The Kakeya Conjecture in Two Dimensions

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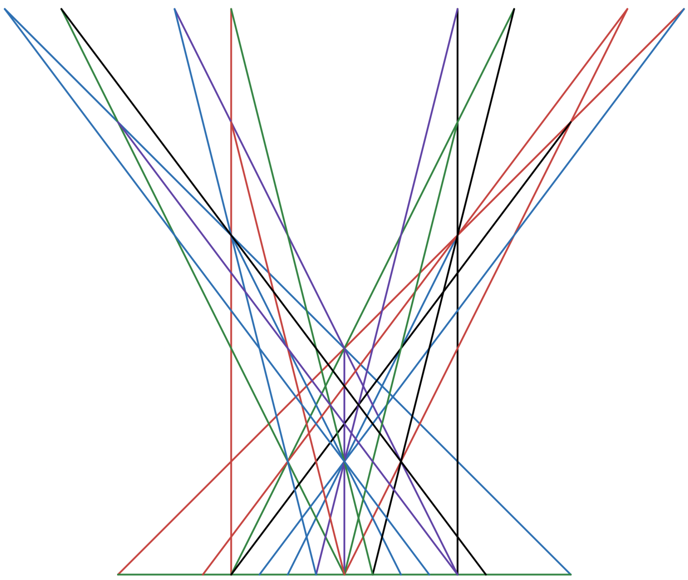

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Plain Language Summary

In this bachelor thesis, we explore a problem related to turning a needle inside a very small space. We start with some intuitive ideas, followed by a solution published by Abram Besicovitch in 1928. After that, we take a deeper look at the kinds of shapes that we can turn this needle in, and their interesting mathematical properties.

In particular, we take a look at its dimension, just like a line has one dimension, a square has two, and a cube three. With specialized mathematical tools, we can give almost any shape a dimension. We could even give the coastline of Britain a dimension. We give a proof that these shapes that allow us to turn a needle, will always have dimension 2. This result is interesting since it shows some similarity with a square, even though these shapes are in many ways very different.

Abstract

This thesis explores the Kakeya conjecture for $n = 2$, which states that every subset of $\mathbb{R}^n$ containing a unit line segment in every direction has Minkowski dimension n. To tackle this problem we explore what the Minkowski dimension is, and use the Kakeya maximal operator.

We start by stating the idea of a Kakeya needle set. We construct a sequence of these sets with arbitrarily small measure, followed by looking at Kakeya sets, which can even have measure equal to zero. We derive a proof of the Kakeya conjecture using the Kakeya maximal operator conjecture and its dual form, where we also prove all necessary implications. Resulting in a complete proof for $n = 2$ with respect to the Minkowski dimension.

Acknowledgements

I would like to thank everyone that helped me write this thesis. First off, a huge thanks to Emiel Lorist. All the weekly meetings helped immensely with digesting and understanding these unfamiliar mathematical concepts. Also a big thanks to my student friends, who helped give me sanity checks on my explanations, and were there to keep me motivated. Lastly thanks to all the great teachers i have had at TU Delft these past four years, as they all, in their own way, helped prepare me for this thesis.

Contents

1

A Needle Drenched in Ink

In this chapter we will introduce a problem from the Japanese mathematician Sōichi Kakeya. We will show some naive ideas, initial candidates for improvement, and hint at the best solution. In the following chapters, we aim to use existing literature on the topic, breaking it down in such a way that this topic becomes digestible for bachelor mathematics students in their final year.

1.1. The problem

In 1917 Sōichi Kakeya proposed a seemingly simple problem in geometry that has sparked areas of mathematics active until present day. Including work of mathematicians Hong Wang and Joshua Zahl [6], who recently released a preprint claiming to prove the Kakeya conjecture for $n = 3$. Note that the Kakeya conjecture is different from the problem that we discuss in this chapter. We formally introduce the Kakeya conjecture in a later chapter.

The problem stated by Kakeya is as follows.

> In the class of figures in which a segment of unit length can be turned around 360°, while always remaining within the figure, which one has the smallest area?

A fun way to think about this problem is to imagine an idealistic needle with unit length and 0 width. Your task is to turn this needle a full 360 degrees. However, the needle is drenched in ink, and when you turn it, any area the needle touches will be colored. Your goal is to find the movement which minimizes this area. A set in which we can perform this rotation of 360 degrees is called a Kakeya needle set.

1.2. Initial ideas

If you are just playing with this needle, something you might try is to hold down one end. While holding this end, you spin the other to make a semi circle. See Figure 1.1

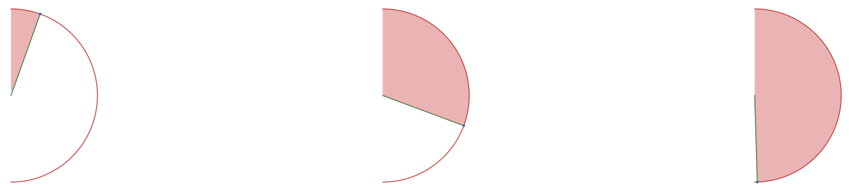


Figure 1.1: The needle and the trace of ink it leaves behind at 3 points in time (circle has radius = 1).

Now that the needle is pointing south, we have turned it 180 degrees. To turn the following 180 degrees, slide the needle back up and repeat the turn. Note that sliding the needle along a straight line leaves no area behind, since the needle has 0 width. This fact will be important next chapter. Now that we have turned the needle 360 degrees, we can calculate the area as half the area of a disc with unit radius, being $\frac{\pi}{2} \approx 1.57$.

Our next idea will turn out to already be a lot better, even though it is a similar solution. If instead of holding down the needle at one of its endpoints, we hold it in the middle, we get a disc with unit diameter. See Figure 1.2. Again a simple calculation shows its area, this time being $\frac{\pi}{4} \approx 0.79$.

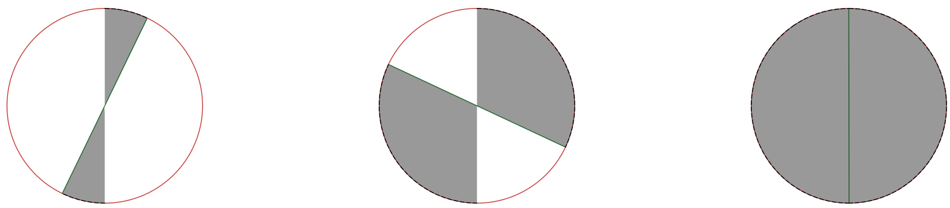


Figure 1.2: The needle and the trace of ink it leaves behind at 3 points in time (circle has radius = $\frac{1}{2}$)

The next Kakeya needle set we show is the equilateral triangle with height 1 and vertices A, B and C. The needle starts as the line starting at C, perpendicular to AB. We then turn it along a circular arc until it hits the side BC. Once the needle hits BC, we slide it along that edge until it touches B. Now, in a similar fashion, we turn it along a circular arc until it touches side AB.

Now we slide it along that edge until it touches A. Once again we turn it along a circular arc until it touches AC, slide it along that edge until it touches C, and finally turn it until it is back in its original position. We have now turned it 180 degrees, and we can simply repeat the process to turn it 180 further degrees to accomplish the desired 360 degree turn. See Figure 1.3.

Figure 1.3: Needle at starting position, and needle with corresponding ink trace after turning along the circular arc.

Now, since the height of this equilateral triangle is 1, we need some trigonometry to calculate its area. Since $\sin(60°) = \frac{1}{|BC|}$ And $|AB| = |BC|$. This leads to an area of $\frac{1}{2} \cdot \frac{1}{\sin(60°)} = \frac{\sqrt{3}}{3} \approx 0.58$ (which is less than $\frac{\pi}{4} \approx 0.79$).

The final set that we will discuss is the hypocycloid. This set is obtained by choosing a fixed point on a small circle and then tracing that point as the circle rolls along the inside of a larger circle. In our case the small circle has diameter $\frac{1}{2}$ and the large circle has diameter $\frac{3}{2}$. See Figure 1.4.

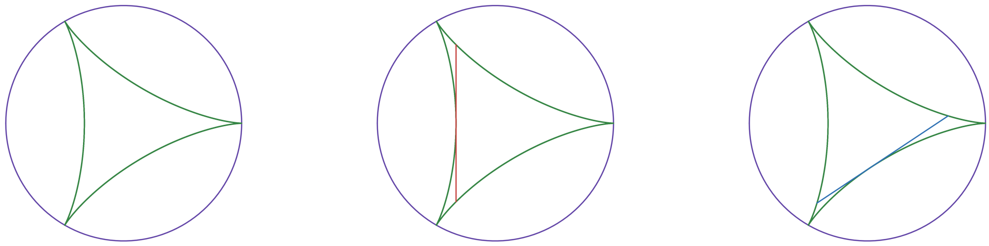


Figure 1.4: Three depictions of the hypocycloid, firstly without any needle, secondly with the needle in upright position, and lastly with the needle after moving along about a third of the hypocycloid. The blue circle around the hypocycloid is the 'large' circle with diameter $\frac{3}{2}$

Of course we are still left with determining the area of this hypocycloid. For this we note that the diameter of the outer circle is $\frac{3}{2}$ thus its radius is $\frac{3}{4}$. Then we use the formula for calculating the area of a hypocycloid which is enclosed in a circle of diameter r: $\frac{2}{9} \cdot \pi \cdot r^2$ [7, (15)] Plugging in $r = \frac{3}{4}$ we arrive at an area of $\frac{\pi}{8} \approx 0.39$. As compared to the equilateral triangle of ≈ 0.58. It was conjectured that this hypocycloid was the set of smallest area.

We have now given a brief description of the problem and provided some initial ideas for how to turn our needle. In the next chapter we make all of these ideas obsolete by introducing a construction of sets that allows arbitrarily small areas to be achieved.

2

The Besicovitch Construction

In this chapter we will construct a sequence of sets with decreasing area, where we can make a Kakeya needle set using (copies of) these sets at any point in the sequence. The construction comes from the work of Abram Besicovitch with help from Julius Pal [1]. We describe this construction, and add some nice visualizations.

2.1. Setup

We will start with a set that is even worse than all the ideas mentioned before. But it will turn out that, with enough refinements, this will still result in a set of arbitrarily small area. So, let us start with a square with side length 4.

