Renewal Processes and Repairable Systems

Renewal Processes and Repairable Systems

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To my wife **Demonti**,
my son **Afif**,
and my daughters **Hana** and **Nadifa**.

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Chapter 1

Introduction

Many situations in our daily life can be modelled as renewal processes. Two examples are arrivals of claims in an insurance company and arrivals of passengers at a train station, if the inter-arrival times between consecutive claims or passengers are assumed to be independent and identically distributed (iid) non-negative random variables. In these examples the number of claims or passengers arrived during the time interval $[0, t]$, $t \geq 0$, is usually taken as a formal definition of a renewal process. A mathematical definition of a renewal process is given in Subsection 1.2.2.

From renewal processes we may construct other stochastic processes. Firstly, consider the renewal-process example of the insurance claims. Consider the claim sizes as random variables and assume that they are independent and identically distributed. If we interpret the claim size as a reward, then the total amount of all claims during the time interval $[0, t]$ is an example of another process which is known as *a renewal reward process*. In this thesis we study renewal reward processes in Chapter 2.

In renewal reward processes it is usually assumed that the reward is earned all at once at the end of the 'inter-arrival times' of the corresponding renewal process. It means that only the rewards earned until the last renewal before time t are considered. The reward may also depend on the inter-arrival time. Motivated by an application in the study of traffic, see Subsection 2.2.1, it is interesting to consider a version of renewal reward processes where the reward is a function of an inter-arrival time length, instantaneously earned, and the reward earned in an *incomplete time interval* is also taken into account. We call this version in this thesis *an instantaneous reward process*.

Now consider the second example of renewal processes about the arrivals of passengers at a train station. Suppose that a train just departed at time 0, and there were no passengers left. We are interested in the waiting time from when passengers arrive (after time 0) until the departure of the next train at some time $t \geq 0$. One of the quantities of interest is the total waiting time of all passengers during the time interval $[0, t]$. This is an example of a class of stochastic processes which we will call *an integrated renewal process*. We will study integrated renewal processes in Chapter 3.

Integrated renewal processes have a connection with *shot noise processes*. In the shot noise processes usually it is assumed that shocks occur according to a Poisson process. Then associated with the ith shock, which occurred at time $S_i > 0, i \geq 1$, is a random variable X_i which represents the 'value' of that shock. The values of the shocks are assumed to be iid and independent of their arrival process. The value of the ith shock at time $t \geq S_i$ equals $X_i\psi(t - S_i)$, where $\psi(x) \geq 0$ for $x \geq 0$, and $\psi(x) = 0$ otherwise. Then the total value of all shocks at time t is a shot noise process. As we will see in Chapter 3, if we take $X_i \equiv 1$ for all $i \geq 1$ and $\psi(x) = x1_{[0,\infty)}(x)$, then the corresponding shot noise process is an 'integrated Poisson process'.

The next topic that we will study in this thesis is the *total downtime of repairable systems*. We first consider a system, regarded as a single component, that can be in two states, either up (in operation) or down (under repair). We suppose that the system starts to operate at time 0. After the first failure the system is repaired, and then functions like the new system. Similarly after the second failure the system is repaired, and functions again as good as the new one, and so on. We will assume that the alternating sequence of up and down times form a so called *alternating renewal process*. One of the interesting quantities to consider is the total downtime of the system during the time interval $[0, t]$. Note that the total downtime can be considered as a reward process on an alternating renewal process. The total downtime is important because it can be used as a performance measure of the system. We will study the total downtime in Chapter 4. In this chapter we also consider repairable systems comprising $n \geq 2$ independent components.

Expressions for the marginal distributions of the renewal reward processes (including the instantaneous reward processes), the integrated renewal processes, and the total downtime of repairable systems, in a *finite time interval* $[0, t]$, are derived in this thesis. Our approach is based on the theory of point processes, especially Poisson point processes. The idea is to represent the processes that we study (including the total downtime of repairable systems) as functionals of Poisson point processes. Important tools we will use are the Palm formula, and the Laplace functional of a Poisson point process. Usually we obtain the marginal distributions of the processes in the form of Laplace transforms.

Asymptotic properties of the processes that we study are also investigated. We use Tauberian theorems to derive asymptotic properties of the expected value of the processes from their Laplace-transform expressions. Other asymptotic properties of the processes like asymptotic variances and asymptotic distributions are studied.

The rest of this chapter is organized as follows. In the next section we give an overview of the literature which has a connection with the study of the renewal and instantaneous reward processes, the integrated renewal processes and the total downtime of repairable systems. We also explain the position of our contributions. In Section 1.2 we introduce some basic notions and facts about point processes and summarize some facts about renewal processes which will be used in the subsequent chapters. Finally in Section 1.3 we give the outline of this thesis.

1.1 Related works

Renewal reward processes have been discussed by many authors. Several results are known in the literature. For example, the renewal-reward theorem and its expected-value version, and the asymptotic normality of the processes can be found in Wolff [49] and Tijms [46]. These asymptotic properties are frequently used in applications of these processes, see for example Csenki [6], Popova and Wu [31], Parlar and Perry [29], and Herz *et al.* [20].

Herz *et al.* [20] modelled the maximal number of cars that can safely cross the traffic stream on a one-way road during the time interval $[0, t]$ as a renewal reward process. For large t the renewal-reward theorem and its expected-value version have been used. In this application it is interesting to consider the case when t is small. So we need to know the distribution properties of renewal reward processes (and also instantaneous reward processes) in every finite time interval $[0, t]$. Several authors have discussed this case. An integral equation for the expected value of renewal reward processes was studied by Jack [23]. Mi [25] gave bounds for the average reward over a finite time interval. Erhardsson [12] studied an approximation of a stationary renewal reward process in a finite time interval by a compound Poisson distribution.

In this thesis we derive expressions for the marginal distributions of renewal and instantaneous reward processes in a *finite* time interval. We give an application to the study of traffic. We also reconsider asymptotic properties of renewal reward processes. A proof of the expected-value version of the renewal-reward theorem by means of a Tauberian theorem is given. Asymptotic normality of instantaneous reward processes is proved. Another result that we derive is the covariance structure of a renewal process, which is a special case of renewal reward processes. We also study a topic about system reliability in a stress-strength model which is closely related to renewal reward processes.

It seems that the terminology 'integrated renewal process' has not been known yet in the literature. But its special case where the corresponding renewal process is a homogeneous Poisson process has been considered by several authors. In Example 2.3(A) of Ross [36] the expected value of an 'integrated homogeneous Poisson process' has been calculated. The probability density function (pdf) of an integrated homogeneous Poisson process can be found in Gubner [19]. In this thesis we derive the marginal distributions of integrated renewal processes. We also consider other natural generalizations of the integrated homogeneous Poisson process, where the homogeneous Poisson process is generalized into the non-homogeneous one and a Cox process (doubly stochastic Poisson process). Asymptotic properties of the integrated Poisson and renewal processes are also investigated.

The distribution of the total downtime of a repairable system has been widely discussed in a number of papers. An expression for the cumulative distribution function (cdf) of the total downtime up to a given time t has been derived by several authors using different methods. In Takács [44] the total probability theorem has been used. The derivation in Muth [26] is based on consideration of the excess time. Finally in Funaki and Yoshimoto [13] the cdf of the total downtime is derived by a conditioning technique. Srinivasan *et al.* [37] derived an expression for the pdf of the total uptime, which has an obvious relation to the total downtime, of a repairable system. They also discussed the covariance structure of the total uptime. For longer time intervals, Takács [44] and Rényi [33] proved that the limiting distribution of the total downtime is a normal distribution. Takács [44] also discussed asymptotic mean and variance of the total downtime. In all these papers it is assumed that *the failure times and the repair times are independent*. Usually it is also assumed that the failure times are iid, and the same for the repair times. An exception is in Takács [45], where the iid assumption has been dropped, but still under the assumption of independence between the failure times and repair times. In his paper [45] Takács discussed some possibilities of asymptotic distributions of the total downtime.

In this thesis we use a different method for the computation of the distribution of the total downtime. We also consider a more general situation where we allow *dependence* of the failure time and the repair time. Some asymptotic properties, which are generalizations of the results of Takács [44] and Rényi [33], of the total downtime are derived. We also discuss the total downtime of repairable systems consisting of $n \geq 2$ independent component.

1.2 Basic notions

1.2.1 Point processes

A point process is a random distribution of points in some space E. In this thesis we will assume that E is a locally compact Hausdorff space with a countable base. As an example E is a subset of an Euclidean space of finite dimension. These assumptions ensure the existence of a Poisson point process in an infinite space, among others, which will be used in the next chapters.

The concept of a point process is formally described as follows. Let $\mathcal{E}$ be the Borel σ-algebra of subsets of E, i.e., the σ-algebra generated by the open sets. For $x \in E$, define the measure δ_x on $(E, \mathcal{E})$ by

$$\delta_x(A) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

for $A \in \mathcal{E}$. The measure δ_x is called the Dirac measure in x. A point measure on $(E, \mathcal{E})$ is a Radon measure μ, a non-negative measure μ with property $\mu(K) < \infty$ for every compact set K, which has a representation:

$$\mu = \sum_{i \in I} \delta_{x_i},$$

where I is a countable index set and $x_i \in E$. A point measure μ is called simple if $\mu(\{x\}) \leq 1$ for all $x \in E$.

Designate by $M_p(E)$ be the space of all point measures on $(E, \mathcal{E})$. Let $\mathcal{M}_p(E)$ be the smallest σ-algebra making all evaluation maps

$$m \in M_p(E) \mapsto m(A) \in \overline{\mathbb{N}}_0, \quad A \in \mathcal{E},$$

measurable, where $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$ and $\mathbb{N}_0$ denotes the set of all non-negative integers.

Definition 1.2.1 *A point process on E is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ into the measurable space $(M_p(E), \mathcal{M}_p(E))$.*

So if N is a point process on E and $\omega \in \Omega$ then $N(\omega)$ is a point measure on $(E, \mathcal{E})$. The probability distribution $\mathbf{P}_N$ of the point process N is the measure $\mathbf{P} \circ N^{-1}$ on $(M_p(E), \mathcal{M}_p(E))$.

The following proposition says that a map N from Ω into $M_p(E)$ is a point process on E if and only if $N(A)$ is an extended non-negative-valued random variable for each $A \in \mathcal{E}$.

Proposition 1.2.1 [34] *Let N be a map from a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ into the space $(M_p(E), \mathcal{M}_p(E))$. The map N is a point process if and only if for every $A \in \mathcal{E}$ the map $\omega \mapsto N(\omega, A)$ from $(\Omega, \mathcal{F}, \mathbf{P})$ into $(\overline{\mathbb{N}}_0, \mathcal{P}(\overline{\mathbb{N}}_0))$ is measurable, where $\mathcal{P}(\overline{\mathbb{N}}_0)$ denotes the power set of $\overline{\mathbb{N}}_0$.*

The intensity measure of a point process N is the measure ν on $(E, \mathcal{E})$ defined, for $A \in \mathcal{E}$, by

$$\begin{aligned} \nu(A) &:= \mathbf{E}[N(A)] \\ &= \int_{\Omega} N(\omega, A)\mathbf{P}(d\omega) \\ &= \int_{M_p(E)} \mu(A)\mathbf{P}_N(d\mu). \end{aligned}$$

Example 1.2.1 *Let $(X_n, n \geq 1)$ be an iid sequence of non-negative random variables. Let $S_n = \sum_{i=1}^{n} X_i$. Then*

$$N := \sum_{n=1}^{\infty} \delta_{S_n}$$

is a point process on $[0,\infty)$. The stochastic process $(N(t), t \geq 0)$ where $N(t) = N([0,t])$, the number of points in the interval $[0,t]$, is called a renewal process. In this context the interval $[0,\infty)$ is usually referred to as the interval of time. The random variable X_n is called the nth *inter-arrival time or* nth *cycle of the renewal process. The intensity measure of the renewal process $N(t)$ is given by $\nu(dt) = dm(t)$, where $m(t)$ is the renewal function, see Subsection 1.2.2.*

Let f be a non-negative measurable function defined on E. Recall that there exist simple functions f_n with $0 \leq f_n \uparrow f$ and f_n is of the form

$$f_n = \sum_{i=1}^{k_n} c_i^{(n)} 1_{A_i^{(n)}}, \quad A_i^{(n)} \in \mathcal{E},$$

where $c_i^{(n)}$ are constants and $A_i^{(n)}$, $i \leq k_n$, are disjoint. Define for the function f and $\omega \in \Omega$

$$N(f)(\omega) := \int_E f(x) N(\omega, dx).$$

This is a random variable since by the monotone convergence theorem

$$N(f)(\omega) = \lim_{n\to\infty} N(f_n)(\omega)$$

and each

$$N(f_n)(\omega) = \sum_{i=1}^{k_n} c_i^{(n)} N(\omega, A_i^{(n)})$$

is a random variable.

The Laplace functional of a point process N is defined as the function ψ_N which takes non-negative measurable functions f on E into $[0,\infty)$ by

$$\begin{aligned}
\psi_N(f) &:= \mathbf{E}\big[\exp\{-N(f)\}\big] \\
&= \int_\Omega \exp\big\{-N(f)(\omega)\big\}\mathbf{P}(d\omega) \\
&= \int_{M_p(E)} \exp\left\{-\int_E f(x)\mu(dx)\right\}\mathbf{P}_N(d\mu).
\end{aligned}$$

Proposition 1.2.2 [34] *The Laplace functional ψ_N of a point process N uniquely determines the probability distribution $\mathbf{P}_N$ of N.*

Poisson point processes

One of the most important examples of point processes is a Poisson point process.

Definition 1.2.2 *Given a Radon measure μ on $(E, \mathcal{E})$, a point process N on E is called a Poisson point process on E with intensity measure μ if N satisfies*

(a) For any $A \in \mathcal{E}$, and any non-negative integer k

$$\mathbf{P}(N(A) = k) = \begin{cases} \frac{e^{-\mu(A)}\mu(A)^k}{k!} & \text{if} \quad \mu(A) < \infty \\ 0 & \text{if} \quad \mu(A) = \infty, \end{cases}$$

(b) If $A_1, ..., A_k$ are disjoint sets in $\mathcal{E}$ then $N(A_1), ..., N(A_k)$ are independent random variables.

Example 1.2.2 *Let $E = [0, \infty)$ and the intensity measure μ satisfies $\mu([0, t]) = \lambda t$ for some positive constant λ and for any $t \geq 0$. Then the Poisson point process N is called a homogeneous Poisson point process on E with intensity or rate λ. If $\mu([0, t]) = \int_0^t \lambda(x)dx$ where $\lambda(x)$ is a non-negative function of x, then the Poisson process N is called a non-homogeneous Poisson process on E with intensity or rate function $\lambda(x)$.*

The following theorem concerning the Laplace functional of a Poisson point process will be used in the subsequent chapters.

Theorem 1.2.1 [34] *The Laplace functional of a Poisson point process N on E with intensity measure μ is given by*

$$\psi_N(f) = \exp\left\{-\int_E (1 - e^{-f(x)})\mu(dx)\right\}.$$

The next theorem says that a renewal process with inter-arrival times iid exponential random variables is a homogeneous Poisson process.

Theorem 1.2.2 [32] *Let $(X_n, n \geq 1)$ be an iid sequence of exponential random variables with parameter 1. Let $S_n = \sum_{i=1}^n X_i$ and set $N = \sum_{n=1}^\infty \delta_{S_n}$. Then N is a homogeneous Poisson process on $[0, \infty)$ with rate 1.*

Starting from a Poisson point process we may construct a new Poisson point process whose points live in a higher dimensional space.

Theorem 1.2.3 [35] *Let E_i, $i = 1, 2$ be two locally compact Hausdorff spaces with countable bases. Suppose $(X_n, n \geq 1)$ are random elements of E_1 such that*

$$\sum_{n=1}^\infty \delta_{X_n}$$

is Poisson point process on E_1 with intensity measure μ. Suppose $(Y_n, n \geq 1)$ are iid random elements of E_2 with common probability distribution $\mathbf{Q}$ and suppose the Poisson point process and the sequence (Y_n) are defined on the same probability space and are independent. Then the point process on $E_1 \times E_2$

$$\sum_{n=1}^{\infty} \delta_{(X_n, Y_n)}$$

is Poisson point process with intensity measure $\mu \times \mathbf{Q}$, where $\mu \times \mathbf{Q}(A_1 \times A_2) = \mu(A_1)\mathbf{Q}(A_2)$ for A_i measurable subset of $E_i, i = 1, 2$.

Palm distribution

Palm distribution plays an important role in the study of point processes. A Palm distribution is defined as a Radon-Nikodym derivative. Let N be a point process on E with distribution $\mathbf{P}_N$ such that $\nu = \mathbf{E}[N]$ a Radon measure. Let $B \in \mathcal{M}_p(E)$ and let $1_B : M_p(E) \mapsto \{0, 1\}$ be the indicator function, i.e.,

$$1_B(\mu) = \begin{cases} 1, & \mu \in B \\ 0, & \mu \notin B \end{cases}.$$

Consider the measure $\mathbf{E}[1_B(N)N]$ which is absolutely continuous with respect to ν. By the Radon-Nikodym theorem there exists a unique almost surely (ν) function $P_x(B) : E \mapsto [0, 1]$ such that

$$\int_A P_x(B)\nu(dx) = \int_{M_p(E)} 1_B(\mu)\mu(A)\mathbf{P}_N(d\mu)$$

for all $A \in \mathcal{E}$. The family $\{P_x(B) : x \in E, B \in \mathcal{M}_p(E)\}$ can be chosen so that P_x is a probability measure on $\mathcal{M}_p(E)$ for all $x \in E$ and so that for each $B \in \mathcal{M}_p(E)$ the function $x \mapsto P_x(B)$ is measurable, see Grandell [17]. The measure P_x is called the Palm distribution (or Palm measure) of the point process N.

Let N be a Poisson point process with intensity measure ν and distribution $\mathbf{P}_\nu$.

Theorem 1.2.4 [17] *For every non-negative measurable function f on $E \times M_p(E)$,*

$$\int_{M_p(E)} \int_E f(x, \mu)\mu(dx)\mathbf{P}_\nu(d\mu) = \int_E \int_{M_p(E)} f(x, \mu + \delta_x)\mathbf{P}_\nu(d\mu)\nu(dx). \quad (1.1)$$

The equation (1.1) is known as the Palm formula for a Poisson point process. We will frequently use this formula later. The Palm distribution for the Poisson point process N can be obtain by taking $f(x,\mu) = 1_A(x)1_B(\mu)$, $A \in \mathcal{E}$, $B \in M_p(E)$, which gives

$$P_x = \Delta_x * \mathbf{P}_\nu$$

where $\Delta_x * \mathbf{P}_\nu$ denotes the convolution between Δ_x, the Dirac measure in δ_x, and $\mathbf{P}_\nu$, i.e.,

$$\begin{aligned}\Delta_x * \mathbf{P}_\nu(B) &:= \int_{M_p(E)}\int_{M_p(E)} 1_B(\mu_1+\mu_2)\Delta_x(d\mu_1)\mathbf{P}_\nu(d\mu_2)\\ &= \int_{M_p(E)} 1_B(\mu+\delta_x)\mathbf{P}_\nu(d\mu).\end{aligned}$$

1.2.2 Renewal processes

In the previous section we saw that a renewal process on the non-negative real line is an example of point processes. In this section we summarize some facts about renewal processes including delayed renewal processes which will be used in the subsequent chapters. Equivalently, the notion of the renewal process in Example 1.2.1 can be stated as follows.

Definition 1.2.3 *Let $(X_i, i \geq 1)$ be an iid sequence of non-negative random variables. A renewal process $(N(t), t \geq 0)$ is a process such that*

$$N(t) = \sup\{n \geq 0 : S_n \leq t\}$$

where

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1 \quad \textit{and} \quad S_0 = 0.$$

The renewal process $(N(t))$ represents the number of occurrences of some event in the time interval $[0,t]$. We commonly interpret X_n as the time between the $(n-1)$st and nth event and call it the nth *inter-arrival time* or nth *cycle* and interpret S_n as the time of nth event or the time of nth arrival. The link between $N(t)$ and the sum S_n of iid random variables is given by

$$N(t) \geq n \quad \text{if and only if} \quad S_n \leq t. \tag{1.2}$$

The distribution of $N(t)$ can be represented in terms of the cdfs of the inter-arrival times. Let F be the cdf of X_1 and F_n be the cdf of S_n. Note that F_n is the n-fold convolution of F with itself. From (1.2) we obtain

$$\begin{aligned}\mathbf{P}(N(t)=n) &= \mathbf{P}(N(t)\geq n) - \mathbf{P}(N(t)\geq n+1)\\ &= \mathbf{P}(S_n \leq t) - \mathbf{P}(S_{n+1}\leq t)\\ &= F_n(t) - F_{n+1}(t). \end{aligned} \tag{1.3}$$

Let $m(t)$ be the expected number of events in the time interval $[0, t]$, i.e., $m(t) = \mathbf{E}[N(t)]$. The function $m(t)$ is called a renewal function. The relationship between $m(t)$ and F is given by the following proposition.

Proposition 1.2.3 [18]

$$m(t) = \sum_{n=1}^{\infty} F_n(t). \tag{1.4}$$

The renewal function $m(t)$ also satisfies the following integral equation.

Proposition 1.2.4 [18] *The renewal function m satisfies the renewal equation*

$$m(t) = F(t) + \int_0^t m(t-x)dF(x). \tag{1.5}$$

Let $\mu = \mathbf{E}(X_1)$ and $\sigma^2 = \mathrm{Var}(X_1)$ be the mean and the variance of X_1, and assume that μ and σ are strictly positif. Some asymptotic properties of the renewal process $(N(t))$ are given in the following.

Theorem 1.2.5 [18] *Assume that $\mu < \infty$. Then,*

(a)

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \textit{with probability 1},$$

(b)

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}, \tag{1.6}$$

(c)

$$m(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1) \tag{1.7}$$

where $o(1)$ denotes a function of t tending to zero as $t \to \infty$, provided σ^2 is finite.

Theorem 1.2.6 [7] *Assume that $\sigma^2 < \infty$. Then*

$$\lim_{t\to\infty} \frac{\mathrm{Var}[N(t)]}{t} = \frac{\sigma^2}{\mu^3}. \tag{1.8}$$

If $\mu_3 = \mathbf{E}(X_1^3) < \infty$, then

$$\mathrm{Var}[N(t)] = \frac{\sigma^2 t}{\mu^3} + \left(\frac{1}{12} + \frac{5\sigma^4}{4\mu^4} - \frac{2\mu_3}{3\mu^3}\right) + o(1). \tag{1.9}$$

For large t the distribution of $N(t)$ is approximately normal, i.e.,

$$\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{d} N(0,1) \quad \textit{as} \quad t \to \infty.$$

Delayed renewal processes

Let $X_0, X_1, X_2, \ldots$ be a sequence of non-negative independent random variables. Let G be the cdf of X_0 and F be the common cdfs of X_i, $i \geq 1$. Let

$$S_n = \sum_{i=0}^{n-1} X_i, \quad n \geq 1 \quad \text{and} \quad \mathrm{S}_0 = 0.$$

The stochastic process $(N_D(t), t \geq 0)$ where

$$N_D(t) = \sup\{n \geq 0 : S_n \leq t\}$$

is called a delayed renewal process. It is easy to see that

$$\mathbf{P}(N_D(t) = n) = G * F_{n-1}(t) - G * F_n(t),$$

where $F * G$ represents the convolution of F and G. The corresponding delayed renewal function satisfies

$$m_D(t) := \mathbf{E}[N_D(t)] = \sum_{n=1}^{\infty} G * F_{n-1}(t).$$

As for ordinary renewal processes, the delayed renewal processes have the following asymptotic properties:

(a) The delayed renewal function $m_D(t)$ is asymptotically a linear function of t, i.e.,

$$\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu}, \tag{1.10}$$

where $\mu = \mathbf{E}(X_1)$, see Ross [36],

(b) If $\sigma^2 = \mathrm{Var}(X_1) < \infty$, $\mu_0 = \mathbf{E}(X_0) < \infty$ and F is a non-lattice distribution, then

$$m_D(t) = \frac{t}{\mu} + \frac{\sigma^2 + \mu^2}{2\mu^2} - \frac{\mu_0}{\mu} + o(1) \tag{1.11}$$

and if $\sigma^2 < \infty$ then

$$\lim_{t\to\infty} \frac{\mathrm{Var}[N_D(t)]}{t} = \frac{\sigma^2}{\mu^3}, \tag{1.12}$$

see Takács [44].

1.3 Outline of the thesis

The subsequent chapters in this thesis can be read independently, and are mostly based on the results in Suyono and van der Weide [38, 39, 40, 41, 42, 43]. Firstly, in Chapter 2 we discuss renewal reward processes. In Section 2.2 we study a version of renewal reward processes, which we call an instantaneous reward process. In Section 2.3 we consider the marginal distribution of renewal reward processes in a finite time interval. Section 2.4 deals with asymptotic properties of the renewal and instantaneous reward processes. The covariance structure of renewal processes is studied in Section 2.5. Finally Section 2.6 is devoted to a study of system reliability in a stress-strength model, where the amplitude of a stress occurring at a time t can be considered as a reward.

Chapter 3 deals with integrated renewal processes. In Section 3.2 we consider the marginal distributions of integrated Poisson and Cox processes. In the next section we consider the marginal distributions of integrated renewal processes. Asymptotic properties of the integrated renewal processes are studied in Section 3.4. Finally in the last section we give an application.

In Chapter 4 we discuss the total downtime of repairable systems. We start with investigating the distribution of the total downtime in Section 4.2. Section 4.3 is devoted to the study of system availability, which is closely related to the total downtime. Section 4.4 concerns the covariance of the total downtime. Asymptotic properties of the total downtime for the dependent case is studied in Section 4.5. Two examples are given in the next section. Finally in Section 4.7 we consider the total downtime of repairable systems consisting of $n \geq 2$ independent components.

Most of the results about the marginal distributions of the processes that we study are in the form of Laplace transforms. In some cases the transforms can be inverted analytically, but mostly the transforms have to be inverted numerically. We give numerical inversions of Laplace transforms in Appendix B.

Chapter 2

Renewal Reward Processes

2.1 Introduction

Consider an iid sequence $(X_n, n \geq 1)$ of strictly positive random variables. Let $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$, $S_0 = 0$, and $(N(t), t \geq 0)$ be the renewal process with renewal cycles X_n. Associated with each cycle length X_n is a reward Y_n where we assume that $((X_n, Y_n), n \geq 1)$ is an iid sequence of random vectors. The variables X_n and Y_n can be dependent. The stochastic process $(R(t), t \geq 0)$ where

$$R(t) = \sum_{n=1}^{N(t)} Y_n \tag{2.1}$$

(with the usual convention that the empty sum equals 0) is called a renewal reward process. Taking $Y_n \equiv 1$, we see that renewal processes can be considered as renewal reward processes.

Motivated by an application in the study of traffic, see Subsection 2.2.1, it is interesting to consider a version of renewal reward processes where the reward is a function of cycle length, i.e,

$$R_\phi(t) = \sum_{n=1}^{N(t)} \phi(X_n) + \phi(t - S_{N(t)}) \tag{2.2}$$

where ϕ is a measurable real-valued function. We will call the process $(R_\phi(t), t \geq 0)$ *an instantaneous reward process.* We will assume that rewards are non-negative, i.e., the function ϕ is a non-negative. Note that in this process we also consider the reward earned in the incomplete renewal cycle $(S_{N(t)}, t]$. If we only consider the reward until the last renewal before time t, and take $Y_n = \phi(X_n)$ then (2.2) reduces to (2.1).

In this chapter we will study these renewal and instantaneous reward processes. Firstly, in Section 2.2, we consider the marginal distribution of the instantaneous reward process defined in (2.2). An application of this process to the study of traffic is given. We give an example where the variables X_n represent the time intervals between consecutive cars on a crossing in a one-way road, and $\phi(x)$ represents the number of cars that can cross the traffic stream safely during a time interval of length x between two consecutive cars.

In Section 2.3 we consider the marginal distributions of the renewal reward process given by formula (2.1). We will only consider the case where the random variables Y_n are non-negative. Asymptotic properties of the renewal and instantaneous reward processes are discussed in Section 2.4. We give an alternative proof for the expected-value version of the renewal-reward theorem using a Tauberian theorem. Section 2.5 deals with the covariance structure of renewal processes. The last section is devoted to the study of system reliability in a stress-strength model, where the amplitude of a stress occurring at a time t can be considered as a reward. Besides considering renewal processes as the occurrence of the stresses, in this last section we also model the occurrence of the stresses as a Cox process (doubly stochastic Poisson process).

We will denote the cdfs of the random variables X_1 and Y_1 by F and G respectively, and denote the joint cdf of X_1 and Y_1 by H, i.e.,

$$H(x, y) = \mathbf{P}(X_1 \le x, Y_1 \le y).$$

The Laplace-Stieltjes transforms of F and H will be denoted by F^* and H^*, i.e.,

$$F^*(\beta) = \int_0^\infty e^{-\beta x} dF(x)$$

and

$$H^*(\alpha, \beta) = \int_0^\infty \int_0^\infty e^{-(\alpha x + \beta y)} dH(x, y).$$

2.2 Instantaneous reward processes

In this section we will consider the marginal distributions of the instantaneous reward process as defined in (2.2). We will use the theory of point processes introduced in Chapter 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which the iid sequence $(X_n, n \ge 1)$ is defined and also an iid sequence $(U_n, n \ge 1)$ of exponentially distributed random variables with parameter 1 such that the sequences (X_n) and (U_n) are independent. Let $(T_n, n \ge 1)$ be the sequence of partial sums of the variables U_n. By Theorem 1.2.2

$$\sum_{n=1}^{\infty} \delta_{T_n}$$

is a Poisson point process on $[0,\infty)$ with intensity measure $\nu(dx) = dx$, and by Theorem 1.2.3 the map

$$\Phi \quad : \quad \omega \quad \mapsto \quad \textstyle\sum_{n=1}^{\infty} \delta_{(T_n(\omega), X_n(\omega))}$$

defines a Poisson point process on $E = [0,\infty) \times [0,\infty)$ with intensity measure $\nu(dtdx) = dtdF(x)$, where F is the cdf of X_1. Note that for almost all $\omega \in \Omega$ $\Phi(\omega)$ is a simple point measure on E satisfying $\Phi(\omega)(\{t\} \times [0,\infty)) \in \{0,1\}$ for every $t \geq 0$. Note also that $\nu([0,t] \times [0,\infty)) < \infty$ for $t \geq 0$. Let $M_p(E)$ be the set of all point measures on E. We will denote the distribution of Φ by $\mathbf{P}_\nu$, i.e., $\mathbf{P}_\nu = \mathbf{P} \circ \Phi^{-1}$.

Define for $t \geq 0$ the functional $\mathbb{A}(t)$ on $M_p(E)$ by

$$\mathbb{A}(t)(\mu) \quad = \quad \int_E 1_{[0,t)}(s) x \mu(dsdx).$$

In the sequel we will write $\mathbb{A}(t,\mu) = \mathbb{A}(t)(\mu)$. Suppose that the point measure μ has the support $\text{supp}(\mu) = ((t_n, x_n))_{n=1}^{\infty}$ with $t_1 < t_2 < \ldots$, see Figure 2.1 (a). It follows that

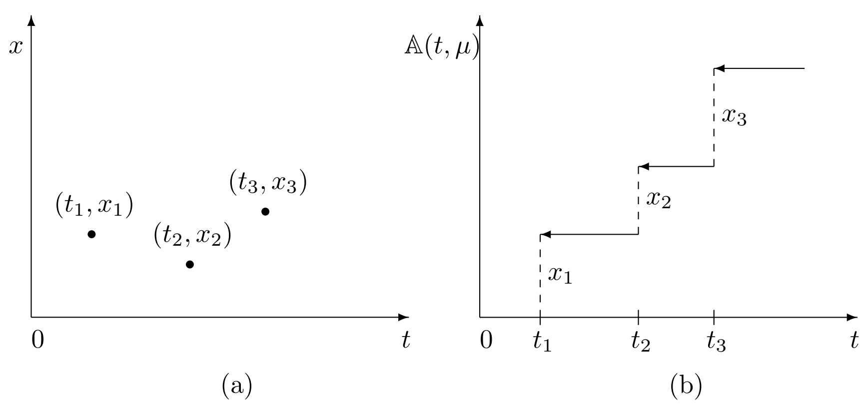


Figure 2.1: (a). Illustration of $\text{supp}(\mu)$. (b). Graph of $\mathbb{A}(t,\mu)$

$$\mu = \sum_{n=1}^{\infty} \delta_{(t_n, x_n)}$$

and $\mathbb{A}(t,\mu)$ can be expressed as

$$\mathbb{A}(t,\mu) \quad = \quad \sum_{n=1}^{\infty} 1_{[0,t)}(t_n) x_n.$$

Note that for every $t \geq 0$, $\mathbb{A}(t,\mu)$ is finite almost surely. Figure 2.1 (b) shows the graph of a realization of $\mathbb{A}(t,\mu)$.

For $(t_n, x_n) \in \text{supp}(\mu)$

$$\begin{aligned} 1_{[0,x_n)}(t - \mathbb{A}(t_n,\mu)) = 1 \quad &\Longleftrightarrow \quad \mathbb{A}(t_n,\mu) \leq t < \mathbb{A}(t_n,\mu) + x_n \\ &\Longleftrightarrow \quad x_1 + \ldots + x_{n-1} \leq t < x_1 + \ldots + x_n. \end{aligned}$$

Hence for a measurable, bounded function f on E we have

$$\int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) f(s,x) \mu(dsdx) = f(t_n, x_n)$$

where n is chosen such that $x_1 + \ldots + x_{n-1} \leq t < x_1 + \ldots + x_n$.

Now define for $t \geq 0$ the functional $\mathbb{R}_\phi(t)$ on $M_p(E)$ by

$$\mathbb{R}_\phi(t)(\mu) \quad = \quad \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \left\{ \mu(1_{[0,s)} \otimes \phi) + \phi(t - \mathbb{A}(s,\mu)) \right\} \mu(dsdx)$$

where

$$\mu(1_{[0,t)} \otimes \phi) \quad = \quad \int_E 1_{[0,t)}(s)\phi(x)\mu(dsdx).$$

Note that if

$$\mu = \sum_{n=1}^{\infty} \delta_{(t_n,x_n)}$$

with $t_1 < t_2 < \ldots$, then

$$\begin{aligned} &\mathbb{R}_\phi(t)(\mu) \\ &\quad = \sum_{n=1}^{\infty} 1_{[0,x_n)}(t - \mathbb{A}(t_n,\mu)) \left\{ \sum_{i=1}^{\infty} 1_{[0,t_n)}(t_i)\phi(x_i) + \phi\Big(t - \sum_{i=1}^{\infty} 1_{[0,t_n)}(t_i)x_i\Big) \right\} \\ &\quad = \sum_{i=1}^{n-1} \phi(x_i) + \phi\Big(t - \sum_{i=1}^{n-1} x_i\Big) \end{aligned} \tag{2.3}$$

where n satisfies $x_1 + \ldots + x_{n-1} \leq t < x_1 + \ldots + x_n$. Then we have a representation for the instantaneous reward process $(R_\phi(t))$ as a functional of the Poisson point process Φ as stated in the following lemma.

Lemma 2.2.1 *With probability 1,*

$$R_\phi(t) = \mathbb{R}_\phi(t)(\Phi).$$

Proof: Let $\omega \in \Omega$. Since $\Phi(\omega) = \sum_{n=1}^{\infty} \delta_{(T_n(\omega), X_n(\omega))}$, then using (2.3) we obtain

$$\mathbb{R}_\phi(t)(\Phi(\omega)) = \sum_{n=1}^{i-1} \phi(X_n(\omega)) + \phi(t - S_{i-1}(\omega))$$

where i satisfies $S_{i-1}(\omega) \leq t < S_i(\omega)$. But this condition holds if and only if $i = N(t, \omega) + 1$, where $N(t) = \sup\{n \geq 0 : S_n \leq t\}$, which completes the proof. $\square$

The following theorem gives the formula for the Laplace transform of the marginal distribution of the instantaneous reward process $(R_\phi(t), t \geq 0)$.

