On the Determination of Approximations of First Integrals *for*, and the Integral–Conservative Numerical Solution *of*

FEW-BODY GRAVITATIONAL PROBLEMS

with Applications to Capture Trajectories

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Abstract

There are 10 algebraic time-independent integrals known for the gravitational n-body problem, of which the 3-body problem is a special case. That problem in turn may be restricted to the circular restricted 3-body problem, which has only the Jacobi-integral as a known time-independent integral. While interesting problems in their own right, in this thesis we further extend the circular restricted 3-body problem with a time-dependent perturbation, modeling the Moon by simple harmonic motion about the Earth in the Earth-Sun rotating coordinate system. This is motivated by our earlier study of ballistic lunar capture trajectories, and serves as a model for the dynamics of a satellite under the influence of the Sun, Earth and Moon. As such it incorporates the necessary features for the study of ballistic lunar capture scenarios.

If these three models are formulated as explicit perturbation problems in a small parameter ε (related to the relative influences of the bodies), the corresponding integrals likewise become structured and a natural hierarchy of effects is found by successive expansion in powers of ε , which is the point–of–view which the work in this thesis will take.

We begin by introducing the major issues relevant to two new approaches to exploiting such integrals for more (numerically and geometrically) accurate trajectory integration in astrodynamics. To this end we review the essentials of Hamiltonian dynamics under perturbation, astrodynamics and the (ballistic) capture problem, a method of integrating vectors for the determination of approximations of first integrals of the motion, and study numerical solution methods which conserve such integrals and integral approximations.

Against this backdrop, the work is split into two parts.

1. In the first part, we apply the method of integrating vectors to our three gravitational problems in turn, demonstrating the feasibility of the *ab initio* construction of the 10 known integrals and proving this reconstruction for the Jacobi integral. The method is not limited to the construction of time–independent integrals, and we derive and discuss the 1st –order linear system of partial differential equations governing integrating factors for general time–dependent integrals, giving perspectives on the solution of the system for each of the three problems.

We find in particular that the capture problem we have introduced may be expected to have a time-dependent analogue of the Jacobi integral, of which the time-independent part we can solve the system of equations for corresponds to the time-averaged motion of the Moon about the Earth. We suggest that methods based on harmonic analysis may provide the tools necessary to solve the full time-independent partial differential equations in further work.

2. In the second part, we implement and compare an algorithm which conserves the Jacobi integral for the restricted 3–body problem, and three variations of an algorithm for the conservation of the energy integral in the planar Jacobi 3–body problem (one of which also conserves angular momentum). We also outline a strategy for incorporating integral–approximations found by the methods of the first part into conservative–integration schemes, and implement these for the restricted 3–body and capture problems.

These schemes have the benefit of explicitly conserving the integrals up to floating–point accuracy, thus essentially eliminating the out–of–manifold error for integrated trajectories, and have shown promise in application to trajectories with sensitive dynamics in particular. We find however that their performance on problems involving large masses is considerably better than on problems involving a small, fast–moving satellite, such as in the ballistic lunar capture problem which motivated their implementation.

There are indications that in the former types of problems, there may be a particular niche for their application in long time–span integration. On the other hand, for the "small satellite, large primaries" scenario which motivated our work, we find that they are performance–limited by their formulation as fixed time–step algorithms, and are likely to exhibit singular perturbation phenomena, which may necessitate a reformulation in terms of explicit force scales. We thus stress the need to extend the methods to variable time–step approaches which implement error–control, and to investigate their long time–scale behavior in further work.

The implementations of algorithms in FORTRAN code, the MATLAB visualization code and the analysis notebooks produced with MATHEMATICA are included on the accompanying CD, together with the raw data sets used to produce the results presented in this thesis.

Preface

This thesis is in a sense a detailed account of my research over the past two years, extending back to an internship at the University of Texas at Austin in the fall semester of 2005, and in I find myself deeply indebted to a number of people.

First, I would like to thank my advisors Ron Noomen and Wim van Horssen for both their guidance in this period, as well as for giving me pretty much a free hand in determining what to study and how to approach it. Likewise, I am deeply indebted to Cesar Ocampo at UT for introducing me to the fascinating study of ballistic capture trajectories, and for the interesting discussions we had during my stay in Austin.

Likewise, I am grateful to a number of friends, among which Jeroen Melman, Kartik Kumar and Federico Gallo in particular, for the numerous interesting and irreverent discussions we had about mutual and mutually exclusive research, and for always being willing to hear me out, no matter how mathematical it got...

Lastly, I am appreciative beyond words of the support of my family, and in particular my loving wife Vanessa, who persevered with me throughout a rather difficult final year, and has given me space both to think and to breathe.

Before proceeding to the work ahead, I'd like to characterize that to which my readers are about to subject themselves.

This work is intended foremost in the spirit of exploratory research and we will approach our problems both with analytical and with computational tools. However, since even the analytical techniques were motivated by a computational problem, it's my deep conviction that the following remark, due to Hamming, can't be emphasized enough, and should be engraved in stone at engineering schools everywhere:

The purpose of computing is insight, not numbers.

Remembering always the physics underlying our models, it is my hope that my work on the problems of this thesis can do the statement some justice, despite the complexity of the discussion and the sea of technical details we must necessarily wade through. If nowhere else, may it shine through in the spirit and the conclusions of this thesis.

Action is the pointer which shows the balance. We must touch not the pointer, but the weight.

- Simone Weil

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Chapter 1

Introduction

This thesis reports on work performed at the faculties of Applied Mathematics and of Aerospace Engineering at Delft University of Technology starting roughly in the summer of 2006 and continuing through the summer of 2007, building on earlier work performed at the Center for Space Research at the University of Texas at Austin, in the fall semester of 2005.

1.1 Background

There are a number of what might be called "classical" problems of celestial mechanics and astrodynamics, among which the full– and restricted 3–body problems feature prominently. Arising from Newton's third law of motion using the straightforward multi–body extension of the gravitational potential $\frac{Gm_1m_2}{r}$, these have been found, in some 300 years of active research, to contain a remarkable range of dynamical behavior. Yet in our age of quick–and–dirty computational approaches to problems, we are often apt to overlook this and focus exclusively on the black–box calculation of orbits.

Moreover, the problems themselves contain very interesting structure. They are essentially Hamiltonian dynamical systems, a fact which has implications for the behavior of orbits, properly flows in a Hamiltonian phase–space. There is also a limited set of 10 known integrals, and since these encapsulate key aspects of the behavior of the system, they too should arguably play a role in any correct simulation, properly a numerical solution (integration) of the equations of motion. Yet with the exception of the astronomy community, practical interest in the design of numerical algorithms incorporating these features has been limited until the late 1990's, and only recently have theoretical approaches again been introduced into what had long become a computational field. This is largely due, however, to the difficulty of treating the problems analytically.

Already in the case of just 3 bodies, we face a coupled system of 18 nonlinear ordinary differential equations. Even after maximally reducing these using the known integrals of motion, we still retain 8 degrees of freedom, which can model anything from simple circular near–Earth satellite orbits, to halo orbits rotating about empty space, and complicated choreographies of trinary star systems and indeed, in this range of complexity lies both the beauty and challenge of the problem.

In particular, the problems that the present work is concerned with share common roots in a study of a technique for constructing ballistic lunar capture transfers in the Earth–Moon system performed by the author. While a numerical solution using a standard technique was the obvious engineering approach, the question was quickly raised as to

whether the solutions thus obtained were correct. This was a particularly relevant question given the characteristic dynamics (linked to the recent work on Weak Stability Boundaries and dynamics on tubular manifolds interacting near the Earth–Moon Lagrange points) and the extreme observed sensitivity to initial conditions.

1.2 Problem Definition

The question, as the present work will make clear, is considerably more difficult to answer than it at first seems, and after considerable reflection, the problems the thesis is concerned with were defined as the following:

1. First, to consider the application of a new method of integrating vectors for the approximation of first integrals [*e.g.* van Horssen, 1999a,b] of the motion to the problems noted above: the 3–body problem (3BP), the (circular) restricted 3–body problem (CR3BP) and the (ballistic) capture problem (CP).

The choice of these dynamical problems is by no means arbitrary; rather, the 3-body problem may be thought of as a prototype of the full dynamics of the n-body problem, the CR3BP as its most useful restriction to manageable levels of complexity, and the CP the simplest extension (or perturbation) of the CR3BP so as to include the essential dynamics involved in ballistic lunar capture.

2. Second, to consider the application of a new framework for the integral–conservative numerical solution of the equations of motion [*e.g.* Shadwick et al., 1999, Kotovych and Bowman, 2002] to the same problems. In particular the framework intended is one which lends itself for direct inclusion of not just the known integrals, but also approximations of integrals such as would be generated by the method of integrating vectors above.

That the two approaches mesh in this way is of particular significance for the capture problem we define in the present work, for which no integrals are in fact known. Moreover, the key motivation for the use of such conservative numerical solution schemes for us lies in a better treatment of the sensitive dynamics we encounter with ballistic capture trajectories, and the present work will examine whether such an approach is indeed worthwhile.

The reader will remark that these problems are somewhat broadly defined, as opposed to *e.g.* the "concrete" optimization of a trajectory. This is consistent with the nature of the present work as exploratory research, and indeed, in the course of this thesis a number of counterintuitive conclusions will be motivated, contrary to the expectations at the outset.

1.3 Approach

Our main goal in this thesis, then, is exploratory: we seek to discover whether the two approaches, theoretical and numerical, can be usefully applied to the 3 problems in astrodynamics we consider, and what insights can be gained using them. Thus:

1. In Part I of this thesis we will lay out the rather extensive theoretical background necessary for the consideration of the problems outlined above. In particular we will reconsider the modeling of the 3-body problems and the 4-body capture problem, paying due attention to their interrelation as a series of successive perturbation problems.

2. Next, with regard to the method of integrating vectors, in Part II of this thesis we will consider the *ab initio* construction of the known integrals of motion for both 3–body problems. Our goal in this is not to reproduce known results *per se*, but rather to get a grip on the types of integrating factors which play a role in the analysis of these problems in astrodynamics.

Using this knowledge, we then consider the problem of generating (approximations of) new integrals of the motion, which is a considerably more complicated problem. The complication stems from the need to solve not just the time–independent parts, but the full time–dependent system of coupled linear *partial* differential equations.

3. Third, in Part III of this thesis, we consider the efficacy of conservative and approximately conservative numerical integration schemes for solving the equations of motion of the problems. We are particularly concerned with the relation of their performance in preserving the integrals of motion to the accuracy of the trajectories integrated, and the scalings in the dynamics which influence this relationship.

In order to perform this comparative study in the absence of existing libraries of code for the simulation, it was found necessary to code a simulation suite ourselves. This IntegrationMethods suite of FORTRAN code, to-gether with MATLAB visualization code and MATHEMATICA analysis code form the foundation for the results presented, and are included on the CD accompanying this thesis.

A few words are in order with regard to a number of intentional choices of approach taken in this work, so as to not take the reader altogether by surprise:

- We have intentionally followed literature conventions as much as possible, to preserve comparability with other work. In consequence, for example, we treat the full 3-body problem in dimensional Jacobi coordinates, while we treat the CR3BP in the conventional normalized (dimensionless) coordinates. Moreover, while both may be derived from appropriate Hamiltonians, we typically use and simulate in "natural" problem-derived coordinates, ¹ rather than in canonical variables.
- While the above remarks apply to the actual modeling of the problems, our numerical results are presented consistently across all models in dimensional kg km s units, and correspond to realistic trajectories for Earth–Moon–Satellite and Sun–Earth–Moon (–Satellite) test cases.
- In contrast to a certain standard approach to problem definitions, however, we encounter a wide diversity of notation in the literature, and have imposed a uniform approach in the present thesis. This is summarized in appendix A, which the reader is asked to briefly skim.
- In the analysis of the chapters of Part II we have intentionally simplified the problems considered in certain ways with regard to expansions in a small parameter ε. This has been done for purposes of clarity and is not in any way essential. It is, moreover, corrected for in Part III when we consider numerical solutions, where we have used full and correct expressions throughout to ensure comparability of results between problems and coordinate systems.

¹Though using the exactly equivalent expressions of the dynamics.

• The numerical solutions for trajectories and orbits found using the methods of Part III are of course themselves intended as the subject of discussion and analysis, but the question of what to benchmark them against is again more involved than it would seem on the surface. A practical approach has been used in benchmarking against a standard RKF7(8) integrator, but we would caution that this choice is nontrivial, and all results are to be considered in the light of the comments of the detailed discussion of chapter 9 on the validity and drawbacks of the approach.

1.4 Document Organization

We summarize here the organization of the thesis document into five parts as follows:

- 1. Part I treats the building blocks necessary to the development of the rest:
 - We begin with a short review of Hamiltonian dynamics in chapter 2, highlighting the main characteristics of this particular class of problems, of which the problems of the present work are an interesting subset.
 - This is then followed by an extended treatment of the necessary astrodynamics including approaches to ballistic capture trajectories in chapter 3. In particular the three core problems of this thesis are introduced here, and discussed extensively from a modeling point–of–view.
 - Next, chapter 4 treats the method of integrating vectors, and concludes with an extended discussion of multiple–scales issues relevant to the particular problems of astrodynamics.
 - Finally, chapter 5 treats the basics of numerical trajectory integration relevant to Part III of this thesis, focusing in particular on single- and multi-step traditional methods and the conservative integration techniques that form the basics of the work done in chapters 10–11.
- 2. Part II treats work done towards the use of the method of integrating vectors for the determination of approximations of first integrals in the general and restricted 3-body problems and the capture problem:
 - Chapter 6 discusses the application of the method to the full 3-body problem, reduced using the known integrals to the planar Jacobi problem. This is the most difficult problem, as it treats the motion of all 3 bodies on an equal footing, but there are 10 known integrals, which we discuss in detail.
 - Chapter 7 does likewise for the restriction of the above to the circular restricted 3-body problem for the motion of only a satellite in the presence of 2 primaries, which assume fixed positions in the rotating coordinate system introduced for this problem. We discuss here in particular the Jacobi integral and the integrating vectors which lead to it, also in connection with the chapter which follows.
 - That is chapter 8, which treats the application of the method to the capture problem, for which there are no known integrals, and we discuss in particular the modifications of the Jacobi integral induced by the perturbation of the CR3BP to the CP. The chapter closes with a discussion of why the problem is particularly interesting as an area for the concentration of future research efforts.

In each of those three cases, the validity of the method in generating the known integrals will be demonstrated (and in one case proven explicitly), after which a discussion of the problems posed encountered in trying to find new integrals will follow.

- 3. Part III then proceeds to the discussion of conservative integration schemes and their performance:
 - Chapter 9 discusses general design considerations and introduces the integration models and methods coded in the IntegrationMethods suite code. The design considerations are split into conceptual issues and coding & validation issues respectively. Additionally we have placed the discussion of performance evaluation here, immediately preceding the presentation and discussion of results.
 - Chapter 10 presents the results of simulations of the 3BP, CR3BP and CP with the different integrators for 2 test cases. We present and compare a number of different implementations of Bowman's algorithms, as well as comparing our methods with the RKF7(8) benchmark and their simple predictor corrector and RKF2(3) "competitors." We close with considerations on the general formulation of multiply–conservative algorithms, which again turns out to pose some challenges in the general setting.
 - Our final chapter 11 then proceeds with the discussion of approximately conservative integrators of differing orders for the CR3BP and CP for a small satellite, focusing on performance relative to order of approximation, and in comparison with the fully conservative integrator of the previous chapter in the CR3BP–case. We also briefly consider the issue of convergence of the integral approximations in the context of conservative solution of actual trajectories.
- 4. Part IV is formed by the conclusions and recommendations of the thesis.
- 5. Part V is composed of the appendices, which treat the notational conventions in appendix A, the derivations of the Hamiltonian formulations (cited in chapter 3) in appendix B, and a brief manual for the IntegrationMethods suite of simulation codes developed for the discussion of Part III in appendix C.

1.5 Division into AE and AM Theses

The core of this work is formed by Parts I–III and can usefully be read as an integral whole: Parts II and III are strongly related, and both draw on the results of Part I, which in particular with respect to astrodynamics issues in chapter 3 and section 4.3 serves as more than just a summary of the literature.

Nonetheless, despite the strong relationships between different areas of the discussion, this work is primarily intended to serve as a thesis for both the faculties of Applied Mathematics and of Aerospace Engineering, and so for organizational purposes it is to be divided into two separate bodies of work as follows:

- 1. The first, which is intended to serve as the body of work submitted to Applied Mathematics, is the union of Parts I and II, together with this introduction and the appropriate remarks in the conclusions and recommendations, as well as appendices A and B.
- 2. The second, intended to serve as the body of work submitted to Aerospace Engineering, is the union of Parts I and III, together with this introduction and the appropriate remarks in the conclusions and recommendations, as well as appendices A and C.

Part I

Preliminaries

Outline of Part I

The first part of this thesis reviews the theory essential to the development in the following two parts. In particular, we desire to formulate the physical problems with due consideration for the nuances of the dynamics under perturbation, and these first three chapters significantly extend the line of thought developed in the earlier literature study. We will proceed as follows:

- 1. In chapter 2, we begin with a consideration of the characteristics of dynamical systems formulated using the framework of Hamiltonian dynamics, and will consider such issues as symplectity and the integrability of Hamiltonian systems in general, as well as discussing some relevant issues in relation to stability and perturbation theory.
- 2. In chapter 3 a review is given of astrodynamics and the capture problem. The *n*-body problem is introduced, after which the 2-body problem is stated and solved, as a baseline against which further problems are introduced as perturbations. In particular:
 - The classical 3-body problem is discussed with its reduction from 18 to 8 first-order ordinary differential equations, and the modifications of the dynamics are discussed from the perspective of perturbations of the 2-body problem.
 - On taking the classical restrictions to the circular restricted 3-body problem, the consequences of the restrictions are demonstrated, and we discuss a framework for other possible restrictions of the 3-body problem as models for different physical phenomena and their consequences.
 - The restricted 3-body problem in turn is also taken as a baseline for further perturbation by a 4th body. In the simplest case an empirical modification is proposed analogous to the quasi-bicircular problem in the literature.
- 3. In chapter 4, the method of integrating vectors introduced in [van Horssen, 1999a,b] is reviewed, and its application as a method of constructing approximations of first integrals is discussed. The issues which arise when treating problems with multiple scales are particularly relevant to the problems of this thesis, and multiple scales issues will be discussed in detail, extending beyond multiple time–scales to mass–, spatial– and force–scales as well.
- 4. Lastly, the discussion of numerical (trajectory) integration given in the literature study is briefly reviewed in chapter 5, both to introduce conventions and to recall the issues relevant to integrator choice and design. In particular the reader is requested to take note of the discussion of conservative integrators, and particularly the consideration of error-propagation and implementation effectiveness, as these schemes lend themselves to the goals of this thesis and will play a prominent role in the third part of the work.

Chapter 2

Hamiltonian Dynamics under Perturbation

This chapter briefly discusses Hamiltonian dynamics in the context of perturbed dynamical systems, and a relevant few of the many results known for the dynamics of such a system (of which class almost all the problems of this thesis are members). An overview of the results may be found in [Verhulst, 2000], while considerably more detailed treatments may be found in *e.g.* [José and Saletan, 1998] from a modern physical perspective or [Arnold, 1989] from the more traditional mathematical physics perspective.

2.1 Hamiltonian Dynamics

A Hamiltonian system is a dynamical system for which a Hamiltonian function $\mathcal{H}(q, p)$ can be defined $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}$, with generalized coordinates q_i and momenta p_i , i = 1, ..., n. The equations of motion are then derived from the Hamiltonian by *Hamilton's canonical equations*:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}.$$
 (2.1)

These equations of motion are a set of $2n \times 1^{st}$ order ODE's.

For the particular case of particles with kinetic energy T in a conservative potential U, it can be shown that the Hamiltonian becomes $\mathcal{H} = T + U$. Hence \mathcal{H} may be thought of as a function in the role of the energy of the system, though the specific relation $\mathcal{H} = T + U$ generally does not hold.

Further, on setting $x = (q, p)^{\top} \in \mathbb{R}^{2n}$, the equations of motion become

$$\begin{cases} \dot{x}_k = \frac{\partial \mathcal{H}}{\partial x_{k+n}} & \text{for } k = 1, \dots, n \\ \dot{x}_k = -\frac{\partial \mathcal{H}}{\partial x_{k-n}} & \text{for } k = n+1, \dots, 2n \end{cases},$$
(2.2)

and thus naturally induce a structure given by the matrix J:

$$\dot{x} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \nabla \mathcal{H} = J \nabla \mathcal{H} .$$
(2.3)

It has the properties that:

$$^{2} = -I_{2n} \qquad \text{and} \qquad (2.4a)$$

$$J^{+} = -J . ag{2.4b}$$

This is a specific example of a symplectic structure, discussed briefly in section 2.1.2 below.

2.1.1 From Lagrangian to Hamiltonian Dynamics

The conventional starting point for theoretical mechanics is typically Lagrangian dynamics, where the concepts of a kinetic energy T and potential energy U have their conventional interpretations and $\mathcal{L} = T - U$. The Hamiltonian formulation above is necessarily linked with Lagrangian dynamics, as it is an equivalent description of the dynamics of a typical mechanical system. This is made more specific as follows.

Consider the Lagrangian $\mathcal{L}(q, \dot{q}, t)$, then:

• Hamilton's canonical momenta p_i are defined by

$$p_j = \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}_j} \,. \tag{2.5}$$

• The relation (for the general case including explicit time-dependence)

$$\mathcal{H}(q_j, p_j, t) = \sum_{j=1}^n p_j \dot{q}_j - \mathcal{L}\left(q_j, \dot{q}_j(q_k, p_k), t\right) \,, \tag{2.6}$$

demonstrates the formal construction of a Hamiltonian from the Lagrangian $\mathcal{L}(q, \dot{q}, t)$, and is a Legendre transform between the two (the canonical momenta p_j come to replace the coordinate derivatives \dot{q}_j as variables).

• Introducing these transformations, and viewing with regard to the formalism (vector) p, q as independent variables, the dynamics which were given by the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \qquad (2.7)$$

as $n \times 2^{nd}$ order equations are now given by equation (2.1) as $2n \times 1^{st}$ order equations instead.

- In the language of dynamical systems, one considers the set of coordinates $\{q\}$ as comprising a space \mathbb{Q} , and associated with each point q is the tangent space $\mathbf{T}_q\mathbb{Q}$. The collection of all these $\mathbf{T}\mathbb{Q} = \bigcup \mathbf{T}_q\mathbb{Q}$ associates to each coordinate on \mathbb{Q} its tangent space, and thus forms a *carrier manifold* for the dynamics (parameterized by q, \dot{q}).
- In Hamiltonian dynamics, the coordinate derivatives *q̇* are replaced by the conjugate momenta *p* which are formally covariant vectors (1-forms), and so *p* ∈ **T**^{*}ℚ = ∪ **T**^{*}_{*q*}ℚ, the *cotangent space* dual to the ordinary tangent

space.

Thus there is a shift in perspective from Lagrangian dynamics on a manifold $T\mathbb{Q}$ to Hamiltonian dynamics on $T^*\mathbb{Q}$, and this $T^*\mathbb{Q}$ is what is formally intended as the phase space of a Hamiltonian system.

2.1.2 Structure of the Phase Space

As outlined in the previous section, Hamiltonian dynamics is a reformulation of the dynamics of a system as a flow in the 2*n* dimensional phase-space $\mathbf{T}^*\mathbb{Q}$ (or on the 2*n* dimensional manifold $\mathbf{T}^*\mathbb{Q}$). Part of the utility of this reformulation stems from the fact that the phase space is structured in interesting ways, which are outlined briefly in the following sections, and summarized in three theorems.

Symplectic Structure

Definition 2.1.1 (*k*-forms and Differential *k*-forms)

A k-form, or exterior form of degree k is a function of k vectors, which is k-linear and antisymmetric. The basic examples are:

- 1-forms, commonly known as covariant vectors, dual to the usual contravariant vectors, which map $\omega : \mathbb{R}^n \to \mathbb{R}$ via the usual inner product $\omega(\mathbf{v}) := (\omega, \mathbf{v}) = \sum_k \omega_k v^k$; and
- 2-forms, $\omega_{[2]} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, typically defined via the exterior product of 1-forms, $\omega_{[2]} := \omega_1 \wedge \omega_2$.