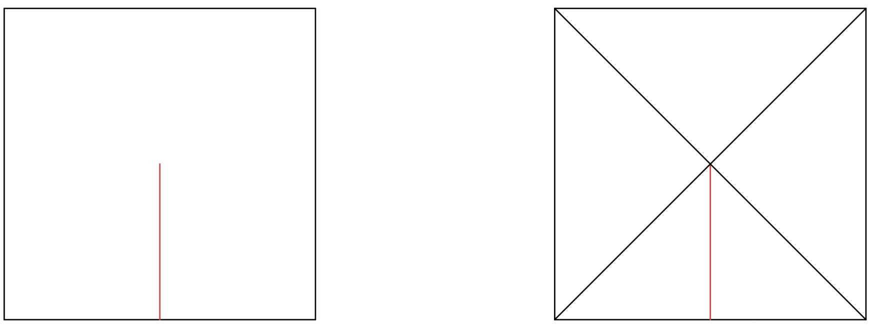


Figure 2.1: Square with sidelength 4, with needle of length 2 (a). In the second figure the square is split up into four triangular segments (b)

It should be apparent that the needle (this time of length 2) can definitely be turned 360 degrees in this figure, since any of the figures shown in Chapter 1 can fit inside this square. For the construction we will split up this square into four triangles. We will construct a sequence of sets, that starts as one of these triangles. Every set in this sequence will allow a 90 degree rotation. Using four copies of these sets we achieve the desired 360 degrees of rotation. Now let us start with the details of this construction.

2.2. Construction

The Besicovitch construction can be seen as a sequence of sets which have smaller and smaller area. To understand its construction, we will show how to, given any set in the sequence, obtain the next set in the sequence. Then we will show that this sequence indeed has the property of decreasing area that we claim.

We denote elements in our sequence as Δ_k, where element k in the sequence has height k. Now let us start with the first set in this sequence (where we start at $k = 2$), which is exactly the quarter of the square we showed earlier. Note that this triangle is isosceles, has height 2 and base of length 4.

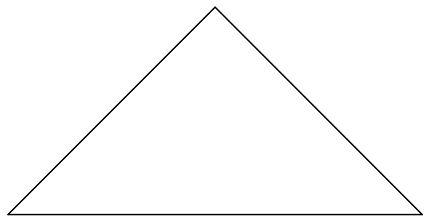
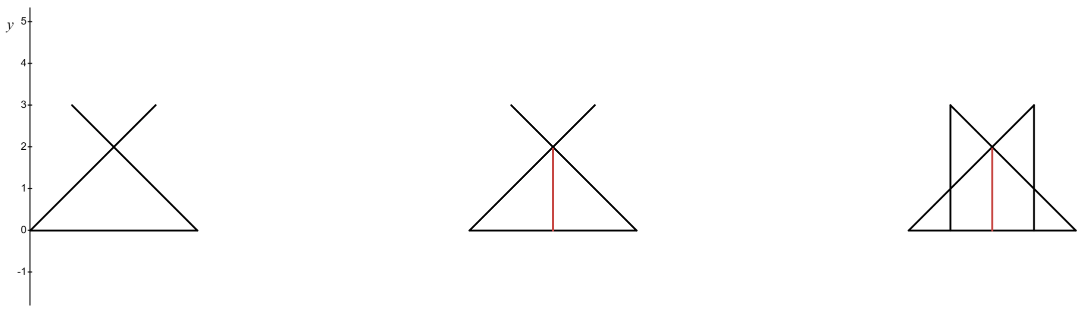


Figure 2.2: The first triangle in the Besicovitch construction, which is one quarter of the 4 by 4 square as shown earlier.

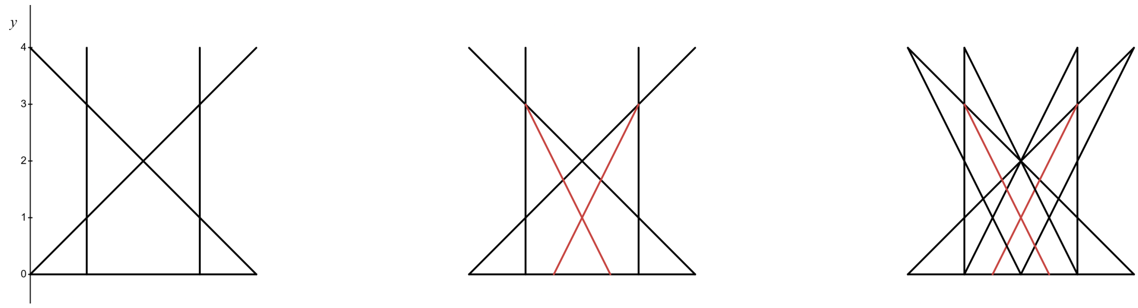
Now that we have shown our starting triangle (also referred to as Δ_2), we want to know how to get the next element of our sequence. We start by taking the 2 lines that meet at the top of our triangle, and continuing them until the height of these lines is 1 higher than the current height. After this we construct the line which bisects the base of our triangle, through the tip of the triangle. Parallel to this bisection, we draw 2 lines connecting the base of our triangle to the extended line segments we just drew, as can be seen in Figure 2.3.



(a) (b) (c)

Figure 2.3: (a) Extending the line segments which intersect at the top of the triangle. (b) Adding the line which bisects the base of the triangle, for all triangles of height 2. (c) Parallel to this bisection, drawing lines that intersect the extension of the line segments created in (a). Ending up with a figure from the Besicovitch construction of height 3.

Note that, by construction, the height of this figure is 1 more than that of Figure 2.2, so now we have Δ_3. Note that Δ_3 can be seen as two overlapping, isosceles, right-angled triangles. Inside each of these triangles, one can easily turn a needle from the hypotenuse onto a leg. This leaves us with the problem of "transferring" our needle from one of these triangles to the other. We can achieve this transfer if we are allowed to make parallel translations. In the next section we will explain that we can do this in arbitrarily small area, with a detailed explanation for Figure 2.5. All in all, this means we can turn a needle of length 3 in the set Δ_3. For now, note the length of our needle has increased from 2 to 3, we will later address how to scale this back down to 1, since the original question is related to a unit line segment. But first, let us provide one more illustration of this construction, showing how to construct Δ_4, see Figure 2.4.



(a) (b) (c)

Figure 2.4: (a) Extending the line segments which intersect at the top of the triangle. (b) Adding the line which bisects the base of the triangle, for all triangles of height 3. (c) Parallel to this bisection, drawing lines that intersect the extension of the line segments created in (a). Ending up with a figure from the Besicovitch construction of height 4.

All other sets in this sequence follow from the repeated application of this construction. Where the bisecting of triangles shown in step (b) of the illustrations should be done for all triangles of height $k-1$ when constructing Δ_k.

2.3. Pal's joins

As we have seen for 2.3, we would like a way of making parallel translations. That's why we now introduce Pal's joins, created by Julius Pal [1].

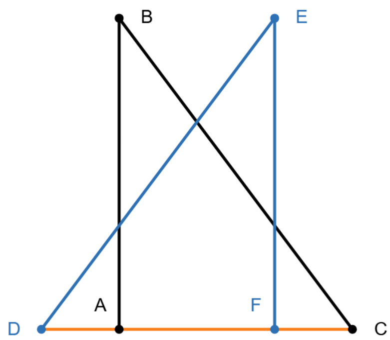


Figure 2.5: Triangle ABC and DEF, overlapping eachother.

In Figure 2.5 we have two triangles, ABC and DEF. Now imagine a needle with the same length as EF. This needle starts along DE, and with some sliding and turning, we can get it to be on top of EF. Now we would like to shift it over to AB, so we can turn it onto BC. This however requires a parallel translation, luckily J. Pal figured out a method to do this in arbitrarily small area.

Let $\epsilon > 0$, we introduce the point G (sitting on line segment AF), such that $\frac{|AG|}{|AF|} < \frac{\epsilon}{8}$. We then draw a line from B through G, and we extend the existing line from E through F, until these two lines meet in point H. And draw the triangle HIJ such that it is congruent with AGB. This results in Figure 2.6.

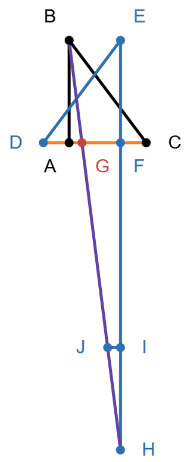


Figure 2.6: Adding the lines BH and EH, and constructing the triangle HIJ.

Since moving along straight lines does not add any surface area to our figure, the only area of interest is that of triangle HIJ. Since the area of a triangle only depends on the length of its base and on its height, we have the inequality $|\triangle ABG| < \frac{\epsilon}{8}|\triangle ABC|$ (Since $|AG| < \frac{\epsilon}{8}|AF|$ and $|AF| < |AC|$). Combining that with the fact that HIJ is congruent with ABG, which is the case by construction, we find that $|\triangle HIJ| < \frac{\epsilon}{8}|\triangle ABC|$. So say we have two congruent triangles $\Delta_1 \cong \Delta_2$, that we join together using a Pal join, we have now shown this adds $\frac{\epsilon}{8}|\Delta_1|$ area. Now say for some n we split up each quarter of the square in Figure 2.1a, in n different triangles. If we do this, each quarter of our square is equivalent to Δ_k of the Besicovitch construction. When we say equivalent, we mean they are the same, up to horizontal translation of these n triangles. This means we use a total of $4n$ triangles, and it would look something like in Figure 2.7.

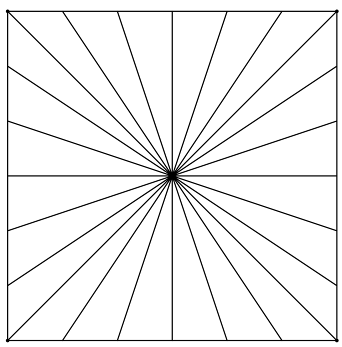


Figure 2.7: Square from Figure 2.1a split up further in triangles

Splitting a square in triangles does not increase its area, so the total area of triangles is 4. This means that the total area added by all the Pál joins, is less than $\frac{\epsilon}{8} \cdot 4 = \frac{\epsilon}{2}$

Now we have described a way to make parallel translations in arbitrarily small area, so let us start with the Besicovitch construction.

2.4. Area calculation

2.4.1. Constant area increase at each step

Now that we have shown how the construction of these intriguing sets works, we shall discuss their areas. First of all we claim that every step in this sequence adds the same amount of area, regardless of k. To prove this, we will look at the set in this sequence which has height k, and we will isolate one of its triangles which also has height k.

(a) (b)

Figure 2.8: (a) Triangle ABC with height k. (b) Next iteration of the Besicovitch construction, with the bisection as dotted line.

