\Bigg[\int_{\lambda_{j-1}}^t (t-s)^{p-1} f_j(s,u(s))ds - \Big(\frac{t-\lambda_{j-1}}{\lambda_1}\Big)^p \int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1} f_j(s,u(s))ds\Bigg]. \tag{24}$$

i.e., it is a unique solution on $t \in J_j$ of the Cauchy problem for the modified FDE:

$$\begin{aligned} {}_{\lambda_{j-1}}^{\;\;C}D_t^p u_j(t) &= f_j(t,u(t)) + \Delta_j(\alpha,\chi,\lambda), \\ u_j(\lambda_{j-1}) &= \alpha_{j-1}, \quad u_j'(\lambda_{j-1}) = \chi_{j-1}, \end{aligned} \tag{25}$$

where $\Delta_j(\alpha,\chi,\lambda) : \Omega \to \mathbb{R}$ is a mapping defined by

$$\Delta_j(\alpha,\chi,\lambda) = \frac{\Gamma(p+1)}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1) - \frac{p}{\lambda_1^p}\int_{\lambda_{j-1}}^{\lambda_j}\Big(\frac{j\lambda}{N} - s\Big)^{p-1} f_j(s,u(s))ds. \tag{26}$$

5. The following error estimate holds:

$$|u_j^\infty(t;\alpha,\chi,\lambda) - u_j^m(t;\alpha,\chi,\lambda)| \leq \frac{\lambda_1^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j, \tag{27}$$

where $t \in J_j$, and M_j and Q_j are defined by (17) and (21).

Next, we state two theorems that establish the connection between the solution to the Cauchy problem (25) and the original FBVP (9)–(11). We show the connection between (25) and the FBVP (13), (14), and the connection between the limit function, defined in (22), and the solution to the original FBVP (9)–(11).

3.3. Connection of the Limit Function to the FIBVP

Consider the Cauchy problem:

$$\begin{aligned} {}_{\lambda_{j-1}}^{\;\;C}D_t^p u_j(t) &= f_j(t,u(t)) + \mu_j, \;\; t \in J_j, \\ u_j(\lambda_{j-1}) &= \alpha_{j-1}, \quad u_j'(\lambda_{j-1}) = \chi_{j-1}, \end{aligned} \tag{28}$$

where $\mu_j \in \mathbb{R}$ is referred to as a control parameter.

The following result holds.

Theorem 3. *Suppose $\alpha_{j-1} \in D_{\beta_j}$, $(\alpha_j, \chi_j, \lambda) \in \Omega_j$ and assume the conditions of Theorem 1 hold. Then, the solution $u_j(\cdot,\alpha,\chi,\lambda;\mu_j)$ of the Cauchy problem (28) also satisfies the boundary conditions in (14) if and only if*

$$\mu_j = \Delta_j(\alpha,\chi,\lambda), \tag{29}$$

where $\Delta_j(\alpha, \chi, \lambda)$ *is given by (26), and in this case*

$$u_j(t, \alpha, \chi, \lambda; \mu_j) = u_j^{\infty}(t; \alpha, \chi, \lambda). \tag{30}$$

For a proof of Theorem 3, we refer to [26].

Theorem 4. *Let the FBVP (13), (14) satisfy Conditions (17)–(21). Then,* $u_j^{\infty}(t; \alpha^*, \chi^*, \lambda^*)$ *is a solution to (13), (14) if and only if the triple* $(\alpha^*, \chi^*, \lambda^*)$ *is a solution to the determining system*

$$\begin{cases} \Delta_j(\alpha^*, \chi^*, \lambda^*) = 0, \\ V_j(\alpha^*, \chi^*, \lambda^*) = 0, \end{cases} \tag{31}$$

where $\Delta_j(\alpha, \chi, \lambda)$ *is given in (26) and* $V_j : \Omega \to \mathbb{R}$ *is a mapping, defined by*

$$V_j(\alpha, \chi, \lambda) = \frac{d}{dt} u_j(\lambda_j; \alpha, \chi, \lambda) - \chi_j, \ \ \text{for} \ \ j = 1, \ldots, N. \tag{32}$$

Proof. First, we note that the second equation in the determining system (31) is derived from the smoothness of the solution $u(t)$ on J. The boundary conditions in (14) prescribe the derivative value of each $u_j(t)$ at the left-end point of the subinterval J_j. Equation (32) requires the derivative of $u_j(t)$ at the right end of the interval J_j to be equal to the derivative of $u_{j+1}(t)$ at the same point, therefore ensuring that the solution $u(t)$ is smooth.

Now, since the conditions of Theorem 2 hold, we can apply Theorem 3 and note that the perturbed equation in (25) coincides with the original FDE in (13), and the solution $u_\infty(t; \alpha^*, \chi^*, \lambda^*)$ satisfies the parameter-dependent boundary conditions in (14) if and only if the pair $(\alpha^*, \chi^*, \lambda^*)$ satisfies (31). That is, $u_\infty(t; \alpha^*, \chi^*, \lambda^*)$ is a solution to FIBVP (13), (14) if and only if (31) holds. □

Remark 2. *Theorem 4 gives necessary and sufficient conditions for the solvability of the FBVP (13), (14) and the construction of its solutions. However, a difficulty in its application arises from the fact that explicit forms of the exact functions* $\Delta(\alpha, \chi, \lambda)$ *and* $V(\alpha, \chi, \lambda)$ *are unknown. To overcome this difficulty, in practice, we solve an approximate determining system*

$$\begin{cases} \Delta_j^m(\alpha, \chi, \lambda) = 0, \\ V_j^m(\alpha, \chi, \lambda) = 0, \end{cases} \tag{33}$$

which depends only on the $(m-1)$*-th and m-th terms of the sequence (15), and can thus be constructed explicitly. In particular, the approximate functions* $\Delta_j^m : \Omega \to \mathbb{R}$ *and* $V_j^m : \Omega \to \mathbb{R}$ *are given by*

$$\begin{aligned} \Delta_j^m(\alpha, \chi, \lambda) = {} & \frac{\Gamma(p+1)}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1) \\ & - \frac{p}{\lambda_1^p} \int_{\lambda_{j-1}}^{\lambda_j} (\lambda_j - s)^{p-1} f_j(s, u^m(s; \alpha, \chi, \lambda)) ds \end{aligned} \tag{34}$$

and

$$
\begin{aligned}
V_j^0(\alpha,\chi,\lambda) &= \frac{d}{dt}u_j^0(\lambda_j;\alpha,\chi,\lambda) - \chi_j = \lambda_j\chi_{j-1} + \frac{p}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1), \\
V_j^m(\alpha,\chi,\lambda) &= \frac{d}{dt}u_j^m(\lambda_j;\alpha,\chi,\lambda) - \chi_j = \lambda_j\chi_{j-1} + \frac{p}{\lambda_1^p}(\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1), \\
&+ \frac{1}{\Gamma(p)}\Bigg[(p-1)\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-2} f_j(s, u^m(s;\alpha,\chi,\lambda))ds \\
&- \frac{p}{\lambda_1}\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1} f_j(s, u^m(s;\alpha,\chi,\lambda))ds\Bigg].
\end{aligned} \tag{35}
$$