Theorem 2.2.1 *Let $(X_n, n \geq 1)$ be an iid sequence of strictly positive random variables with common cdf F. Let $(S_n, n \geq 0)$ be the sequence of partial sums of the variables X_n and $(N(t), t \geq 0)$ be the corresponding renewal process: $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Let $\phi : [0, \infty) \to [0, \infty)$ be a measurable function. Let $(R_\phi(t), t \geq 0)$ be the instantaneous reward process defined in* (2.2)*. Then for $\alpha, \beta > 0$*

$$\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)}) e^{-\beta t} dt = \frac{\int_0^\infty [1 - F(t)] e^{-\beta t - \alpha \phi(t)} dt}{1 - \int_0^\infty e^{-\beta t - \alpha \phi(t)} dF(t)}. \tag{2.4}$$

Proof: By Lemma 2.2.1

$$\begin{aligned}
\mathbf{E}(e^{-\alpha R_\phi(t)}) &= \int_{M_p(E)} e^{-\alpha \mathbb{R}_\phi(t)(\mu)} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \exp\Big\{ -\alpha \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \\
&\quad \Big[\mu(1_{[0,s)} \otimes \phi) + \phi(t - \mathbb{A}(s,\mu))\Big] \mu(dsdx) \Big\} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \\
&\quad \exp\Big\{ -\alpha \Big[\mu(1_{[0,s)} \otimes \phi) + \phi(t - \mathbb{A}(s,\mu))\Big] \Big\} \mu(dsdx) \mathbf{P}_\nu(d\mu).
\end{aligned}$$

Applying the Palm formula for Poisson point processes, see Theorem 1.2.4, we obtain

$$
\begin{aligned}
&\mathbf{E}(e^{-\alpha R_\phi(t)}) \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} 1_{[0,x)}(t - \mathbb{A}(s, \mu + \delta_{(s,x)})) \\
&\quad \exp\Big\{ - \alpha \Big[(\mu + \delta_{(s,x)})(1_{[0,s)} \otimes \phi) + \phi(t - \mathbb{A}(s, \mu + \delta_{(s,x)})) \Big] \Big\} \\
&\quad \mathbf{P}_\nu(d\mu) dF(x) ds \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} 1_{[0,x)}(t - \mathbb{A}(s, \mu)) \\
&\quad \exp\Big\{ - \alpha \Big[\mu(1_{[0,s)} \otimes \phi) + \phi(t - \mathbb{A}(s, \mu)) \Big] \Big\} \mathbf{P}_\nu(d\mu) dF(x) ds.
\end{aligned}
$$

Using Fubini's theorem and a substitution we obtain

$$
\begin{aligned}
&\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)}) e^{-\beta t} dt \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} \int_0^\infty 1_{[0,x)}(t - \mathbb{A}(s, \mu)) \\
&\quad \exp\Big\{ - \alpha \Big[\mu(1_{[0,s)} \otimes \phi) + \phi(t - \mathbb{A}(s, \mu)) \Big] \Big\} e^{-\beta t} dt \mathbf{P}_\nu(d\mu) dF(x) ds \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} \exp\Big\{ - \Big[\beta \mathbb{A}(s, \mu) + \alpha \mu(1_{[0,s)} \otimes \phi) \Big] \Big\} \mathbf{P}_\nu(d\mu) \\
&\quad \Big[\int_0^x \exp\Big\{ - \beta t - \alpha \phi(t) \Big\} dt \Big] dF(x) ds.
\end{aligned}
$$

The integral with respect to $\mathbf{P}_\nu$ can be calculated as follows. Note that

$$
\beta \mathbb{A}(s, \mu) + \alpha \mu(1_{[0,s)} \otimes \phi) = \int_E 1_{[0,s)}(r) [\beta u + \alpha \phi(u)] \mu(dr du).
$$

So we can apply the formula for the Laplace functional of Poisson point processes, see Theorem 1.2.1, to obtain

$$
\begin{aligned}
&\int_{M_p(E)} \exp\Big\{ - \Big[\beta \mathbb{A}(s, \mu) + \alpha \mu(1_{[0,s)} \otimes \phi) \Big] \Big\} \mathbf{P}_\nu(d\mu) \\
&= \exp\Big\{ - \int_0^\infty \int_0^\infty \Big[1 - e^{-1_{[0,s)}(r)[\beta u + \alpha \phi(u)]} \Big] dF(u) dr \Big\} \\
&= \exp\Big\{ -s \int_0^\infty [1 - e^{-\beta u - \alpha \phi(u)}] dF(u) \Big\}.
\end{aligned}
$$

It follows that

$$\begin{aligned}\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)})e^{-\beta t}dt &= \int_0^\infty \int_0^\infty \exp\Big\{-s\int_0^\infty [1-e^{-\beta u-\alpha\phi(u)}]dF(u)\Big\} \\ &\quad \Big[\int_0^x \exp\Big\{-\beta t-\alpha\phi(t)\Big\}dt\Big]dF(x)ds \\ &= \int_0^\infty \exp\Big\{-s\int_0^\infty [1-e^{-\beta u-\alpha\phi(u)}]dF(u)\Big\}ds \\ &\quad \int_0^\infty \int_0^x \exp\Big\{-\beta t-\alpha\phi(t)\Big\}dtdF(x) \\ &= \frac{\int_0^\infty [1-F(t)]e^{-\beta t-\alpha\phi(t)}dt}{1-\int_0^\infty e^{-\beta u-\alpha\phi(u)}dF(u)}. \quad \square\end{aligned}$$

We can take derivatives with respect to α in (2.4) to find Laplace transforms of the moments of $R_\phi(t)$. For example the Laplace transforms of the first and second moments of $R_\phi(t)$ are given in the following proposition.

Proposition 2.2.1 *Suppose that the same assumptions as in the Theorem 2.2.1 hold. Assume also that the function $\phi(t)$ is continuous or piecewise continuous in every finite interval $(0,T)$. Then*

(a) If $\mathbf{E}\left[\phi(X_1)e^{-\beta X_1}\right] < \infty$ for some $\beta > 0$ and $\phi(t) = o(e^{-\gamma t})$, $\gamma > 0$, as $t \to \infty$, then for $\beta > \gamma$

$$\begin{aligned}&\int_0^\infty \mathbf{E}[R_\phi(t)]e^{-\beta t}dt \\ &= \frac{\int_0^\infty \phi(t)e^{-\beta t}dF(t) + \beta\int_0^\infty [1-F(t)]e^{-\beta t}\phi(t)dt}{\beta[1-F^*(\beta)]}. \qquad (2.5)\end{aligned}$$

(b) If $\mathbf{E}\left[\phi^2(X_1)e^{-\beta X_1}\right] < \infty$ for some $\beta > 0$ and $\phi(t) = o(e^{-\gamma t/2})$, $\gamma > 0$, as $t \to \infty$, then for $\beta > \gamma$

$$\begin{aligned}\int_0^\infty \mathbf{E}[R_\phi^2(t)]e^{-\beta t}dt &= \frac{\int_0^\infty \phi^2(t)e^{-\beta t}dF(t) + \beta\int_0^\infty [1-F(t)]\phi^2(t)e^{-\beta t}dt}{\beta[1-F^*(\beta)]} \\ &+ \frac{2\int_0^\infty \phi(t)e^{-\beta t}dF(t)\int_0^\infty [1-F(t)]\phi(t)e^{-\beta t}dt}{[1-F^*(\beta)]^2} \\ &+ \frac{2\left[\int_0^\infty \phi(t)e^{-\beta t}dF(t)\right]^2}{\beta[1-F^*(\beta)]^2}. \qquad (2.6)\end{aligned}$$

Corollary 2.2.1 *If we only consider the rewards until the last renewal before time t, then* (2.2) *simplifies to*

$$R_\phi(t) = \sum_{n=1}^{N(t)} \phi(X_n) \qquad (2.7)$$

and

$$\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)})e^{-\beta t}dt \;=\; \frac{1-F^*(\beta)}{\beta[1-\int_0^\infty e^{-\beta t-\alpha\phi(t)}dF(t)]}.$$

As an application of Corollary 2.2.1 consider the function $\phi(t) = t$. In this case $R_\phi(t) = S_{N(t)} = \sum_{n=1}^{N(t)} X_n$ and the double Laplace transform of $S_{N(t)}$ is given by

$$\int_0^\infty \mathbf{E}(e^{-\alpha S_{N(t)}})e^{-\beta t}dt \;=\; \frac{1-F^*(\beta)}{\beta[1-F^*(\alpha+\beta)]}.$$

As another application take $\phi(t) \equiv 1$ in (2.7). Then $R_\phi(t) = N(t)$ which is a renewal process. In this case we have

$$\int_0^\infty \mathbf{E}(e^{-\alpha N(t)})e^{-\beta t}dt \;=\; \frac{1-F^*(\beta)}{\beta[1-e^{-\alpha}F^*(\beta)]}, \tag{2.8}$$

from which, using the uniqueness theorem for power series, see Bartle [3], we can derive

$$\int_0^\infty \mathbf{P}[N(t)=n]e^{-\beta t}dt \;=\; \frac{1}{\beta}[1-F^*(\beta)]F^*(\beta)^n. \tag{2.9}$$

Also, from (2.8) we can easily deduce that

$$\int_0^\infty \mathbf{E}[N(t)]e^{-\beta t}dt \;=\; \frac{F^*(\beta)}{\beta[1-F^*(\beta)]}. \tag{2.10}$$

Formulae (2.9) and (2.10) are standard, see for example Grimmett and Stirzaker [18], and can be derived directly using (1.3) and (1.4) or (1.5) by taking their Laplace transforms.

2.2.1 A civil engineering example

Consider a traffic stream on a one-way road. The number of cars that can cross the stream on an intersection depends on the time intervals between consecutive cars in the traffic stream. Civil engineers usually model the traffic stream as a homogeneous Poisson process, which means that the distances between consecutive cars are assumed to be independent random variables all with the same exponential distribution, see Herz *et al.* [20]. The number of cars that can safely cross the traffic stream between the n^{th} and the $(n+1)^{\text{th}}$ cars of the traffic stream equals $\lfloor x_{n+1}/a \rfloor$, where x_{n+1} is the time distance between the two cars, $a > 0$ some parameter, and $\lfloor x \rfloor$ represents the integer part of x. As a more general and more realistic model we consider a renewal process as a model for the traffic stream, i.e. the time intervals between consecutive are iid with some arbitrary distribution. The number of cars that can safely cross the traffic stream during the time between two consecutive cars in the traffic stream can be considered as a reward and the total number of cars that can cross the traffic stream up to time t is an instantaneous reward process. We will calculate the distribution of the maximal number of cars that can safely cross the traffic stream during the time interval $[0, t]$.

Suppose that we have 100 synthetic data of the inter-arrival times of cars as in Table 2.1. The average of the data is equal to 5.7422. If we assume that the

Table 2.1: The synthetic data of the inter-arrival times of cars.

1.2169	1.3508	1.5961	1.6633	2.5308
2.5696	2.6021	2.6447	2.6762	2.6783
2.6913	2.7065	2.8696	3.2053	3.4394
3.5028	3.5474	3.5577	3.6191	3.7724
3.9254	3.9400	4.0549	4.0759	4.1093
4.1170	4.1417	4.2162	4.2280	4.2526
4.4784	4.7046	4.7171	4.7174	4.7585
4.7814	4.8284	4.8364	4.8691	5.0278
5.1833	5.2221	5.2357	5.3068	5.3291
5.3864	5.4620	5.4675	5.4865	5.4907
5.5378	5.6410	5.6628	5.6834	5.7610
5.7809	5.8106	5.8397	5.8755	6.1123
6.2104	6.2269	6.3748	6.6107	6.6587
6.6626	6.6807	6.6835	6.7116	6.7283
6.7373	6.7529	6.7672	6.9731	7.0478
7.1100	7.1933	7.3344	7.6249	7.6311
7.9067	8.0114	8.0606	8.2526	8.3095
8.3575	8.3931	8.4245	8.8314	9.0008
9.0716	9.1862	9.4143	9.4661	9.5850
9.6002	10.2193	10.2391	10.7850	11.8890

data is exponentially distributed with parameter λ, then the estimate for λ is equal to 0.1741 (=1/5.7422). Suppose that the reward function ϕ is given by

$$\phi(t) = \lfloor t/2 \rfloor. \tag{2.11}$$

In this case

$$\int_0^\infty \mathbf{E}[R_\phi(t)]e^{-\beta t}dt \quad = \quad \frac{(\beta+\lambda)e^{-2(\beta+\lambda)}}{\beta^2\left[1-e^{-2(\beta+\lambda)}\right]} \tag{2.12}$$

and

$$\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)})e^{-\beta t}dt = \frac{1-e^{-2(\lambda+\beta)}}{[\lambda+\beta][1-e^{-\alpha-2(\lambda+\beta)}]-\lambda[1-e^{-2(\lambda+\beta)}]} \tag{2.13}$$

with $\lambda = 0.1741$. Using numerical inversions of Laplace transforms, see Appendix B, we obtain the graph of the mean of $R_\phi(t)$, see Figure 2.2 (dashed line), and the distribution of $R_\phi(t)$ for $t = 10$, see the first column of Table 2.2.

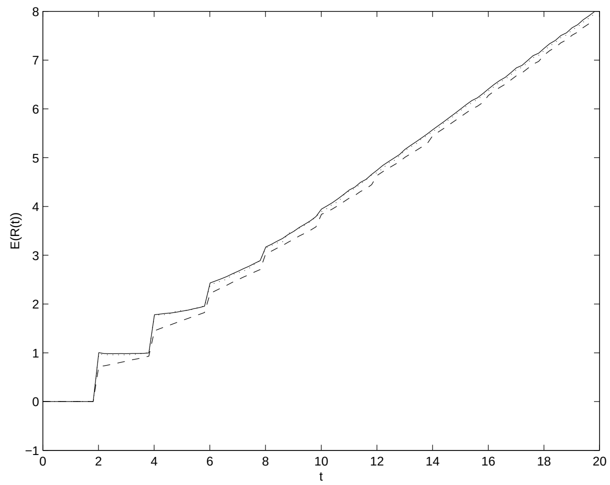


Figure 2.2: Graphs of the mean of $R_\phi(t)$: solid line for Gamma(1,6), dotted line for empirical distribution and dashed line for exp(0.1741).

Table 2.2: Distributions of $R_\phi(10)$ with $X_n \sim \exp(0.1741)$, $X_n \sim \text{Gamma}(1,6)$ and with $F(x) = \mathbb{F}_n(x)$, the empirical distribution function of the data set in Table 2.1.

k	$\mathbf{P}(R_\phi(10) = k)$; $X_n \sim \exp(0.1741)$	$\mathbf{P}(R_\phi(10) = k)$; $F(x) = \mathbb{F}_n(x)$	$\mathbf{P}(R_\phi(10) = k)$; $X_n \sim \text{Gamma}(1,6)$
0	0.0001	0.0000	0.0000
1	0.0032	0.0007	0.0003
2	0.0452	0.0026	0.0013
3	0.2597	0.1543	0.1344
4	0.6040	0.8223	0.8301
5	0.0877	0.0200	0.0338

If we look at the histogram of the data, see Figure 2.3, it does not seem reasonable to assume that the data is exponentially distributed. Without assuming that the data has come from a certain family of parametric distributions we can calculate the distribution of the instantaneous reward process using the

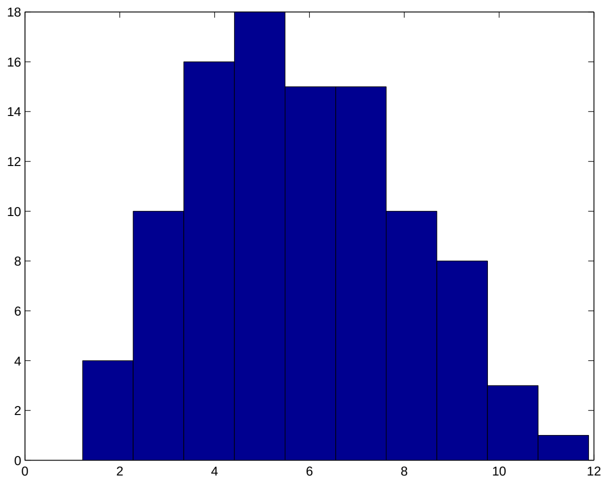


Figure 2.3: Histogram of the data set in Table 2.1.

empirical distribution $\mathbb{F}_n$ of the data:

$$\mathbb{F}_n(x) = \frac{\#\{X_i \leq x : i = 1, ..., n\}}{n}. \tag{2.14}$$

Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics corresponding to $X_i, i = 1, 2, ..., n$, and let $X_0 = 0$. We denote by $x_{j:n}$ the realizations of $X_{j:n}$. Using (2.14) we obtain

$$\begin{aligned} &\int_0^\infty \mathbf{E}[R_\phi(t)]e^{-\beta t}dt \\ &= \frac{\frac{1}{n}\sum_{k=1}^n \phi(x_{k:n})e^{-\beta x_{k:n}} + \beta \sum_{k=1}^n \int_{x_{k-1:n}}^{x_{k:n}} \left[1 - \frac{k-1}{n}\right] e^{-\beta t}\phi(t)dt}{\beta\left[1 - \frac{1}{n}\sum_{k=1}^n e^{-\beta x_{k:n}}\right]} \end{aligned} \tag{2.15}$$

Based on the data in Table 2.1,

$$\begin{aligned} &\frac{1}{n}\sum_{k=1}^n \phi(x_{k:n})e^{-\beta x_{k:n}} \\ &= \frac{1}{100}\Bigg[\sum_{k=5}^{22} e^{-\beta x_{k:100}} + 2\sum_{k=23}^{59} e^{-\beta x_{k:100}} + 3\sum_{k=60}^{81} e^{-\beta x_{k:100}} \\ &\quad + 4\sum_{k=82}^{96} e^{-\beta x_{k:100}} + 5\sum_{k=97}^{100} e^{-\beta x_{k:100}}\Bigg] \\ &=: K_1(\beta) \end{aligned}$$

and

$$\begin{aligned}
&\beta \sum_{k=1}^{n} \int_{x_{k-1:n}}^{x_{k:n}} \left[1 - \frac{k-1}{n}\right] e^{-\beta t}\phi(t)dt \\
&= \frac{1}{100}\left[96e^{-2\beta} + 78e^{-4\beta} + 41e^{-6\beta} + 19e^{-8\beta} + 4e^{-10\beta}\right] - K_1(\beta).
\end{aligned}$$

So the numerator of (2.15) equals

$$\frac{1}{100}\left[96e^{-2\beta} + 78e^{-4\beta} + 41e^{-6\beta} + 19e^{-8\beta} + 4e^{-10\beta}\right].$$

It follows that

$$\int_0^\infty \mathbf{E}[R_\phi(t)]e^{-\beta t}dt = \frac{96e^{-2\beta} + 78e^{-4\beta} + 41e^{-6\beta} + 19e^{-8\beta} + 4e^{-10\beta}}{\beta\left[100 - \sum_{k=1}^{100} e^{-\beta x_{k:100}}\right]}. \quad (2.16)$$

Inverting this transform numerically we obtain the graph of the mean of $R_\phi(t)$, see Figure 2.2 (dotted line).

Next we calculate the double Laplace transform of $R_\phi(t)$ using the empirical distribution of the inter-arrival times. Substituting (2.14) into (2.4) we obtain

$$\begin{aligned}
\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)})e^{-\beta t}dt &= \frac{\int_0^\infty [1 - \mathbb{F}_n(t)]e^{-\alpha\phi(t)-\beta t}dt}{1 - \int_0^\infty e^{-\alpha\phi(t)-\beta t}d\mathbb{F}_n(t)} \\
&= \frac{\sum_{k=1}^{n} \int_{x_{k-1:n}}^{x_{k:n}} \left[1 - \frac{k-1}{n}\right] e^{-\alpha\phi(t)-\beta t}dt}{1 - \frac{1}{n}\sum_{k=1}^{n} e^{-\alpha\phi(x_{k:n})-\beta x_{k:n}}}. \quad (2.17)
\end{aligned}$$

Based on the data in Table 2.1, the numerator of (2.17) is equal to

$$\begin{aligned}
&\sum_{k=1}^{100} \int_{x_{k-1:100}}^{x_{k:100}} \left[1 - \frac{k-1}{n}\right] e^{-\alpha\phi(t)-\beta t}dt \\
&= \frac{1}{100\beta}\Big[K_2(\beta) - \Big(96 + 78e^{-(\alpha+2\beta)} + 41e^{-2(\alpha+2\beta)} + 19e^{-3(\alpha+2\beta)} \\
&\qquad + 4e^{-4(\alpha+2\beta)}\Big)\Big(1 - e^{-\alpha}\Big)e^{-2\beta}\Big]
\end{aligned}$$

and the denominator of (2.17) is equal to

$$1 - \frac{1}{100}\sum_{k=1}^{100} e^{-\alpha\phi(x_{k:100})-\beta x_{k:100}} = \frac{K_2(\beta)}{100},$$

where

$$K_2(\beta) \quad = \quad 100 - \sum_{k=1}^{4} e^{-\beta x_{k:100}} - e^{-\alpha} \sum_{k=5}^{22} e^{-\beta x_{k:100}} - e^{-2\alpha} \sum_{k=23}^{59} e^{-\beta x_{k:100}}$$
$$-e^{-3\alpha} \sum_{k=60}^{81} e^{-\beta x_{k:100}} - e^{-4\alpha} \sum_{k=82}^{96} e^{-\beta x_{k:100}} - e^{-5\alpha} \sum_{k=97}^{100} e^{-\beta x_{k:100}}.$$

The distribution of $R_\phi(t)$ for $t = 10$ can be seen in the second column of Table 2.2.

The data set in Table 2.1 was generated from a Gamma(1,6) random variable which has a pdf

$$f(x; 1, 6) = \frac{1}{120} x^5 e^{-x}, \quad x \geq 0.$$

Based on this cycle length distribution the graph of the mean of $R_\phi(t)$ can be seen in Figure 2.2 (solid line) and the distribution of $R_\phi(t)$ for $t = 10$ can be seen in the last column of Table 2.2. From this table we see that the Kolmogorov-Smirnov distance (see e.g., Dudewicz [10]) between the cdfs of $R_\phi(t)$ based on the exponential and the Gamma cycles equals 0.2183, whereas the Kolmogorov-Smirnov distance between the cdfs of $R_\phi(t)$ based on the empirical distribution function and the Gamma cycle equals 0.0199. So we conclude in this example that approximation for the distribution of $R_\phi(t)$ based on the use of the empirical distribution function of the data is better than the use of an exponential distribution with parameter estimated from the data.

2.3 Renewal reward processes

Consider the renewal reward process defined in (2.1), i.e.,

$$R(t) = \sum_{n=1}^{N(t)} Y_n.$$

We assume that Y_1 is a non-negative random variable. In this section we will derive an expression for the distribution of $R(t)$ for finite t.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which the iid sequence (X_n, Y_n) of random vectors is defined and also an iid sequence $(U_n, n \geq 1)$ of exponentially distributed random variables with parameter 1 such that the sequences (X_n, Y_n) and (U_n) are independent. Let $(T_n, n \geq 1)$ be the sequence of partial sums of the variables U_n. Then the map

$$\Phi : \quad \omega \in \Omega \mapsto \sum_{n=1}^{\infty} \delta_{(T_n(\omega), X_n(\omega)), Y_n(\omega))},$$

where $\delta_{(x,y,z)}$ is the Dirac measure in (x,y,z), defines a Poisson point process on $E = [0,\infty) \times [0,\infty) \times [0,\infty)$ with intensity measure $\nu(dtdxdy) = dtdH(x,y)$, where H is the joint cdf of X_1 and Y_1. Let $M_p(E)$ be the set of all point measures on E. We will denote the distribution of Φ over $M_p(E)$ by $\mathbf{P}_\nu$.

Define for $t \geq 0$ the functionals $\mathbb{A}_X(t)$ and $\mathbb{A}_Y(t)$ on $M_p(E)$ by

$$\mathbb{A}_X(t)(\mu) = \int_E 1_{[0,t)}(s) x \mu(dsdxdy)$$

and

$$\mathbb{A}_Y(t)(\mu) = \int_E 1_{[0,t)}(s) y \mu(dsdxdy).$$

In the sequel we will write $\mathbb{A}_X(t,\mu) = \mathbb{A}_X(t)(\mu)$ and $\mathbb{A}_Y(t,\mu) = \mathbb{A}_Y(t)(\mu)$. If

$$\mu = \sum_{i=1}^{\infty} \delta_{(t_i,x_i,y_i)}$$

with $t_1 < t_2 < \ldots$, then

$$\mathbb{A}_X(t,\mu) = \sum_{i=1}^{\infty} 1_{[0,t)}(t_i) x_i \quad \text{and} \quad \mathbb{A}_Y(t,\mu) = \sum_{i=1}^{\infty} 1_{[0,t)}(t_i) y_i.$$

Note that with probability 1, $\mathbb{A}_X(t,\mu)$ and $\mathbb{A}_Y(t,\mu)$ are finite.

Define also for $t \geq 0$ the functional $\mathbb{R}(t)$ on $M_p(E)$ by

$$\mathbb{R}(t)(\mu) \quad = \quad \int_E 1_{[0,x)}(t - \mathbb{A}_X(s,\mu)) \mathbb{A}_Y(s,\mu) \mu(dsdxdy).$$

Then we can easily prove the following lemma:

Lemma 2.3.1 *With probability 1,*

$$R(t) = \mathbb{R}(t)(\Phi).$$

The following theorem gives the formula for the distribution of $R(t)$ in the form of double Laplace transform.

Theorem 2.3.1 *Let $((X_n, Y_n), n \geq 1)$ be an iid sequence of random vectors with joint cdf H, where X_n are strictly positive and Y_n are non-negative random variables. Let $(N(t), t \geq 0)$ be the renewal process with renewal cycles X_n. Define for $t \geq 0$*

$$R(t) = \sum_{n=1}^{N(t)} Y_n.$$

Then for $\alpha, \beta > 0$

$$\int_0^\infty \mathbf{E}(e^{-\alpha R(t)}) e^{-\beta t} dt = \frac{1 - F^*(\beta)}{\beta[1 - H^*(\beta, \alpha)]}. \tag{2.18}$$

Proof: By Lemma 2.3.1

$$\begin{aligned}
&\mathbf{E}(e^{-\alpha R(t)}) \\
&= \int_{M_p(E)} e^{-\alpha \mathbb{R}(t)(\mu)} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \exp\Big\{ - \alpha \int_E 1_{[0,x)}(t - \mathbb{A}_X(s,\mu)) \mathbb{A}_Y(s,\mu) \mu(dsdxdy) \Big\} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \int_E 1_{[0,x)}(t - \mathbb{A}_X(s,\mu)) \exp\Big\{ - \alpha \mathbb{A}_Y(s,\mu) \Big\} \mu(dsdxdy) \mathbf{P}_\nu(d\mu).
\end{aligned}$$

Applying the Palm formula for Poisson point processes we obtain

$$\begin{aligned}
\mathbf{E}(e^{-\alpha R(t)}) &= \int_0^\infty \int_0^\infty \int_0^\infty \int_{M_p(E)} 1_{[0,x)}(t - \mathbb{A}_X(s,\mu)) \\
&\quad \exp\Big\{ - \alpha \mathbb{A}_Y(s,\mu) \Big\} \mathbf{P}_\nu(d\mu) dH(x,y) ds.
\end{aligned}$$

Using Fubini's theorem we obtain

$$\begin{aligned}
&\int_0^\infty \mathbf{E}(e^{-\alpha R(t)}) e^{-\beta t} dt \\
&= \frac{1}{\beta}[1 - F^*(\beta)] \int_0^\infty \int_{M_p(E)} \exp\Big\{ - \Big[\beta \mathbb{A}_X(s,\mu) + \alpha \mathbb{A}_Y(s,\mu) \Big] \Big\} \mathbf{P}_\nu(d\mu) ds.
\end{aligned}$$

Using the Laplace functional of Poisson point processes we obtain

$$\begin{aligned}
&\int_{M_p(E)} \exp\Big\{ - \Big[\beta \mathbb{A}_X(s,\mu) + \alpha \mathbb{A}_Y(s,\mu) \Big] \Big\} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \exp\Big\{ - \int_E 1_{[0,s)}(r)(\beta u + \alpha v) \mu(drdudv) \Big\} \mathbf{P}_\nu(d\mu) \\
&= \exp\Big\{ - \int_0^\infty \int_0^\infty \int_0^\infty \Big[1 - e^{-1_{[0,s)}(r)[\beta u + \alpha v]} \Big] dH(u,v) dr \Big\} \\
&= \exp\Big\{ - s[1 - H^*(\beta, \alpha)] \Big\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_0^\infty \mathbf{E}(e^{-\alpha R(t)}) e^{-\beta t} dt &= \frac{1}{\beta}[1 - F^*(\beta)] \int_0^\infty \exp\Big\{ - s[1 - H^*(\beta, \alpha)] \Big\} ds \\
&= \frac{1 - F^*(\beta)}{\beta[1 - H^*(\beta, \alpha)]}. \quad \Box
\end{aligned}$$

The following proposition concerns the Laplace transforms of the first and second moments of $R(t)$, which can be derived by taking derivatives with respect to α in (2.18), and then setting $\alpha = 0$.

Proposition 2.3.1 *Under the same assumptions as in the Theorem 2.3.1 we have*

(a) If $\mathbf{E}\left[Y_1 e^{-\beta X_1}\right] < \infty$ *for some* $\beta > 0$*, then*

$$\int_0^\infty \mathbf{E}[R(t)]e^{-\beta t}dt = \frac{\int_0^\infty \int_0^\infty y e^{-\beta x} dH(x,y)}{\beta[1 - F^*(\beta)]}, \tag{2.19}$$

(b) If $\mathbf{E}\left[Y_1^2 e^{-\beta X_1}\right] < \infty$ *for some* $\beta > 0$*, then*

$$\begin{aligned}\int_0^\infty & \mathbf{E}[R^2(t)]e^{-\beta t}dt \\ &= \frac{\int_0^\infty \int_0^\infty y^2 e^{-\beta x} dH(x,y)}{\beta[1 - F^*(\beta)]} + \frac{2\left[\int_0^\infty \int_0^\infty y e^{-\beta x} dH(x,y)\right]^2}{\beta[1 - F^*(\beta)]^2}.\end{aligned} \tag{2.20}$$

Remark 2.3.1 *If* $(X_n, n \geq 1)$ *and* $(Y_n, n \geq 1)$ *are independent then* (2.18), (2.19)*, and* (2.20) *reduce to*

(a)

$$\int_0^\infty \mathbf{E}(e^{-\alpha R(t)})e^{-\beta t}dt = \frac{1 - F^*(\beta)}{\beta[1 - F^*(\beta)G^*(\alpha)]},$$

(b)

$$\int_0^\infty \mathbf{E}[R(t)]e^{-\beta t}dt = \frac{\mu_Y F^*(\beta)}{\beta[1 - F^*(\beta)]},$$

(c)

$$\int_0^\infty \mathbf{E}[R^2(t)]e^{-\beta t}dt = \frac{F^*(\beta)}{\beta[1 - F^*(\beta)]}\left[\sigma_Y^2 + \mu_Y^2 + \frac{2\mu_Y^2 F^*(\beta)}{1 - F^*(\beta)}\right],$$

where $\mu_Y = \mathbf{E}(Y_1)$ *and* $\sigma_Y^2 = \mathrm{Var}(Y_1)$.

2.4 Asymptotic properties

Asymptotic properties of renewal reward processes like the renewal-reward theorem and its expected-value version, and asymptotic normality of the processes are well known. In this section we will reconsider some of them. We will use Tauberian theorems to derive expected-value version of the renewal reward theorem. We will also investigate other asymptotic properties of the renewal reward processes including asymptotic normality of the instantaneous reward process defined in (2.2). We first consider the renewal reward process $(R(t), t \geq 0)$ with renewal cycles X_n and rewards non-negative random variables Y_n as defined in (2.1), i.e.,

$$R(t) = \sum_{n=1}^{N(t)} Y_n. \tag{2.21}$$

If $\mu_X = \mathbf{E}(X_1)$ and $\mu_Y = \mathbf{E}(Y_1)$ are finite, then

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mu_Y}{\mu_X} \quad \text{with probability 1}, \tag{2.22}$$

and

$$\lim_{t\to\infty} \frac{\mathbf{E}[R(t)]}{t} = \frac{\mu_Y}{\mu_X}, \tag{2.23}$$

which are well known as the renewal-reward theorem and its expected-value version respectively, see Tijms [46] for example. The renewal reward theorem can easily be proved using the strong law of large numbers. A proof of (2.23) can be found for example in Ross [36], where he used Wald's equation. We will give a proof for (2.23) using the following Tauberian theorem which can be found in Widder [48].

Theorem 2.4.1 (Tauberian theorem) *If $\alpha(t)$ is non-decreasing and such that the integral*

$$f(s) = \int_0^\infty e^{-st} d\alpha(t)$$

converges for $s > 0$ and if for some non-negative number γ and some constant C

$$f(s) \sim \frac{C}{s^\gamma} \quad \text{as} \quad s \longrightarrow 0$$

then

$$\alpha(t) \sim \frac{Ct^\gamma}{\Gamma(\gamma+1)} \quad \text{as} \quad t \longrightarrow \infty.$$

The proof of (2.23):

In Section 2.3 we have proved that the Laplace transform of $\mathbf{E}[R(t)]$ is given by

$$\int_0^\infty \mathbf{E}[R(t)]e^{-\beta t} dt = \frac{\int_0^\infty \int_0^\infty v e^{-\beta u} dH(u,v)}{\beta[1 - F^*(\beta)]}, \tag{2.24}$$

see (2.19). Assuming μ_Y is finite, we obtain from this equation

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[R(t)] = \frac{\int_0^\infty \int_0^\infty v e^{-\beta u} dH(u,v)}{1 - F^*(\beta)}.$$

Using dominated convergence we can prove that

$$\int_0^\infty \int_0^\infty v e^{-\beta u} dH(u,v) = \mu_Y + o(1) \quad \text{as} \quad \beta \longrightarrow 0.$$

Similarly if μ_X is finite,

$$F^*(\beta) = 1 - \beta\mu_X + o(\beta) \quad \text{as} \quad \beta \longrightarrow 0.$$

It follows that

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[R(t)] \sim \frac{\mu_Y}{\mu_X} \frac{1}{\beta} \quad \text{as} \quad \beta \longrightarrow 0.$$

Obviously $\mathbf{E}[R(t)]$ is non-decreasing. So we can apply the Tauberian theorem with $\gamma = 1$. □

A stronger version of (2.23) can be derived for the case where X_1 has a density in some interval. Assume that $\sigma_X^2 = \text{Var}(X_1)$ and $\sigma_{XY} = \text{Cov}(X_1, Y_1)$ are finite. Let f_X and f_{XY} denote the density function of X_1 and the joint density of X_1 and Y_1 respectively. Let $M(t) = \mathbf{E}[R(t)]$. Conditioning on X_1, from (2.21) we obtain

$$M(t) = K(t) + \int_0^t M(t-x) f_X(x) dx, \tag{2.25}$$

where $K(t) = \int_0^t \int_0^\infty y f_{XY}(x,y) dy dx$.

Define for $t \geq 0$ the function

$$Z(t) = M(t) - \frac{\mu_Y}{\mu_X} t.$$

From (2.25), we find that $Z(t)$ satisfies the following integral equation

$$Z(t) = a(t) + \int_0^t Z(t-x) f_X(x) dx,$$

where

$$a(t) = K(t) - \mu_Y + \frac{\mu_Y}{\mu_X} \int_t^\infty (x-t) f_X(x) dx.$$

We see that $a(t)$, $t \geq 0$, is a finite sum of monotone functions. So we can use the *key renewal theorem*, see e.g., Tijms [46], to obtain

$$\begin{aligned} \lim_{t\to\infty} Z(t) &= \frac{1}{\mu_X} \int_0^\infty a(x) dx \\ &= \frac{1}{\mu_X} \left[\int_0^\infty [K(x) - \mu_Y] dx + \frac{\mu_Y}{\mu_X} \int_0^\infty \int_x^\infty (s-x) f_X(s) ds dx \right]. \end{aligned}$$

It can easily be verified that

$$\int_0^\infty [K(x) - \mu_Y]dx = -\mathbf{E}[X_1 Y_1] = -(\sigma_{XY} + \mu_X \mu_Y)$$

and

$$\int_0^\infty \int_x^\infty (s-x) f_X(s) ds dx = \frac{1}{2}\mathbf{E}(X_1^2) = \frac{1}{2}(\sigma_X^2 + \mu_X^2).$$

It follows that

$$\begin{aligned} \lim_{t\to\infty} Z(t) &= \frac{1}{\mu_X}\left[-(\sigma_{XY} + \mu_X\mu_Y) + \frac{\mu_Y}{2\mu_X}(\sigma_X^2 + \mu_X^2)\right] \\ &= \frac{\sigma_X^2 \mu_Y - 2\mu_X \sigma_{XY} - \mu_X^2 \mu_Y}{2\mu_X^2}. \end{aligned}$$

From this we conclude that

$$\mathbf{E}[R(t)] = \frac{\mu_Y}{\mu_X} t + \frac{\sigma_X^2 \mu_Y - 2\mu_X \sigma_{XY} - \mu_X^2 \mu_Y}{2\mu_X^2} + o(1) \quad \text{as} \quad t \to \infty. \tag{2.26}$$

Now we will consider asymptotic properties of the instantaneous reward process defined in (2.2), i.e.,

$$R_\phi(t) = \sum_{n=1}^{N(t)} \phi(X_n) + \phi(t - S_{N(t)}).$$

Putting $\mu_Y = \mathbf{E}[\phi(X_1)]$, we can prove that (2.22) and (2.23) look exactly the same (the contribution of the reward earned in the incomplete renewal cycle $(S_{N(t)}, t]$ disappear in the limit). Next, to obtain a formula close to (2.26) we put $\sigma_{XY} = \mathrm{Cov}(X_1, \phi(X_1))$. A similar argument as for proving (2.26) can be used to deduce

$$\mathbf{E}[R_\phi(t)] = \frac{\mu_Y}{\mu_X} t + \frac{\sigma_X^2 \mu_Y - 2\mu_X \sigma_{XY} - \mu_X^2 \mu_Y}{2\mu_X^2} + A + o(1) \quad \text{as} \quad t \to \infty,$$

where $A = \frac{1}{\mu_X}\int_0^\infty [1 - F(t)]\phi(t)dt$, provided the function $\phi(t)$ is Laplace transformable. The extra constant A can be interpreted as a contribution of the reward earned in the incomplete renewal cycle.