First of all, note that we now have in Figure 2.8b that E and D are at height $k+1$, and G and F are at height $k-1$. Now note that, by parallel angles and opposite angles, we find the congruence relationship $\triangle CEG \cong \triangle CFD$.

Secondly, let us look at the area of $\triangle CEG$, if we take EC as base, then the height of this triangle is the length of FG. So $|\triangle CEG| = \frac{1}{2}|CE| \cdot |FG|$. Now since $|CE| = |CF| = 1$ we have that $|\triangle CEG| = |\triangle CFG|$. Combining this with the congruence relationship we found we get the result $|\triangle CEG| = |\triangle CDF| = |\triangle CFG|$.

Which means that for any triangle of height k, the area added by constructing the next layer of the Besicovitch construction, is equal to 2 times the area enclosed by the triangle between height k and $k-1$. In our case this is equal to $2 \cdot |\triangle CFG|$.

Now this is where the beauty happens. All of these triangles in the set combined can be placed next to eachother (after parallel translation) to form 1 isosceles triangle of height k. And the area that an isosceles triangle encloses between height k and $k-1$, is that of and isosceles triangle of height 1 and base 2. This triangle is denoted by Δ_1. This can be seen as the blue shaded area between height 1 and 2, in Figure 2.9 for $k=3$ and $k=4$.

So the total area added by increasing the set of height k in the sequence to a set of height $k+1$ is at most equal to $2 \cdot |\Delta_1|$. We write at most since the triangles akin to $\triangle CEG$ and $\triangle CDF$ might overlap with other parts of the figure. The most important takeaway here is that $2 \cdot |\Delta_1|$ is *not* dependent on k, so the area added by the construction is not dependent on which step of the construction we are in.

To get more precise in the area of our Δ_k for arbitrary k, we will need to get back to our premise of a unit line segment.

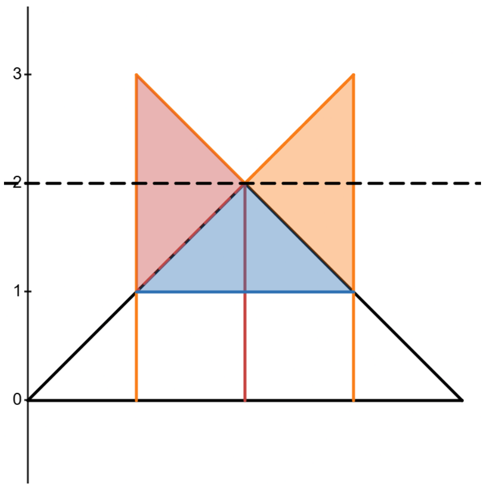
2.4.2. Normalizing to unit length

We now want to be able to say something about the area of our figure, in relation to turning a unit length line segment in it. For this we will need our Δ_k to have a height of 1.

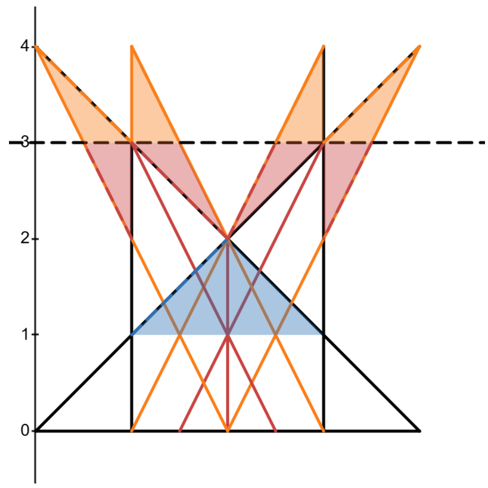
To achieve this we simply scale the whole set down by a factor k. You can imagine this by repeating the construction up to Δ_k from scratch, but instead start with an isosceles triangle of height $\frac{2}{k}$ (akin to Δ_2), and at each following step the height of the set will increase by $\frac{1}{k}$.

To now calculate the area of Δ_k, recall that we have just shown each step in our sequence adds at most $2 \cdot |\Delta_1|$ area. This means we only need to know the area of Δ_1 and Δ_2.

Both of these are isosceles triangles with height $\frac{1}{k}$ and $\frac{2}{k}$ respectively. Where each of their bases



(a) $k = 3$



(b) $k = 4$

Figure 2.9: Showcasing that the added area is at most equal to $2 \cdot |\Delta_1|$.

is double the length of their height. This leads us to the following equations.

$$|\Delta_1| = \frac{1}{2} \cdot \frac{1}{k} \cdot \frac{2}{k} = \frac{1}{k^2}.$$
$$|\Delta_2| = \frac{1}{2} \cdot \frac{2}{k} \cdot \frac{4}{k} = \frac{4}{k^2}.$$
$$|\Delta_k| \leq |\Delta_2| + (k-2) \cdot 2|\Delta_1| = \frac{4}{k^2} + (k-2)\frac{2}{k^2} = \frac{4}{k^2} + \frac{2k}{k^2} - \frac{4}{k^2} = \frac{2}{k}.$$

Now recall we need 4 of these Δ_k to turn our line segment the full 360 degrees. These 4 combined will have area $\leq \frac{8}{k}$.

Now we would like $\frac{8}{k} < \frac{\epsilon}{2}$. We can achieve this by choosing $k > \frac{16}{\epsilon}$. Finally combining 4 of these Δ_k, and the Pal joins we described in Section 2.3, we arrive at a set in which we can turn a line segment of length 1, 360 degrees, with an area of less than ϵ. This is what we wanted to show.

3

Preliminaries

For the following chapter we will need some heavier machinery to tackle the Kakeya conjecture. Therefore we introduce the necessary techniques and definitions here.

3.1. Balls, spheres and tubes

Definition 3.1 (Unit Sphere)**.** The unit sphere S^{n-1} is defined as :

$$S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

Definition 3.2 ((Unit) Ball)**.** Let $a \in \mathbb{R}^n, \delta > 0$. The closed ball $B^n(a, \delta)$ is defined as :

$$B^n(a, \delta) := \{x \in \mathbb{R}^n : |x - a| \leq \delta\}.$$

We will refer to $B^n(0, 1)$ (also known as the unit ball) as simply B^n. We also often just write $B(a, \delta)$, when it is clear what n is.

Definition 3.3 (δ-tubes)**.** Let $\delta > 0$, then an example of an n-dimensional δ-tube with length 1 is

$$\{x \in \mathbb{R}^n : x_1^2 + \dots + x_{n-1}^2 \leq (\frac{\delta}{2})^2, 0 \leq x_n \leq 1\}.$$

All other n-dimensional δ-tubes can be made from this one using translation and rotation.

Definition 3.4 (Sphere coefficients)**.** Coefficients β_n and γ_n are defined such that, for a n-dimensional ball with radius δ :

$$\beta_n \cdot \delta^n := |B^n|,$$

and for a n-dimensional sphere with radius δ

$$\gamma_n \cdot \delta^n := |S^{n-1}|.$$

3.2. Measure theory and L^p-spaces

Definition 3.5 (Diameter of a set)**.** Let $A \subset \mathbb{R}^n$ non-empty, with the Euclidean metric $d : A \times A \to \mathbb{R}$. Then the diameter of A is defined as

$$d(A) := \sup\{d(x, y) : x, y \in A\}.$$

Definition 3.6 (δ-neighbourhood)**.** Let $E \subset \mathbb{R}^n$, with metric $d : E \times E \to \mathbb{R}$, let $\delta > 0$. Then the δ-neighbourhood E_δ is defined as

$$E_\delta := \{x \in \mathbb{R}^n : \exists e \in E : d(e, x) < \delta\}.$$

Definition 3.7 (L_p norm)**.** Let (Ω, Σ, μ) be a measure space. The L^p norm of the measurable function $f : \Omega \to \mathbb{R}$, is defined as:

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

In this thesis our set Ω will either be $\mathbb{R}^n$ or S^{n-1}, these sets require different measures for us to get meaningful results. When we are integrating over $\mathbb{R}^n$ we will use the Lebesgue measure, and when we are integrating over S^{n-1} we will use the Hausdorff measure, see Definition 3.10. This choice of measure holds for any integral we do in this thesis.

Definition 3.8 ($L^1_{\text{loc}}(\mathbb{R}^n)$)**.** If for every compact subset $K \subset \mathbb{R}^n$

$$\int_K |f| d\mu < \infty,$$

then $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Definition 3.9 ($\lesssim$)**.** The symbol $\lesssim_{a,b,c}$ will be used a lot throughout the next chapter. It is defined as follows

$$X \lesssim_{a,b,c} Y \iff X \leq C(a,b,c)Y$$

Where $0 < C < \infty$ So, X is less than Y, up to a constant, which is allowed to depend on anything in the subscript of $\lesssim$. In the next chapter, the most important thing is usually that this constant does not depend on δ, but we will be as explicit as possible, naming the parameters the constant is allowed to depend on. If there is nothing in the subscript, then the constant is not dependent on any parameters.

Definition 3.10 (Hausdorff measure)**.** The $n-1$-dimensional Hausdorff measure on a set $A \subset \mathbb{R}^n$ with countable covering $A \subset \bigcup_{i \in \mathbb{N}} W_i$, $\delta > 0$, is defined as:

$$\mathcal{H}^{n-1}(A) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_i d(W_i)^{n-1} : A \subset \bigcup_{i \in \mathbb{N}} W_i, \; d(W_i) \leq \delta \right\}$$

We won't concern ourselves with this definition too much, but it is needed to define L^p norm on S^{n-1} in a meaningful way. We will provide one example calculation which we will need later.

Example 3.11 (Hausdorff measure of $S^{n-1} \cap B(a, \delta)$)**.** Let $a \in S^{n-1}, \delta > 0$, then we have that under the Hausdorff measure

$$|S^{n-1} \cap B(a, \delta)| \lesssim_n \delta^{n-1}$$

Proof. Before we start our proof a quick visual for $n = 2$.

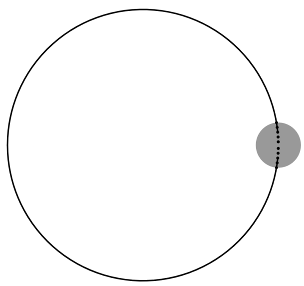


Figure 3.1: The shaded area is $B(a, \delta)$, the dotted line is $S^{n-1} \cap B(a, \delta)$

Here we find that the intersection, in the case of $n = 2$, is a small arc length of the unit circle. We will give a proof for this case. The higher dimensional case follows by a similar argument but more cumbersome notation.

