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ORIGINAL PAPER

On the competition of two conflicting messages

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Abstract There are plenty of conflicting messages in online social networks. This paper addresses the competition of two conflicting messages. Based on a novel individual-level competing spreading model (the generic UABU model), three criteria for one or two messages to terminate are presented. These criteria manifest the influence of the two message-spreading networks on the evolution of the two messages. Extensive computer simulations show that when a message terminates, the dynamics of a simplified UABU model (the linear UABU model) fits well with the expected evolutionary process of the message. These findings help in understanding the competing spreading process of two conflicting messages.

Keywords Conflicting messages · Competitive spreading · Spreading network · Exact spreading model · Generic spreading model · Linear spreading model

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1 Introduction

The rapidly popularized online social networks (OSNs) provide a global environment for us to spread and acquire information quickly [1,2]. However, there are lots of untrue messages in OSNs, and some of them could induce huge economic losses or serious social panic [3,4]. For example, Syrian hackers once broke into the Twitter account of Associated Press (AP) and dispersed the misinformation that explosions at White House had injured Obama, leading to 10 billion USD losses before the rumor was clarified [5]. Therefore, false messages with serious consequences must be clarified timely. In such scenarios, there are often two conflicting messages: the fake message and the message clarifying it. In order to assess the effectiveness of the effort to suppress misinformation, one must first gain insight into the competing spreading behavior of two conflicting messages [6].

The spreading dynamics is devoted to modeling and studying a variety of spreading phenomena by employing the qualitative analysis methodology of differential dynamical systems. In many network-related scenarios, there often occur competing spreading phenomena, such as competing viruses [7–10], competing ideas [11–13], conflicting messages [14–20], epidemics and alerts [21,22], malware and patches [23–28] and cyber attacks and defenses [29], to name a few. The competing spreading dynamics, which is a branch of the spreading dynamics, is committed to modeling and studying a variety of competing spreading phenomena. As another branch of the spreading dynamics, the individual-level spreading dynamics casts the evolutionary process of the state of each individual involved in a spreading phenomenon as a separate differential equation, resulting in a higher-dimensional differential dynamical system. One striking advantage of the individual-level spreading models lies in that they accommodate the structure of the spreading networks and hence help in understanding the influence of the network structures on the spreading processes. In recent years, the individual-level spreading modeling technique has been successfully applied to virus spreading [30–33], malware spreading [34–39] and cyber attack-defense [40,41]. To the best of our knowledge, the competing spreading dynamics of conflicting messages has not yet been explored from the individual-level perspective.

Inspired by the above-mentioned efforts and the works in Refs. [42–45], this paper addresses the competition of two conflicting messages. An individual-level competing spreading model (the generic UABU model) is introduced. Then, three criteria for one or two messages to terminate are presented. These criteria manifest the influence of the two message-spreading networks on the evolutionary process of the two messages. Extensive computer simulations show that when a message terminates, the dynamics of a simplified UABU model (the linear UABU model) fits well with the expected evolutionary process of the message. These findings contribute to the understanding of the competing spreading process of two conflicting messages.

The subsequent materials are organized in this fashion. Sections 2 and 3 describe and study the generic UABU model, respectively. Section 4 experimentally examines the validity of the linear UABU model. This work is summarized by Sect. 5.

2 A generic individual-level competing spreading model

This section is dedicated to establishing a generic individual-level continuous-time dynamic model capturing the competing spreading process of two conflicting messages.

2.1 Notions, notations and hypotheses

Suppose there are two conflicting messages, A and B, that emerge at time $t = 0$ and spread among a population of N persons labeled $1, 2, \ldots, N$. Let $V = \{1, 2, \ldots, N\}$. Suppose message A propagates through an OSN $G_A = (V, E_A)$ known as the *A-spreading network*, where $(i, j) \in E_A$ if and only if person j can deliver message A directly to person i. Likewise, suppose message B is circulated through an OSN $G_B = (V, E_B)$ known as the *B-spreading network*. In what follows, it is always assumed that the two networks are strongly connected.

Depending on personal judgment, every person may either believe message A (A-believing), or believe message B (B-believing), or be uncertain. Let $X_i(t) = 0, 1$ and 2 denote that at time t, person i is uncertain, A-believing and B-believing, respectively. Then, the state of the population at time t is represented by the vector

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_N(t))^T. \tag{1}$$

Next, let us introduce a set of hypotheses as follows.

(H$_1$) Due to the influence of A-believer j, at any time uncertain person i turns to believe message A at rate $\beta_{ij}^{UA} \geq 0$. Here, $\beta_{ij}^{UA} > 0$ if and only if $(i, j) \in E_A$. β_{ij}^{UA} is proportional to (a) the rate at which person j delivers message A to person i, and (b) the probability of person i believing message A when receiving it. Let $\mathbf{M}_{UA} = \left(\beta_{ij}^{UA}\right)_{N\times N}$. This hypothesis captures the influence of the spread of message A on uncertain persons.

(H$_2$) Due to the influence of A-believer j, at any time B-believer i turns to believe message A at rate $\beta_{ij}^{BA} \geq 0$. Here, $\beta_{ij}^{BA} > 0$ if and only if $(i, j) \in E_A$. β_{ij}^{BA} is proportional to (a) the rate at which person j delivers message A to person i, and (b) the probability of person i believing message A when receiving it. Let $\mathbf{M}_{BA} = \left(\beta_{ij}^{BA}\right)_{N\times N}$. This hypothesis captures the influence of the spread of message A on B-believers. Certainly, we have $\beta_{ij}^{BA} \le \beta_{ij}^{UA}$.

(H$_3$) Due to the influence of B-believer j, at any time uncertain person i turns to believe message B at rate $\beta_{ij}^{UB} \ge 0$, Here, $\beta_{ij}^{UB} > 0$ if and only if $(i, j) \in E_B$. β_{ij}^{UB} is proportional to (a) the rate at which person j delivers message B to person i and (b) the probability of person i believing message B when receiving it. Let $\mathbf{M}_{UB} = \left(\beta_{ij}^{UB}\right)_{N\times N}$. This hypothesis captures the influence of the spread of message B on uncertain persons.

(H$_4$) Due to the influence of B-believer j, at any time A-believer i turns to believe message B at rate $\beta_{ij}^{AB} \ge 0$, Here, $\beta_{ij}^{AB} > 0$ if and only if $(i, j) \in E_B$. β_{ij}^{AB} is proportional to (a) the rate at which person j delivers message B to person i, and (b) the probability of person i believing message B when receiving it. Let $\mathbf{M}_{AB} = \left(\beta_{ij}^{AB}\right)_{N\times N}$. This hypothesis captures the influence of the spread of message B on A-believers. Certainly, we have $\beta_{ij}^{AB} \le \beta_{ij}^{UB}$.

(H$_5$) Due to the forgetfulness or the loss of interest, A-believer i turns to be uncertain at rate $\delta_i^A > 0$. Let $\mathbf{D}_A = diag\left(\delta_i^A\right)$.

(H$_6$) Due to the forgetfulness or the loss of interest, B-believer i turns to be uncertain at rate $\delta_i^B > 0$. Let $\mathbf{D}_B = diag\left(\delta_i^B\right)$.

All the forthcoming competitively spreading models are assumed to comply with these hypotheses.

2.2 The original UABU model

For fundamental knowledge on continuous-time Markov chains, see Ref. [46].

Another way of representing the population state at time t is by the decimal number $a(t) = \sum_{k=1}^N X_k(t)\, 3^{k-1}$. In this context, there are totally 3^N possible population states: $0, 1, \ldots, 3^N - 1$. According to the previous hypotheses, the infinitesimal generator $\mathbf{Q} = [q_{ab}]_{3^N\times 3^N}$ for the competing spreading process is given as

$$q_{ab} = \begin{cases} \delta_m^A, & \text{if } a = b + 3^{m-1}, \quad m = 1, 2, \ldots, N, \quad x_m = 1; \\ \delta_m^B, & \text{if } a = b + 2\cdot 3^{m-1}, m = 1, 2, \ldots, N, x_m = 2; \\ \sum_{k=1}^N \beta_{mk}^{UA} 1_{\{x_k=1\}}, & \text{if } a = b - 3^{m-1}, \quad m = 1, 2, \ldots, N, \quad x_m = 0; \\ \sum_{k=1}^N \beta_{mk}^{BA} 1_{\{x_k=1\}}, & \text{if } a = b + 3^{m-1}, \quad m = 1, 2, \ldots, N, \quad x_m = 2; \\ \sum_{k=1}^N \beta_{mk}^{UB} 1_{\{x_k=2\}}, & \text{if } a = b - 2\cdot 3^{m-1}, m = 1, 2, \ldots, N, x_m = 0; \\ \sum_{k=1}^N \beta_{mk}^{AB} 1_{\{x_k=2\}}, & \text{if } a = b - 3^{m-1}, m = 1, 2, \ldots, N, x_m = 1; \\ -\sum_{c=0, c\neq a}^{N-1} q_{ac}, & \text{if} \quad a = b; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

where $a = \sum_{k=1}^N x_k 3^{k-1}$, 1_S stands for the indicator function of set S. The continuous-time Markov chain model with the infinitesimal generator $\mathbf{Q}$ is referred to as the *original Uncertain-A-B-Uncertain (UABU) model*. See Fig. 1 for the state transition rates of a person under the original UABU model.