In the following section, we analyze the solvability of the "model-type" problems (13), (14). In particular, we use topological degree theory to show the existence of parameters $(\alpha,\chi,\lambda) \in \Omega$, which determine the solution to each FBVP "model" problem (13) and (14) (Lemma 3, Theorem 5). We establish bounds on the approximate determining functions (34) and (35), which are required for the solvability of our problem (Lemma 5, Theorem 6). This provides the basis for a search algorithm for the parameters $(\alpha,\chi,\lambda) \in \Omega$, see Remark 3. In addition, we prove two results which estimate the distance between two limit functions $u_j^\infty(t;\alpha',\chi',\lambda')$ and $u_j^\infty(t;\alpha'',\chi'',\lambda'')$ for two different vectors $(\alpha',\chi',\lambda'), (\alpha'',\chi'',\lambda'') \in \Omega$ (Lemma 4) and the deviation between the approximate and exact solutions of each "model" FBVP (13), (14) (Theorem 7).

4. Solvability Analysis

We begin by estimating the difference between the exact and approximate determining functions, (26) and (34), and (32) and (35). This will be used along with the Brouwer topological degree theory in order to show the existence of $(\alpha,\chi,\lambda) \in \Omega$, which defines the solution of each (13), (14).

4.1. Sufficient Conditions

Lemma 3. *Suppose the conditions of Theorem 2 are satisfied. Then, for arbitrary $m \geq 1$ and $(\alpha,\chi,\lambda) \in \Omega$ for the exact and approximate determining functions $\Delta_j : \Omega \to \mathbb{R}$, $\Delta_j^m : \Omega \to \mathbb{R}$, $V_j : \Omega \to \mathbb{R}$, and $V_j^m : \Omega \to \mathbb{R}$, defined by (26), (34), (32), and (35), respectively, the following inequalities hold:*

$$
\begin{aligned}
|\Delta_j(\alpha,\chi,\lambda) - \Delta_j^m(\alpha,\chi,\lambda)| &\leq \frac{Q_j^m M_j}{1 - Q_j}, \\
|V_j(\alpha,\chi,\lambda) - V_j^m(\alpha,\chi,\lambda)| &\leq 2\frac{Q_j^m}{1 - Q_j} M_j \lambda_1^{p-1},
\end{aligned} \tag{36}
$$

where M_j, K_j, and Q_j are given in (17), (18), and (21).

Proof. Fix an arbitrary pair $(\alpha,\chi,\lambda) \in \Omega$. Then, by virtue of the Lipschitz condition (18) and the estimates (21) and (27), we have

$$
\begin{aligned}
&|\Delta_j(\alpha,\chi,\lambda) - \Delta_j^m(\alpha,\chi,\lambda)| \\
&\leq \frac{K_j}{\lambda_1^p}\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1}|u_j^m(s;\alpha,\chi,\lambda) - u_j^\infty(s;\alpha,\chi,\lambda)|ds \\
&\leq \frac{K_j p}{\lambda_1^p}\frac{\lambda_1^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1 - Q_j} M_j \int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1}ds \leq \frac{Q_j^m}{1 - Q_j} M_j.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&|V_j(\alpha,\chi,\lambda) - V_j^m(\alpha,\chi,\lambda)| \\
&\leq \frac{K_j}{\Gamma(p)}\left[(p-1)\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s))^{p-1}|u_j^m(s;\alpha,\chi,\lambda) - u_j^\infty(s;\alpha,\chi,\lambda)|ds\right. \\
&\left. + \frac{p}{\lambda_1}\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1}|u_j^m(s;\alpha,\chi,\lambda) - u_j^\infty(s;\alpha,\chi,\lambda)|ds\right] \\
&\leq \frac{K_j}{\Gamma(p)}\frac{\lambda_1^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j\left[(p-1)\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1}ds\right. \\
&\left. + \frac{p}{\lambda_1}\int_{\lambda_{j-1}}^{\lambda_j}(\lambda_j - s)^{p-1}ds\right] \leq 2\frac{Q_j^m}{1-Q_j}M_j\lambda_1^{p-1}.
\end{aligned}
$$

This proves the lemma. □

On the basis of the exact and approximate determining functions (26), (34), (32), and (35), we introduce the mappings $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Phi_m : \mathbb{R} \to \mathbb{R}$, defined by

$$
\Phi_j(\alpha,\chi,\lambda) := \begin{pmatrix} \Phi_j^1(\alpha,\chi,\lambda) \\ \Phi_j^2(\alpha,\chi,\lambda) \end{pmatrix}, \tag{37}
$$

$$
\Phi_j^m(\alpha,\chi,\lambda) := \begin{pmatrix} \Phi_j^{1,m}(\alpha,\chi,\lambda) \\ \Phi_j^{2,m}(\alpha,\chi,\lambda) \end{pmatrix}, \tag{38}
$$

with $\Phi_j^1(\alpha,\chi,\lambda)$, $\Phi_j^2(\alpha,\chi,\lambda)$, $\Phi_j^{1,m}(\alpha,\chi,\lambda)$, and $\Phi^{2,m}(\alpha,\chi,\lambda)$, defined as $\Delta_j(\alpha,\chi,\lambda)$, $V_j(\alpha,\chi,\lambda)$, $\Delta_j^m(\alpha,\chi,\lambda)$, and $V_j^m(\alpha,\chi,\lambda)$, respectively.

The following results hold.