Without loss of generality. take $a = (0, 1)$. Now note any point $x \in S^1$ can be written as $(\cos\ \theta, \sin\theta)$. We then find

$$d(x, a) = \sqrt{\cos(\theta)^2 + (\sin(\theta) - 1)^2} = \sqrt{2 - 2\sin\theta}.$$

Which means $d(x, a) \leq \delta \iff \sin\theta \geq 1 - \frac{\delta^2}{2}$. This is the case if $|\theta| \leq \arcsin(1 - \frac{\delta^2}{2})$. Using a Taylor expansion, on the sine around 0, we see this is equal to $\delta + O(\delta^3)$. Now we have shown that the arc length is $2\delta + O(\delta^3)$. Now using the definition of Hausdorff measure (choosing $d(W_i) \leq \epsilon$, instead of $\leq \delta$ to avoid using the same letter twice) fix $0 < \epsilon < 2\delta$. We can now cover $|S^1 \cap B(a, \delta)|$ with $\lceil \frac{2\delta+O(\delta^3)}{\epsilon} \rceil$ sets of diameter ϵ. Since the definition takes the infimum over all coverings, and we have just given a covering, we find that, taking $\epsilon \downarrow 0$, the 1-dimensional Hausdorff measure is smaller than

$$\sum_i d(W_i)^1 \leq \sum_i \epsilon \leq \lceil \frac{2\delta + O(\delta^3)}{\epsilon} \rceil \cdot \epsilon \lesssim_n \delta.$$

□

Theorem 3.12 (Interpolating between L^p bounds)**.** Let $\theta \in (0, 1)$, let $1 \leq p_0 < \infty$, let $p_\theta = \frac{p_0}{1-\theta}$, let $f \in L^{p_0}(\Omega) \cap L^\infty(\Omega)$. Then

$$\|f\|_{L^{p_\theta}(\Omega)} \leq \|f\|^{\theta}_{L^\infty(\Omega)} \cdot \|f\|^{1-\theta}_{L^{p_0}(\Omega)}.$$

Proof.

$$\begin{aligned}\|f\|_{L^{p_\theta}(\Omega)} = \Big(\int_\Omega |f|^{p_\theta} d\mu\Big)^{\frac{1}{p_\theta}} = \Big(\int_\Omega |f|^{p_0} \cdot |f|^{p_\theta - p_0} d\mu\Big)^{\frac{1}{p_\theta}} &\leq \Big(\int_\Omega |f|^{p_0} \cdot \|f\|^{p_\theta - p_0}_{L^\infty(\Omega)} d\mu\Big)^{\frac{1}{p_\theta}} \\ &\leq \Big(\|f\|^{p_\theta - p_0}_{L^\infty(\Omega)} \int_\Omega |f|^{p_0} d\mu\Big)^{\frac{1}{p_\theta}} \\ &\leq \|f\|^{1-\frac{p_0}{p_\theta}}_{L^\infty(\Omega)} \cdot \|f\|^{\frac{p_0}{p_\theta}}_{L^{p_0}(\Omega)} \\ &\leq \|f\|^{\theta}_{L^\infty(\Omega)} \cdot \|f\|^{1-\theta}_{L^{p_0}(\Omega)}\end{aligned}$$

□

Theorem 3.13 (Hölder's inequality)**.** Let (Ω, Σ, μ) be a measure space, $p, q \in [1, \infty]$ $s.t. \frac{1}{p} + \frac{1}{q} = 1$. Let $f, g \in L^p(\Omega) \cap L^q(\Omega)$. Then

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|f\|_{L^q(\Omega)}.$$

A proof can be found in [4].

Theorem 3.14 (Duality)**.** Let $p, q \in (1, \infty)$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Let $a_k \geq 0$ for $1 \leq k \leq m$. Then we have that

$$\Big(\sum_{k=1}^m a_k^p\Big)^{\frac{1}{p}} = \max\Big\{\sum_{k=1}^m a_k b_k\ ;\ b_k \geq 0 \text{ and } \sum_{k=1}^m b_k^q = 1\Big\}.$$

Proof. Let $S = \{1, ..., m\}, \Sigma = P(S)$, and let μ the counting measure. Then we find by Hölder (Theorem 3.13) that

$$\sum_{k=1}^{m} a_k b_k \leq \Big(\sum_{k=1}^{m} a_k^p\Big)^{\frac{1}{p}} \cdot \Big(\sum_{k=1}^{m} b_k^q\Big)^{\frac{1}{q}} = \Big(\sum_{k=1}^{m} a_k^p\Big)^{\frac{1}{p}},$$

where the equality is true since $\sum_{k=1}^{m} b_k^q = 1$. To show "≥" we have freedom over which b_k we choose. So, let us choose

$$b_k = \frac{a_k^{p-1}}{(\sum_{j=1}^{m} a_j^p)^{\frac{1}{q}}}.$$

Now let us verify $\sum_{k=1}^{m} b_k^q = 1$, where we note that by assumption on p, q we have $(p-1)q = p$

$$\sum_{k=1}^{m} b_k = \sum_{k=1}^{m}\left(\frac{a_k^{p-1}}{(\sum_{j=1}^{m} a_j^p)^{\frac{1}{q}}}\right)^q = \frac{\sum_{k=1}^{m} a_k^{(p-1)q}}{\sum_{j=1}^{m} a_j^p} = \frac{\sum_{k=1}^{m} a_k^p}{\sum_{k=1}^{m} a_k^p} = 1.$$

Let us look at $\sum_{k=1}^{m} a_k b_k$ for this chosen b_k

$$\sum_{k=1}^{m} a_k b_k = \sum_{k=1}^{m} a_k \cdot \frac{a_k^{p-1}}{(\sum_{j=1}^{m} a_j^p)^{\frac{1}{q}}} = \frac{\sum_{k=1}^{m} a_k^p}{(\sum_{j=1}^{m} a_j^p)^{\frac{1}{q}}} = \Big(\sum_{k=1}^{m} a_k^p\Big)^{1-\frac{1}{q}}.$$

Now since $1 - \frac{1}{q} = \frac{1}{p}$ we find for this choice of b_k

$$\sum_{k=1}^{m} a_k b_k = \Big(\sum_{k=1}^{m} a_k\Big)^{\frac{1}{p}}.$$

So, when taking the maximum over b_k, we will find ≥ instead of equality. □

3.3. Minkowski dimension

In this section we will elaborate on the Minkowski dimension. Since the main premise of this thesis is to say something about the Minkowski dimension of Kakeya sets, it would be nice to have an idea of how the Minkowski dimension works. First off let us state the definition.

Definition 3.15 (Upper Minkowski dimension). Let A a non-empty, bounded subset of $\mathbb{R}^n$, for $\epsilon > 0$ let $N(A, \epsilon)$ be the smallest amount of ϵ-balls needed to cover A. Let $0 \leq C < \infty$ Then the upper Minkowski dimension $\overline{\dim}_M(A)$ is defined as:

$$\overline{\dim}_M(A) := \inf\{s : \limsup_{\epsilon \downarrow 0} N(A, \epsilon)\epsilon^s \leq C\}.$$

Here it is only important that C is not infinite, any definition with $0 \leq C < \infty$ is equivalent. The lower Minkowkski dimension ($\underline{\dim}_M(A)$) is defined almost the same but $\limsup$ is replaced by $\liminf$. When the lower and upper Minkowski dimension are equal, we call it the Minkowski dimension. We denote this as $\dim_M(A)$.

Example 3.16 (Minkowski dimension of a square). Let $A := [0, 1]^2 \subset \mathbb{R}^2$ To cover it in squares with side length ϵ you would need at least $\frac{1}{\epsilon^2}$ squares. So since a ball with diameter ϵ is smaller than a square with sidelength ϵ we find $N(A, \epsilon) \geq \frac{1}{\epsilon^2}$. Using this we find

$$\overline{\dim}_M(A) \geq \inf\{s : \limsup_{\epsilon \downarrow 0} \epsilon^{s-2} = 0\}.$$

In our case we find that $\limsup = \liminf = 0$ for $s = 2$ so we find $\dim_M(A) \geq 2$ (≥ since we have only shown a lower bound for $N(A, \epsilon)$, not shown that this bound can be attained). To confirm that indeed

$\dim_M(A) = 2$, we should give a covering of A. For this we divide A into an evenly spaced grid, of $\lceil \frac{2}{\epsilon^2} + 1 \rceil$ by $\lceil \frac{2}{\epsilon^2} + 1 \rceil$ points. So the points have coordinates $(\frac{i}{\lceil \frac{2}{\epsilon^2}+1 \rceil}, \frac{j}{\lceil \frac{2}{\epsilon^2}+1 \rceil})$ for $i, j \in \{0, 1, ..., \lceil \frac{2}{\epsilon^2} + 1 \rceil\}$. Then at each point in this grid place a ball with diameter ϵ. For $\epsilon = 1$ this looks like this:

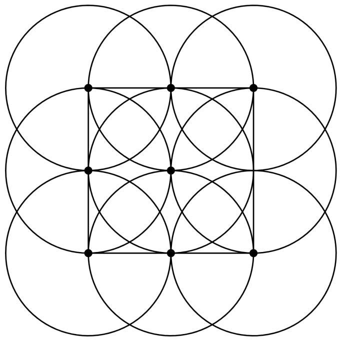


Figure 3.2: Covering of a square, with balls of diameter 1

Now we have shown that $N(A, \epsilon) \leq \lceil \frac{2}{\epsilon^2} + 1 \rceil$ and if we apply the definition in a similar fashion as we did for $\frac{1}{\epsilon^2}$ we now confirm that $\dim_M(C) = 2$.

A nice property of the Minkowski dimension is that it is not limited to being an integer. To show this, let us look at the next example.

Example 3.17. Let $A := \{0\} \cup \{\frac{1}{k},\ k \in \mathbb{N}\}$. We will show that the Minkowski dimension is $\frac{1}{2}$. To show this, it suffices to both show that we need at least $\frac{1}{\sqrt{\epsilon}}$ balls of diameter ϵ and also that we can actually give a covering in $\lesssim \frac{1}{\sqrt{\epsilon}}$ balls of the same size.