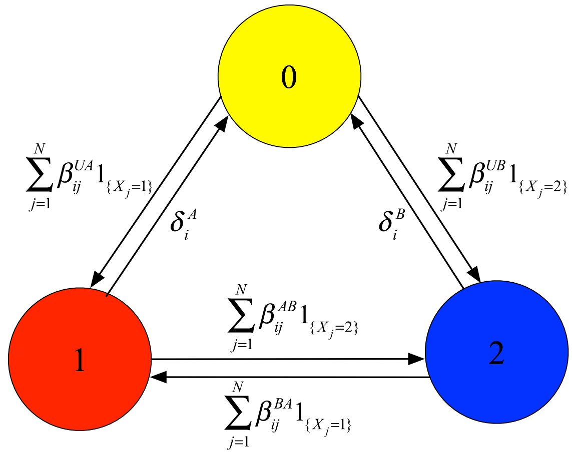

Fig. 1 State transition rates of person i under the original UABU model

2.3 The exact UABU model

For $0 \leq a \leq 3^N - 1, a = \sum_{k=1}^{N} x_k 3^{k-1}$, let $S_a(t)$ denote the probability that the state of the population at time t is $(x_1, \dots, x_N)$.

$$S_a(t) = \Pr\{X_1(t) = x_1, \dots, X_N(t) = x_N\}. \tag{3}$$

Let $\mathbf{S}(t)$ denote the probability distribution of the population state at time t.

$$\mathbf{S}(t) = [S_0(t), \dots, S_{3^N-1}(t)]^T. \tag{4}$$

Then, $\mathbf{S}(t)$ obeys

$$\frac{\mathrm{d}\mathbf{S}^T(t)}{\mathrm{d}t} = \mathbf{S}^T(t)\mathbf{Q}. \tag{5}$$

We refer to the continuous-time Markov chain as the *exact UABU model*, because it accurately characterizes the expected competing spreading process of the two conflicting messages. The state transition rates of a person under the exact UABU model cannot be clearly shown as a diagram.

Although the exact UABU model is a linear differential system, with the solution $\mathbf{s}^T(t) = \mathbf{s}^T(0)e^{\mathbf{Q}t}$, its dimensionality grows exponentially with the increasing population size, leading to mathematical intractability.

Let

$$A_i(t) = Pr\{X_i(t) = 1\}, \tag{6}$$

$$B_i(t) = Pr\{X_i(t) = 2\}. \tag{7}$$

Obviously, $A_i(t)$ and $B_i(t)$ together capture the expected state of person i at time t. The following lemma gives an equivalent form of the exact UABU model.

Lemma 1 *The exact UABU model is equivalent to the model*

$$\begin{cases} \frac{dA_i(t)}{dt} = \sum_{j=1}^{N} \beta_{ij}^{UA} Pr\{X_i(t) = 0, X_j(t) = 1\} \\ \quad + \sum_{j=1}^{N} \beta_{ij}^{BA} Pr\{X_i(t) = 2, X_j(t) = 1\} \\ \quad - \sum_{j=1}^{N} \beta_{ij}^{AB} Pr\{X_i(t) = 1, X_j(t) = 2\} - \delta_i^A A_i(t), \\ \frac{dB_i(t)}{dt} = \sum_{j=1}^{N} \beta_{ij}^{UB} Pr\{X_i(t) = 0, X_j(t) = 2\} \\ \quad + \sum_{j=1}^{N} \beta_{ij}^{AB} Pr\{X_i(t) = 1, X_j(t) = 2\} \\ \quad - \sum_{j=1}^{N} \beta_{ij}^{BA} Pr\{X_i(t) = 2, X_j(t) = 1\} - \delta_i^B B_i(t), \\ \quad i = 1, 2, \dots, N. \end{cases} \tag{8}$$

The proof of this lemma is left to "Appendix A." The equivalent model is not closed. If one attempted to close the equivalent model by adding some joint probability terms, the resulting model would be of dimensionality 3^N again, which is still mathematically intractable.

2.4 The linear UABU model

In order to simplify the exact UABU model, it is necessary to reduce its dimensionality while keeping its closedness. To this end, let us make an added set of independence hypotheses as follows ($1 \leq i \leq N$, $1 \leq j \leq N, i \neq j$).

(H_7) $Pr\{X_i(t) = 0, X_j(t) = 1\} = (1 - A_i(t) - B_i(t))A_j(t)$.
(H_8) $Pr\{X_i(t) = 0, X_j(t) = 2\} = (1 - A_i(t) - B_i(t))B_j(t)$.
(H_9) $Pr\{X_i(t) = 1, X_j(t) = 2\} = A_i(t)B_j(t)$.
(H_{10}) $\Pr\{X_i(t) = 2, X_j(t) = 1\} = B_i(t)A_j(t)$.

Based on the independence hypotheses and the equivalent model (7), we obtain the following approximation model of the exact UABU model.

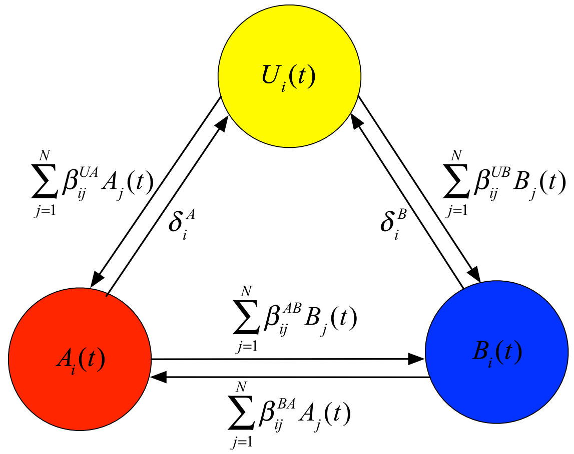

Fig. 2 State transition rates of person i under the linear URTU model

$$\begin{cases} \frac{\mathrm{d}A_i(t)}{\mathrm{d}t} = (1 - A_i(t) - B_i(t)) \sum_{j=1}^{N} \beta_{ij}^{UA} A_j(t) \\ \quad + B_i(t) \sum_{j=1}^{N} \beta_{ij}^{BA} A_j(t) \\ \quad - A_i(t) \sum_{j=1}^{N} \beta_{ij}^{AB} B_j(t) - \delta_i^A A_i(t), \\ \frac{\mathrm{d}B_i(t)}{\mathrm{d}t} = (1 - A_i(t) - B_i(t)) \sum_{j=1}^{N} \beta_{ij}^{UB} B_j(t) \\ \quad + A_i(t) \sum_{j=1}^{N} \beta_{ij}^{AB} B_j(t) \\ \quad - B_i(t) \sum_{j=1}^{N} \beta_{ij}^{BA} A_j(t) - \delta_i^B B_i(t), \\ i = 1, 2, \ldots, N. \end{cases} \tag{9}$$

We refer to this model as the *linear UABU model*, because the A-spreading rates, $\sum_{j=1}^{N} \beta_{ij}^{UA} A_j(t)$ and $\sum_{j=1}^{N} \beta_{ij}^{BA} A_j(t)$, are linear in $A_1(t), \ldots, A_N(t)$, and the B-spreading rates, $\sum_{j=1}^{N} \beta_{ij}^{UB} B_j(t)$ and $\sum_{j=1}^{N} \beta_{ij}^{AB} B_j(t)$, are linear in $B_1(t), \ldots, B_N(t)$. See Fig. 2 for the state transition rates of a person under the linear UABU model.

The linear UABU model is a closed $2N$-dimensional dynamical system and hence is mathematically tractable. In reality, the dynamics of the model may deviate from the expected competing spreading process of the two conflicting messages, because the independence hypotheses may fail.

2.5 The generic UABU model

For the purpose of approximating the exact UABU model more accurately, let us consider a more general competing spreading model as follows.

$$\begin{cases} \frac{\mathrm{d}A_i(t)}{\mathrm{d}t} = (1 - A_i(t) \\ \quad - B_i(t)) f_i^{UA} (A_1(t), \ldots, A_N(t)) \\ \quad + B_i(t) f_i^{BA} (A_1(t), \ldots, A_N(t)) \\ \quad - A_i(t) f_i^{AB} (B_1(t), \ldots, B_N(t)) \\ \quad - \delta_i^A A_i(t), \\ \frac{\mathrm{d}B_i(t)}{\mathrm{d}t} = (1 - A_i(t) \\ \quad - B_i(t)) f_i^{UB} (B_1(t), \ldots, B_N(t)) \\ \quad + A_i(t) f_i^{AB} (B_1(t), \ldots, B_N(t)) \\ \quad - B_i(t) f_i^{BA} (A_1(t), \ldots, A_N(t)) \\ \quad - \delta_i^B B_i(t), \\ i = 1, 2, \ldots, N. \end{cases} \tag{10}$$

Here, f_i^{UA} stands for the rate at which the uncertain person i turns to believe message A, f_i^{UB} the rate at which the uncertain person i turns to believe message B, f_i^{AB} the rate at which the A-believer i turns to believe message B, and f_i^{BA} the rate at which the B-believer i turns to believe message A. These spreading rates are assumed to satisfy the following generic conditions.

(C_1) (Proximity) An uncertain person or a B-believer can and can only be influenced by those A-believers that can deliver message A to him through the A-spreading network. That is, f_i^{UA} or f_i^{BA} is dependent upon $A_j(t)$ if and only if $(i, j) \in E_A$. Likewise, an uncertain person or a A-believer can and can only be influenced by those B-believers that can deliver message B to him through the B-spreading network. That is, f_i^{UB} or f_i^{AB} is dependent upon $B_j(t)$ if and only if $(i, j) \in E_B$.