Theorem 5. *Suppose the conditions of Theorem 2 hold, and one can find an $m \geq 1$ and a set $\Omega \subset \mathbb{R}$, such that the following relation is true:*

$$
|\Phi_j^m| \rhd_{\partial\Omega_j} \begin{pmatrix} Q_j^m M_j(1-Q_j)^{-1} \\ 2Q_j^m M_j(1-Q_j)^{-1}\lambda_1^p \end{pmatrix}, \tag{39}
$$

where $\partial\Omega_j$ is the boundary of the set Ω_j, and the definition of the relation $\rhd$ is given in [22]. If the Brouwer degree of the mapping Φ_m satisfies

$$
deg(\Phi_j^m, \Omega_j, 0) \neq 0, \tag{40}
$$

then there exists a triple $(\alpha_j^*, \chi_j^*, \lambda^*) \in \Omega_j$, such that

$$
u_j^*(t) = u_j^*(t;\alpha_j^*,\chi_j^*,\lambda^*) = \lim_{m\to\infty} u_j^m(t;\alpha_j^*,\chi_j^*,\lambda^*) \tag{41}
$$

is the solution to the nonlinear FIBVP (13), (14), defined on $J^* := [\lambda_{j-1}^*, \lambda_j^*]$, which satisfies

$$
u_j^*(\lambda_j^*) = \alpha_j^*. \tag{42}
$$

Proof. We first show that the vector fields Φ and Φ_m are homotopic. Let us introduce the family of vector mappings for $\theta \in [0,1]$

$$P(\theta,\alpha,\chi,\lambda) = \Phi_j^m(\alpha,\chi,\lambda) + \theta[\Phi_j(\alpha,\chi,\lambda) - \Phi_j^m(\alpha,\chi,\lambda)], \ \ (\alpha,\chi,\lambda) \in \partial\Omega, \tag{43}$$

Then, $P(\theta,\alpha,\chi,\lambda)$ is continuous for all $(\alpha,\chi,\lambda) \in \partial\Omega, \ \theta \in [0,1]$. We have

$$P(0,\alpha,\chi,\lambda) = \Phi_j^m(\alpha,\chi,\lambda), \ \ P(1,\alpha,\chi,\lambda) = \Phi_j(\alpha,\chi,\lambda)$$

and for any $(\alpha,\chi,\lambda) \in \partial\Omega$,

$$\begin{aligned} |P(\theta,\alpha,\chi,\lambda)| =& |\Phi_j^m(\alpha,\chi,\lambda) + \theta[\Phi_j(\alpha,\chi,\lambda) - \Phi_j^m(\alpha,\chi,\lambda)]| \\ \geq& |\Phi_j^m(\alpha,\chi,\lambda)| - |\Phi_j(\alpha,\chi,\lambda) - \Phi_j^m(\alpha,\chi,\lambda)|. \end{aligned} \tag{44}$$

From the other side, by virtue of (37), (38), and the relations in (36), we have

$$|\Phi_j(\alpha,\chi,\lambda) - \Phi_j^m(\alpha,\chi,\lambda)| \leq \begin{pmatrix} Q_j^m M_j (1-Q_j)^{-1} \\ 2Q_j^m M_j (1-Q_j)^{-1} \lambda_1^p, \end{pmatrix}. \tag{45}$$

From (39), (44), and (45), it follows that

$$|P(\theta,\alpha,\chi,\lambda)| \rhd_{\partial\Omega} 0, \ \ \theta \in [0,1],$$

which means that $P(\theta,\alpha,\chi,\lambda) \neq 0$ for all $\theta \in [0,1]$ and $(\alpha,\chi,\lambda) \in \Omega$, i.e., the mappings (43) are nondegenerate, and thus the vector fields Φ_j and Φ_j^m are homotopic. Since relation (40) holds and the Brouwer degree is preserved under homotopies, it follows that

$$\deg(\Phi_j,\Omega,0) = \deg(\Phi_j^m,\Omega,0) \neq 0,$$

which implies that there exists $(\alpha_j^*,\chi_j^*,\lambda^*) \in \Omega$ such that $\Phi_j(\alpha_j^*,\chi_j^*,\lambda^*) = 0$ by the classical topological result in [45].

Hence, the triple $(\alpha_j^*,\chi_j^*,\lambda^*)$ satisfies the determining system (31).

By Theorem 4, it follows that the function defined in (41) is a solution to the FBVP (13), (14) and satisfies (42). □

The following lemma gives the "closeness" of the limit functions $u_j^\infty(t;\alpha',\chi',\lambda')$ and $u_j^\infty(t;\lambda'',\chi'',\lambda'')$ for two different sets of parameters $(\alpha',\chi',\lambda'), (\alpha'',\chi'',\lambda'') \in \Omega$.

Lemma 4. *Suppose that the conditions of Theorem 2 are satisfied for an FIBVP (13), (14) with parameter-dependent boundary conditions. Then, for arbitrary pairs*

$$(\alpha',\chi',\lambda'), \ (\alpha'',\chi'',\lambda'') \in \Omega,$$

the limit functions $u_j^\infty(t;\alpha',\chi',\lambda')$ *and* $u_j^\infty(t;\alpha'',\chi'',\lambda'')$ *of the sequences* $u_j^m(t;\alpha',\chi',\lambda')$ *and* $u_j^m(t;\alpha'',\chi'',\lambda'')$ *of the form (15) satisfy the inequality*

$$|u_j^\infty(t;\alpha',\chi',\lambda') - u_j^\infty(t;\alpha'',\chi'',\lambda'')| \leq \frac{1}{1-Q_j}\left(L_j + \frac{4M_j(\lambda_1')^p}{\Gamma(p+1)}\right), \tag{46}$$

where Q is defined in (21), and

$$L_j := L_j(\alpha'_{j-1,j}, \chi'_{j-1}, \lambda', \alpha''_{j-1,j}, \chi''_{j-1}, \lambda'')$$
$$= |\alpha'_{j-1} - \alpha''_{j-1}| + \lambda'|\chi'_{j-1} - \chi''_{j-1}| + \gamma^p(|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1| + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|),$$
$$\lambda_{max} := \max(\lambda', \lambda'').$$

Proof. Let us first estimate the difference $|u_j^m(t;\alpha',\chi',\lambda') - u_j^m(t;\alpha'',\chi'',\lambda'')|$. Consider first $m = 0$:

$$|u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha',\chi',\lambda'')| \le |\alpha'_{j-1} - \alpha''_{j-1}| + t|\chi'_{j-1} - \chi''_{j-1}|$$
$$+ \left(\frac{t}{\lambda'\lambda''_1}\right)^p [|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1|(\lambda'')^p + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|(\lambda')^p]$$

Assume without loss of generality that $\lambda_{max} = \lambda'$. Then,

$$|u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha'',\chi'',\lambda'')| \le |\alpha'_{j-1} - \alpha''_{j-1}| + \lambda'|\chi'_{j-1} - \chi''_{j-1}|$$
$$+ \left(\frac{\lambda'}{\lambda''_1}\right)^p (|\alpha'_j - \alpha'_{j-1} - \chi'_{j-1}\lambda'_1| + |\alpha''_j - \alpha''_{j-1} - \chi''_{j-1}\lambda''_1|) = L_j \tag{47}$$