First off let us fix a $\epsilon > 0$, and then choose $n \in \mathbb{N}$ such that $\frac{1}{(n+1)^2} < \epsilon \leq \frac{1}{n^2}$. We can cover the interval $[0, \frac{1}{n+1}]$ in $n+1$ balls of diameter ϵ since $(n+1)\epsilon > \frac{1}{n+1}$. We can cover $[\frac{1}{n}, 1] \cap A$ in n balls, since there are exactly n points in $[\frac{1}{n}, 1] \cap A$. So we find a covering in $2n+1$ balls. Now observe that

$$\epsilon \leq \frac{1}{n^2} \iff 2n + 1 \leq 2\epsilon^{-\frac{1}{2}} + 1.$$

If one applies this in the definition of the Minkowski dimension, we find that the dimension is at most $\frac{1}{2}$. If we take the same ϵ, we observe that

$$\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > \frac{1}{n^2}.$$

So for all n such that $\epsilon \leq \frac{1}{n^2}$, we find that each ball can not cover both $\frac{1}{n}$ and $\frac{1}{n-1}$. So we truly need at least n ϵ-balls to cover $[\frac{1}{n}, 1]$, where by choice of ϵ, $\epsilon^{-\frac{1}{2}} - 1 < n$. Now we find that we need at least $\epsilon^{-\frac{1}{2}} - 1$ balls with diameter ϵ. Which is equivalent to the Minkowski dimension being at least $\frac{1}{2}$. So we find that the Minkowski dimension of A is $\frac{1}{2}$.

4

The Kakeya conjecture

In this chapter we will state the Kakeya conjecture, the Kakeya maximal operator conjecture, show that the Kakeya maximal operator conjecture implies the Kakeya conjecture (with regards to Minkowski dimension), and prove the Kakeya maximal operator conjecture for $n = 2$. So in total, we will show the Minkowski part of the Kakeya conjecture for $n = 2$. First a graphic to show the structure of the proof we give.

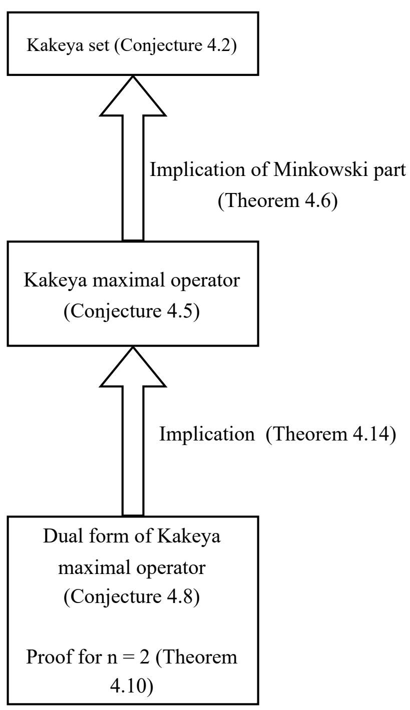


Figure 4.1: Structure of proof

4.1. Conjecture and status

We will start with stating the conjecture, but for that first we need to define what a Kakeya set is.

Definition 4.1 (Kakeya set)**.** A Kakeya set, is a subset of $\mathbb{R}^n$ which contains a unit line segment in every direction.

Conjecture 4.2 (The Kakeya conjecture)**.** Every bounded Kakeya set in $\mathbb{R}^n$ has Minkowski and Hausdorff dimension n

Since we will not show anything about the conjecture with regards to its Hausdorff dimension we omit its definition here. We will only show something about the Minkowski dimension, since proving it for the Hausdorff dimension would require more definitions and work. Given the time constraints of this thesis, we decided to focus on just the Minkowski dimension.

We will construct a proof of the conjecture (regarding Minkowski dimension) for $n = 2$. For $n = 3$ Hong Wang and Joshua Zahl [6] have recently published a preprint which, if correct, proves that case (for Hausdorff dimension, and thus also for Minkowski dimension). For all greater n the conjecture is open.

4.2. Kakeya Maximal Operator Conjecture

To state the Kakeya maximal operator conjecture, we first need to define our Kakeya maximal operator.

Definition 4.3. (Kakeya maximal operator) For $\delta > 0$, $\omega \in S^{n-1}$, $a \in \mathbb{R}^n$, let $T_\delta^\omega(a)$ denote the tube in $\mathbb{R}^n$, centred at a, oriented in the ω direction, of length 1, with cross-sectional radius δ. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Kakeya maximal operator is defined as

$$f_\delta^*(\omega) := \sup_{a\in\mathbb{R}^n} \frac{1}{|T_\delta^\omega(a)|} \int_{T_\delta^\omega} |f| d\mu.$$

We will now start by proving a relatively simple statement using this definition

Example 4.4. Let $n > 1$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\delta > 0$, then

$$\|f_\delta^*\|_{L^\infty(S^{n-1})} \leq \beta_{n-1}^{-1} \cdot \delta^{-(n-1)} \cdot \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. First let us plug in the definition of L^∞ and L^1 norm, and the definition of $f_\delta^*(\omega)$. This gives us :

$$\operatorname*{ess\,sup}_{\omega\in S^{n-1}} \sup_{a\in\mathbb{R}^n} \frac{1}{|T_\delta^\omega(a)|} \int_{T_\delta^\omega(a)} |f| d\mu \leq \frac{1}{\beta_{n-1} \cdot \delta^{n-1}} \int_{\mathbb{R}^n} |f| d\mu$$

Since the volume of $T_\delta^\omega(a)$ is independent of ω and a, we can take it out of both suprema. Then we note that, by definition of β_n we have:

$$\frac{1}{|T_\delta^\omega(a)|} = \frac{1}{\beta_{n-1} \cdot \delta^{n-1}}$$

Now since they are both positive we can divide them out of our inequality, giving us:

$$\operatorname*{ess\,sup}_{\omega\in S^{n-1}} \sup_{a\in\mathbb{R}^n} \int_{T_\delta^\omega(a)} |f| d\mu \leq \int_{\mathbb{R}^n} |f| d\mu$$

Now if we prove this inequality we have proven the original statement. Finally since $T_\delta^\omega(a) \subset \mathbb{R}^n$, we find, by monotonicity with respect to the domain, that indeed the inequality holds, since the integrand is the same in both integrals. □

Now that we have shown this, let us get on to the actual Kakeya maximal operator conjecture.

Conjecture 4.5 (Kakeya Maximal Operator Conjecture)**.** Let $n \geq 2$, $f \in L^1_{loc}(\mathbb{R}^n)$, $\delta > 0$, $a \in \mathbb{R}^n$, $\omega \in S^{n-1}$, then for all $\epsilon > 0$ and $n \leq p \leq \infty$

$$\|f_\delta^*\|_{L^p(S^{n-1})} \lesssim_{(n,p,\epsilon)} \delta^{-\epsilon} \|f\|_{L^p(\mathbb{R}^n)}.$$

Why are we introducing this new conjecture? We will show that this maximal operator conjecture implies the Kakeya set conjecture. We will also prove this maximal operator conjecture is true in the case of $n = 2$.

4.2.1. The Kakeya Maximal Operator Conjecture Implies the Kakeya Conjecture

To show this implication holds, we will assume the Kakeya maximal operator conjecture is true, and use this to show the Kakeya conjecture (at least with respect to Minkowski dimension) is true.

Theorem 4.6 (The Kakeya Maximal Operator Conjecture Implies the Kakeya Conjecture)**.**
Let $\delta > 0,\ n > 1,\ f \in L^1_{loc}(\mathbb{R}^n),\ a \in \mathbb{R}^n,\ \omega \in S^{n-1}$.If for a p with $n \leq p < \infty$ we have that for all $\epsilon > 0$

$$\|f^*_\delta\|_{L^p(S^{n-1})} \lesssim_{n,p,\epsilon} \delta^{-\epsilon}\|f\|_{L^p(\mathbb{R}^n)},$$

then any Kakeya set has Minkowski dimension n.

Proof. Consider any bounded Kakeya set B. We define B_δ as the δ-neighbourhood of B. And as function f we take the indicator function χ_{B_δ}

$$f_\delta := \chi_{B_\delta}(x) = \begin{cases} 1, & x \in B_\delta \\ 0, & x \notin B_\delta \end{cases}$$

Recall the definition of a δ-neighbourhood.

Definition (δ-neighbourhood)**.** Let $E \subset \mathbb{R}^n$, with metric $d : E \times E \to \mathbb{R}$, let $\delta > 0$. Then the δ-neighbourhood E_δ is defined as

$$E_\delta := \{x \in \mathbb{R}^n : \exists e \in E : d(e, x) < \delta\}.$$

Now note that $f^*_\delta = 1$ for all $\omega \in S^{n-1}$. This is true since, by definition of the Kakeya set, it contains a line segment in any direction, and since we defined B_δ as a δ-neighbourhood around B, it will contain the tube $T^\omega_\delta(b)$ (for some $b \in \mathbb{R}^n$). So when we then integrate the characteristic function over it, and then divide by the volume of that tube, we get 1.

Next, let us think about $\|f^*_\delta\|_{L^p(S^{(n-1)})}$ where we have

$$\|f^*_\delta\|_{L^p(S^{(n-1)})} = \left(\int_{S^{n-1}} |1|^p d\mu\right)^{\frac{1}{p}}.$$

This will always be some constant depending on n and p. For example when $n = p = 2$ the domain is the unit circle, and we take the square root of its circumference so $\sqrt{2\pi}$. This constant is equal to $\gamma_n^{\frac{1}{p}}$. Now let us look at

$$\|f\|_{L^p(\mathbb{R}^n)}, \text{ or equivalently } \left(\int_{\mathbb{R}^n} |f|^p d\mu\right)^{\frac{1}{p}}.$$

Since f is an indicator function, raising it to the p-th power doesn't change it. So we are integrating the indicator function, which means we get the size of our domain, $|B_\delta|$. Then we raise it to the power $\frac{1}{p}$ So we end up with.

$$\|f\|_{L^p(\mathbb{R}^n)} = |B_\delta|^{\frac{1}{p}}.$$

Now, using the Kakeya Maximal Operator Conjecture, we arrive at the following expression:

$$\gamma_n^{\frac{1}{p}} \leq C(n, \epsilon) \cdot \delta^{-\epsilon}|B_\delta|^{\frac{1}{p}}.$$

After rearranging we get:

$$\frac{\gamma_n^{\frac{1}{p}} \cdot \delta^\epsilon}{C(n, \epsilon)} \leq |B_\delta|^{\frac{1}{p}}.$$

Now, to prove that B_δ has Minkowski dimension n, we will assume that it has Minkowski dimension d, and then show that $n = d$. Assuming B_δ has Minkowski dimension d there must exist a covering using $\lesssim_n \delta^{-d}$ n-dimensional balls. This constant depending on n can be included in the term $C(n, \epsilon)$. Each of these balls has a volume of $\beta_n \cdot \delta^n$. Since these balls cover the set B_δ, we know that the actual measure is upper bounded by the volume of this covering. So we choose to estimate $|B_\delta|$ by $\delta^{-d} \cdot \delta^n \cdot \beta_n$. This results in :

$$\frac{\gamma_n^{\frac{1}{p}} \cdot \delta^\epsilon}{C(n,\epsilon)} \leq (\delta^{n-d} \cdot \beta_n)^{\frac{1}{p}}.$$