(C_2) (Nullity) Message A cannot spread unless there is a A-believer in the population. That is, $f_i^{UA}(0, \ldots, 0) = f_i^{BA}(0, \ldots, 0) = 0$. Likewise,

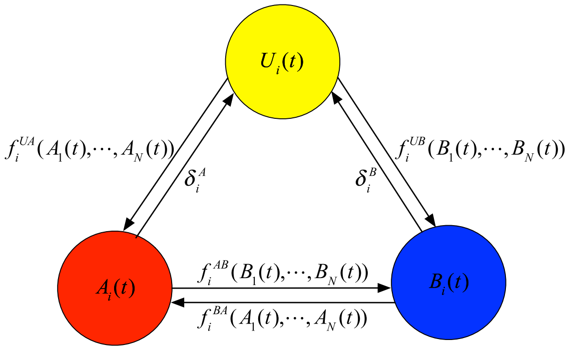

Fig. 3 State transition rates of person i under the generic UABU model

message B cannot spread unless there is a B-believer in the population. That is, $f_i^{UB}(0,\ldots,0) = f_i^{AB}(0,\ldots,0) = 0$.

(C_3) (Ordering) An uncertain person is easier to believe message A than a B-believer, and an uncertain person is easier to believe message B than a A-believer. That is, $f_i^{UA} \geq f_i^{BA}$, $f_i^{UB} \geq f_i^{AB}$.

(C_4) (Smoothness) The spreading rates are sufficiently smooth. Technically speaking, f_i^{UA}, f_i^{UB}, f_i^{AB} and f_i^{BA} are twice continuously differentiable.

(C_5) (Monotonicity) The spreading rates are strictly increasing with respect to every relevant argument. That is, $\frac{\partial f_i^{UA}(x_1,\ldots,x_N)}{\partial x_j} > 0$ if f_i^{UA} is dependent upon x_j, $\frac{\partial f_i^{UB}(x_1,\ldots,x_N)}{\partial x_j} > 0$ if f_i^{UB} is dependent upon x_j, $\frac{\partial f_i^{AB}(x_1,\ldots,x_N)}{\partial x_j} > 0$ if f_i^{AB} is dependent upon x_j, and $\frac{\partial f_i^{BA}(x_1,\ldots,x_N)}{\partial x_j} > 0$ if f_i^{AB} is dependent upon x_j.

(C_6) (Concavity) The spreading rates flatten out and tend to saturation. That is, $\frac{\partial^2 f_i^{UA}(x_1,\ldots,x_N)}{\partial x_j \partial x_k} \leq 0$, $\frac{\partial^2 f_i^{UB}(x_1,\ldots,x_N)}{\partial x_j \partial x_k} \leq 0$, $\frac{\partial^2 f_i^{AB}(x_1,\ldots,x_N)}{\partial x_j \partial x_k} \leq 0$, and $\frac{\partial^2 f_i^{BA}(x_1,\ldots,x_N)}{\partial x_j \partial x_k} \leq 0$.

We refer to the model (10) as the *generic UABU model*. See Fig. 3 for the state transition rates of a person under the generic URTU model. Obviously, this model subsumes the linear UABU model as well as many other UABU models with nonlinear spreading rates.

Let

$$\Omega = \Big\{(x_1,\ldots,x_{2N}) \in \mathbb{R}_+^{2N} \mid x_i + x_{N+i} \leq 1, 1 \leq i \leq N\Big\}. \tag{11}$$

Certainly, the initial state of the generic UABU model lies in Ω. And it is easily shown that the state of the model always stays within Ω.

Let us introduce the following vector and matrix notations.

$$\begin{aligned}
\mathbf{A}(t) &= (A_1(t),\ldots,A_N(t))^T,\\
\mathbf{B}(t) &= (B_1(t),\ldots,B_N(t))^T,\\
\text{diag}\mathbf{A}(t) &= \text{diag}\,(A_i(t)),\\
\text{diag}\mathbf{B}(t) &= \text{diag}\,(B_i(t)),\\
\mathbf{f}_{UA}(\mathbf{A}(t)) &= \left(f_1^{UA}(\mathbf{A}(t)),\ldots,f_N^{UA}(\mathbf{A}(t))\right)^T,\\
\mathbf{f}_{UB}(\mathbf{B}(t)) &= \left(f_1^{UB}(\mathbf{B}(t)),\ldots,f_N^{UB}(\mathbf{B}(t))\right)^T,\\
\mathbf{f}_{AB}(\mathbf{B}(t)) &= \left(f_1^{AB}(\mathbf{B}(t)),\ldots,f_N^{AB}(\mathbf{B}(t))\right)^T,\\
\mathbf{f}_{BA}(\mathbf{A}(t)) &= \left(f_1^{BA}(\mathbf{A}(t)),\ldots,f_N^{BA}(\mathbf{A}(t))\right)^T.
\end{aligned} \tag{12}$$

Then, the generic UABU model can be recast as

$$\begin{cases}
\frac{d\mathbf{A}(t)}{dt} = (\mathbf{I}_N - \text{diag}\mathbf{A}(t) - \text{diag}\mathbf{B}(t))\mathbf{f}_{UA}(\mathbf{A}(t)) \\
\qquad + \text{diag}\mathbf{B}(t)\mathbf{f}_{BA}(\mathbf{A}(t)) \\
\qquad - \text{diag}\mathbf{A}(t)\mathbf{f}_{AB}(\mathbf{B}(t)) - \mathbf{D}_A\mathbf{A}(t),\\
\frac{d\mathbf{B}(t)}{dt} = (\mathbf{I}_N - \text{diag}\mathbf{A}(t) - \text{diag}\mathbf{B}(t))\mathbf{f}_{UB}(\mathbf{B}(t)) \\
\qquad + \text{diag}\mathbf{A}(t)\mathbf{f}_{AB}(\mathbf{B}(t)) \\
\qquad - \text{diag}\mathbf{B}(t)\mathbf{f}_{BA}(\mathbf{A}(t)) - \mathbf{D}_B\mathbf{B}(t),
\end{cases} \tag{13}$$

where $\mathbf{I}_N$ stands for the identity matrix of order N.

3 Dynamics of the generic UABU model

Consider the generic UABU model (10). Let $A(t)$ and $B(t)$ denote the expected fraction at time t of A-believers and B-believers, respectively.

$$A(t) = \frac{1}{N}\sum_{i=1}^{N} A_i(t), \tag{14}$$

$$B(t) = \frac{1}{N}\sum_{i=1}^{N} B_i(t). \tag{15}$$

The main goal of this work is to determine the evolution tendency of $A(t)$ and $B(t)$ over time. For that purpose, we need some preliminary knowledges, which are listed below.

3.1 Preliminaries

For fundamental knowledge on matrix theory, see Ref. [47]. In what follows, we consider only real square matrices. Given a matrix $\mathbf{M}$, let $s(\mathbf{M})$ denote the maximum real part of an eigenvalue of $\mathbf{M}$, $\rho(\mathbf{M})$ the spectral radius of $\mathbf{M}$, i.e., the maximum modulus of an eigenvalue of $\mathbf{M}$. $\mathbf{M}$ is *Metzler* if its off-diagonal entries are all nonnegative.

Lemma 2 (Corollary 8.1.30 in [47]) *Let* $\mathbf{M}$ *be a nonnegative matrix. If* $\mathbf{M}$ *has a positive eigenvector* $\mathbf{x}$*, then* $\rho(\mathbf{M})$ *is an eigenvalue of* $\mathbf{M}$*, and* $\mathbf{x}$ *belongs to* $\rho(\mathbf{M})$.

Lemma 3 (Lemma 2.3 in [48]) *Let* $\mathbf{M}$ *be an irreducible Metzler matrix. Then,* $s(\mathbf{M})$ *is a simple eigenvalue of* $\mathbf{M}$*, and, up to scalar multiple,* $\mathbf{M}$ *has a unique positive eigenvector* $\mathbf{x}$ *belonging to* $s(\mathbf{M})$.

A matrix $\mathbf{M}$ is *Hurwitz stable* or simply *Hurwitz* if its eigenvalues all have negative real parts, i.e., $s(\mathbf{M}) < 0$.

Lemma 4 (Chapter 2 in [49]) *A matrix* $\mathbf{A}$ *is Hurwitz if and only if there is a positive definite matrix* $\mathbf{P}$ *such that* $\mathbf{A}^T\mathbf{P} + \mathbf{PA}$ *is negative definite.*

A matrix $\mathbf{M}$ is *diagonally stable* if there is a positive definite diagonal matrix $\mathbf{D}$ such that $\mathbf{M}^T\mathbf{D} + \mathbf{DM}$ is negative definite. Obviously, a diagonally stable matrix is Hurwitz.

Lemma 5 (Section 2 in [50]) *A Metzler matrix is diagonally stable if it is Hurwitz.*

Lemma 6 (Lemma A.1 in [51]) *Let* $\mathbf{M}$ *be an irreducible Metzler matrix. If* $s(\mathbf{M}) = 0$*, then there is a positive definite diagonal matrix* $\mathbf{D}$ *such that* $\mathbf{M}^T\mathbf{D} + \mathbf{DM}$ *is negative semi-definite.*

For fundamental theory on differential dynamical systems, see Ref. [52].