Under some conditions on the function ϕ, the limiting distribution of the instantaneous reward process $(R_\phi(t))$ is a normal distribution. To prove this we need the following lemmas.

Lemma 2.4.1 [8] *If $\chi(t)$, $\varepsilon(t)$ and $\delta(t)$ are random functions $(0 < t < \infty)$, and are such that the asymptotic distribution of $\chi(t)$ exists, $\varepsilon(t)$ converges in probability to 1 and $\delta(t)$ converges in probability to 0 for $t \to \infty$, then the asymptotic distribution of $\chi(t)\varepsilon(t) + \delta(t)$ exists and coincides with that of $\chi(t)$.*

Lemma 2.4.2 [33] *If* $(X_n^{(1)})$, (X_n), *and* $(X_n^{(2)})$ *are sequences of random variables such that* $X_n^{(1)} \leq X_n \leq X_n^{(2)}$ *and the sequences* $(X_n^{(1)})$ *and* $(X_n^{(2)})$ *have the same asymptotic distribution for* $n \to \infty$, *then* X_n *has also that asymptotic distribution.*

Lemma 2.4.3 [33] *Let* ξ_n *denote a sequence of identically distributed random variables having finite second moment. Let* $\nu(t)$ *denote a positive integer-valued random variable for* $t > 0$, *for which* $\frac{\nu(t)}{t}$ *converges in probability to* $c > 0$ *for* $t \to \infty$. *Then* $\frac{\xi_{\nu(t)}}{\sqrt{\nu(t)}}$ *converges in probability to 0.*

Assume that the function ϕ is non-negative and non-decreasing, or bounded. Let $Y_n := \phi(X_n)$. Then

$$\sum_{n=1}^{N(t)} Y_n = \sum_{n=1}^{N(t)} \phi(X_n) \leq R_\phi(t) \leq \sum_{n=1}^{N(t)+1} \phi(X_n) = \sum_{n=1}^{N(t)+1} Y_n. \tag{2.27}$$

Assume that σ_X^2 and σ_Y^2 are finite. Then using the Central Limit Theorem for random sums, see Embrechts *et al.* [11] we obtain, as $t \to \infty$,

$$\left[\mathrm{Var}\left(Y_1 - \frac{\mu_Y}{\mu_X} X_1\right) \frac{t}{\mu_X}\right]^{-1/2} \left(\sum_{n=1}^{N(t)} Y_n - \frac{\mu_Y}{\mu_X} t\right) \xrightarrow{d} N(0,1), \tag{2.28}$$

where

$$\mathrm{Var}\left(Y_1 - \frac{\mu_Y}{\mu_X} X_1\right) \frac{t}{\mu_X} = \left(\frac{\mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 - 2\mu_X \mu_Y \sigma_{XY}}{\mu_X^3}\right) t.$$

Now we will consider the limiting distribution of $\sum_{n=1}^{N(t)+1} Y_n$. Let

$$C = \frac{\mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 - 2\mu_X \mu_Y \sigma_{XY}}{\mu_X^3}.$$

Note that

$$\frac{\sum_{n=1}^{N(t)+1} Y_n - \frac{\mu_Y}{\mu_X} t}{\sqrt{Ct}} = \frac{\sum_{n=1}^{N(t)} Y_n - \frac{\mu_Y}{\mu_X} t}{\sqrt{Ct}} + \frac{Y_{N(t)+1}}{\sqrt{Ct}}$$

where the first term in the right-hand side converges in distribution to the standard normal random variable. If we can prove that $\frac{Y_{N(t)+1}}{\sqrt{t}} \xrightarrow{p} 0$ as $t \to \infty$ then by Lemma 2.4.1

$$\frac{\sum_{n=1}^{N(t)+1} Y_n - \frac{\mu_Y}{\mu} t}{\sqrt{Ct}} \xrightarrow{d} N(0,1) \quad \text{as } t \to \infty. \tag{2.29}$$

But since $\sigma_Y^2 < \infty$ it follows, by Lemma 2.4.3 and the fact that $\frac{N(t)}{t} \xrightarrow{a.s.} \frac{1}{\mu_X} (> 0)$ as $t \to \infty$,

$$\begin{aligned} \frac{Y_{N(t)+1}}{\sqrt{t}} &= \frac{Y_{N(t)+1}}{\sqrt{N(t)+1}} \sqrt{\frac{N(t)+1}{t}} \\ &\xrightarrow{p} 0 \quad \text{as } t \to \infty. \end{aligned}$$

Finally, combining (2.27), (2.28), (2.29), and Lemma 2.4.2, it follows that

$$\frac{R_\phi(t) - \frac{\mu_Y}{\mu_X} t}{\sqrt{Ct}} \xrightarrow{d} N(0,1) \quad \text{as } t \to \infty.$$

2.5 Covariance structure of renewal processes

Basically, the covariance structure of renewal reward processes can be derived using point processes, but it will involve more complicated calculations. In this section we will derive the covariance structure of a renewal process, a special case of renewal reward processes. Using the notations in Section 2.2 define for $t \geq$ the functional $\mathbb{N}(t)$ on $M_p(E)$ by

$$\mathbb{N}(t)(\mu) = \int_E \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) 1_{[0,s)}(u) \mu(dudv) \mu(dsdx).$$

Let $\omega \in \Omega$. Then

$$\begin{aligned} \mathbb{N}(t)(\Phi(\omega)) &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \left\{ 1_{[0,X_n(\omega))}(t - \mathbb{A}(T_n(\omega), \Phi(\omega))) 1_{[0,T_n(\omega))}(T_i(\omega)) \right\} \\ &= \sum_{i=1}^{\infty} 1_{[0,T_{N(t,\omega)+1}(\omega))}(T_i(\omega)) \\ &= N(t,\omega). \end{aligned}$$

So we have, with probability 1,

$$N(t) = \mathbb{N}(t)(\Phi).$$

Using this functional expression of $N(t)$ we derive the double Laplace transform of $\mathbf{E}[N(t_1)N(t_2)]$ which is stated in the following theorem. The proof is given in Appendix A.

Theorem 2.5.1 *Let $(X_n, n \geq 1)$ be an iid sequence of strictly positive random variables with common cdf F. Let $(N(t), t \geq 0)$ be the renewal process with renewal cycles X_n. Then for $\alpha, \beta > 0$,*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \mathbf{E}[N(t_1)N(t_2)] e^{-\alpha t_1 - \beta t_2} dt_1 dt_2 \\ &= \frac{[1 - F^*(\alpha)F^*(\beta)]F^*(\alpha+\beta)}{\alpha\beta[1 - F^*(\alpha)][1 - F^*(\beta)][1 - F^*(\alpha+\beta)]}. \end{aligned} \tag{2.30}$$

Example 2.5.1 *Let* $(N(t), t \geq 0)$ *be a renewal process with inter-arrival times* X_n *having Gamma distribution with common pdf*

$$f(x; \lambda, m) = \frac{\lambda e^{-\lambda x}(\lambda x)^{m-1}}{\Gamma(m)}, \quad \lambda > 0, \quad x \geq 0.$$

Using (2.10) *and* (2.30) *we obtain*

$$\int_0^\infty \mathbf{E}[N(t)]e^{-\beta t}dt = \frac{\lambda^m}{\beta[(\beta+\lambda)^m - \lambda^m]}$$

and

$$\begin{aligned} &\int_0^\infty \int_0^\infty \mathbf{E}[N(t_1)N(t_2)]e^{-\alpha t_1}e^{-\beta t_2}dt_1 dt_2 \\ &= \frac{[\lambda(\alpha+\lambda)^m - \lambda^{3m}]}{\alpha\beta[(\alpha+\lambda)^m - \lambda^m][(\beta+\lambda)^m - \lambda^m][(\alpha+\beta+\lambda)^m - \lambda^m]}. \end{aligned}$$

As an example, take $m = 2$*. Transforming back these Laplace transforms we obtain*

$$\mathbf{E}[N(t)] = \frac{1}{2}\lambda t - \frac{1}{4} + \frac{1}{4}e^{-2\lambda t}$$

and for $t_1 \leq t_2$

$$\begin{aligned} \mathbf{E}[N(t_1)N(t_2)] &= \frac{1}{16}[1 - 2\lambda(t_2 - t_1) + 4\lambda^2 t_1 t_2 - (1 + 4\lambda t_1 - 2\lambda t_2)e^{-2\lambda t_1} \\ &\quad -(1 + 2\lambda t_1)e^{-2\lambda t_2} + e^{-2\lambda(t_2 - t_1)}]. \end{aligned}$$

Hence for $t_1 \leq t_2$

$$\mathrm{Cov}[N(t_1), N(t_2)] = \frac{1}{4}\lambda(t_1 - e^{-2\lambda t_1} - e^{-2\lambda t_2}) + \frac{1}{16}(e^{-2\lambda(t_2-t_1)} - e^{-2\lambda(t_1+t_2)}).$$

Note that for $m = 1$ *the process* $(N(t), t \geq 0)$ *is a homogeneous Poisson process with rate* λ*, and in this case*

$$\mathbf{E}[N(t_1)N(t_2)] = \lambda^2 t_1 t_2 + \lambda \min\{t_1, t_2\}.$$

This result can also be obtained using (2.30)*. Moreover the covariance between* $N(t_1)$ *and* $N(t_2)$ *for* $t_1 < t_2$ *is given by*

$$\mathrm{Cov}[N(t_1), N(t_2)] = \lambda t_1.$$

2.6 System reliability in a stress-strength model

In this section we consider a system which is supposed to function during a certain time after which it fails. The failures of the system are of two kinds: proper failures which are due to own wear-out and occur even in an unstressed environment, and failures which are due to random environmental stresses. We will only consider the latter. The system we consider here is generic, and models all kinds of varieties of products, subsystems, or components.

The study about system reliability where the failures are only due to random environmental stresses is known as a stress-strength interference reliability or a stress-strength model. Many authors have paid attention to the study of such a model, see for example Xue and Yang [50], Chang [4], and Gaudoin and Soler [14]. In other literature this model is also called a load-capacity interference model, see Lewis and Chen [24].

In Gaudoin and Soler [14] three types of stresses are considered: point, alternating and diffused stresses. The systems they considered may have a memorization: a stress which occurs at a given time can influence the future failures if the system has kept it in memory. Two types of stochastic influence models are proposed: stress-strength duration models (type I) and random environment lifetime models (type II). The type I models are based on the assumption that a system failure occurs at time t if the accumulation of memory of all stresses occurred before time t exceeds some strength threshold of the system whereas the type II models are models for which, conditionally on all stresses that occurred before, the cumulative failure (hazard) rate at time t is proportional to the accumulation of memory of all stresses occurring before time t. In type II models the stresses weaken the system. The accumulation of stresses will cause the system to fail, but this failure is not associated to a given strength threshold.

In this section we will consider extensions of some models proposed in Gaudoin and Soler. We will restrict ourselves to point stresses. The point stresses are impulses which occur at random times with random amplitudes. In the type I models Gaudoin and Soler assumed that the occurrence times of the stresses is modelled respectively as a homogeneous Poisson process, a non-homogeneous Poisson process and a renewal process. The two kinds of memory they considered are: (i) the system keeps no memory of the stresses, and (ii) the system keeps a permanent memory of all stresses occurred before. The amplitudes of the stresses and their occurrence times are assumed to be independent. We give two generalizations. Firstly, the occurrence times of the stresses are modelled as a Cox process (doubly stochastic Poisson process) and we keep the independence assumption. Secondly, the occurrence times of the stresses is modelled as a renewal process, but they may depend on the amplitudes of the stresses. We discuss this in Subsection 2.6.1.

In the type II models Gaudoin and Soler model the occurrence times of the stresses respectively as a homogeneous Poisson process and a non-homogeneous

Poisson process. They assumed that the amplitudes of the stresses are independent of their occurrence times, and considered any kind of memory. We give a generalization where the occurrence times of the stresses is modelled as a Cox process. We give also a further generalization where the occurrence times of the stresses are modelled as a renewal process and may depend on their amplitudes, but we only assume that the system keeps a permanent memory of the stresses which occurred. We discuss this in Subsection 2.6.2.

2.6.1 Type I models

Suppose that a system operated at time $t_0 = 0$ is exposed to stresses occurring at random time points $S_1, S_2, S_3, \ldots$ where $S_0 := 0 < S_i < S_{i+1}$, $\forall i \geq 1$. Let $N(t) = \sup\{n \geq 0 : S_n \leq t\}$ be the number of stresses that occurred in the time interval $[0, t]$. Let the amplitude of the stress at time S_n be given by the non-negative random variable Y_n. Assume that the sequence $(Y_n, n \geq 1)$ is iid with a common distribution function G and independent of the sequence $(S_n, n \geq 1)$. After the occurrence of a stress the system may keeps the stress into its memory. In Gaudoin and Soler the memory of the system is represented by a deterministic Stieltjes measure. Here we will represent the memory of the system in terms of a recovery rate. We call a function h the recovery rate of the system if it is non-negative, non-increasing, bounded above from 1, and it vanishes on $(-\infty, 0)$. We will assume that at time t the contribution of the stress that has occurred at time $S_n \leq t$ has an amplitude $Y_n h(t - S_n)$. So the accumulation of the stresses at time t is given by

$$\begin{aligned} L(t) &= \sum_{n=1}^{\infty} Y_n h(t - S_n) \\ &= \sum_{n=1}^{N(t)} Y_n h(t - S_n). \end{aligned} \tag{2.31}$$

If the strength threshold of the system equals a positive constant u, then the reliability of the system at time t is given by

$$\tilde{R}(t) = \mathbf{P}(\sup_{0 \leq s \leq t} L(s) \leq u). \tag{2.32}$$

In general it is difficult to calculate $\tilde{R}(t)$. In the case the system keeps no memory of the stresses, the equation (2.32) simplifies to

$$\tilde{R}(t) = \mathbf{P}(\max\{Y_1, Y_2, ..., Y_{N(t)}\} \leq u) \tag{2.33}$$

and in the case of the system keeps a permanent memory of the stresses (without recovery), equation (2.31) reduces to

$$L(t) = \sum_{n=1}^{N(t)} Y_n \tag{2.34}$$

and (2.32) simplifies to

$$\tilde{R}(t) \quad = \quad \mathbf{P}(L(t) \leq u). \tag{2.35}$$

We see that if $(N(t))$ is a renewal process, then $(L(t))$ is a renewal reward process.

Gaudoin and Soler [14] consider homogeneous Poisson processes, non-homogeneous Poisson processes and renewal processes as models for $(N(t))$. A generalization of the non-homogeneous Poisson process is the Cox process, see e.g., Grandell [17]. A Cox process can be considered as a non-homogeneous Poisson process with randomized intensity measure. For a non-homogeneous Poisson process with intensity measure ν we have

$$\mathbf{P}(N(t) = k) \quad = \quad \frac{(\nu[0,t])^k}{k!} e^{-\nu[0,t]}, \quad k = 0, 1, 2, \ldots$$

For a Cox process the intensity measure ν is chosen according to some probability measure Π and

$$\mathbf{P}(N(t) = k) \quad = \quad \int \frac{(\nu[0,t])^k}{k!} e^{-\nu[0,t]} \Pi(d\nu), \quad k = 0, 1, 2, \ldots.$$

So if $(N(t))$ is a Cox process then, by conditioning on the number of stresses in the time interval $[0, t]$, the reliability in (2.33) can be expressed as

$$\tilde{R}(t) = \int e^{-[1-G(u)]\nu[0,t]} \Pi(d\nu).$$

As an example let $\nu[0, t] = \Lambda t$ where Λ is chosen according to uniform distribution in $[0, 1]$. Then

$$\tilde{R}(t) \quad = \quad \frac{1 - e^{-[1-G(u)]t}}{[1 - G(u)]t}.$$

In the case without recovery, note that the reliability $\tilde{R}(t)$ is just the cdf of $L(t)$ at point u. It follows that we only need to calculate the distribution of $L(t)$ in (2.34). By conditioning on the number of stresses in the time interval $[0, t]$ we obtain the Laplace transform of $L(t)$ which is given by

$$\begin{aligned} \psi(t, \alpha) \quad &:= \quad \mathbf{E}\left(e^{-\alpha L(t)}\right) \\ &= \quad \int e^{-[1-G^*(\alpha)]\nu[0,t]} \Pi(d\nu) \\ &= \quad \int e^{-[1-G^*(\alpha)] \int_0^t \nu(ds)} \Pi(d\nu). \end{aligned}$$

As an example let $\nu(ds) = X(s)ds$ where $(X(t), t \geq 0)$ is a continuous time Markov chain on $\{0,1\}$. Suppose that the chain starts at time 0 in state 1 where it stays an exponential time with mean $1/\lambda_1$. Then it jumps to state 0 where it stays an exponential time with mean $1/\lambda_0$, and so on. It follows that

$$\psi(t,\alpha) = \mathbf{E}\left(e^{-c(\alpha)\int_0^t X(s)ds}\right)$$

where $c(\alpha) = 1 - G^*(\alpha)$.

Let

$$\tau_i = \inf\{t \geq 0 | X(t) \neq i\}.$$

Starting from 1, the random variable τ_1 is the time at which the process leaves the state 1 and

$$\mathbf{P}_1(\tau_1 > t) = e^{-\lambda_1 t}.$$

Similarly for τ_0 we have

$$\mathbf{P}_0(\tau_0 > t) = e^{-\lambda_0 t}.$$

Then

$$\begin{aligned}\psi_1(t,\alpha) &= \mathbf{E}_1\left(e^{-c(\alpha)\int_0^t X(s)ds}, \tau_1 > t\right) + \mathbf{E}_1\left(e^{-c(\alpha)\int_0^t X(s)ds}, \tau_1 < t\right) \\ &= e^{-(\lambda_1 + c(\alpha))t} + \int_0^t \lambda_1 e^{-(\lambda_1 + c(\alpha))x}\psi_0(t-x, c(\alpha))dx, \qquad (2.36)\end{aligned}$$

and

$$\begin{aligned}\psi_0(t,\alpha) &= \mathbf{E}_0\left(e^{-c(\alpha)\int_0^t X(s)ds}, \tau_0 > t\right) + \mathbf{E}_0\left(e^{-c(\alpha)\int_0^t X(s)ds}, \tau_0 < t\right) \\ &= e^{-\lambda_0 t} + \int_0^t \lambda_0 e^{-\lambda_0 x}\psi_1(t-x, c(\alpha))dx. \qquad (2.37)\end{aligned}$$

Define for $\beta > 0$ and $i = 0, 1$

$$\hat{\psi}_i(\beta,\alpha) = \int_0^\infty e^{-\beta t}\psi_i(t,\alpha)dt.$$

From (2.36) and (2.37) we get the system of equations

$$\begin{cases} \hat{\psi}_1(\beta,\alpha) = \frac{1}{\lambda_1 + c(\alpha) + \beta} + \frac{\lambda_1}{\lambda_1 + c(\alpha) + \beta}\hat{\psi}_0(\beta,\alpha) \\ \hat{\psi}_0(\beta,\alpha) = \frac{1}{\lambda_0 + \beta} + \frac{\lambda_0}{\lambda_0 + \beta}\hat{\psi}_1(\beta,\alpha) \end{cases}$$

It follows that

$$\hat{\psi}_1(\beta,\alpha) = \frac{\lambda_1 + \lambda_0 + \beta}{\mu c(\alpha) + (\lambda_1 + \lambda_0 + c(\alpha))\beta + \beta^2}.$$

Transforming back this transform we obtain

$$\psi_1(t,\alpha) = e^{-\frac{1}{2}(\lambda+c(\alpha))t}\left[\cos\left(\frac{\sqrt{b}t}{2}\right) + \frac{\lambda - c(\alpha)}{\sqrt{b}}\sin\left(\frac{\sqrt{b}t}{2}\right)\right] \tag{2.38}$$

where $\lambda = \lambda_0 + \lambda_1$ and $b = 4\lambda_0 c(\alpha) - [\lambda + c(\alpha)]^2$. To find the distribution function of $L(t)$ we transform back numerically the Laplace transform in (2.38) with respect to α. For example if $\lambda_0 = \lambda_1 = 1$ and $G(x) = 1 - e^{-x}$ then we get the distribution function of $L(10)$ as in Figure 2.4. Note that there is a mass

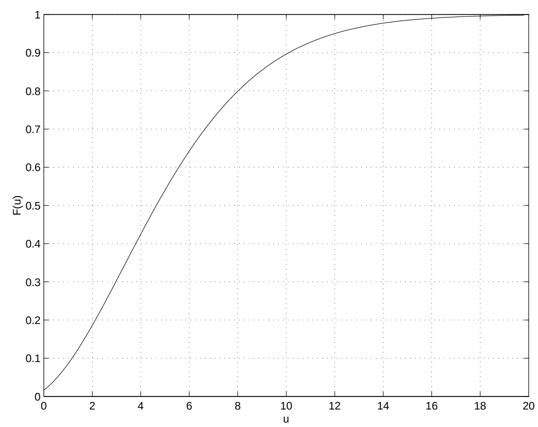


Figure 2.4: The graph of the distribution function of $L(10)$.

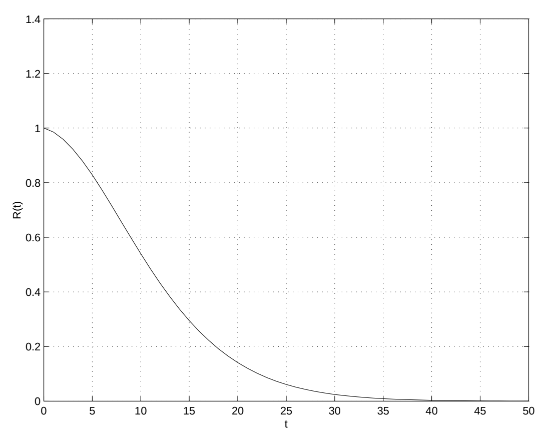


Figure 2.5: The graph of $\tilde{R}(t)$ for $u = 5$.

at 0 which corresponds to the event that no stress occurs in the time interval

$[0,t]$. The graph of $\tilde{R}(t)$, for $u=5$, is given in Figure 2.5.

Next we will consider the second generalization of the reliability in (2.35), i.e.,

$$\tilde{R}(t) = \mathbf{P}(L(t) \le u)$$

where $L(t) = \sum_{n=1}^{N(t)} Y_n$. As in Gaudoin and Soler [14] we assume that $N(t)$ is a renewal process, but we allow a dependence between the sequence (S_n) and (Y_n). Since $N(t)$ is a renewal process then $X_n = S_n - S_{n-1}$, $n = 1, 2, \ldots$, where $S_0 = 0$ are iid random variables. We will assume that $((X_n, Y_n), n \ge 1)$ is an iid sequence of random vectors. Note that in this case $L(t)$ is a renewal reward process. So we can use Theorem 2.3.1 to determine the distribution of $L(t)$:

$$\int_0^\infty \mathbf{E}(e^{-\alpha L(t)})e^{-\beta t} dt \quad = \quad \frac{1 - F^*(\beta)}{\beta[1 - H^*(\beta,\alpha)]}$$

where F is the cdf of X_1, and H is the joint cdf of X_1 and Y_1.

2.6.2 Type II models

Using the notation in Subsection 2.6.1 we now consider a model for the lifetime of the system where the cumulative failure rate at time t is proportional to the accumulation of all stresses occurred before time t. In this case the system reliability is given by

$$\tilde{R}(t) = \mathbf{E}\left(e^{-\alpha \sum_{n=1}^{\infty} Y_n h(t-S_n)}\right)$$

where $\alpha > 0$ is a proportionality constant and h is an arbitrary recovery function, see Subsection 2.6.1. We see in this case that the reliability is the Laplace transform of $L(t) := \sum_{n=1}^{\infty} Y_n h(t - S_n)$. The case where the occurrence times (S_n) is a non-homogeneous Poisson process has been discussed by Gaudoin and Soler [14]. Here firstly we will consider a generalization where (S_n) is a Cox process. We assume that the sequences (S_n) and (Y_n) are independent. We will express $L(t)$ as a functional of a Poisson point process.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which the random variables S_n and Y_n are defined such that the sequences (S_n) and (Y_n) are independent. Since we assume that (S_n) is a Cox process, then the map

$$\omega \in \Omega \mapsto \sum_{n=1}^{\infty} \delta_{S_n(\omega)}$$

defines a Poisson point process on $[0,\infty)$ with intensity measure ν where the measure ν is chosen randomly according to some probability distribution Π. More formally, the intensity measure ν is chosen from the set $M_+([0,\infty))$ of all

Radon measures on $[0,\infty)$. Moreover, since (S_n) and (Y_n) are independent, the map

$$\Phi : \omega \in \Omega \mapsto \sum_{n=1}^{\infty} \delta_{(S_n(\omega), Y_n(\omega))}$$

defines a Poisson point process on $E = [0,\infty) \times [0,\infty)$ with intensity measure $\nu \times G(dtdy) = \nu(dt)dG(y)$, where G is the cdf of Y_1. Let $M_p(E)$ be the set of simple point measures μ on E. Denote the distribution of Φ over $M_p(E)$ by $\mathbf{P}_{\nu\times G}$, i.e., $\mathbf{P}_{\nu\times G} = \mathbf{P} \circ \Phi^{-1}$.

As in Lemma 2.2.1 we have, with probability 1,

$$L(t) = \mathbb{L}(t)(\Phi) \tag{2.39}$$

where $\mathbb{L}(t)(\mu) = \int_E yh(t-s)\mu(dsdy)$. Using the formula for the Laplace functional of Poisson point processes, see Theorem 1.2.1, we obtain

$$\begin{aligned}
\tilde{R}(t) &= \int_{M_+([0,\infty))} \int_{M_p(E)} e^{-\int_E \alpha y h(t-s) 1_{[0,t]}(s)\mu(dsdy)} \mathbf{P}_{\nu\times G}(d\nu)\Pi(d\nu) \\
&= \int_{M_+([0,\infty))} \exp\left\{ -\int_E \left[1 - e^{-\alpha y h(t-s)1_{[0,t]}(s)}\right] \nu \times G(dsdy) \right\} \Pi(d\nu) \\
&= \int_{M_+([0,\infty))} \exp\left\{ -\int_0^t \left[1 - G^*(\alpha h(t-s))\right] \nu(ds) \right\} \Pi(d\nu).
\end{aligned}$$

The last equality follows from the independence assumption between (S_n) and (Y_n).

As an example let $M_+([0,\infty)) = \{\mu : \mu(dt) = \lambda dt, \lambda \in [0,\infty)\}$ and Π be a probability distribution of an exponential random variable with parameter η on $M_+([0,\infty))$. Then

$$\begin{aligned}
\tilde{R}(t) &= \int_0^\infty \exp\left\{ -\int_0^t \lambda\left[1 - G^*(\alpha h(t-s))\right] ds \right\} \eta e^{-\eta\lambda} d\lambda \\
&= \eta \int_0^\infty \exp\left\{ -\left[t + \eta - \int_0^t G^*(\alpha h(t-s)) ds\right] \lambda \right\} d\lambda \\
&= \frac{\eta}{\eta + t - \int_0^t G^*(\alpha h(t-s)) ds}.
\end{aligned}$$

Moreover, if we assume that $h(t) = e^{-t}$ and $G(u) = 1 - e^{-\gamma u}$, $\gamma > 0$, then we get

$$\tilde{R}(t) = \frac{\eta}{\eta - \ln\left(\frac{\alpha e^{-t} + \gamma}{\alpha + \gamma}\right)}.$$

Now consider the second generalization where the system keeps a permanent memory of the stresses ($h(t) \equiv 1$). Suppose that the occurrence times (S_n) are modelled as a renewal process. As in the end of the previous subsection, if $X_n = S_n - S_{n-1}, n = 1, 2, \ldots$ where $S_0 = 0$ represent the inter-arrival times of the renewal process and (X_n, Y_n) is assumed to be an iid sequence of random vectors then

$$\int_0^\infty \tilde{R}(t)e^{-\beta t}dt = \frac{1 - F^*(\beta)}{\beta[1 - H^*(\beta, \alpha)]}$$

where F is the cdf of the X_1 and H is the joint cdf of X_1 and Y_1. As a special case, when the sequences (X_n) and (Y_n) are independent, we have

$$\int_0^\infty \tilde{R}(t)e^{-\beta t}dt = \frac{1 - F^*(\beta)}{\beta[1 - F^*(\beta)G^*(\alpha)]}$$

where G is the cdf of the Y_1.

As an example assume that (X_n, Y_n) is an iid sequence of random vectors. Suppose that X_1 and Y_1 have a joint bivariate exponential distribution with

$$\mathbf{P}(X_1 > x, Y_1 > y) = e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12}\max(x,y))}; \quad x, y \geq 0; \quad \lambda_1, \lambda_2, \lambda_{12} > 0.$$

The marginals are given by

$$\mathbf{P}(X_1 > x) = e^{-(\lambda_1 + \lambda_{12})x} \tag{2.40}$$

and

$$\mathbf{P}(Y_1 > y) = e^{-(\lambda_2 + \lambda_{12})y}. \tag{2.41}$$

The correlation coefficient ρ between X_1 and Y_1 is given by

$$\rho = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}.$$

In this case

$$\int_0^\infty \tilde{R}(t)e^{-\beta t}dt = \frac{C_1\beta + C_2}{C_1\beta^2 + C_3\beta + C_4},$$

where $C_1 = \lambda_2 + \lambda_{12} + \alpha$, $C_2 = C_1(C_1 + \lambda_1)$, $C_3 = \lambda_1\alpha + C_2$, and $C_4 = \alpha(\lambda_1 + \lambda_{12})(C_1 + \lambda_1)$. Inverting this transform we obtain

$$\begin{aligned}\tilde{R}(t) &= e^{-\frac{C_3}{2C_1}t}\left[\cos\left(\frac{1}{2C_1}\sqrt{4C_1C_4 - C_3^2}t\right)\right.\\ &\quad \left. + \frac{2C_2 - C_3}{\sqrt{4C_1C_4 - C_3^2}}\sin\left(\frac{1}{2C_1}\sqrt{4C_1C_4 - C_3^2}t\right)\right].\end{aligned}$$

In case the sequences (X_n) and (Y_n) are independent where X_1 and Y_1 satisfy (2.40) and (2.41),

$$\int_0^\infty \tilde{R}(t)e^{-\beta t}dt = \frac{\alpha + \lambda_2 + \lambda_{12}}{\beta(\alpha + \lambda_2 + \lambda_{12}) + \alpha(\lambda_1 + \lambda_{12})}.$$

Inverting this transform we obtain

$$\tilde{R}(t) \quad = \quad \exp\left\{ -\frac{\alpha(\lambda_1+\lambda_{12})}{\alpha+\lambda_2+\lambda_{12}}t \right\}.$$

Now we will observe the effect of the dependence between X_1 and Y_1 on the reliability $\tilde{R}(t)$ in this example. As examples, firstly take $\alpha = 1$, $\lambda_1 = \lambda_2 = \frac{2}{3}$, $\lambda_{12} = \frac{1}{3}$. Then for the dependent case, $\rho = 0.2$ and

$$\tilde{R}(t) \quad = \quad e^{-\frac{3}{2}t}\Big[\cosh(0.9574t) + 1.2185\sinh(0.9574t)\Big],$$

and for the independent case

$$\tilde{R}(t) \quad = \quad e^{-\frac{1}{2}t}.$$

The graphs of $\tilde{R}(t)$ can be seen in Figure 2.6.

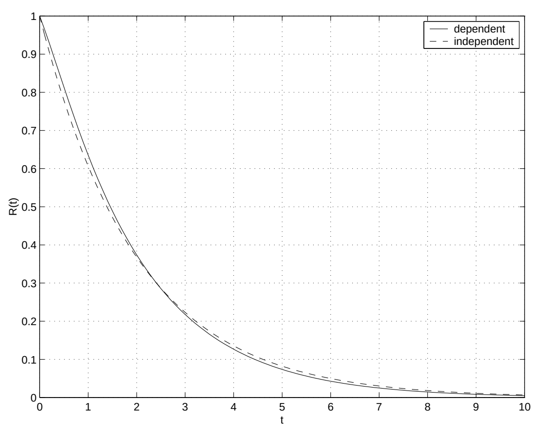


Figure 2.6: The graphs of $\tilde{R}(t)$ for $\alpha = 1$, $\lambda_1 = \lambda_2 = \frac{2}{3}$, $\lambda_{12} = \frac{1}{3}$.

Secondly, take $\alpha = 1$, $\lambda_1 = \lambda_2 = \frac{1}{3}$, $\lambda_{12} = \frac{2}{3}$. Then for the dependent case, $\rho = 0.5$ and

$$\tilde{R}(t) \quad = \quad e^{-1.25t}\Big[\cosh(0.6292t) + 1.7219\sinh(0.6292t)\Big],$$

and for the independent case

$$\tilde{R}(t) \quad = \quad e^{-\frac{1}{2}t}.$$

The graphs of $\tilde{R}(t)$ can be seen in Figure 2.7.

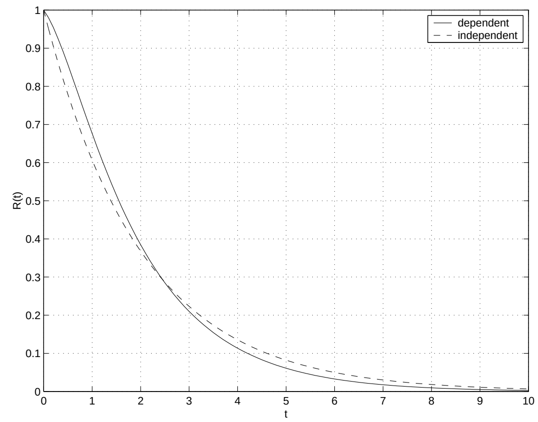


Figure 2.7: The graphs of $\tilde{R}(t)$ for $\alpha = 1$, $\lambda_1 = \lambda_2 = \frac{1}{3}$, $\lambda_{12} = \frac{2}{3}$.

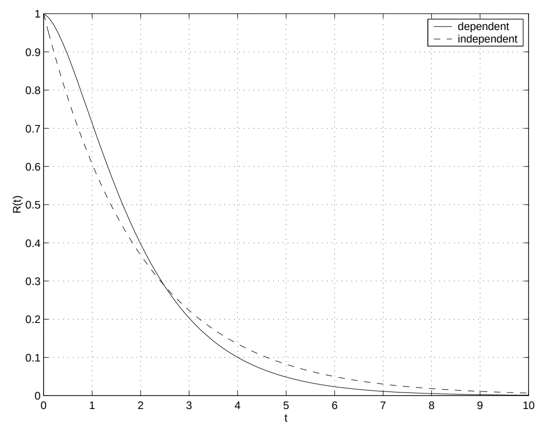


Figure 2.8: The graphs of $\tilde{R}(t)$ for $\alpha = 1$, $\lambda_1 = \lambda_2 = \frac{1}{9}$, $\lambda_{12} = \frac{8}{9}$.

Finally, take $\alpha = 1$, $\lambda_1 = \lambda_2 = \frac{1}{9}$, $\lambda_{12} = \frac{8}{9}$. Then for the dependent case, $\rho = 0.8$ and

$$\tilde{R}(t) = e^{-1.0833t}\Big[\cosh(0.3436t) + 2.9913\sinh(0.3436t)\Big],$$

and for the independent case

$$\tilde{R}(t) = e^{-\frac{1}{2}t}.$$

The graphs of $\tilde{R}(t)$ can be seen in Figure 2.8.

Chapter 3

Integrated Renewal Processes

3.1 Notations and Definitions

Consider a locally finite point process on the positive half line $[0, \infty)$. Denote the ordered sequence of points by $0 < S_1 < S_2 < \ldots$. We will think of the points S_n as arrival times. We define $S_0 := 0$, but this does not mean that we assume that there is a point in 0. Let $N(t)$ be the number of arrivals in the time interval $[0, t]$, i.e., $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Define for $t \geq 0$

$$Y(t) = \int_0^t N(s)ds.$$

If $(N(t), t \geq 0)$ is a renewal process, we call the stochastic process $(Y(t), t \geq 0)$ *an integrated renewal process*. Note that we can express $Y(t)$ as

$$Y(t) = \sum_{i=1}^{N(t)} (t - S_i) = tN(t) - Z(t), \tag{3.1}$$

where

$$Z(t) = \sum_{i=1}^{N(t)} S_i. \tag{3.2}$$

Figure 3.1 shows the graphs of $Y(t)$ and $Z(t)$.

In this chapter we will discuss the distributions of $Y(t)$ and $Z(t)$. In Section 3.2 we discuss the distributions of $Y(t)$ and $Z(t)$ when $(N(t))$ is a Poisson or a Cox process. In Section 3.3 we discuss the distributions of $Z(t)$ and $Y(t)$ when $(N(t))$ is a renewal process. Their asymptotic properties are studied in Section 3.4. Finally an application is given in Section 3.5.