A differential *k*-form at a point *x* of a manifold *M* is an exterior *k*-form on the tangent space $\mathbf{T}_x M$, i.e. a skew-symmetric function of *k* vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ tangent to *M* at **x**. Note also that the geometrical interpretation of a differential *k*-form is that of an oriented area-element in the space on which it is defined. On \mathbb{R}^n , we typically take coordinates x_i and have differential *k*-forms of the form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$; that every such differential *k*-form can be uniquely written in this way is a theorem ([Arnold, 1989, section 34]).

Example 2.1.2 (k–Forms)

Some examples of k-forms are:

- The differential 1–form corresponding to unit length on a curve: dx
- The differential 2-form corresponding to unit area on a manifold: $dx_1 \wedge dx_2$
- The volume or 3-current in electrodynamics: $\mathbf{J} = J^a \varepsilon_{abcd} dx_b \wedge dx_c \wedge dx_d$, with ε_{abcd} the Levi-Civita permutation symbol and a, b, c, d running through the coordinates.

Definition 2.1.3 (Symplectic Structure)

Let M^{2n} be an even-dimensional differentiable manifold, e.g. $\mathbf{T}^*\mathbb{Q}$ in the case of a Hamiltonian system. A *symplectic* structure on M^{2n} is a closed non-degenerate differential 2–form $\omega_{[2]}$ on M^{2n} .¹ The set $(M^{2n}, \omega_{[2]})$ is referred to as a *differential manifold*.

Lemma 2.1.4 (Natural Symplectic Structure of a Hamiltonian System)

The tangent bundle T*Q introduced for Hamiltonian systems has a natural symplectic structure. It has local coordinates

¹*i.e.* a bilinear antisymmetric function of 2 vectors on a 2n-dimensional manifold M^{2n} which is both closed and non-degenerate in a topological sense.

(q, p) at all points in $\mathbf{T}^*\mathbb{Q}$, on which the symplectic structure is given by

$$\boldsymbol{\omega}_{[2]} = \mathbf{d}\mathbf{p} \wedge \mathbf{d}\mathbf{q} = \sum_{i} \mathbf{d}p_{i} \wedge \mathbf{d}q_{i} .$$
(2.8)

Proof. See [Arnold, 1989, section 37].

Example 2.1.5 (Recovery of the Hamiltonian Vector Field)

As an example (due to [Arnold, 1989]) one can construct an isomorphism $J : T^*\mathbb{Q} \to T\mathbb{Q}$ by associating with each $\xi \in T\mathbb{Q}$:

$$\omega_{[1,\xi]}(\eta) = \omega_{[2]}(\eta,\xi) \qquad \forall \, \eta \in \mathbf{T}\mathbb{Q} \,, \tag{2.9}$$

which takes vectors from the tangent space to the cotangent space using the natural symplectic structure $\omega_{[2]}$.

Then with \mathcal{H} the Hamiltonian, define the vector field $J d\mathcal{H}$ using the differential 1-form $d\mathcal{H} \in \mathbf{T}^* \mathbb{Q}$ and J in the role of $\omega_{[1,\xi]}$ as a mapping to $\mathbf{T}\mathbb{Q}$. Taking $\mathbb{R}^{2n} = \{x = (q,p)^{\mathsf{T}}\}$, one obtains in this way:

$$\dot{x} = J \,\mathrm{d}\mathcal{H}(x) \longleftrightarrow \dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \ \dot{p} = -\frac{\partial \mathcal{H}}{\partial q},$$
(2.10)

which we refer to as the Hamiltonian vector field.

This shows, equivalently, that the isomorphism J is (in the (q, p) basis) precisely the matrix $\begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ introduced in section 2.1.

Thus $J d\mathcal{H} \equiv J \nabla \mathcal{H}$ as introduced earlier in section 2.1, and from the symplectic structure induced by the Hamiltonian, one recovers precisely Hamilton's equations of motion in this way.

In particular, note that in applications (almost exclusively in real coordinates), one uses the structure J to define a matrix A or vector function f(x) to be symplectic if it holds that:

 $A^{\top}JA = J \qquad \text{or} \tag{2.11a}$

 $\nabla f(x)^{\top} J \nabla f(x) = J$ respectively. (2.11b)

Volume Preserving Flow

The flow in a Hamiltonian system is also volume preserving, as the following theorem asserts.

Theorem 2.1.6 (Liouville's Theorem)

Consider the equation $\dot{x} = f(x)$ and a domain $D(0) \in \mathbb{R}^n$ which has a volume v(0). The flow of the system defines the

"time advance mapping" $g_t : \mathbb{R}^n \to \mathbb{R}^n$ such that $D(t) = g_t D(0)$. For a volume element v(t) then

$$\left. \frac{\mathrm{d}v}{\mathrm{d}t} \right|_{t=0} = \int_{D(0)} \nabla \cdot f(x) \,\mathrm{d}x \,. \tag{2.12}$$

When this flow is generated by a time-independent Hamiltonian system, it is volume preserving.

Proof. In such a system, we find that

$$\nabla \cdot f(x) = \sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) = \sum_{i=1}^{n} \left(\frac{\partial^2 \mathcal{H}}{\partial q_i \partial p_i} - \frac{\partial^2 \mathcal{H}}{\partial p_i \partial q_i} \right) = 0.$$
(2.13)

In consequence, the system cannot have attractive equilibrium points (in positive or negative time),³ as these imply the shrinking resp. expansion of a volume element as time progresses. Of course this does not prohibit a lower-dimensional sub-manifold exhibiting attraction; an example would be the stable manifold of a saddle point.

Note that this theorem, understood as a geometric proposition about the flow in a Hamiltonian system, is also important in the numerical study of *any* Hamiltonian system or class of flows therein (as opposed to the evolution of a single trajectory) in that any computation should likewise not contain attractive equilibria, which would be un-physical. This is a key issue discussed further in chapter 5. This result is in turn linked to the notion of a symplectic structure by the next.

Theorem 2.1.7 (Preservation of Symplectic Structure by Hamiltonian Flow)

If we use the Hamiltonian vector field $J d\mathcal{H}$ to define a phase flow, *i.e.* the vector field fulfills the role of f(x) in the usual

$$\dot{x} = f(x) = J\nabla\mathcal{H} \,,$$

then this phase flow preserves the natural symplectic structure $\omega_{[2]}$. This can also be stated explicitly in terms of the symplectic linear structure (remark that this is indeed an antisymmetric function of 2 vectors):

$$[\mathbf{v}_1,\mathbf{v}_2]:=(J\mathbf{v}_1,\mathbf{v}_2),$$

with (,) the usual inner product, as:

$$\underbrace{[g_t(\mathbf{x}_1), g_t(\mathbf{x}_2)]}_{\text{structure advanced by flow}} = \underbrace{[\mathbf{x}_1, \mathbf{x}_2]}_{\text{original structure}}, \qquad (2.14)$$

with g_t the time advance map of the Hamiltonian flow.

Proof. See [Arnold, 1989, section 38].

²This can be defined in terms of the stroboscopic map of dynamical systems theory. Alternatively, one could think of this operation as the propagation operator $D(t) = \mathcal{P}^{t_0 \to t}(D(0))$ as introduced in [Verzijl, 2006, 2005].

³Nodes or foci, stable (attraction in forward time) resp. unstable (attraction in negative time).

Corollary 2.1.8

In the case n = 1 such that $M^{2n} = \mathbb{R}^2$, this asserts that Hamiltonian flow preserves area, which is Liouville's theorem 2.1.6 in 2 dimensions. For n > 1 it can likewise be shown that preservation of the symplectic structure corresponds precisely to conservation of (oriented) volume, which is intuitive in light of the geometric interpretation of the structure as an area element noted earlier.

A practical application of this theory is in the use of symplectic integrators which preserve this structure of the Hamiltonian dynamics, and thus phase–space volume in numerical integration. As discussed in [Verzijl, 2006] there is an inherent tradeoff involved in choosing between the conservative integration schemes discussed in this thesis and symplectic integrators, though both are motivated by the preservation of the underlying properties and structures of the dynamical system in numerical integration.

2.2 Integrability

The issue of the analytical integrability of a system is the question of whether a system can be reduced to a set of known functions of time. Ideally these are analytic functions by which the number of unknowns may be reduced such that all are fixed by a single independent variable (typically the time t), but convergent series are also often admitted, as well as the reduction to a form which can be integrated by quadratures.

However, while the concept is simple, the details are often quite technical, and the reader is referred the following chapter on astrodynamics, as well as to [Szebehely, 1967] for an extended discussion in the context of the 3–body problem of celestial mechanics. The concept is introduced here only in a general sense, applicable to any dynamical system.

2.2.1 First Integrals

Definition 2.2.1 (First Integral)

Recall first the orbital derivative of a function F(y), where $y = (x,t)^{\top}$ and $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $F : \mathbb{R}^{n+1} \to \mathbb{R}$:

$$L_t(F) = \frac{\partial F}{\partial y} \dot{y} = \sum_{i=1}^n \frac{\partial F}{\partial x_i} \dot{x}_i + \frac{\partial F}{\partial t} .$$
(2.15)

Then a first integral of a system is a function I(y) such that $L_t(I) \equiv \frac{dI}{dt} = 0$.

Example 2.2.2 (Hamiltonian as First Integral)

An example, as noted above, is the Hamiltonian.

$$L_t(\mathcal{H}(p,q)) = \sum_{i=1}^n \left(\frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i \right) = \sum_{i=1}^n \left(-\dot{q}_i \dot{p}_i + \dot{p}_i \dot{q}_i \right) = 0.$$

Example 2.2.3 (Integrals in Action-Angle Variables)

Another example is the motivation for a canonical transformation to so-called action-angle variables. If the system

can (via a suitable coordinate transformation) be written as

$$\dot{I} = 0, \quad \dot{\varphi} = \omega(I) ,$$

then clearly each (constant) action variable I_i is a first integral,

$$L_t(I_i) = \sum_j \frac{\partial I_i}{\partial I_j} \dot{I}_j + \frac{\partial I_i}{\partial \varphi_j} \dot{\varphi}_j = 0, \qquad (2.16)$$

as the first set of terms is zero on substituting the Hamiltonian equations, while the second is zero by the hypothesis that the *I* are constants independent of the time-varying φ . Moreover, in this formulation there are *n* such integrals, and the system is integrable, as it is fully specified by the *I* and the *n* equations $\dot{\varphi} = \omega(I)$ given initial conditions $\varphi(0)$.

In the above an explicit time dependence was considered for the first integral; this is often omitted in the literature, but this need not be the case. It is important to note that when the literature speaks of a limited number of first integrals existing for a given system, the writer is usually referring to integrals of the form I(x) and not I(x,t). Appendix C.3 of [Verzijl, 2006] gives an example for a simple 2^{nd} order system, the harmonic oscillator, which is completely characterized by two first integrals, one of which must be explicitly time–dependent.

For the particular case of the n-body problem-class, it is known that there is a limit of precisely this kind (*i.e.* considering time–independence) on the number of first integrals, due to Bruns and Poincaré, which is discussed in section 3.1.2 in the context of the non-integrability of such systems.

Another result which should be noted in the context of Hamiltonian systems is the following.

Lemma 2.2.4 (First Integrals and Involution)

It follows from the Poisson bracket:

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$$

that for an autonomous system with Hamiltonian \mathcal{H} ,

$$0 = L_t(I(p,q)) = \{ I, \mathcal{H} \},$$
(2.17)

is an equivalent definition of a first integral, which is said to be *in involution* with the Hamiltonian \mathcal{H} .

An issue to which we return in chapter 3 is that of the integrability or reducibility of a system by means of first integrals. It is a theorem due to Liouville that a system with *n* degrees of freedom⁴ is integrable by quadratures if one can find *n* functionally independent first integrals in involution, that is $I_{i,j}$ such that:

$$\{I_i(p,q), I_j(p,q)\} = 0 \qquad \forall i, j \in \{1, \dots, n\}.$$
(2.18)

Proof. The first part of this lemma is evident on using the definition of $\{I, \mathcal{H}\}$ the second is considerably more complicated, and we refer the reader to [Arnold, 1989, section 40 & 49], or alternatively [Katok and Hasselblatt, 1996, section 5.5].

⁴*i.e.* a 2n-dimensional phase–space.

2.2.2 Symmetry and First Integrals

Before continuing, it should be mentioned that Noether's theorem relates the symmetries of a dynamical system to its first integrals in a very elegant way, and may be useful as a guide in the investigation of a system with as-yet-unknown first integrals, by instead searching for the generating symmetries and invariance principles. While not central to present work, we note the result here in connection with a remark at the end of section 6.4.2.

Theorem 2.2.5 (Noether's Theorem)

We give the standard (Lagrangian) formulation following Arnold, which for the case of *n*-body dynamics is equally applicable though the formulation is Hamiltonian. The principle itself is in fact more general.

Given a symmetry expressed by some mapping $h: M \to M$,

$$\mathcal{L}(h(\mathbf{x})) = \mathcal{L}(\mathbf{x}) , \qquad (2.19)$$

with a Lagrangian H and a vector $\mathbf{x} \in \mathbf{T}\mathbb{Q}$; if the system admits the one parameter group of diffeomorphisms $h^s : \mathbb{Q} \to \mathbb{Q}, s \in \mathbb{R}$, then the Lagrangian system of equations corresponding to L has a first integral $I : \mathbf{T}\mathbb{Q} \to \mathbb{R}$. In local coordinates it takes the form:

$$I(q,\dot{q}) = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} \frac{dh^{s}(\mathbf{q})}{ds} \bigg|_{s=0} = \mathbf{p} \frac{dh^{s}(\mathbf{q})}{ds} \bigg|_{s=0}.$$
(2.20)

Proof. Mathematically, this expresses the physical concept that a diffeomorphism to an alternate path via the symmetry should not change the Lagrangian, or:

$$\frac{\mathrm{d}\mathcal{L}\left(h^{s}(\mathbf{q})\right)}{\mathrm{d}s}=0$$

The result follows on substituting the Euler–Lagrange equations. Evaluating the expression at s = 0 expresses the choice of taking the value of the integral in the untranslated system. For details, see *e.g.* [Arnold, 1989, José and Saletan, 1998].

Examples of symmetries generated in this way are the conservation of linear momentum due to translational symmetry, of angular momentum due to rotational symmetry and of energy due to translational symmetry in time.⁵ The interested reader is also referred to [Golubitsky and Stewart, 2000] for a much more detailed discussion of new and ongoing research in this area.

2.3 Stability & Perturbation Theory

This section reviews a few key notions from the theory of stability analysis and perturbation theory for ordinary differential equations. We follow largely the treatment of [Verhulst, 2000], but also borrow from [Katok and Hasselblatt,

⁵For specific mechanical systems, we do not mean to assert that this can be assumed a priori.

1996] and other sources where appropriate, with the goal of briefly introducing the minimum necessary basic definitions and theorems needed further on.

It has also been attempted to impose a uniform convention and notation which will be followed throughout the thesis. The reader with a grounding in these techniques may prefer to skim the chapter briefly for notational conventions only.⁶

2.3.1 Flows and Mappings

Consider the autonomous system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.21}$$

and let \mathbf{x}_c be some critical (equilibrium) point, which we may take to be the origin without loss of generality. We also consider in the following periodic solutions $\phi(t)$ which exhibit the property that $\exists T < \infty$ such that $\phi(t) = \phi(t+T)$.⁷

Definition 2.3.1 (Time–Advance and Stroboscopic Map)

The stroboscopic map of a continuous flow x(t) in some space M, is the map $g_{t_0}: M \to M$, such that

$$g_{t_1}(x(t_0)) = x(t_0 + t_1) \quad \text{and consequently}$$

$$(2.22a)$$

$$e^{(k)}(x(t_0)) = x(t_0 + t_1) \quad \text{for the iterated men}$$

$$(2.22b)$$

$$g_{t_1}^{(r)}(x(t_0)) = x(t+kt_1)$$
 for the iterated map. (2.22b)

It is the most straightforward discretization of a flow, returning the solution to a continuous system of differential equations advanced by time t_0 . The case $t_0 = 1$ is also called the time-one map, and taking t_0 arbitrary it is usually referred to as the *time-advance map* g_t :

$$g_t(x(t_0)) = x(t_0 + t)$$
(2.23)

Definition 2.3.2 (Poincaré Sections & Maps)

A Poincaré section is an at most (n-1)-dimensional transversal to the flow described by equation (2.21). Let this transversal be denoted V, a manifold which is punctured by the (periodic) orbit we consider and nowhere tangent to it.



Figure 2.1: Example of a Poincaré section V and flow-induced map P

The Poincaré map of a flow, then, is the return map $P: V \to V$ induced by the flow, often denoted the "map of first return" (illustrated in figure 2.1). Note that if at a moment of intersection the *point* $\phi(t_0) \in V$ then for a periodic or-

⁶See also appendix A in this regard.

⁷This is only some notation for the special case of a periodic orbit $x(t;x_0,t_0) \equiv \phi(t)$, of course. The same holds for $\gamma(x_0)$ below, which are orbits $x(t;x_0,t_0)$ puncturing the transversal manifold V at time t_0 at $x_0 \in V$.

bit $\phi(t_0 + T) = \phi(t_0) \in V$ as well, and we have a return of the flow, though in general it may return at some earlier $t_0 < t_1 < t_0 + T$ as well.

By the continuous dependence of solutions on initial conditions, this defines the mapping P for some neighborhood in V punctured by orbits $\gamma(x_0)$ passing near $\phi(t_0)$, which also return to V if not exactly periodic. As usual, P² denotes the twice iterated Poincaré map, or the map of second return, and in general Pⁿ the map of n^{th} return to the section V.

This introduction of Poincaré maps allows the formulation of geometric approaches to stability which complements the usual notions near a critical point, though these will not be pursued further here.⁸

2.3.2 Perturbation Theory

The goal of perturbation theory is to consider the effect of a small perturbation of a system. The classic example is a disturbance δx to a solution $x \to x + \delta x$ of the differential equation $\dot{x} = f(x)$. In engineering, this is typically done in the context of stability analysis of some particular solution to the differential equations, though the sense in which it is discussed here is the study of small perturbations of the system itself, which may lead to qualitatively different behavior.

As motivated further on, this is in fact also the approach on which much of the work in this thesis will be based, though both perspectives are useful in different contexts. Two typical examples of these kinds of perturbations are:

- 1. The perturbation of a simple Hamiltonian \mathcal{H}_0 to a more complex one modulated by some small parameter ε : $\mathcal{H}_0 \rightarrow \mathcal{H}_0 + \varepsilon \mathcal{H}_1$; however, as the equations of motion are derived from the Hamiltonian, this is of course essentially the same as the following:
- 2. The perturbation of a system of differential equations, e.g. the harmonic oscillator, by a small nonlinearity in the system, *e.g.* $\ddot{x} + x = \varepsilon (1 x^2) \dot{x}$ (the Van der Pol oscillator).

Before considering these, we first define what it means for a perturbation to be $\mathcal{O}(\epsilon)$.

Definition 2.3.3 (Landau Order Symbol) The Landau "big O" notation for the order O(...) is defined as follows:

$$\begin{split} \delta_1(\varepsilon) &= \mathcal{O}(\delta_2(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0 \quad \text{if} \\ \exists \ k \in \mathbb{R} \quad \text{such that} \quad \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} \leq k \quad \text{as} \quad \varepsilon \to 0 \end{split}$$

The qualification "as $\varepsilon \to 0$ " is usually omitted, but always implicit. Remark that by this definition, $a\varepsilon^3 + b\varepsilon^4 = O(\varepsilon^3)$ but not $O(\varepsilon^4)$, as the term $a\varepsilon^3$ would lead to a bound k/ε which is no bound at all as $\varepsilon \to 0$. However, remark that the converse *is* true: $a\varepsilon^3 + b\varepsilon^4 = O(\varepsilon^2)$ for example, as now any real number of the correct sign is a sufficiently strong bound as $\varepsilon \to 0$.

There is a number of approaches to the study of perturbations of this latter kind above, *i.e.* of a system of differential equations. We briefly discuss two in particular, for their relation to the approaches to problems in astrodynamics con-

⁸The interested reader may consult [Verzijl, 2006, chapter 2].

sidered later:

- 1. Expansions of the equations and perturbation terms on a given timescale;
- 2. Normalization transformations which seek to recast the equations to a form as close as possible to a linear system, which is in general obstructed by resonances in the equations.

Expansions

The starting point for the study of a perturbed system is usually its expansion in powers of the small perturbing parameter ε . Consider $\dot{x} = f(x, t, \varepsilon)$.

A naive expansion⁹ would be to simply take both f and x expanded as a series in powers of ε :

$$f(x,t,\varepsilon) = f_0(x,t) + \varepsilon f_1(x,t) + \varepsilon^2 \cdots, \qquad (2.24a)$$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 \cdots .$$
(2.24b)

Similarly, in seeking periodic solutions, one may take for the problem $\dot{x} = f(x,t,\varepsilon)$ corresponding to the unperturbed problem $\dot{x}_0 = f(x,t,0)$

$$x(t_0) = x_0(t_0) + \mu \qquad \text{with } \mu \text{ accounting for unknown deviations in initial condition.}$$
(2.25a)
$$x(t) = y(t) + x_0(t) \qquad \text{is then a translation used to obtain}$$
(2.25b)

$$\dot{y} = f(y + x_0, t, \varepsilon) - f(x_0, t, 0)$$

$$\dot{y} \equiv F(y, t, \varepsilon) \qquad \text{with initial condition } y(t_0) = \mu.$$
(2.25c)

However, there is a problem with the expansion (2.24) and the formulation for periodic solutions so far presented, in that such an expansion to order $O(\varepsilon^m)$ provides an approximation of x(t) to order $O(\varepsilon^{m+1})$ accuracy, but *only* on a timescale of order O(1) (in some appropriate units).

Theorem 2.3.4 (Poincaré Expansion Theorem)

Noting that the second formulation above contains the first, consider the problem

$$\dot{\mathbf{y}} = F(\mathbf{y}, t, \mathbf{\epsilon}), \quad \mathbf{y}(t_0) = \boldsymbol{\mu} \,. \tag{2.26}$$

Further let $|t - t_0| < h$, $y \in D \subset \mathbb{R}^n$, $0 < \varepsilon \le \varepsilon_0$, $0 < \mu \le \mu_0$.

If $F(y,t,\varepsilon)$ is continuous with respect to its arguments for $||y|| < \rho$ for some radius ρ and ε bounded as above, then the solution y(t) can be expanded in a convergent power series with respect to ε and μ in some neighborhood of the unperturbed $\varepsilon = \mu = 0$, convergent on a timescale of O(1).

Proof. See [Verhulst, 2000, section 9.4]

⁹See also [Verhulst, 2000, theorem 9.1 & 9.2] for details on the existence and convergence of such expansions.

This result ensures that the naive approach works, but only in a very limited sense. What we typically seek, however, is an expansion or approximation that is accurate on a longer timescale, e.g. $O(1/\epsilon^k)$ for *k* as large as possible. To this end other techniques have been developed which are summarized here, of which the third will be discussed in detail later in the thesis.

- The Poincaré-Lindstedt technique, also known as continuation.¹⁰ This technique seeks convergent series approximations of *periodic* solutions to a system. Part of its power lies in the ability to construct approximations on time–scales considerably longer than that of the period T of the orbit being approximated. However, it is beyond the scope of the present report to discuss the method in detail here.
- The method of averaging is another method, pioneered by Lagrange, specifically in the context of his study of the 3-body problem as a perturbation of the 2-body problem.¹¹ The approach is to seek asymptotic series¹² solutions for an equation in *Lagrange standard form*, $\dot{y} = \varepsilon f(y, t)$.