Lemma 7 (Chaplygin Lemma, see Theorem 31.4 in [53]) *Consider a smooth n-dimensional system of differential equations*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}((\mathbf{x}(t)), \quad t \geq 0 \tag{16}$$

and the corresponding system of differential inequalities

$$\frac{d\mathbf{y}(t)}{dt} \leq \mathbf{f}((\mathbf{y}(t)), \quad t \geq 0 \tag{17}$$

with $\mathbf{x}(0) = \mathbf{y}(0)$*. Suppose that for any* $a_1, \ldots, a_n \geq 0$*, there hold*

$$\begin{aligned} & f_i(x_1 + a_1, \ldots, x_{i-1} + a_{i-1}, x_i, x_{i+1} \\ & \quad + a_{i+1}, \ldots, x_n + a_n) \\ & \geq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \end{aligned} \tag{18}$$

Then, $\mathbf{y}(t) \leq \mathbf{x}(t), t \geq 0$.

Lemma 8 (Strauss-Yorke theorem, see Corollary 3.3 in [54]) *Consider a differential dynamical system*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}((\mathbf{x}(t)) + \mathbf{g}(t, \mathbf{x}(t)), \quad t \geq 0, \tag{19}$$

with $\mathbf{g}(t, \mathbf{x}(t)) \to \mathbf{0}$ *when* $t \to \infty$*. Let*

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}((\mathbf{y}(t)), \quad t \geq 0 \tag{20}$$

denote the limit system of this system. If the origin is a global attractor for the limit system, and every solution to the original system is bounded on $[0, \infty)$*, then the origin is also a global attractor for the original system.*

For fundamental knowledge on fixed-point theory, see Ref. [55].

Lemma 9 (Brouwer fixed-point theorem, see Theorem 4.10 in [55]) *Let* $C \subset \mathbb{R}^n$ *be nonempty, bounded, closed and convex. Let* $f : C \to C$ *be a continuous function. Then,* f *has a fixed point.*

3.2 The equilibria

The first step to understand the dynamics of a differential dynamical system is to examine all of its equilibria. The generic UABU model might admit four different types of equilibria, which are defined as follows.

Definition 1 Let $\mathbf{E} = (\mathbf{A}^T, \mathbf{B}^T)^T$ be an equilibrium of the generic UABU model (10).

(a) $\mathbf{E}$ is *uncertain* if $\mathbf{A} = \mathbf{B} = \mathbf{0}$, which stands for the steady population state in which all persons are uncertain almost surely.
(b) $\mathbf{E}$ is *A-dominant* if $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$, which stands for a steady population state in which some persons believe message A with positive probability and, meanwhile, no person believes message B almost surely.
(c) $\mathbf{E}$ is *B-dominant* if $\mathbf{A} = \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$, which stands for a steady population state in which some persons believe message B with positive probability and, meanwhile, no person believes message A almost surely.
(d) $\mathbf{E}$ is *coexistent* if $\mathbf{A} \neq \mathbf{0}$ and $\mathbf{B} \neq \mathbf{0}$, which stands for a steady population state in which some persons believe message A with positive probability and, meanwhile, some persons believe message B with positive probability.

Obviously, the generic UABU model always admits the uncertain equilibrium $\mathbf{E}_U = (0, \ldots, 0)^T$. Due to the complexity of the model, we are unable to figure out its any other equilibria. For our purpose, define a pair of Metzler matrices as follows.

$$\mathbf{C}_A = \frac{\partial \mathbf{f}_{UA}(\mathbf{0})}{\partial \mathbf{x}} - \mathbf{D}_A, \quad \mathbf{C}_B = \frac{\partial \mathbf{f}_{UB}(\mathbf{0})}{\partial \mathbf{x}} - \mathbf{D}_B, \quad (21)$$

where $\frac{\partial \mathbf{f}_{UA}(\mathbf{0})}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{f}_{UB}(\mathbf{0})}{\partial \mathbf{x}}$ stand for the Jacobian matrix of $\mathbf{f}_{UA}$ and $\mathbf{f}_{UB}$ evaluated at the origin, respectively. As the two spreading networks, G_A and G_B, are strongly connected, the two matrices are both irreducible.

We are ready to present the following fundamental result about the equilibria of the generic UABU model.

Theorem 1 *Consider the model* (10). *The following claims hold.*

(a) *If $s(\mathbf{C}_A) > 0$, then the model admits a unique A-dominant equilibrium. Let $\mathbf{E}_A = (\mathbf{A}^{*T}, \mathbf{0}^T)^T$ denote the equilibrium, $\mathbf{A}^* = (A_1^*, \ldots, A_N^*)^T$. Then, $\mathbf{0} < \mathbf{A}^* < \mathbf{1}$.*
(b) *If $s(\mathbf{C}_B) > 0$, then the model admits a unique B-dominant equilibrium. Let $\mathbf{E}_B = (\mathbf{0}^T, \mathbf{B}^{*T})^T$ denote the equilibrium, $\mathbf{B}^* = (B_1^*, \ldots, B_N^*)^T$. Then, $\mathbf{0} < \mathbf{B}^* < \mathbf{1}$.*

The proof of the theorem is left to "Appendix B." This theorem implies that the existence and location of a dominant equilibrium of the generic UABU model are dependent in a complex way upon the basic parameters as well as the network structures. Henceforth, let

$$A^* = \frac{1}{N}\sum_{i=1}^{N} A_i^*, \quad B^* = \frac{1}{N}\sum_{i=1}^{N} B_i^*. \quad (22)$$

Then, $A^* > 0$, $B^* > 0$.

3.3 Attractivity analysis

Now, let us examine the attractivity of the equilibria of the generic URTU model. First, we have the following criterion for the attractivity of the uncertain equilibrium.

Theorem 2 *Consider the model* (10). *Suppose $s(\mathbf{C}_A) \leq 0$ and $s(\mathbf{C}_B) \leq 0$. Then, the uncertain equilibrium $\mathbf{E}_U$ attracts Ω. Hence, $A(t) \to 0$ and $B(t) \to 0$ as $t \to \infty$.*

The proof of the theorem is left to "Appendix C." This theorem has the following useful corollary.

Corollary 1 *The uncertain equilibrium $\mathbf{E}_U$ of the model* (10) *attracts Ω if one of the following conditions is satisfied.*

(a) $\rho(\mathbf{C}_A\mathbf{D}_A^{-1} + \mathbf{I}_N) < 1$, $\rho(\mathbf{C}_B\mathbf{D}_B^{-1} + \mathbf{I}_N) < 1$.
(b) $\rho(\mathbf{M}_{UA}\mathbf{D}_A^{-1}) < 1$, $\rho(\mathbf{M}_{UB}\mathbf{D}_B^{-1}) < 1$.
(c) $\sum_{i=1}^{N} \beta_{ij}^{UA} < \delta_j^A$, $\sum_{i=1}^{N} \beta_{ij}^{UB} < \delta_j^B$, $j = 1, 2 \ldots, N$.
(d) $\sum_{j=1}^{N} \frac{\beta_{ij}^{UA}}{\delta_j^A} < 1$, $\sum_{j=1}^{N} \frac{\beta_{ij}^{UB}}{\delta_j^B} < 1$, $i = 1, 2, \ldots, N$.

The proof of this corollary is left to "Appendix D." The following theorem offers a criterion for the global attractivity of the A-dominant equilibrium.

Theorem 3 *Consider the model* (10). *Suppose $s(\mathbf{C}_A) > 0$ and $s(\mathbf{C}_B) \leq 0$. Then, the A-dominant equilibrium $\mathbf{E}_A$ attracts $\{(\mathbf{A}, \mathbf{B}) \in \Omega : \mathbf{A} \neq \mathbf{0}\}$. Hence, if $A(0) \neq 0$, then $A(t) \to A^*$ and $B(t) \to 0$ as $t \to \infty$.*

The proof of the theorem is left to "Appendix E." In parallel, we have the following criterion for the attractivity of the B-dominant equilibrium.

Theorem 4 *Consider the model* (10). *Suppose $s(\mathbf{C}_A) \leq 0$ and $s(\mathbf{C}_B) > 0$. Then, the B-dominant equilibrium $\mathbf{E}_B$ attracts $\{(\mathbf{A}, \mathbf{B}) \in \Omega : \mathbf{B} \neq \mathbf{0}\}$. Hence, if $B(0) \neq 0$, then $A(t) \to 0$ and $B(t) \to B^*$ as $t \to \infty$.*

The argument for the theorem is analogous to that for Theorem 3 and hence is omitted.

Theorems 2 and 4 demonstrate that when (a) $s(\mathbf{C}_A) \leq 0$ and $s(\mathbf{C}_B) \leq 0$, or (b) $s(\mathbf{C}_A) \leq 0$, $s(\mathbf{C}_B) > 0$ and $B(0) \neq 0$, nobody would believe message A in a long run. Similarly, Theorems 2 and 3 imply that when (a) $s(\mathbf{C}_A) \leq 0$ and $s(\mathbf{C}_B) \leq 0$, or (b) $s(\mathbf{C}_A) > 0$, $s(\mathbf{C}_B) \leq 0$ and $A(0) \neq 0$, nobody would believe message B in a long run.

In practice, if a message A is undesirable, the following measures are recommended.

(a) Release a message B that refutes message A.
(b) Enhance the B-spreading rates by providing more evidences that support message B.
(c) Reduce the A-spreading rates by providing more evidences that refute message A.
(d) Expand channels of spreading message B such as mass media and official announcement.
(e) Lessen channels of spreading message A by improving the quality of people.