Next, using (17), (18), (47), and the results of Lemmas 1 and 2 in [23], we obtain for $m = 1$:

$$|u_j^1(t;\alpha',\chi',\lambda') - u_j^1(t;\alpha'',\chi'',\lambda'')| \le |u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha'',\chi'',\lambda'')|$$
$$+ \frac{K_j}{\Gamma(p)} \int_{\lambda'_{j-1}}^{t} (t-s)^{p-1} |u^0(s;\alpha',\chi',\lambda') - u^0(s;\alpha'',\chi'',\lambda'')| ds$$
$$- \left(\frac{t - \lambda'_{j-1}}{\lambda'_1}\right)^p \int_{\lambda'_{j-1}}^{\lambda'_{j-1}} (\lambda'_j - s)^{p-1} |u^0(s;\alpha',\chi',\lambda') - u^0(s;\alpha'',\chi'',\lambda'')| ds$$
$$+ \frac{M_j}{\Gamma(p)} \left[\left(\frac{t - \lambda'_{j-1}}{\lambda''_1}\right)^p \int_{\lambda''_{j-1}}^{\lambda''_j} (\lambda''_j - s)^{p-1} ds \right.$$
$$\left. + \left(\frac{\lambda''_{j-1} - t}{\lambda'_1}\right)^p \int_{\lambda'_{j-1}}^{\lambda'_j} (\lambda'_j - s)^{p-1} ds + \int_{\lambda''_{j-1}}^{\lambda'_{j-1}} (t-s)^{p-1} ds \right]$$
$$\le L_j + L_j K_j \frac{(\lambda'_1)^p}{2^{2p-1}\Gamma(p+1)} + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)} = L_j + L_j Q_j + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}.$$

We will use induction to show that the following estimate holds for m:

$$|u_j^m(t;\alpha',\chi',\lambda') - u_j^m(t;\alpha'',\chi'',\lambda'')| \le L_j \sum_{i=0}^{m} Q_j^i + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)} \sum_{i=0}^{m-1} Q_j^i. \tag{48}$$

Assume that (48) holds for $m-1$, i.e.

$$|u_j^{m-1}(t;\alpha',\chi',\lambda') - u_j^{m-1}(t;\alpha'',\chi'',\lambda'')| \le L_j \sum_{i=0}^{m-1} Q_j^i + \frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)} \sum_{i=0}^{m-2} Q_j^i,$$

and consider

$$
\begin{aligned}
|u_j^m(t;\alpha',\chi',\lambda') - u_j^m(t;\alpha'',\chi'',\lambda'')| &\le |u_j^0(t;\alpha',\chi',\lambda') - u_j^0(t;\alpha'',\chi'',\lambda'')| \\
&+\frac{K_j}{\Gamma(p)}\int_{\lambda'_{j-1}}^{t}(t-s)^{p-1}|u^{m-1}(s;\alpha',\chi',\lambda') - u^{m-1}(s;\alpha'',\chi'',\lambda'')|ds \\
&-\Big(\frac{t-\lambda'_{j-1}}{\lambda'_1}\Big)^p\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j-s)^{p-1}|u^{m-1}(s;\alpha',\chi',\lambda') - u^{m-1}(s;\alpha'',\chi'',\lambda'')|ds \\
&+\frac{M_j}{\Gamma(p)}\Bigg[\Big(\frac{t-\lambda''_{j-1}}{\lambda''_1}\Big)^p\int_{\lambda''_{j-1}}^{\lambda''_j}(\lambda''_j-s)^{p-1}ds \\
&+\Big(\frac{t-\lambda'_{j-1}}{\lambda'_1}\Big)^p\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j-s)^{p-1}ds+\int_{\lambda''_{j-1}}^{\lambda'_{j-1}}(t-s)^{p-1}ds\Bigg] \\
&\le L_j+\Bigg[L_j\sum_{i=0}^{m-1}Q_j^i+\frac{4M_j(\lambda'_1)^p)}{\Gamma(p+1)}\sum_{i=0}^{m-2}Q_j^i\Bigg]K_j\frac{(\lambda'_1)^p}{2^{2p-1}\Gamma(p+1)}+\frac{4M_j(\lambda')^p)}{\Gamma(p+1)N^p} \\
&= L_j\sum_{i=0}^{m}Q_j^i+\frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i,
\end{aligned}
$$

that is, (48) holds. Passing to the limit $m\to\infty$ in (48) and using (21) yields (46), as required. □

4.2. Necessary Conditions

Next, we prove the following Lemma which will be used to establish an upper bound for (26) and (32), required for the existence of parameters $(\alpha,\chi,\lambda)\in\Omega$, which determines the solution $u_j(t;\alpha,\chi,\lambda)$ of each of the problems (13), (14).

Lemma 5. *Suppose the conditions of Theorem 2 are satisfied. Then, the functions $\Delta:\Omega\to\mathbb{R}$ and $V:\Omega\to\mathbb{R}$ satisfy the following estimates for arbitrary pairs (α',χ',λ'), $(\alpha'',\chi'',\lambda'')\in\Omega$:*

$$
\begin{aligned}
|\Delta_j(\alpha',\chi',\lambda') - \Delta_j(\alpha'',\chi'',\lambda'')| &\le \frac{\Gamma(p+1)}{(\lambda''_1)^p}(|\alpha'_{j-1}-\alpha'_j-\chi'_{j-1}\lambda'_1| \\
&+|\alpha''_{j-1}-\alpha''_j-\chi''_{j-1}\lambda''_1|)+\frac{2K_j}{1-Q_j}\Big(L_j+\frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}\Big)+2M_j
\end{aligned} \tag{49}
$$

and

$$
\begin{aligned}
|V_j(\alpha',\chi',\lambda')) - V_j(\alpha'',\chi'',\lambda'')| &\le \frac{j}{N}|\lambda'\chi'_{j-1}-\lambda''\chi''_{j-1}| \\
&+\frac{p}{\lambda''_1}(|\alpha'_j-\alpha'_{j-1}-\chi'_{j-1}\lambda'_1|+|\alpha''_j-\alpha''_{j-1}-\chi''_{j-1}\lambda''_1|) \\
&+\frac{4K_j(\lambda'_1)^{p-1}}{1-Q_j}\Big(L_j+\frac{4M_j(\lambda'_1)^p}{\Gamma(p+1)}\Big)+4(\lambda'_1)^{p-1}M_j
\end{aligned} \tag{50}
$$