This is shorthand for

$$\dot{y} = \varepsilon \Phi^{-1} g(\Phi(t)y, t)$$
 with (2.27a)

$$x = \Phi(t)y$$
 a transform using $\Phi(t)$, the fundamental matrix of solutions (2.27b)
to the unperturbed ($\varepsilon = 0$) case of the system:

$$\dot{x} = A(t)x + \varepsilon g(x,t) . \tag{2.27c}$$

Once in this form, the equations may be averaged over t by the argument that the variations are small enough in magnitude, due to the leading ε term, to warrant this. This typically leads to $O(\varepsilon)$ accurate approximations on a time-scale of $O(1/\varepsilon)$, and is detailed in [Verhulst, 2000, chapter 11], where the method is also illustrated for the study of periodic orbits.

• The method of multiple time-scales, which recasts the equations of a problem in a form which makes the (often different) time-scales explicit, through transformations of the type $\tau = \varepsilon^k t$ in the expansions. This then leads to different validity time-scales $O(1/\varepsilon^k)$ for each characteristic scale of the dynamics. We shall have considerably more to say about this method and its extensions beyond application specifically to *time* in section 4.3, and so will postpone the discussion to there.

2.3.3 The KAM Theorem

Next, we give a brief discussion of the KAM theorem, due to Kolmogorov, Arnold and Moser, which essentially describes what happens to the solutions of an integrable Hamiltonian system under small perturbations. While a very technical theorem with regard to proof, what it asserts, in essence, is simple: that sufficiently small perturbations of a dynamical system do not drastically alter the structure of the phase space.

The following summary is very brief; for deeper discussion, many texts are available, such as e.g. [Verhulst, 2000, Arnold, 1989, José and Saletan, 1998] in general and [Siegel and Moser, 1995] with a detailed direct application to

¹⁰Which finds specialized application in software tools for the visualization of trajectories, periodic orbits and manifolds in such tools as AUTO, Content, etc. in particular.

¹¹The reader is referred also to the discussion of sections 3.3 and 8.2.4 in particular in connection with this approach.

¹²These are series which are not *necessarily* convergent, though truncation to a finite order may give (often arbitrarily) good results.
celestial mechanics.

Theorem 2.3.5 (Kolmogorov-Arnold-Moser)

Suppose that for a Hamiltonian system with $\mathcal{H}(p,q)$ containing terms small to $\mathcal{O}(\varepsilon)$, one has a canonical transform to action-angle coordinates (I, φ) such that

$$\dot{t} = 0 + \varepsilon f(I, \varphi) \tag{2.28a}$$

$$\dot{\mathbf{\phi}} = \mathbf{\omega}(I) + \mathbf{\varepsilon}g(I, \mathbf{\phi})$$
 (2.28b)

This system is of course integrable when $\varepsilon \to 0$, with *n* first integrals *I*, and can be studied by the methods of Birkhoff-Gustavson normalization.¹³ As noted above, the motion of the system takes place on invariant tori parameterized by the angle variables φ_i and frequencies $\omega(I)$. Suppose now additionally that¹⁴

$$\mathcal{H}(I, \varphi) = \mathcal{H}_0(I) + \varepsilon \mathcal{H}_1(I, \varphi) . \tag{2.29}$$

The KAM theorem then asserts that for $\varepsilon > 0$, many of the invariant tori which exist in the integrable case persist, though they may be somewhat deformed by the perturbation. Specifically, if \mathcal{H}_0 is non-degenerate:

$$\left|\frac{\partial^2 \mathcal{H}_0}{\partial I^2}\right| \neq 0, \qquad (2.30)$$

then most of the invariant tori which exist in the integrable case of the unperturbed system persist for $\varepsilon > 0$ and sufficiently small; moreover the Lebesgue measure¹⁵ of the complement of the set of tori tends to zero as $\varepsilon \to 0$.

As the system is perturbed, we distinguish of the original invariant tori those resonant (with $\frac{\omega_i}{\omega_j} \in \mathbb{Q}$) and those non-resonant (with $\frac{\omega_i}{\omega_j} \notin \mathbb{Q}$). The KAM theorem further asserts specifically that the resonant tori will in general disintegrate into chaotic regions known as *resonance gaps* while it is the non-resonant tori which mostly persist with slight deformation.

Proof. See e.g. [Arnold, 1989, José and Saletan, 1998]

Note that as $\omega = \frac{\partial \mathcal{H}_0}{\partial I}$, the non-degeneracy requirement is essentially that the frequencies be decoupled from one another when considering the different tori, characterized by *I*, in phase space.

2.3.4 Small Denominators

In general, lacking the means to find a simple closed analytical expression for the solution of a given problem (whether an *n*-body problem or a derived– or sub–problem of some form in the context of this thesis), we attempt to find a series expansion for the solution instead, which solves the problem asymptotically, and ideally converges quickly.

¹³A normalization method specifically for Hamiltonians, see e.g. [Verhulst, 2000, chapter 13].

¹⁴The normalization to $O(\varepsilon)$ ensures that we can eliminate φ (likewise to $O(\varepsilon)$) as a variable, from the term $\mathcal{H}_0(I)$.

¹⁵Lebesgue measure might, for the unfamiliar reader be thought of as a formalization of the intuitive notion of the *size* of a set; for details at an introductory level, see *e.g.* [Capinski and Kopp, 2004].

In the case of the problems of celestial mechanics however, the search for general solutions in terms of series expansions is often quickly obstructed by the presence of small denominators of the form:

$$\frac{1}{((x-x_i)^2+(y-y_i)^2+(z-z_i)^2)^{k/2}}$$

(k integer) which explode on close approaches to bodies in the problem (e.g. body i in this example, when $x, y, z \rightarrow x_i, y_i, z_i$ during a close approach).

Likewise in the more general case of Hamiltonian systems with resonances (of which the above are a special case), we typically encounter problems when we try to analyze the system by casting it in normal form via some canonical transformation (see *e.g.* [Verhulst, 2000, section 13.3] and [Tuwankotta, 2002]). Non–resonant terms can be transformed away, but corresponding to each resonance we get small denominators as well.¹⁶

In the present work our concern will primarily be with the fact that since forces typically take the form introduced here, we may expect that force terms may grow by orders of magnitude during close approaches, and this is all the more relevant given our interest in ballistic lunar capture,¹⁷ There is, however, a suitable method for dealing with the resulting discrepancies in force magnitudes, which we will introduce in section 4.3.3 further on in this thesis.

 $^{^{16}}$ An interesting question, in modeling solar system dynamics, for example, is whether such resonances actually lead to instability of the *n*-body system, and has been studied since the 1800's. For a further discussion coupled to an informative historical review, we refer the reader to [Giorgilli, 1998].

 $^{^{17}}$ In which a satellite will typically depart from a near–Earth orbit, possibly encounter the Moon on an outbound leg, and return to it with a very close approach on the inbound leg leading to capture; *cf.* section 3.4.

Chapter 3

Astrodynamics and the Capture Problem

This chapter will treat a progression of relevant problems in astrodynamics, leading up to a model for considering the 4–body ballistic lunar capture problem. The emphasis will be on viewing the problems from the point–of–view of perturbation theory, specifically as subsequent perturbations of the 2–body problem.

We first introduce the n-body problem as a framework and consider the reducibility of the system via first integrals. After discussing the 2-body problem, the only case in which there is a closed, analytical solution, we discuss the 3-body problems and the 4-body capture problem as perturbations in some detail, as well as reductions, restrictions and the rationale for them.

Treatments of all but the last problem may be found in *e.g.* [Wakker, 2002a,b], [Vallado, 2004] or [Szebehely, 1967], while the literature on the latter capture problem in particular is referenced separately in sections 3.4 and 3.5.

3.1 The *n*-Body Problem

Consider first the *n*-body problem, which we approach here¹ as generalizing Newton's law of gravitation for the attractive force k/r^2 between two bodies at a distance $r = ||\mathbf{r}||$ to *n* bodies, *cf.* figure 3.1. The force between any 2 bodies may be derived from a potential *U*:

$$U = -\frac{Gm_im_j}{r_{ij}} \quad \text{and} \quad \mathbf{F} = -\nabla U = \frac{Gm_im_j}{r_{ij}^3} \mathbf{r}_{ij} , \quad \text{where}$$

$$r_{ij} = \|\mathbf{r}_{ij}\| = \|\mathbf{r}_j - \mathbf{r}_i\| \quad \text{"from } i \text{ to } j \text{"such that}$$

$$\|\mathbf{r}_{ij}\| = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} .$$

$$(3.1)$$

¹It may equivalently be approached from the Lagrangian or Hamiltonian perspectives, and a number of the problems of this chapter are given and/or derived from those in appendix B.

In the *n*-body setting, this is generalized to the potential U_i for a body *i*, given as

$$U_i = -\sum_{j=1,\neq i}^n \frac{Gm_i m_j}{r_{ij}} \quad \text{where}$$
(3.2a)

$$\mathbf{F}_{i} = -\nabla_{i} U_{i} = -\frac{\partial U_{i}}{\partial \mathbf{r}_{i}}, \qquad (3.2b)$$

with the convention of [Wakker, 2002a, Szebehely, 1967] that \mathbf{F}_i is the force on body *i*, and thus a sum of components, each aligned with the vector $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ for each pair of bodies *i*, *j* (*cf.* figure 3.1). Then with Newton's second law for *n* forces acting on body *i*, one obtains the *n*-body equations of motion as:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1,\neq j}^n \frac{Gm_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \quad \text{or with } \mathbf{\rho}_j := Gm_j$$
(3.3a)

$$\ddot{\mathbf{r}}_i = \sum_{j=1,\neq j}^n \frac{\rho_j}{r_{ij}^3} \mathbf{r}_{ij} \quad \text{for} \quad i = 1, \dots, n \,.$$
(3.3b)



Figure 3.1: Inertial system and orientation for *n*-body problem

Throughout, the factor m_i in the potential for body *i* will generally be omitted, and one instead considers the acceleration potential $U_i := -\sum_{j \neq i} \frac{\rho_j}{r_{ij}}$. Thus, alternatively one can derive the accelerations from this scaled potential as:

$$\ddot{\mathbf{r}}_i = -\nabla_i U_i = -\frac{\partial}{\partial \mathbf{r}_i} \left(-\sum_{j=1,\neq i}^n \frac{\rho_j}{r_{ij}} \right) = \sum_{j=1,\neq i}^n \frac{\rho_j}{r_{ij}^3} \mathbf{r}_{ij} \,. \tag{3.4}$$

These are the equations of motion in an inertial reference frame, and may need to be transformed to another reference frame to facilitate analysis or numerical solution as in [Verzijl, 2005]. Rather than proposing a general reformulation here, we will discuss these as needed in the following sections.

3.1.1 Integrals of Motion

An important result in the study of celestial mechanics has traditionally been the derivation of (first) integrals of motion. 10 are known for the general n-body problem (see *e.g.* [Vallado, 2004], [Wakker, 2002a]), which we formulate as a theorem.

Theorem 3.1.1 (Known Integrals)

There are 10 known integrals of the n-body problem in an inertial coordinate system: linear momentum (6), angular

momentum (3) and conservation of energy (1). That these are indeed first integrals will now be shown by constructive proof.

Proof.

• Summing (3.3a) over all *i*, we obtain:

$$\sum_{i} m_i \, \ddot{\mathbf{r}}_i = \sum_{i} \sum_{j, \neq j} \frac{G m_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \, ,$$

which is zero as each of $\frac{n(n-1)}{2}$ pair terms is canceled by the relation $\mathbf{r}_{ij} = -\mathbf{r}_{ji}$ (anti-symmetry), whence with constant masses:

$$\frac{d^2}{dt^2} \left(\sum_{i=1}^n m_i \mathbf{r}_i \right) = \sum_{i=1}^n m_i \dot{\mathbf{r}}_i = 0, \text{ and on integrating twice with respect to time,}$$
(3.5a)
$$\sum_{i=1}^n m_i \mathbf{r}_i = \mathbf{a}t + \mathbf{b}.$$
(3.5b)

The 2×3 scalar components of the linear momentum **a** and the constant **b** corresponding to the original (t=0) center of mass are the first 6 integrals.

• A further 3 follow from the conservation of angular momentum, vector multiplying (3.3a) by \mathbf{r}_i and summing

$$\sum_{i} m_{i} \mathbf{r}_{i} \times \ddot{\mathbf{r}}_{i} = \sum_{i} \mathbf{r}_{i} \times \sum_{j, i \neq j} \frac{Gm_{i}m_{j}}{r_{ij}^{3}} \mathbf{r}_{ij}$$
$$= \sum_{i} \sum_{j, i \neq j} \frac{Gm_{i}m_{j}}{r_{ij}^{3}} \mathbf{r}_{i} \times \mathbf{r}_{j},$$

by the properties of the cross product. This is null by anti-symmetry, yielding:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i} m_{i} \mathbf{r}_{i} \times \dot{\mathbf{r}}_{i} \right) = 0, \text{ which on integrating yields}$$
(3.6a)
$$\sum_{i} m_{i} \mathbf{r}_{i} \times \dot{\mathbf{r}}_{i} \equiv \mathbf{H} \to \text{const}.$$
(3.6b)

• Lastly, there is the conservation of energy. Taking the potentials $U_i = -\sum_{j,\neq i} \frac{\rho_j}{r_{ij}}$ as before, note that the potential $U = \sum_i U_i$, while still conservative, does *not* correspond to a central force field, and this effective potential at any point **r** will in general be *time-varying*, in contrast to the 2–body case.

It is, consequently, certainly not the case that the sum of kinetic and potential energy is *a priori* constant. Nonetheless, an energy integral is found as follows: taking the inner product of (3.3a) with $\dot{\mathbf{r}}_i$, and summing as usual,

$$\sum_{i} m_{i} \, \ddot{\mathbf{r}}_{i} \cdot \dot{\mathbf{r}}_{i} = \sum_{i} \sum_{j, j \neq i} \frac{Gm_{i}m_{j}}{r_{ij}^{3}} \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_{i} \,. \tag{3.7}$$

Then using:

$$\ddot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \,,$$

and remarking that since U_i depends on time only through position coordinates:

$$\sum_{i} \sum_{j,j\neq i} \frac{Gm_i m_j}{r_{ij}^3} \mathbf{r}_{ij} \cdot \dot{\mathbf{r}}_i = \sum_{i} \frac{\mathrm{d}U_i}{\mathrm{d}\mathbf{r}_i} \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_{i} \frac{\mathrm{d}U_i}{\mathrm{d}t}$$

one arrives (again assuming constant masses) at:

$$\frac{1}{2}\sum_{i}m_{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{r}}_{i}\cdot\dot{\mathbf{r}}_{i}\right)=\sum_{i}\sum_{j,j\neq i}Gm_{i}m_{j}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r_{ij}}\right).$$
(3.8)

Integrating with respect to time, we finally obtain the following conserved expression (where we associate the first term with the total kinetic energy and the latter with the total potential energy):

$$\mathcal{E} = T + U = \sum_{i} m_i \left(\frac{\dot{r}_i^2}{2} - \sum_{j, j \neq i} \frac{\rho_j}{r_{ij}} \right) = C.$$
(3.9)

Note, however, that this expression for the conserved energy holds only in inertial coordinate frames, as the velocities as seen in other frames contain components due to the rotation of the frame, the choice of which is arbitrary. We return to the issue in section 3.3 below.

Note further that clearly each of these also satisfies the formal condition $L_t(I) = 0$ following the formal definition of first integrals in section 2.2.

3.1.2 Non-Integrability

[Szebehely, 1967] discusses the notion of integrability in some detail. Essentially, the question is, for a general system, whether a dynamical system such as those we treat here, can have all its variables expressed as known analytic functions of time.

This is the case, for some problems in celestial mechanics (e.g. the planar restricted 3-body problem), when one allows as 'known functions' convergent infinite series, and uses regularization techniques to remove singularities. However, not much insight is gained from the complex expressions which result, as they do not lead to a better understanding of such issues as stability, and the behavior of the system in general terms.

What one is typically interested in is the reduction of the complex problem to a simpler one, preferably one which has an intuitive physical meaning. Nonetheless, a key negative result is known, (actually two results) due to Bruns and Poincaré, which precludes further first integrals of the 'nice' variety we demonstrated in the previous section.

Theorem 3.1.2 (Bruns' Nonexistence Theorem)

In the problem of n bodies, the only integrals which involve the coordinates and velocities algebraically, and which do not involve the time explicitly, are the integrals of the center of mass, the angular momentum, and of energy.

Proof. The proof may be found in the original paper by Bruns in [Acta Mathematica 11, 25] or the more recent [Whittaker, 1988, section 164]. \Box

Theorem 3.1.3 (Poincaré Nonexistence Theorem)

Stated for the restricted 3-body problem, let *H* be the Hamiltonian of the system, which is expanded in a small parameter ε as

$$H(q,p) = H_0(p) + \varepsilon H_1(q,p) + \varepsilon^2 H_2(q,p) + \dots,$$
(3.10)

where (p,q) are the Hamiltonian momenta and coordinates with components $\{p_1, p_2, q_1, q_2\}$. The Hamiltonian is periodic in q_1, q_2 with period 2π . Let now $\varphi = \varphi(p,q,\varepsilon)$ be a likewise periodic function of the coordinates, and then for sufficiently small ε , we expand in a convergent power series as

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{\varepsilon} \boldsymbol{\varphi}_1 + \boldsymbol{\varepsilon}^2 \boldsymbol{\varphi}_2 + \dots \tag{3.11}$$

Poincaré's theorem states that the restricted problem has no integral except the Jacobian integral, which is of the form $\phi = \text{const.}$

Proof. The full theorem and proof may be found in [Poincaré, 1957].

For a discussion of both, and their extensions (and relaxations of some conditions on the way the coordinates enter the integral) by Painlevé and Siegel, see [Szebehely, 1967, p.43]. These have also been further generalized in the following theorem.

Theorem 3.1.4 (Generalized Bruns' Theorem)

In the Newtonian (n+1)-body problem in \mathbb{R}^p with $n \ge 2$ and $1 \le p \le n+1$, every first integral which is algebraic with respect to positions, linear momenta and time, is an algebraic function of the classical first integrals: the energy, the p(p-1)/2 components of angular momentum and the 2p integrals that come from the uniform linear motion of the center of mass.

Proof. A detailed proof is given for this generalized theorem in [Juillard-Tosel, 2000]. \Box

Two final remarks may be made with regard to this discussion:

- 1. The theorems' results preclude the existence of other algebraic integrals of motion, but they do not preclude non-algebraic integrals, and more specifically, our construction of approximations to such;
- 2. The theorem, the reader will note, also does not preclude first integrals which are explicit functions of time see also our discussion of a toy problem in appendix C of [Verzijl, 2006].

3.2 The 2–Body Problem

The basic 2-body problem is well documented in the literature and will be discussed in some detail here, as it will form the basis of our approaches to more difficult problems using perturbation methods.

3.2.1 Standard Formulation

The 2-body problem is the restriction of the n-body problem to the following set of equations for bodies 1 and 2:

$$\ddot{\mathbf{r}}_{1} = \frac{\rho_{2}}{r_{12}^{3}} \mathbf{r}_{12} , \qquad (3.12a)$$
$$\ddot{\mathbf{r}}_{2} = \frac{\rho_{1}}{r_{21}^{3}} \mathbf{r}_{21} . \qquad (3.12b)$$



Figure 3.2: Basic 2-body problem dynamics

Viewed in isolation, there is no preferred coordinate system, and the most convenient is typically quasi-inertial and centered at one of the bodies. This is traditionally done by translating via $\mathbf{r} := \mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ to:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = +\frac{\rho_1}{r_{21}^3} \mathbf{r}_{21} - \frac{\rho_2}{r_{12}^3} \mathbf{r}_{12} \quad \text{or}
\ddot{\mathbf{r}} = -\frac{\rho_1 + \rho_2}{r^3} \mathbf{r} \,.$$
(3.13)

When $\rho_2 \ll \rho_1$ (e.g. with body 1 a planet and body 2 a satellite), one can additionally drop ρ_2 from the above to good approximation, but this is not essential. It does, however, raise the following interesting point.

In the relative formulation (3.13), one still considers the full 2–body problem , but if one were to restrict this in a way similar to the restriction of the 3–body problem (discussed in section 3.3.5), by simply assuming that the effect on the primary is negligible and dropping ρ_2 , *the equations of motion remain the same up to a constant*. In this sense the 2–body problem and what we might term a "restricted 2–body problem" or "1–body problem" are equivalent.

However, conservation of linear momentum itself motivates a similar reformulation of the problem. Instead of dealing with two 2^{nd} –order equations in arbitrary inertial space, why not refer them to their common center of mass, which is simply given by:

$$\mathbf{r}_{c} = \frac{m_{1}\mathbf{r}_{1} + m_{2}\mathbf{r}_{2}}{m_{1} + m_{2}} = \frac{\sum_{i=1}^{N=2} m_{i}\mathbf{r}_{i}}{\sum_{i=1}^{N=2} m_{i}}.$$
(3.14)

By the demonstration in section 3.1.1, this is simply a scaled version of the linear momentum, and is conserved. In fact, by simple Galilean relativity, we can take for our inertial system precisely the one moving with velocity **a** and located at **b** at time t = 0, and so set $\mathbf{r}_c \equiv \mathbf{0}$. Combining this with a description of the dynamics in relative terms allows one to reduce the system from a coupled 2^{nd} order system to a single 2^{nd} order ODE, as in equation (3.13).

This gives the motion of the 2 primaries relative to each other as a single ODE, but note that the position of either primary can be reconstructed if the ODE solution and the position of the center of mass are both known. To wit, define:

$$\mathbf{r}_{c1} = \mathbf{r}_1 - \mathbf{r}_c = \mathbf{r}_1 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{m_2 (\mathbf{r}_1 - \mathbf{r}_2)}{m_1 + m_2}$$
(3.15a)

$$\mathbf{r}_{c2} = \mathbf{r}_2 - \mathbf{r}_c = \mathbf{r}_2 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{m_1 (\mathbf{r}_2 - \mathbf{r}_1)}{m_1 + m_2} , \qquad (3.15b)$$

and then with the definition of **r**, note that the reconstruction follows from:

$$\mathbf{r}_{1} = \mathbf{r}_{c} + \mathbf{r}_{c1} = \mathbf{0} - \frac{m_{2}}{m_{1} + m_{2}} \mathbf{r} = -\mu \mathbf{r}$$
(3.16a)

$$\mathbf{r}_{2} = \mathbf{r}_{c} + \mathbf{r}_{c2} = \mathbf{0} + \frac{m_{1}}{m_{1} + m_{2}} \mathbf{r} = (1 - \mu)\mathbf{r}, \quad \text{where}$$

$$\mu = \frac{m_{2}}{m_{1} + m_{2}}.$$
(3.16b)

Remark that effectively what is being done here is the introduction of a uniformly moving inertial coordinate system in which the positions of the two primaries lie along a single vector \mathbf{r} at positions determined by their relative masses. This observation is precisely what motivates the introduction of the parameters μ , $1-\mu$ in the restricted 3-body problem.

Example 3.2.1 (Earth-orbiting Satellite)

As a special case of the above analysis, take for example $m_1 \gg m_2$ as is typical for a satellite '2' orbiting the Earth '1'; then $\mathbf{r}_2 \approx \mathbf{r}$ and $\mathbf{r}_1 \approx \mathbf{0}$, and the motion is determined by $\ddot{\mathbf{r}} \approx -\frac{Gm_1}{r^3}\mathbf{r}$ where m_1 is the mass of the Earth.

This serves to underline the fact that we need only solve equation (3.13) to describe the solutions of the 2-body problem (3.12), on introducing a coordinate transform motivated by the conservation of linear momentum. Indeed, while it is usually not thought of in this way, we have in fact reduced the problem using the integral, and this perspective is one to which we will return further on.

Lagrangian and Hamiltonian Formulations 3.2.2

In addition to the standard inertial formulation above, the 2-body problem may also be formulated using either the Lagrangian or Hamiltonian formalism. These are given for reference, and may also be found in appendix B together with the Hamiltonians for a number of other problems discussed in this chapter.