4 The accuracy of the linear UABU model

As was mentioned in Sect. 2, the exact UABU model accurately describes the expected competing spreading process of two conflicting messages, and the linear UABU model is an approximation of the exact UABU model. We wonder under what conditions the linear UABU model is satisfactory. This section is committed to answer the question through computer simulations.

For the comparison purpose, we need to numerically solve the exact UABU model, because its closed-form solution is far beyond our reach. Based on the standard Gillespie algorithm for numerically solving continuous-time Markov chain models [56], we develop a numerical algorithm for solving the exact UABU model. The basic idea of the algorithm is to take the average of $M = 10^4$ sample paths of the original UABU model as an approximation to the solution to the exact UABU model. In the following experiments, a randomly chosen person is initialized as being A-believing, a randomly chosen person is initialized as being B-believing, and all the remaining persons in the population are initialized as being uncertain.

Example 1 Scale-free networks are a large class of networks having widespread applications [57]. Take a randomly generated scale-free network with 100 nodes as both the A-spreading network and the B-spreading network. By taking random combinations of the basic parameters, we get 4096 linear UABU models, which are divided into the following four collections.

(a) 94 linear UABU models for which both $A(t)$ and $B(t)$ approach zero. By observations, we find that for each of the models, its dynamics fits with the expected competing spreading process. See Fig. 4a for a pair of examples.
(b) 1764 linear UABU models for which $A(t)$ approaches a nonzero value, but $B(t)$ approaches zero. By observations, we find that for each of the models, its dynamics about message B accords with the expected spreading process of message B, but its dynamics about message A deviates significantly from the expected spreading process of message A. See Fig. 4b for a pair of examples.
(c) 1770 linear UABU models for which $A(t)$ approaches zero, but $B(t)$ approaches a nonzero value. By observations, we find that for each of the models, its dynamics about message A conforms to the expected spreading process of message A, but its dynamics about message B disagrees with the expected spreading process of message B. See Fig. 4c for a pair of examples.
(d) 468 linear UABU models for which both $A(t)$ and $B(t)$ approach nonzero values. By observations, we find that for each of the models, its dynamics obviously deviates from the expected competing spreading process. See Fig. 4d for a pair of examples.

Example 2 Small-world networks are another large class of networks having widespread applications [59]. Take a randomly generated small-world network with 100 nodes as both the A-spreading network and the B-spreading network. By taking random combinations of the parameters, we get 4096 pairs of linear and exact UABU models, which are divided into the following four collections.

(a) 151 linear UABU models for which both $A(t)$ and $B(t)$ approach zero. By observations, we find that for each of the models, its dynamics fits with the expected competing spreading process. See Fig. 5a for a pair of examples.
(b) 1657 linear UABU models for which $A(t)$ approaches a nonzero value, but $B(t)$ approaches zero. By observations, we find that for each of the

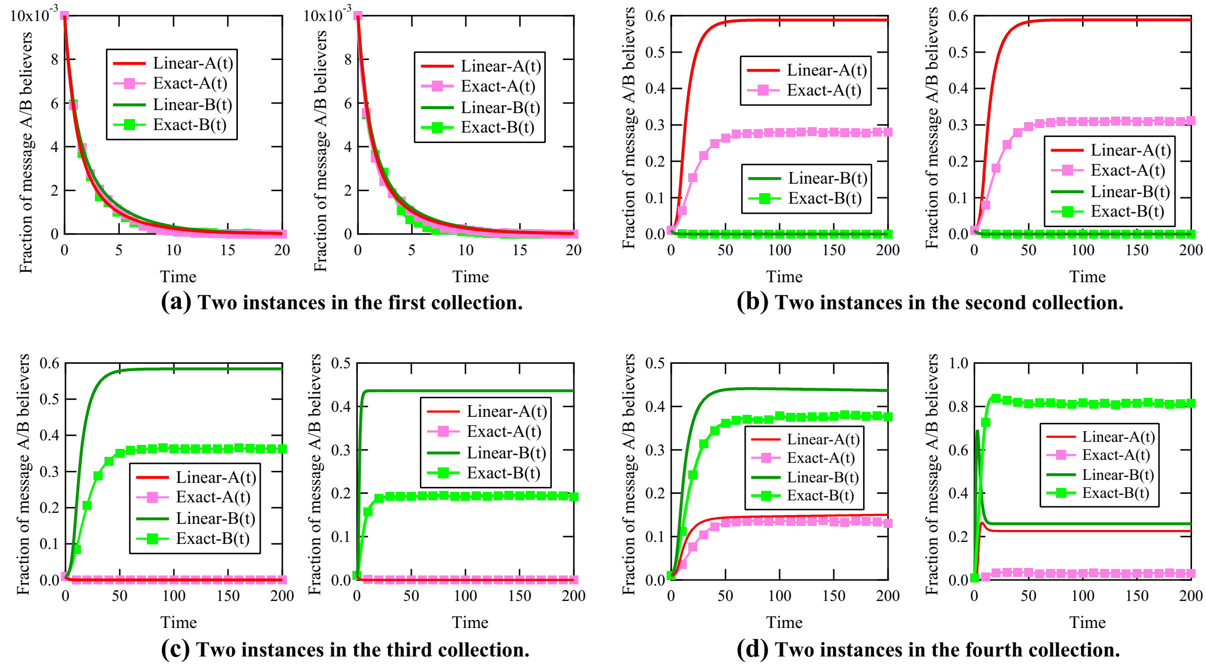

Fig. 4 Comparison between the linear UABU models in Example 1 and the corresponding exact UABU models

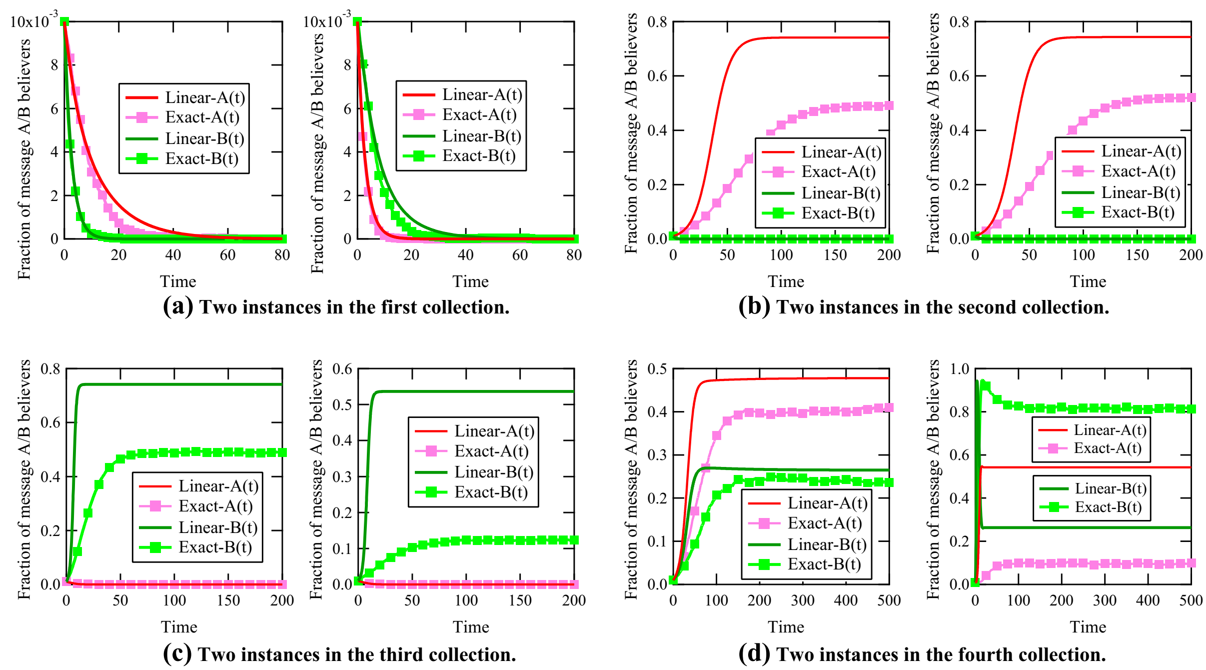

Fig. 5 Comparison between the linear UABU models in Example 2 and the corresponding exact UABU models

models, its dynamics about message B accords with the expected spreading process of message B, but its dynamics about message A deviates significantly from the expected spreading process of message A. See Fig. 5b for a pair of examples.

(c) 1639 linear UABU models for which $A(t)$ approaches zero, but $B(t)$ approaches a nonzero value. By observations, we find that for each of the models, its dynamics about message A conforms to the expected spreading process of message A, but its dynamics about message B disagrees with the expected spreading process of message B. See Fig. 5c for a pair of examples.

(d) 649 linear UABU models for which both $A(t)$ and $B(t)$ approach nonzero values. By observations, we find that for each of the models, its dynamics obviously deviates from the expected competing spreading process. See Fig. 5d for a pair of examples.

The following conclusions are drawn from the previous examples and many similar examples.

(a) When $A(t)$ approaches zero, the dynamics of the linear UABU model fits well with the expected spreading process of message A. However, when $A(t)$ approaches a nonzero value, there is a significant difference between the dynamics of the linear UABU model and the expected spreading process of message A.

(b) When $B(t)$ approaches zero, the dynamics of the linear UABU model fits perfectly with the expected spreading process of message B. However, when $B(t)$ approaches a nonzero value, there is a remarkable difference between the dynamics of the linear UABU model and the expected spreading process of message B.

In the case where the linear UABU model works well, it can be employed to quickly predict the expected evolutionary process of message A or/and message B.