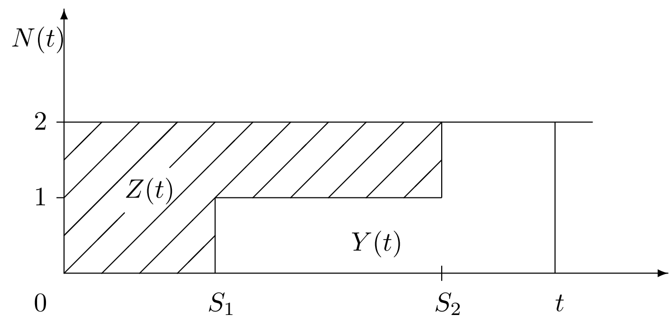


Figure 3.1: Graphs of $Y(t)$ and $Z(t)$.

3.2 $(N(t))$ a Poisson or Cox process

Firstly, suppose that the process $(N(t), t \geq 0)$ is a homogeneous Poisson process with rate λ. It is well known that given $N(t) = n$, the n arrival times $S_1, ..., S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the time interval $[0, t]$ (see e.g. Ross [36]). Conditioning on the number of arrivals in the time interval $[0, t]$ we obtain

$$\begin{aligned} \mathbf{E}(e^{-\alpha Y(t)}) &= \sum_{n=0}^{\infty} \mathbf{E}\left[e^{-\alpha \sum_{i=1}^{n}(t-S_i)} | N(t) = n\right] \mathbf{P}(N(t) = n) \\ &= \sum_{n=0}^{\infty} e^{-\alpha n t} \mathbf{E}\left[e^{-\alpha \sum_{i=1}^{n} V_i}\right] \frac{(\lambda t)^n}{n!} e^{-\lambda t} \end{aligned}$$

where $V_i, i = 1, 2, ..., n$ are independent and identically uniform random variables on $[0, t]$. Since

$$\mathbf{E}\left[e^{-\alpha V_1}\right] = \frac{1}{\alpha t}\left[e^{\alpha t} - 1\right]$$

it follows that

$$\begin{aligned} \mathbf{E}(e^{-\alpha Y(t)}) &= \sum_{n=0}^{\infty} e^{-\alpha n t} \left[\frac{1}{\alpha t}\left[e^{\alpha t} - 1\right]\right]^n \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\ &= \sum_{n=0}^{\infty} \frac{\left[\frac{\lambda(1-e^{-\alpha t})}{\alpha}\right]^n}{n!} e^{-\lambda t} \\ &= \exp\left\{\frac{\lambda(1 - \alpha t - e^{-\alpha t})}{\alpha}\right\}. \end{aligned} \tag{3.3}$$

From (3.3) we deduce that $\mathbf{E}[Y(t)] = \frac{1}{2}\lambda t^2$ and $\text{Var}[Y(t)] = \frac{1}{3}\lambda t^3$. Using a similar argument we can prove that $Z(t)$ has the same Laplace transform as

$Y(t)$. So by uniqueness theorem for Laplace transforms we conclude that $Z(t)$ has the same distribution as $Y(t)$.

The distribution of $Y(t)$ has mass at zero with

$$\mathbf{P}(Y(t) = 0) = e^{-\lambda t}.$$

The density function $f_{Y(t)}$ of the continuous part of $Y(t)$ can be obtained by inverting the Laplace transform in (3.3). Note that we can express (3.3) as

$$\begin{aligned}\mathbf{E}(e^{-\alpha Y(t)}) &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n (1-e^{-\alpha t})^n}{n! \alpha^n} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} e^{-kt\alpha}}{\alpha^n}.\end{aligned}$$

Inverting this transform we obtain, for $x > 0$,

$$\begin{aligned} f_{Y(t)}(x) & \\ &= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left[\frac{x^{n-1}}{(n-1)!} 1_{(0,\infty)}(x) + \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{(x-kt)^{n-1}}{(n-1)!} 1_{(kt,\infty)}(x) \right] \\ &= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda^n x^{n-1}}{n!(n-1)!} 1_{(0,\infty)}(x) \\ &\quad + \lambda e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{n=k}^{\infty} \frac{[\lambda(x-kt)]^{n-1}}{(n-1)!(n-k)!} 1_{(kt,\infty)}(x) \\ &= \frac{\sqrt{\lambda}}{\sqrt{x}} e^{-\lambda t} I_1(2\sqrt{\lambda x}) 1_{(0,\infty)}(x) \\ &\quad + \lambda e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} [\lambda(x-kt)]^{\frac{1}{2}(k-1)} I_{k-1}(2\sqrt{\lambda(x-kt)}) 1_{(kt,\infty)}(x) \end{aligned}$$

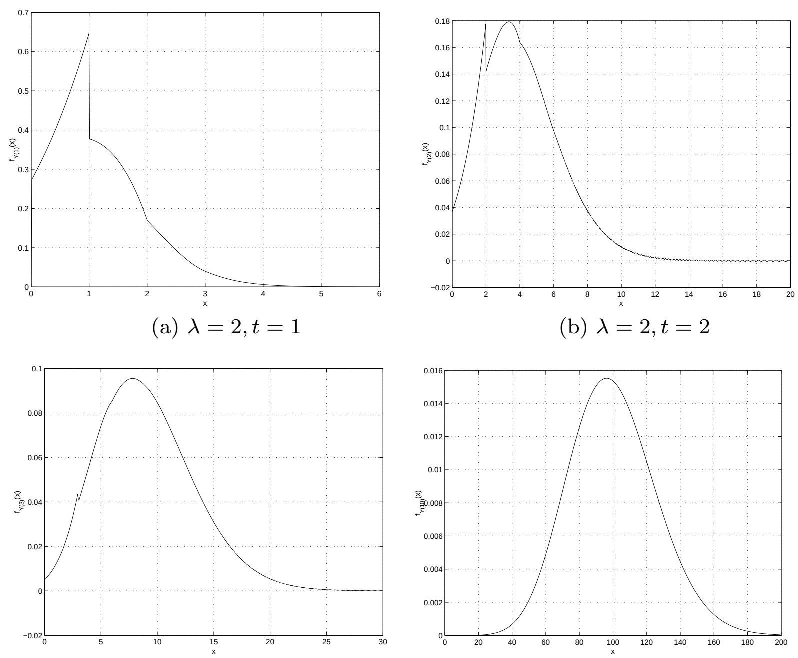
where $I_k(x)$ is the Modified Bessel function of the first kind, i.e.,

$$I_k(x) = (1/2x)^k \sum_{m=0}^{\infty} \frac{(1/4x^2)^m}{m! \Gamma(k+m+1)},$$

see Gradshteyn and Ryzhik [16]. The graphs of the pdf of $Y(t)$ for $\lambda = 2$, $t = 1, 2, 3$ and $t = 10$ can be seen in Figure 3.2.

For large t, the distribution of $Y(t)$ can be approximated with a normal distribution having mean $\frac{1}{2}\lambda t^2$ and variance $\frac{1}{3}\lambda t^3$. To prove this we will consider the characteristic function of the normalized $Y(t)$. Firstly, note that

$$\mathbf{E}\left(e^{-i\alpha \frac{Y(t) - \frac{1}{2}\lambda t^2}{\sqrt{\frac{1}{3}\lambda t^3}}} \right) = e^{\frac{1}{2} i\alpha\sqrt{3\lambda t}} \mathbf{E}\left(e^{-\frac{i\alpha\sqrt{3}}{t\sqrt{\lambda t}} Y(t)} \right).$$



(a) $\lambda = 2, t = 1$ (b) $\lambda = 2, t = 2$

(c) $\lambda = 2, t = 3$ (d) $\lambda = 2, t = 10$

Figure 3.2: The graphs of the pdf of $Y(t)$ for $\lambda = 2$, $t = 1, 2, 3$ and $t = 10$.

Using (3.3) with α is replaced by $\frac{i\alpha\sqrt{3}}{t\sqrt{\lambda t}}$ and an expansion we obtain

$$\begin{aligned}
\mathbf{E}\left(e^{-\frac{i\alpha\sqrt{3}}{t\sqrt{\lambda t}}Y(t)}\right) &= \exp\left\{\frac{\lambda t\sqrt{\lambda t}}{i\alpha\sqrt{3}}\left[1 - \frac{i\alpha\sqrt{3}}{\sqrt{\lambda t}} - e^{\frac{-i\alpha\sqrt{3t}}{\sqrt{\lambda}}}\right]\right\} \\
&= \exp\left\{\frac{\lambda t\sqrt{\lambda t}}{i\alpha\sqrt{3}}\left[\frac{3\alpha^2}{2\lambda t} - \frac{i\alpha^3\sqrt{3}}{2\lambda t\sqrt{\lambda t}} + o(t^{-3/2})\right]\right\} \\
&= \exp\left\{-\frac{1}{2}i\alpha\sqrt{3\lambda t} - \frac{1}{2}\alpha^2 + \frac{\lambda t\sqrt{\lambda t}}{i\alpha\sqrt{3}}o(t^{-3/2})\right\}
\end{aligned}$$

as $t \to \infty$. It follows that

$$\mathbf{E}\left(e^{-i\alpha\frac{Y(t)-\frac{1}{2}\lambda t^2}{\sqrt{\frac{1}{3}\lambda t^3}}}\right) \longrightarrow e^{-\frac{1}{2}\alpha^2} \quad \text{as} \quad t \to \infty$$

which is the characteristic function of the standard normal distribution.

Now consider the case where $(N(t))$ is a non-homogeneous Poisson process with intensity measure ν. Given $N(t) = n$, the arrival times $S_i, i = 1, 2, ..., n$

have the same distribution as the order statistics of n iid random variables having a common cdf

$$G(x) = \begin{cases} \frac{\nu([0,x])}{\nu([0,t])}, & x \le t \\ 1, & x > t. \end{cases}$$

In this case the Laplace transform of $Z(t)$ is given by

$$\begin{aligned} \mathbf{E}\left(e^{-\alpha Z(t)}\right) &= \sum_{n=0}^{\infty} \mathbf{E}\left(e^{-\alpha \sum_{i=1}^{n} S_i} | N(t) = n\right) \mathbf{P}(N(t) = n) \\ &= \sum_{n=0}^{\infty} \left[\frac{\int_0^t e^{-\alpha x} d\nu([0,x])}{\nu([0,t])}\right]^n \frac{\nu([0,t])^n}{n!} e^{-\nu([0,t])} \\ &= \exp\left\{\int_0^t e^{-\alpha x} d\nu([0,x])\right\} e^{-\nu([0,t])} \\ &= \exp\left\{\int_0^t \left[e^{-\alpha x} - 1\right] d\nu([0,x])\right\}. \end{aligned} \tag{3.4}$$

From this Laplace transform we deduce that

$$\mathbf{E}[Z(t)] = \int_0^t x d\nu([0,x])$$

and

$$\mathrm{Var}[Z(t)] = \int_0^t x^2 d\nu([0,x]).$$

Similarly, we can prove that

$$\mathbf{E}(e^{-\alpha Y(t)}) = \exp\left\{\int_0^t [e^{-\alpha(t-x)} - 1] d\nu([0,x])\right\}, \tag{3.5}$$

$$\mathbf{E}[Y(t)] = \int_0^t (t-x) d\nu([0,x]),$$

and

$$\mathrm{Var}[Y(t)] = \int_0^t (t-x)^2 d\nu([0,x]).$$

Note that in general $Y(t)$ has a different distribution from $Z(t)$ in case $(N(t))$ is a non-homogeneous Poisson process.

Next we consider the distributions of $Y(t)$ and $Z(t)$ when $(N(t))$ is a Cox process. A Cox process is a generalization of a Poisson process. In a Cox process the intensity measure ν of the Poisson process is chosen randomly according to some probability distribution Π. So if $(N(t))$ is a Cox process then from (3.4) and (3.5) we obtain

$$\mathbf{E}\left(e^{-\alpha Z(t)}\right) = \int \exp\left\{-\int_0^t [1-e^{-\alpha x}]d\nu([0,x])\right\}\Pi(d\nu)$$

and

$$\mathbf{E}\left(e^{-\alpha Y(t)}\right) = \int \exp\left\{-\int_0^t [1-e^{-\alpha(t-x)}]d\nu([0,x])\right\}\Pi(d\nu).$$

As an example let the intensity measure ν satisfy $\nu([0,t]) = \Lambda t$ for some positive random variable Λ. If Λ is exponentially distributed with parameter η, then

$$\mathbf{E}\left(e^{-\alpha Y(t)}\right) = \mathbf{E}\left(e^{-\alpha Z(t)}\right) = \frac{\alpha\eta}{\alpha(\eta+t)-(1-e^{-\alpha t})}.$$

3.3 $(N(t))$ a renewal process

In this section we will consider the distribution of the processes $(Y(t))$ and $(Z(t))$ defined in (3.1) and (3.2) for the case that $(N(t))$ is a renewal process. Let $X_n = S_n - S_{n-1}$, $n \geq 1$, be the inter-arrival times of the renewal process. Note that $(X_n, n \geq 1)$ is an iid sequence of strictly positive random variables. Let F denote the cdf of X_1. As usual we will denote Laplace-Stieltjes transform of F by F^*. First we consider the process $(Z(t))$. Obviously we can express $Z(t)$ as

$$Z(t) = \sum_{i=1}^{N(t)} [N(t)+1-i]X_i. \tag{3.6}$$

We will use point processes to derive the distribution of $Z(t)$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which the iid sequence (X_n) is defined and also an iid sequence $(U_n, n \geq 1)$ of exponentially distributed random variables with parameter 1 such that the sequences (X_n) and (U_n) are independent. Let $(T_n, n \geq 1)$ be the sequence of partial sums of the variables U_n. Then the map

$$\Phi \;:\; \omega \;\mapsto\; \textstyle\sum_{n=1}^{\infty} \delta_{(T_n(\omega), X_n(\omega))}, \tag{3.7}$$

where $\delta_{(x,y)}$ is the Dirac measure in (x,y), defines a Poisson point process on $E = [0,\infty) \times [0,\infty)$ with intensity measure $\nu(dtdx) = dtdF(x)$. Let $M_p(E)$ be the set of all point measures on E. We will denote the distribution of Φ by $\mathbf{P}_\nu$, i.e., $\mathbf{P}_\nu = \mathbf{P} \circ \Phi^{-1}$.

Define for $t \geq 0$ the functional $\mathbb{A}(t)$ on $M_p(E)$ by

$$\mathbb{A}(t)(\mu) = \int_E 1_{[0,t)}(s) x \mu(dsdx).$$

In the sequel we write $\mathbb{A}(t,\mu) = \mathbb{A}(t)(\mu)$. Define also for $t \geq 0$ the functional $\mathbb{Z}(t)$ on $M_p(E)$ by

$$\mathbb{Z}(t)(\mu) = \int_E \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu))\mu([r,s) \times [0,\infty)) u 1_{[0,s)}(r)\mu(drdu)\mu(dsdx).$$

Lemma 3.3.1 *With probability 1,*

$$Z(t) \quad = \quad \mathbb{Z}(t)(\Phi).$$

Proof: Let $\omega \in \Omega$. Then

$$\begin{aligned}
&\mathbb{Z}(t)(\Phi(\omega)) \\
&= \sum_{n=1}^{\infty} 1_{[0,X_n(\omega))}(t - \mathbb{A}(T_n(\omega),\Phi(\omega))) \\
&\quad \sum_{i=1}^{\infty} \Phi(\omega)([T_i(\omega),T_n(\omega)) \times [0,\infty)) X_i(\omega) 1_{[0,T_n(\omega))}(T_i(\omega)) \\
&= \sum_{i=1}^{\infty} \Phi(\omega)([T_i(\omega),T_{N(t,\omega)+1}(\omega)) \times [0,\infty)) X_i(\omega) 1_{[0,T_{N(t,\omega)+1}(\omega))}(T_i(\omega)) \\
&= \sum_{i=1}^{N(t,\omega)} [N(t,\omega)+1-i] X_i(\omega). \quad \square
\end{aligned}$$

Theorem 3.3.1 *Let $(X_n, n \geq 1)$ be an iid sequence of strictly positive random variables with common distribution function F. Let $(S_n, n \geq 0)$ be the sequence of partial sums of the variables X_n and $(N(t), t \geq 0)$ be the corresponding renewal process: $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Let*

$$Z(t) \quad = \quad \sum_{i=1}^{N(t)} [N(t)+1-i] X_i.$$

Then for $\alpha, \beta > 0$

$$\int_0^\infty \mathbf{E}(e^{-\alpha Z(t)}) e^{-\beta t} dt \quad = \quad \frac{1}{\beta}[1 - F^*(\beta)] \sum_{n=0}^{\infty} \prod_{i=1}^{n} F^*(\alpha[n+1-i]+\beta) \tag{3.8}$$

(with the usual convention that the empty product equals 1).

Proof: By Lemma 3.3.1

$$\begin{aligned}
\mathbf{E}(e^{-\alpha Z(t)}) &= \int_{M_p(E)} e^{-\alpha \mathbb{Z}(t)(\mu)} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \exp\Big\{ -\alpha \int_E \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \\
&\quad \mu([r,s) \times [0,\infty)) u 1_{[0,s)}(r) \mu(drdu) \mu(dsdx) \Big\} \mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)} \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \\
&\quad \exp\Big\{ -\alpha \int_E \mu([r,s) \times [0,\infty)) u 1_{[0,s)}(r) \mu(drdu) \Big\} \mu(dsdx) \mathbf{P}_\nu(d\mu)
\end{aligned}$$

Applying the Palm formula for Poisson point processes, see Theorem 1.2.4, we obtain

$$\begin{aligned}
&\mathbf{E}(e^{-\alpha Z(t)}) \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} 1_{[0,x)}(t - \mathbb{A}(s, \mu + \delta_{(s,x)})) \\
&\quad \exp\Big\{ - \int_E \alpha(\mu + \delta_{(s,x)})([r,s) \times [0,\infty)) u 1_{[0,s)}(r) (\mu + \delta_{(s,x)})(drdu) \Big\} \\
&\quad \mathbf{P}_\nu(d\mu) dF(x) ds \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \\
&\quad \exp\Big\{ - \int_E \alpha\mu([r,s) \times [0,\infty)) u 1_{[0,s)}(r) \mu(drdu) \Big\} \mathbf{P}_\nu(d\mu) dF(x) ds.
\end{aligned}$$

Using Fubini's theorem and a substitution we obtain

$$\begin{aligned}
&\int_0^\infty \mathbf{E}(e^{-\alpha Z(t)}) e^{-\beta t} dt \\
&= \int_0^\infty \int_0^\infty \int_{M_p(E)} \int_0^\infty 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \\
&\quad \exp\Big\{ - \int_E \alpha\mu([r,s) \times [0,\infty)) u 1_{[0,s)}(r) \mu(drdu) \Big\} e^{-\beta t} dt \mathbf{P}_\nu(d\mu) dF(x) ds \\
&= \frac{1}{\beta}[1 - F^*(\beta)] \int_0^\infty \int_{M_p(E)} \\
&\quad \exp\Big\{ - \int_E \Big[\alpha\mu([r,s) \times [0,\infty)) + \beta\Big] u 1_{[0,s)}(r) \mu(drdu) \Big\} \mathbf{P}_\nu(d\mu) ds.
\end{aligned}$$

The integral with respect to $\mathbf{P}_\nu$ can be written as a sum of integrals over the sets $B_n := \{\mu \in M_p(E) : \mu([0,s) \times [0,\infty)) = n\}$, $n = 0, 1, 2, ...$ Fix a value of n and let $\mu \in M_p(E)$ be such that $\mu([0,s) \times [0,\infty)) = n$ and $\mathrm{supp}(\mu) = ((t_i, x_i))_{i=1}^\infty$. So $t_n < s \le t_{n+1}$. For such a measure μ the integrand with respect to $\mathbf{P}_\nu$ can be written as

$$\begin{aligned}
&\int_E \Big[\alpha\mu([r,s) \times [0,\infty)) + \beta\Big] u 1_{[0,s)}(r)\mu(drdu) \\
&= \sum_{i=1}^\infty \Big[\alpha\mu([t_i,s) \times [0,\infty)) + \beta\Big] x_i 1_{[0,s)}(t_i) \\
&= \sum_{i=1}^n (\alpha[n+1-i] + \beta)x_i.
\end{aligned}$$

Now the measure $\mathbf{P}_\nu$ is the image measure of $\mathbf{P}$ under the map Φ, see (3.7). Expressing the integral with respect to $\mathbf{P}_\nu$ over B_n as an integral with respect to $\mathbf{P}$ over the subset $A_n := \{\omega \in \Omega : T_n(\omega) < s \le T_{n+1}(\omega)\}$ of Ω, and using independence of (T_n) and (X_n), we obtain

$$\begin{aligned}
&\int_{B_n} e^{-\int_E \big[\alpha\mu([r,s)\times[0,\infty))+\beta\big] u 1_{[0,s)}(r)\mu(drdu)} \mathbf{P}_\nu(d\mu) \\
&= \int_{A_n} \exp\Big\{ - \sum_{i=1}^n (\alpha[n+1-i] + \beta)X_i(\omega)\Big\}\mathbf{P}(d\omega) \\
&= \mathbf{E}\Big[\exp\Big\{ - \sum_{i=1}^n (\alpha[n+1-i] + \beta)X_i\Big\}\Big]\mathbf{P}(A_n) \\
&= \prod_{i=1}^n \mathbf{E}\Big[\exp\Big\{ - (\alpha[n+1-i] + \beta)X_i\Big\}\Big]\frac{s^n}{n!}e^{-s} \\
&= \prod_{i=1}^n F^*(\alpha[n+1-i] + \beta)\frac{s^n}{n!}e^{-s}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{M_p(E)} e^{-\int_E [\alpha\mu([r,s)\times[0,\infty))+\beta] u 1_{[0,s)}(r)\mu(drdu)} \mathbf{P}_\nu(d\mu) \\
&= \sum_{n=0}^\infty \prod_{i=1}^n F^*(\alpha[n+1-i] + \beta)\frac{s^n}{n!}e^{-s}.
\end{aligned}$$

Since for each n, $\int_0^\infty \frac{s^n}{n!}e^{-s}ds = 1$, the theorem follows. □

The Laplace transform of the mean of $Z(t)$ can be derived from (3.8) as follows. Let $\varphi(\alpha) = \mathbf{E}\left(e^{-\alpha Z(t)}\right)$ and define

$$W_{n,i}(\alpha) = F^*(\alpha[n+1-i] + \beta).$$

Then

$$V_n(\alpha) = \prod_{i=1}^{n} F^*(\alpha[n+1-i]+\beta) = \prod_{i=1}^{n} W_{n,i}(\alpha).$$

Hence

$$V_n'(\alpha) = \sum_{j=1}^{n} W_{n,j}'(\alpha) \prod_{i=1, i\neq j}^{n} W_{n,i}(\alpha), \tag{3.9}$$

where

$$W_{n,j}'(\alpha) \quad = \quad -(n+1-j)\int_0^\infty xe^{-(\alpha[n+1-j]+\beta)x}dF(x)$$

if $\mathbf{E}\left[X_1 e^{-\beta X_1}\right] < \infty$ for some $\beta > 0$. Since $W_{n,i}(0) = F^*(\beta)$ and

$$W_{n,j}'(0) = -(n+1-j)\int_0^\infty xe^{-\beta x}dF(x),$$

it follows that

$$\begin{aligned} V_n'(0) &= \sum_{j=1}^{n} -(n+1-j)\int_0^\infty xe^{-\beta x}dF(x)F^*(\beta)^{n-1} \\ &= -\int_0^\infty xe^{-\beta x}dF(x)F^*(\beta)^{n-1}\sum_{j=1}^{n}(n+1-j) \\ &= -\frac{1}{2}\int_0^\infty xe^{-\beta x}dF(x)F^*(\beta)^{n-1}n(n+1). \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt \\ &= \int_0^\infty -\varphi'(0)e^{-\beta t}dt \\ &= \frac{1}{2\beta}[1-F^*(\beta)]\int_0^\infty xe^{-\beta x}dF(x)\sum_{n=0}^{\infty}F^*(\beta)^{n-1}n(n+1) \\ &= \frac{1}{\beta}[1-F^*(\beta)]\int_0^\infty xe^{-\beta x}dF(x)\frac{1}{[1-F^*(\beta)]^3} \\ &= \frac{\int_0^\infty xe^{-\beta x}dF(x)}{\beta[1-F^*(\beta)]^2}. \end{aligned}$$

Thus we have the following proposition:

Proposition 3.3.1 *Under the same assumptions as Theorem 3.3.1, and if* $\mathbf{E}\left[X_1 e^{-\beta X_1}\right] < \infty$ *for some* $\beta > 0$*, then*

$$\int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt \quad = \quad \frac{\int_0^\infty xe^{-\beta x}dF(x)}{\beta[1-F^*(\beta)]^2}. \tag{3.10}$$

Now we will derive the Laplace transform of the second moment of $Z(t)$. From (3.9) we obtain

$$\begin{aligned} V_n''(\alpha) \quad &= \quad \sum_{j=1}^n W_{n,j}''(\alpha) \prod_{i=1,i\neq j}^n W_{n,i}(\alpha) \\ &\quad + \sum_{j=1}^n W_{n,j}'(\alpha) \sum_{k=1,k\neq j}^n W_{n,k}'(\alpha) \prod_{i=1,i\neq j\neq k}^n W_{n,i}(\alpha), \end{aligned}$$

where

$$W_{n,j}''(\alpha) \quad = \quad (n+1-j)^2 \int_0^\infty x^2 e^{-(\alpha[n+1-j]+\beta)x} dF(x)$$

if $\mathbf{E}\left[X_1^2 e^{-\beta X_1}\right] < \infty$ for some $\beta > 0$,. Since $W_{n,i}(0) = F^*(\beta)$, $W_{n,j}'(0) = -(n+1-j)\int_0^\infty xe^{-\beta x}dF(x)$, and $W_{n,j}''(0) = (n+1-j)^2 \int_0^\infty x^2 e^{-\beta x}dF(x)$, then

$$\begin{aligned} V_n''(0) \quad &= \quad \sum_{j=1}^n (n+1-j)^2 \int_0^\infty x^2 e^{-\beta x} dF(x) F^*(\beta)^{n-1} \\ &\quad + \sum_{j=1}^n -(n+1-j) \int_0^\infty xe^{-\beta x} dF(x) \\ &\quad \sum_{k=1,k\neq j}^n -(n+1-k) \int_0^\infty xe^{-\beta x} dF(x) F^*(\beta)^{n-2} \\ &= \quad \int_0^\infty x^2 e^{-\beta x} dF(x) F^*(\beta)^{n-1} \sum_{j=1}^n (n+1-j)^2 \\ &\quad + \Big[\int_0^\infty xe^{-\beta x} dF(x)\Big]^2 F^*(\beta)^{n-2} \sum_{j=1}^n (n+1-j) \sum_{k=1,k\neq j}^n (n+1-k). \end{aligned}$$

So

$$\int_0^\infty \mathbf{E}[Z^2(t)]e^{-\beta t}dt$$

$$= \int_0^\infty \varphi''(0)e^{-\beta t}dt$$

$$= \frac{1}{\beta}[1-F^*(\beta)]\sum_{n=0}^{\infty}\left(\int_0^\infty x^2e^{-\beta x}dF(x)F^*(\beta)^{n-1}\sum_{j=1}^{n}(n+1-j)^2\right.$$

$$\left.+\Big[\int_0^\infty xe^{-\beta x}dF(x)\Big]^2F^*(\beta)^{n-2}\sum_{j=1}^{n}(n+1-j)\sum_{k=1,k\neq j}^{n}(n+1-k)\right)$$

$$= \frac{1}{\beta}[1-F^*(\beta)]\left(\int_0^\infty x^2e^{-\beta x}dF(x)\frac{F^*(\beta)+1}{[1-F^*(\beta)]^4}\right.$$

$$\left.+2\Big[\int_0^\infty xe^{-\beta x}dF(x)\Big]^2\frac{2+F^*(\beta)}{[1-F^*(\beta)]^5}\right)$$

$$= \frac{1}{\beta[1-F^*(\beta)]^3}\left([F^*(\beta)+1]\int_0^\infty x^2e^{-\beta x}dF(x)\right.$$

$$\left.+\frac{4+2F^*(\beta)}{1-F^*(\beta)}\Big[\int_0^\infty xe^{-\beta x}dF(x)\Big]^2\right).$$

Thus we have the following proposition:

Proposition 3.3.2 *Under the same assumptions as Theorem 3.3.1, and if* $\mathbf{E}\left[X_1^2e^{-\beta X_1}\right]<\infty$ *for some* $\beta>0$,

$$\int_0^\infty \mathbf{E}[Z^2(t)]e^{-\beta t}dt = \frac{[1+F^*(\beta)]\int_0^\infty x^2e^{-\beta x}dF(x)}{\beta[1-F^*(\beta)]^3} + \frac{2[2+F^*(\beta)][\int_0^\infty xe^{-\beta x}dF(x)]^2}{\beta[1-F^*(\beta)]^4}. \tag{3.11}$$

Remark 3.3.1 *If* X_1 *is exponentially distributed with parameter* λ, *then using* (3.10) *and* (3.11) *we obtain*

$$\int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt = \frac{\lambda}{\beta^3}$$

and

$$\int_0^\infty \mathbf{E}[Z^2(t)]e^{-\beta t}dt = \frac{2\lambda[3\lambda+\beta]}{\beta^5}.$$

Inverting these transforms we obtain $\mathbf{E}[Z(t)]=\frac{1}{2}\lambda t^2$, $\mathbf{E}[Z^2(t)]=\frac{1}{4}\lambda^2t^4+\frac{1}{3}\lambda t^3$, *and hence* $\mathrm{Var}[Z(t)]=\frac{1}{3}\lambda t^3$. *These results are the same as those in the previous section.*

Now we will consider the marginal distribution of the process $(Y(t))$ when $(N(t))$ is a renewal process. It is easy to see that

$$Y(t) = \sum_{i=1}^{N(t)} (i-1)X_i + N(t)[t - S_{N(t)}].$$

Define for $t \geq 0$ the functional $\mathbb{Y}(t)$ on $M_p(E)$ by

$$\mathbb{Y}(t)(\mu) = \int_E \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu)) \left\{ \mu([0,r) \times [0,\infty)) u 1_{[0,s)}(r) + \mu([0,s) \times [0,\infty))(t - \mathbb{A}(s,\mu)) \right\} \mu(drdu)\mu(dsdx).$$

Then as in Lemma 3.3.1, with probability 1, $Y(t) = \mathbb{Y}(t)(\Phi)$. The following theorem can be proved using arguments as for $Z(t)$. We omit the proof.

Theorem 3.3.2 *Let $(X_n, n \geq 1)$ be an iid sequence of strictly positive random variables with common distribution function F. Let $(S_n, n \geq 0)$ be the sequence of partial sums of the variables X_n and $(N(t), t \geq 0)$ be the corresponding renewal process: $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Let*

$$Y(t) = \sum_{i=1}^{N(t)} (i-1)X_i + N(t)[t - S_{N(t)}].$$

Then

(a)

$$\int_0^\infty \mathbf{E}(e^{-\alpha Y(t)}) e^{-\beta t} dt = \sum_{n=0}^\infty \frac{1 - F^*(\alpha n + \beta)}{\alpha n + \beta} \prod_{i=1}^n F^*(\alpha[i-1] + \beta),$$

(b)

$$\int_0^\infty \mathbf{E}[Y(t)] e^{-\beta t} dt = \frac{F^*(\beta)}{\beta^2[1 - F^*(\beta)]},$$

(c) If $\mathbf{E}\left[X_1 e^{-\beta X_1}\right] < \infty$ for some $\beta > 0$, then

$$\int_0^\infty \mathbf{E}[Y^2(t)] e^{-\beta t} dt = \frac{2F^*(\beta)[1 - F^*(\beta)^2 + \beta \int_0^\infty t e^{-\beta t} dF(t)]}{\beta^3[1 - F^*(\beta)]^3}.$$

3.4 Asymptotic properties

In this section we will discuss asymptotic properties of $(Y(t))$ and $(Z(t))$ as defined in Section 3.1 for the case that $(N(t))$ is a renewal process having inter-arrival times X_n with common cdf F. We first consider asymptotic properties of the mean of $Z(t)$.

Theorem 3.4.1 *If* $\mu_1 = \mathbf{E}[X_1] < \infty$ *then as* $t \to \infty$

$$\mathbf{E}[Z(t)] \quad \sim \quad \frac{t^2}{2\mu_1}.$$

Proof: In Section 3.3 we have proved that the Laplace transform of $\mathbf{E}[Z(t)]$ is given by

$$\int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt \quad = \quad \frac{\int_0^\infty xe^{-\beta x}dF(x)}{\beta[1-F^*(\beta)]^2}.$$

Note that

$$\int_0^\infty e^{-\beta t}d\mathbf{E}[Z(t)] \quad = \quad \lim_{t\to\infty} \mathbf{E}[Z(t)]e^{-\beta t} + \beta \int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt.$$

Since $0 \le Z(t) \le tN(t)$ where $N(t)$ denotes the renewal process corresponding to the sequence (X_n), it follows that

$$\begin{aligned} 0 \le \lim_{t\to\infty} \mathbf{E}[Z(t)]e^{-\beta t} &\le \lim_{t\to\infty} t\mathbf{E}[N(t)]e^{-\beta t} \\ &= \lim_{t\to\infty} t\left[\frac{t}{\mu_1} + o(1)\right]e^{-\beta t} \\ &= 0. \end{aligned}$$

This implies

$$\begin{aligned} \int_0^\infty e^{-\beta t}d\mathbf{E}[Z(t)] &= \beta \int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt \\ &= \frac{\int_0^\infty xe^{-\beta x}dF(x)}{[1-F^*(\beta)]^2}. \end{aligned}$$

By dominated convergence it is easy to see that

$$\int_0^\infty xe^{-\beta x}dF(x) \quad = \quad \mu_1 + o(1)$$

and

$$F^*(\beta) \quad = \quad 1 - \mu_1\beta + o(\beta)$$

as $\beta \to 0$. Hence

$$\int_0^\infty e^{-\beta t}d\mathbf{E}[Z(t)] \quad \sim \quad \frac{1}{\mu_1\beta^2} \quad \text{as} \quad \beta \to 0.$$

Obviously $\mathbf{E}[Z(t)]$ is non-decreasing. So we can apply Theorem 2.4.1 (Tauberian theorem) with $\gamma = 2$ to get the result. □

Next we will derive a stronger version for the asymptotic form of $\mathbf{E}[Z(t)]$. We will assume that the inter-arrival times X_n are continuous random variables. We also assume that the Laplace transform of $\mathbf{E}[Z(t)]$ given in Theorem 3.3.1 is a rational function, i.e.,

$$\frac{\int_0^\infty xe^{-\beta x}dF(x)}{\beta[1-F^*(\beta)]^2} \tag{3.12}$$

is a rational function of β. This situation holds for example when X_1 has a gamma distribution.

Since the Laplace transform of $\mathbf{E}[Z(t)]$ is a rational function, we can split (3.12) into partial fractions. To do this, firstly observe that $F^*(0) = 1$ and $F^{*\prime}(0) = -\mu_1 < 0$. So we conclude that the equation

$$1 - F^*(\beta) = 0 \tag{3.13}$$

has a simple root at $\beta = 0$. Hence the partial fraction expansion of (3.12) contains terms proportional to $1/\beta^2$ and $1/\beta$. For now, we will consider β as a complex variable and denote its real part by $\Re(\beta)$. Assuming $\mu_i = \mathbf{E}(X_1^i)$, $i = 1, 2, 3$ are finite we can express the Laplace transform of $\mathbf{E}[Z(t)]$ as

$$\int_0^\infty \mathbf{E}[Z(t)]e^{-\beta t}dt \;=\; \frac{1}{\mu_1\beta^3} + \frac{1}{\mu_1^2}\left(\frac{\mu_3}{6} + \frac{\mu_2^2}{4\mu_1}\right)\frac{1}{\beta} + r(\beta). \tag{3.14}$$

where $r(\beta)$ is a rational function of β with non-zero poles at $\beta_1, \beta_2, ...$ It follows from (3.13) that for every j, $\Re(\beta_j) < 0$. Also, since F is continuous, then there can be no purely imaginary roots of (3.13). There are some other remarks about the roots β_j. If β_j is a simple real root then it gives a term proportional to $1/(\beta-\beta_j)$ in the partial fraction of $r(\beta)$, inverting into $e^{\beta_j t}$. If β_j is a multiple root then it leads to a term proportional to $t^r e^{\beta_j t}$. Since $\Re(\beta_j) < 0$ these terms tend to zero exponentially fast as $t \to \infty$. Note also that the roots β_j must occur in conjugate pairs. Otherwise $\mathbf{E}[Z(t)]$ contains a term which is a complex number. So from (3.14) we obtain

$$\mathbf{E}[Z(t)] \;=\; \frac{1}{2\mu_1}t^2 + \frac{1}{\mu_1^2}\left(\frac{\mu_3}{6} + \frac{\mu_2^2}{4\mu_1}\right) + o(1) \quad \text{as} \quad t \to \infty. \tag{3.15}$$

In (3.15) the term $o(1)$ tends to 0 as $t \to \infty$ exponentially fast.