Now raising both sides to the power of p

$$\frac{(\gamma_n^{\frac{1}{p}})^p \cdot \delta^{\epsilon p}}{C(n,\epsilon)^p} \leq \delta^{n-d} \cdot \beta_n.$$

Now we take the logarithm on both sides, and apply logarithm rules to obtain

$$p(\log \gamma_n^{\frac{1}{p}} + \epsilon \log \delta - \log C(n,\epsilon)) \leq (n-d)\log \delta + \log(\beta_n).$$

When applying the definition of Minkowski dimension, we take the limit of δ to 0. We do this since for any delta we find $B \subset B_\delta$, thus any covering of B_δ is a covering of B, so the Minkowski dimension of B is upper bounded by that of B_δ. Thus let us consider 0 < δ < 1, which means that $\log \delta$ < 0. Now we will divide by $\log \delta$, and since this is dividing by a negative number we must flip the inequality sign.

$$p\Big(\frac{\log \gamma_n^{\frac{1}{p}}}{\log \delta} + \epsilon - \frac{\log C(n,\epsilon)}{\log \delta}\Big) \geq n - d + \frac{\log \beta_n}{\log \delta}.$$

Now as we take the limit of δ to 0, a lot of terms vanish and we end up with

$$p\epsilon \geq n-d$$
$$d + p\epsilon \geq n.$$

Since the assumed Maximal operator conjecture is valid for all ϵ > 0, we must have that $d \geq n$. Finally to see that also $d \leq n$, it suffices to show that any bounded subset of $\mathbb{R}^n$ has dimension $\leq n$. Note that in Example 3.16 we covered a square in $\lesssim_n \delta^{-n}$ balls. We can do this in a similar fashion for any bounded set. Thus we find $d = n$. □

Now that we have shown this implication from one conjecture to the next, we would like to prove the maximal operator conjecture, which we will show for $n = 2$.

4.2.2. Proving The Kakeya Maximal Operator Conjecture for n = 2

To prove this we will use a proof based on a master thesis by R. Stedman [5], who himself used results from Córdoba. This was the first full proof of the Kakeya maximal operator conjecture for $n = 2$. Allthough Roy Davies gave a direct proof of the Kakeya conjecture (regarding Hausdorff dimension) in 1971 [2]. For our proof we will first introduce a dual form of the Kakeya maximal operator conjecture. We will then prove that this dual form implies the Kakeya maximal operator conjecture.

For this dual form we first need one more definition.

Definition 4.7. A set of orientations $\Omega \subset S^{n-1}$ is said to be δ-separated if $|\omega - \omega'| > \delta$ for all $\omega, \omega' \in \Omega$

Conjecture 4.8 (Dual form of Kakeya Maximal Operator Conjecture)**.** Let $\delta > 0$, $n > 1$, $\mathbb{T}$ any collection of n-dimensional tubes of length 1 and cross-sectional radius δ (δ-tubes for short) whose orientations are δ-separated. Then for $\frac{n}{n-1} \leq p \leq \infty$ and $\epsilon > 0$

$$\Big\| \sum_{T \in \mathbb{T}} \chi_T \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p,\epsilon} \delta^{\frac{n-1}{p} - \epsilon} (\#\mathbb{T})^{\frac{1}{p}}.$$

So loosely speaking, the conjecture is stating a relationship between the overlap of tubes (denoted by the L^p norm of the summation of tubes) and the size of the set $\mathbb{T}$ (denoted by $(\#\mathbb{T})^{\frac{1}{p}}$ times the size of the tubes (denoted by $\delta^{\frac{n-1}{p}-\epsilon}$).

Lemma 4.9 (Proving the Dual form for $L^{p_0}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, implies the whole conjecture)**.** Let $\delta > 0$, let $\mathbb{T}$ be any collection of tubes of length 1 and cross-sectional radius δ (δ-tubes for short) whose orientations are δ-separated. Let $\epsilon > 0$, let $n \geq 2$. If Conjecture 4.8 is true for $p_0 = \frac{n}{n-1}$ and for $p = \infty$, then for $\frac{n}{n-1} \leq p \leq \infty$

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p,\epsilon} \delta^{\frac{n-1}{p}-\epsilon} (\#\mathbb{T})^{\frac{1}{p}}.$$

Proof. For ease of writing we will write $\sum_{T\in\mathbb{T}} \chi_T = f$. Let $\theta \in (0,1)$, let $p_\theta = \frac{\frac{n}{n-1}}{1-\theta}$. (This lets us achieve any value between $\frac{n}{n-1}$ and infinity by varying θ.) Now using the assumption we immediately get

$$\|f\|_{L^{\frac{n}{n-1}}} \lesssim_{n,p,\epsilon} \delta^{\frac{(n-1)^2}{n}-\epsilon} \cdot (\#\mathbb{T})^{\frac{n-1}{n}}$$
$$\|f\|_{L^\infty} \lesssim_{n,p,\epsilon} \delta^{-\epsilon}.$$

Now we use Theorem 3.12 which gives us

$$\|f\|_{L^{p_\theta}} \leq \|f\|_{L^{\frac{n}{n-1}}}^{1-\theta} \|f\|_{L^\infty}^{\theta},$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{\frac{n}{n-1}}, \qquad p_\theta = \frac{\frac{n}{n-1}}{1-\theta}, \qquad 1-\theta = \frac{\frac{n}{n-1}}{p_\theta}, \qquad \theta = 1 - \frac{\frac{n}{n-1}}{p_\theta}.$$

Using this we find

$$\|f\|_{L^{\frac{n}{n-1}}}^{1-\theta} \lesssim_{n,p,\epsilon} \delta^{\frac{n-1}{p_\theta}-\epsilon\left(\frac{\frac{n}{n-1}}{p_\theta}\right)} \cdot (\#\mathbb{T})^{\frac{1}{p_\theta}} \tag{4.1}$$

$$\|f\|_{L^\infty}^{\theta} \lesssim_{n,p,\epsilon} \delta^{-\epsilon\left(1-\frac{\frac{n}{n-1}}{p_\theta}\right)}. \tag{4.2}$$

Now multiplying (4.1) and (4.2) gives us

$$\|f\|_{L^{p_\theta}} \lesssim_{n,p,\epsilon} \delta^{\frac{n-1}{p_\theta}-\epsilon} \cdot (\#\mathbb{T})^{\frac{1}{p_\theta}}.$$

□

Theorem 4.10 (The Kakeya Maximal Operator Conjecture is true for $n = 2$)**.** Let $n = 2$, fix $\epsilon > 0$ and $\mathbb{T}$, then take $\delta_0 > 0$ s.t. when $\delta < \delta_0$, we have $\#\mathbb{T} \lesssim \delta^{-\epsilon}$ (where $\mathbb{T}$ is a collection of δ_0-seperated δ_0-tubes). Then for $\frac{2}{1} \leq p \leq \infty$, $0 < \delta < \delta_0$

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|_{L^p(\mathbb{R}^2)} \lesssim_{p,\epsilon} \delta^{\frac{1}{p}-\epsilon} (\#\mathbb{T})^{\frac{1}{p}}.$$

Proof. According to Lemma 4.9 we need only prove the theorem for $p = \frac{n}{n-1} = 2$ and for $p = \infty$. We will start with $p = 2$ (which is the harder case). First note that we can write

$$\#\mathbb{T} = \frac{1}{2\delta} \sum_{T\in\mathbb{T}} |T|.$$

Using this we can rewrite Conjecture 4.8 (which is what we are trying to prove) as

$$\Big\| \sum_{T\in\mathbb{T}} \chi_T \Big\|_{L^2(\mathbb{R}^2)} \lesssim_{p,\epsilon} \delta^{\frac{1}{2}-\epsilon} \Big(\frac{1}{2\delta} \sum_{T\in\mathbb{T}} |T|\Big)^{\frac{1}{2}} = \frac{\sqrt{2}}{2} \delta^{-\epsilon} \Big(\sum_{T\in\mathbb{T}} |T|\Big)^{\frac{1}{2}}. \tag{4.3}$$

In fact we will prove a stronger statement (note it is in fact a stronger statement since $\log(x) < p(x)$ when x tends to infinity for any power function $p(x) = x^k$, where $k \in (0, \infty)$). We will show

$$\Big\|\sum_{T\in\mathbb{T}}\chi_T\Big\|_{L_2(\mathbb{R}^2)} \lesssim (\log\frac{1}{\delta})^{\frac{1}{2}}\Big(\sum_{T\in\mathbb{T}}|T|\Big)^{\frac{1}{2}}.$$

Let us first examine the left hand side of this inequality, while squaring it

$$\begin{aligned}\Big\|\sum_{T\in\mathbb{T}}\chi_T\Big\|^2_{L^2(\mathbb{R}^n)} &= \int\Big(\sum_{T\in\mathbb{T}}\chi_T(x)\Big)^2 dx\\ &= \int\sum_{T\in\mathbb{T}}\sum_{T'\in\mathbb{T}}\chi_T(x)\chi_{T'}(x)dx\\ &= \sum_{T\in\mathbb{T}}\sum_{T'\in\mathbb{T}}|T\cap T'|.\end{aligned}$$

Now since

$$\sum_{T\in\mathbb{T}}(\log\frac{1}{\delta}|T|) = (\log\frac{1}{\delta})\sum_{T\in\mathbb{T}}|T|,$$

we need only show that

$$\sum_{T'\in\mathbb{T}}|T\ \cap\ T'| \lesssim (\log\frac{1}{\delta})|T|. \tag{4.4}$$

To get a bound on this overlap between T and T', suppose their orientations differ in angle θ, with $\delta \leq \theta \leq \frac{\pi}{2}$. Now divide all possible θ in intervals of the form $I_j := [2^{j-1}, 2^j)$, where $j_{min} := \lceil\log\delta\rceil$, $j_{max} := \lceil\log\frac{\pi}{2}\rceil$ and $j_{min} \leq j \leq j_{max}$. Now note that every θ is in an I_j for some j. If we now take a look at 2 of these tubes their overlap might look something like in Figure 4.2

(a) Tubes T and T' and their overlap (b) Tubes T and T' with a larger angle difference

Figure 4.2: Examples of overlap between T and T'