Proof. By virtue of the definition of $\Delta(z,\lambda)$ in (26), the boundedness and Lipschitz-continuity of $f(t,u(t))$ (17), (18), and the estimate in Lemma 4, we obtain

$$
\begin{aligned}
&|\Delta_j(\alpha',\chi',\lambda') - \Delta_j(\alpha'',\chi'',\lambda'')| \\
&\le \Gamma(p+1)\Bigg[\frac{|\alpha'_{j-1}-\alpha'_j-\chi'_{j-1}\lambda'_1|}{(\lambda'_1)^p}+\frac{|\alpha''_{j-1}-\alpha''_j-\chi''_{j-1}\lambda''_1|}{(\lambda''_1)^p}\Bigg] \\
&+p\Bigg[\Bigg|\frac{1}{(\lambda'_1)^p}\int_{\lambda'_{j-1}}^{\lambda'_j}(\lambda'_j-s)^{p-1}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))-f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]ds\Bigg|
\end{aligned}
$$

$$+\left|\frac{1}{(\lambda_1'')^p}\int_{\lambda_{j-1}''}^{\lambda_j''}(\lambda_j''-s)^{p-1}[f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))-f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]ds\right|$$
$$+\left|\frac{1}{(\lambda_1')^p}\int_{\lambda_{j-1}'}^{\lambda_j'}(\lambda_j'-s)^{p-1}f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))ds\right|$$
$$+\left|\frac{1}{(\lambda_1')^p}\int_{\lambda_{j-1}''}^{\lambda_j''}(\lambda_j''-s)^{p-1}f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))ds\right|\Bigg]$$
$$\leq\frac{\Gamma(p+1)}{(\lambda_1'')^p}(|\alpha_{j-1}'-\alpha_j'-\chi_{j-1}'\lambda_1'|+|\alpha_{j-1}''-\alpha_j''-\chi_{j-1}''\lambda_1''|)$$
$$+\frac{2K_j}{1-Q_j}\left(L_j+\frac{4M_j(\lambda_1')^p}{\Gamma(p+1)}\right)+2M_j,$$

as required in (49).

Now, from the definition of $V(\alpha,\chi,\lambda)$, (32), we derive:

$$|V_j(\alpha',\chi',\lambda')-V_j(\alpha'',\chi'',\lambda'')|\leq|j\lambda_1'\chi_{j-1}'+\frac{p}{\lambda_1'}(\alpha_j'-\alpha_{j-1}'-\chi_{j-1}'\lambda_1')$$
$$-j\lambda_2''\chi_{j-1}''-\frac{p}{\lambda_1''}(\alpha_j''-\alpha_{j-1}''-\chi_{j-1}''\lambda_1'')|$$

$$+\frac{1}{\Gamma(p)}\Bigg[(p-1)\left|\int_{\lambda_{j-1}'}^{\lambda_j'}(\lambda_j'-s)^{p-2}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))-f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]\right|$$
$$+\frac{p}{\lambda_1'}\left|\int_{\lambda_{j-1}'}^{\lambda_j'}(\lambda_j'-s)^{p-2}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))-f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))]\right|$$
$$+(p-1)\left|\int_{\lambda_{j-1}''}^{\lambda_j''}(\lambda_j''-s)^{p-2}[f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))-f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))]\right|$$
$$+\frac{p}{\lambda_1''}\left|\int_{\lambda_{j-1}''}^{\lambda_j''}(\lambda_j''-s)^{p-2}[f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))-f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))]\right|$$
$$+(p-1)\left|\int_{\lambda_{j-1}'}^{\lambda_j'}(\lambda_j'-s)^{p-2}f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))ds\right|$$
$$+\frac{p}{\lambda_1'}\left|\int_{\lambda_{j-1}'}^{\lambda_j'}(\lambda_j'-s)^{p-2}[f_j(s,u_j^\infty(s;\alpha',\chi',\lambda'))ds\right|$$
$$+(p-1)\left|\int_{\lambda_{j-1}''}^{\lambda_j''}(\lambda_j''-s)^{p-2}f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))ds\right|$$
$$+\frac{p}{\lambda_1''}\left|\int_{\lambda_{j-1}''}^{\lambda_j''}(\lambda_j''-s)^{p-2}[f_j(s,u_j^\infty(s;\alpha'',\chi'',\lambda''))ds\right|\Bigg]$$
$$\leq\frac{j}{N}|\lambda'\chi_{j-1}'-\lambda''\chi_{j-1}''|+\frac{p}{\lambda_1''}(|\alpha_j'-\alpha_{j-1}'-\chi_{j-1}'\lambda_1'|+|\alpha_j''-\alpha_{j-1}''-\chi_{j-1}''\lambda_1''|)$$
$$+\frac{4K_j(\lambda_1')^{p-1}}{1-Q_j}\left(L_j+\frac{4M_j(\lambda_1')^p}{\Gamma(p+1)}\right)+4(\lambda_1')^{p-1}M_j$$

This proves the lemma. □

Theorem 6. *Suppose the conditions of Theorem 2 are satisfied. Then, in order for the domain Ω to contain a pair of parameters $(\alpha^*,\chi^*,\lambda^*)$, it is necessary that for all $m\geq 1$, $(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})\in\Omega$. the following inequalities hold:*

$$|\Delta_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \leq \sup_{(\alpha,\chi,\lambda)\in\Omega} \left\{ \frac{\Gamma(p+1)}{\lambda_1^p}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| \right.$$

$$\left. +|\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) + \frac{2K_j}{1-Q_j}\left(\tilde{L}_{j,2} + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) \right\} + 2M_j + \frac{Q_j^m M_j}{1-Q_j}, \quad (51)$$

$$|V_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \leq \sup_{(\alpha,\chi,\lambda)\in\Omega} \left\{ \frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda\chi_{j-1}| \right.$$

$$+\frac{p}{\lambda_1}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|)$$

$$\left. +\frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda}_1)^{p-1}M_j \right\} + 2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda}_1)^{p-1} \quad (52)$$

where

$$\tilde{L}_j := L_j(\tilde{\alpha}_{j-1,j}, \tilde{\chi}_{j-1}, \tilde{\lambda}, \alpha^*_{j-1,j}, \chi^*_{j-1}, \lambda^*),$$
$$\tilde{L}_{j,2} := L_j(\tilde{\alpha}_{j-1,j}, \tilde{\chi}_{j-1}, \tilde{\lambda}, \alpha_{j-1,j}, \chi_{j-1}, \lambda).$$

Proof. Assume that the determining functions vanish at $\alpha = \alpha^*, \chi = \chi^*, \lambda = \lambda^*$, that is, $\Delta_j(\alpha^*,\chi^*,\lambda^*) = 0$ and $V_j(\alpha^*,\chi^*,\lambda^*) = 0$. Applying Lemma 5 with $(\alpha',\chi',\lambda') = (\tilde{\alpha},\tilde{\chi},\tilde{\lambda})$ and $(\alpha'',\chi'',\lambda'') = (\alpha^*,\chi^*,\lambda^*)$ yields

$$|\Delta_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda}) - \Delta_j(\alpha^*,\chi^*,\lambda^*)| = |\Delta_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})|$$
$$\leq \frac{\Gamma(p+1)}{(\lambda_1^*)^p}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j^* - \alpha_{j-1}^* - \chi_{j-1}^*\lambda_1^*|) + \frac{2K_j}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 2M_j.$$