Lagrangian Formulation

Introducing:

$$M := m_1 + m_2$$
 and $\bar{\mu} := \frac{m_1 m_2}{m_1 + m_2}$,

the following generalized coordinates may be introduced:^{2,3}

²The subscript on the center-of-mass coordinate \mathbf{r}_b carries the meaning binary, which will be used in the 3-body problem and 4-body problems to distinguish it from the true center of mass \mathbf{r}_0 in those cases. In the 2-body problem the two coincide. ³Note also that the use of $\bar{\mu}$ is a convention which is used only here, and is not to be confused with the parameter $\mu = \frac{m_2}{m_1+m_2}$ will

be a dimensionless parameter in the 3-body problem, cf. section 3.3.5.

$$\mathbf{r} := \mathbf{r}_2 - \mathbf{r}_1 \;, \tag{3.17a}$$

$$\mathbf{r}_{b} := \frac{m_{1}\mathbf{r}_{1} + m_{2}\mathbf{r}_{2}}{m_{1} + m_{2}} = \frac{\sum_{i=1}^{N=2} m_{i}\mathbf{r}_{i}}{\sum_{i=1}^{N=2} m_{i}}.$$
(3.17b)

With these it is found (cf. appendix B) that

$$T = \frac{1}{2}M\dot{r}_b^2 + \frac{1}{2}\bar{\mu}\dot{r}^2 \quad \text{and} \quad U = -G\frac{m_1m_2}{r} , \qquad (3.18)$$

and so it follows that the Lagrangian is given by:

$$L = T - U = \frac{1}{2}M\dot{r}_b^2 + \frac{1}{2}\bar{\mu}\dot{r}^2 + G\frac{m_1m_2}{r} .$$
(3.19)

Hamiltonian Formulation

The Hamiltonian formulation approaches the problem as in the previous section, but introduces a slightly different set of generalized coordinates, to match the problem description in terms of \mathbf{r} , \mathbf{r}_b rather than $\mathbf{r}_{1,2}$. Introduce the same mass conventions and kinetic and potential energy as in the previous section to obtain (3.19).

Note that $M\bar{\mu} \equiv m_1m_2$. Now, intuitively the Hamiltonian is simply T + U (and expressed in generalized coordinates and momenta respectively), but this is demonstrated explicitly using the Legendre transform:

$$H = \sum_{j} p_{j} \dot{q}_{j} - L(q, p) .$$
(3.20)

With the same \mathbf{r} , \mathbf{r}_b as generalized coordinates, the generalized momenta become:

$$q_1 = \mathbf{r}_b, \quad p_1 = \frac{\partial L}{\partial \dot{q}_1} = M \dot{\mathbf{r}}_b , \qquad (3.21)$$

$$q_2 = \mathbf{r}, \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = \mu \dot{\mathbf{r}} , \qquad (3.22)$$

and then as might be expected,

$$H = M\dot{r}_{b}\dot{r}_{b} + \bar{\mu}\dot{r}\dot{r} - \left(\frac{1}{2}M\dot{r}_{b}^{2} + \frac{1}{2}\bar{\mu}\dot{r}^{2} + G\frac{m_{1}m_{2}}{r}\right)$$

$$= \frac{1}{2}M\dot{r}_{b}^{2} + \frac{1}{2}\bar{\mu}\dot{r}^{2} - G\frac{m_{1}m_{2}}{r} = \frac{p_{1}^{2}}{2M} + \frac{p_{2}^{2}}{2\bar{\mu}} - G\frac{M\bar{\mu}}{\|q_{2}\|}.$$
 (3.23)

The equations of motion are derived from the relations $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, which give:

$$\dot{q}_1 = \frac{p_1}{M} \longrightarrow \dot{\mathbf{r}}_b = \dot{\mathbf{r}}_b , \qquad (3.24a)$$

$$\dot{q}_2 = \frac{p_2}{\bar{\mu}} \longrightarrow \dot{\mathbf{r}} = \dot{\mathbf{r}},$$
 (3.24b)

$$\dot{p}_1 = 0 \longrightarrow M\ddot{\mathbf{r}}_b = 0,$$
(3.24c)

 $GM\bar{u}$
 $GM\bar{u}$

$$\dot{p}_2 = -\frac{GM\mu}{\|q_2\|^3} \longrightarrow \bar{\mu}\ddot{\mathbf{r}} = -\frac{GM\mu}{r^3}\mathbf{r}.$$
(3.24d)

In addition to simply reformulating the problem, two conclusions may also be drawn:

- 1. The motion of the center of mass is indeed given by $\mathbf{r}_b(t) = \mathbf{a}t + \mathbf{b}$, *i.e.* uniform motion, corresponding to conservation of the system's linear momentum. Consequently we can take a uniformly moving inertial system and set the origin to $\mathbf{r}_b = 0$.
- 2. Further, the relative motion of the bodies,⁴ is given by the equation of motion $\ddot{\mathbf{r}} = -\frac{G(m_1 + m_2)}{r^3}\mathbf{r}$, as was previously asserted.

3.2.3 Solution of the 2–Body Problem

This 2^{nd} order nonlinear differential equation (3.13) is of course the staple of elementary orbital mechanics, whose solutions [see *e.g.* Wakker, 2002a, Vallado, 2004] are conic sections given by:

$$r(\theta) = \frac{p}{1 + e\cos(\theta)}, \qquad (3.25)$$

where *e* is the eccentricity, θ the true anomaly and *p* the semi-parameter H^2/ρ with *H* the magnitude of the angular momentum $\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}}$, which is of course normal to the orbital plane. It gives the relation between the pair (*r*, θ) which describes the motion in the orbital plane. For a stable orbit *H* and *e* are constants and the time dependence enters only via the true anomaly θ .



Figure 3.3: Definition of conic sections for a satellite m_2 orbiting a primary m_1 whose position is the center of the inertial frame of reference ($r_{p,a}$ are pericenter and apocenter distances respectively, and *a* is the semi-major axis)

⁴and by extension relative to the center of mass, as explained above.

The description of such a (planar) orbit in 3 dimensional space is conventionally done by means of Kepler elements; this, as well as a far more detailed discussion of the properties of these solutions may be found in *e.g.* [Wakker, 2002a, Vallado, 2004].

However, the proof that this is indeed the solution is instructive both for our purposes later on, and for the way in which reduction of the solution can be performed explicitly using first integrals, and so it will be discussed here in full. A derivation of the Runge–Lenz vector will be given first, after which the 2–body problem solution is formulated as a theorem.

Lemma 3.2.2 (Runge-Lenz Vector)

The Runge–Lenz vector $\dot{\mathbf{r}} \times \mathbf{H} - \rho \frac{\mathbf{r}}{r}$ is a vector integral of the motion in the 2–body problem.

Proof. The proof begins by setting $\rho := G(m_1 + m_2)$ for simplicity. Then taking the cross product of the equation of motion with the angular momentum,

$$\ddot{\mathbf{r}} \times \mathbf{H} = -\frac{\rho}{r^3} \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) ; \qquad (3.26)$$

Observing that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, this can be rewritten as

$$\ddot{\mathbf{r}} \times \mathbf{H} = -\frac{\rho}{r^3} \left((\mathbf{r} \cdot \dot{\mathbf{r}}) \mathbf{r} - (\mathbf{r} \cdot \mathbf{r}) \dot{\mathbf{r}} \right)$$
(3.27a)

$$= -\frac{\rho}{r^2} \left(\dot{r} \mathbf{r} - r \dot{\mathbf{r}} \right) \,, \tag{3.27b}$$

where the identity $\mathbf{r} \cdot \dot{\mathbf{r}} = r\dot{r}$ has been used.⁵ Recalling that **H** was an integral and so constant in time, we can integrate with respect to time to find

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\mathbf{r}} \times \mathbf{H} \right) = \rho \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathbf{r}}{r} \right) \qquad \text{and so}$$
$$\dot{\mathbf{r}} \times \mathbf{H} = \rho \frac{\mathbf{r}}{r} + \mathbf{c} , \qquad (3.28)$$

with **c** an constant of integration.

The integral $\dot{\mathbf{r}} \times \mathbf{H} - \rho_r^{\mathbf{r}}$ is normal to the angular momentum and so in the plane of motion. This implies that in scalar components, it provides two independent integrals, which are precisely necessary to the full reduction of the 2–body problem, equation (3.13).

Theorem 3.2.3 (Solution of the 2–Body Problem) The 2–body problem is given by:

$$\ddot{\mathbf{r}} = -\frac{G(m_1+m_2)}{r^3}\mathbf{r} \,.$$

Its solution is given in a plane normal to the angular momentum (vector) integral $\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}}$ by

$$r = \frac{p}{1 + e \cos(\theta)}$$

where $p, e, \theta(t)$ are the semi-parameter, eccentricity and true anomaly as above.

⁵This identity concerns polar coordinates, where $\dot{\mathbf{r}} = (\dot{r}, r\dot{\phi})^{\mathsf{T}}$ and the component $\dot{r}//\mathbf{r}$ while $r\dot{\phi} \perp \mathbf{r}$.

Proof. The proof, following [Wakker, 2002a] and using the Runge–Lenz vector is as follows.⁶

First, noting that the angular momentum vector is conserved, it can be used to define a normal plane to which the dynamics are confined, which will be referred to as the *orbital plane (cf.* figure 3.3). Let the *z* axis be aligned with the vector **H**, and then observe that with some reference in the plane, we can introduce polar coordinates *r*, ϕ :

$$r = \|\mathbf{r}\| = \sqrt{r_x^2 + r_y^2}$$
 and $\phi = \operatorname{atan}(r_y/r_x)$.

In these coordinates,

$$\mathbf{H} = (0, 0, H)$$
 where $H = r^2 \phi$. (3.29)

Next, taking the outer product of the equation of motion with **H**, one may obtain as before the Runge–Lenz vector $\dot{\mathbf{r}} \times \mathbf{H} - \rho \frac{\mathbf{r}}{r} = \mathbf{c}$, as shown above. Taking the inner product of this expression in turn with **r**:

$$\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{H}) = \rho \left(r + \mathbf{r} \cdot \mathbf{c} \right) \,. \tag{3.30}$$

Observe that:

$$H^{2} = \mathbf{H} \cdot \mathbf{H} = \left(\mathbf{r} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right) \cdot \mathbf{H} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{H}\right) \cdot \mathbf{r} .$$
(3.31)

Substituted into the Runge-Lenz vector:

$$H^2 = \rho(r + \mathbf{r} \cdot \mathbf{c})$$
 or on rearranging and working out the inner product: (3.32)

$$r = \frac{H^2/\rho}{1+c\cos(\phi-\omega)},$$
(3.33)

where ω is the offset of a reference direction from which ϕ is measured relative to the *x*-axis. The constant *c* can be shown to be precisely the magnitude of the eccentricity vector $e = ||\mathbf{e}|| cf$. [Wakker, 2002a, section 6.3], while the true anomaly $\theta := \phi - \omega$ is taken measured from the *x* axis aligned with the eccentricity vector as in figure 3.3 (see also *e.g.* [Wakker, 2002a, chapter 6]).

With the notation for the semi–parameter introduced earlier, the proof is complete. \Box

It should be noted that one begins with 2×2^{nd} order vector equations in 3 coordinates, for a total of 12×1^{st} order equations after reduction. There are 10 known integrals of motion, as discussed above, and thus the system is reducible if 2 additional, functionally independent integrals are known.

For the 2-body problem, these are the in-plane components of the Runge–Lenz vector. In the general n-body problem by contrast, $n \ge 3$, no such remaining integrals present themselves, and this is precisely why only the 2-body problem can be solved analytically.

3.3 The 3–Body Problems

This section will make the first step in increasing complexity from the basis of the 2–body problem. After setting the stage with some alternative formulations of the general 3–body problem, two alternative approaches will be taken.

⁶There is an alternative proof via the differential equations themselves which might be considered 'traditional,' but it does not make the role of first integrals in the reduction of the problem as clear as the approach taken here.

On the one hand, it will be shown explicitly how the general problem can be reduced using the known integrals to the planar Jacobi 3–body problem (pJ3BP). On the other hand, we will introduce a number of simplifying assumptions (*restrictions*) which will motivate the planar circular restricted 3–body problem (pCR3BP).

3.3.1 Discussion of Perturbative Effects

Before moving on to the standard approaches to the 3-body problem, however, it is worthwhile to pause for a moment and consider a modeling question. Suppose that we have all the integrals, and by extension a solution to the 2-body problem. Then adding a small third body to the system can be thought of as a perturbation, but what *kinds* of perturbations are *a priori* possible?⁷

A reasonable answer to this question is shown in table 3.1, which is based on the interactions with the 3^{rd} body illustrated in figure 3.4; our intention is to show the different possible classes of perturbation of 2 interacting primaries by a 3^{rd} body.



Figure 3.4: Possible Interactions in 3–Body Model

The top row indicates the 4 possible mutual influences between bodies 1 and 3 in a candidate model, and taking any of these fixed, there are likewise 4 possibilities for the interactions between 2 and 3 (for a total of $4 \times 4 = 16$ possibilities).

With regard to the contents, remark the following:

• With regard to the top row, body 1 with the options Y/Y or Y/N is perfectly plausible, corresponding to the 1st primary influencing the small body and itself either being perturbed or not by the presence of the small body in return.

By contrast, the options N/Y and N/N are inconsistent by what is intended by the label *primary*: if the 1st primary (body 1) is influenced but does not influence, we are assigning it the role of a secondary body, while if

⁷The reader will note further that while we are presently concerned with the 3-body problem, this question can be asked for each step from an *n*-body problem to an (n+1)-body problem, and so forms a useful guide to modeling with what we might call the "perturbation perspective."

	Influences Body 3 1, 2 \rightarrow 3	/	Influenced by Body 3 $1, 2 \leftarrow 3$	
Body 1	Y/Y	Y/N	N/Y	N/N
Body 2	Y/Y	Y/Y	NR	NR
	Y/N	Y/N	NR	NR
	N/Y	N/Y	NR	NR
	N/N	N/N	NR	NR

Table 3.1: Matrix of Possible Modeling Choices in Perturbing the 2–Body Problem – Read as "Body 1 (or 2) Influences Body 3? (Yes, No or Not Relevant) / Is Influenced By Body 3? (Yes, No or Not Relevant)." 2 Classes of interaction marked Not Relevant due to the inconsistency of their implication with the modeling assumptions, as explained in the text.

it neither influences nor is influenced it may as well be neglected in the discussion of multi-body gravitational effects altogether.

- Having thus eliminated 2 classes of interaction for the 1^{st} primary, it is clear that with regard to the choices for the 2^{nd} primary (body 2), likewise presumed to be a dominant mass when compared with the small 3^{rd} body, the following 8 scenarios are possible:
 - **1:Y/Y, 2:Y/Y** corresponds to the full 3-body problem, taking all influences into account, which may be reduced to the planar Jacobi 3-body problem used in present work as a model. We designate this [Case A].
 - **1:Y/N, 2:Y/Y** corresponds to the 1st primary influencing the small body 3 but not being influenced by it, while the 2^{nd} primary both influences and is influenced. In principle there is nothing wrong with this as long as it is interpreted as the 2^{nd} primary being perturbed about its nominal 2–body orbit, rather than this perturbation also affecting the primary. We designate this [Case B].
 - **1:Y/N, 2:Y/N** corresponds to the restricted 3-body problem: bodies 1 and 2, the primaries, are not influenced by the small body 3 (and so may be assumed to satisfy a 2-body problem solution) while it is influenced by them both. This may of course be further restricted to the pCR3BP as we will do in the present work. We designate this [Case C].

The other cases are effectively reducible to these three, or irrelevant, as we proceed to show:

- **1:Y/Y, 2:Y/N** where the 2^{nd} primary influences the small body 3 but is not influenced by it is essentially the same as Case B, which is easily seen by swapping the bodies $1 \leftrightarrow 2$ here.
- **1:Y/Y, 2:N/Y** is inconsistent in that the 2^{nd} primary does *not* influence the small body 3 but *is* influenced by it; if we here swap the bodies $2 \leftrightarrow 3$ however, this becomes the same as the previous scenario and thus falls under Case B as well.
- **1:Y/N, 2:N/Y** in turn is likewise inconsistent in the same way as the previous case, and on reordering $2 \leftrightarrow 3$ to resolve this, this is seen to be the equivalent of Case C.
- **1:Y/Y, 2:N/N** however, is pointless in the same way as [1:N/N] above, as the 2nd primary now contributes nothing to the discussion. Strictly taken, [2:N/N] only rules out mutual influence between bodies 2 and 3, but that is inconsistent with [1:Y/Y] where the small body 3 influences the 1st primary, which in turn interacts with body 2 via the 2–body problemsolution: one cannot have it both ways.

1:Y/N, 2:N/N likewise corresponds to [1:N/N] above, and while not impossible on physical grounds,⁸ the 2nd primary does not participate in the 3–body problem dynamics, making it inconsistent with our modeling assumptions (a 3–body perturbation of the 2–body problem).

Thus under the assumption of perturbation of the 2-body problem by a third small mass, there are effectively only 3 model choices possible, as summarized in table 3.2.

Case	Body $1 \rightleftharpoons 3$	Body $2 \rightleftharpoons 3$	Problem Characterization
А	Y/Y	Y/Y	Full 3–body problem
В	Y/N	Y/Y	Perturbed 2 nd Primary
C	Y/N	Y/N	Restricted 3-body problem

Table 3.2: Summary of Consistent Modeling Choices in Perturbing the 2-Body Problem

In light of this analysis, we choose in the present work to consider only Case A (in section 3.3.3) and Case C (in section 3.3.5) as the limiting perturbations of the 2–body problem, and intentionally ignore B, as by studying cases A and C we capture the extremes of a range in which B is expected to hold the middle.

3.3.2 The General 3–Body Problem

The equations for the general problems are simply (3.4) with the summation limit n = 3.

$$\ddot{\mathbf{r}}_{i} = -\nabla_{i} U_{i} = \sum_{j=1,\neq i}^{n=3} \frac{\rho_{j}}{r_{ij}^{3}} \mathbf{r}_{ij}, \qquad i = 1, \dots, 3.$$
(3.34)

There are also, of course, Lagrangian and Hamiltonian formulations of this problem, and these are discussed briefly in appendix B. For present purposes, however, it will serve better to set that issue aside, and to work from the standard formulation towards a simpler one, first through reduction and then through restriction.

3.3.3 Reduction by First Integrals

It is worth pointing out that our point of departure is the unrestricted 3–body problem, which is a system of 3 coupled 2^{nd} order equations for 3 vector components of each body's position, which may be written as a system of 18×1^{st} order ODE–equations.

Further, there are 10 known integrals (for this as a particular case of the *n*-body problem) in an inertial coordinate system: linear momentum (6), angular momentum (3) and conservation of energy (1). Due to the complexity of the equations as they stand, in theory, reduction is a sensible approach, and this system can be reduced to $18 - 10 = 8 \times 1^{st}$ order ordinary differential equations (ODEs).

The following sections will develop these ideas properly, and the standard reference in what follows is the combination of [Wintner, 1947] and [Szebehely, 1967].

⁸This option would correspond to a 2^{nd} 2-body problem in which the 1^{st} primary so dominates the 3^{rd} small body that it is essentially a quasi-1-body problem.

Reduction by Linear Momentum

Consider first conservation of linear momentum for reduction. Taking inspiration from section 3.2's approach, let us introduce Jacobi coordinates. The usual equations (3.34) in the inertial frame are the starting point. The center of mass (barycenter) of the system is given by:

$$\mathbf{r}_0 = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_1 + m_2 + m_3} , \qquad (3.35)$$

while the binary barycenter *only* is likewise given by:

$$\mathbf{r}_b = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \,. \tag{3.36}$$

Conservation of linear momentum, *i.e.* $\sum_{i} m_i \dot{\mathbf{r}}_i = \mathbf{a}$ implies by integration that $\frac{\sum_{i} m_i \mathbf{r}_i}{\sum_{i} m_i} = \mathbf{r}_0(t) = \mathbf{a}t + \mathbf{b}$ where the constants \mathbf{a}, \mathbf{b} are fixed at time $t = t_0 = 0$, determining the evolution for all time. On changing to barycentric coordinates, *i.e.* with origin at $\mathbf{r}_0(t)$ for all t, the resulting system is inertial because it is moving uniformly, and the same equations hold (are invariant) under the transformation $\mathbf{r}'_i := \mathbf{r}_i - \mathbf{r}_0$.

This barycentric system is now taken as the new point of departure, and all vectors are referred to the new origin (dropping the convention of the prime on \mathbf{r}'_i).

Jacobi Coordinates

Remark first that the above is a coordinate transform, and so would be valid even without conservation of linear momentum. The trick to the conventional approach to reduction lies in subsequently changing coordinates to a new set $\{ \mathbf{q}, \mathbf{Q}, \mathbf{r}_0 \}$ such that while one still has 3 coordinate vectors, the third obeys $\ddot{\mathbf{r}}_0 = 0$ and so is redundant, reducing the equations to $3 \times 2 = 6$.

To that end, let:

$$\mathbf{q} = \mathbf{r}_2 - \mathbf{r}_1 \tag{3.37a}$$
$$\mathbf{O} = \mathbf{r}_3 - \mathbf{r}_b \ . \tag{3.37b}$$

The reconstruction from these new coordinates is given by:

$$\mathbf{r}_1 = \mathbf{r}_b - \mu \mathbf{q} \tag{3.38a}$$

$$\mathbf{r}_2 = \mathbf{r}_b + (1 - \mu)\mathbf{q} \tag{3.38b}$$

$$\mathbf{r}_3 = \mathbf{r}_b + \mathbf{Q} \,, \tag{3.38c}$$

and we will also have need of the relations

$$\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{q} \tag{3.39a}$$

$$\mathbf{r}_{13} = \mathbf{r}_3 - \mathbf{r}_1 = \mathbf{Q} + \mu \mathbf{q} \tag{3.39b}$$

$$\mathbf{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2 = \mathbf{Q} - (1 - \mu)\mathbf{q}$$
, (3.39c)

where as in our discussion above, one introduces $\mu = \frac{m_2}{m_1 + m_2}$ and $1 - \mu = \frac{m_1}{m_1 + m_2}$.

These definitions of the coordinates are illustrated in figure 3.5. Note in particular the similarity to the restricted 3– body problem discussed in section 3.3.5 where $\mathbf{r}_b = \mathbf{r}_0 = 0$, which comes down to 2 primaries at $-\mu$ and $1 - \mu$ along



Figure 3.5: Definition of Jacobi Coordinates

the q vector through the origin, and \mathbf{r}_3 free to move in their fixed gravity field (*cf.* our case C above).