Similar arguments as before can be used to obtain the asymptotic variance of $Z(t)$. Assume that $\mu_4 = \mathbf{E}[X_1^4] < \infty$. If the Laplace transform of $\mathbf{E}[Z^2(t)]$ in (3.11) is a rational function of β, then we can prove that

$$\int_0^\infty \mathbf{E}[Z^2(t)]e^{-\beta t}dt \;=\; \frac{6}{\mu_1^2\beta^5} + \frac{2(\mu_2 - \mu_1^2)}{\mu_1^3\beta^4} + \frac{C}{\beta} + r(\beta)$$

where C is a constant depending on $\mu_i, i = 1,2,3,4$ and $r(\beta)$ is a rational function of β having non-zero poles. Inverting this transform we obtain

$$\mathbf{E}[Z^2(t)] = \frac{1}{4\mu_1^2}t^4 + \frac{\mu_2 - \mu_1^2}{3\mu_1^3}t^3 + C + o(1) \quad \text{as} \quad t \to \infty. \tag{3.16}$$

It follows, from (3.15) and (3.16), that

$$\frac{\mathrm{Var}[Z(t)]}{t^3} \to \frac{\mu_2 - \mu_1^2}{3\mu_1^3} \quad \text{as} \quad t \to \infty.$$

For the process $(Y(t))$ we have the following asymptotic properties. The proof is quite similar to that of $(Z(t))$, and is therefore omitted.

Proposition 3.4.1 *Let $N(t)$ be a renewal process with inter-arrival times X_n. Let $\mu_i = \mathbf{E}[X_1^i]$. Let $Y(t) = \int_0^t N(s)ds$ be the corresponding integrated renewal process. Then*

(a) If $\mu_1 < \infty$ then

$$\mathbf{E}[Y(t)] \sim \frac{t^2}{2\mu_1} \quad \textit{as} \quad t \to \infty.$$

(b) If μ_i, $i = 1,2,3$ are finite and if the Laplace transform of $\mathbf{E}[Y(t)]$ stated in Theorem 3.3.2 is a rational function of β, then

$$\mathbf{E}[Y(t)] = \frac{1}{2\mu_1}t^2 + \left(\frac{\mu_2}{2\mu_1^2} - 1\right)t + \left(\frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2}\right) + o(1) \quad \textit{as} \quad t \to \infty.$$

(c) If μ_i, $i = 1,...,5$ are finite and if the Laplace transform of $\mathbf{E}[Y^2(t)]$ stated in Theorem 3.3.2 is a rational function of β, then

$$\begin{aligned}\mathbf{E}[Y^2(t)] = {} & \frac{1}{4\mu_1^2}t^4 + \left(\frac{5\mu_2 - 2\mu_1^2}{6\mu_1^3}\right)t^3 + \frac{1}{\mu_1^2}\left(1 + \frac{3\mu_2^2}{2\mu_1^2} - \frac{21\mu_2}{2} - \frac{2\mu_3}{3\mu_1}\right)t^2 \\ & + \frac{1}{\mu_1^2}\left(\frac{2\mu_3}{3} - \frac{\mu_2^2}{\mu_1} + \frac{\mu_4}{4\mu_1} - \frac{3\mu_2\mu_3}{2\mu_1^2} + \frac{3\mu_2^3}{2\mu_1^5}\right)t \\ & + C + o(1) \quad \textit{as} \quad t \to \infty,\end{aligned}$$

where C is a constant depending on $\mu_i, i = 1,2,3,4,5$.

(d) Under the assumptions as in (c),

$$\frac{\mathrm{Var}[Y(t)]}{t^3} \to \frac{\mu_2 - \mu_1^2}{3\mu_1^3} \quad \textit{as} \quad t \to \infty.$$

3.5 An application

Suppose that travellers arrive at a train depot according to a renewal process. Suppose that a train just departed at time 0, and there were no travellers left. If the next train departs at some time $t \geq 0$ then the sum of the waiting times of all the travellers arriving in the time interval $[0, t]$, i.e., $\sum_{i=1}^{N(t)}(t - S_i)$, is an integrated renewal process. In Ross [36] Example 2.3(A) the special case of this process where the arrival process of the travellers is a homogeneous Poisson process with rate λ has been considered. He showed that the expected sum of the waiting times of the travellers arriving in the time interval $[0, t]$ is equal to $1/2\lambda t^2$. The calculation is based on conditioning on the number of arrivals in the interval $[0, t]$. Using the results in the preceding sections in this example, we give more information about the process.

Let $Y(t) = \sum_{i=1}^{N(t)}(t - S_i)$. As stated in Section 3.2 the variance of $Y(t)$ is equal to $1/3\lambda t^3$, the distribution of $Y(t)$ has mass at 0 with $\mathbf{P}(Y(t) = 0) = e^{-\lambda t}$ and the continuous part of $Y(t)$ has a density function, for $x > 0$,

$$\begin{aligned} f_{Y(t)}(x) & \\ &= \frac{\sqrt{\lambda}}{\sqrt{x}} e^{-\lambda t} I_1(2\sqrt{\lambda x}) 1_{(0,\infty)}(x) \\ &+ \lambda e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (\lambda(x - kt))^{\frac{1}{2}(k-1)} I_{k-1}(2\sqrt{\lambda(x - kt)}) 1_{(kt,\infty)}(x) \end{aligned}$$

where $I_k(x)$ is the Modified Bessel function of the first kind. The graphs of the mean and the variance of $Y(t)$ for $\lambda = 0.5$ can be seen in Figure 3.3 and 3.4 (solid line). The graphs of the density function of $Y(t)$ for $\lambda = 0.5$, $t = 3$ and $t = 10$ can be seen in Figure 3.5 and 3.6 (solid line).

Now suppose that the inter-arrival times X_n have a common Gamma(γ,2) distribution having a density function

$$f(x; \gamma, 2) = \gamma^2 x e^{-\gamma x}, \quad \gamma > 0, \quad x \geq 0.$$

Note that if $\gamma = 2\lambda$ then these inter-arrival times have the same mean as the exponential random variable with parameter λ. Using Theorem 3.3.2 we obtain

$$\int_0^\infty \mathbf{E}[Y(t)] e^{-\beta t} dt = \frac{\gamma^2}{\beta^3(\beta + 2\gamma)},$$

and

$$\int_0^\infty \mathbf{E}[Y^2(t)] e^{-\beta t} dt = \frac{2\gamma^2(\beta^3 + 4\gamma\beta^2 + 8\gamma^2\beta + 6\gamma^3)}{\beta^5(\beta + 2\gamma)^3}.$$

Inverting these transforms we obtain

$$\mathbf{E}[Y(t)] = \frac{\gamma}{4} t^2 - \frac{1}{4} t + \frac{1}{8\gamma} - \frac{1}{8\gamma} e^{-2\gamma t}$$

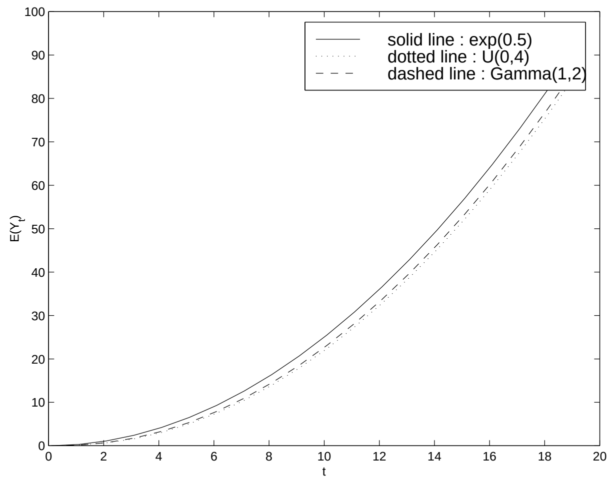


Figure 3.3: Graphs of the mean of $Y(t)$ when the (X_n) iid exp(0.5), uniform(0,4) and Gamma(1,2).

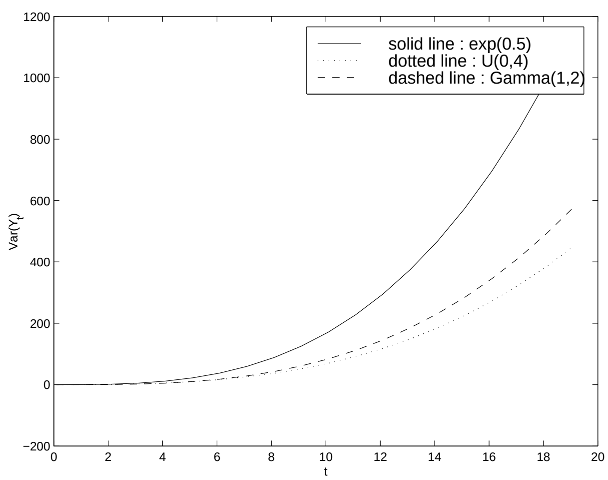


Figure 3.4: Graphs of the variance of $Y(t)$ when the (X_n) iid exp(0.5), uniform(0,4) and Gamma(1,2).

and

$$\mathbf{E}[Y^2(t)] = \frac{\gamma^2}{16}t^4 - \frac{\gamma}{24}t^3 + \frac{1}{8}t^2 - \frac{1}{8\gamma}t + \frac{1}{32\gamma^2} + (\frac{1}{16}t^2 - \frac{1}{16\gamma}t - \frac{1}{32\gamma^2})e^{-2\gamma t}.$$

Hence the variance of $Y(t)$ is given by

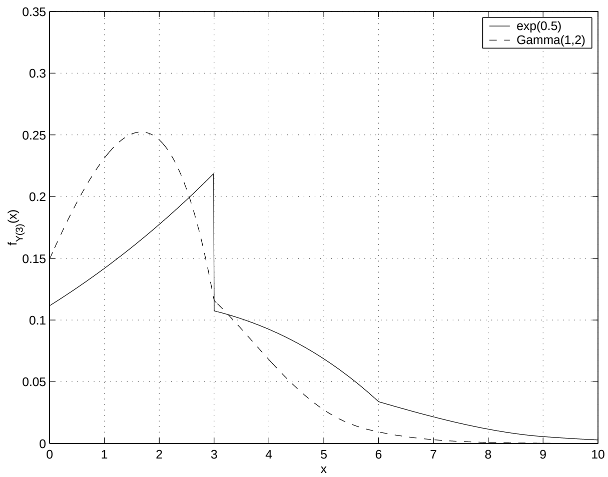


Figure 3.5: Graphs of the probability density function of $Y(3)$ when the (X_n) are iid exp(0.5) and Gamma(1,2).

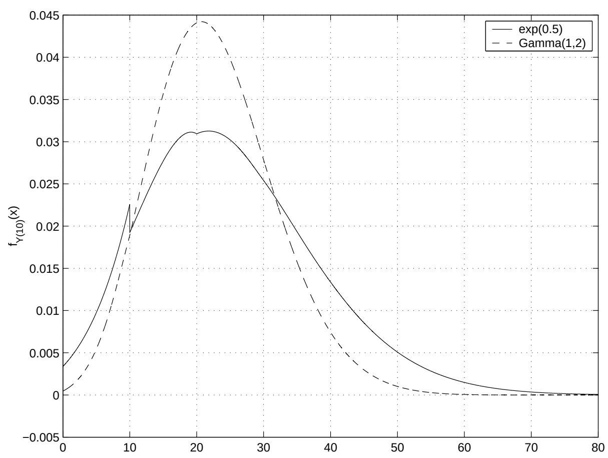


Figure 3.6: Graphs of the probability density function of $Y(10)$ when the (X_n) are iid exp(0.5) and Gamma(1,2).

$$\mathrm{Var}[Y(t)] \quad = \quad \frac{\gamma}{12}t^3 - \frac{1}{6\gamma}t - \frac{1}{64\gamma^2} + \frac{1}{8}te^{-2\gamma t} - \frac{1}{64\gamma^2}e^{-4\gamma t}.$$

The graphs of the mean and the variance of $Y(t)$ for $\gamma = 1$ (hence X_1 has the same mean as the exponential random variable with parameter 0.5) can be seen in Figure 3.3 and 3.4 (dashed line).

The double Laplace transform of $Y(t)$ when the inter-arrival times X_n have a common Gamma$(\gamma, 2)$ distribution is given by

$$\int_0^\infty \mathbf{E}(e^{-\alpha Y(t)})e^{-\beta t}dt$$
$$= \gamma^2 \sum_{n=0}^{\infty} \frac{(\alpha n+\beta+\gamma)^2-\gamma^2}{(\alpha n+\beta+\gamma)^2(\alpha n+\beta)} \prod_{i=1}^{n} \frac{1}{[\alpha(i-1)+\beta+\gamma]^2}.$$

The pdf of $Y(t)$ can be approximated by first truncating the infinite sum in this transform and then by inverting the truncated transform. The graphs of the pdfs of $Y(3)$ and $Y(10)$ for $\gamma = 1$ can be seen in Figure 3.5 and 3.6 (dashed line).

If we assume that the inter-arrival times X_n, are independent and uniformly distributed on $[0, 4]$ (hence X_1 has the same mean as an exponential random variable with parameter 0.5), then the Laplace transforms of the first and second moments of $Y(t)$ are given by

$$\int_0^\infty \mathbf{E}[Y(t)]e^{-\beta t}dt = \frac{1-e^{-4\beta}}{\beta^2[4\beta-1+e^{-4\beta}]},$$

and

$$\int_0^\infty \mathbf{E}[Y^2(t)]e^{-\beta t}dt = \frac{[1-e^{-4\beta}][4\beta^2+\beta-\frac{1}{4}-(4\beta^2+\beta-\frac{1}{2})e^{-4\beta}-\frac{1}{4}e^{-8\beta}]}{\beta^3[2\beta-\frac{1}{2}(1-e^{-4\beta})]^3}.$$

Using numerical inversions of Laplace transforms in Appendix B we get the graphs of $\mathbf{E}[Y(t)]$ and Var$[Y(t)]$, see Figure 3.3 and 3.4 (dotted line). The marginal pdf of $Y(t)$ is more complicated to obtain in this case.

Chapter 4

Total Downtime of Repairable Systems

4.1 Introduction

Consider a repairable system which is at any time either in operation (up) or under repair (down) after failure. The effectiveness of the system can be measured by the total downtime, i.e., the total amount of time the system is down during a given time interval. An expression for the cumulative distribution function (cdf) of the total downtime in the time interval $[0, t]$ has been derived by several authors using different methods. In Takács [44] the total probability theorem has been used. The derivation in Muth [26] is based on consideration of the excess time. Finally in Funaki and Yoshimoto [13] the cdf of the total downtime is derived by a conditioning technique. Srinivasan *et al.* [37] derived an expression for the probability density function (pdf) of the total uptime of the system in the time interval $[0, t]$. They also discussed its covariance structure. For longer time intervals, Takács [44] and Rényi [33] proved that the distribution of the total downtime approaches a normal distribution. Takács [44] also discussed the asymptotic mean and variance of the total downtime. In all these papers it is assumed that *the failure time and the repair time are independent*.

We use a different method for computation of the distribution of the total downtime. We also consider a more general situation where we allow *dependence* of the failure time and the repair time. Our derivation is based on a representation of the total downtime as a functional of a Poisson point process.

This chapter is organized as follows. In Section 4.2 we define the total downtime and derive its distribution in a fixed time interval. In Section 4.3 we discuss the system availability which is closely related to the total downtime.

In Section 4.4 we study the covariance structure of the total downtime for a dependent case. Asymptotic properties of the total downtime for a dependent case are derived in Section 4.5. We give examples in Section 4.6 and finally in Section 4.7 we consider repairable systems consisting of $n \geq 2$ independent components.

4.2 Distribution of total downtime

We consider a repairable system which is at any time either in operation (up) or under repair (down) after failure, denoted as 1 and 0 respectively. Suppose that the system starts to operate at time $t = 0$. Let (X_i) and (Y_i), $i \geq 1$, denote the time spent in the state 1 and 0 respectively during the ith visit to that state. The random variables X_i and Y_i are known as the failure time and the repair time respectively. We assume that the sequence (X_i, Y_i) of random vectors is iid with strictly positive components. However, our set up is more general than that in Takács [44], Muth [26], Funaki and Yoshimoto [13], Rényi [33], and Srinivasan *et al.* [37], as we allow that X_i and Y_i are dependent.

Let $S_n = \sum_{i=1}^{n}(X_i + Y_i), n \geq 1$, $S_0 = 0$, and $N(t) = \sup\{n \geq 0 : S_n \leq t\}$. Then the total downtime $D(t)$ can be expressed (with the usual convention that the empty sum equals 0) as

$$D(t) \quad = \quad \begin{cases} \sum_{i=1}^{N(t)} Y_i, & \text{if } S_{N(t)} \leq t < S_{N(t)} + X_{N(t)+1} \\ t - \sum_{i=1}^{N(t)+1} X_i, & \text{if } S_{N(t)} + X_{N(t)+1} \leq t < S_{N(t)+1}. \end{cases} \tag{4.1}$$

Denote the state of the system at time t by $Z(t)$. Then the total downtime $D(t)$ can also be expressed as

$$D(t) = \int_0^t 1_{\{0\}}(Z(s))ds. \tag{4.2}$$

We will assume that $Z(t)$ is right continuous.

Throughout this chapter we will use the following notation for cdfs:

$$F(x) = \mathbf{P}(X_1 \leq x),$$

$$G(y) = \mathbf{P}(Y_1 \leq y),$$

$$H(x, y) = \mathbf{P}(X_1 \leq x, Y_1 \leq y),$$

$$K(w) = \mathbf{P}(X_1 + Y_1 \leq w).$$

We denote by F_n and G_n the cdfs of $\sum_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} Y_i$, respectively. The Laplace-Stieltjes transforms of a cdf F and a joint cdf H will be denoted by F^* and H^*, i.e.,

$$F^*(\beta) = \int_0^{\infty} e^{-\beta x} F(x)dx$$

and

$$H^*(\alpha, \beta) = \int_0^\infty \int_0^\infty e^{-(\alpha x + \beta y)} dH(x, y).$$

We will use point processes for the derivation of the distribution of the total downtime $D(t)$. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability space on which the iid sequence (X_i, Y_i) is defined and also an iid sequence $(U_i, i \geq 1)$ of exponentially distributed random variables with parameter 1 such that the sequences (U_i) and (X_i, Y_i) are independent. Let $(T_n, n \geq 1)$ be the sequence of partial sums of the variables U_i. Then the map

$$\Phi \quad : \quad \omega \quad \mapsto \quad \textstyle\sum_{n=1}^{\infty} \delta_{(T_n(\omega), X_n(\omega), Y_n(\omega))},$$

where $\delta_{(x,y,z)}$ is the Dirac measure in (x, y, z), defines a Poisson point process on $E = [0, \infty) \times [0, \infty) \times [0, \infty)$ with intensity measure $\nu(dtdxdy) = dtdH(x, y)$. Note that for almost all $\omega \in \Omega$, $\Phi(\omega)$ is a simple point measure on E such that there is at most one point from the support of $\Phi(\omega)$ on each fibre $\{t\} \times [0, \infty)$ and $\Phi(\omega)([0, t] \times [0, \infty)) < \infty$ for every $t \geq 0$. Let $M_p(E)$ be the set of all point measures on E. We will denote by $\mathbf{P}_\nu$ the distribution of Φ over $M_p(E)$.

Define on $M_p(E)$, for $t \geq 0$, the functionals

$$\mathbb{A}_X(t)(\mu) = \int_E x 1_{[0,t)}(s) \mu(dsdxdy),$$

$$\mathbb{A}_Y(t)(\mu) = \int_E y 1_{[0,t)}(s) \mu(dsdxdy),$$

and

$$\mathbb{A}(t)(\mu) = \mathbb{A}_X(t)(\mu) + \mathbb{A}_Y(t)(\mu).$$

So for example $\mathbb{A}(t)(\mu)$ is the sum of the x- and the y-coordinates of the points in the support of μ up to time t. In the sequel it is convenient to write an expression like $\mathbb{A}_X(t)(\mu)$ as $\mathbb{A}_X(t, \mu)$. Define also for $t \geq 0$

$$\begin{aligned} \mathbb{D}(t)(\mu) \quad = \quad & \int_E \Big\{ 1_{[0,x)}(t - \mathbb{A}(s, \mu)) \mathbb{A}_Y(s, \mu) \\ & \quad + 1_{[x,x+y)}(t - \mathbb{A}(s, \mu))[t - \mathbb{A}_X(s+, \mu)] \Big\} \mu(dsdxdy). \end{aligned}$$

The next lemma motivates the definition of $\mathbb{D}(t)$.

Lemma 4.2.1 *With probability 1,*

$$D(t) = \mathbb{D}(t)(\Phi).$$

Proof: Let $\omega \in \Omega$. Then

$$\begin{aligned}
&\mathbb{D}(t)(\Phi(\omega)) \\
&= \sum_{i=1}^{\infty}\Big[1_{[0,X_i(\omega))}(t-\mathbb{A}(T_i(\omega),\Phi(\omega)))\mathbb{A}_Y(T_i(\omega),\Phi(\omega)) \\
&+ 1_{[X_i(\omega),X_i(\omega)+Y_i(\omega))}(t-\mathbb{A}(T_i(\omega),\Phi(\omega)))[t-\mathbb{A}_X(T_i(\omega)+,\Phi(\omega))]\Big].
\end{aligned}$$

Note that $1_{[0,X_i(\omega))}(t-\mathbb{A}(T_i(\omega),\Phi(\omega))) = 1$ if and only if $S_{N(t,\omega)}(\omega) \le t < S_{N(t,\omega)}(\omega) + X_{N(t,\omega)+1}(\omega)$, and this last statement implies that $i = N(t,\omega)+1$. The same conclusion is true when $1_{[X_i(\omega),X_i(\omega)+Y_i(\omega))}(t-\mathbb{A}(T_i(\omega),\Phi(\omega))) = 1$. Since the intervals $\{[S_{i-1}, S_{i-1}+X_i), [S_{i-1}+X_i, S_{i-1}+X_i+Y_i) : i \ge 1\}$ partition $[0,\infty)$, for any $t>0$ one and only one of the indicators in the sum will be non-zero. So if $1_{[0,X_i(\omega))}(t-\mathbb{A}(T_i(\omega),\Phi(\omega))) = 1$ then $i = N(t,\omega)+1$ and

$$\begin{aligned}
\mathbb{D}(t)(\Phi(\omega)) &= \mathbb{A}_Y(T_{N(t,\omega)+1}(\omega),\Phi(\omega)) \\
&= \begin{cases} 0, & \text{if } N(t,\omega)=0 \\ \sum_{j=1}^{N(t,\omega)} Y_j(\omega), & \text{if } N(t,\omega)\ge 1, \end{cases}
\end{aligned}$$

and if $1_{[X_i(\omega),X_i(\omega)+Y_i(\omega))}(t-\mathbb{A}(T_i(\omega),\Phi(\omega))) = 1$ then

$$\begin{aligned}
\mathbb{D}(t)(\Phi(\omega)) &= t-\mathbb{A}_X(T_{N(t,\omega)+1}(\omega)+,\Phi(\omega)) \\
&= t-\sum_{j=1}^{N(t,\omega)+1} X_j(\omega). \qquad \square
\end{aligned}$$

The following theorem gives the distribution of the total downtime $D(t)$ in the form of a double Laplace transform.

Theorem 4.2.1 *Let $D(t)$ be as defined in* (4.1). *Then for $\alpha, \beta > 0$*

$$\int_0^{\infty} \mathbf{E}[e^{-\alpha D(t)}]e^{-\beta t}dt = \frac{\alpha[1-F^*(\beta)]+\beta[1-H^*(\beta,\alpha+\beta)]}{\beta(\alpha+\beta)[1-H^*(\beta,\alpha+\beta)]}. \quad (4.3)$$

Proof: By Lemma 4.2.1

$$
\begin{aligned}
&\mathbf{E}(e^{-\alpha D(t)})\\
&= \int_{M_p(E)} e^{-\alpha \mathbb{D}(t)(\mu)} \mathbf{P}_\nu(d\mu)\\
&= \int_{M_p(E)} \exp\Big\{ -\alpha \int_E \Big[1_{[0,x)}(t-\mathbb{A}(s,\mu))\mathbb{A}_Y(s,\mu)\\
&\quad +1_{[x,x+y)}(t-\mathbb{A}(s,\mu))(t-\mathbb{A}_X(s+,\mu))\Big]\mu(dsdxdy)\Big\}\mathbf{P}_\nu(d\mu)\\
&= \int_{M_p(E)}\int_E \Big[1_{[0,x)}(t-\mathbb{A}(s,\mu))e^{-\alpha\mathbb{A}_Y(s,\mu)}\\
&\quad +1_{[x,x+y)}(t-\mathbb{A}(s,\mu))e^{-\alpha(t-\mathbb{A}_X(s+,\mu))}\Big]\mu(dsdxdy)\mathbf{P}_\nu(d\mu)\\
&=: C_1(\alpha,t)+C_2(\alpha,t).
\end{aligned}
$$

Applying the Palm formula for Poisson point processes, see Theorem 1.2.4, we obtain

$$
\begin{aligned}
C_1(\alpha,t) &:= \int_{M_p(E)}\int_E 1_{[0,x)}(t-\mathbb{A}(s,\mu))e^{-\alpha\mathbb{A}_Y(s,\mu)}\mu(dsdxdy)\mathbf{P}_\nu(d\mu)\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)} 1_{[0,x)}(t-\mathbb{A}(s,\mu+\delta_{(s,x,y)}))\\
&\quad \exp\Big\{-\alpha\mathbb{A}_Y(s,\mu+\delta_{(s,x,y)})\Big\}\mathbf{P}_\nu(d\mu)dH(x,y)ds\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)} 1_{[0,x)}(t-\mathbb{A}(s,\mu))e^{-\alpha\mathbb{A}_Y(s,\mu)}\\
&\quad \mathbf{P}_\nu(d\mu)dH(x,y)ds
\end{aligned}
$$

and

$$
\begin{aligned}
C_2(\alpha,t) &:= \int_{M_p(E)}\int_E 1_{[x,x+y)}(t-\mathbb{A}(s,\mu))e^{-\alpha(t-\mathbb{A}_X(s+,\mu))}\mu(dsdxdy)\mathbf{P}_\nu(d\mu)\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)} 1_{[x,x+y)}(t-\mathbb{A}(s,\mu+\delta_{(s,x,y)}))\\
&\quad \exp\Big\{-\alpha\Big[t-\mathbb{A}_X(s+,\mu+\delta_{(s,x,y)})\Big]\Big\}\mathbf{P}_\nu(d\mu)dH(x,y)ds\\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)} 1_{[x,x+y)}(t-\mathbb{A}(s,\mu))\\
&\quad \exp\Big\{-\alpha\Big[t-\mathbb{A}_X(s+,\mu)-x\Big]\Big\}\mathbf{P}_\nu(d\mu)dH(x,y)ds.
\end{aligned}
$$

Using Fubini's theorem and a substitution we obtain

$$\begin{aligned}
&\int_0^\infty C_1(\alpha,t)e^{-\beta t}dt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)}\Big[\int_0^x e^{-\beta t}dt\Big] \\
&\quad \exp\Big\{-\Big[\alpha\mathbb{A}_Y(s,\mu)+\beta\mathbb{A}(s,\mu)\Big]\Big\}\mathbf{P}_\nu(d\mu)dH(x,y)ds \\
&= \frac{1}{\beta}[1-F^*(\beta)]\int_0^\infty\int_{M_p(E)}\exp\Big\{-\Big[\alpha\mathbb{A}_Y(s,\mu)+\beta\mathbb{A}(s,\mu)\Big]\Big\}\mathbf{P}_\nu(d\mu)ds.
\end{aligned}$$

Note that

$$\alpha\mathbb{A}_Y(s,\mu)+\beta\mathbb{A}(s,\mu)=\int_E 1_{[0,s)}(\tilde{s})(\alpha\tilde{y}+\beta(\tilde{x}+\tilde{y}))\mu(d\tilde{s}d\tilde{x}d\tilde{y}).$$

So we can use the formula for the Laplace functional of Poisson point processes, see Theorem 1.2.1, to obtain

$$\begin{aligned}
&\int_{M_p(E)}\exp\Big\{-\Big[\alpha\mathbb{A}_Y(s,\mu)+\beta\mathbb{A}(s,\mu)\Big]\Big\}\mathbf{P}_\nu(d\mu) \\
&= \exp\Big\{-\int_0^\infty\int_0^\infty\int_0^\infty\Big[1-e^{-1_{[0,s)}(\tilde{s})(\alpha\tilde{y}+\beta(\tilde{x}+\tilde{y}))}\Big]dH(\tilde{x},\tilde{y})d\tilde{s}\Big\} \\
&= \exp\Big\{-s\Big[1-H^*(\beta,\alpha+\beta)\Big]\Big\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_0^\infty C_1(\alpha,t)e^{-\beta t}dt &= \frac{1}{\beta}[1-F^*(\beta)]\int_0^\infty\exp\Big\{-s\Big[1-H^*(\beta,\alpha+\beta)\Big]\Big\}ds \\
&= \frac{1-F^*(\beta)}{\beta[1-H^*(\beta,\alpha+\beta)]}. \qquad (4.4)
\end{aligned}$$

Similarly we calculate the Laplace transform of $C_2(\alpha, t)$ as follows:

$$\begin{aligned}
&\int_0^\infty C_2(\alpha,t)e^{-\beta t}dt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)}\int_0^\infty 1_{[x,x+y)}(t-\mathbb{A}(s,\mu)) \\
&\quad \exp\Big\{-\alpha\Big[t-\mathbb{A}_X(s+,\mu)-x\Big]\Big\}e^{-\beta t}dt\mathbf{P}_\nu(d\mu)dH(x,y)ds \\
&= \int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)}\Big[\int_x^{x+y}e^{-(\alpha+\beta)t}dt\Big] \\
&\quad \exp\Big\{-(\alpha+\beta)\mathbb{A}(s,\mu)+\alpha[\mathbb{A}_X(s+,\mu)+x]\Big\}\mathbf{P}_\nu(d\mu)dH(x,y)ds \\
&= \frac{1}{\alpha+\beta}\int_0^\infty\int_0^\infty\int_0^\infty\int_{M_p(E)}\Big[e^{-(\alpha+\beta)x}-e^{-(\alpha+\beta)(x+y)}\Big]e^{\alpha x} \\
&\quad \exp\Big\{-(\alpha+\beta)\mathbb{A}(s,\mu)+\alpha\mathbb{A}_X(s+,\mu)\Big\}\mathbf{P}_\nu(d\mu)dH(x,y)ds \\
&= \frac{1}{\alpha+\beta}\Big[\int_0^\infty\int_0^\infty\Big[e^{-\beta x}-e^{-(\beta x+(\alpha+\beta)y)}\Big]dH(x,y)\Big] \\
&\quad \int_0^\infty\int_{M_p(E)}\exp\Big\{-(\alpha+\beta)\mathbb{A}(s,\mu)+\alpha\mathbb{A}_X(s+,\mu)\Big\}\mathbf{P}_\nu(d\mu)ds \\
&= \frac{1}{\alpha+\beta}[F^*(\beta)-H^*(\beta,\alpha+\beta)] \\
&\quad \int_0^\infty\int_{M_p(E)}\exp\Big\{-(\alpha+\beta)\mathbb{A}(s,\mu)+\alpha\mathbb{A}_X(s+,\mu)\Big\}\mathbf{P}_\nu(d\mu)ds.
\end{aligned}$$

The integral with respect to $\mathbf{P}_\nu$ can be calculated using the Palm formula for Poisson point process as follows:

$$\begin{aligned}
&\int_{M_p(E)}\exp\Big\{-(\alpha+\beta)\mathbb{A}(s,\mu)+\alpha\mathbb{A}_X(s+,\mu)\Big\}\mathbf{P}_\nu(d\mu) \\
&= \int_{M_p(E)}\exp\Big\{-\int_E\Big[1_{[0,s)}(\tilde{s})(\alpha+\beta)(\tilde{x}+\tilde{y})-1_{[0,s]}(\tilde{s})\alpha\tilde{x}\Big]\mu(d\tilde{s}d\tilde{x}d\tilde{y})\Big\}\mathbf{P}_\nu(d\mu) \\
&= \exp\Big\{-\int_0^\infty\int_0^\infty\int_0^\infty\Big[1-e^{-[1_{[0,s)}(\tilde{s})(\alpha+\beta)(\tilde{x}+\tilde{y})-1_{[0,s]}(\tilde{s})\alpha\tilde{x}]}\Big]dH(\tilde{x},\tilde{y})d\tilde{s}\Big\} \\
&= \exp\Big\{-s\int_0^\infty\int_0^\infty\Big[1-e^{-[\beta\tilde{x}+(\alpha+\beta)\tilde{y}]}\Big]dH(\tilde{x},\tilde{y})\Big\} \\
&= \exp\Big\{-s\Big[1-H^*(\beta,\alpha+\beta)\Big]\Big\}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^\infty C_2(\alpha,t)e^{-\beta t}dt \\
= \quad & \frac{1}{\alpha+\beta}[F^*(\beta)-H^*(\beta,\alpha+\beta)]\int_0^\infty \exp\Big\{-s\Big[1-H^*(\beta,\alpha+\beta)\Big]\Big\}ds \\
= \quad & \frac{F^*(\beta)-H^*(\beta,\alpha+\beta)}{(\alpha+\beta)[1-H^*(\beta,\alpha+\beta)]}. \qquad (4.5)
\end{aligned}$$

Summing (4.4) and (4.5) we get the result. □

Taking derivatives with respect to α in (4.3) and setting $\alpha = 0$ we get the Laplace transforms of $\mathbf{E}[D(t)]$ and $\mathbf{E}[D^2(t)]$ as stated in the following proposition:

Proposition 4.2.1 *For $\beta > 0$,*

(a)

$$\int_0^\infty \mathbf{E}[D(t)]e^{-\beta t}dt \quad = \quad \frac{F^*(\beta)-H^*(\beta,\beta)}{\beta^2[1-H^*(\beta,\beta)]}, \qquad (4.6)$$

(b)

$$\begin{aligned}
\int_0^\infty \mathbf{E}[D^2(t)]e^{-\beta t}dt \quad = \quad & \frac{2}{\beta^3}\left[\frac{F^*(\beta)-H^*(\beta,\beta)}{1-H^*(\beta,\beta)}\right. \\
- \quad & \left.\frac{\beta[1-F^*(\beta)]\int_0^\infty\int_0^\infty ye^{-\beta(x+y)}dH(x,y)}{[1-H^*(\beta,\beta)]^2}\right] \qquad (4.7)
\end{aligned}$$

Remark 4.2.1 *For the case that (X_i) and (Y_j) are independent (4.3) simplifies to*

$$\int_0^\infty \mathbf{E}(e^{-\alpha D(t)})e^{-\beta t}dt = \frac{\alpha[1-F^*(\beta)]+\beta[1-F^*(\beta)G^*(\alpha+\beta)]}{\beta(\alpha+\beta)[1-F^*(\beta)G^*(\alpha+\beta)]}. \qquad (4.8)$$

Takács [44], Muth [26], Funaki and Yoshimoto [13] derived for the independent case the following formula for the distribution function of the total downtime:

$$\mathbf{P}(D(t)\le x) = \begin{cases} \sum_{n=0}^\infty G_n(x)[F_n(t-x)-F_{n+1}(t-x)], & t > x \\ 1, & t \le x. \end{cases} \qquad (4.9)$$

Taking double Laplace transforms on both sides of (4.9) we obtain (4.8).

4.3 System availability

This section concerns the system availability of repairable systems, which is closely related to the total downtime. The system availability $A_{11}(t)$ at time t is defined as the probability that the system is working at time t, i.e.,

$$A_{11}(t) = \mathbf{P}(Z(t) = 1).$$

The relationship between the system availability $A_{11}(t)$ and the total downtime $D(t)$ is given by the following equation:

$$\mathbf{E}[D(t)] = t - \int_0^t A_{11}(s)ds, \tag{4.10}$$

which can easily verified using (4.2).

In Pham-Gia and Turkkan [30] the system availability of a repairable system where both uptime and downtime are gamma distributed has been considered. They calculate the system availability by computing numerically the renewal density of a renewal process with inter-arrival times the sum of two gamma random variables, and then using the following integral equation:

$$A_{11}(t) = \bar{F}(t) + \int_0^t \bar{F}(t-u)dm(u), \tag{4.11}$$

where $m(t) = \mathbf{E}[N(t)]$. This equation can be found for example in Barlow [2].