From Lemma 3, we know that

$$|\Delta_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda}) - \Delta_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \leq \frac{Q_j^m M_j}{1-Q_j}.$$

Hence,

$$|\Delta_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \leq |\Delta_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| + |\Delta_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda}) - \Delta_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})|$$
$$\leq \frac{\Gamma(p+1)}{(\lambda_1^*)^p}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j^* - \alpha_{j-1}^* - \chi_{j-1}^*\lambda_1^*|)$$
$$+\frac{2K_j}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 2M_j + \frac{Q_j^m M_j}{1-Q_j}$$
$$\leq \sup_{(\alpha,\chi,\lambda)\in\Omega} \left\{ \frac{\Gamma(p+1)}{\lambda_1}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) \right.$$
$$\left. +\frac{2K_j}{1-Q_j}\left(\tilde{L}_{j,2} + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) \right\} + 2M_j + \frac{Q_j^m M_j}{1-Q_j},$$

as stated in (51). Applying again Lemma 5, now to $V_j(\alpha,\chi,\lambda)$, we have

$$|V_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda}) - V_j(\alpha^*,\chi^*,\lambda^*)| = |V_j(\tilde{\alpha},\tilde{\chi}\tilde{\lambda})|$$
$$\leq \frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda^*\chi^*_{j-1}| + \frac{p}{\lambda_1^*}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j^* - \alpha^*_{j-1} - \chi^*_{j-1}\lambda_1^*|)$$
$$+\frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda})^{p-1}M_j$$

From Lemma 3, we know that

$$|V_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda}) - V_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \leq 2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda})^{p-1},$$

thus, combining the two yields

$$|V_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \leq |V_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| + |V_j^m(\tilde{\alpha},\tilde{\chi},\tilde{\lambda}) - V_j(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})|$$
$$\leq \frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda^*\chi^*_{j-1}| + \frac{p}{\lambda_1^*}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1| + |\alpha_j^* - \alpha^*_{j-1} - \chi^*_{j-1}\lambda_1^*|)$$
$$+\frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda}_1)^{p-1}M_j + 2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda}_1)^{p-1}$$
$$\leq \sup_{(\alpha,\chi,\lambda)\in\Omega}\left\{\frac{j}{N}|\tilde{\lambda}\tilde{\chi}_{j-1} - \lambda\chi_{j-1}| + \frac{p}{\lambda_1}(|\tilde{\alpha}_j - \tilde{\alpha}_{j-1} - \tilde{\chi}_{j-1}\tilde{\lambda}_1|\right.$$
$$\left.+|\alpha_j - \alpha_{j-1} - \chi_{j-1}\lambda_1|) + \frac{4K_j(\tilde{\lambda}_1)^{p-1}}{1-Q_j}\left(\tilde{L}_j + \frac{4M_j(\tilde{\lambda}_1)^p}{\Gamma(p+1)}\right) + 4(\tilde{\lambda}_1)^{p-1}M_j\right\}$$
$$+2\frac{Q_j^m}{1-Q_j}M_j(\tilde{\lambda}_1)^{p-1}.$$

This proves the theorem. □

Remark 3. *On the basis of Theorem 6, we can establish an algorithm of approximate search for the set of 2N parameters* $(\alpha_1^*,...,\alpha_N^*,\chi_1^*,...,\chi_{N-1}^*,\lambda^*)$*, which define the solution* $u(\cdot)$ *of the FBVP (9)–(11). Let* $\Omega = \cup_{j=1}^N\Omega_j$ *and let us represent the product of open sets* $\Omega \subset \mathbb{R}\times\mathbb{R}\times\mathbb{R}$ *as the finite union of subsets* Ω^i $(\Omega^i = D^i \times X^i \times \Lambda^i)$*:*

$$\Omega = \cup_{i=1}^N\Omega^i. \tag{53}$$

In each subset Ω^i*, we pick a set* $(\alpha_1^i,...,\alpha_N^i,\chi_1^i,...,\chi_{N-1}^i,\lambda^i)$ *and calculate the approximate solution* $u_j^m(t;\alpha_1^i,...,\alpha_j^i,\chi_1^i,...,\chi_j^i,\lambda^i)$ *on each subinterval* J_j *using the recurrence Formula (15). Then, we find the values of the determining functions* $\Delta_j^m(\alpha_1^i,...,\alpha_j^i,\chi_1^i,...,\chi_j^i,\lambda^i)$ *and* $V_j^m(\alpha_1^i,...,\alpha_j^i,\chi_1^i,...,\chi_j^i,\lambda^i)$*, according to (34) and (35), and exclude from (53) subsets* Ω^i *for which the inequality does not hold. According to Theorem 6, these subsets cannot contain a set* $(\alpha_1^i,...,\alpha_N^i,\chi_1^i,...,\chi_{N-1}^i,\lambda^i)$ *that determines the solution* $u(\cdot)$*. The remaining subsets* $\Omega^{i_1},...,\Omega^{i_s}$ *form a set* $\Omega^{m,k}$*, such that only* $(\tilde{\alpha}_1,...,\tilde{\alpha}_N,\tilde{\chi}_1,...,\tilde{\chi}_{N-1},\tilde{\lambda}) \in \Omega^{m,k}$ *can determine* $u(\cdot)$*. As* $k,m\to\infty$*, the set* $\Omega^{m,k}$ *"follows" the set* Ω*, which may contain a set* $(\alpha_1^*,...,\alpha_N^*,\chi_1^*,...,\chi_{N-1}^*,\lambda^*)$ *and defines a solution to (13), (14). Each set* $(\tilde{\alpha}_1,...,\tilde{\alpha}_N,\tilde{\chi}_1,...,\tilde{\chi}_{N-1},\tilde{\lambda})$ *can be seen as an approximation of* $(\alpha_1^*,...,\alpha_N^*,\chi_1^*,...,\chi_{N-1}^*,\lambda^*)$*, which determines solution of the FBVP (13), (14). It is clear that*

$$|\tilde{\alpha}_j - \alpha_j^*| \leq \sup_{\alpha\in D^{m,k}}|\tilde{\alpha}_j - \alpha_j|, \quad |\tilde{\chi}_j - \chi_j^*| \leq \sup_{\chi\in X^{m,k}}|\tilde{\chi}_j - \chi_j|, \quad |\tilde{\lambda} - \lambda^*| \leq \sup_{\lambda\in\Lambda^{m,k}}|\tilde{\lambda} - \lambda|,$$

and the function $u_m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})$, given by (16), where each $u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})$ is calculated using the iterative formula (15), can be seen as an approximate solution to the FIBVP (9)–(11).