In the case where the linear UABU model does not work well, we have to resort to a generic UABU model with proper nonlinear spreading rates to achieve the goal of accurate prediction. In this context, the deep learning methodology may be employed to accurately estimate the spreading rates [60].

5 Concluding remarks

This paper has investigated the competing spreading dynamics of two conflicting messages. Based on a novel individual-level competing spreading model (the generic UABU model), three criteria for one or two messages to terminate have been presented. These criteria manifest the influence of the two message-spreading networks on the evolution of the two messages. Extensive simulation experiments have shown that when a message terminates, the dynamics of a simplified UABU model fits well with the expected evolutionary process of the message.

Toward the direction, lots of problems have yet to be resolved. Under the generic UABU model, a criterion for the existence/attractivity of a coexistent equilibrium should be figured out, and the cost paid for restraining an undesirable message must be minimized [61–63]. It is known that quarantining the influential persons who are spreading an undesirable message is an effective measure of containing the prevalence of the message [16]. Hence, it is valuable to develop a competing spreading model that accommodates the quarantine effect. In the context of individual-level competing spreading models, it is of practical importance to understand the influence of a variety of real-world factors on the spread of conflicting messages. Last, it is rewarding to investigate the formation of the dynamic pattern of the population state [64–68].

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Appendix A: Proof of Lemma 1

Given a sufficiently small time interval $\Delta t > 0$, it follows from the total probability formula that

$$\begin{aligned} &A_i(t+\Delta t) \\ &= (1 - A_i(t) - B_i(t)) \Pr\{X_i(t+\Delta t) = 1 \mid X_i(t) = 0\} \\ &\quad + A_i(t) \Pr\{X_i(t+\Delta t) = 1 \mid X_i(t) = 1\} \\ &\quad + B_i(t) \Pr\{X_i(t+\Delta t) = 1 \mid X_i(t) = 2\}, 1 \le i \le N. \end{aligned} \tag{A.1}$$

By the conditional total probability formula and in view of the model (2), we get that

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=0\}\\
&\quad=\sum_{\mathbf{x}\in\{0,1,2\}^N, x_i=0} \Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=0,\\
&\qquad \mathbf{X}(t)=\mathbf{x}\}\cdot \Pr\{\mathbf{X}(t)=\mathbf{x} \mid X_i(t)=0\}\\
&\quad=\frac{\Delta t}{1-A_i(t)-B_i(t)}\cdot \sum_{\mathbf{x}\in\{0,1,2\}^N, x_i=0}\sum_{j=1}^N \beta_{ij}^{UA} 1_{\{x_j=1\}}\\
&\qquad \cdot \Pr\{\mathbf{X}(t)=\mathbf{x}\}+o(\Delta t)\\
&\quad=\frac{\Delta t}{1-A_i(t)-B_i(t)}\cdot \sum_{j=1}^N \beta_{ij}^{UA} \sum_{\mathbf{x}\in\{0,1,2\}^N} 1_{\{x_i=0, x_j=1\}}\\
&\qquad \cdot \Pr\{\mathbf{X}(t)=\mathbf{x}\}+o(\Delta t)\\
&\quad=\frac{\Delta t}{1-A_i(t)-B_i(t)} \sum_{j=1}^N \beta_{ij}^{UA} \Pr\{X_i(t)=0,\\
&\qquad X_j(t)=1\}+o(\Delta t), \quad 1\le i\le N. \qquad \text{(A.2)}
\end{aligned}$$

Similarly, we can derive that

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=2 \mid X_i(t)=1\}\\
&\quad=\frac{\Delta t}{A_i(t)}\cdot \sum_{j=1}^N \beta_{ij}^{AB} \Pr\{X_i(t)=1, X_j(t)=2\}\\
&\qquad +o(\Delta t), \quad 1\le i\le N \qquad \text{(A.3)}
\end{aligned}$$

and that

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=0 \mid X_i(t)=1\}=\delta_i^A \Delta t+o(\Delta t),\\
&\quad 1\le i\le N. \qquad \text{(A.4)}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=1\}\\
&\quad=1-\frac{\Delta t}{A_i(t)}\cdot \sum_{j=1}^N \beta_{ij}^{AB} \Pr\{X_i(t)=1, X_j(t)=2\}\\
&\qquad -\delta_i^A \Delta t+o(\Delta t), \quad 1\le i\le N. \qquad \text{(A.5)}
\end{aligned}$$

Besides, we have

$$\begin{aligned}
&\Pr\{X_i(t+\Delta t)=1 \mid X_i(t)=2\}\\
&\quad=\frac{\Delta t}{B_i(t)}\cdot \sum_{j=1}^N \beta_{ij}^{BA} \Pr\{X_i(t)=2, X_j(t)=1\}\\
&\qquad +o(\Delta t), \quad 1\le i\le N. \qquad \text{(A.6)}
\end{aligned}$$

Substituting these equations into Eq. (A.1), rearranging the terms, dividing both sides by Δt and letting $\Delta t\to 0$, we get that

$$\begin{aligned}
\frac{dA_i(t)}{dt}&=\sum_{j=1}^N \beta_{ij}^{UA} \Pr\{X_i(t)=0, X_j(t)=1\}\\
&\quad+\sum_{j=1}^N \beta_{ij}^{BA} \Pr\{X_i(t)=2, X_j(t)=1\}\\
&\quad-\sum_{j=1}^N \beta_{ij}^{AB} \Pr\{X_i(t)=1, X_j(t)=2\}\\
&\quad-\delta_i^A A_i(t), \quad 1\le i\le N. \qquad \text{(A.7)}
\end{aligned}$$

The last N equations in Lemma 1 can be derived in an analogous way. The proof is complete.

Appendix B: Proof of Theorem 1

(a) Suppose the model (10) admits a A-dominant equilibrium $\mathbf{E}=(A_1,\ldots,A_N,0,\ldots,0)^T$. Let $\mathbf{A}=(A_1,\ldots,A_N)^T$. We show that $\mathbf{0}<\mathbf{A}<\mathbf{1}$. It follows from the model that

$$A_i=\frac{f_i^{UA}(\mathbf{A})}{\delta_i^A+f_i^{UA}(\mathbf{A})}<1, \quad 1\le i\le N. \qquad \text{(B.1)}$$

Hence, $\mathbf{A}<\mathbf{1}$. On the contrary, suppose that some $A_k=0$. It follows from the model (10) that $f_k^{UA}(\mathbf{A})=0$. As G_A is strongly connected, we get that some $\beta_{kl}^{UA}>0$, implying that $A_l=0$. Repeating this argument, we finally get that $\mathbf{A}=\mathbf{0}$, contradicting the assumption that $\mathbf{E}$ is a A-dominant equilibrium. Hence, $\mathbf{A}>\mathbf{0}$.

Define a continuous mapping $\mathbf{H}=(H_1,\ldots,H_N)^T: (0,1]^N\to(0,1]^N$ by

$$H_i(\mathbf{x})=\frac{f_i^{UA}(\mathbf{x})}{\delta_i^A+f_i^{UA}(\mathbf{x})}, \quad \mathbf{x}=(x_1,\ldots,x_N)^T\in(0,1]^N. \qquad \text{(B.2)}$$

It suffices to show that $\mathbf{H}$ has a unique fixed point. Let $\mathbf{B}(t)\equiv\mathbf{0}$ and rewrite the model (10) as

$$\frac{d\mathbf{A}(t)}{dt}=\mathbf{C}_A\mathbf{A}(t)+\mathbf{G}(\mathbf{A}(t)), \qquad \text{(B.3)}$$

where $\mathbf{G}(\mathbf{A}(t))=o(\|\mathbf{A}(t)\|)$. By Lemma 3, $\mathbf{C}_A$ has a positive eigenvector $\mathbf{v}=(v_1,\ldots,v_N)^T$ belonging to the eigenvalue $s(\mathbf{C}_A)$. As $s(\mathbf{C}_A)>0$, we have $\mathbf{C}_A\mathbf{v}=s(\mathbf{C}_A)\mathbf{v}>\mathbf{0}$. Hence, there is a small $\varepsilon>0$ such that

$$\mathbf{C}_A \cdot (\varepsilon \mathbf{v}) + \mathbf{G}(\varepsilon \mathbf{v}) = \varepsilon s(\mathbf{C}_A)\mathbf{v} + \mathbf{G}(\varepsilon \mathbf{v}) \geq \mathbf{0}, \quad \text{(B.4)}$$

which is equivalent to $\mathbf{H}(\varepsilon \mathbf{v}) \geq \varepsilon \mathbf{v}$. On the other hand, it is easily verified that $\mathbf{H}$ is monotonically increasing, i.e., $\mathbf{u} \geq \mathbf{w}$ implies $\mathbf{H}(\mathbf{u}) \geq \mathbf{H}(\mathbf{w})$. Define a compact convex set as $K = \prod_{i=1}^{N} [\varepsilon v_i, 1]$. Then, $\mathbf{H}|_K$ maps K into K. It follows from Lemma 9 that $\mathbf{H}$ has a fixed point in K. Denote this fixed point by $\mathbf{A}^* = (A_1^*, \ldots, A_N^*)^T$.