As can be seen in the figure above, $T\ \cap\ T'$ is in the shape of a parallelogram with height δ and width dependent on θ. Consider a right angled triangle with base length L, height δ and angle θ (this represents half of the parallelogram). Then we get the relationship $\sin(\theta) = \frac{\delta}{L}$. For small δ we can use that $\sin(\theta) \approx \theta$ Resulting in $L \approx \frac{\delta}{\theta}$. And, since for any θ there is a j such that $\theta < 2^j$ (we choose the smallest such j), we have

$$|T\ \cap\ T'| \lesssim 2^j\delta|T|.$$

Now we need to sum over all possible angles to get the total overlap that we want ($\sum_{T'\in\mathbb{T}}|T\ \cap\ T'|$). We write this as

$$\sum_{j_{min}\leq j\leq j_{max}}\ \sum_{T'\in\mathbb{T},\ \theta\in I_j} 2^j\delta|T|.$$

Now using that $|T|$ does not depend on j, we can take it out of the sum, and we just want to show the rest of the sum $\lesssim (\log \frac{1}{\delta})$. Now since the set of tubes is δ-separated, we have that for each j there are $O(\frac{2^{-j}}{\delta})$ tubes whose orientations are within $O(2^{-j})$ of tube T. So the value of the inner sum is $O(1)$. In the outer sum we add this $O(\log \frac{1}{\delta})$ times (this how many j we need since $\log \frac{\pi}{2} - \log \delta \lesssim \log(\frac{1}{\delta})$). So we find

$$\sum_{j_{min} \leq j \leq j_{max}} \sum_{T' \in \mathbb{T},\ \theta \in I_j} 2^j \delta \lesssim \log(\frac{1}{\delta}).$$

This proves (4.4) which was sufficient for the case of $p = 2$.

Now we still have to prove Conjecture 4.8 for $p = \infty$. So we have to prove that

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{L^\infty} \lesssim_{n,\epsilon} \delta^{-\epsilon}$$

Note that the L^∞ norm is the essential supremum (supremum which ignores sets of measure zero) the largest this could be is if all tubes overlap in one area, then the norm would be $\#\mathbb{T}$. So it suffices to show:

$$\#\mathbb{T} \lesssim_{n,\epsilon} \delta^{-\epsilon}$$

We chose our δ to be small enough for this to hold. Thus the conjecture holds for $p = 2$ and for $p = \infty$. □

4.2.3. Proving Duality

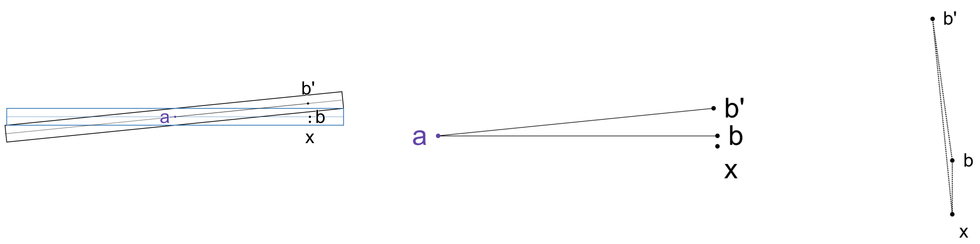
To prove the Kakeya maximal operator conjecture, we proved the dual form of this conjecture. So to properly complete our proof of the Kakeya set conjecture for $n = 2$, we are still left with showing this is in fact the dual. Or at least showing that this dual form implies the Kakeya maximal operator conjecture. The proofs provided here are based on the notes from Itamar Oliveira [3]. Before we dive straight into the proof, we need some additional building blocks. First of, a more specific type of δ-separated set, a maximal δ-separated set.

Definition 4.11 (Maximal δ-separated)**.** A set of orientations Ω is said to be a maximal δ-separated set if in addition to being δ-separated we have that, for any $\omega \in S^{n-1}$ there exist a $\omega' \in \Omega$ $s.t.$ $|\omega - \omega'| < \delta$.

Lemma 4.12. Let $0 < \delta < 1$, let $f \in L^1_{loc}(\mathbb{R}^n)$. If $\omega, \omega' \in S^{n-1}$ and $|\omega - \omega'| \leq \delta$, then

$$f^*_\delta(\omega) \lesssim_n f^*_\delta(\omega').$$

Proof. Let $a \in \mathbb{R}^n$ be arbitrary, and fix x on one of the far ends of the tube (for our tube in Example 3.3 this would mean fixing $x_n \in \{0, 1\}$). Let b be the projection of x onto the main axis of $T^\omega_\delta(a)$ and b' the projection of x onto the main axis of $T^{\omega'}_\delta(a)$. Let ξ be the angle $\angle bab'$. Now note $\sin(\xi) < \xi \leq \delta$ ($\xi \leq \delta$ by our assumption $|\omega - \omega'| \leq \delta$).



(a) Sketch of the situation (b) Showcasing the angle ξ (c) Zoom on the triangle inequality

Basic trigonometry on the triangle $\triangle b'ab$ says that $\sin(\xi) = \sin(\angle bb'a)\frac{|b-b'|}{|a-b|}$. (Here it is important to note that for small δ even when $\angle bb'a$ is not exactly 90 degrees, it will be close to 90, which means $\sin(\angle bb'a)$ can be replaced by a constant.) So we find that

$$|b - b'| = |a - b| \sin(\xi) \lesssim \delta.$$

We also have by the triangle inequality that

$$|x - b'| \leq |x - b| + |b - b'| \lesssim 2\delta,$$

so $x \in T^{\omega'}_{M\delta}$. Here M is a constant due to the fact that $\angle bb'a$ is not exactly 90 degrees. Since both tubes are convex sets, we have that by proving the boundary of the tube is a subset of $T^{\omega'}_{M\delta}$ that the whole tube is a subset. Thus we find $T^{\omega}_{\delta}(a) \subset T^{\omega'}_{M\delta}(a)$. Now note we can cover $T^{\omega'}_{M\delta}(a)$ with $C(n)$ tubes of the form $T^{\omega'}_{\delta}(y_k)$ with $y_k \in \mathbb{R}^n$ and $C(n) \leq 5^n$.Since the tubes have the same orientation, it is equivalent to covering a (n-1)-dimensional ball with (n-1)-dimensional smaller balls, which is somewhat similar to Example 3.16. Finally we get the following inequalities,

$$\begin{aligned}
\frac{1}{|T^{\omega}_{\delta}(a)|}\int_{T^{\omega}_{\delta}(a)} |f(x)|dx &\leq \frac{1}{|T^{\omega'}_{\delta}(a)|}\int_{T^{\omega'}_{M\delta}(a)} |f(x)|dx \\
&\leq \sum_{k=1}^{C(n)} \frac{1}{|T^{\omega'}_{\delta}(y_k)|}\int_{T^{\omega'}_{\delta}(y_k)} |f(x)|dx \\
&\leq C(n) f^*_{\delta}(\omega'),
\end{aligned}$$

where the first inequality is due to monotonicity of the integral with respect to domain since $T^{\omega}_{\delta}(a) \subset T^{\omega'}_{M\delta}(a)$. The second inequality uses linearity of the integral using that the union of $T^{\omega'}_{\delta}(y_k)$ is a covering of $T^{\omega'}_{M\delta}(a)$. The last inequality is a direct result of the definition of the Kakeya maximal operator.

Finally to get the required result of this lemma, we need only take the supremum over a on the left hand side of the inequality. □

We will now prove a lemma which does most of the heavy lifting for the proof of duality. The proof of duality comes down to showing that we are able to use the assumption in the following lemma.

Lemma 4.13. Let $1 < p < \infty$, $q = \frac{p}{p-1}$, $0 < \delta < 1$ and $0 < M < \infty$. Suppose that

$$\left\| \sum_{k=1}^{m} t_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \leq M$$

for all $\mathbb{T} := \{T_1, ..., T_m\}$ where $\mathbb{T}$ is a δ-separated set of δ-tubes, then for all $t_1, ...t_m$ positive numbers such that

$$\sum_{k=1}^{m} t_k^q \leq \delta^{1-n}.$$

This implies

$$\|f^*_{\delta}\|_{L^p(S^{n-1})} \leq C(n) M \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let $\Omega := \{\omega_1, ..., \omega_m\}$ be a maximal δ-separated subset of S^{n-1}. First of we claim that $\bigcup_{k=1}^{m}\{S^{n-1} \cap B(\omega_k, \delta)\}$ covers S^{n-1}. To see why this is true imagine an orientation ω_j which is not covered by this union of balls, this would mean that for any $\omega_k \in \Omega$, $|\omega_j - \omega_k| > \delta$ (Since it is not covered by the union of balls). This however contradicts maximality of the set Ω.

Now let $\omega \in S^{n-1} \subset \bigcup_{k=1}^{m}\{S^{n-1} \cap B(\omega_k, \delta)\}$ arbitrary. We have by Lemma 4.12 that for some $1 \leq k \leq m$

$$f^*_{\delta}(\omega) \leq C(n) f^*_{\delta}(\omega_k).$$

Using this we build up an estimate on $\|f_\delta^*\|_{L^p}(S^{n-1})$.

$$\begin{aligned}\|f_\delta^*\|_{L^p(S^{n-1})}^p &\leq \sum_{k=1}^{m} \int_{S^{n-1}\cap B(\omega_k,\delta)} |f_\delta^*(\omega)|^p d\mu \\ &\leq \sum_{k=1}^{m} C(n)^p \int_{S^{n-1}\cap B(\omega_k,\delta)} |f_\delta^*(\omega_k)|^p d\mu \\ &\leq \sum_{k=1}^{m} C(n)^p |f_\delta^*(\omega_k)|^p |S^{n-1} \cap B(\omega_k,\delta)| \\ &\lesssim_n \sum_{k=1}^{m} (|f_\delta^*(\omega_k)|\delta^{\frac{n-1}{p}})^p.\end{aligned}$$

Here the final inequality ($\lesssim$) uses the fact that each sphere cap (intersection between S^{n-1} and $B(\omega_k,\delta)$) has Hausdorff measure approximately δ^{n-1}. (An idea as to why this is true is given in Example 3.11.)