Finally, we estimate the deviation between the exact solution to the FBVP (13), (14), $u_j^\infty(t;\alpha^*,\chi^*,\lambda^*)$, and its approximate solution $u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})$.

Theorem 7. *Suppose the conditions of Theorem 2 hold, and the pair $(\alpha^*,\chi^*,\lambda^*)\in\Omega$ is a solution to the exact determining system (31), and $(\tilde{\alpha},\tilde{\chi},\tilde{\lambda})$ is an arbitrary point in $\Omega_{m,N}$. Then, the following estimate holds*

$$\begin{aligned}|u_j^\infty(t;\alpha^*,\chi^*,\lambda^*)-u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \le{} & \frac{(\lambda_1^*)^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j \\ & + \sup_{(\alpha,\chi,\lambda)\in\Omega_{m,N}}\left[\tilde{L}_j\sum_{i=0}^{m}Q_j^i+\frac{4M_j(\lambda_1)^p)}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i\right].\end{aligned} \tag{54}$$

Proof. From the estimates in (27) and (48), we have

$$\begin{aligned}|u_j^\infty(t;\alpha^*,\chi^*,\lambda^*)-u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| &\le |u_j^\infty(t;\alpha^*,\chi^*,\lambda^*)-u_j^m(t;\alpha^*,\chi^*,\lambda^*)| \\ &\quad +|u_j^m(t;\alpha^*,\chi^*,\lambda^*)-u_j^m(t;\tilde{\alpha},\tilde{\chi},\tilde{\lambda})| \\ &\le \frac{(\lambda_1^*)^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Qj}M_j+L_j\sum_{i=0}^{m}Q_j^i+\frac{4M_j(\lambda_1')^p}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i \\ &\le \frac{(\lambda_1^*)^p}{2^{2p-1}\Gamma(p+1)}\frac{Q_j^m}{1-Q_j}M_j+\sup_{(\alpha,\chi,\lambda)\in\Omega_{m,N}}\left[\tilde{L}_j\sum_{i=0}^{m}Q_j^i+\frac{4M_j(\lambda_1)^p}{\Gamma(p+1)}\sum_{i=0}^{m-1}Q_j^i\right].\end{aligned}$$

This proves the theorem. □

5. Example

In this section, we apply the numerical-analytic method from Section 3 to a model example in a finite interval setting.

Let us consider the differential equation

$${}_0^C D_t^p u(t) = \frac{1}{(\cosh t)^2}F(u(t)) - \frac{2\omega\sinh t}{(\cosh t)^3} \;\; (:= f(t,u(t))),\;\; t\in[0,\lambda], \tag{55}$$

for two values of p, $p=2,\ 3/2$, and subject to parameter-dependent boundary conditions

$$\begin{aligned}&u(0)=1000, \quad u'(0)=1500, \\ &u(0)+u(\lambda)+u'(\lambda)=1000,\end{aligned} \tag{56}$$

i.e., $A=B=C=1$, $d=1000$, $\alpha_0=1000$, $\chi_0=1500$, and ω is given.

Remark 4. *The function $f(t,u(t))$ in Equation (55) is an example of a nonlinear function satisfying the conditions of Theorem 1.*

Let the nonlinearity in Equation (55) be $F(u(t))=\sin(u(t))/10$. For simplicity of computations, we construct an approximating sequence directly on the entire interval $[0,\lambda]$. However, it is possible to apply the interval-splitting method.

When $p=2$, Equation (55), coupled with asymptotic conditions of the type (2), is derived as a mathematical model of Arctic gyres with a vanishing azimuthal velocity and oceanic vorticity $F(u(t))$. Then, $\omega=4649.56$ is taken as the dimensionless Coriolis parameter [36], and $u(t)$ is a stream function representing radially symmetric solutions.

The first boundary condition in (2) can be interpreted as assigning some constant ϕ_0 to the stream function at the North Pole, and the second boundary condition means that the flow is stagnant at the North Pole, see [46,47] for details.

In this case, the FIBVP (55), (56) is considered on the domain

$$D := \{530 \leq u(t) \leq 1952\}, \quad t \in [0, \lambda].$$

The right-hand side function $f(t, u(t))$ satisfies the Lipschitz condition (18) with constant $K = 0.1$.

Implementing (15) and solving the corresponding system of approximate determining equations for six iterations yields the parameter values shown in Table 1.

Table 1. Computed parameter values of α_1^m and λ_m for $m = 0, \ldots, 5$, $p = 2$.

m	α_1^m	λ_m
0	1182.4028437371185	1.148642801991218
1	1331.576257823826	1.0473603198657913
2	1331.5762578223805	1.0473603198648302
3	1331.57625782238	1.0473603198648305
4	1331.5762578223798	1.0473603198648311
5	1331.57625782238	1.0473603198648307

It is clear from these calculations that the values we obtain for α_1^m and λ_m converge to the exact parameter values. With the computed value of $\lambda \approx 1.05$, we find that $Q \approx 0.01$, i.e., the inequality in (21) is satisfied, which guarantees the convergence of the approximating sequence. Plots of the first 6 terms of the sequence are shown in the left panel of Figure 1.

The right panel of the same figure shows a comparison between the left- and right-hand sides of Equation (55) with $u_5(t; \alpha_1, \lambda)$ plugged in. From this comparison we see that the left- and right-hand sides of Equation (55) are in good agreement for $m = 5$. The error between the last two iterations, defined as

$$E = |u_5(t; \alpha_1, \lambda) - u_4(t; \alpha_1, \lambda)|, \tag{57}$$

is $E = 5.2 \times 10^{-12}$. If necessary, the iteration process can be continued until the desired precision of computation is obtained.

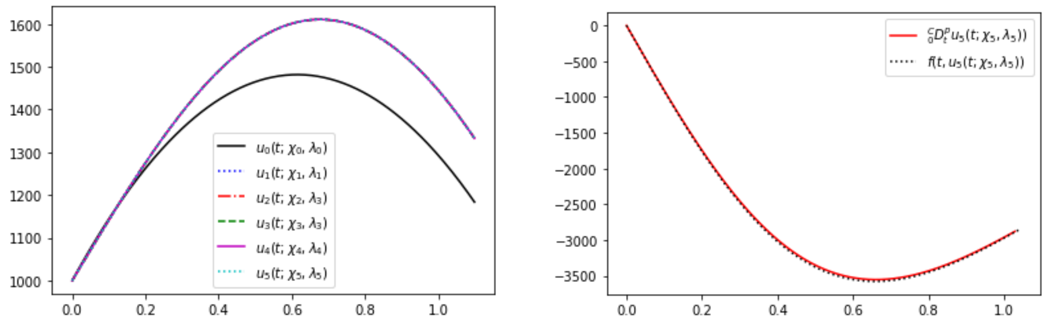

Figure 1. Plots of the first 6 terms of the sequence in (15) (left panel) and of the left- and right-hand sides of Equation (55) with the last term $u_5(t; \alpha_1, \lambda)$ plugged in (right panel) for $p = 2$.