Suppose $\mathbf{H}$ has a fixed point $\mathbf{A}^{**} = (A_1^{**}, \ldots, A_N^{**})^T$ other than $\mathbf{A}^*$. Let

$$\theta = \max_{1 \leq i \leq N} \frac{A_i^*}{A_i^{**}}, \quad i_0 = \arg \max_{1 \leq i \leq N} \frac{A_i^*}{A_i^{**}}. \quad \text{(B.5)}$$

Without loss of generality, assume $\theta > 1$. It follows that

$$\begin{aligned} A_{i_0}^* &= H_{i_0}(\mathbf{A}^*) \leq H_{i_0}(\theta \mathbf{A}^{**}) \frac{f_{i_0}^{UA}(\theta \mathbf{A}^{**})}{\delta_{i_0}^A + f_{i_0}^{UA}(\theta \mathbf{A}^{**})} \\ &< \frac{f_{i_0}^{UA}(\theta \mathbf{A}^{**})}{\delta_{i_0}^A + f_{i_0}^{UA}(\mathbf{A}^{**})} \leq \frac{\theta f_{i_0}^{UA}(\mathbf{A}^{**})}{\delta_{i_0}^A + f_{i_0}^{UA}(\mathbf{A}^{**})} \\ &= \theta H_{i_0}(\mathbf{A}^{**}) = \theta A_{i_0}^{**}, \end{aligned} \quad \text{(B.6)}$$

where $f_{i_0}^{UA}(\theta \mathbf{A}^{**}) \leq \theta f_{i_0}^{UA}(\mathbf{A}^{**})$ follows from the concavity of $f_{i_0}^{UA}$. This contradicts the assumption that $A_{i_0}^* = \theta A_{i_0}^{**}$. Hence, $\mathbf{A}^*$ is the unique fixed point of $\mathbf{H}$. The proof is complete.

(b) The argument is analogous to that for Claim (a) and hence is omitted.

Appendix C: Proof of Theorem 2

Let $(\mathbf{A}(t)^T, \mathbf{B}(t)^T)^T$ be a solution to the model (10). It follows from the first N equations of the model (13), which is an equivalent form of the model (10), that

$$\frac{d\mathbf{A}(t)}{dt} \leq (\mathbf{I}_N - diag\mathbf{A}(t))\mathbf{f}_{UA}(\mathbf{A}(t)) - \mathbf{D}_A\mathbf{A}(t). \quad \text{(C.1)}$$

Consider the comparison system

$$\frac{d\mathbf{u}(t)}{dt} = (\mathbf{I}_N - diag\mathbf{u}(t))\mathbf{f}_{UA}(\mathbf{u}(t)) - \mathbf{D}_A\mathbf{u}(t) \quad \text{(C.2)}$$

with $\mathbf{u}(0) = \mathbf{A}(0)$. This system admits the trivial equilibrium $\mathbf{0}$. Moreover, it follows from Lemma 7 that $\mathbf{u}(t) \geq \mathbf{A}(t) \geq \mathbf{0}$. We proceed by distinguishing two possibilities.

Case 1 $s(\mathbf{C}_A) < 0$. By Lemma 5, there is a positive definite diagonal matrix $\mathbf{P}_1$ such that $\mathbf{C}_A^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{C}_A$ is negative definite. Let $\mathbf{u} = (u_1, \ldots, u_N)^T$, and define a positive definite function as

$$V_1(\mathbf{u}) = \mathbf{u}^T\mathbf{P}_1\mathbf{u}. \quad \text{(C.3)}$$

By calculations, we get that

$$\begin{aligned} \frac{dV_1(\mathbf{u}(t))}{dt} \Big|_{(C.2)} &= 2\mathbf{u}(t)^T\mathbf{P}_1 \frac{d\mathbf{u}(t)}{dt} \\ &\leq 2\mathbf{u}(t)^T\mathbf{P}_1 \left[\mathbf{f}_{UA}(\mathbf{u}(t)) - \mathbf{D}_R\mathbf{u}(t)\right] \\ &\leq 2\mathbf{u}(t)^T\mathbf{P}_1\mathbf{C}_A\mathbf{u}(t) \\ &= \mathbf{u}(t)^T[\mathbf{C}_A^T\mathbf{P}_1 + \mathbf{P}_1\mathbf{C}_A]\mathbf{u}(t) \leq 0. \end{aligned} \quad \text{(C.4)}$$

Here, the second inequality follows from the concavity of $\mathbf{f}_{UA}(\mathbf{x}) - \mathbf{D}_A\mathbf{x}$. Furthermore, $\frac{dV_1(\mathbf{u}(t))}{dt}\big|_{(C.2)} = 0$ if and only if $\mathbf{u}(t) = \mathbf{0}$. According to the LaSalle invariance principle (Corollary 4.1 in [52]), the trivial equilibrium $\mathbf{0}$ of the system (C.2) is asymptotically stable for $[0, 1]^N$.

Case 2: $s(\mathbf{C}_A) = 0$. By Lemma 6, there is a positive definite diagonal matrix $\mathbf{P}_2$ such that $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$ is negative semi-definite. Define a positive definite function as

$$V_2(\mathbf{u}) = \mathbf{u}^T\mathbf{P}_2\mathbf{u}. \quad \text{(C.5)}$$

Similarly, we have

$$\frac{dV_2(\mathbf{u}(t))}{dt} \Big|_{(C.2)} \leq \mathbf{u}(t)^T[\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A]\mathbf{u}(t) \leq 0. \quad \text{(C.6)}$$

If $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$ is negative definite, the subsequent argument is analogous to that for Case 1. Now, assume $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$ is not negative definite, which implies

$$s(\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A) = 0. \quad \text{(C.7)}$$

As $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$ is Metzler and irreducible, it follows from Lemma 3 that (a) 0 is a simple eigenvalue of $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$, and (b) up to scalar multiple, $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$ has a positive eigenvector belonging to eigenvalue 0. Obviously, $\frac{dV_2(\mathbf{u}(t))}{dt}\big|_{(C.2)} = 0$ if

$\mathbf{u}(t) = \mathbf{0}$. On the contrary, suppose $\frac{\mathrm{d}V_2(\mathbf{u}(t))}{\mathrm{d}t}|_{(C.2)} = 0$ for some $\mathbf{u}(t) \geq \mathbf{0}$. If $\mathbf{u}(t) > \mathbf{0}$, then $\mathbf{f}_{UA}(\mathbf{u}(t)) > \mathbf{0}$, implying $\frac{\mathrm{d}V_2(\mathbf{u}(t))}{\mathrm{d}t}|_{(C.2)} < 0$, a contradiction. If $\mathbf{u}(t)$ has a zero component, then $\mathbf{u}(t)$ is not an eigenvector of $\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A$ belonging to eigenvalue 0. It follows from the Rayleigh formula (Theorem 4.2.2 in [47]) that

$$\mathbf{u}(t)^T[\mathbf{C}_A^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{C}_A]\mathbf{u}(t) < 0, \tag{C.8}$$

implying $\frac{\mathrm{d}V_2(\mathbf{u}(t))}{\mathrm{d}t}|_{(C.2)} < 0$, again a contradiction. Hence, $\mathbf{u}(t) = \mathbf{0}$ if $\frac{\mathrm{d}V_2(\mathbf{u}(t))}{\mathrm{d}t}|_{(C.2)} = 0$. It follows from the LaSalle invariance principle that the trivial equilibrium $\mathbf{0}$ of the system (C.2) is asymptotically stable with respect to $[0, 1]^N$.

Combining Cases 1 and 2, we get $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$. According to Lemma 7, we get $\mathbf{A}(t) \leq \mathbf{u}(t)$, which implies $\mathbf{R}(t) \to \mathbf{0}$ as $t \to \infty$.

Similarly, we can derive that $\mathbf{B}(t) \to \mathbf{0}$ as $t \to \infty$. The proof is complete.

Appendix D: Proof of Corollary 1

(a) We first show $s(\mathbf{C}_A) < 0$. As $\mathbf{C}_A\mathbf{D}_A^{-1}$ is Metzler and irreducible, and it follows from Lemma 3 that $\mathbf{C}_A\mathbf{D}_A^{-1}$ has a positive eigenvector $\mathbf{x}$ belonging to eigenvalue $s(\mathbf{C}_A\mathbf{D}_A^{-1})$. So,

$$\left(\mathbf{C}_A\mathbf{D}_A^{-1} + \mathbf{I}_N\right)\mathbf{x} = \left[s(\mathbf{C}_A\mathbf{D}_A^{-1}) + 1\right]\mathbf{x}. \tag{D.1}$$

That is, $\mathbf{x}$ is an eigenvector of $\mathbf{C}_A\mathbf{D}_A^{-1} + \mathbf{I}_N$ belonging to eigenvalue $s(\mathbf{C}_A\mathbf{D}_A^{-1}) + 1$. It follows from Lemma 2 that

$$s(\mathbf{C}_A\mathbf{D}_A^{-1}) = \rho(\mathbf{C}_A\mathbf{D}_A^{-1} + \mathbf{I}_N) - 1 < 0. \tag{D.2}$$

By Lemma 5, there is a positive definite diagonal matrix $\mathbf{D}$ such that the matrix

$$\mathbf{P} = (\mathbf{C}_A\mathbf{D}_A^{-1})^T\mathbf{D} + \mathbf{D}(\mathbf{C}_A\mathbf{D}_A^{-1}) \tag{D.3}$$

is negative definite. Direct calculations give

$$\left[\mathbf{D}_A^{\frac{1}{2}}\mathbf{C}_A\mathbf{D}_A^{-\frac{1}{2}}\right]^T\mathbf{D} + \mathbf{D}\left[\mathbf{D}_A^{\frac{1}{2}}\mathbf{C}_A\mathbf{D}_A^{-\frac{1}{2}}\right] = \mathbf{D}_A^{\frac{1}{2}}\mathbf{P}\mathbf{D}_A^{\frac{1}{2}}. \tag{D.4}$$

As $\mathbf{D}_A^{\frac{1}{2}}\mathbf{P}\mathbf{D}_A^{\frac{1}{2}}$ is negative definite, $\mathbf{D}_A^{\frac{1}{2}}\mathbf{C}_A\mathbf{D}_A^{-\frac{1}{2}}$ is diagonally stable and hence Hurwitz. It follows that

$$s(\mathbf{C}_A) = s(\mathbf{D}_A^{\frac{1}{2}}\mathbf{C}_A\mathbf{D}_A^{-\frac{1}{2}}) < 0. \tag{D.5}$$

Similarly, we have $s(\mathbf{C}_B) < 0$. The declared result follows from Theorem 2.