In general an expression for the system availability can be derived using our result about the expected value of the total downtime given in (4.6), and using (4.10). Taking Laplace transforms on both sides of (4.10) we obtain

$$\int_0^\infty \mathbf{E}[D(t)]e^{-\beta t}dt = \frac{1}{\beta^2} - \frac{1}{\beta}\int_0^\infty A_{11}(t)e^{-\beta t}dt.$$

Taking (4.6) into consideration we obtain

$$\int_0^\infty A_{11}(t)e^{-\beta t}dt = \frac{1 - F^*(\beta)}{\beta[1 - H^*(\beta, \beta)]}. \tag{4.12}$$

In particular, if (X_i) and (Y_i) are independent then

$$\int_0^\infty A_{11}(t)e^{-\beta t}dt = \frac{1 - F^*(\beta)}{\beta[1 - F^*(\beta)G^*(\beta)]}. \tag{4.13}$$

Remark 4.3.1 *The Laplace transform of $A_{11}(t)$ can also be derived from* (4.11)*. Taking Laplace transform on both sides of this equation we obtain*

$$\int_0^\infty A_{11}(t)e^{-\beta t}dt = \frac{1}{\beta}[1 - F^*(\beta)][1 + m^*(\beta)] \tag{4.14}$$

where m^* is the Laplace-Stieltjes transform of $m(t)$. But it is well known that

$$m^*(\beta) \quad = \quad \frac{K^*(\beta)}{1-K^*(\beta)},$$

where K is the cdf of X_1+Y_1. Substituting this equation into (4.14) and using the fact that $K^*(\beta)=H^*(\beta,\beta)$ we get (4.12).

Example 4.3.1 *Let $(X_i, i \geq 1)$ be an iid sequence of non-negative random variables having a common Gamma(λ, m) distribution with a pdf*

$$f(x;\lambda,m) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(m)}, \quad x \geq 0.$$

Let $(Y_i, i \geq 1)$ be an iid sequence of non-negative random variables having a common Gamma(μ, n) distribution. Assume that (X_i) and (Y_i) are independent. Then using (4.13) we obtain

$$\int_0^\infty A_{11}(t)e^{-\beta t}dt \quad = \quad \frac{(\mu+\beta)^n[(\lambda+\beta)^m-\lambda^m]}{\beta[(\lambda+\beta)^m(\mu+\beta)^n-\lambda^m\mu^n]}. \tag{4.15}$$

The system availability $A_{11}(t)$ can be obtained by inverting this transform.

As an example let $m=n=1$. Then X_1 and Y_1 are exponentially distributed with parameter λ and μ respectively. The system availability is given by

$$A_{11}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}. \tag{4.16}$$

As another example let $m=n=2$, $\lambda=1$ and $\mu=2$. In this case

$$A_{11}(t) = \frac{2}{3} + \frac{1}{12}e^{-3t} + \frac{1}{4}e^{-3t/2}\cos(\sqrt{7}t/2) + \frac{5}{28}\sqrt{7}e^{-3t/2}\sin(\sqrt{7}t/2).$$

For non-integers m and n we can invert numerically the transform in (4.15). As an example let $m=7.6$, $n=2.4$, $\lambda=2$ and $\mu=1.1765$. In this case

$$\int_0^\infty A_{11}(t)e^{-\beta t}dt = \frac{(1.1765+\beta)^{2.4}((2+\beta)^{7.6}-2^{7.6})}{\beta[(2+\beta)^{7.6}(1.1765+\beta)^{2.4}-2^{7.6}1.1765^{2.4}]}.$$

The graph of $A_{11}(t)$ can be seen in Figure 4.1 which is the same as Figure 1 in Pham-Gia and Turkkan [30].

Example 4.3.2 *Let $((X_n, Y_n), n \geq 1)$ be an iid sequence of non-negative random vectors having a common joint bivariate exponential distribution given by*

$$\mathbf{P}(X_1 > x, Y_1 > y) \quad = \quad e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12}\max(x,y))}; \quad x,y \geq 0; \quad \lambda_1,\lambda_2,\lambda_{12} > 0.$$

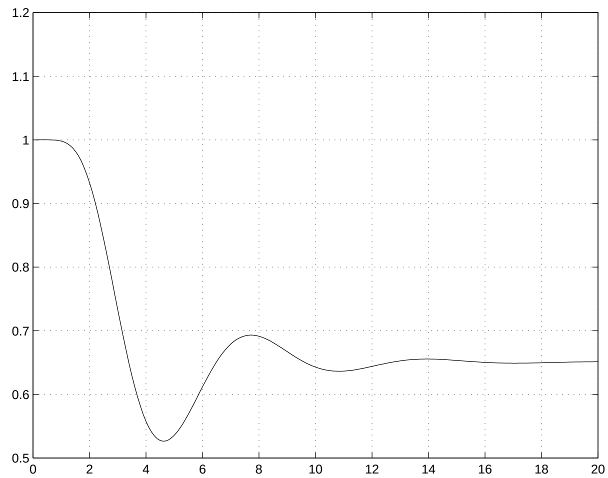


Figure 4.1: Graph of $A_{11}(t)$.

Obviously

$$\mathbf{P}(X_1 > x) \quad = \quad e^{-(\lambda_1+\lambda_{12})x} \tag{4.17}$$

and

$$\mathbf{P}(Y_1 > y) \quad = \quad e^{-(\lambda_2+\lambda_{12})y}. \tag{4.18}$$

The correlation coefficient ρ_{XY} *between* X_1 *and* Y_1 *is given by*

$$\rho_{XY} = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}.$$

The Laplace-Stieltjes transform of the cdf F *of* X_1 *and the joint cdf* H *of* X_1 *and* Y_1 *are given by*

$$F^*(\beta) = \frac{\lambda_1 + \lambda_{12}}{\beta + \lambda_1 + \lambda_{12}}$$

and

$$H^*(\alpha, \beta) = \frac{(\lambda_1 + \lambda_2 + \lambda_{12} + \alpha + \beta)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + \lambda_{12}\alpha\beta}{(\lambda_1 + \lambda_2 + \lambda_{12} + \alpha + \beta)(\lambda_1 + \lambda_{12} + \alpha)(\lambda_2 + \lambda_{12} + \beta)},$$

see Barlow [2]*. It follows that*

$$\int_0^\infty A_{11}(t)e^{-\beta t}dt \quad = \quad \frac{(\lambda + 2\beta)(\lambda_2 + \lambda_{12})}{\beta[2\beta^2 + (3\lambda + \lambda_{12})\beta + \lambda(\lambda + \lambda_{12})]}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. *Inverting this transform we obtain*

$$A_{11}(t) = \frac{\lambda_2 + \lambda_{12}}{\lambda + \lambda_{12}} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda t} + \frac{\lambda_{12}(\lambda_2 - \lambda_1)}{(\lambda_1 + \lambda_2)(\lambda + \lambda_{12})} e^{-(\lambda + \lambda_{12})t/2}.$$

4.4 Covariance of total downtime

Let $U(t) = t - D(t)$ be the total uptime of the system up to time t. Obviously

$$\mathrm{Cov}(D(t_1), D(t_2)) = \mathrm{Cov}(U(t_1), U(t_2)).$$

So we might as well study $\mathrm{Cov}(U(t_1), U(t_2))$.

Let $0 \le t_1 \le t_2 < \infty$. Then

$$\begin{aligned} \mathbf{E}[U(t_1)U(t_2)] &= \mathbf{E}\left[\int_{x=0}^{t_1}\int_{y=0}^{t_2} 1_{\{1\}}(Z(x))1_{\{1\}}(Z(y))dydx\right] \\ &= 2\int_{x=0}^{t_1}\int_{y=x}^{t_1} \mathbf{P}(Z(x)=1, Z(y)=1)dydx \\ &\quad + \int_{x=0}^{t_1}\int_{y=t_1}^{t_2} \mathbf{P}(Z(x)=1, Z(y)=1)dydx. \end{aligned} \tag{4.19}$$

Let $\varphi(x, y) = \mathbf{P}(Z(x) = 1, Z(y) = 1)$. For $0 \le x \le y < \infty$,

$$\begin{aligned} \varphi(x,y) &= \mathbf{P}(Z(x)=1, Z(y)=1, y < X_1) \\ &\quad + \mathbf{P}(Z(x)=1, Z(y)=1, x < X_1 < y) \\ &\quad + \mathbf{P}(Z(x)=1, Z(y)=1, X_1 < x). \end{aligned} \tag{4.20}$$

Obviously

$$\mathbf{P}(Z(x) = 1, Z(y) = 1, y < X_1) = 1 - F(y).$$

For the second term, note that the event "$Z(x) = 1, Z(y) = 1, x < X_1 < y$" is equivalent to the event "$x < X_1$ and for some $n \ge 1$, $S_n < y < S_n + X_{n+1}$", where $S_n = \sum_{i=1}^{n}(X_i + Y_i)$. Let $R_n = \sum_{i=2}^{n}(X_i + Y_i)$, $n \ge 2$. Then (X_1, Y_1),

R_n and X_{n+1} are independent. Denote by K_n the cdf of R_n. Then

$$\begin{aligned}
&\mathbf{P}(Z(x)=1, Z(y)=1, x<X_1<y) \\
&= \sum_{n=1}^{\infty} \mathbf{P}(x<X_1, S_n \le y < S_n + X_{n+1}) \\
&= \mathbf{P}(x<X_1, X_1+Y_1 \le y < X_1+Y_1+X_2) \\
&\quad + \sum_{n=2}^{\infty} \mathbf{P}(x<X_1, (X_1+Y_1)+R_n \le y < (X_1+Y_1)+R_n+X_{n+1}) \\
&= \int_{x_1\in(x,y]} \int_{y_1\in[0,y-x_1]} \int_{x_2\in(y-x_1-y_1,\infty)} dF(x_2)dH(x_1,y_1) \\
&\quad + \sum_{n=2}^{\infty} \int_{x_1\in(x,y]} \int_{y_1\in[0,y-x_1]} \int_{r_n\in[0,y-x_1-y_1]} \int_{x_{n+1}\in(y-x_1-y_1-r_n,\infty)} \\
&\quad dF(x_{n+1})dK_n(r_n)dH(x_1,y_1) \\
&= \int_{x_1\in(x,y]} \int_{w\in[x_1,y]} \left\{ \int_{x_2\in(y-w,\infty)} dF(x_2) \right. \\
&\quad \left. + \sum_{n=2}^{\infty} \int_{r_n\in[0,y-w]} \int_{x_{n+1}\in(y-w-r_n,\infty)} dF(x_{n+1})dK_n(r_n) \right\} dH(x_1, w-x_1) \\
&= \int_{x_1\in(x,y]} \int_{w\in[x_1,y]} \sum_{n=1}^{\infty} \mathbf{P}(R_n \le y-w < R_n + X_{n+1}) dH(x_1, w-x_1) \\
&= \int_{x_1\in(x,y]} \int_{w\in[x_1,y]} \mathbf{P}(z(y-w)=1) dH(x_1, w-x_1) \\
&= \int_{x_1\in(x,y]} \int_{w\in[x_1,y]} A_{11}(y-w) dH(x_1, w-x_1),
\end{aligned}$$

where $A_{11}(t)$ denotes the availability of the system at time t starting in state 1 at time 0. Finally, the last term in (4.20) can be obtained by conditioning on X_1+Y_1, i.e.,

$$\begin{aligned}
&\mathbf{P}(Z(x)=1, Z(y)=1, X_1 \le x) \\
&= \mathbf{P}(Z(x)=1, Z(y)=1, X_1+Y_1 \le x) \\
&= \int_0^{\infty} \mathbf{P}(Z(x)=1, Z(y)=1, X_1+Y_1 \le x | X_1+Y_1=w) dK(w) \\
&= \int_0^{x} \mathbf{P}(Z(x-w)=1, Z(y-w)=1) dK(w) \\
&= \int_0^{x} \varphi(x-w, y-w) dK(w).
\end{aligned}$$

So we obtain

$$\begin{aligned}\varphi(x,y) &= 1-F(y)+\int_{x_1\in(x,y]}\int_{w\in[x_1,y]} A_{11}(y-w)dH(x_1,w-x_1)\\ &+ \int_0^x \varphi(x-w,y-w)dK(w). \end{aligned} \tag{4.21}$$

Taking double Laplace transforms on both sides of (4.21) we obtain

$$\begin{aligned}\hat{\varphi}(\alpha,\beta) &:= \int_0^\infty\int_0^\infty \varphi(x,y)e^{-\alpha x-\beta y}dxdy\\ &= \frac{\alpha[1-F^*(\beta)]-\beta[F^*(\beta)-F^*(\alpha+\beta)]}{\alpha\beta(\alpha+\beta)[1-K^*(\alpha+\beta)]}\\ &\quad+\frac{\hat{A}_{11}(\beta)[H^*(\beta,\beta)-H^*(\alpha+\beta,\beta)]}{\alpha[1-K^*(\alpha+\beta)]}.\end{aligned}$$

where

$$\hat{A}_{11}(\beta) := \int_0^\infty A_{11}(t)e^{-\beta t}dt = \frac{1-F^*(\beta)}{\beta[1-H^*(\beta)]},$$

see (4.12). This formula is a generalization of the result in Srinivasan [37], since $H^*(\beta) = F^*(\beta)G^*(\beta)$ when X_i and Y_i are independent.

Now from (4.19) we obtain

$$\begin{aligned}&\int_{t_1=0}^\infty\int_{t_2=t_1}^\infty \mathbf{E}[U(t_1)U(t_2)]e^{-\alpha t_1-\beta t_2}dt_2dt_1\\ &= \frac{2\hat{\varphi}(0,\alpha+\beta)}{\beta(\alpha+\beta)}+\frac{\hat{\varphi}(\alpha,\beta)-\hat{\varphi}(0,\alpha+\beta)}{\alpha\beta}\\ &= \frac{\hat{\varphi}(\alpha,\beta)}{\alpha\beta}+\frac{[\alpha-\beta]\hat{\varphi}(0,\alpha+\beta)}{\alpha\beta(\alpha+\beta)}.\end{aligned}$$

It follows that

$$\int_0^\infty\int_0^\infty \mathbf{E}[U(t_1)U(t_2)]e^{-\alpha t_1-\beta t_2}dt_1dt_2 = \frac{1}{\alpha\beta}[\hat{\varphi}(\alpha,\beta)+\hat{\varphi}(\beta,\alpha)].$$

4.5 Asymptotic properties

In this section we want to address asymptotic properties of the total downtime $D(t)$. To this end we use a method in Takács [44] which is based on a comparison with the asymptotic properties of a delayed renewal process related to the process that we are studying. First we summarize some known results about delayed renewal processes $(\tilde{N}(t), t \geq 0)$ which will be used in the following.

Let $(V_n, n \geq 1)$ be an i.i.d. sequence of non-negative random variables. Let V_0 be a non-negative random variable which is independent of the sequence (V_n). Define $\tilde{S}_0 = 0$, $\tilde{S}_n = \sum_{i=0}^{n-1} V_i, n \geq 1$ and $\tilde{N}(t) = \sup\{n \geq 0 : \tilde{S}_n \leq t\}$. The Laplace-Stieltjes transforms of the first and the second moments of $\tilde{N}(t)$ are given by

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[\tilde{N}(t)] = \frac{\mathbf{E}(e^{-\beta V_0})}{1 - \mathbf{E}(e^{-\beta V_1})} \tag{4.22}$$

and

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[\tilde{N}^2(t)] = \frac{2\mathbf{E}(e^{-\beta V_0})}{[1 - \mathbf{E}(e^{-\beta V_1})]^2} - \frac{\mathbf{E}(e^{-\beta V_0})}{1 - \mathbf{E}(e^{-\beta V_1})}, \tag{4.23}$$

respectively, see Takács [44].

Now we are in a position to derive the asymptotic properties of $D(t)$. The same argument as used in Takács [44] for the independent case can now be employed to derive asymptotic properties of the total downtime for the dependent case.

Let $\mu_X = \mathbf{E}(X_1)$, $\mu_Y = \mathbf{E}(Y_1)$, $\sigma_X^2 = \mathrm{Var}(X_1)$, $\sigma_Y^2 = \mathrm{Var}(Y_1)$ and $\sigma_{XY} = \mathrm{Cov}(X_1, Y_1)$. Let

$$V_n = X_n + Y_n, \quad n = 1, 2, 3.... \tag{4.24}$$

Lemma 4.5.1 *Let $(\tilde{N}(t), t \geq 0)$ be the delayed renewal process determined by the random variables $(V_n), n = 0, 1, 2, ...,$ where V_0 has the distribution*

$$\mathbf{P}(V_0 \leq x) = \begin{cases} \frac{1}{\mu_X} \int_0^x [1 - F(y)] dy, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{4.25}$$

and $V_n, n = 1, 2, ...,$ are defined as in (4.24). Then

$$\mathbf{E}[D(t)] + \mu_X \mathbf{E}[\tilde{N}(t)] = t. \tag{4.26}$$

Proof: It is easy to verify that $\mathbf{E}(e^{-\beta V_0}) = \frac{1-F^*(\beta)}{\beta \mu_X}$ and $\mathbf{E}(e^{-\beta V_1}) = H^*(\beta, \beta)$. Substitution in (4.22) yields

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[\tilde{N}(t)] = \frac{1 - F^*(\beta)}{\beta \mu_X [1 - H^*(\beta, \beta)]}. \tag{4.27}$$

Now from (4.6) we obtain

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[D(t)] = \frac{F^*(\beta) - H^*(\beta, \beta)}{\beta [1 - H^*(\beta, \beta)]}. \tag{4.28}$$

Taking the Laplace-Stieltjes transform on the left-hand side of (4.26) and taking (4.27) and (4.28) into consideration we obtain

$$\begin{aligned}\int_0^\infty e^{-\beta t} d\mathbf{E}[D(t)] + \mu_X \int_0^\infty e^{-\beta t} d\mathbf{E}[\tilde{N}(t)] \\ = \frac{F^*(\beta) - H^*(\beta,\beta)}{\beta[1 - H^*(\beta,\beta)]} + \frac{1 - F^*(\beta)}{\beta[1 - H^*(\beta,\beta)]} \\ = \frac{1}{\beta}\end{aligned}$$

which is equal to the Laplace-Stieltjes transform of t. □

Remark 4.5.1 *The relation* (4.26) *has been proved by Takács* [44] *for the case that* (X_i) *and* (Y_j) *are independent.*

Theorem 4.5.1 *If* $\mu_X + \mu_Y < \infty$, *then*

$$\lim_{t\to\infty} \frac{\mathbf{E}[D(t)]}{t} = \frac{\mu_Y}{\mu_X + \mu_Y}, \tag{4.29}$$

and if σ_X^2 *and* σ_Y^2 *are finite, and* $X_1 + Y_1$ *is a non-lattice random variable, then*

$$\lim_{t\to\infty}\left(\mathbf{E}[D(t)] - \frac{\mu_Y t}{\mu_X + \mu_Y}\right) = \frac{\mu_Y \sigma_X^2 - \mu_X \sigma_Y^2 - 2\mu_X \sigma_{XY}}{2(\mu_X + \mu_Y)^2} - \frac{\mu_X \mu_Y}{2(\mu_X + \mu_Y)}. \tag{4.30}$$

Proof: From Lemma 4.5.1 and (1.10) we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbf{E}[D(t)]}{t} &= 1 - \mu_X \lim_{t\to\infty} \frac{\mathbf{E}[\tilde{N}(t)]}{t} \\ &= 1 - \frac{\mu_X}{\mu_X + \mu_Y} \\ &= \frac{\mu_Y}{\mu_X + \mu_Y}.\end{aligned}$$

For the second part, from (4.26) we obtain

$$\lim_{t\to\infty}\left(\mathbf{E}[D(t)] - \frac{\mu_Y t}{\mu_X + \mu_Y}\right) = -\mu_X \lim_{t\to\infty}\left(\mathbf{E}[\tilde{N}(t)] - \frac{t}{\mu_X + \mu_Y}\right) \tag{4.31}$$

Now if $X_1 + Y_1$ is a non-lattice random variable, then using (1.11) we obtain

$$\begin{aligned}&\lim_{t\to\infty}\left(\mathbf{E}[\tilde{N}(t)] - \frac{t}{\mu_X + \mu_Y}\right) \\ &= \frac{\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} + (\mu_X + \mu_Y)^2}{2(\mu_X + \mu_Y)^2} - \frac{\mu_0}{\mu_X + \mu_Y}\end{aligned} \tag{4.32}$$

Next it is easy to see that

$$\mu_0 = \mathbf{E}(V_0) = \frac{\sigma_X^2 + \mu_X^2}{2\mu_X}. \tag{4.33}$$

Substitution (4.32) and(4.33) into (4.31) completes the proof. □

Remark 4.5.2 *The first result* (4.29) *of Theorem 4.5.1 can be proved using a Tauberian theorem. From* (4.28) *if* μ_X *and* μ_Y *are finite, we obtain*

$$\int_0^\infty e^{-\beta t} d\mathbf{E}[D(t)] \quad \sim \quad \frac{\mu_Y}{\beta(\mu_X + \mu_Y)} \quad as \quad \beta \to 0.$$

Obviously $\mathbf{E}[D(t)]$ *is non-decreasing. So we can use Theorem 2.4.1 to conclude* (4.29).

Now we will derive the asymptotic variance of $D(t)$.

Lemma 4.5.2 *Let* $(\tilde{N}(t), t \geq 0)$ *be the delayed renewal process determined by the random variables* $(V_n), n = 0, 1, 2, \ldots$, *where* V_0 *has the Laplace transform*

$$\mathbf{E}(e^{-\beta V_0}) \quad = \quad \frac{[1 - F^*(\beta)] \int_0^\infty \int_0^\infty y e^{-\beta(x+y)} dH(x, y)}{\beta \mu_X \mu_Y}$$

and $V_n, n = 1, 2 \ldots$, *are defined as in* (4.24). *Then*

$$\mathbf{E}[D^2(t)] = 2 \int_0^t \mathbf{E}[D(u)] du - \mu_X \mu_Y (\mathbf{E}[\tilde{N}(t)] + \mathbf{E}[\tilde{N}^2(t)]). \tag{4.34}$$

Proof: Firstly, it is easy to see from the Laplace transform of V_0 that

$$\mu_0 = \mathbf{E}(V_0) = \mu_X + \frac{\sigma_X^2 + \mu_X^2}{2\mu_X} + \frac{\sigma_{XY} + \sigma_Y^2 + \mu_Y^2}{\mu_Y}. \tag{4.35}$$

Then take the Laplace-Stieltjes transforms on both sides of (4.34) and use (4.6), (4.7), (4.22), (4.23), and (4.35). □

Theorem 4.5.2 *If* σ_X^2 *and* σ_Y^2 *are finite, and* $X_1 + Y_1$ *is a non-lattice random variable, then*

$$\lim_{t \to \infty} \frac{\mathrm{Var}[D(t)]}{t} \quad = \quad \frac{\mu_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 - 2\mu_X \mu_Y \sigma_{XY}}{(\mu_X + \mu_Y)^3}.$$

Proof: If $X_1 + Y_1$ is a non-lattice random variable then, using (1.11) and (1.12) respectively, we can show that the delayed renewal process defined in Lemma

4.5.2 has properties as $t \to \infty$

$$\begin{aligned}\mathbf{E}[\tilde{N}(t)] &= \frac{t}{\mu_X + \mu_Y} + \frac{\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY} + (\mu_X + \mu_Y)^2}{2(\mu_X + \mu_Y)^2} \\ &- \frac{\mu_Y(\sigma_X^2 + \mu_X^2) + 2\mu_X^2\mu_Y + 2\mu_X(\sigma_{XY} + \sigma_Y^2 + \mu_Y^2)}{2\mu_X\mu_Y(\mu_X + \mu_Y)} + o(1)\end{aligned}$$

and

$$\mathbf{E}[\tilde{N}^2(t)] = \mathbf{E}[\tilde{N}(t)]^2 + \frac{\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}}{(\mu_X + \mu_Y)^3} t + o(t)$$

From (4.30) we deduce

$$\begin{aligned}2\int_0^t \mathbf{E}[D(u)]du &= \frac{\mu_Y t^2}{\mu_X + \mu_Y} + \left[\frac{\mu_Y\sigma_X^2 - \mu_X\sigma_Y^2 - 2\mu_X\sigma_{XY}}{(\mu_X + \mu_Y)^2} - \frac{\mu_X\mu_Y}{\mu_X + \mu_Y}\right] t \\ &+ o(t) \quad \text{as} \quad t \to \infty.\end{aligned}$$

It follows, using Lemma 4.5.2, that

$$\begin{aligned}\mathbf{E}[D^2(t)] &= \frac{\mu_Y^2 t^2}{(\mu_X + \mu_Y)^2} \\ &- \left[\frac{\mu_X\mu_Y^3 + (\mu_X^2 - 2\sigma_X^2)\mu_Y^2 + (\sigma_Y^2 + 4\sigma_{XY})\mu_X\mu_Y - \mu_X^2\sigma_Y^2}{(\mu_X + \mu_Y)^3}\right] t \\ &+ o(t).\end{aligned}$$

and hence, by taking (4.30) into consideration,

$$\begin{aligned}\mathrm{Var}[D(t)] &= \mathbf{E}[D^2(t)] - \mathbf{E}[D(t)]^2 \\ &= \frac{\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 - 2\mu_X\mu_Y\sigma_{XY}}{(\mu_X + \mu_Y)^3} t + o(t). \quad \square\end{aligned}$$

Now we will consider the asymptotic distribution of the total downtime. For the case that (X_i) and (Y_j) are independent the limiting distribution of the total downtime $D(t)$ is normal as $t \to \infty$, i.e.,

$$\frac{D(t) - \frac{\mu_Y t}{\mu_X + \mu_Y}}{\sqrt{\frac{\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2}{(\mu_X + \mu_Y)^3} t}} \xrightarrow{d} N(0,1), \tag{4.36}$$

provided σ_X^2 and σ_Y^2 are finite, see Takács [44] and Rényi [33]. In the following theorem we give the limiting distribution of the total downtime $D(t)$ for the dependent case.

Theorem 4.5.3 *If σ_X^2 and σ_Y^2 are finite then*

$$\frac{D(t) - \frac{\mu_Y t}{\mu_X + \mu_Y}}{\sqrt{\frac{\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 - 2\mu_X\mu_Y\sigma_{XY}}{(\mu_X + \mu_Y)^3} t}} \xrightarrow{d} N(0,1) \quad \textit{as} \quad t \to \infty.$$

Proof: First note that

$$\sum_{i=1}^{N(t)} Y_i \leq D(t) \leq \sum_{i=1}^{N(t)+1} Y_i,$$

where $N(t) = \sup\{n \geq 0 : \sum_{j=1}^{n}(X_j + Y_j) \leq t\}$. Using the Central Limit Theorem for random sums, see Embrechts *et al.* [11], we obtain

$$\left[\operatorname{Var}\left(Y_1 - \frac{\mu_Y(X_1+Y_1)}{\mu_X+\mu_Y}\right)\frac{t}{\mu_X+\mu_Y}\right]^{-1/2}\left(\sum_{i=1}^{N(t)} Y_i - \frac{\mu_Y t}{\mu_X+\mu_Y}\right) \xrightarrow{d} N(0,1),$$

where

$$\begin{aligned}
&\operatorname{Var}\left(Y_1 - \frac{\mu_Y(X_1+Y_1)}{\mu_X+\mu_Y}\right)\frac{t}{\mu_X+\mu_Y} \\
&\quad= \left(\sigma_Y^2 + \frac{\mu_Y^2}{(\mu_X+\mu_Y)^2}\operatorname{Var}(X+Y) - \frac{2\mu_Y}{\mu_X+\mu_Y}\operatorname{Cov}(Y, X+Y)\right)\frac{t}{\mu_X+\mu_Y} \\
&\quad= \frac{\sigma_Y^2(\mu_X+\mu_Y)^2 + \mu_Y^2(\sigma_X^2+\sigma_Y^2+2\sigma_{XY}) - 2\mu_Y(\mu_X+\mu_Y)(\sigma_{XY}+\sigma_Y^2)}{(\mu_X+\mu_Y)^3}t \\
&\quad= \frac{\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 - 2\mu_X\mu_Y\sigma_{XY}}{(\mu_X+\mu_Y)^3}t.
\end{aligned}$$

The proof is complete if we can show that

$$\frac{Y_{N(t)+1}}{\sqrt{t}} \xrightarrow{P} 0 \quad \text{as} \quad t \to \infty. \tag{4.37}$$

But by the fact that $\frac{N(t)}{t} \xrightarrow{P} \frac{1}{\mu_X+\mu_Y}(> 0)$ and the assumption that $\sigma_Y^2 < \infty$, then using Lemma 2.4.3 we obtain

$$\frac{Y_{N(t)}}{\sqrt{N(t)}} \xrightarrow{P} 0.$$

Hence (4.37) follows. □

4.6 Examples

In this section we give two examples. In the first example we will see the effect of dependence of the failure and repair times on the distribution of the total downtime. In the second example we will see that for some cases we have analytic expressions for the first and second moments of the total downtime.

Example 4.6.1 *Let* $(X_i, i \geq 1)$ *and* $(Y_i, i \geq 1)$ *be the sequences of the failure times and repair times respectively, of a repairable system such that* $((X_i, Y_i), i \geq 1)$ *is an iid sequence of non-negative random vectors. Let* X_1 *and* Y_1 *have a joint bivariate exponential distribution given by*

$$\mathbf{P}(X_1 > x, Y_1 > y) = e^{-(\lambda_1 x + \lambda_2 y + \lambda_{12} \max(x,y))}; \quad x, y \geq 0; \quad \lambda_1, \lambda_2, \lambda_{12} > 0.$$

The marginals are given by

$$\mathbf{P}(X_1 > x) = e^{-(\lambda_1 + \lambda_{12})x} \tag{4.38}$$

and

$$\mathbf{P}(Y_1 > y) = e^{-(\lambda_2 + \lambda_{12})y}. \tag{4.39}$$

In this case we have $\mu_X = \frac{1}{\lambda_1 + \lambda_{12}}$, $\mu_Y = \frac{1}{\lambda_2 + \lambda_{12}}$, $\sigma_X^2 = \frac{1}{(\lambda_1 + \lambda_{12})^2}$, $\sigma_Y^2 = \frac{1}{(\lambda_2 + \lambda_{12})^2}$, $\sigma_{XY} = \frac{\lambda_{12}}{(\lambda_1 + \lambda_2 + \lambda_{12})(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})}$, *and the correlation coefficient* ρ_{XY} *between* X_1 *and* Y_1

$$\rho_{XY} = \frac{\lambda_{12}}{\lambda_1 + \lambda_2 + \lambda_{12}}.$$

Using (4.6) *we obtain*

$$\int_0^\infty \mathbf{E}[D(t)] e^{-\beta t} dt = \frac{(2\lambda_1 + \lambda_{12})\beta + (\lambda_1 + \lambda_{12})^2 + \lambda_2(\lambda_1 + \lambda_{12})}{\beta^2 [2\beta^2 + (3\lambda_1 + 3\lambda_2 + 4\lambda_{12})\beta + (\lambda_1 + \lambda_2)^2 + 3\lambda_{12}(\lambda_1 + \lambda_2) + 2\lambda_{12}^2]}.$$

This transform can be inverted analytically. As an example for $\lambda_1 = 1$, $\lambda_2 = 2$, *and* $\lambda_{12} = 3$ *we obtain*

$$\mathbf{E}[D(t)] = \frac{4}{9}t - \frac{13}{162} + \frac{2}{81}e^{-9t/2} + \frac{1}{18}e^{-6t}.$$

The distribution of $D(t)$ *has mass at 0 with*

$$\mathbf{P}(D(t) = 0) = \mathbf{P}(X_1 > t) = e^{-(\lambda_1 + \lambda_{12})t}.$$

The pdf of the continuous part of $D(t)$ *can be obtained by inverting its double Laplace transform. As an example let* $\lambda_1 = 1, \lambda_2 = 2$, *and* $\lambda_{12} = 3$. *In this case*

$$\int_0^\infty \mathbf{E}[e^{-\alpha D(t)}] e^{-\beta t} dt = \frac{\alpha + \beta - \frac{4\alpha}{4+\beta} - \beta C(\alpha, \beta)}{\beta(\alpha + \beta)\left[1 - C(\alpha, \beta)\right]} \tag{4.40}$$

where

$$C(\alpha, \beta) = \frac{20(6 + \alpha + 2\beta) + 3\beta(\alpha + \beta)}{(6 + \alpha + 2\beta)(4 + \beta)(5 + \beta)}.$$

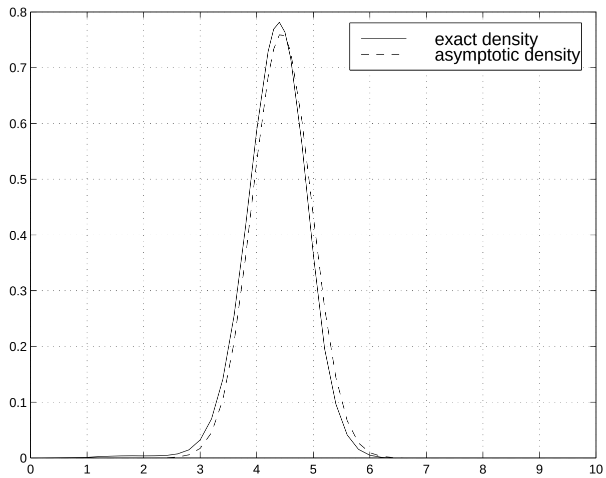


Figure 4.2: The graph of the density of $D(10)$ with $\lambda_1 = 1$ $\lambda_2 = 2$ and $\lambda_{12} = 3$. Solid line for exact density; dashed line for normal approximation.

Using the numerical inversion of a double Laplace transform, see Appendix B*, we get the graph of the pdf of* $D(10)$*, see Figure 4.2 (solid line). In this Figure we also compare the pdf of* $D(10)$ *with its normal approximation (dashed line). We see that the pdf of* $D(10)$ *is close to its normal approximation.*

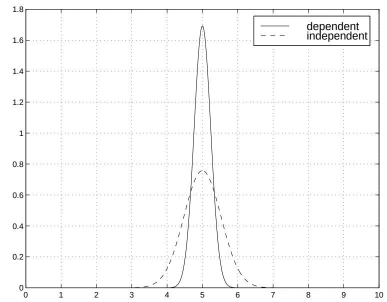
The effect of dependence between the failure and the repair times can be seen in Figure 4.3. In this figure we compare the graphs of the normal approximations of $D(10)$ *where* (X_i) *and* (Y_j) *are independent and satisfy* (4.38) *and* (4.39) *with the normal approximations of* $D(10)$ *for various correlation coefficients* ρ_{XY}*. We see that the smaller their correlation coefficient the closer their normal approximations.*

Example 4.6.2 *Let* $(X_i, i \geq 1)$ *and* $(Y_i, i \geq 1)$ *denote the sequences of the failure times and repair times respectively, of a repairable system such that* (X_i) *are iid non-negative random variables having a common Gamma*(λ, m) *distribution with a pdf*

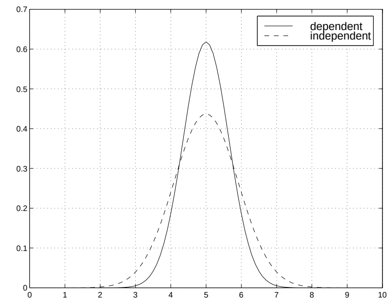
$$f_{X_1}(x) = \frac{\lambda^m}{\Gamma(m)} x^{m-1} e^{-\lambda x}, \quad x \geq 0$$

and (Y_i) *are iid non-negative random variables having a common Gamma*(μ, n) *distribution.*

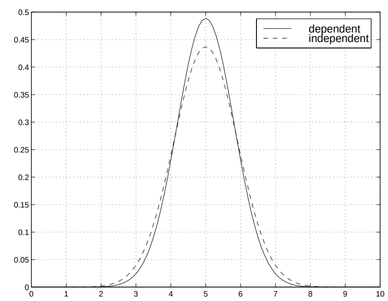
Firstly we will consider the case where $m = n = 1$*. In this case* X_1 *and* Y_1 *are exponentially distributed with parameter* λ *and* μ *respectively. Using* (4.6)*,*



(a) $\sigma_{XY} = 0.8$ ($\lambda_1 = \lambda_2 = 1, \lambda_{12} = 8$)



(b) $\sigma_{XY} = 0.5$ ($\lambda_1 = \lambda_2 = 1, \lambda_{12} = 2$)



(c) $\sigma_{XY} = 0.2$ ($\lambda_1 = \lambda_2 = 2, \lambda_{12} = 1$)

Figure 4.3: The graphs of the normal approximations of $D(10)$. Solid line for dependent cases and dashed line for the independent cases.