Here however $\mathbf{r}_b \neq 0$, and in fact:

$$\begin{split} \mathbf{r}_{0} - \mathbf{r}_{b} &= \frac{\left(m_{1}\mathbf{r}_{1} + m_{2}\mathbf{r}_{2} + m_{3}\mathbf{r}_{3}\right)\left(m_{1} + m_{2}\right) - \left(m_{1}\mathbf{r}_{1} + m_{2}\mathbf{r}_{2}\right)\left(m_{1} + m_{2} + m_{3}\right)}{\left(m_{1} + m_{2} + m_{3}\right)\left(m_{1} + m_{2}\right)} \\ &= \frac{m_{3}(m_{1} + m_{2})\mathbf{r}_{3} - m_{3}(m_{1}\mathbf{r}_{1} + m_{2}\mathbf{r}_{2})}{\left(m_{1} + m_{2} + m_{3}\right)\left(m_{1} + m_{2}\right)} \\ &= \frac{m_{1}m_{3}(\mathbf{r}_{3} - \mathbf{r}_{1}) + m_{2}m_{3}(\mathbf{r}_{3} - \mathbf{r}_{2})}{\left(m_{1} + m_{2} + m_{3}\right)\left(m_{1} + m_{2}\right)} \,, \end{split}$$

which can be rewritten in the new coordinates as:

$$\mathbf{r}_{0} - \mathbf{r}_{b} = m_{3} \cdot \frac{m_{1}(\mathbf{Q} + \mu \mathbf{q}) + m_{2}(\mathbf{Q} - (1 - \mu)\mathbf{q})}{(m_{1} + m_{2} + m_{3})(m_{1} + m_{2})} .$$
(3.40)

This can always be computed, and in the barycentric system is a vector from the binary center of mass to the origin. However, if one expands the products in the expression $\frac{m_3(m_1+m_2)\mathbf{r}_3-m_3(m_1\mathbf{r}_1+m_2\mathbf{r}_2)}{(m_1+m_2+m_3)(m_1+m_2)}$ and cancels terms, what remains is considerably more insightful:

$$\mathbf{r}_0 - \mathbf{r}_b = -\frac{m_3}{m_1 + m_2 + m_3} \mathbf{r}_b + \frac{m_3}{m_1 + m_2 + m_3} \mathbf{r}_3 ,$$

which on rewriting becomes:

$$\mathbf{r}_0 - \frac{m_1 + m_2}{m_1 + m_2 + m_3} \mathbf{r}_b = \frac{m_3}{m_1 + m_2 + m_3} \mathbf{r}_3 .$$
(3.41)

Then recalling that in these coordinates the true 3-body barycenter is at the origin: $\mathbf{r}_0 = \mathbf{0}$, and

$$\mathbf{r}_{b} = -\frac{m_{3}}{m_{1} + m_{2}} \mathbf{r}_{3} \quad \text{and thus:} \mathbf{Q} = \mathbf{r}_{3} - \mathbf{r}_{b} = \frac{m_{1} + m_{2} + m_{3}}{m_{1} + m_{2}} \mathbf{r}_{3} .$$
(3.42)

With this and letting $v = \frac{m_3}{m_1 + m_2}$, the reconstruction equations become:

$$\mathbf{r}_{1} = -\frac{\mathbf{v}\mathbf{Q}}{1+\mathbf{v}} - \mu\mathbf{q} , \qquad (3.43a)$$

$$\mathbf{r}_2 = -\frac{\mathbf{v}\mathbf{Q}}{1+\mathbf{v}} + (1-\mu)\mathbf{q} , \qquad (3.43b)$$

$$\mathbf{r}_b = -\frac{\upsilon \mathbf{Q}}{1+\upsilon} \,, \tag{3.43c}$$

$$\mathbf{r}_3 = +\frac{\mathbf{Q}}{1+\upsilon} \,. \tag{3.43d}$$

Interpretation of Jacobi Coordinates

Keep in mind that the vector **q** through this binary barycenter at $-\frac{v\mathbf{Q}}{1+v}$ generates a line along which m_1, m_2 lie at fixed positions determined by their mass ratio ($-\mu$, $1-\mu$ respectively). In a sense we begin to see a precursor of the approach taken for the restricted 3-body problem in building on a 2-body problem, but without the restrictions used there.

The restriction that naturally suggests itself is roughly equivalent to setting $\mathbf{r}_b \approx \mathbf{0}$ in equation (3.43)), though based on the treatment above one sees that in reality the two bodies are moving along a fixed line parallel to \mathbf{q} which is slightly offset from the origin and itself moving aligned with \mathbf{Q} in time, its motion determined by the perturbing effect of the third body (3.43d). In the cases we will consider, this will likely be more–or–less rotating; and we illustrate this breakdown into a composite of two vectors in figure 3.6 below.

Jacobi Coordinate Equations of Motion

Using the Jacobi coordinates, and armed with an understanding of what the difference is between the full 2–body and 3–body problems in terms of the center of mass and its perturbation by the third body, we can now derive the equations of motion. Straightforwardly, one simply rewrites the equations (3.34) for $\ddot{\mathbf{r}}_{1,2}$ in terms of the Jacobi coordinates using the relations (3.39), to obtain:



Figure 3.6: Motion in the 3-body problem as perturbed 2-body motion.

$$\begin{split} \ddot{\mathbf{r}}_{1} &= + \frac{\rho_{2}\mathbf{q}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{3}(\mathbf{Q} + \mu\mathbf{q})}{\|\mathbf{Q} + \mu\mathbf{q}\|\|^{3}} \\ \ddot{\mathbf{r}}_{2} &= - \frac{\rho_{1}\mathbf{q}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{3}(\mathbf{Q} - (1 - \mu)\mathbf{q})}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^{3}}, \quad \text{which are subtracted from each other to obtain} \\ \ddot{\mathbf{q}} &= -(\rho_{1} + \rho_{2})\frac{\mathbf{q}}{\|\mathbf{q}\|^{3}} + \mu_{3}\left(\frac{\mathbf{Q} - (1 - \mu)\mathbf{q}}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^{3}} - \frac{\mathbf{Q} + \mu\mathbf{q}}{\|\mathbf{Q} + \mu\mathbf{q}\|\|^{3}}\right). \end{split}$$
(3.44a)
$$\ddot{\mathbf{r}}_{3} &= -\frac{\rho_{1}(\mathbf{Q} + \mu\mathbf{q})}{\|\mathbf{Q} + \mu\mathbf{q}\|^{3}} - \frac{\rho_{2}(\mathbf{Q} - (1 - \mu)\mathbf{q})}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^{3}}, \quad \text{and recalling the link between } \mathbf{r}_{3} \text{ and } \mathbf{Q}, \\ \ddot{\mathbf{Q}} &= -\frac{\rho_{1}(1 + \upsilon)(\mathbf{Q} + \mu\mathbf{q})}{\|\mathbf{Q} + \mu\mathbf{q}\|^{3}} - \frac{\rho_{2}(1 + \upsilon)(\mathbf{Q} - (1 - \mu)\mathbf{q})}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^{3}}, \qquad (3.44b) \end{split}$$

which checks against [Belbruno, 2004, sec 1.2] as well. Remark that these are indeed 2×2^{nd} order equations for 3 vector components, and so correspond to 12×1^{st} order equations, a reduction by 6 from the original 18 as expected.

Reduction by Angular Momentum

The next reduction to consider is that via conservation of angular momentum, $\sum_{i} \mathbf{r}_{i} \times \dot{\mathbf{r}}_{i} = \mathbf{H}_{0}$ (a constant), which would seem simple enough to implement. Two approaches suggest themselves:

1. One can try to straightforwardly eliminate 3 variables by writing them in terms of the others, and use these expressions for later reconstruction in place of the corresponding differential equations. With regard to this however, it is quickly found that we are hindered by the facts that the the eliminations are not unique, and give extremely complicated expressions in terms of coupled differential equations involving the unknown coordi-

nates, as we see in:

$$h_1 = m_1(y_1\dot{z}_1 - z_1\dot{y}_1) + m_2(y_2\dot{z}_2 - z_2\dot{y}_2) + m_3(y_3\dot{z}_3 - z_3\dot{y}_3), \qquad (3.45a)$$

$$h_2 = m_1(z_1\dot{x}_1 - x_1\dot{z}_1) + m_2(z_2\dot{x}_2 - x_2\dot{z}_2) + m_3(z_3\dot{x}_3 - x_3\dot{z}_3), \qquad (3.45b)$$

$$h_3 = m_1(x_1\dot{y}_1 - y_1\dot{x}_1) + m_2(x_2\dot{y}_2 - y_2\dot{x}_2) + m_3(x_3\dot{y}_3 - y_3\dot{x}_3), \qquad (3.45c)$$

with $\mathbf{H}_0 = (h_1, h_2, h_3)^{\mathsf{T}}$ constant. If we use an extended state vector and treat positions and velocities on equal footing this need not be a problem per se, but it is generally accepted that such substitutions complicate rather than clarify the situation.

2. Alternatively, the intuition developed with reduction by linear momentum suggests that it might be easier to find a coordinate transformation analogous to the conservation of linear momentum approach above, such that the reduction is implicit in the transformed system. Historically, this task has proven considerably harder than one might initially imagine, and in fact no coordinate transformation has been found to effect such a reduction for the general *n*-body problem, as detailed in [Wintner, 1947].

What has been found to be feasible is partial reduction by very complicated changes of variables (*e.g.* Poincaré's rectangular coordinates, Jacobi's approach) and so–called *contact transformations*, *cf.* [Malige et al., 2002] and *cf.* [Arnold et al., 1997]. Beyond their complexity, these carry the additional downside of costing us much of our physical insight into the problem, in stark contrast to the introduction of Jacobi coordinates previously.

Due to what might be termed their effective complexity-to-insight ratio, we will not consider them further here. We do point out, however, in the specific case of the 3-body problem that there is the additional geometric fact that the 3 bodies at each point in time form a triangle and thus span a plane. This can be exploited to find a coordinate transformation which Wintner [Wintner, 1947] gives for non-collinear configurations of the 3-body problem, *i.e.* general triangle configurations $\Delta(t)$. The transform and resulting Hamiltonian is discussed briefly in appendix B.4.

What is particularly interesting, however, is that such a triangle's motion, on reflection, always determines an instantaneous plane, and in the more general setting, we recall that the constancy of angular momentum likewise determines a plane perpendicular to \mathbf{H}_0 , which is the analogue of the orbital plane for the 2–body problem. Now while this latter perpendicular plane can always be defined, the dynamics need not necessarily be restricted to it. Turning that thought around, we note that we effect explicit conservation of the angular momentum if we *restrict* motion to such a plane, and by appropriate orientation of the coordinate system, we may choose the invariant z = 0 plane.

It is easily verified from the equations of motion that this is indeed an invariant plane for the 3-body problem, and that the angular momentum vector becomes a single *z*-component perpendicular to it, h_3 in equation (3.45) above. While this is a special, restricted case, unlike the triangle-configuration approach just described it *does* scale beyond 3 bodies.⁹

Reduction by Energy Integral

The last of the 10 algebraic integrals is the conservation of energy, and its discussion will be brief as it does not help to simplify the problem very much. The expression considered, expanded in full to demonstrate the roles of all variables

⁹Which is useful provided that the problem we wish to model can be considered planar to good approximation.

involved is:

$$\begin{split} E_0 &= \frac{1}{2} \left(m_1 (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + m_2 (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) + m_3 (\dot{x}_3^2 + \dot{y}_3^2 + \dot{z}_3^2) \right) \\ &- \frac{Gm_1 m_2}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} - \frac{Gm_1 m_3}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}} \\ &- \frac{Gm_2 m_3}{\sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2}} \,. \end{split}$$

This would theoretically be of use in reducing the order of our equations by 1, and the method that comes to mind is direct substitution, after isolating some variable. However, as before, the isolation of a velocity \dot{x}_i would result in a square-root of a complicated expression involving potential terms and which would complicate our expressions significantly.

Conversely the isolation of a position x_i may not even be possible in closed form, given that each occurs in a complicated manner in the *denominator* of *two* of the potential terms. While a change of coordinates simplifying the form sufficiently for our purposes may exist, none is available, to the best knowledge of the author.

Feasible and Infeasible Reductions

Against this backdrop, we might summarize as follows:

- Reducing by conservation of linear momentum is feasible and makes sense, but is effected not by direct substitution but by a change of coordinates (to barycentric inertial Jacobi coordinates) which gives conservation implicitly.
- Reducing by angular momentum *is* at least partially feasible by very complicated transformations after which we are left with a system which is *more* complicated and offers little or no physical interpretation. The same can be said of the reduction via the energy integral, which offers at most the possibility of isolation of a velocity.
- Reducing by angular momentum by a relatively simple coordinate transform analogous to that used for reduction by linear momentum is *infeasible*, and in fact "no such algebraic representation has ever been devised for n > 3" [Wintner, 1947, §391]. For the n = 3 case, the only 'nontrivial' approach is by analyzing the time-evolution of a triangle with the 3 masses at its vertices [Wintner, 1947, §394], for which Wintner gives a Hamiltonian, *cf.* appendix B.4. This approach, however, does not scale beyond 3 bodies.
- As Wintner describes motion *in a plane*, the approach suggests an appropriate planar problem (which reduces the problem to $2 \times 2 \times 2 = 8 \times 1^{st}$ -order equations), which as we have shown does explicitly conserve angular momentum, taken perpendicular to the plane.¹⁰ Thus this is a feasible approach, though formally it is a restriction rather than a reduction, albeit to the case of motion on an invariant manifold interesting in its own right.

We remark the following with regard to the latter approach, which we will take. For the Earth–Sun–Moon system (which is for the moment our primary interest as the setting for ballistic lunar capture trajectories), the inclination between the bicircular orbits (ES and EM) is approximately 23° . The problem cannot strictly be said to be planar, but

¹⁰Moreover this is *a priori* an invariant plane of the system, as is easily demonstrated, and so interesting in its own right.

the wealth of useful results, in particular those of Koon, Marsden *et al.* (*e.g.* [Koon et al., 2000, 2001, Gómez et al., 2001]) using the pCR3BP give us considerable reason to expect that it is a useful restriction for the problems we are interested in. We return to their work in section 3.4 below.

3.3.4 Planar Jacobi Problem Summarized

With the last 4 integrals having proven to be of little use, the most promising option is to return to the particular counterpoint of the planar problem, which has effectively been reduced to 8 equations from 12, by dropping 2×2^{nd} order equations. Remark, however, that from the perspective of the full problem this is not really a reduction by integrals so much as a restriction.

Indeed from the perspective of the planar problem one has indeed reduced its $12 (= 3 \times 2 \times 2^{nd} \text{ order})$ original equations by using the 4 (rather than 6) equations corresponding to conservation of linear momentum, *without* making use of the conservation of angular momentum (now 1 equation) and energy. However, the barycentric system in the plane now conserves both linear momentum and angular momentum.

Thus on these considerations we will consider as the maximally reduced system the following planar equivalent of the equations (3.44), which we denote the planar Jacobi 3–body problem (pJ3BP):

$$\ddot{\mathbf{q}} = -(\rho_1 + \rho_2) \frac{\mathbf{q}}{\|\mathbf{q}\|^3} + \rho_3 \left(\frac{\mathbf{Q} - (1 - \mu)\mathbf{q}}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} - \frac{\mathbf{Q} + \mu\mathbf{q}}{\|\mathbf{Q} + \mu\mathbf{q}\|\|^3} \right)$$
(3.46a)

$$\ddot{\mathbf{Q}} = -\frac{\rho_1(1+\upsilon)(\mathbf{Q}+\mu\mathbf{q})}{\|\mathbf{Q}+\mu\mathbf{q}\|^3} - \frac{\rho_2(1+\upsilon)(\mathbf{Q}-(1-\mu)\mathbf{q})}{\|\mathbf{Q}-(1-\mu)\mathbf{q}\|^3} \,.$$
(3.46b)

3.3.5 The Restricted 3–Body Problem

Next we turn to the restricted 3-body problem, and in particular its circular, planar restriction which will be considered in connection with new techniques for approximating first integrals further on. A full derivation of the problem is also given, as it is instructive for the capture problems which are derived further on.

Restriction

The problem is referred to as restricted on the basis of a number of simplifying assumptions that distinguish it from the general 3–body problem of the previous section, which will be discussed here.

- 1. To begin, one considers a system of 3 bodies, 2 of which have masses $m_1, m_2 \gg m_3$ and which are referred to as primaries; the third body with its much smaller mass is sometimes referred to as the secondary (*cf.* case C, table 3.2).
- 2. One then assumes, on the basis of these mass ratios, that the gravitational field of the 2 primaries is unaffected by the third body (and hence they orbit as the analytical solutions of the 2–body problem above), while their gravitational field (time-varying in inertial coordinates) in turn wholly determines the motion of the third body.

Nothing has yet been said about the nature of the motion of the primaries:

- If this is restricted to circular motion, *i.e.* setting e = 0 in the solutions of the 2-body problem, the result is the *circular restricted 3-body problem*, which is perhaps best tackled in the rotating coordinate system introduced in the next section. Note however that:
 - 1. This restriction does not force one of the primaries to rotate about the other per se: both still rotate (circularly) about the common center of mass, and this will only (roughly) coincide with one of the primaries for sufficiently large mass ratio m_1/m_2 .
 - 2. Also, the reader will note that this motion of the primaries is being explicitly prescribed, and so the usual formulation in the literature effectively studies only the motion of the third body in the presence of the gravity field induced by the primaries. Hence, one studies not a coupled system of 3 vector equations, but only a single vector (or 3 scalar) equations.
- If one does not restrict the motion of the primaries to circular orbits, but rather only to those which are closed, $0 \le e < 1$, the problem is called the *elliptic restricted 3–body problem*, and it is likewise best tackled in a rotating coordinating system, with the additional modification of periodically pulsating axes. This is not the approach which will be taken here, and indeed, though potentially a more realistic model, is not that common in the literature due to the small qualitative difference with the circular case for typical (e.g. solar system) configurations having only small eccentricities.
- It should also be noted that the problem of the motion of the third body may be restricted to the plane; whether this is a large or a small qualitative change likely depends on the application.

In principle this thesis will make the restriction on the grounds that as the rotation of the primaries is in the x, y-plane, the effects of the rotating coordinate system are still captured in the planar problem, while one may reasonably expect that the consideration of an additional dimension of motion will add nothing crucial to the discussion, while its omission likewise does not oversimplify.¹¹

Circular Restricted Problem Formulation

On these considerations it is proposed to consider the planar circular restricted 3–body problem in rotating coordinates in detail. The derivation will follow [Szebehely, 1967]. Note that following his conventions, in this section z, Z will denote complex variables for the planar problem, and not a third coordinate.

Rotating Coordinates

Recall that for the primaries, the motion is determined by the single vector **r** relative to the origin of the uniformly moving inertial system, *cf.* equations (3.16). The primaries are located at $-\frac{m_2}{m_1+m_2}\mathbf{r}$, $\frac{m_1}{m_1+m_2}\mathbf{r}$ respectively, as shown above, which becomes $-\mu\mathbf{r}$, $(1-\mu)\mathbf{r}$ on introducing the usual $\mu = \frac{m_2}{m_1+m_2}$.

The solutions of the 2-body problem for the vector \mathbf{r} describe a Kepler-ellipse with constant angular velocity $\boldsymbol{\omega}$. A direct consequence then, is that on keeping the same point of origin and introducing a co-rotating coordinate frame,

¹¹The reader may want to compare in this regard the work of the Caltech group [Koon et al., 2000, 2001], where as section 3.5 will briefly illustrate, a framework for ballistic capture has been developed using the techniques of Dynamical Systems Theory. Indeed, the core of their work rests on just the planar circular restricted problem, which was later differentially corrected using ephemerides for the planning of realistic solar system trajectories.



Figure 3.7: Psuedo-inertial system and rotating coordinates for the restricted 3-body problem

also with ω , then in this system the primaries will take the above positions along a *fixed* **r**, which may be taken as the *x*-axis. Letting $r_0 := \|\mathbf{r}(t = t_0)\| = \|\mathbf{r}\|$, this choice locates the primaries at

$$\begin{pmatrix} -\mu r_0 \\ 0 \end{pmatrix}, \begin{pmatrix} (1-\mu)r_0 \\ 0 \end{pmatrix}, \quad \text{respectively.}$$
(3.47)

This considerably simplifies the form of the gravitational field, as it is no longer time-dependent. The price paid is that the effects of the rotating coordinate system appear in the new equations of motion. To this end, consider a rotation given by the matrix:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ with inverse } \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix}, \quad (3.48)$$

where x, y are used for positions in the ω -rotating frame and X, Y in the inertial axes, as illustrated in figure 3.7. Let us denote the rotation matrix

$$R^{ir} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \quad \text{and observe that:}$$
(3.49)

$$R^{ri} = R^{ir^{-1}} = R^{ir^{\top}}, (3.50)$$

where the superscripts i and r refer to *inertial* and *rotating* frames respectively. The relative position vectors for the third body are then given as:

$$\mathbf{r}_{13} = (x + \mu r_0, y)^{\top}, \quad \mathbf{r}_{23} = (x - (1 - \mu)r_0, y)^{\top}$$
(3.51)

in the rotating frame, and equation (3.48) gives the transformation back to inertial coordinates.

Equations of Motion

To simplify the analysis, introduce at this point complex coordinates to effectively deal with both vector components in a single expression. Let complex z = x + iy be the position in rotating coordinates, and a complex factor $e^{i\omega t}$ account for the rotation, *i.e.* $Z = ze^{i\omega t} = X + iY$ in inertial coordinates. Observe that all these coordinates refer to the third body of mass m_3 . Now note that:

$$\frac{\partial^2 Z}{\partial t^2} = \left(\frac{\partial^2 z}{\partial t^2} + 2i\omega \frac{\partial z}{\partial t} - \omega^2 z\right) e^{i\omega t} , \qquad (3.52)$$

and one requires that:

$$\frac{\partial^2 Z}{\partial t^2} = \frac{\partial^2 X}{\partial t^2} + i \frac{\partial^2 Y}{\partial t^2} . \tag{3.53}$$

For the latter the accelerations in the inertial system are needed:

$$\ddot{X} = -\left(\frac{\rho_1(X + \mu r_0 \cos \omega t)}{r_1^3} + \frac{\rho_2(X - (1 - \mu)r_0 \cos \omega t)}{r_2^3}\right),$$
(3.54a)

$$\ddot{Y} = -\left(\frac{\rho_1(Y + \mu r_0 \sin \omega t)}{r_1^3} + \frac{\rho_2(Y - (1 - \mu)r_0 \sin \omega t)}{r_2^3}\right) \quad \text{where:}$$
(3.54b)

$$r_1 = |Z - Z_1| = |z + \mu r_0| = \sqrt{(x + \mu r_0)^2 + y^2},$$

$$r_2 = |Z - Z_2| = |z - (1 - \mu)r_0| = \sqrt{(x - (1 - \mu)r_0)^2 + y^2}$$

However, noting that:

- $(x + \mu r_0, y)$ in the rotating system refers to the same point as the above $(X + \mu r_0 \cos \omega t, Y + \mu r_0 \sin \omega t)$,
- and also keeping in mind that distances are invariant under these transformations,¹²

it is of course evident that one can simply write the equations of motion in the complex inertial frame as:

$$\ddot{Z} = -\left(\frac{\rho_1(z+\mu r_0)}{r_1^3} + \frac{\rho_2(z-(1-\mu)r_0)}{r_2^3}\right)e^{i\omega t}, \qquad (3.55)$$

And consequently, equating the above to expression (3.52) and canceling $e^{i\omega t}$,

$$\frac{\partial^2 z}{\partial t^2} + 2i\omega \frac{\partial z}{\partial t} - \omega^2 z = \left(\frac{\partial^2 x}{\partial t^2} + 2i\omega \frac{\partial x}{\partial t} - \omega^2 x\right) + i\left(\frac{\partial^2 y}{\partial t^2} + 2i\omega \frac{\partial y}{\partial t} - \omega^2 y\right).$$
(3.56)

On collecting real and imaginary terms separately yields:

$$\frac{\partial^2 x}{\partial t^2} - 2\omega \frac{\partial y}{\partial t} - \omega^2 x = -\frac{\rho_1(x + \mu r_0)}{|z + \mu r_0|^3} - \frac{\rho_2(x - (1 - \mu)r_0)}{|z - (1 - \mu)r_0|^3},$$
(3.57a)

$$\frac{\partial^2 y}{\partial t^2} + 2\omega \frac{\partial x}{\partial t} - \omega^2 y = -\frac{\rho_1 y}{|z + \mu r_0|^3} - \frac{\rho_2 y}{|z - (1 - \mu) r_0|^3} .$$
(3.57b)

 $^{^{12}}$ Formally, the matrix is an isometric transformation between the two coordinate systems: it preserves distances with respect to the Euclidean norm's induced metric.