Next up we use Theorem 3.14, where we let b_k s.t.

$$\Big(\sum_{k=1}^{m} (|f_\delta^*(\omega_k)|\delta^{\frac{n-1}{p}})^p\Big)^{\frac{1}{p}} = \sum_{k=1}^{m} (|f_\delta^*(\omega_k)|\delta^{\frac{n-1}{p}}) b_k \; ; \; b_k \geq 0 \text{ and} \sum_{k=1}^{m} b_k^q = 1,$$

this gives us

$$\|f_\delta^*\|_{L^p(S^{n-1})} \lesssim_n \sum_{k=1}^{m} |f_\delta^*(\omega_k)|\delta^{\frac{n-1}{p}} b_k.$$

Now we choose $t_k = \delta^{\frac{1-n}{q}} b_k$. Next let us take a look at how we can rewrite $\frac{n-1}{p}$

$$\begin{aligned}\frac{n-1}{p} &= n-1+(1-n)+\frac{n-1}{p} \\ &= n-1+(1-n)(1-\frac{1}{p}) \\ &= n-1+(1-n)(\frac{1}{q}) \\ &\Leftrightarrow \\ \delta^{\frac{n-1}{p}} b_k &= \delta^{n-1}\cdot\delta^{\frac{1-n}{q}} b_k = \delta^{n-1} t_k.\end{aligned}$$

This results in

$$\sum_{k=1}^{m} |f_\delta^*(\omega_k)|\delta^{\frac{n-1}{p}} b_k = \delta^{n-1} \sum_{k=1}^{m} |f_\delta^*(\omega_k)| t_k.$$

We also find that

$$\delta^{n-1} \sum_{k=1}^{m} t_k^q = \delta^{n-1} \sum_{k=1}^{n} (\delta^{\frac{1-n}{q}} b_k)^q = \sum_{k=1}^{m} b_k = 1.$$

This satisfies the condition we posed on t_k.

Now let $\epsilon > 0$ small. Then there exists $a_k \in \mathbb{R}^n$ such that

$$|f_\delta^*(\omega_k)| - \epsilon \leq \frac{1}{|T_\delta^{\omega_k}(a_k)|} \int_{T_\delta^{\omega_k}(a_k)} |f(x)| dx.$$

Using this we find

$$\begin{aligned}
\|f_\delta^*\|_{L^p(S^{n-1})} &\lesssim_n \delta^{n-1}\sum_{k=1}^{m}(|f_\delta^*(\omega_k)|-\epsilon+\epsilon)t_k \\
&\le \delta^{n-1}\sum_{k=1}^{m}\Big(\frac{1}{|T_\delta^{\omega_k}(a_k)|}\int_{T_\delta^{\omega_k}(a_k)}|f(x)|dx\Big)t_k+\epsilon \\
&\lesssim_n \sum_{k=1}^{m} t_k\int_{T_\delta^{\omega_k}(a_k)}|f(x)|dx+\epsilon \\
&= \int_{\mathbb{R}^n}\Big(\sum_{k=1}^{m}t_k\chi_{T_\delta^{\omega_k}(a_k)}\Big)|f(x)|dx+\epsilon \\
&\le \Big\|\sum_{k=1}^{m}t_k\chi_{T_\delta}^{\omega_k}(a_k)\Big\|_{L^q(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}+\epsilon \\
&\le M\|f\|_{L^p(\mathbb{R}^n)})+\epsilon.
\end{aligned}$$

Note on the second line we write $+\,\epsilon$ and not $+\,t_k\epsilon$, this is allowed since $t_k\epsilon\le\epsilon$. For the third line we use that the volume of a tube of length 1 with cross-sectional diameter δ is approximately δ^{n-1}. The fifth line is just applying Hölder's inequality (Theorem 3.13). To get the final inequality we directly use our assumption. The desired result is then obtained by letting $\epsilon\downarrow 0$. □

Theorem 4.14 (Proof of duality)**.** Let $1<p<\infty,\ q=\frac{p}{p-1},\ 1\le M<\infty$ and $0<\delta<1$. Then

$$\|f_\delta^*\|_{L^q(S^{n-1})}\lesssim_{n,p,\epsilon} M\delta^{-\epsilon}\|f\|_{L^p}(\mathbb{R}^n), \tag{4.5}$$

for all $f\in L^p(\mathbb{R}^n)$ and $\epsilon>0$ if and only if

$$\Big\|\sum_{k=1}^{m}\chi_{T_k}\Big\|_{L^q(\mathbb{R}^n)}\lesssim_{n,q,\epsilon} M\delta^{\frac{n-1}{q}-\epsilon}(\#\mathbb{T})^{\frac{1}{q}}, \tag{4.6}$$

for all $\epsilon>0$ and all δ-separated tubes $\{T_1,...,T_m\}=\mathbb{T}$.

Proof. We will only prove (4.6) $\implies$ (4.5) since this is all we need for our proof of the Kakeya set conjecture to be valid. The converse is proven in [3]. We assume (4.6), and let $\{T_1,...,T_m\}=\mathbb{T}$ a set of δ-separated δ-tubes and let $t_1,...t_m>0$ such that $\delta^{n-1}\sum_{k=1}^{m}t_k^q\le 1$. By Lemma 4.13 it suffices to show

$$\Big\|\sum_{k=1}^{m}t_k\chi_{T_k}\Big\|_{L^q(\mathbb{R}^n)}\lesssim_{n,q,\epsilon} M\delta^{-\epsilon}.$$

First of we note that $\#\mathbb{T}\cdot\delta^{n-1}\lesssim_n 1$ (S^{n-1} can be covered using $\lesssim_n \delta^{1-n}$ δ–balls, since the measure of $S^{n-1}\lesssim_n 1$ and the measure of a (n-1) dimensional δ-ball $\lesssim_n \delta^{n-1}$ and $\frac{1}{\delta^{n-1}}=\delta^{1-n}$, so we can have at most $\lesssim_n \delta^{1-n}$ δ-seperated tubes). Using this while multiplying (4.6) by δ^{n-1} and choosing an $\epsilon_0\le 1$ resulting in $\delta^{n-1-\epsilon_0}<1$. We find that

$$\begin{aligned}
\Big\|\sum_{k=1}^{m}\delta^{n-1}\chi_{T_k}\Big\|_{L^q(\mathbb{R}^n)} &\lesssim_{n,q,\epsilon} \delta^{n-1}M\delta^{-\epsilon_0}(\delta^{n-1}\cdot(\#\mathbb{T}))^{\frac{1}{q}} \\
&\lesssim_{n,p,\epsilon} \delta^{n-1-\epsilon_0}M\cdot 1^{\frac{1}{q}} \\
&\le C.
\end{aligned}$$

Here $0 < C < \infty$ is a constant that does not depend on δ. Another key observation will be that $t_k \leq \delta^{\frac{1-n}{q}}$ for all $1 \leq k \leq m$. To see why this is true, note that by our assumption on t_k we get $\sum_{k=1}^{m} t_k^q \leq \delta^{1-n}$ .Thus $t_k^q \leq \delta^{1-n} \iff t_k \leq \delta^{\frac{1-n}{q}}$.

We have just shown that if in the place of t_k we have δ^{n-1} the norm is less than C. So we just need to prove the inequality for the $t_k \geq \delta^{n-1}$. Thus, we are interested in $\delta^{n-1} \leq t_k \leq \delta^{\frac{1-n}{q}}$. We will divide this interval into intervals of the form $I_j = \{k : 2^{j-1} \leq t_k < 2^j\}$, with $j_{min} := \lceil \log \delta^{n-1} \rceil, j_{max} := \lceil \log \delta^{\frac{1-n}{q}} \rceil$ and $j_{min} \leq j \leq j_{max}$. For this we need $O(\log \frac{1}{\delta})$ intervals ($|\log \delta^{n-1} - \log \delta^{C}| \lesssim_n \log(\frac{1}{\delta})$ for any constant C). Let m_j denote the amount of k in I_j. (This also means there are m_j tubes associated with I_j.) Observe also that for any I_j, $2^{jq} \leq (2t_k)^q$ (since $2^j \leq 2t_k$). Using this we find

$$m_j 2^{jq} \leq \sum_{k \in I_j} (2t_k)^q \leq \sum_{k=1}^{m} (2t_k)^q \leq 2^q \delta^{1-n}. \tag{4.7}$$

Finally we have everything we need to make our final claim.

$$\begin{aligned}
\left\| \sum_{k=1}^{m} t_k \chi_{T_k} \right\| \leq \sum_{j_{min} \leq j \leq j_{max}} \left\| \sum_{k \in I_j} t_k \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} &\leq \sum_{j_{min} \leq j \leq j_{max}} \left\| \sum_{k \in I_j} 2^j \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \\
&= \sum_{j_{min} \leq j \leq j_{max}} 2^j \left\| \sum_{k \in I_j} \chi_{T_k} \right\|_{L^q(\mathbb{R}^n)} \\
&\lesssim_{n,q,\epsilon} \sum_{j_{min} \leq j \leq j_{max}} 2^j M \delta^{\frac{-\epsilon}{2}} (m_j \delta^{n-1})^{\frac{1}{q}} \\
&\lesssim M \delta^{\frac{-\epsilon}{2}} \sum_{j_{min} \leq j \leq j_{max}} 1 \\
&\lesssim_{n,q} M \log(\frac{1}{\delta}) \delta^{\frac{-\epsilon}{2}} \\
&\lesssim_{\epsilon} M \delta^{-\epsilon}.
\end{aligned}$$

Firstly just using the triangle inequality and that $t_k < 2^j$ by definition of I_j. In the first $\lesssim_{n,q,\epsilon}$ we are simply applying our assumption (4.6) using $\frac{\epsilon}{2}$. The next $\lesssim$ uses (after pulling 2^j into the parentheses) (4.7) directly. Finally we use that there are $O(\log \frac{1}{\delta})$ intervals, and that $\log(\frac{1}{\delta}) \lesssim_{\epsilon} \delta^{-\epsilon}$. (This is equivalent to the observation that log(x) < p(x) when x tends to infinity for any power function p(x).) □

5 Outlook

This section explores some results related to the Kakeya conjecture that were not shown in this thesis, but are known. We will also highlight which parts of this problem are still open.

As mentioned previously in this thesis, we ignored the case of the Hausdorff dimension. For $n = 2$, the Kakeya conjecture has been proven for the Hausdorff dimension. An example of a proof can be seen in the notes by Itamar Oliveira [3]. This proof also uses the Kakeya maximal operator conjecture. More recent is the work by Hong Wang and Joshua Zahl [6], with a proposed proof for $n = 3$ (for both Hausdorff and Minkowski dimension). Their proof only covers the Kakeya set conjecture, so the Kakeya maximal operator conjecture is still open for $n = 3$. Another related conjecture, which implies the Kakeya maximal operator conjecture, is the restriction conjecture. This conjecture is related to Fourier transforms, and a more detailed exploration of the topic can be found in the thesis by Richard Steadman [5]. We cited this thesis before when we gave a proof of Kakeya for $n = 2$, but it also goes more in depth on the restriction conjecture.

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