(4.7)*, and* (4.3) *we obtain*

$$\int_0^\infty \mathbf{E}[D(t)]e^{-\beta t}dt = \frac{\lambda}{\beta^2[\beta + \lambda + \mu]},$$

$$\int_0^\infty \mathbf{E}[D^2(t)]e^{-\beta t}dt \quad = \quad \frac{2\lambda(\lambda + \beta)}{\beta^3[\beta + \lambda + \mu]^2},$$

and

$$\int_0^\infty \mathbf{E}[e^{-\alpha D(t)}]e^{-\beta t}dt \quad = \quad \frac{\alpha + \beta + \lambda + \mu}{(\lambda + \beta)(\alpha + \beta) + \mu\beta}.$$

Inverting these transforms we obtain

$$\mathbf{E}[D(t)] = \frac{\lambda t}{\lambda + \mu} - \frac{\lambda}{(\lambda + \mu)^2}(1 - e^{-(\lambda + \mu)t}),$$

$$\mathbf{E}[D^2(t)] = \frac{\lambda^2 t^2}{(\lambda+\mu)^2} + \frac{2\lambda(\mu-\lambda)t}{(\lambda+\mu)^3} + \frac{2\lambda(\lambda-2\mu)+2\lambda[2\mu-\lambda+\mu(\lambda+\mu)t]e^{-(\lambda+\mu)t}}{(\lambda+\mu)^4},$$

and

$$\mathbf{E}[e^{-\alpha D(t)}] = e^{-(\alpha+\lambda+\mu)t/2}\left(\cos(\sqrt{c}t/2) + \frac{\alpha+\lambda+\mu}{\sqrt{c}}\sin(\sqrt{c}t/2)\right) \quad (4.41)$$

where $c = 4\lambda\alpha - (\alpha+\lambda+\mu)^2$. *The graph of the pdf of* $D(t)$ *can be obtain by inverting numerically the Laplace transform in* (4.41). *As an example the graph of the pdf of* $D(20)$ *for* $\lambda = 1$ *and* $\mu = 5$ *can be seen in Figure 4.4 (dashed line).*

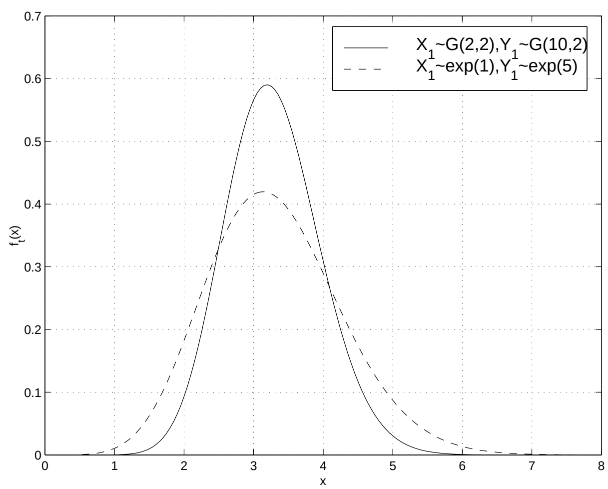


Figure 4.4: The graph of the pdf of $D(20)$ with $X_1 \sim \exp(1)$, $Y_1 \sim \exp(2)$ (solid line) and $X_2 \sim \text{Gamma}(2,2)$, $Y_2 \sim \text{Gamma}(10,2)$ (dashed line).

For integers m *and* n *we have explicit expressions for the first and second moments of* $D(t)$. *As an example, if* $m = n = 2$, $\lambda = 2$ *and* $\mu = 10$, *then using* (4.6) *and* (4.7) *we obtain*

$$\mathbf{E}[D(t)] = \frac{1}{6}t - \frac{1}{18} + \frac{1}{180}e^{-12t} + \frac{1}{20}\cos(2t)e^{-6t} + \frac{1}{10}\sin(2t)e^{-6t},$$

and

$$\mathbf{E}[D^2(t)] = \frac{1}{36}t^2 + \frac{1}{216}t - \frac{1}{864} + \frac{1}{864}e^{-12t} + \frac{1}{108}te^{-12t} + \frac{1}{4}t\cos(2t)e^{-6t} - \frac{1}{8}\sin(2t)e^{-6t}.$$

In this case we also have explicit expressions for $\mathbf{E}[e^{-\alpha D(t)}]$. *The graph of the pdf of* $D(20)$ *for* $m = n = 2$, $\lambda = 2$ *and* $\mu = 10$ *can be seen in Figure 4.4 (solid line). Note that in this case* X_1 *and* Y_1 *have means 1 and 0.2 respectively, which are the same as the means of exponential random variables with parameter 1 and 5 respectively.*

4.7 Systems consisting of n independent components

In this section we will consider the distribution of the total uptime (downtime) of a system consisting of $n \geq 2$ stochastically independent components. The system we will discuss can be a series, a parallel or a k-out-of-n system, but we will formulate the results only for a series system. The results only concern the total uptime. The corresponding results for the total downtime can be derived similarly, or using the obvious relation between the total uptime and total downtime. Firstly we will discuss the case where both the failure and repair times of the components are exponentially distributed, and later on we will consider the case where the failure or the repair times are arbitrarily distributed.

4.7.1 Exponential failure and repair times

Consider a series system comprising $n \geq 2$ stochastically independent two-state components, each of which can be either up or down, denoted as 1 and 0 respectively. Suppose that system starts to operate at time 0. If a component fails, it is repaired and put into operation again. During the repair the unfail component may fail. There are no capacity constrains at the repair shop.

Denote by $Z_i(t), i = 1, 2, ..., n$ the state of the ith component at time t. Then the total uptime of the system in the time interval $[0, t]$ is given by

$$U(t) = \int_0^t 1_{\mathbf{1_n}}(Z_1(s), ..., Z_n(s))ds \tag{4.42}$$

where $\mathbf{1_n}$ denotes the vector of ones of length n.

Let X_{ij} and Y_{ij}, $j = 1, 2, ...,$ denote the consecutive uptimes and downtimes, respectively, of the ith component. Assume that the sequences (X_{ij}) and (Y_{ij}) are independent. Assume also that for each i, the random variables $X_{ij}, j = 1, 2, ...,$ have a common exponential distribution with parameter λ_i, and $Y_{ij}, j = 1, 2, ...,$ have a common exponential distribution with parameter μ_i. Then $(Z_i(t), t \geq 0), i = 1, 2, \ldots, n$ are independent, continuous time Markov chains on $\{0, 1\}$ with generators

$$Q_i = \begin{pmatrix} -\mu_i & \mu_i \\ \lambda_i & -\lambda_i \end{pmatrix}, \quad \lambda_i, \mu_i > 0.$$

Let
$$Y_n(t) = (Z_1(t), Z_2(t), \dots, Z_n(t)).$$
Then $Y_n = (Y_n(t), t \geq 0)$ is a continuous-time Markov chain on
$$I = \{0,1\}^n,$$
the set of row vectors of length n with entries zeros and or ones.

Let $a \in I$ be a state of the Markov chain Y_n. Then a has the form
$$a = (\epsilon_1(a), \epsilon_2(a), \dots, \epsilon_n(a))$$
where $\epsilon_j(a) \in \{0,1\}$, $j = 1, 2, \dots, n$. The generator
$$Q = (q_{ab})_{a,b \in I}$$
of the Markov chain Y_n has the following properties. Suppose
$$b = (\epsilon_1(b), \epsilon_2(b), \dots, \epsilon_n(b)).$$
If a and b have two or more different entries, then $q_{ab} = q_{ba} = 0$. Suppose now a and b have only one different entry. Then there exists an index j such that
$$\epsilon_i(a) = \epsilon_i(b) \quad \text{for all } i \neq j$$
and
$$\epsilon_j(a) = 1 - \epsilon_j(b).$$
Let
$$\nu_{i,0} = \lambda_i \quad \text{and} \quad \nu_{i,1} = \mu_i.$$
Then
$$q_{ab} = \nu_{j,\epsilon_j(b)} \quad \text{and} \quad q_{ba} = \nu_{j,\epsilon_j(a)}.$$

Lemma 4.7.1 *The vector $\pi = (\pi_a)_{a \in I}$ where*
$$\pi_a = \nu_{1,\epsilon_1(a)} \nu_{2,\epsilon_2(a)} \dots \nu_{n,\epsilon_n(a)} \prod_{i=1}^{n} \frac{1}{\lambda_i + \mu_i}$$
is the stationary distribution of the Markov chain Y_n.

Proof: It is clear that π_a is non-negative for every $a \in I$. The fact that
$$\sum_{a \in I} \pi_a = 1$$
can be proved by induction. The proof is complete if we can show that
$$\pi_a q_{ab} = \pi_b q_{ba} \quad \text{for all } a, b \in I.$$

If a and b have two or more different entries then

$$q_{ab} = q_{ba} = 0.$$

Now suppose that a and b have only one different entry, say at jth entry. In this case

$$\nu_{i,\epsilon_i(a)} = \nu_{j,\epsilon_j(b)} \quad \text{for all } i \neq j$$

and if $\nu_{j,\epsilon_j(a)} = \lambda_j$ then $\nu_{j,\epsilon_j(b)} = \mu_j$ and vice versa. The entries π_a and π_b of π are given by

$$\pi_a = C\nu_{j,\epsilon_j(a)}$$

and

$$\pi_b = C\nu_{j,\epsilon_j(b)}$$

where

$$C = \nu_{1,\epsilon_1(a)} \ldots \nu_{j-1,\epsilon_{j-1}(a)}\nu_{j+1,\epsilon_{j+1}(a)} \ldots \nu_{n,\epsilon_n(a)} \prod_{i=1}^{n} \frac{1}{\lambda_i + \nu_i}.$$

It follows that

$$\begin{aligned} \pi_a q_{ab} &= C\nu_{j,\epsilon_j(a)}\nu_{j,\epsilon_j(b)} \\ &= C\nu_{j,\epsilon_j(b)}\nu_{j,\epsilon_j(a)} \\ &= \pi_b q_{ba}. \quad \square \end{aligned}$$

As a consequence of the fact that the Markov chain Y_n has the stationary distribution π, we have the following proposition:

Proposition 4.7.1 *Let $U(t)$ be the total uptime of a series system with n stochastically independent components. Suppose that the ith, $i = 1, 2, \ldots, n$ component has up and down times which are independent and exponentially distributed with parameter λ_i and μ_i respectively. Then*

(a)

$$\mathbf{E}^{\pi}[U(t)] = \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i + \mu_i} t,$$

where $\mathbf{E}^{\pi}$ denotes the expectation under initial distribution the stationary distribution π,

(b) with probability 1,

$$\frac{U(t)}{t} \longrightarrow \prod_{i=1}^{n} \frac{\mu_i}{\lambda_i + \mu_i}, \quad \text{as} \quad t \longrightarrow \infty.$$

Proof: Use the fact that if the initial distribution of the Markov chain Y_n is its stationary distribution then the distribution of the chain at any time equals the stationary distribution, and the fact that the fraction of time the chain in a state approximately equals the stationary distributions at that state, see e.g. Wolff [49]. □

Next we will derive the variance of $U(t)$ for the case $n = 2$. For the bigger n the calculations become more complicated. For $n = 2$ the Markov chain Y_2 has the state space $I = \{00, 01, 10, 11\}$ and stationary distribution

$$\pi = (\pi_a)_{a \in I} = \frac{1}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)} (\lambda_1\lambda_2 \quad \lambda_1\mu_2 \quad \mu_1\lambda_2 \quad \mu_1\mu_2).$$

The transition matrix of the Markov chain Y_2 can be calculated explicitly:

$$P(t) \quad = \quad \frac{P_1 + P_2 e^{-(\lambda_1+\mu_1)t} + P_3 e^{-(\lambda_2+\mu_2)t} + P_4 e^{-(\lambda_1+\mu_1+\lambda_2+\mu_2)t}}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

where

$$P_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \pi,$$

$$P_2 = \begin{pmatrix} \mu_1\lambda_2 & \mu_1\mu_2 & -\mu_1\lambda_2 & -\mu_1\mu_2 \\ \mu_1\lambda_2 & \mu_1\mu_2 & -\mu_1\lambda_2 & -\mu_1\mu_2 \\ -\lambda_1\lambda_2 & -\lambda_1\mu_2 & \lambda_1\lambda_2 & \lambda_1\mu_2 \\ -\lambda_1\lambda_2 & -\lambda_1\mu_2 & \lambda_1\lambda_2 & \lambda_1\mu_2 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} \lambda_1\mu_2 & -\lambda_1\mu_2 & \lambda_1\mu_2 & -\mu_1\mu_2 \\ -\lambda_1\lambda_2 & \lambda_1\lambda_2 & -\mu_1\lambda_2 & \mu_1\lambda_2 \\ \lambda_1\mu_2 & -\lambda_1\mu_2 & \lambda_1\mu_2 & -\mu_1\mu_2 \\ -\lambda_1\lambda_2 & \lambda_1\lambda_2 & -\mu_1\lambda_2 & \mu_1\lambda_2 \end{pmatrix},$$

and

$$P_4 = \begin{pmatrix} \mu_1\mu_2 & -\mu_1\mu_2 & -\mu_1\mu_2 & \mu_1\mu_2 \\ -\mu_1\lambda_2 & \mu_1\lambda_2 & \mu_1\lambda_2 & -\mu_1\lambda_2 \\ -\lambda_1\mu_2 & \lambda_1\mu_2 & \lambda_1\mu_2 & -\lambda_1\mu_2 \\ \lambda_1\lambda_2 & -\lambda_1\lambda_2 & -\lambda_1\lambda_2 & \lambda_1\lambda_2 \end{pmatrix}.$$

In particular the transition probability from state 11 at time 0 into state 11 at time t is given by

$$\begin{aligned} P_{11,11}(t) \quad = \quad & \frac{1}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)} \Big[\mu_1\mu_2 + \lambda_1\mu_2 e^{-(\lambda 1+\mu_1)t} + \mu_1\lambda_2 e^{-(\lambda 2+\mu_2)t} \\ & +\lambda_1\lambda_2 e^{-(\lambda 1+\lambda_2+\mu_1+\mu_2)t}\Big]. \end{aligned} \qquad (4.43)$$

Under the stationary distribution we have

$$\mathbf{E}^{\pi}[U(t)] = \pi_{11}t = \frac{\mu_1\mu_2 t}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}.$$

For the second moment of $U(t)$, under the stationary distribution,

$$\begin{aligned}
\mathbf{E}^{\pi}[U^2(t)] &= \mathbf{E}^{\pi}\left[\int_0^t 1_{\{11\}}Y_2(s)ds\right]^2 \\
&= 2\int_0^t\int_0^s \mathbf{P}^{\pi}(Y(r)=11, Y(s)=11)drds \\
&= 2\frac{\mu_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}\int_0^t\int_0^s P_{11,11}(s-r)drds.
\end{aligned}$$

Using (4.43) we obtain

$$\mathbf{E}^{\pi}[U^2(t)] = (\pi_{11}t)^2 + \sigma^2 t + r(t)$$

where

$$\sigma^2 = \frac{2\mu_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}\left[\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2+\mu_1+\mu_2)^2} + \frac{\lambda_1\mu_2}{\lambda_1+\mu_1} + \frac{\mu_1\lambda_2}{\lambda_2+\mu_2}\right]$$

and

$$r(t) = \frac{\lambda_1\lambda_2[e^{-(\lambda_1+\lambda_2+\mu_1+\mu_2)t}-1]}{(\lambda_1+\lambda_2+\mu_1+\mu_2)^2} + \frac{\lambda_1\mu_2[e^{-(\lambda_1+\mu_1)t}-1]}{(\lambda_1+\mu_1)^2} + \frac{\mu_1\lambda_2[e^{-(\lambda_2+\mu_2)t}-1]}{(\lambda_2+\mu_2)^2}.$$

It follows that the variance of $U(t)$, under the stationary distribution,

$$\mathrm{Var}^{\pi}[U(t)] = \sigma^2 t + r(t).$$

From this expression we conclude

$$\lim_{t\to\infty}\frac{\mathrm{Var}^{\pi}[U(t)]}{t} = \sigma^2.$$

This limit is also valid under any initial distribution of the Markov chain Y_2, and can be obtained using a formula (8.11) of Iosifescu [21]. Moreover, using another formula on page 256 of Iosifescu, under any initial distribution of the Markov chain Y_n, the limiting distribution of $U(t)$ is normal, i.e.,

$$\frac{U(t)-\pi_{11}t}{\sigma\sqrt{t}} \xrightarrow{d} N(0,1) \quad \text{as} \quad t\to\infty.$$

Now we will consider the probability distribution of $U(t)$. Define for $\alpha, \beta > 0$ and $a \in I$

$$\hat{\psi}_a(\alpha,\beta) := \int_0^{\infty}\mathbf{E}^a[e^{-\alpha U(t)}]e^{-\beta t}dt$$

and starting from $t = 0$ in state a

$$\tau_a := \inf\{t \geq 0 : Y_n(t) \neq a\}.$$

Starting from $\mathbf{1_n}$, the random variable $\tau_{\mathbf{1_n}}$ is the time at which the chain leaves the state $\mathbf{1_n}$ and

$$\mathbf{P}^{\mathbf{1_n}}(\tau_{\mathbf{1_n}} > t) = e^{-\sum_{i=1}^n \lambda_i t}.$$

Conditioning on $T_{\mathbf{1_n}}$ we obtain the system equations

$$\begin{cases} \hat{\psi}_{\mathbf{1_n}}(\alpha,\beta) & = & \frac{1}{\alpha+\beta+\sum_{i=1}^n \lambda_i}[1 + \sum_{b \neq \mathbf{1_n}} q_{\mathbf{1_n b}} \hat{\psi}_b(\alpha,\beta)] \\ \hat{\psi}_a(\alpha,\beta) & = & \frac{1}{\beta+q_a}[1 + \sum_{b\neq a} q_{ab}\hat{\psi}_b(\alpha,\beta)] \quad \text{if } a \neq \mathbf{1_n} \end{cases}$$

where $q_a = \sum_{i=1}^n \nu_{i,1-\epsilon_i(a)}$ whenever $a = (\epsilon_1(a), \epsilon_2(a), \ldots, \epsilon_n(a))$. Solving this system equations we get the double Laplace transform of $U(t)$. As an example for $n = 2$, the solution for $\hat{\psi}_{11}(\alpha,\beta)$ is given by

$$\hat{\psi}_{11}(\alpha,\beta) = \frac{ABC + D\beta + C\beta^2 + \beta^3}{\mu_1\mu 2C\alpha + (ABC + E\alpha)\beta + (D + F\alpha)\beta^2 + (2C + \alpha)\beta^3 + \beta^4} \tag{4.44}$$

where $A = \lambda_1 + \mu_1$, $B = \lambda_2 + \mu_2$, $C = A + B$, $D = A^2 + B^2 + 3AB$, $E = AB + \mu_1 A + \mu_2 B + 2\mu_1\mu_2$ and $F = \mu_1 + \mu_2 + C$. Note that the left-hand side of (4.44) can be written as

$$\int_0^\infty \int_0^t f(x,t) e^{-(\alpha x + \beta t)} dx dt$$

where $f(x,t)$ is the density function of $U(t)$ at time t. Transforming back the double Laplace transform (4.44) with respect to α, we get

$$\int_0^\infty f(x,t)e^{-\beta t}dt = \frac{(ABC + D\beta + C\beta^2 + \beta^3)}{\mu_1\mu_2 C + E\beta + F\beta^2 + \beta^3} \; e^{-\frac{(ABC\beta + D\beta^2 + 2C\beta^3 + \beta^4)x}{\mu_1\mu_2 C + E\beta + F\beta^2 + \beta^3}}.$$

The probability density function of $U(t)$ can be obtained by inverting numerically this transform.

4.7.2 Arbitrary failure or repair times

In general it is complicated to obtain an explicit expression for the distribution of the total uptime of systems comprising $n \geq 2$ components when the failure or repair times of the components are arbitrarily distributed. In some cases it is possible to derive the expression for the mean of the total uptime.

Consider the series system in the previous subsection. Assume that for each i, the random variables $X_{ij}, j = 1, 2, ...,$ have a common distribution function

F_i, and the random variables $Y_{ij}, j = 1, 2, ...$, have a common distribution function G_i. Denote by F_i^* and G_i^* the Laplace-Stieltjes transforms of F_i and G_i respectively. Let $A_{11}^{(i)}(t)$ be the availability of the ith component at time t. Then from (4.42), the mean of the total uptime of the series system can be formulated as

$$\mathbf{E}[U(t)] = \int_0^t \prod_{i=1}^n A_{11}^{(i)}(s)ds. \tag{4.45}$$

In some cases we have analytic expressions for the availability, which can be obtained by inverting its Laplace transform given by

$$\int_0^\infty A_{11}^{(i)}(t)e^{-\beta t}dt = \frac{1 - F_i^*(\beta)}{\beta[1 - F_i^*(\beta)G_i^*(\beta)]}, \tag{4.46}$$

see (4.13).

As an example let $n = 2$. Suppose that X_{1j} and Y_{1j} are exponentially distributed with parameter 1 and 2 respectively. Suppose also that $X_{2j} \sim$ Gamma(1,2) having a pdf

$$f(x) = xe^{-x}, \quad x \geq 0$$

and $Y_{2j} \sim$ Gamma(2,2). Then using (4.46), we obtain

$$A_{11}^{(1)}(t) = \frac{2}{3} + \frac{1}{3}e^{-3t}$$

and

$$A_{11}^{(2)}(t) = \frac{2}{3} + \frac{1}{12}e^{-3t} + \frac{1}{28}e^{-3t/2}[7\cos(\sqrt{7}t/2) + 5\sqrt{7}\sin(\sqrt{7}t/2)].$$

Using (4.45) we obtain

$$\begin{aligned}
\mathbf{E}[U(t)] &= \frac{4}{9}t + \frac{115}{396} - \frac{1}{216}e^{-6t} - \frac{5}{54}e^{-3t} \\
&\quad - \left[\frac{7}{264}e^{-9t/2} + \frac{1}{6}e^{-3t/2}\right]\cos(\sqrt{7}t/2) \\
&\quad - \left[\frac{19}{1848}e^{-9t/2} + \frac{1}{42}e^{-3t/2}\right]\sqrt{7}\sin(\sqrt{7}t/2).
\end{aligned}$$

Appendix A

The proof of Theorem 2.5.1

In Section 2.5 we have proved that, with probability 1, $N(t) = \mathbb{N}(t)(\Phi)$ where Φ is a Poisson point process having intensity measure $\nu(dsdx) = dsdF(x)$ and

$$\mathbb{N}(t)(\mu) \quad = \quad \int_E \int_E 1_{[0,x)}(t - \mathbb{A}(s,\mu))1_{[0,s)}(u)\mu(dudv)\mu(dsdx), \qquad \text{(A.1)}$$

where

$$\mathbb{A}(s,\mu) = \int_E 1_{[0,s)}(y)z\mu(dydz)$$

and $E = [0,\infty) \times [0,\infty)$. In the sequel we will write $I^n = [0,\infty)^n$ and $M = M_p(E)$, the set of all point measures on E. The distribution of Φ on M is denoted by $\mathbf{P}_\nu$.

The main tools or arguments that we will use in this proof are:

(1) : Fubini's theorem
(2) : substitution
(3) : The Palm formula for Poisson point processes (see Theorem 1.2.4)
(4) : The Laplace functional of Poisson point processes (see Theorem 1.2.1).

We indicate the use of these arguments by writing the corresponding numbers over equality signs. For example the notation $\stackrel{(1,2)}{=}$ means that we have used Fubini's theorem and substitution one or several times. We will also use the following notations: For fix $\alpha, \beta \geq 0$ we put

$$\begin{aligned} P &= 1 - F^*(\alpha) \\ Q &= 1 - F^*(\beta) \\ R &= 1 - F^*(\alpha + \beta) \\ S &= F^*(\alpha) - F^*(\alpha + \beta) \\ T &= F^*(\beta) - F^*(\alpha + \beta) \end{aligned}$$

where F^* denotes the Laplace-Stieltjes transform of F. Note that $P+S=Q+T=R$.

Define for fix $\alpha, \beta \geq 0$

$$L(\alpha,\beta,s,\tilde{s}) := \int_M e^{-\alpha\mathbb{A}(s,\mu)-\beta\mathbb{A}(\tilde{s},\mu)}\mathbf{P}_\nu(d\mu), \quad s,\tilde{s}\geq 0.$$

For $s>\tilde{s}$,

$$\begin{aligned}
&L(\alpha,\beta,s,\tilde{s})\\
&= \int_M \exp\Big\{-\int_E \Big[1_{[0,s)}(y)\alpha z + 1_{[0,\tilde{s})}(y)\beta z\Big]\mu(dydz)\Big\}\mathbf{P}_\nu(d\mu)\\
&\overset{(4)}{=} \exp\Big\{-\int_0^\infty\int_0^\infty \Big[1-e^{1_{[0,s)}(y)\alpha z+1_{[0,\tilde{s})}(y)\beta z}\Big]dF(z)dy\Big\}\\
&= \exp\Big\{-\int_0^{\tilde{s}}\int_0^\infty [1-e^{-(\alpha+\beta)z}]dF(z)dy - \int_{\tilde{s}}^s\int_0^\infty [1-e^{-\alpha z}]dF(z)dy\Big\}\\
&= \exp\Big\{-\tilde{s}[1-F^*(\alpha+\beta)]-(s-\tilde{s})[1-F^*(\alpha)]\Big\}\\
&= \exp\Big\{-s[1-F^*(\alpha)]-\tilde{s}[F^*(\alpha)-F^*(\alpha+\beta)]\Big\}\\
&= e^{-sP-\tilde{s}S}.
\end{aligned}$$

In the sequel we will write $L(\alpha,\beta,s,\tilde{s};s>\tilde{s}) = L(\alpha,\beta,s,\tilde{s})$ when $s>\tilde{s}$. Hence

$$L(\alpha,\beta,s,\tilde{s};s>\tilde{s}) = e^{-sP-\tilde{s}S}.$$

Similarly, it can be proved that for $s<\tilde{s}$

$$L(\alpha,\beta,s,\tilde{s};s<\tilde{s}) = e^{-\tilde{s}Q-sT},$$

and for $s=\tilde{s}$

$$L(\alpha,\beta,s,s) := L(\alpha,\beta,s,\tilde{s};s=\tilde{s}) = e^{-sR}.$$

Now we will calculate the double Laplace transform of $\mathbf{E}[N(t_1)N(t_2)]$. Using (A.1) we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \mathbf{E}[N(t_1)N(t_2)]e^{-\alpha t_1-\beta t_2}dt_1dt_2\\
&= \int_{I^2}\int_M \mathbb{N}(t_1,\mu)\mathbb{N}(t_2,\mu)\mathbf{P}_\nu(d\mu)e^{-\alpha t_1-\beta t_2}dt_1dt_2
\end{aligned}$$

$$\stackrel{(2)}{=} \int_{I^2}\int_M \left[\int_{I^4} 1_{[0,x)}(t_1 - \mathbb{A}(s,\mu))1_{[0,s)}(u)\mu(dudv)\mu(dsdx)\right.$$
$$\left.\int_{I^4} 1_{[0,\tilde{x})}(t_2 - \mathbb{A}(\tilde{s},\mu))1_{[0,\tilde{s})}(\tilde{u})\mu(d\tilde{u}d\tilde{v})\mu(d\tilde{s}d\tilde{x})\right] e^{-\alpha t_1 - \beta t_2}$$
$$\mathbf{P}_\nu(d\mu)dt_1 dt_2$$
$$\stackrel{(1,2,3)}{=} \frac{1}{\alpha}\int_{I^3}\int_M\int_{I^6} 1_{[0,\tilde{x})}(t_2 - \mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}))1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})$$
$$[1-e^{-\alpha x}]e^{-\beta t_2}e^{-\alpha\mathbb{A}(s,\mu)}(\mu+\delta_{(s,x)})(d\tilde{u}d\tilde{v})(\mu+\delta_{(s,x)})(d\tilde{s}d\tilde{x})$$
$$\mu(dudv)\mathbf{P}_\nu(d\mu)dF(x)dsdt_2.$$

Note that this integral can be split into four terms. For one of these terms, the integration over I^6 is with respect to $\delta_{(s,x)}(d\tilde{s}d\tilde{x})\delta_{(s,x)}(d\tilde{u}d\tilde{v})$. This integral equals 0, because the integrand contains the factor $1_{[0,\tilde{s})}(\tilde{u})$ which with respect to these measures integrates to $1_{[0,s)}(s)$. So we only need to calculate the three remaining integrals.

Case 1: The integral with $\mu(d\tilde{s}d\tilde{x})$ and $\delta_{(s,x)}(d\tilde{u}d\tilde{v})$.
In this case

$$T_1 := \frac{1}{\alpha}\int_{I^3}\int_M\int_{I^6} 1_{[0,\tilde{x})}(t_2 - \mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}))1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}]$$
$$e^{-\beta t_2}e^{-\alpha\mathbb{A}(s,\mu)}\delta_{(s,x)}(d\tilde{u}d\tilde{v})\mu(d\tilde{s}d\tilde{x})\mu(dudv)\mathbf{P}_\nu(d\mu)dF(x)dsdt_2$$
$$\stackrel{(1,2)}{=} \frac{1}{\alpha\beta}\int_{I^2}\int_M\int_{I^4} 1_{[0,s)}(u)1_{[0,\tilde{s})}(s)[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}]$$
$$e^{-\alpha\mathbb{A}(s,\mu)-\beta\mathbb{A}(\tilde{s},\mu)-\beta x}\mu(dudv)\mu(d\tilde{s}d\tilde{x})\mathbf{P}_\nu(d\mu)dF(x)ds$$
$$\stackrel{(3)}{=} \frac{T}{\alpha\beta}\int_{I^3}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,\tilde{s})}(s)[1-e^{-\beta\tilde{x}}]e^{-\alpha\mathbb{A}(s,\mu)-\beta\mathbb{A}(\tilde{s},\mu)}$$
$$\mu(dudv)\mathbf{P}_\nu(d\mu)dF(\tilde{x})d\tilde{s}ds$$
$$\stackrel{(3)}{=} \frac{QT}{\alpha\beta}\int_0^\infty\int_s^\infty\int_{I^2}\int_M 1_{[0,s)}(u)e^{-\alpha\mathbb{A}(s,\mu)-\alpha v-\beta\mathbb{A}(\tilde{s},\mu)-\beta v}$$
$$\mathbf{P}_\nu(d\mu)dF(v)dud\tilde{s}ds$$
$$= \frac{QT}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_s^\infty sL(\alpha,\beta,s,\tilde{s},s<\tilde{s})d\tilde{s}ds$$
$$\stackrel{(4)}{=} \frac{QT}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_s^\infty se^{-\tilde{s}Q-sT}d\tilde{s}ds$$
$$= \frac{QT}{\alpha\beta}F^*(\alpha+\beta)\frac{1}{QR^2}$$
$$= \frac{F^*(\alpha+\beta)[F^*(\beta)-F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^2}.$$

Case 2: The integral with $\delta_{(s,x)}(d\tilde{s}d\tilde{x})$ and $\mu(d\tilde{u}d\tilde{v})$.
In this case

$$\begin{aligned}
T_2 \quad := \quad & \frac{1}{\alpha}\int_{I^3}\int_M\int_{I^6} 1_{[0,\tilde{x})}(t_2 - \mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}))1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}] \\
& e^{-\beta t_2}e^{-\alpha\mathbb{A}(s,\mu)}\mu(d\tilde{u}d\tilde{v})\delta_{(s,x)}(d\tilde{s}d\tilde{x})\mu(dudv)\mathbf{P}_\nu(d\mu)dF(x)dsdt_2 \\
= \quad & \frac{1}{\alpha}\int_{I^3}\int_M\int_{I^4} 1_{[0,x)}(t_2 - \mathbb{A}(s,\mu))1_{[0,s)}(u)1_{[0,s)}(\tilde{u})[1-e^{-\alpha x}] \\
& e^{-\alpha\mathbb{A}(s,\mu)}e^{-\beta t_2}\mu(d\tilde{u}d\tilde{v})\mu(dudv)\mathbf{P}_\nu(d\mu)dF(x)dsdt_2 \\
\overset{(1,2,3)}{=} \quad & \frac{1}{\alpha\beta}\int_{I^4}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,s)}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta x}] \\
& e^{-(\alpha+\beta)\mathbb{A}(s,\mu)-(\alpha+\beta)v}(\mu+\delta_{(u,v)})(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(v)dudF(x)ds \\
= \quad & \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\int_{I^2}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,s)}(\tilde{u})e^{-(\alpha+\beta)\mathbb{A}(s,\mu)} \\
& (\mu+\delta_{(u,v)})(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)duds.
\end{aligned}$$

This integral can be split into two terms.

Subcase 21: Using the measure $\mu(d\tilde{u}d\tilde{v})$ in the inner integral.
In this case

$$\begin{aligned}
T_{21} \quad := \quad & \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\int_{I^2}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,s)}(\tilde{u})e^{-(\alpha+\beta)\mathbb{A}(s,\mu)} \\
& \mu(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)duds \\
\overset{(3)}{=} \quad & \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\int_{I^3}\int_M 1_{[0,s)}(\tilde{u})se^{-(\alpha+\beta)\mathbb{A}(s,\mu)-(\alpha+\beta)\tilde{v}} \\
& \mathbf{P}_\nu(d\mu)dF(\tilde{v})d\tilde{u}ds \\
= \quad & \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)^2\int_0^\infty s^2L(\alpha,\beta,s,s)ds \\
\overset{(4)}{=} \quad & \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)^2\int_0^\infty s^2e^{-sR}ds \\
= \quad & \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)^2\frac{2}{R^3} \\
= \quad & \frac{2F^*(\alpha+\beta)^2[1-F^*(\alpha)-F^*(\beta)+F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^3}.
\end{aligned}$$

Subcase 22: Using the measure $\delta_{(u,v)}(d\tilde{u}d\tilde{v})$ in the inner integral.
In this case

$$\begin{aligned}
T_{22} &:= \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\int_{I^2}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,s)}(\tilde{u})e^{-(\alpha+\beta)\mathbb{A}(s,\mu)} \\
&\quad \delta_{(u,v)}(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)duds \\
&= \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\int_{I^2} 1_{[0,s)}(u)1_{[0,s)}(u)L(\alpha,\beta,s,s)duds \\
&\overset{(4)}{=} \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty se^{-sR}ds \\
&= \frac{P-T}{\alpha\beta}F^*(\alpha+\beta)\frac{1}{R^2} \\
&= \frac{F^*(\alpha+\beta)[1-F^*(\alpha)-F^*(\beta)+F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
T_2 &:= T_{21}+T_{22} \\
&= \frac{F^*(\alpha+\beta)[1+F^*(\alpha+\beta)][1-F^*(\alpha)-F^*(\beta)+F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^3}.
\end{aligned}$$

Case 3: The integral with $\mu(d\tilde{s}d\tilde{x})$ and $\mu(d\tilde{u}d\tilde{v})$.
In this case

$$\begin{aligned}
T_3 &:= \frac{1}{\alpha}\int_{I^3}\int_M\int_{I^6} 1_{[0,\tilde{x})}(t_2-\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)})) \\
&\quad 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}]e^{-\beta t_2}e^{-\alpha\mathbb{A}(s,\mu)} \\
&\quad \mu(d\tilde{u}d\tilde{v})\mu(d\tilde{s}d\tilde{x})\mu(dudv)\mathbf{P}_\nu(d\mu)dF(x)dsdt_2 \\
&\overset{(1,2)}{=} \frac{1}{\alpha\beta}\int_{I^2}\int_M\int_{I^6} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}]e^{-\alpha\mathbb{A}(s,\mu)} \\
&\quad e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)})}\mu(d\tilde{u}d\tilde{v})\mu(dudv)\mu(d\tilde{s}d\tilde{x})\mathbf{P}_\nu(d\mu)dF(x)ds \\
&\overset{(3)}{=} \frac{1}{\alpha\beta}\int_{I^4}\int_M\int_{I^4} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})}e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)})} \\
&\quad \mu(d\tilde{u}d\tilde{v})(\mu+\delta_{(\tilde{s},\tilde{x})})(dudv)\mathbf{P}_\nu(d\mu)dF(\tilde{x})d\tilde{s}dF(x)ds.
\end{aligned}$$

We can be split this integral into two terms.