Normalization

The convention in the literature, which is useful, is to take a step further in normalizing the equations by making them dimensionless. This will also allow us further on to reuse the analysis and simulation code for both Sun-Earth–Moon and Earth–Moon–Satellite configurations. To normalize following the standard convention [see *e.g.* Szebehely, 1967] one takes:

$$M = m_1 + m_2 \quad \text{as a mass-scale, such that:}$$
$$\frac{m_1}{m_1 + m_2} = 1 - \mu \quad \text{and}$$
$$\frac{m_2}{m_1 + m_2} = \mu \quad \text{which sum to 1;}$$
$$L = r_0 = \langle \|\mathbf{r}\| \rangle \quad \text{as a distance scale, and}$$
$$T = \frac{1}{\omega} \quad \text{as a time scale.}$$

With these scaling factors, note in particular the resulting acceleration terms:

$$\frac{\rho_1(x+\mu r_0)}{\left((x+\mu r_0)^2+y^2\right)^{3/2}} \longrightarrow \frac{(1-\mu)(x+\mu)}{\left((x+\mu)^2+y^2\right)^{3/2}},$$

$$\frac{\rho_2(x-(1-\mu)r_0)}{\left((x-(1-\mu)r_0)^2+y^2\right)^{3/2}} \longrightarrow \frac{\mu(x-(1-\mu))}{\left((x-(1-\mu))^2+y^2\right)^{3/2}} \quad \text{where the relation:}$$

$$\frac{G(m_1+m_2)}{\omega^2 r_0^3} \equiv 1 \qquad \text{has been exploited.} \qquad (3.58)$$

This is in fact simply a restatement of Kepler's third law for the primaries, which the reader will recall are indeed taken to satisfy the 2-body problem exactly. The reader will note also that we *do not* take $T = \frac{2\pi}{\omega}$ as a time-scale, which is the period associated with angular frequency ω . This is by no means essential, and our choice is nonetheless a correct choice of time-scale.

Consequently, the equations of motion of the planar circular restricted 3-body problem (pCR3BP) in dimensionless coordinates become:

$$\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x} = x - \frac{1 - \mu}{r_1^3} (x + \mu) - \frac{\mu}{r_2^3} (x - (1 - \mu)), \qquad (3.59a)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y} = y - \frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y \qquad \text{where:}$$
(3.59b)

$$\begin{aligned} r_1 &= |z + \mu| = \sqrt{(x + \mu)^2 + y^2} \\ r_2 &= |z - (1 - \mu)| = \sqrt{(x - (1 - \mu))^2 + y^2} \,. \end{aligned}$$

It should be emphasized that as indicated, the forces can indeed be derived from the scaled potential:¹³

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} .$$
(3.60)

¹³That this should be so is by no means *a priori* certain, but is an expected consequence of the gravitational field becoming conservative in the new rotating coordinates. Remark also at the same time that the left-hand-side expressions now contain velocities as well as accelerations, which is never the case for a conservative field in inertial coordinates.

For reference, the equations of motion in the full 3-dimensional rather than planar case, in the rotating coordinate system become (*cf.* [Wakker, 2002a]):

$$\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x} = x - \frac{1 - \mu}{r_1^3} (x + \mu) - \frac{\mu}{r_2^3} (x - (1 - \mu)), \qquad (3.61a)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y} = y - \frac{1 - \mu}{r_1^3} y - \frac{\mu}{r_2^3} y, \qquad (3.61b)$$

$$\ddot{z} = \frac{\partial\Omega}{\partial z} = -\frac{1-\mu}{r_1^3} z - \frac{\mu}{r_2^3} z, \qquad (3.61c)$$

which is intuitive as the *z*-component is unaffected by the rotation.

Jacobi Integral

Note that the 10 integrals of the general problem do not actually hold in the circular restricted 3–body problem per se, due to the assumptions placed on the third body and the restrictions of the role it plays for the motion of the other two. There will always be an error term proportional to m_3 with respect to exact conservation. However, conservation of energy, though violated, has a clear analogue in the conservation of the Jacobi integral, which is formulated next, and is the only known integral for the circular restricted 3–body problem.

Lemma 3.3.1 (Jacobi Integral)

Formulated for the full 3-dimensional case, there exists in these rotating coordinates a further constant of the motion due to Jacobi, given in dimensionless coordinates by:

$$\mathcal{I} = V^2 - 2\Omega$$
 or equivalently: (3.62)

$$\mathcal{I} = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(x^2 + y^2\right) - \frac{2(1-\mu)}{r_1} - \frac{2\mu}{r_2}, \qquad (3.63)$$

where in the planar case z may be dropped from both the V term and the r_1 , r_2 .

Proof. Multiplying (3.61) by \dot{x}_i and summing one obtains:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \dot{x}\frac{\partial\Omega}{\partial x} + \dot{y}\frac{\partial\Omega}{\partial y} + \dot{z}\frac{\partial\Omega}{\partial z} \quad \text{or}$$

$$\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = L_t(\Omega) = \frac{d\Omega}{dt} \quad \text{and so, on integrating:}$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = V^2 = 2\Omega - C, \qquad (3.64)$$

where *C* is determined by the initial conditions of the 3rd body. $C = -\mathcal{J}$, known as Jacobi's constant, is a first integral of the system in rotating coordinates, and plays a role analogous to the energy in inertial coordinates, as it can be shown that $C = -2\mathcal{E}$ [*cf.* Wakker, 2002a].

This new integral is also linked to two other issues which will not be discussed in detail here, but are worth mentioning:

- 1. The discussion of Belbruno's Weak Stability Boundaries in section 3.4.1, which are also to a certain extent determined by the value of Jacobi's constant *C*, [see *e.g.* Belbruno, 2004, chapter 3].
- 2. Hill's surfaces, or the surfaces of zero velocity, follow from setting $V = 0 = 2\Omega C$, which gives an equation

for manifolds which determine the boundary of the real-space regions accessible to a third body starting with given initial velocity, via the set $\mathcal{I}^{-1}(C)$. This issue is intimately linked with the Lagrange points, as discussed in [Wakker, 2002a, Deurloo, 2003] among others, and is introduced briefly in section 3.4.2.

Hamiltonian Formulation

There is also a corresponding Hamiltonian formulation of the problem, as shown in appendix B. In 2 dimensions:

$$H(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{x^2 + y^2}{2} - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2},$$
(3.65)

which with canonical coordinates

$$q_1 = x, \quad q_2 = y \quad \text{and}$$
 (3.66a)

$$p_1 = \dot{x} - y, \qquad p_2 = \dot{y} + x,$$
 (3.66b)

becomes

$$H(q, p) = \frac{1}{2}(p_1^2 + p_2^2) + p_1 q_2 - p_2 q_1 - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2}.$$
(3.67)

The corresponding equations of motion are:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1 + q_2 , \qquad (3.68a)$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2 - q_1 , \qquad (3.68b)$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = p_2 - \frac{1-\mu}{r_1^3}(q_1 - \mu) - \frac{\mu}{r_2^3}(q_1 + 1 - \mu), \qquad (3.68c)$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -p_1 - \frac{1-\mu}{r_1^3} q_2 - \frac{\mu}{r_2^3} q_2 \,. \tag{3.68d}$$

3.4 Ballistic Capture and Transfer Techniques

This section discusses the two classes of Earth–Moon transfer techniques based on Ballistic (Lunar) Capture (BLC), which we (as shorthand) refer to as ballistic capture transfers. While the underlying mathematical-physical structures are the same, the issue of finding such transfer trajectories can be approached from two distinct points of view: that of Weak Stability Boundaries (WSB's) on the one hand, and that of the manifold structure associated with the Lagrange point using the techniques of Dynamical Systems Theory (DST) on the other.

Note however, that

- Our focus here is not to give a comprehensive presentation of the issues. On the subject of WSB approaches, the most up to date source for this is [Belbruno, 2004], while for the DST view, we refer to [Koon et al., 2000, 2001]. We also draw on the discussion in [Deurloo, 2002, chapter 3], but will focus here primarily on the underlying concepts in continuity with the previous chapters.
- Consequently, our discussion will be largely qualitative, and our goal is to further motivate forward targeting within the WSB context as an algorithmic approach, while noting the direct connection with the DST techniques which underlie it.

3.4.1 WSB-based Transfers

In this discussion we follow Belbruno's presentation in Belbruno [2004], discussing first the concept of capture, and then its relation to the WSB. We then discuss the framing of the problem as arising from a degenerate case of the CR3BP which motivates an approach to finding and targeting the WSB, though a more robust approach exists which obviates the need for this.

Capture

Definition 3.4.1 (Capture)

By capture we intend a certain type of bounded motion with respect to a body or a system of bodies. In particular we define *permanent capture* as the case when as $t \to \infty$ the relative position of the captured body is bounded $\|\mathbf{r}\| < \infty$ while as $t \to -\infty$, $\|\mathbf{r}\| \to \infty$ may occur.

In practice, note that one may need to specify a more realistic bound $||\mathbf{r}|| \le r_{max} < \infty$ to formulate a tractable problem. As an example, consider ballistic capture around the Moon: a body which transits the Earth–Moon region and remains bounded at the Jupiter-radius about the Sun should not be considered captured for the purposes of such a problem, despite the fact that it satisfies the definition based only on bounded-ness.

[Belbruno, 2004] shows that the set of orbits leading to permanent capture (barring maneuvers) is a non-empty set of measure zero.¹⁴ Thus we may in general expect that in targeting capture we cannot expect to achieve permanent capture.

With respect to a specific body (which we take w/o loss of generality to be the second primary), we may link the concept of capture more concretely to orbital mechanics by noting that an object *i* at r_{i2} from it is 'captured' because its kinetic energy cannot overcome the energy of gravitational well in which it is located; *i.e.* ignoring other bodies than the secondary:

$$\mathcal{E} = \sum_{i} m_{i} \left(\frac{\dot{r}_{i}^{2}}{2} - \sum_{j, j \neq i} \frac{\rho_{j}}{r_{ij}} \right) \rightarrow$$

$$\mathcal{E}_{2}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m_{i}}{2} \dot{r}_{i2}^{2} - \frac{Gm_{i}m_{2}}{r_{i2}}, \quad \text{such that with:} \qquad (3.69a)$$

$$\frac{\dot{r}_{i2}^{2}}{2} < \frac{\rho_{2}}{r_{i2}} \quad \text{it follows that:} \quad \mathcal{E}_{2}(\mathbf{r}, \dot{\mathbf{r}}) < 0. \qquad (3.69b)$$

This provides a convenient definition of (instantaneous¹⁵) capture with respect to a body, even in an *n*-body model of the solar system. Note that, in the context of conic sections, parabolic orbits are precisely those which escape the central body with zero velocity at infinity, achieving equality, $\mathcal{E}_2 = 0$.

Thus let:

$$\Sigma_1 = \{ \mathbf{r}, \dot{\mathbf{r}} \,|\, \mathcal{E}_2 \le 0 \} \tag{3.70}$$

define the set (in phase space) on which the body we consider, typically a satellite, has negative energy with respect to

¹⁴A nonempty set of measure zero is typically a set consisting of disconnected mathematical points which form no contiguous area, thus making them in practice impossible/useless to target due to the inherent uncertainties involved in the mechanics of spaceflight. Actually hitting a point leading to stable capture would have more to do with luck than with planning, and for our purposes we need only consider those leading to temporary capture.

¹⁵By this we intend 'at the moment of evaluation.' Belbruno also terms this ballistic capture, as the state of capture is achieved without the use of a maneuver.

the Moon in this case, and is instantaneously captured. Belbruno further considers the set of points

$$\Sigma_2 = \{ \mathbf{r}, \dot{\mathbf{r}} \,|\, \dot{r}_{i2} = 0 \} \,, \tag{3.71}$$

where the radial velocity is zero, *i.e.* those points which are locally peri- or apoapse points. These are useful as a practical consideration for defining the moment of capture as occurring at periselene (perilune).

Weak Stability Boundary

Noting that in a 3-body approximation of the dynamics, the points accessible to a satellite are bounded by the set $\mathcal{J}^{-1}(C)$, i.e. those where the Jacobi integral has the value *C* corresponding roughly to the current energy and angular momentum of the satellite in the rotating system. On this set $\mathcal{J}^{-1}(C)$'s boundary, the velocity of a satellite arriving from said current state goes to zero, and thus the set is the limit that can be reached with a given energy.

Thus he defines the set

$$W = \mathcal{J}^{-1}(C) \cap \Sigma_1 \cap \Sigma_2 \tag{3.72}$$

as the set on which ballistic capture occurs: the Weak Stability Boundary.

This set can be extended (as \overline{W}) by allowing points where \mathcal{E}_2 , \dot{r} marginally larger than 0, and a key result of Belbruno's work has been to establish, for the pCR3BP model, that the intersection of \overline{W} with a hyperbolic network Λ associated with near-parabolic orbits near the central body 2, is non-empty:

$$\bar{W} \cap \Lambda \neq \emptyset \,. \tag{3.73}$$

This implies that there exists a set of points on \overline{W} on which both permanent capture and unbounded oscillatory motion occur, and further that there exists an invariant set on \overline{W} which gives rise to chaotic motion. We note these results in particular because they underlie two qualitative phenomena observed:

- 1. The chaotic nature of the orbits and their sensitivity to small perturbations of their initial conditions. This is understandable if we accept that the chaotic nature in these simple models extends to the more complicated ones which are typically used, with our satellite typically transiting or being captured on or near \overline{W}_2 of the secondary.
- 2. The fact that all ballistic capture trajectories found numerically exhibit temporary capture only¹⁶ (and so prove unstable over time), and require the execution of a (relatively small) maneuver in order to obtain permanent capture. This is understandable given that the set on which permanent capture occurs is non-empty, but has measure zero, implying an infinitesimally small chance of actually finding a permanent–capture trajectory that requires no correction.

Patched Methods

The first approaches to the construction of ballistic capture trajectories involved the notion of patched methods, also known as backward targeting. The basic idea in these approaches is

¹⁶This is both the author's experience, as well as that of Belbruno and Ocampo, as related in private communications.

• To first estimate the location of the WSB numerically. This is typically [*cf*. Belbruno, 1987, Belbruno and Miller, 1993] done by noting that in a 2–body situation, the local velocity is coupled to the eccentricity of the orbit, and thus the boundary for the stability of an orbit (in the 3–body situation) around the secondary body can be estimated by increasing the eccentricity at given radial distance and velocity direction.

A typical criterium recurring in Belbruno's work is the consideration of an orbit as stable when it completes at least one full orbit of the secondary in the presence of the full 3–body gravitational field. Others are possible, as [Deurloo, 2002] notes.

• Having determined the location of the WSB as a function W(r, e), to use *e.g.* a trajectory arc originating near the secondary integrated backwards to a suitable point $\mathbf{r}_p \in W$ and an arc integrated forward from the primary to the same \mathbf{r}_p ,¹⁷ and then incorporating a maneuver with $\Delta V = |\dot{\mathbf{r}}^+ - \dot{\mathbf{r}}^-|$ to patch the two arcs together in velocity as well as in position.



Figure 3.8: Patched (Backward) targeting principle; departing trajectory patched to capture trajectory in WSB, with maneuver $\Delta V = \|\dot{\mathbf{r}}^+ - \dot{\mathbf{r}}^-\|$

For a relatively detailed discussion of the implementation of a patched method in combination with optimization via genetic algorithms, we refer to [Biesbroek, 1999, Biesbroek and Ockels, 1999] in particular. We remark also that their results corroborate a number of qualitative features found using forward targeting (see below) and differential correction alone in [Verzijl, 2005].

Forward Targeting

If one thinks of the above as a "backward" method, then a forward method would be an approach which avoids the need to patch trajectory arcs by targeting directly from initial conditions. As discussed in [Verzijl, 2005], it has been demonstrated by Belbruno and others, in addition to our own work, that with a suitable algorithm capture can successfully be targeted directly from initial conditions (*i.e.* from a parking orbit and initial epoch).

This method has been reported to be more robust than pure patched approaches (*cf.* [Belbruno and Carrico, 2000]), though as noted above these have been shown to work well in combination with genetic algorithms, which go further by than 'simple' patching by framing the problem as an exercise in optimization.

In [Verzijl, 2005] specifically however, we discussed the search for WSB-based ballistic capture transfers from the perspective of forward targeting with differential correction algorithms, which was found there to be somewhat lacking

¹⁷This is typically hard to 'get right the first time,' and usually a differential correction algorithm is used on a candidate solution found by trial and error.



Figure 3.9: Example Ballistic Capture Trajectory (red) found in [Verzijl, 2005] – results in conventional units (kg–km–s) in an Earth–Sun rotating coordinate system, generated using an analytical targeting algorithm and a DE405 ephemeride–based force–model [Standish, 1997]. Diamonds indicate initial conditions of solution, squares indicate end–points.

in robustness. This was thought to be mainly a result of the differential correction approach itself, being a first-order method applied iteratively to a highly nonlinear problem transiting an essentially chaotic region.

3.4.2 DST-based Transfers

In addition to establishing a framework for dynamics in the restricted 3–body problem in [Koon et al., 2000], the group of Marsden et al. also discusses ballistic lunar capture from within said framework in [Koon et al., 2001]. The discussion will highlight the salient points here, though without going into depth.¹⁸

In treating the pCR3BP, [Koon et al., 2000] find among many other results the following, which we use to briefly sketch their approach to ballistic capture, which is detailed in [Koon et al., 2001].

1. Starting with the equations of motion in rotating coordinates (3.61) and the energy integral $\mathcal{I} = -C$ introduced in section 3.3, the zero-velocity surfaces or *Hill's regions* given by $\mathcal{I}^{-1}(C)$ are the limits to the domain of mo-

¹⁸The reason for this is that while their results and techniques are essentially based on the methods of DST (reviewed in [Verzijl, 2006]), the motivation for present research was the efficient solution of the forward targeting problem. While the framework of DST-based methods is powerful, it does not lend itself to this formulation as well, and the methods they propose are largely based on patched-arc approaches.



tion, and for critical values of the energy \mathcal{E} , dissolve into the 5 Lagrange equilibrium points.¹⁹ This is illustrated in figure 3.10.

Figure 3.10: Hill's regions illustrated for increasing energy in the Sun-Earth-satellite pCR3BP in rotating coordinates, adapted from [Koon et al., 2000]. As the energy of the massless 3^{rd} body (parameterized by *C*) increases, the grey forbidden regions become accessible. In the bottom left figure, two necks about the Earth region are visible: the Lagrange point L_1 is at the center of the left neck and L_2 is at the center of the right neck.

2. The Lagrange points Koon et al. are concerned with are L_1 (center of right/inner neck) and L_2 (center of left-/outer neck), which are of course simply 2 of the 3 collinear equilibrium points of the problem in rotating coordinates.

With these equilibrium points are associated stable $W_{L_{1,2}}^s$ and unstable $W_{L_{1,2}}^u$ manifolds, which are tubes in phasespace. In their study, Koon et al. show that in a region around each Lagrange point there are a number of different types of orbits determined by the initial conditions of the trajectory (which determine the values of the constants $\alpha_{1,2}$ below in figure 3.11).

3. Using this knowledge, their approach is to construct a ballistic capture trajectory in two parts. First, a suitable trajectory is constructed starting within the stable tube of the Sun-Earth L_2 point near some Earth parking orbit, which transits the region and leaves on the unstable manifold.

Then, by finding a suitable patch point in a Poincaré section as in figure 3.12, it is ensured that this orbit follows the stable manifold associated with the Earth–Moon L_2 point,²⁰ leading to ballistic capture near the moon, as illustrated in figure 3.13.

¹⁹Bear in mind that this does not contradict the earlier discussion on Hamiltonian systems, as equilibrium points in the rotating system correspond to orbits in the inertial system. Remark also that in practice, a satellite will orbit this mathematical point in a Lyapunov orbit, appearing to be orbiting empty space from the perspective of the Earth.

²⁰The Moon is of course also within the Sun-Earth neck region, given the scale of the relative orbits.



x (nondimensional units, rotating frame)

Figure 3.11: Types of orbits in a neck region, from [Koon et al., 2000]. $\alpha_1 = \alpha_2 = 0$ corresponds to a periodic Lyapunov orbit, $\alpha_1 \alpha_1 = 0$ correspond to orbits asymptotic to the periodic orbit, $\alpha_1 \alpha_2 < 0$ are transit orbits which flow in on a stable manifold and leave on an unstable manifold in forward time, and $\alpha_1 \alpha_2 > 0$ are non-transit orbits which return outwards without traversing the region.



Figure 3.12: Figure on the left shows the Poincaré mapping demonstrating what Koon et al. refer to as 'twisting,' whereby a small strip of variation in velocity near the unstable manifold maps to hug nearly the entire (section of) the stable manifold. The geometry in the neck in which this can used for considerably flexibility in targeting is shown in the figure on the right, both taken from [Koon et al., 2001].



Figure 3.13: Figure on the left shows an overlapping patch region found in the Poincaré section where an initial condition may be found that goes from $W_{L_2,SE}^u$ to $W_{L_2,EM}^s$, and allows the construction of the orbit on the right, in which the trajectory continues all the way to capture near the Moon's neck region; both taken from [Koon et al., 2001].

4. Note that here by patch-point we mean a point where ideally the manifolds' Poincaré sections naturally overlap, because in that case no maneuver is needed for the ballistic trajectory. In other cases, such as *e.g.* that treated in [Elvik, 2004], a maneuver may be necessary to bridge the velocity-mismatch determining the locations of the manifolds, also as in section 3.4.1.

The approach taken has the advantage of working directly with the geometric structures which determine the flow in the region in which it is desired to construct ballistic capture transfers. However, due to the need to construct and analyze the geometry of the structures (which is comparatively simple *only* for the planar problem), pursuing their techniques further was not envisioned in the motivation for present research.

3.5 The Capture Problem

This final section in this ongoing discussion of astrodynamics will treat a comparatively simple model for ballistic capture devised by the author. It may be thought of as a stripped-down version of the bicircular problem which retains the essentials of the quasi-bicircular problem introduced in [Andreu, 1998], an interpretation to which the discussion returns at the end of the section.

3.5.1 A Model for Ballistic Lunar Capture

The model which will be considered is a simple restricted 4-body problem formulation, and will be derived as a straightforward perturbation of the circular restricted 3-body problem. Of course, in general, one can simply model the problem using the equations of the 4-body problem by setting n = 4 in equation (3.4). However, these carry even more complexity than the 3-body problem discussed previously, and are less reducible (realistically, from $4 \times 2 \times 2 = 16 \times 1^{st}$ order planar equations to 16 - 4 = 12 planar equations, if one proceeds in parallel with the earlier discussion).

When studying ballistic *lunar* capture, however, there is already a large body of knowledge of the dynamics of the Earth–Moon region motivating much simpler models for the relative motion of the Earth, Moon and Sun, while what one is really interested in is of course the motion of the satellite. This leads to the idea, under almost the same assumptions as the circular restricted 3–body problem, of making a simple extension there to include the Moon in a prescribed periodic orbit around the Earth.
Perturbing the pCR3BP

To that end, recall the (non-normalized) planar circular restricted 3-body formulation above, and extend it in the natural way for the equations of a massless 4^{th} body (the new "secondary") in the gravitational field of 3 primaries:

$$\ddot{X} = -\left(\frac{\rho_1(X - X_1)}{r_1^3} + \frac{\rho_2(X - X_2)}{r_2^3} + \frac{\rho_3(X - X_3)}{r_3^3}\right),$$
(3.74a)

$$\ddot{Y} = -\left(\frac{\rho_1(Y-Y_1)}{r_1^3} + \frac{\rho_2(Y-Y_2)}{r_2^3} + \frac{\rho_3(Y-Y_3)}{r_3^3}\right);$$
(3.74b)

which again leads, on the choice of a preferred ω -rotating frame to:

$$Z = ze^{i\omega t} ,$$

$$\ddot{Z} = \left(\ddot{z} + 2i\omega\dot{z} - \omega^{2}z\right)e^{i\omega t}$$

$$= -\left(\frac{\rho_{1}(z - z_{1})}{r_{1}^{3}} + \frac{\rho_{2}(z - z_{2})}{r_{2}^{3}} + \frac{\rho_{3}(z - z_{3})}{r_{3}^{3}}\right)e^{i\omega t} .$$
(3.75)

Above, $Z_j = X_j + iY_j$, j = 1, ..., 3 are the locations of the primaries, which we will return to in a moment.

Considerations

The following considerations are pertinent to the choices to be made in formulating the problem.

- Succinctly, the model desired is something that both:
 - 1. works as an "extended perturbation"²¹ of the 2-body situation which is thoroughly understood, and
 - 2. is formulated along the lines of the restricted 3-body problem, in line with the similar assumptions and restriction, motivated by the success of that model in the Sun-Earth-Moon configuration.
- One must then consider the following physical motivation for this intuitive idealization.