(b) By the concavity of $f_i^{UA}(\mathbf{x})$, we have $\frac{\partial f_i^{UA}(\mathbf{0})}{\partial x_j} \leq \beta_{ij}^{UA}$. That is, $\mathbf{C}_A + \mathbf{D}_A \leq \mathbf{M}_{UA}$. Hence,

$$\rho(\mathbf{C}_A\mathbf{D}_A^{-1} + \mathbf{I}_N) \leq \rho(\mathbf{M}_{UA}\mathbf{D}_A^{-1}) < 1. \tag{D.6}$$

Similarly, we have $\rho(\mathbf{C}_B\mathbf{D}_B^{-1} + \mathbf{I}_N) < 1$. The claim follows from Claim (a) of this corollary.

(c) The claim follows from Claim (b) of this corollary and the well-known inequality $\rho(\mathbf{M}) \leq ||\mathbf{M}||_1$.

(d) The claim follows from Claim (b) of this corollary and the well-known inequality $\rho(\mathbf{M}) \leq ||\mathbf{M}||_\infty$.

Appendix E: Proof of Theorem 3

Let $(\mathbf{A}(t)^T, \mathbf{B}(t)^T)^T$ be a solution to the model (10) with $\mathbf{A}(0) \neq \mathbf{0}$. It follows from the last N equations of the model (9) that

$$\frac{\mathrm{d}\mathbf{B}(t)}{\mathrm{d}t} \leq (\mathbf{I}_N - diag(\mathbf{B}(t)))\mathbf{f}_{UB}(\mathbf{B}(t)) - \mathbf{D}_B\mathbf{B}(t). \tag{E.1}$$

By an argument analogous to that for Theorem 2, we get $\mathbf{B}(t) \to \mathbf{0}$. Consider the following limit system of model (5).

$$\frac{\mathrm{d}\mathbf{u}(t)}{\mathrm{d}t} = (\mathbf{I}_N - diag\mathbf{u}(t))\mathbf{f}_{UA}(\mathbf{u}(t)) - \mathbf{D}_A\mathbf{u}(t) \tag{E.2}$$

with $\mathbf{u}(0) = \mathbf{A}(0)$. Theorem 1 confirms that the system admits a unique nonzero equilibrium $\mathbf{A}^* = (A_1^*, \ldots, A_N^*)$. By Lemma 8, it suffices to show that $\mathbf{A}^*$ is asymptotically stable for $(0, 1]^N$. Given a solution $\mathbf{u}(t) = (u_1(t), \ldots, u_N(t))^T$ to the system (E.2) with $\mathbf{u}(0) > \mathbf{0}$. First, let us show the following claim.

Claim 1 $\mathbf{u}(t) > \mathbf{0}$ for all $t > 0$.

Proof of Claim 1 On the contrary, suppose there is $t_0 > 0$ such that (a) $\mathbf{u}(t) > \mathbf{0}$, $0 < t < t_0$, and (b) $u_i(t_0) = 0$ for some i. According to the smoothness of

$\mathbf{u}(t)$, we get $\frac{du_i(t_0)}{dt} = 0$, implying that $f_i^{UA}(\mathbf{u}(t_0)) = 0$. As G_A is strongly connected, there is j such that $\beta_{ij}^{UA} > 0$, which implies $u_j(t_0) = 0$. Working inductively, we conclude that $\mathbf{u}(t_0) = 0$. This contradicts the uniqueness of the solution to the system (E.2) with given initial condition. Claim 1 is proven. □

For $t > 0$, let

$$Z(\mathbf{u}(t)) = \max_{1\le k\le N} \frac{u_k(t)}{A_k^*}, \quad z(\mathbf{u}(t)) = \min_{1\le k\le N} \frac{u_k(t)}{A_k^*}. \tag{E.3}$$

Define a function V_3 as

$$V_3(\mathbf{u}(t)) = \max\{Z(\mathbf{u}(t))-1, 0\}+\max\{1-z(\mathbf{u}(t)), 0\}. \tag{E.4}$$

It is easily verified that V_3 is positive definite with respect to $\mathbf{A}^*$, i.e., (a) $V_3(\mathbf{u}(t)) \ge 0$, and (b) $V_3(\mathbf{u}(t)) = 0$ if and only if $\mathbf{u}(t) = \mathbf{A}^*$. Next, let us show that $D^+V_3(\mathbf{u}(t)) \le 0$, where D^+ stands for the upper right Dini derivative. To this end, we need to show the following two claims for $t > 0$.

Claim 2 $D^+Z(\mathbf{u}(t)) \le 0$ if $Z(\mathbf{u}(t)) \ge 1$. Moreover, $D^+Z(\mathbf{u}(t)) < 0$ if $Z(\mathbf{u}(t)) > 1$.

Claim 3 $D_+z(\mathbf{u}(t)) \ge 0$ if $z(\mathbf{u}(t)) \le 1$. Moreover, $D_+z(\mathbf{u}(t)) > 0$ if $z(\mathbf{u}(t)) < 1$. Here, D_+ stands for the lower right Dini derivative.

Proof of Claim 2 Choose k_0 such that

$$Z(\mathbf{u}(t)) = \frac{u_{k_0}(t)}{A_{k_0}^*}, \quad D^+Z(\mathbf{u}(t)) = \frac{u'_{k_0}(t)}{A_{k_0}^*}. \tag{E.5}$$

Then,

$$\frac{A_{k_0}^*}{u_{k_0}(t)} u'_{k_0}(t) = \left(1-u_{k_0}(t)\right) \frac{A_{k_0}^*}{u_{k_0}(t)} f_{k_0}^{UA}(\mathbf{u}(t)) - \delta_{k_0}^A A_{k_0}^*. \tag{E.6}$$

If $f_{k_0}^{UA}(\mathbf{u}(t)) = 0$, then $\frac{A_{k_0}^*}{u_{k_0}(t)} u'_{k_0}(t) < 0$, which implies $D^+Z(\mathbf{u}(t)) < 0$. Now assume $f_{k_0}^{UA}(\mathbf{u}(t)) > 0$, then

$$\begin{aligned}\frac{A_{k_0}^*}{u_{k_0}(t)} u'_{k_0}(t) &\le (1 - A_{k_0}^*) \frac{A_{k_0}^*}{u_{k_0}(t)} f_{k_0}^{UA}(\mathbf{u}(t)) - \delta_{k_0}^A A_{k_0}^* \\ &\le (1 - A_{k_0}^*) f_{k_0}^{UA}\left(\frac{A_{k_0}^*}{u_{k_0}(t)}\mathbf{u}(t)\right) - \delta_{k_0}^A A_{k_0}^* \\ &\le (1 - A_{k_0}^*) f_{k_0}^{UA}\left(\mathbf{A}^*\right) - \delta_{k_0}^A A_{k_0}^* = 0,\end{aligned} \tag{E.7}$$

where the second inequality follows from the concavity of $f_{k_0}^{UA}$, and the third inequality follows from the monotonicity of $f_{k_0}^{UA}$. This implies $D^+Z(\mathbf{u}(t)) \le 0$. Noting that the first inequality is strict if $Z(\mathbf{u}(t)) > 1$, we get that $D^+Z(\mathbf{u}(t)) < 0$ if $Z(\mathbf{u}(t)) > 1$. Claim 2 is proven. □

The argument for Claim 3 is analogous to that for Claim 2 and hence is omitted. Next, consider three possibilities.

Case 1: $Z(\mathbf{u}(t)) < 1$. Then, $z(\mathbf{u}(t)) < 1$ and $V_3(\mathbf{u}(t)) = 1 - z(\mathbf{u}(t))$. Hence, $D^+V_3(\mathbf{u}(t)) = -D_+z(\mathbf{u}(t)) < 0$.
Case 2: $z(\mathbf{u}(t)) > 1$. Then, $Z(\mathbf{u}(t)) > 1$ and $V_3(\mathbf{u}(t)) = Z(\mathbf{u}(t)) - 1$. Hence, $D^+V_3(\mathbf{u}(t)) = D^+Z(\mathbf{u}(t)) < 0$.
Case 3 If $Z(\mathbf{u}(t)) \ge 1$, $z(\mathbf{u}(t)) \le 1$. Then, $V_3(\mathbf{u}(t)) = Z(\mathbf{u}(t)) - z(\mathbf{u}(t))$ and $D^+V_3(\mathbf{u}(t)) = D^+Z(\mathbf{u}(t)) - D_+z(\mathbf{u}(t)) \le 0$. Moreover, the equality holds if and only if $\mathbf{u}(t) = \mathbf{A}^*$.

The declared result follows from the LaSalle invariance principle.

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