Subcase 31: Using the measure $\delta_{(\tilde{s},\tilde{x})}(dudv)$.
In this case

$$\begin{aligned}
T_{31} &:= \frac{1}{\alpha\beta}\int_{I^4}\int_M\int_{I^4} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}]e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})} \\
&\quad e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)})}\mu(d\tilde{u}d\tilde{v})\delta_{(\tilde{s},\tilde{x})}(dudv)\mathbf{P}_\nu(d\mu)dF(\tilde{x})d\tilde{s}dF(x)ds \\
&= \frac{1}{\alpha\beta}\int_{I^4}\int_M\int_{I^2} 1_{[0,s)}(\tilde{s})1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu)-\alpha\tilde{x}}e^{-\beta\mathbb{A}(\tilde{s},\mu)}\mu(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(\tilde{x})d\tilde{s}dF(x)ds \\
&\overset{(3)}{=} \frac{PS}{\alpha\beta}\int_0^\infty\int_0^s\int_{I^2}\int_M 1_{[0,\tilde{s})}(\tilde{u})e^{-\alpha\mathbb{A}(s,\mu)-\alpha\tilde{v}}e^{-\beta\mathbb{A}(\tilde{s},\mu)-\beta\tilde{v}} \\
&\quad \mathbf{P}_\nu(d\mu)dF(\tilde{v})d\tilde{u}d\tilde{s}ds \\
&= \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_0^s \tilde{s}L(\alpha,\beta,s,\tilde{s};s>\tilde{s})d\tilde{s}ds \\
&\overset{(4)}{=} \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_0^s \tilde{s}e^{-sP-\tilde{s}S}d\tilde{s}ds \\
&= \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\frac{1}{PR^2} \\
&= \frac{F^*(\alpha+\beta)[F^*(\alpha)-F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^2}.
\end{aligned}$$

Subcase 32: Using the measure $\mu(dudv)$.
In this case

$$\begin{aligned}
T_{32} &:= \frac{1}{\alpha\beta}\int_{I^4}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}]e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})} \\
&\quad e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)})}\mu(dudv)\int_E \mu(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(\tilde{x})d\tilde{s}dF(x)ds \\
&\overset{(3)}{=} \frac{1}{\alpha\beta}\int_{I^6}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}+\delta_{(u,v)})} \\
&\quad (\mu+\delta_{(u,v)})(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(v)dudF(\tilde{x})d\tilde{s}dF(x)ds.
\end{aligned}$$

This integral can be split into two terms.
Sub-subcase 321: Using the measure $\mu(d\tilde{u}d\tilde{v})$ in the inner integral.
In this case

$$\begin{aligned}
T_{321} &:= \frac{1}{\alpha\beta}\int_{I^6}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}+\delta_{(u,v)})} \\
&\quad \mu(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(v)dudF(\tilde{x})d\tilde{s}dF(x)ds \\
&= A+B
\end{aligned}$$

where

$$
\begin{aligned}
A &= \frac{1}{\alpha\beta}\int_{I^2}\int_0^s\int_{I^3}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}+\delta_{(u,v)})} \\
&\quad \mu(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(v)dudF(\tilde{x})d\tilde{s}dF(x)ds \\
&\overset{(3)}{=} \frac{1}{\alpha\beta}\int_{I^2}\int_0^s\int_{I^5}\int_M 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu)-\alpha\tilde{v}-\alpha\tilde{x}-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(u,v)})-\beta\tilde{v}} \\
&\quad \mathbf{P}_\nu(d\mu)dF(\tilde{v})d\tilde{u}dF(v)dudF(\tilde{x})d\tilde{s}dF(x)ds \\
&= \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_0^s\int_0^{\tilde{s}}\int_0^\infty\int_M \tilde{s}e^{-\alpha\mathbb{A}(s,\mu)-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu)-\beta v} \\
&\quad \mathbf{P}_\nu(d\mu)dF(v)dud\tilde{s}ds \\
&+ \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_0^s\int_{\tilde{s}}^s\int_0^\infty\int_M \tilde{s}e^{-\alpha\mathbb{A}(s,\mu)-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu)} \\
&\quad \mathbf{P}_\nu(d\mu)dF(v)dud\tilde{s}ds \\
&= \frac{PS}{\alpha\beta}F^*(\alpha+\beta)^2\int_0^\infty\int_0^s \tilde{s}^2L(\alpha,\beta,s,\tilde{s};s>\tilde{s})d\tilde{s}ds \\
&+ \frac{PS}{\alpha\beta}F^*(\alpha)F^*(\alpha+\beta)\int_0^\infty\int_0^s \tilde{s}(s-\tilde{s})L(\alpha,\beta,s,\tilde{s};s>\tilde{s})d\tilde{s}ds \\
&\overset{(4)}{=} \frac{PS}{\alpha\beta}F^*(\alpha+\beta)^2\int_0^\infty\int_0^s \tilde{s}^2e^{-sP-\tilde{s}S}d\tilde{s}ds \\
&+ \frac{PS}{\alpha\beta}F^*(\alpha)F^*(\alpha+\beta)\int_0^\infty\int_0^s \tilde{s}(s-\tilde{s})e^{-sP-\tilde{s}S}d\tilde{s}ds \\
&= \frac{PS}{\alpha\beta}F^*(\alpha+\beta)^2\frac{2}{PR^3}+\frac{PS}{\alpha\beta}F^*(\alpha)F^*(\alpha+\beta)\frac{1}{P^2R^2} \\
&= \frac{F^*(\alpha+\beta)[F^*(\alpha)-F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^2}\left[\frac{2F^*(\alpha+\beta)}{1-F^*(\alpha+\beta)}+\frac{F^*(\alpha)}{1-F^*(\alpha)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
B &= \frac{1}{\alpha\beta}\int_{I^2}\int_s^\infty\int_{I^3}\int_M\int_{I^2} 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}+\delta_{(u,v)})-\beta x} \\
&\quad \mu(d\tilde{u}d\tilde{v})\mathbf{P}_\nu(d\mu)dF(v)dudF(\tilde{x})d\tilde{s}dF(x)ds \\
&\overset{(3)}{=} \frac{1}{\alpha\beta}\int_{I^2}\int_s^\infty\int_{I^5}\int_M 1_{[0,s)}(u)1_{[0,\tilde{s})}(\tilde{u})[1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{u},\tilde{v})})-\alpha v}e^{-\beta\mathbb{A}(\tilde{s},\mu)-\beta x-\beta v-\beta\tilde{v}} \\
&\quad \mathbf{P}_\nu(d\mu)dF(\tilde{v})d\tilde{u}dF(v)dudF(\tilde{x})d\tilde{s}dF(x)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{QT}{\alpha\beta} F^*(\alpha+\beta) \int_0^\infty \int_s^\infty \int_0^\infty \int_0^s \int_0^\infty \int_M s e^{-\alpha \mathbb{A}(s,\mu)-\alpha\tilde{v}} e^{-\beta\mathbb{A}(\tilde{s},\mu)-\beta\tilde{v}} \\
&\quad \mathbf{P}_\nu(d\mu) dF(\tilde{v}) d\tilde{u} dF(v) d\tilde{s} ds \\
&+ \frac{QT}{\alpha\beta} F^*(\alpha+\beta) \int_0^\infty \int_s^\infty \int_0^\infty \int_s^{\tilde{s}} \int_0^\infty \int_M s e^{-\alpha \mathbb{A}(s,\mu)} e^{-\beta\mathbb{A}(\tilde{s},\mu)-\beta\tilde{v}} \\
&\quad \mathbf{P}_\nu(d\mu) dF(\tilde{v}) d\tilde{u} dF(v) d\tilde{s} ds \\
&= \frac{QT}{\alpha\beta} F^*(\alpha+\beta)^2 \int_0^\infty \int_s^\infty s^2 L(\alpha,\beta,s,\tilde{s}; s<\tilde{s}) d\tilde{s} ds \\
&+ \frac{QT}{\alpha\beta} F^*(\beta) F^*(\alpha+\beta) \int_0^\infty \int_s^\infty s(\tilde{s}-s) L(\alpha,\beta,s,\tilde{s}; s<\tilde{s}) d\tilde{s} \\
&\overset{(4)}{=} \frac{QT}{\alpha\beta} F^*(\alpha+\beta)^2 \int_0^\infty \int_s^\infty s^2 e^{-\tilde{s}Q - sT} d\tilde{s} ds \\
&+ \frac{QT}{\alpha\beta} F^*(\beta) F^*(\alpha+\beta) \int_0^\infty \int_s^\infty s(\tilde{s}-s) e^{-\tilde{s}Q - sT} d\tilde{s} ds \\
&= \frac{QT}{\alpha\beta} F^*(\alpha+\beta)^2 \frac{2}{QR^3} + \frac{QT}{\alpha\beta} F^*(\beta) F^*(\alpha+\beta) \frac{1}{Q^2R^2} \\
&= \frac{F^*(\alpha+\beta)[F^*(\beta)-F^*(\alpha+\beta)]}{\alpha\beta[1-F^*(\alpha+\beta)]^2} \left[\frac{2F^*(\alpha+\beta)}{1-F^*(\alpha+\beta)} + \frac{F^*(\beta)}{1-F^*(\beta)} \right].
\end{aligned}
$$

Sub-subcase 322: Using the measure $\mu(d\tilde{u}d\tilde{v})$ in the inner integral.
In this case

$$
\begin{aligned}
T_{322} &:= \frac{1}{\alpha\beta} \int_{I^6} \int_M \int_{I^2} 1_{[0,s)}(u) 1_{[0,\tilde{s})}(\tilde{u}) [1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu+\delta_{(\tilde{s},\tilde{x})})-\alpha v} e^{-\beta\mathbb{A}(\tilde{s},\mu+\delta_{(s,x)}+\delta_{(u,v)})} \\
&\quad \delta_{(u,v)}(d\tilde{u}d\tilde{v}) \mathbf{P}_\nu(d\mu) dF(v) du dF(\tilde{x}) d\tilde{s} dF(x) ds \\
&= \frac{1}{\alpha\beta} F^*(\alpha+\beta) \int_{I^2} \int_0^s \int_0^\infty \int_0^{\tilde{s}} du \int_M [1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu)-\alpha\tilde{x}} e^{-\beta\mathbb{A}(\tilde{s},\mu)} \mathbf{P}_\nu(d\mu) du dF(\tilde{x}) d\tilde{s} dF(x) ds \\
&+ \frac{1}{\alpha\beta} F^*(\alpha+\beta) \int_{I^2} \int_s^\infty \int_0^\infty \int_0^s du \int_M [1-e^{-\alpha x}][1-e^{-\beta\tilde{x}}] \\
&\quad e^{-\alpha\mathbb{A}(s,\mu)} e^{-\beta\mathbb{A}(\tilde{s},\mu)-\beta x} \mathbf{P}_\nu(d\mu) du dF(\tilde{x}) d\tilde{s} dF(x) ds \\
&= \frac{PS}{\alpha\beta} F^*(\alpha+\beta) \int_0^\infty \int_0^s \tilde{s} L(\alpha,\beta,s,\tilde{s}; s>\tilde{s}) d\tilde{s} ds \\
&+ \frac{QT}{\alpha\beta} F^*(\alpha+\beta) \int_0^\infty \int_s^\infty s L(\alpha,\beta,s,\tilde{s}; s<\tilde{s}) d\tilde{s} ds
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(4)}{=} \quad \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_0^s \tilde{s}e^{-sP-\tilde{s}S}d\tilde{s}ds \\
&+ \quad \frac{QT}{\alpha\beta}F^*(\alpha+\beta)\int_0^\infty\int_s^\infty se^{-\tilde{s}Q-sT}d\tilde{s}ds \\
&= \quad \frac{PS}{\alpha\beta}F^*(\alpha+\beta)\frac{1}{PR^2}+\frac{QT}{\alpha\beta}F^*(\alpha+\beta)\frac{1}{QR^2} \\
&= \quad \frac{F^*(\alpha+\beta)}{\alpha\beta[1-F^*(\alpha+\beta)]}\Big[F^*(\alpha)-F^*(\alpha+\beta)+F^*(\beta)-F^*(\alpha+\beta)\Big].
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
T_3 &= T_{31}+A+B+T_{322} \\
&= \frac{F^*(\alpha+\beta)}{\alpha\beta[1-F^*(\alpha+\beta)]}\left\{\frac{[1+F^*(\alpha+\beta)][F^*(\alpha)+F^*(\beta)-2F^*(\alpha+\beta)]}{1-F^*(\alpha+\beta)}\right. \\
&\qquad \left.\frac{F^*(\alpha)[F^*(\alpha)-F^*(\alpha+\beta)]}{1-F^*(\alpha)}+\frac{F^*(\beta)[F^*(\beta)-F^*(\alpha+\beta)]}{1-F^*(\beta)}\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \mathbf{E}[N(t_1)N(t_2)]e^{-\alpha t_1-\beta t_2}dt_1dt_2 \\
&\quad = \quad T_1+T_2+T_3 \\
&\quad = \quad \frac{[1-F^*(\alpha)F^*(\beta)]F^*(\alpha+\beta)}{\alpha\beta[1-F^*(\alpha)][1-F^*(\beta)][1-F^*(\alpha+\beta)]}. \quad \square
\end{aligned}
$$

Appendix B

Numerical inversions of Laplace transforms

B.1 Single Laplace transform

Let f be a real-valued function defined on the positive half-line. The Laplace transform of f is defined to be

$$\hat{f}(\beta) = \int_0^\infty f(t)e^{-\beta t}dt, \tag{B.1}$$

where β is a complex variable, whenever this integral exists. Given $\hat{f}$, we can retrieve the original function f using the following inversion formula:

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{t\beta}\hat{f}(\beta)d\beta \\ &= \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} e^{itu}\hat{f}(a+iu)du. \end{aligned} \tag{B.2}$$

where a is a real number chosen such that $\hat{f}(\beta)$ has no singularity on or to the right of the vertical line $\beta = a$, see e.g. Abate and Whitt [1]. For some Laplace transforms $\hat{f}$ we have analytic expressions for f, a table for these is available, see for example Oberhettinger [28]. When the transform cannot be inverted analytically, we can approximate the function f numerically. Several numerical inversion algorithms have been proposed by several authors, see for example Abate and Whitt [1], Weeks [47] and Iseger [22]. Following Abate and Whitt, we will use the trapezoidal rule to approximate the integral in (B.2) and analyze the corresponding discretization error using the *Poisson summation formula.*

The trapezoidal rule approximates the integral of a function g over the bounded interval $[c,d]$ by the integral of the piecewise linear function obtained by connecting the $n+1$ evenly spaced points $g(c+kh)$, $0 \le k \le n$ where $h=(d-c)/n$, i.e.,

$$\int_c^d g(x)dx \approx h\left[\frac{g(c)+g(d)}{2}+\sum_{k=1}^{n-1} g(c+kh)\right],$$

see Davis and Rabinowitz [9]. In case $c=-\infty$ and $d=\infty$ we approximate the integral of g over the real line as

$$\int_{-\infty}^{\infty} g(x)dx \approx h_1 \sum_{k=-\infty}^{\infty} g(kh_1) \tag{B.3}$$

where h_1 is a small positive constant. This formula can also be obtained using the trapezoidal rule with obvious modifications.

Applying (B.3) to (B.2) with step size $h_1=\pi/t$, $t>0$, and letting $a=A/t$ at the same time, we get

$$f(t) \approx \frac{e^A}{2t} \sum_{k=-\infty}^{\infty} (-1)^k \hat{f}([A+i\pi k]/t). \tag{B.4}$$

This approximation can also be obtained by using the Poisson summation formula: For an integrable function g

$$\sum_{k=-\infty}^{\infty} g(t+2\pi k/h_2) = \frac{h_2}{2\pi} \sum_{k=-\infty}^{\infty} \varphi(kh_2)e^{-ih_2tk} \tag{B.5}$$

where h_2 is some positive constant and $\varphi(u)=\int_{-\infty}^{\infty} g(x)e^{iux}dx$, the Fourier transform of g. Taking $g(x)=e^{-a_1x}f(x)1_{[0,\infty)}(x)$ in (B.5) where a_1 is chosen such that the function g is integrable, we obtain

$$\sum_{k=0}^{\infty} e^{-a_1(t+2\pi k/h_2)} f(t+2\pi k/h_2) \quad = \quad \frac{h_2}{2\pi} \sum_{k=-\infty}^{\infty} \hat{f}(a_1-ikh_2)e^{-ih_2tk} \tag{B.6}$$

where $\hat{f}$ is the Laplace transform of f, see (B.1). Letting $a_1=A/t$ and $h_2=\pi/t$ in (B.6) we obtain

$$f(t) \quad = \quad \frac{e^A}{2t} \sum_{k=-\infty}^{\infty} (-1)^k \hat{f}([A+i\pi k]/t) - e_d \tag{B.7}$$

where

$$e_d = \sum_{k=1}^{\infty} e^{-2kA} f([2k+1]t).$$

Comparing (B.4) and (B.7), we conclude that e_d is an explicit expression for the discretization error associated with the trapezoidal rule approximation. This discretization error can easily be bounded whenever f is bounded. For example if $|f(x)| \le C$ then $|e_d| \le Ce^{-2A}/(1-e^{-2A})$, and if $|f(x)| \le Cx$ then

$$|e_d| \le \frac{(3e^{-2A} - e^{-4A})Cx}{(1-e^{-2A})^2}.$$

We used (B.4) to invert numerically Laplace transforms in this thesis. Note that the formula (B.4) can be expressed as

$$f(t) \quad \approx \quad \frac{e^A}{2t}\hat{f}(A/t) + \frac{e^A}{t}\sum_{k=1}^{\infty}(-1)^k \Re(\hat{f}([A+i\pi k]/t)) \tag{B.8}$$

by using the fact that $\hat{f}(\beta) + \hat{f}(\bar{\beta}) = 2\Re(\hat{f}(\beta))$, where $\bar{\beta}$ and $\Re(\beta)$ denote the complex conjugate and real part of β respectively.

Below we give a numerical-inversion example of Laplace transform of the expected value of the instantaneous reward process in Chapter 2. We write the programs in Matlab.

Example B.1.1 *In Subsection 2.2.1 equation* (2.12) *we have the Laplace transform of the expected value of an instantaneous reward process:*

$$\int_0^{\infty} \mathbf{E}[R_\phi(t)]e^{-\beta t}dt = \frac{(\beta+\lambda)e^{-2(\beta+\lambda)}}{\beta^2[1-e^{-2(\beta+\lambda)}]}. \tag{B.9}$$

Denote the right-hand side of (B.9) *by* $\hat{f}(\beta)$*. Using* (B.8) *we get*

$$\mathbf{E}[R_\phi(t)] \approx \frac{e^A}{2t}\hat{f}(A/t) + \frac{e^A}{t}\sum_{k=1}^{M}(-1)^k \Re(\hat{f}([A+i\pi k]/t)).$$

The following is a Matlab program for approximating $\mathbf{E}[R_\phi(t)]$ *for* $\lambda = 0.1741$.

```
function [f]=expmean(t,M)

A=5;
P=exp(A)/(2*t);
B=A/t;
C=i*pi/t;
lambda=0.1741;
D=B+lambda;
```

```
T1=P*D*exp(-2*D)/(B^2*(1-exp(-2*D)));

m=0;
for k=1:M
   beta=B+C*k;
   U=(beta+lambda)*exp(-2*(beta+lambda));
   V=beta^2*(1-exp(-2*(beta+lambda)));
   R=real(U/V);
   m=m+(-1)^k*R;
end

T2=2*P*m;
f=T1+T2; %f=E[R_\phi(t)].
```

B.2 Double Laplace transform

This section concerns a generalization of the result in the previous section. We refer to Choudhury *at al.* [5] with some modifications. Let $f(t_1, t_2)$ be a real-valued function of non-negative real variables t_1 and t_2. Denote its double Laplace transform by

$$\hat{f}(\alpha, \beta) = \int_0^\infty \int_0^\infty f(t_1, t_2) e^{-(\alpha t_1 + \beta t_2)} dt_1 dt_2$$

where α and β are complex variables, provided the integral exists. To retrieve numerically the function f from $\hat{f}$, we use the two-dimensional Poisson summation formula: For a real-valued function g on $\mathbb{R}^2$,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} & g(t_1 + 2\pi j/h_1, t_2 + 2\pi k/h_2) \\ = & \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{h_1 h_2}{4\pi^2} \varphi(jh_1, kh_2) e^{-i(h_1 t_1 j + h_2 t_2 k)}, \end{aligned} \tag{B.10}$$

where φ denotes the bivariate Fourier transform of g, i.e.,

$$\varphi(u, v) = \int_{-\infty}^\infty \int_{-\infty}^\infty g(x, y) e^{i(xu + yv)} dx dy,$$

Taking $g(x,y) = f(x,y)e^{-(a_1x+a_2y)}$ when $x, y \geq 0$ and $g(x,y) = 0$ otherwise in (B.10), we obtain

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-[a_1(t_1+2\pi j/h_1)+a_2(t_2+2\pi k/h_2)]} f(t_1 + 2\pi j/h_1, t_2 + 2\pi k/h_2)$$
$$= \frac{h_1 h_2}{4\pi^2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{f}(a_1 - ijh_1, a_2 - ikh_2)e^{-i(h_1t_1j+h_2t_2k)}.$$

Letting $h_1 = \pi/t_1$, $h_2 = \pi/t_2$, $a_1 = A_1/t_1$, $a_2 = A_2/t_2$, we obtain, after some simplifications,

$$f(t_1,t_2) = \frac{e^{A_1+A_2}}{4t_1t_2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-1)^{j+k} \hat{f}([A_1 - i\pi j]/t_1, [A_2 - i\pi k]/t_2) - e_d \tag{B.11}$$

where

$$\begin{aligned} e_d &= \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} e^{-(A_1j+A_2k)} f([2j+1]t_1, [2k+1]t_2) \\ &+ \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} e^{-(A_1j+A_2k)} f([2j+1]t_1, [2k+1]t_2). \end{aligned}$$

If $|f(t_1,t_2)| \leq C$ for some constant C, then the error e_d can be bounded:

$$|e_d| \leq \frac{C(e^{-A_1} + e^{-A_2} - e^{-(A_1+A_2)})}{(1 - e^{-A_1})(1 - e^{-A_2})}.$$

By noting that $\hat{f}(\alpha,\beta) + \hat{f}(\bar{\alpha},\bar{\beta}) = 2\Re(\hat{f}(\alpha,\beta))$, from (B.11), for $A_1 = A_2 = A$, we get an approximation for $f(t_1,t_2)$:

$$\begin{aligned} f(t_1,t_2) & \\ \approx \frac{e^{2A}}{4t_1t_2} \Bigg\{ & -\hat{f}(A/t_1) + 2\sum_{j=0}^{M}\sum_{k=0}^{N}(-1)^{j+k}\Re(\hat{f}([A - i\pi j]/t_1, [A - i\pi k]/t_2)) \\ & + 2\sum_{j=1}^{M}\sum_{k=0}^{N}(-1)^{j+k}\Re(\hat{f}([A - i\pi j]/t_1, [A + i\pi k]/t_2)) \Bigg\}. \end{aligned} \tag{B.12}$$

We used this formula to obtain the numerical inversion of the double Laplace transform in (4.40) of the total downtime.

So far we have assumed that the variables t_1 and t_2 are continuous. It is also possible to consider the case where t_1 or t_2 are discrete variables. Choudhury *at al.* [5] have discussed the cases where t_1 or t_2 are non-negative integers. We will discuss the case where one of the variable is discrete and the other is continuous in the following example.

In Chapter 2 we have the following formula: For $\alpha, \beta > 0$,

$$\int_0^\infty \mathbf{E}(e^{-\alpha R_\phi(t)})e^{-\beta t}dt = \frac{\int_0^\infty [1-F(t)]e^{-\alpha\phi(t)-\beta t}dt}{1-\int_0^\infty e^{-\alpha\phi(t)-\beta t}dF(t)} \tag{B.13}$$

where

$$R_\phi(t) = \sum_{n=1}^{N(t)} \phi(X_n) + \phi(t - S_{N(t)})$$

where ϕ is a non-negative measurable function, see (2.4). It can be proved that this formula remains true if we replace α by $i\alpha$.

The random variable $R_\phi(t)$ possibly takes values $x + l\lambda$, $l = 0, \pm1, \pm2, \ldots$, depend on the choice of the function ϕ. In this case, by replacing α with $i\alpha$, we can write the equation (B.13) as

$$\int_0^\infty \sum_{l=0}^\infty \mathbf{P}(R_\phi(t) = x + l\lambda)e^{-i\alpha(x+l\lambda)-\beta t}dt = \frac{\int_0^\infty [1-F(t)]e^{-i\alpha\phi(t)-\beta t}dt}{1-\int_0^\infty e^{-i\alpha\phi(t)-\beta t}dF(t)}.$$

Denote the right-hand side of this equation by $\psi(\alpha, \beta)$. Using the inversion formula for Laplace transforms and discrete Fourier transforms, see Abate and Whitt [1], we obtain

$$\mathbf{P}(R_\phi(t) = x + l\lambda) = \frac{\lambda e^{at}}{4\pi^2}\int_0^{2\pi/\lambda}\int_{-\infty}^\infty \psi(\alpha, a + i\beta)e^{i[(x+l\lambda)\alpha + t\beta]}d\beta d\alpha \tag{B.14}$$

for suitably constant a. Applying the trapezoidal rule twice to (B.14) with step sizes $2\pi/(\lambda n)$ and π/t for the integrals with respect to α and β respectively, and letting $a = A/t$ at the same time, we obtain

$$\begin{aligned}\mathbf{P}&(R_\phi(t) = x + l\lambda)\\ &\approx \frac{e^A}{2nt}\left[\frac{1}{2}\sum_{k=-\infty}^{\infty}(-1)^k\left(\psi(0, (A+i\pi k)/t) + \psi(2\pi/\lambda, (A+i\pi k)/t)e^{i2\pi x/\lambda}\right)\right.\\ &\qquad\left.+\sum_{j=1}^{n-1}\sum_{k=-\infty}^{\infty}(-1)^k\psi(2\pi j/(\lambda n), (A+i\pi k)/t)e^{i2\pi j(l+x/\lambda)/n}\right].\end{aligned} \tag{B.15}$$

We used this formula to approximate the probability mass function of $R_\phi(t)$ in Subsection 2.2.1 as described in the following example.

Example B.2.1 *We have the following transform of $R_\phi(t)$, after replacing α with $i\alpha$,:*

$$\int_0^\infty \mathbf{E}[e^{-i\alpha R_\phi(t)}]e^{-\beta t}dt$$
$$= \frac{1-2e^{-2(\beta+\lambda)}}{(\beta+\lambda)[1-e^{-i\alpha}e^{-2(\beta+\lambda)}]-\lambda[1-e^{-2(\beta+\lambda)}]}, \quad \text{(B.16)}$$

see (2.13). *In this example $R_\phi(t)$ is a non-negative-integer-valued random variable. Denote the right-hand side of* (B.16) *by $\psi(\alpha,\beta)$. Since $\psi(0,(A+i\pi k)/t) = \psi(2\pi,(A+i\pi k)/t)$, using* (B.15) *we get a formula for approximating the pdf of $R_\phi(t)$:*

$$\mathbf{P}(R_\phi(t)=l) \approx \frac{e^A}{2nt}\sum_{j=0}^{n-1}\sum_{k=-M}^{M}(-1)^k\psi(2\pi j/n,(A+i\pi k)/t)e^{i2\pi jl/n}.$$

The following is a Matlab program for approximating the pdf of $R_\phi(t)$ for $\lambda = 0.1741$.

```
function [f]=exppmf(l,t,n,M)

A=5;
P=exp(A)/(2*n*t);
B=A/t;
C=i*pi/t;
h=2*pi/n;
D=i*l*h;
lambda=0.1741;

m=0;
for j=0:(n-1)
   E=exp(D*j);
   alpha=j*h;
   F=exp(-i*alpha);
   for k=-M:M
      beta=B+C*k;
      U1=beta+lambda;
      U2=exp(-2*U1);
      U=1-U2;
      V1=U1*(1-F*U2);
      V2=lambda*U;
      V=V1-V2;
      R=E*U/V;
```

```
      m=m+(-1)^k*R;
   end
end

f=real(P*m); %P(R_\phi(t)=l)
```

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Samenvatting

Vernieuwingsprocessen en repareerbare systemen

In dit proefschrift bespreken we de volgende onderwerpen.

1. **Renewal reward processen**

   Wij geven een berekening van de marginale verdelingen van renewal reward processen en van een klasse van processen die we in dit proefschrift aanduiden als instantaneous reward processen. Onze benadering is gebaseerd op de theorie van puntprocessen, in het bijzonder Poisson puntprocessen. Hierbij wordt het renewal reward proces (instantaneous reward proces) gerepresenteerd als een functionaal van een Poisson punt proces. Belangrijke hulpmiddelen die gebruikt worden bij de afleiding zijn de Palm formule en de formule voor de Laplace getransformeerde van een Poisson puntproces. Als resultaat vinden we de Laplace getransformeerden van de marginale verdelingen. We geven een toepassing van instantaneous reward processen in een verkeersprobleem.

   Een aantal asymptotische eigenschappen van renewal reward processen wordt opnieuw onder de loupe genomen. Zo geven we met behulp van een Tauber stelling een bewijs van de versie van de renewal reward stelling met verwachtingswaarden. We leiden vervolgens een tweede orde term af voor de versie van de renewal reward stelling met verwachtingswaarden. Gelijksoortige resultaten onderzoeken we voor instantaneous reward processen. Verder bewijzen we asymptotische normaliteit voor instantaneous reward processen.

   We berekenen de covariantie structuur van een renewal proces, dat kan worden beschouwd als een bijzonder geval van een renewal reward proces. Verder bestuderen we systeem betrouwbaarheid voor een stress-strength model, waarbij de omvang van de stress geïnterpreteerd wordt als een "reward". De tijdstippen waarop stress optreedt modelleren met behulp van vernieuwingsprocessen en Cox processen. Met de resultaten die we afgeleid hebben voor renewal reward processen onderzoeken we de invloed van de afhankelijkheid tussen stress en strength op de systeem betrouwbaarheid.

2. **Geïntegreerde vernieuwingsprocessen**

   In de literatuur is de kansdichtheid van een geïntegreerd homogeen Poisson proces bekend. Het is natuurlijk om dit te generaliseren voor niet-homogene Poisson processen, Cox processen en vernieuwingsprocessen. In dit proefschrift leiden we formules af voor de marginale verdelingen met behulp van conditionering voor geïntegreerde Poisson en Cox processen. Voor geïntegreerde vernieuwingsprocessen gebruiken we puntproces representaties. De resultaten worden gegeven in de vorm van Laplace getransformeerden. De asymptotische eigenschappen van geïntegreerde vernieuwingsprocessen worden onderzocht. Tenslotte is er een toepassing op een verkeersprobleem.

3. **Totale downtime van reprareerbare systemen**

   Verschillende auteurs hebben met een aantal verschillende methodes formules afgeleid voor verdelingsfunctie van de toatle downtime van een repareerbaar systeem, dat bij deze studies als één component wordt beschouwd. Wij leiden formules af met nog een andere methode (gebaseerd op puntprocessen) en we beschouwen ook het algemenere geval waarbij afhankelijkheid wordt toegelaten tussen faal en reparatie tijden.

   De covariantie structuur en de asymptotische eigenschappen van de totale downtijd zijn voor het onafhankelijke geval bekend in de literatuur. Wij leiden resultaten af voor het afhankelijke geval. We geven voorbeelden van het effect van afhankelijkheid tussen faaltijd en reparatietijd op de totale downtijd.

   We bespreken ook de totale downtijd voor repareerbare systemen die uit twee of meer stochastisch onafhankelijke componenten bestaan. We leiden ee uitdrukking af voor de marginale verdeling van de totale uptijd van het systeem als faaltijden en reparatietijden van iedere component exponentieel verdeeld zijn. Voor willekeurige faal- en reparatietijden geven we een uitdrukking voor de verwachte totale uptijd.

Summary

Renewal processes and repairable systems

In this thesis we discuss the following topics.

1. **Renewal reward processes**

   The marginal distributions of renewal reward processes and its version, which we call in this thesis instantaneous reward processes, are derived. Our approach is based on the theory of point processes, especially Poisson point processes. The idea is to represent the renewal reward processes and its version as functionals of Poisson point processes. Important tools we use are the Palm formula and the Laplace functional of Poisson point processes. The results are presented in the form of Laplace transforms. An application of the instantaneous reward processes to the study of traffic is given.

   Some asymptotic properties of the renewal reward processes are reconsidered. A proof of the expected-value version of the renewal reward theorem using the Tauberian theorem is given. A second order term in the expected-value version of the renewal reward theorem is obtained. Similar results for the instantaneous reward processes are investigated. Asymptotic normality of the instantaneous reward processes is proved.

   The covariance structure of renewal processes, which can be considered as a special case of renewal reward processes, is derived. As an addition, we study system reliability in a stress-strength model, where the amplitudes of stresses can be considered as rewards. We consider renewal and Cox processes as models for the occurrences of the stresses. Using our result on renewal reward processes we investigate the effect of dependence between stress and strengths on system reliability.

2. **Integrated renewal processes**

   The marginal probability density function of an integrated homogeneous Poisson Process is known in the literature. It is natural to generalize the

integrated homogeneous Poisson process into integrated non homogeneous Poisson, Cox, and renewal processes. In this thesis we derive expressions for the marginal distributions of integrated Poisson and Cox processes using conditioning arguments, and derive the marginal distributions of integrated renewal processes using the theory of point processes. The results are presented in the form of Laplace transforms. Asymptotic properties of the integrated renewal processes are also investigated. An application to the study of traffic is given.

3. **Total downtime of repairable systems**

   An expression for the cumulative distribution function of the total downtime of a repairable system, which is regarded as a single component, under an assumption that the failure and the repair times of the system are independent has been derived by several authors using different methods. We use a different method (using point processes) to compute the distribution function of the total downtime. We also consider a more general situation where we allow dependence of the failure and the repair times of the system.

   The covariance structure and asymptotic properties of the total downtime for the independent case are also known in the literature. We derive the similar results for the dependent case. Examples are given to see the effect of dependence between the failure and the repair times on the total downtime.

   We also discuss the total downtime of repairable systems consisting of $n \geq 2$ stochastically independent components. We derive an expression for the marginal distribution of the total uptime of the system for the case the failure and the repair times of each component are exponentially distributed. For arbitrary failure or repair times of the components we derive an expression for the mean of the total uptime.

Ringkasan

Proses renewal dan sistem-sistem tereparasi

Di tesis ini di bahas topik-topik berikut.

1. **Proses renewal reward**

   Distribusi marginal dari *proses renewal reward* (renewal reward process) dan versinya, yang dinamakan *proses instantaneous reward* (instantaneous reward process), diturunkan. Dasar teori yang dipakai adalah teori tentang *proses titik* (point process), khususnya proses titik Poisson. Cara yang digunakan adalah dengan menyatakan proses renewal reward dan versinya sebagai fungsional dari proses-proses titik Poisson. Untuk perhitungan digunakan rumus-rumus Palm dan fungsional Laplace dari proses titik Poisson. Hasil-hasil yang diperoleh disajikan dalam bentuk transformasi Laplace. Penerapan proses instantaneous reward dalam studi tentang lalu lintas diberikan.

   Beberapa sifat asimtotik dari proses renewal reward dipelajari kembali. Bukti dengan teorema Tauber untuk versi harga harapan dari teorema renewal reward diberikan. Suku konstan dalam versi harga harapan dari teorema renewal reward ditemukan. Sifat-sifat serupa untuk proses instantaneous reward juga di selidiki. Distribusi asimtotik normal untuk proses instantaneous reward dibuktikan.

   Kovariansi dari proses renewal, yang juga dapat dipandang sebagai proses renewal reward, diturunkan. Sebagai tambahan dipelajari reliabilitas sistem dalam model *stress-strength*, dimana besarnya stress dapat dipandang sebagai reward. Proses kejadian dari stress dimodelkan dengan proses Cox dan renewal. Efek dari dependen antara stress dan strength terhadap reliabilitas sistem diselidiki menggunakan hasil tentang proses renewal reward.

2. **Proses integrated renewal**

   Fungsi kepadatan probabilitas marginal dari sebuah *proses integrated renewal* (integrated renewal process) dengan proses Poisson homogen sebagai proses dasarnya telah dikenal di literatur. Adalah alami untuk menggeneralisasi proses Poisson homogen sebagai proses dasar ke proses Poisson tak homogen, proses Cox, dan ke proses renewal. Di tesis ini distribusi marginal dari proses integrated renewal diturunkan dengan teknik kondisional untuk proses Poisson dan Cox sebagai proses dasarnya, dan dengan memakai teori tentang proses titik untuk proses renewal sebagai proses dasarnya. Hasil-hasilnya disajikan dalam bentuk transformasi Laplace. Sifat-sifat asimtotik dari proses integrated renewal juga diselidiki. Sebuah penerapan dalam studi lalu lintas diberikan.

3. **Total downtime sistem-sistem tereparasi**

   Sebuah ekspresi untuk fungsi distribusi kumulatif dari total *downtime* sebuah sistem yang dapat direparasi, di mana sistem tersebut dipandang sebagai sebuah komponen, telah diturunkan oleh beberapa penulis dengan metode yang berbeda di bawah asumsi bahwa waktu-waktu bekerja dan reparasi dari sistem saling independen. Di tesis ini sebuah metode yang berbeda (yakni dengan menggunakan teori tentang proses titik) digunakan untuk menentukan fungsi distribusi dari total downtime. Situasi yang dipelajari juga lebih umum karena diperbolehkan adanya dependensi antara waktu bekerja dan reparasi dari sistem.

   Kovariansi dan sifat-sifat asimtotik dari total downtime untuk kasus independen juga telah diketahui di literatur. Hasil-hasil yang serupa untuk kasus dependen diturunkan dalam tesis ini. Contoh-contoh untuk melihat efek dependen antara waktu bekerja dan reparasi terhadap total downtime diberikan.

   Di tesis ini juga dibahas total downtime dari sistem-sistem dengan dua komponen atau lebih dimana komponen-komponen tersebut saling independen secara stokastik. Sebuah ekspresi untuk distribusi marginal dari total uptime sistem dimana waktu bekerja dan reparasi dari setiap komponennya berdistribusi eksponensial diturunkan. Untuk kasus distribusi waktu bekerja dan reparasi dari komponennya sembarang, harga harapan dari total uptime diturunkan.

Curriculum Vitae

Suyono was born on December 18, 1967 in Purworejo, Indonesia. After finishing his secondary high school in Purworejo in 1986, he studied Mathematics Education at the Jakarta Institute for Teachers Training and Education and graduated in 1991. In August 1994 he started his Master Program in Mathematics at Gadjah Mada University in Yogyakarta, Indonesia, and obtained his degree in 1998. From October 1998 until February 2003 he carried out a doctoral research at the Department of Control, Risk, Optimization, Stochastic and Systems (CROSS), Faculty of Information Technology and Systems (ITS), Delft University of Technology. Currently he is a lecturer at the Department of Mathematics, Faculty of Mathematics and Natural Sciences, State University of Jakarta, Indonesia.