Naively, supposing the desire to model specifically lunar capture, it would seem that the key is the motion between the Earth and the Moon, with a small gravitational perturbation due to the Sun which is very far away.²²

There are a number of objections to this idea taken at face value:

- First, a technical objection: when constructing a perturbation formulation, the assumption underlying the formulation is that the perturbation terms will be small. If the Sun is the perturbation term, one must deal with its exceedingly large mass and sphere of influence somehow.²³

²¹Perhaps a better term would be a "perturbation of a perturbation," as (strictly taken) one perturbs the 3-body problem for which there are limited integrals and no analytically closed solutions.

 $^{^{22}}$ The validity of this interpretation of the problem is not in question; extensive simulation by the group of Prof. Ocampo at the University of Texas at Austin show that in a quasi-inertial frame co-moving with the Earth, a ballistic capture transfer resembles very much a bi-elliptic transfer with the ΔV boost at apoapse provided by the gravitational interaction with the Sun. However, as a perspective it does not lend itself well to the modeling approach taken in this thesis. ²³The reader will note that we consider the issue in a more general setting in section 4.3.3, even though as we motivate here, it will

not be our approach.

- Second, a more fundamental objection is that the Moon arguably orbits the Sun, not the Earth, from the gravitational perspective. As a thought-experiment: remove the Earth, and ask what the Moon will do relative to its current orbit. It is reasonable to expect, of course, that it will continue to orbit the Sun, and will do so in roughly the same orbit, to good approximation.
- Third, a deeper observation, due to the work of the Caltech group, is that from the perspective of DST, the mechanism for ballistic capture is, in reality, the flow on manifolds near the Earth–Sun $L_{1,2}$ Lagrange-points and neck regions, which happens to also be the region in which the Earth–Moon $L_{1,2}$ points also lie (and not the other way around, *cf.* figure 3.13).

Given the latter 2 considerations, it is not unreasonable to begin with the Earth–Sun rotating system, and then consider the effect of the Moon (the smallest primary mass) as the perturbation of the system. However, the first point requires some deeper consideration, and will be treated accordingly in section 4.3 during the discussion of multiple scales inherent in the perturbation formulation of the problem.

Given the mass ratios, roughly 10³⁰: 10²⁴: 10²² or as powers of 10, it is likewise not unreasonable to further assume that the motion of the Moon, with mass ratio slightly over μ ≈ 0.0125, does not noticeably disturb the Earth's orbit about the Sun. To wit, the Earth essentially orbits the Sun as it would without the presence of the Moon. The common barycenter is located approximately 3000 km from the Earth's internal center of mass, or at roughly half the Earth's radius. Thus, it makes sense to further take the Moon as relevant to the satellite orbit for capture purposes, but itself not significantly perturbing the Earth–Sun mutual gravitation.

Note also that this in no way changes the fact that the accelerations are proportional to $1/r^2$ for each body, and so a relatively small mass may yet dominate depending on the distance between the bodies, i.e. the satellite's motion near the Moon is of course still dominated by the Moon in this formulation, reflected in equations with terms like $\frac{\varepsilon^n m_0}{r_0^2}$.

- This then motivates a formulation based on the periodic motion of the bodies, in which:
 - the frame rotates with the Earth–Sun ω (period ~ 365.25 d), cf. [Wakker, 2002a, ch. 7]

$$\omega = \sqrt{\frac{\rho_1 + \rho_2}{r_{0,ES}^3}}; \tag{3.76}$$

(see also appendix C for the details of the constants used for simulation purposes; the subscripts 1, 2 refer to Earth and Sun, as $r_{0,ES}$ refers to the Earth–Sun system).

- the Moon moves within the inertial system with $\bar{\omega}$ (period ~ 27.28 d), determined by

$$\omega = \sqrt{\frac{\rho_2 + \rho_3}{r_{0,EM}^3}}; \qquad (3.77)$$

where ρ_3 corresponding to the Moon could arguably be omitted, just as with ρ_2 in the Earth–Sun case. This is the sidereal angular velocity.

 In addition, a distinction must be made with the Moon's motion in the rotating system based on the Earth– Sun line, relative to which the Moon moves not with the sidereal but with the synodic angular velocity such that the period ~ 29.35 d. The relation between the two is given by [?]:

$$\frac{1}{T_{syn}} = \left| \frac{1}{T_{Earth}} - \frac{1}{T_{syn}} \right|,$$
(3.78)

from which one obtains $\omega_{syn} = 2\pi/T_{syn}$.

The Earth–Sun system is considered a 2–body problem with a closed circular solution, whose gravitational field determines the motion of the Moon and satellite. The Moon does not influence these two, but orbits the Earth and Sun circularly, and affects the satellite, which in turn orbits in the gravitational field of all 3 primaries, influencing none. We propose to refer to this as the *Ballistic Lunar Capture Problem* formulation or simply, for purposes of this thesis, the *Capture Problem* (CP).

Luckily this proposed formulation is easy to parameterize, first by noting the appropriate transformations to inertial, based on the Sun as m_1 at $-\mu r_0$ along the *x*-axis at t = 0, the Earth as m_2 at $(1 - \mu)r_0$, and the Moon m_3 at some $\eta = \frac{r_{0.EM}}{r_0}$ relative to the position of the Earth (typically with an additional phase offset ϕ_0 at t = 0, omitted here for clarity) :

$$Z_1 = -\mu r_0 e^{i\omega t} , \qquad (3.79a)$$

$$Z_2 = (1 - \mu) r_0 e^{i\omega t} , \qquad (3.79b)$$

$$Z_{3} = \left(1 - \mu + \eta e^{i\bar{\omega}t}\right) r_{0}e^{i\omega t} = (1 - \mu)r_{0}e^{i\omega t} + \eta r_{0}e^{i(\bar{\omega} + \omega)t} .$$
(3.79c)

Note that the latter implies that even in an Earth–Sun rotating coordinate system, the Moon indeed orbits the Earth circularly in the model. This is borne out as a good first approximation by the simulations using ephemerides performed in [Verzijl, 2005], supporting the approximations of circularity made here.

Equations of Motion

The equations of motion, then, are derived along the same lines as in section 3.3.5, with the Sun and Earth forming the 2-body problem basis with $r_0 = r_{0,ES}$ and taking the restriction to motion in the plane. The key difference now is that there are for present purposes²⁴ *two* angular velocities ω , $\bar{\omega}$ in play. Substituting the parametrization of the bodies as just motivated into expression (3.74), to find

$$\ddot{Z} = \left(\ddot{z} + 2i\omega\dot{z} - \omega^2 z\right)e^{i\omega t}$$

$$= -\left(\frac{\rho_1(z + \mu r_0)}{r_1^3} + \frac{\rho_2(z - (1 - \mu)r_0)}{r_2^3} + \frac{\rho_3(z - (1 - \mu + \eta e^{i\tilde{\omega} t})r_0)}{r_3^3}\right)e^{i\omega t}.$$
(3.80)

Expanding with z = x + iy, canceling $e^{i\omega t}$ and collecting terms real and imaginary:

$$\ddot{x} - 2\omega \dot{y} - \omega^2 x = -\frac{\rho_1 (x+\mu)r_0}{r_1^3} - \frac{\rho_2 (x - (1-\mu)r_0)}{r_2^3} - \frac{\rho_3 (x - (1-\mu+\eta\cos\bar{\omega}t)r_0)}{r_3^3}, \qquad (3.81a)$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{\rho_1 y}{r_1^3} - \frac{\rho_2 y}{r_2^3} - \frac{\rho_3 (y - \eta r_0 \sin \bar{\omega} t)}{r_3^3} .$$
(3.81b)

²⁴The Sun–Earth angular velocity which are denoted ω and the Earth–Moon angular velocity, which we will denote $\bar{\omega}$. One could of course introduce another for the Sun–Moon rotation, but this would be some periodic $\hat{\omega}(t)$ due to the back and forth about the Earth, and needlessly complicate matters.



Figure 3.14: Definition of Capture Problem Coordinates

For the motion in the rotating frame, however, the distance $r_3 = r_3(t)$ explicitly:

$$r_{3} = |z - (1 - \mu + \eta e^{i\bar{\omega}t})r_{0}| = \sqrt{(x - (1 - \mu + \eta \cos\bar{\omega}t)r_{0})^{2} + (y - \eta r_{0}\sin\bar{\omega}t)^{2}}.$$
(3.82)

This complicates matters because it introduces a time-dependence, albeit the simplest periodic one possible, but which enters nonlinearly both in the numerator and denominator making the gravitational field explicitly time-dependent.

Normalization

As with the circular restricted 3-body problem, normalization is carried out here using:

$$M = m_1 + m_2$$
 as a mass scale, such that:

$$\begin{aligned} \frac{m_1}{m_1 + m_2} &= 1 - \mu ,\\ \frac{m_2}{m_1 + m_2} &= \mu ,\\ \frac{m_3}{m_1 + m_2} &= \upsilon ;\\ L &= r_0 &= \langle \| \mathbf{r}_{ES} \| \rangle \qquad \text{as a length scale (using the Earth–Sun distance), and} \\ T &= \frac{1}{\omega_{FS}} \qquad \text{as a time scale.} \end{aligned}$$

Note that it is the ω_{ES} which is used to scale the time, and that there is a distinction between this and the synodic angular velocity of the Moon about the Earth ω_{syn} which is also necessary (and must be taken into account during

normalization as well). One thus introduces $\omega' := \omega_{syn}/\omega_{ES} > 1$ to account for the scaling with reference to the Moon.

Thus one finally obtains:

$$\ddot{x} - 2\dot{y} = x - \frac{1 - \mu}{r_1^3} (x + \mu) - \frac{\mu}{r_2^3} (x - (1 - \mu)) - \frac{\upsilon}{r_3^3} \left(x - (1 - \mu + \eta \cos \omega' t) \right) , \qquad (3.83a)$$

$$\ddot{y} + 2\dot{x} = y - \frac{1-\mu}{r_1^3}y - \frac{\mu}{r_2^3}y - \frac{\nu}{r_3^3}(y - \eta\sin\omega' t)$$
 where: (3.83b)

$$\begin{split} r_1 &= |z + \mu| = \sqrt{(x + \mu)^2 + y^2} ,\\ r_2 &= |z - (1 - \mu)| = \sqrt{(x - (1 - \mu))^2 + y^2} ,\\ r_3 &= |z - (1 - \mu + \eta \, e^{i\omega' t})| = \sqrt{(x - (1 - \mu + \eta \cos \omega' t))^2 + (y - \eta \sin \omega' t)^2} . \end{split}$$

3.5.2 Remarks on the Model

The attentive reader will note that this is no longer a case of simply restricting the dynamics inherent in the 3–body problem: it is rather an extension based on empirical grounds: the observed motion of the Moon around the Earth and the relatively small effect that this has on the motion of the Earth around the Sun. It is worth remarking, in fact, that a more accurate approach exists which is more closely tied to the actual dynamics: the model introduced as the quasi-bicircular problem by the group of Simó in Barcelona: [Andreu, 1998].

The bicircular problem proper considers 2 coupled 3—body problems, wherein the Earth and Moon orbit their common barycenter, which in turn orbits the Sun (or strictly, orbits the Earth-system–Sun barycenter, which is essentially the same). The problem with this is that it is a stitching together of two 2—body problem solutions and not a true solution of the 3—body problem.

The approach of Andreu's work, instead, was to find a true 3-body problem solution for the Moon's motion in terms of Fourier series, which could then become the basis of his 4-body problem work. The model proposed here might be thought of as the lowest harmonic of precisely such a Fourier series solution, restricting the lengthy approximations found by Andreu using computer algebra to a single periodic oscillation about the Earth's position. Indeed, we remark that the lowest order harmonic listed at [Andreu, 1998, p. 33] has a weight of 0.990930, making this interpretation quite reasonable

Clearly the approach to modeling above is then not the only approach, but we remark, anticipating the further development in Part II of this thesis using perturbation methods for first integrals, that it might be expected that results found using the above model based on simple harmonic oscillation promise to scale to the quasi-bicircular problem more easily than to a patched model.

Chapter 4

Method of Integrating Vectors

This chapter will discuss the method of integrating vectors as a framework for the search for approximations of analytical integrals of the equations of motion of a given problem (in this thesis those of chapter 3).

The method is best thought of as a generalization of the use of an integrating factor to make a scalar ordinary differential equation exact and hence integrable, *cf.* [Boyce and DiPrima, 2001, chapter 2]. Our discussion is based on [van Horssen, 1997, 1999a,b, Waluya, 2003], and their notation is more or less adopted here.¹

4.1 Exact Equations and Integrating Factors

In beginning this discussion a bit of perspective is useful, and it is prudent to consider what motivates the method introduced in the next section. Consider an ordinary differential equation (ODE) of the form:

$$M(x,t) + N(x,t)\frac{dx}{dt} = 0.$$
 (4.1)

This equation is exact if [see e.g. Boyce and DiPrima, 2001, section 2.1]

$$\exists \Psi(x,t) \quad \text{such that} \quad \begin{cases} \partial_t \Psi = M(x,t) \\ \partial_x \Psi = N(x,t) \end{cases}$$

It is straightforward to show that this is equivalent to requiring that:

$$L_t\left(\Psi(x,t)\right) = \frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \qquad (4.2)$$

and has as consequence the requirement (obtained by taking equality of mixed partial derivatives) that:

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t} \,. \tag{4.3}$$

This latter condition is in fact wholly equivalent as a definition of exactness.

The usefulness of exact differential equations lies in the implication that for a single scalar 1^{st} -order ODE as suggested here, the equation's solution is (possibly implicitly) contained in any first integral; equation (4.2) then demonstrates

¹Though the method is discussed only for the purpose of such approximations, note that it has been applied successfully there to determine the existence and stability of periodic solutions as well.

that $\Psi(x,t)$ fulfills precisely this role.

In general, a given ODE will not be exact, and thus one seeks an *integrating factor* ν which will make it exact:

$$\nu M(x,t) + \nu N(x,t)\frac{\mathrm{d}x}{\mathrm{d}t} = 0.$$
(4.4)

In turn, the relevant conditions become:

$$\frac{\partial(\nu M)}{\partial x} = \frac{\partial(\nu N)}{\partial t} \quad \text{or equivalently} \tag{4.5}$$

$$M \,\partial_x \mathbf{v} - N \,\partial_t \mathbf{v} + \mathbf{v} (\,\partial_x M - \,\partial_t N) = 0 \,; \tag{4.6}$$

the latter a *partial* differential equation (PDE) for the unknown factor v which is typically as difficult to solve as the original equations.

The method of integrating vectors considered in the references above can then schematically be described as the combination of a generalization of an equation made exact by an integrating factor to a vector case, combined with getting around the problematic PDE (now a system of $\frac{1}{2}n(n+1)$ PDE's in fact, as we shall show in the next section) by constructing successive approximations in a small parameter ε which is inherent in the original ODE system.

Before outlining the method in detail, consider the following two examples which show the usual and a generalized approach to the solution of a simple ordinary differential equation.

Example 4.1.1 (Exact Equations Approach)

Consider:

$$\frac{dx}{dt} + 2x = 3$$
 with $x(0) = \frac{3}{2}$, (4.7)

which phrased as above implies N(x,t) = 1 and M(x,t) = 2x - 3. Multiplying by v := v(t):

$$v \frac{dx}{dt} = (3 - 2x)v$$
, and we now require that:
 $\frac{dxv}{dt} = v \frac{dx}{dt} + \frac{dv}{dt}x$.

From this it follows on comparing with expression (4.7) that:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 2\mathbf{v} \; ,$$

and consequently we readily find that:

$$v(x,t) = c_1(x) e^{2t} \text{ and on letting } c_1(x) = 1 \text{ have found:}$$

$$3e^{2t} = e^{2t} \frac{dx}{dt} + 2e^{2t} x, \text{ or:}$$

$$\frac{d}{dt} \left(e^{2t} x \right) = 3e^{2t}.$$

$$(4.8)$$

On integrating, we obtain:

$$e^{2t} x = \frac{3}{2}e^{2t} + k \quad \text{or:}$$

$$x(t) = \frac{3}{2} + ke^{-2t}, \quad \text{a family of solutions satisfying the initial condition, in fact.}$$
(4.9)

Example 4.1.2 (Integrating Vectors Formalism)

We reprise:

$$\frac{dx}{dt} + 2x = 3 \quad \text{with} \quad x(0) = \frac{3}{2} \quad \text{and multiplied by } v \text{ to obtain:}$$
$$v\frac{dx}{dt} = (3 - 2x)v.$$

Now, using the formalism of the method of integrating vectors for the 1-dimensional case:

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla(\mathbf{v} \cdot f)$$

$$= -(3 - 2x)\frac{\partial \mathbf{v}}{\partial x} + 2\mathbf{v}, \quad \text{which we rewrite as:}$$

$$2\mathbf{v} = \partial_t \mathbf{v} + (3 - 2x)\partial_x \mathbf{v}. \quad (4.10)$$

This is a linear 1^{st} –order PDE which is easily solved using the method of characteristics (see *e.g.* [Evans, 1997, section 3.2]). On a characteristic parameterized by a variable s, v(s) = const and so it follows that, on comparing the above:

$$\frac{d\mathbf{v}}{ds} = 0 = \partial_x \mathbf{v} \frac{dx}{ds} + \partial_t \mathbf{v} \frac{dt}{ds} \quad \text{from which it follows that:}$$

$$\begin{cases} \frac{d\mathbf{v}}{ds} &= 2\mathbf{v} , \\ \frac{dx}{ds} &= 3 - 2x , \\ \frac{dt}{ds} &= 1 . \end{cases}$$

Solving these 3 equations:

$$s = t + c_1 ,$$

$$v = c_2(x)e^{2s} , \text{ and}$$

$$\frac{d(3-2x)}{3-2x} = -2ds \text{ whence it follows that:}$$

$$3-2x = c_3e^{-2s} \text{ or } c_3 = (3-2x)e^{2s} .$$

Specifying $c_1 = 1$ for simplicity, and setting $c_2(x) = g(c_3)$ with g an arbitrary function, the general solution is:

.

$$\mathbf{v}(x,t) = g\left((3-2x)e^{2t}\right)e^{2t} . \tag{4.11}$$

This is now substituted into the equations for the integral:

$$\frac{\partial \Psi}{\partial x} = \mathbf{v} = g\left((3-2x)e^{2t}\right)e^{2t},\tag{4.12}$$

$$\frac{\partial \Psi}{\partial t} = -\mathbf{v} \cdot f = (2x - 3)g\left((3 - 2x)e^{2t}\right)e^{2t}.$$
(4.13)

A general solution will be of the form $\psi(x,t) = e^{2t} \int_x g\left((3-2z)e^{2t}\right) e^{2t} dz + (2x-3) \int_t g\left((3-2x)e^{2z}\right) e^{2z} dt + k$, but for this problem we instead will specify a simpler form for g, namely set $g(\ldots) = 1$.

With this then,

$$\frac{\partial \Psi}{\partial x} = e^{2t},$$

$$\frac{\partial \Psi}{\partial t} = (2x - 3)e^{2t}, \text{ and it follows that}$$

$$\Psi = \frac{1}{2}(2x - 3)e^{2t} + k.$$
(4.14)

This form is precisely what we need, and now recalling that since $\frac{\partial \Psi}{\partial x} \frac{dx}{dt} + \frac{\partial \Psi}{\partial t} = 0$, this is by construction an integral; thus we can reduce the single 1st-order equation *fully* with just this single expression, defining the solution implicitly by $\Psi(x,t) = const$.

$$\Psi = \frac{1}{2}(3-2x)e^{2t} + k = const , \text{ from which it is easily found that the solution is:}$$

$$x(t) = \frac{3}{2} + ce^{-2t} , \text{ again a family of solutions.}$$
(4.15)

This example illustrates a relatively complicated method for a relatively simple problem, but we would like to point out that the problems introduced in the previous chapter, to which we intend to apply it, are considerably more complicated, and cannot be handled using the naive approach.

The more complicated second method, by contrast, does generalizes satisfactorily, and it is useful to think of it as mapping from the concepts for exact equations, as it were to the concepts we introduce properly in the next section:

$$\begin{array}{rcl} x & \longrightarrow & \mathbf{x} \,, \\ t & \longrightarrow & t \,, \\ & \frac{M(x,t)}{N(x,t)} & \longrightarrow & \mathbf{f}(\mathbf{x},t) \,, \\ & \Psi(x,t) & \longrightarrow & I(\mathbf{x},t) \,, \\ & L_t \left(\Psi(x,t) \right) & \longrightarrow & L_t \left(I(\mathbf{x},t) \right) \quad \text{and} \\ & \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad \longrightarrow & \frac{\partial I(\mathbf{x},t)}{\partial t} + \nabla I(\mathbf{x},t) \cdot \dot{\mathbf{x}} = 0 \,: \quad \text{term for term equivalent to:} \\ & \frac{\partial (\nabla f)}{\partial t} + \mathbf{v} \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad \longrightarrow & -\mathbf{v} \cdot \mathbf{f} + \mathbf{v} \cdot \dot{\mathbf{x}} = 0 \,. \end{array}$$

4.2 A Nonlinear Method based on Integrating Factors

Consider a system of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, where for this section the vector notation is kept explicit. An integrating factor is a factor v by which the equations may be multiplied such that the resulting new equations are exact and thus integrable.

[van Horssen, 1999a] extends this approach to a general vector $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, with $\mathbf{x}, \mathbf{f} \in \mathbb{R}^n$, and further it is assumed that each element f_1, \ldots, f_n is sufficiently smooth such that a twice continuously differentiable first integral $I(x_1, \ldots, x_n, t; c) = 0$ exists with *c* an arbitrary constant of integration.

Element-wise one multiplies the vector DE by (continuously differentiable) integrating factors $v_i(\mathbf{x},t)$ which may

conveniently be expressed using the inner product with the vector v:

$$\mathbf{v} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v} \cdot \mathbf{f}(\mathbf{x}, t) \,. \tag{4.16}$$

It is then required that this be equal to the expression obtained by differentiating the first integral, *i.e.*

$$\begin{cases} \nu_1 \frac{dx_1}{dt} + \dots + \nu_n \frac{dx_n}{dt} - \nu \cdot \mathbf{f} = 0 \\ \frac{\partial I}{\partial x_1} \frac{dx_1}{dt} + \dots + \frac{\partial I}{\partial x_n} \frac{dx_n}{dt} + \frac{\partial I}{\partial t} = 0, \quad \text{or equivalently} \end{cases}$$
(4.17a)

$$\begin{cases} \nabla I &= \mathbf{v} \\ \frac{\partial I}{\partial t} &= -\mathbf{v} \cdot \mathbf{f} \,. \end{cases}$$
(4.17b)

Now using the differentiability assumed above, note that on taking derivatives and using equality of mixed partials, one obtains from the above that:

$$\frac{\partial \mathbf{v}_1}{\partial x_2} = \frac{\partial (\nabla I)_1}{\partial x_2} = \frac{\partial^2 I}{\partial x_2 \partial x_1} \equiv \frac{\partial^2 I}{\partial x_1 \partial x_2} \longrightarrow \frac{\partial \mathbf{v}_2}{\partial x_1} = \frac{\partial \mathbf{v}_1}{\partial x_2}, \quad \text{and:} \\ -\frac{\partial (\mathbf{v} \cdot \mathbf{f})}{\partial x_1} = \frac{\partial^2 I}{\partial x_1 \partial t} \equiv \frac{\partial^2 I}{\partial t \partial x_1} \longrightarrow \frac{\partial \mathbf{v}_1}{\partial t} = -\frac{\partial (\mathbf{v} \cdot \mathbf{f})}{\partial x_1},$$

and so working through the relations (4.17) above:

$$\begin{cases} \frac{\partial \mathbf{v}_{1}}{\partial x_{2}} &= \frac{\partial \mathbf{v}_{2}}{\partial x_{1}}, \\ \frac{\partial \mathbf{v}_{1}}{\partial x_{3}} &= \frac{\partial \mathbf{v}_{3}}{\partial x_{1}}, \frac{\partial \mathbf{v}_{2}}{\partial x_{3}} = \frac{\partial \mathbf{v}_{3}}{\partial x_{2}}, \\ \vdots & & & \\ \frac{\partial \mathbf{v}_{1}}{\partial x_{n}} &= \frac{\partial \mathbf{v}_{n}}{\partial x_{1}}, \dots, \frac{\partial \mathbf{v}_{n-1}}{\partial x_{n}} = -\frac{\partial \mathbf{v}_{n}}{\partial x_{n-1}}, \\ \frac{\partial \mathbf{v}_{1}}{\partial t} &= -\frac{\partial (\mathbf{v} \cdot \mathbf{f})}{\partial x_{1}}, \dots, \frac{\partial \mathbf{v}_{n-1}}{\partial t} = -\frac{\partial (\mathbf{v} \cdot \mathbf{f})}{\partial x_{n-1}}, \frac{\partial \mathbf{v}_{n}}{\partial t} = -\frac{\partial (\mathbf{v} \cdot \mathbf{f})}{\partial x_{n}}, \end{cases}$$
(4.18)

or equivalently

$$\begin{cases} \frac{\partial \mathbf{v}_i}{\partial x_j} &= \frac{\partial \mathbf{v}_j}{\partial x_i} ,\\ \frac{\partial \mathbf{v}}{\partial t} &= -\nabla(\mathbf{v} \cdot \mathbf{f}) . \end{cases}$$
(4.19)

Thus the integrating vector v must satisfy this system of $\frac{1}{2}n(n+1)$ 1st -order, linear partial differential equations. On determining v from these, a first integral follows by integration of equation (4.17).