When $p = 3/2$, we consider the FIBVP (55), (56) on the domain

$$D := \{-29073.12 \leq u(t) \leq 33187.76\}, \quad t \in [0, \lambda].$$

As before, the sequence (15) is implemented and the corresponding system of approximate determining equations is solved for six iterations. The obtained parameter values shown in Table 2, from which it is clear that α_1^m and λ_m converge to the exact values.

Table 2. Computed parameter values of α_1^m and λ_m for $m = 0, ..., 5$, $p = 3/2$.

m	α_1^m	λ_m
0	909.1619919658109	0.8560901404160095
1	−546.1015115169015	8.602557635546512
2	−546.101514252212	8.60255774462411
3	−546.1015142479218	8.602557744453811
4	−546.1015142479329	8.602557744454153
5	−546.1015142479316	8.602557744454158

Plots of the first six terms of the sequence are shown in the left panel of Figure 2 and the right panel of the same figure shows a comparison between the left- and right-hand sides of Equation (55) with $u_5(t; \alpha_1, \lambda)$ plugged in. From this, we see that the left- and right-hand sides of Equation (55) are in good agreement for $m = 5$. The error between the last two iterations, defined as

$$E = |u_5(t; \alpha_1, \lambda) - u_4(t; \alpha_1, \lambda)|, \tag{58}$$

is $E = 1.7 \times 10^{-9}$.

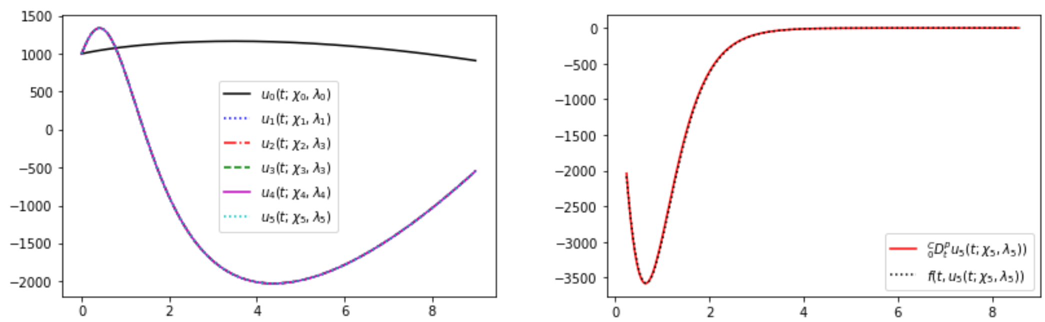

Figure 2. Plots of the first 6 terms of the sequence in (15) (left panel) and of the left- and right-hand sides of Equation (55), with the last term $u_5(t; \alpha_1, \lambda)$ plugged in (right panel) for $p = 3/2$.

6. Discussion

The numerical–analytic method discussed in this paper facilitates the analysis of the existence of solutions to FBVPs with parameter-dependent boundary conditions over intervals of arbitrary length. It also provides a practical framework for the construction of approximations to their solutions, enabling their visualization, which is valuable for gaining qualitative insights. Compared to numerical methods and other approximation approaches, such as, for instance, series expansion and Grünwald-Lentikov methods (see [1,48]), the technique presented here has several advantages. While purely numerical methods rely on discretizations that introduce numerical errors, the numerical–analytic scheme uses a closed-form approximating sequence. This eliminates the discretization error, thus improving accuracy. In addition, the numerical–analytic method serves as a tool for analyzing FBVPs, addressing the existence and uniqueness of solutions and the solvability of the problem. Moreover, the method is computationally efficient since it only requires the numerical solution of a system of algebraic equations. Another advantage is its ability to accommodate various types of boundary constraints. In contrast to classical numerical approaches, the numerical–analytic does not require explicit knowledge or estimation of the initial conditions. When such data is unavailable, it is instead determined as the root of an algebraic equation at each iteration. Furthermore, the method's applicability can be extended beyond the three-point parameter-dependent boundary conditions considered here, encompassing Dirichlet, multipoint, nonlinear, and integral boundary conditions through a suitable parametrization.

The generalized dichotomy-based approach discussed here allows for the extension of the interval of definition of the problem and broadens the applicability of the numerical–analytic technique. While Lipschitz continuity of the right-hand side function in the FDE is required for convergence, the ability to split the interval of definition enables the method to be applied to problems where the convergence conditions are not satisfied on the entire domain. Furthermore, the interval-splitting technique can be used to enhance the speed of convergence, which depends on the length of the interval, as evidenced by the estimate in (27). This interval partitioning method also allows for extending the interval of definition of the BVP, making it well-suited for studying the long-term or asymptotic behavior of solutions to fractional-order boundary value problems, which are crucial in modeling complex physical, biological, and engineering systems.

Future work will focus on conducting a detailed comparative study of classical numerical and numerical–analytic techniques applied to differential equations involving generalized fractional-order operators. This investigation aims to assess the accuracy, efficiency, and robustness of the methods. A comprehensive report on this comparative analysis is ongoing work. For further details, the reader is referred to [49].

7. Conclusions

In this paper, we study FIBVPs in two different settings. First, we consider an FDE defined on a semi-infinite domain, subject to asymptotic conditions, in order to establish conditions for the existence of bounded solutions to the problem. In the second problem setting, we consider an FDE defined on a domain of unknown finite length, subject to initial and three-point boundary conditions. We use the numerical–analytic method in combination with an interval splitting technique to construct a sequence of approximations and prove the existence and uniqueness of solutions to the FIBVP. Furthermore, we give a detailed analysis of the solvability of the problem.

The method presented in this paper is well-suited for the analysis and approximation of solutions to nonlinear fractional boundary value problems (FBVPs) with nonfixed boundaries, which arise in various applied contexts. It is adaptable to a range of fractional operators and can accommodate various types of boundary conditions. The BVP under consideration is inherently complex due to the nonlinearity in the right-hand side function, the unknown domain length, and the parameter-dependent boundary conditions—even when classical derivatives are used. Given that the numerical–analytic technique is also applicable to the integer-order case, and that the system's behavior can vary depending on the order of the derivative, it is therefore natural to to compare the method's performance across different orders, such as $p \in (0,1]$ and $p \in (1,2]$ in future works.

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