This formalism is applied to a perturbed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \mathbf{\epsilon})$ as follows.

Assume that \mathbf{f} has the form:

$$\mathbf{f}(\mathbf{x},t;\mathbf{\varepsilon}) = \mathbf{f}_0(\mathbf{x},t) + \mathbf{\varepsilon}\mathbf{f}_1(\mathbf{x},t,\mathbf{\varepsilon}), \qquad (4.20)$$

and then expand the integrating factor along the same lines, but explicitly in ϵ as:

$$\mathbf{v}(\mathbf{x},t;\mathbf{\varepsilon}) = \mathbf{v}_0(\mathbf{x},t) + \mathbf{\varepsilon}\mathbf{v}_1(\mathbf{x},t) + \dots + \mathbf{\varepsilon}^m \mathbf{v}_m(\mathbf{x},t) + \dots, \qquad (4.21)$$

where the subscripts now indicate the order of expansion of the vector v (and not the scalar components²). On substituting this expansion into equations (4.17) and (4.19) one obtains what will be referred to throughout this thesis as the "equations of condition"³ which the integrating factor must satisfy at each order:

$$\begin{cases} \nabla I = \mathbf{v} \\ \frac{\partial I}{\partial t} = -\mathbf{v} \cdot \mathbf{f} \end{cases} \longrightarrow \begin{cases} \nabla I_k = \mathbf{v}_k \\ \frac{\partial I_k}{\partial t} = -[\mathbf{v} \cdot \mathbf{f}]_k \end{cases} k = 0, 1, 2, \dots, \qquad (4.22)$$

$$\begin{cases} \frac{\partial \mathbf{v}_{i}}{\partial x_{j}} &= \frac{\partial \mathbf{v}_{j}}{\partial x_{i}} \\ \frac{\partial \mathbf{v}}{\partial t} &= -\nabla [\mathbf{v} \cdot \mathbf{f}] \end{cases} \longrightarrow \begin{cases} \frac{\partial \mathbf{v}_{i,k}}{\partial x_{j}} &= \frac{\partial \mathbf{v}_{j,k}}{\partial x_{i}} \\ \frac{\partial \mathbf{v}_{k}}{\partial t} &= -\nabla [\mathbf{v} \cdot \mathbf{f}]_{k} \end{cases}$$
(4.23)

In these equations the key expression is $v \cdot \mathbf{f}$. Its gradients determine the time–dependent part of the equations of condition at the "low–level" (4.23), as well as the time–dependence of the final integral in the "high–level" (4.22).⁴

This latter point will be shown to be particularly relevant in constructing the time–independent integrals of the astrodynamics problems of chapter 3 in Part II of this thesis. In general finding an integrating factor will prove decidedly complicated, but even there the "high–level" equations sometimes allow considerable limitation of the search-space for candidates.

When the factors thus obtained are substituted into expression (4.17), one has the approximation truncated at order m:

$$\begin{cases} \nabla \hat{I} = \mathbf{v}_0(\mathbf{x}, t) + \ldots + \mathbf{\varepsilon}^m \mathbf{v}_m(\mathbf{x}, t) \\ \frac{\partial \hat{I}}{\partial t} = -\left[\left(\mathbf{v}_0(\mathbf{x}, t) + \ldots + \mathbf{\varepsilon}^m \mathbf{v}_m(\mathbf{x}, t) \right) \cdot \mathbf{f} \right]_*, \end{cases}$$
(4.24)

where \hat{I} denotes the approximation of the first integral *I*, and the * is used to indicate that terms of order higher than *m* are neglected.⁵ \hat{I} then is obtained by setting

$$\hat{I} = I_0(\mathbf{x}, t) + \varepsilon I_1(\mathbf{x}, t) + \dots + \varepsilon^m I_m(\mathbf{x}, t) , \qquad (4.25)$$

and this forms an integral approximation accurate to $\mathcal{O}(\varepsilon^{m+1})$, as demonstrated in the following lemma.

Lemma 4.2.1 (Approximation Accuracy)

It may be noted here that this approximation of a first integral I is exact up to $O(\varepsilon^{m+1})$, *i.e.*

²These would be expanded as $v_{i,0} + \varepsilon v_{i,1} + \cdots + \varepsilon^m v_{i,m} + \cdots$ etc.

³Strictly taken, these are nothing of the sort: they are simply the equations which an integral, integral expansion or integral approximation and its integrating vectors must satisfy. However, we find it useful throughout this thesis to use the term for the reader's easy recollection of the system (4.22)-(4.23) to which we refer, and so we ask the reader's patience with our sloppy use of it.

⁴The terms high-level and low-level are here likewise chosen for easy reference, and are motivated by the conceptual separation between the integrating factor (which is determined by the so-called "low-level equations") which is used as a building block for the actual integral approximation (which must satisfy the so-called "high-level equations").

⁵We note this explicitly, as there are terms in ε 'hidden' in **f** as well; *i.e.* it would not be sufficient to truncate the expansion of v only and then take the inner product.

$$\frac{\mathrm{d}\hat{I}}{\mathrm{d}t} = 0 + \varepsilon^{m+1} R_{m+1} \qquad \text{and so} \tag{4.26a}$$

$$\hat{I}(\mathbf{x},t;\varepsilon) = \hat{I}(\mathbf{x}(0),0;\varepsilon) + \varepsilon^{m+1} \int_{t_0}^t R_{m+1} \mathrm{d}t , \qquad (4.26b)$$

for some remainder term $R_{m+1}(\mathbf{x}, t, \mathbf{v}_0, \dots, \mathbf{v}_m; \varepsilon)$. Consequently on a timescale $t \leq T = L$ this gives an $O(\varepsilon^{m+1})$ approximation of a first integral, while on a timescale $t \leq T = \frac{L}{\varepsilon}$ the approximation is good to $O(\varepsilon^m)$.

Proof. This is proven in *e.g.* [van Horssen, 1999a], and follows from a straightforward substitution of the expansion into the formal expression $\frac{d\hat{t}}{dt}$.

In the references given above, the method is applied to a number of instructive examples of both mildly and highly nonlinear equations, and it is this success which motivates its consideration for the problems introduced in earlier chapters.

4.3 Treating Multiple Scales

As the final tool needed in approaching part II of this thesis, this section will discuss the relevant issues when dealing with multiple scales in the problems to be considered. In particular, the reader will note that the discussion is extended beyond the usual analysis of multiple time–scales to spatial–, mass– and force–scales as well.

4.3.1 Time–Scales

In [van Horssen, 1999b] the method described in the previous section is extended to problems involving multiple time scales, and here we take the opportunity to introduce this approach as well.

Definition 4.3.1 (Time-Scale)

Consider a constant *C* and $g_{1,2}(\varepsilon)$ functions of a small parameter ε , typically $g_{1,2} = \varepsilon^{m_{1,2}}$.

By a timescale, one intends some scale $1/g_2(\varepsilon)$, such that for a function *f*, one has that $f(x,t;\varepsilon) = O(g_1(\varepsilon))$ as $\varepsilon \longrightarrow 0$, for $0 \le t \le C/g_2(\varepsilon)$.

The reason that multiple time scales are introduced is typically to gain a better understanding of the secular terms which may appear in naive approximations such as those for the integrating vector introduced above. These terms are relevant on a long timescale, and may reflect an issue with the modeling of a problem, such as the problem of small denominators, first noted in section 2.3.4.

Example 4.3.2

As an example, [van Horssen, 1999a,b] treats the Van der Pol oscillator:

$$\frac{d^2 y}{dt^2} + y = \varepsilon (1 - y^2) \frac{dy}{dt} .$$
(4.27)

Changing to polar coordinates and using the approach discussed in the previous section, it is found that:

• An expansion of the form

$$I_1 = r + \varepsilon \left(-\left(\frac{1}{2} - \frac{3}{8}r^2\right)\varphi - \frac{r}{4}\sin(2\varphi) - \frac{r^2}{32}\sin(4\varphi) \right) + O\left(\varepsilon^2\right)$$

holds for a first integral.⁶ However, since $\varphi = \varphi(0) + t + O(\varepsilon)$, for time scales of the order $O(\varepsilon^{-1})$, the 'small' term $\varepsilon(\frac{1}{2} - \frac{3}{8}r^2)\varphi$ in the integrating factor clearly becomes O(1).

- The multiple scales approach instead tries to make the approximation less naive by explicitly incorporating a second (long) timescale $\tau = \varepsilon t$, into a new expression⁷ $x_0(t, \tau) + \varepsilon x_1(t, \tau) + \ldots + \varepsilon^n x_n(t, \tau)$, computed such that each term $x_i(t, \tau)$ no longer contains secular terms.
- The method outlined in the previous section is then approached by first extending the system of equations with $\frac{d\tau}{dt} = \varepsilon$. Likewise adding an element to the integrating vector v_{n+1} , the approximation method is carried out as before, now keeping explicit track of the τ terms.

This approach now yields

$$I_1 = -e^{\tau} \left(\frac{1}{2r^2} - \frac{1}{8}\right) - \varepsilon \frac{e^{\tau}}{r^3} \left(\frac{r}{4}\sin(2\varphi) + \frac{r^3}{32}\sin(4\varphi)\right) + \mathcal{O}\left(\varepsilon^2\right) \ .$$

Here, clearly there is no longer a 'hidden' secular term, but rather the behavior on long time scales has been made explicit in the e^{τ} pre-factors.

Also, in contrast to the first approximation, this is valid to $O(\varepsilon^2)$ on a $O(1/\varepsilon)$ timescale, whereas the former was valid to the same order but only on an O(1) timescale.

For details of the execution of the method, the reader is referred to the papers referenced above and Waluya's dissertation [Waluya, 2003] for these and a number of other examples.

It is also incumbent upon us to point out, as the astute reader has no doubt noted, that in the circular restricted 3-body problem treated in the previous chapter, we have effectively introduced just such a timescale in normalizing the equations with $T = \frac{1}{\omega}$, an issue to which we will return in detail in Parts II and III of this thesis.

4.3.2 Mass- and Spatial-Scales

In formulating a perturbation problem one can analogously introduce both mass and spatial scales, by scaling the equations of motion relative to a reference mass m_0 or reference distance r_0 . The former also has consequences for the gravitational parameters ρ_i of course.

The formalism used to introduce the scaling is analogous to that used in setting $\tau = \varepsilon^{j}t$ as a time scale, though this must now be modified to be *e.g.* $\rho_2 = a\varepsilon^{j_2}\rho_0$, where *a* is some dimensionless constant that is used to correct to the true values rather than just order of magnitude accuracy (which will be necessary for simulation purposes).

 $^{^{6}}$ The secularity is expressed via ϕ and already occurs in the integrating factors.

⁷In the case that the method is used to calculate an approximation to the solution of the system. We will continue to be concerned mainly with approximations of first integrals in the present work.

Thus, introduce the notations \bar{m}_i and $\bar{\mathbf{r}}_i$ for the scaled (dimensionless) masses and vector coordinates. For the latter, one scales by some (scalar) r_0 which has units of distance.

$$m_i = \varepsilon^{j_i} \bar{m}_i m_0$$
 where $\bar{m}_i = \frac{m_i}{m_0 \varepsilon^{j_i}}$ or (4.28a)

$$\rho_i = \varepsilon^{j_i} \bar{\rho}_i \rho_0 \quad \text{where} \quad \bar{\rho}_i = \frac{G m_i}{G m_0 \varepsilon^{j_i}} \,, \tag{4.28b}$$

and likewise

$$\mathbf{r}_i = \varepsilon^{j_i} \, \bar{\mathbf{r}}_i \, r_0 \quad \text{where} \quad \bar{\mathbf{r}}_i = \frac{\mathbf{r}_i}{r_0 \, \varepsilon^{j_i}} \, .$$

$$(4.29)$$

Here, a scaling factor ε is chosen first, as above, and a given mass or position's magnitude is reflected in the factors $m_0 \varepsilon^{j_i}$, $r_0 \varepsilon^{k_i}$ for each term i = 1, ..., n in the problem. This only gives the correct order of magnitude, though, and is corrected by the dimensionless \bar{m}, \bar{r} so that the true value can be recovered.

In particular, note that the dimensionless vector $\bar{\mathbf{r}}_i$ still plays the same role as before, also with respect to differentiation and integration. However, as motivated in the next section, it makes more sense to consider these factors together in a single force–scale rather than scaling them out separately.

4.3.3 Force–Scales

Taking the above considerations a step further, we recall that in problems typical of astrodynamics, it will not be a mass– or distance–scale in isolation that determines the problem, but rather a force–scale, as reflected in the "competition" between acceleration terms. Introducing a structuring based on small factors not only for the mass (ε^{α}), but also for the distance (ε^{β}), one arrives at a net factor corresponding to the acceleration along the lines of $\varepsilon^{\alpha-2\beta}$.

The motivation for this is as follows.

- As long as the masses are representative for the forces/accelerations, this structuring isn't strictly necessary, but the potential for small-denominators issues implies that this may not always be the case; indeed, in the region determining lunar ballistic capture it most certainly is not, due to the balance of Earth, lunar and solar accelerations discussed in [Verzijl, 2005, 2006].
- In particular, with the goal of discussing the implications for ballistic capture, it's important that the model makes explicit exactly what's going on in the capture region, and precisely this case is sensitive to the balancing of accelerations mentioned in the previous point. To wit, the contributions are actually S:E:M → O(ε⁶ : ε⁶ : ε⁸), as will be shown below. The reader will that this fact is *not* readily apparent from the simple scaling by masses that one would naively assume.

Let, for the moment, the order factors be (α_i, β_i) , i = 1, ..., 3, and the references be the primary mass $m_0 = m_1$ and some distance–scale r_0 . Then, for the acceleration terms of *e.g.* the 4–body equations one arrives at:

Sun:
$$-\frac{Gm_1\mathbf{r}_{14}}{\|\mathbf{r}_{14}\|^3} = -\frac{Gm_0}{r_0^2} \frac{\bar{m}_1\bar{\mathbf{r}}_{14}}{\|\bar{\mathbf{r}}_{14}\|^3} \epsilon^{\alpha_1 - 2\beta_1},$$
 (4.30a)

Earth:
$$-\frac{Gm_2\mathbf{r}_{24}}{\|\mathbf{r}_{24}\|^3} = -\frac{Gm_0}{r_0^2} \frac{\bar{m}_2\bar{\mathbf{r}}_{24}}{\|\bar{\mathbf{r}}_{24}\|^3} \epsilon^{\alpha_2 - 2\beta_2},$$
 (4.30b)

Moon:
$$-\frac{Gm_3\mathbf{r}_{34}}{\|\mathbf{r}_{34}\|^3} = -\frac{Gm_0}{r_0^2} \frac{\bar{m}_3\bar{\mathbf{r}}_{34}}{\|\bar{\mathbf{r}}_{34}\|^3} \epsilon^{\alpha_3 - 2\beta_3} .$$
 (4.30c)

As with multiple time scales, the new formulation is still dimensional, though the further step to normalization is trivial in that one simply cancels the terms $-\frac{Gm_0}{r_0^2}$ in the normalization process. In particular, for *e.g.* the CR3BP one takes precisely this step, canceling the dimensional factors, and as such $\bar{m}_1 \equiv 1$ on normalizing by $m_0 = m_1$ (as in section 3.3.5). We give two examples.

Example 4.3.3 (CR3BP)

We note first, that in normalizing the circular restricted 3–body problem, we effectively chose a force scale following the literature, by taking:

$$r_0 = \left\langle \|\mathbf{r}_{12}\|\right\rangle,$$
$$m_0 = m_1 + m_2,$$

with indices 1, 2 referring to the primaries in a 2-body problem orbit, following literature conventions. We shall have more to say about this later, in particular in chapters 7 and 8 and in the performance considerations of chapters 10 and 11 of Part III of this thesis.

Example 4.3.4 (Near–Moon Region) Suppose one considers the case that:

$$r_0 = \langle \|\mathbf{r}_{23}\|\rangle, \tag{4.31}$$

corresponding to scaling by the average Earth-Moon distance. Further, note that $r_{23} \approx 2.57 \cdot 10^{-3} r_{12}$, and if one now assumes that the satellite (body 4) is relatively near the Moon (say $O(10^4 \text{ km})$), then with $\varepsilon = 10^{-1}$,

$$\begin{split} r_{14} &\approx \varepsilon^{-3} r_0 \; , \\ r_{24} &\approx \varepsilon^0 r_0 \; , \\ r_{34} &\approx \varepsilon^1 r_0 \; . \end{split}$$

Consequently using these with the following mass ratios:

$$\begin{aligned} &\alpha_1=0, \quad \beta_1=-3, \\ &\alpha_2=6, \quad \beta_2=0, \\ &\alpha_3=8, \quad \beta_3=1; \end{aligned}$$

one arrives at:

$$\ddot{\mathbf{r}}_{4} = -\frac{Gm_{0}}{r_{0}^{2}} \left(\frac{\bar{m}_{1}\bar{\mathbf{r}}_{14}}{\|\bar{\mathbf{r}}_{14}\|^{3}} \,\epsilon^{6} + \frac{\bar{m}_{2}\bar{\mathbf{r}}_{24}}{\|\bar{\mathbf{r}}_{24}\|^{3}} \,\epsilon^{6} + \frac{\bar{m}_{3}\bar{\mathbf{r}}_{34}}{\|\bar{\mathbf{r}}_{34}\|^{3}} \,\epsilon^{6} \right)$$
(4.32)

as the "correct"⁸ expansion in terms of acceleration scales in this region, which we may remark corresponds more or less to the illustration of the Weak Stability Boundary in figure 3.8 in section 3.4.1. If we were instead to take the body quite near the Moon (say $O(10^3 \text{ km})$, $\beta_3 = 2$), then the attraction of the Moon would come to dominate as an $O(\epsilon^4)$ –term instead:

$$\ddot{\mathbf{r}}_{4} = -\frac{Gm_{0}}{r_{0}^{2}} \left(\frac{\bar{m}_{1}\bar{\mathbf{r}}_{14}}{\|\bar{\mathbf{r}}_{14}\|^{3}} \, \varepsilon^{6} + \frac{\bar{m}_{2}\bar{\mathbf{r}}_{24}}{\|\bar{\mathbf{r}}_{24}\|^{3}} \, \varepsilon^{6} + \frac{\bar{m}_{3}\bar{\mathbf{r}}_{34}}{\|\bar{\mathbf{r}}_{34}\|^{3}} \, \varepsilon^{4} \right) \,.$$

$$(4.33)$$

Physical Regions

The preceding example illustrates that based on this choice of scalings, in the *near-Moon region*⁹ it is expected that the Moon's gravitation dominates (though falling off quickly), while the Earth's influence is at most perhaps two orders of magnitude smaller and so usually forms the dominant perturbation, while the Sun is possibly (though not necessarily) less important. When very close, this motivates a characterization as the *captured* regime, though the picture looks quite different in other regions, and the scaling indeed reflects this explicitly.

As noted above, there are two assumptions being made in the choice of scaling.

1. The first is the choice of a scaling factor for the distance, and in modeling two possibilities for the best choice arise. On the one hand, if one is interested in capture, the most logical choice is the Earth-Moon distance, even if this isn't aesthetically consistent with normalizing by the largest mass in the numerator.

However it should be remarked that when performing numerical calculations in this formulation, the issue of minimizing the amplification of errors by small denominators may become relevant, and it may become preferable to instead scale such that the dimensionless terms in denominators are multiplied by the smallest possible factor ε^k , to avoid unnecessarily amplifying the machine error. This has not, however, been investigated by the author, and will not be pursued further in this thesis.

2. The second assumption being made (albeit implicitly) is about the region in which one assumes the satellite to operate. In example 4.3.4, near the Moon, it was found that S:E:M influences are approximately $O(\varepsilon^6 : \varepsilon^6 : \varepsilon^6)$ or $O(\varepsilon^6 : \varepsilon^6 : \varepsilon^4)$ depending on the proximity to the Moon. By analogous calculation, one finds the results in table 4.1. Note in particular the ratios asserted earlier where the forces due to Earth and Sun approximately balance, leading to ballistic lunar capture.

		Near-Earth		Near-Moon (1)		Near-Moon (2)		Capture Region	
Body	α	β	$\alpha - 2\beta$	β	$\alpha - 2\beta$	β	$\alpha - 2\beta$	β	$\alpha - 2\beta$
r ₁₄ : Sun	0	-3	6	-3	6	-3	6	-3	6
r_{24} : Earth	6	+1	4	0	6	0	6	0	6
<i>r</i> ₃₄ : Moon	8	0	6	+1	6	+2	4	0	8

Table 4.1: Summary of force ratios corresponding to a single force-scaling in 4 different physical regions; capture refers to the region where forces balance and exhibit capture dynamics, not the actual capture which occurs in the near–Moon (1) region (after which ideally the satellite would proceed to and remain in the near–Moon (2) region.)

⁸Correct for a certain purpose, but in no way unique.

⁹Specifically one might have in mind here a sphere of ca. 50 000 km, which is the threshold for capture used in [Verzijl, 2005], though it is apparent from the above that this threshold itself corresponds roughly to entering the region of triple balance described in the preceding example, rather than to Moon–dominance, for which we need to be an order of magnitude closer.

Remark in particular with regard to the above table that in the second and fourth cases there is an (order of magnitude) balance between the dominant forces,¹⁰ suggesting the possibility of an actual balance of forces. It should then come as no surprise that the remarks of section 3.4 fall into these two categories:

1. Exterior ballistic capture transfers could be effected in the capture region in the table, where the dominant Earth and Sun forces would balance and a correct alignment with the Moon, and/or a small ΔV -correction would then be sufficient to propel the spacecraft along a capture trajectory;

The reader will remark also that in this region, the Moon–Satellite distance has been estimated at $O(\varepsilon^0 r_0)$, which is justified by the fact that during a typical ballistic capture trajectory, the distance to the Moon never varies beyond ~ 3 × r_{EM} , as is easily observed in figure 3.9 (or the equivalent figures in *e.g.* [Belbruno and Carrico, 2000]).

2. Interior ballistic capture transfers on the other hand could be effected in the near-Moon (1) region where all three forces could conceivably balance, but it should be remarked that the requirements for balance would now be more complex, as the balance would be between 3 rather than 2 bodies of equal importance.

In fact this latter observation may be the reason why it has proven much easier to find exterior ballistic capture trajectories, despite the initial results found by Belbruno [Belbruno, 1987, 1990] in the late 1980's. In particular, as observed in [Belbruno and Carrico, 2000] the practical requirements for successful targeting of exterior ballistic captures are relatively few (mainly limits on the relative Sun-Earth-Moon configuration at departure), while very few interior trajectories are known in the literature.

In light of the above discussion, the reader is asked to note again that the scalings might be improperly chosen for a particular region, or become improper as a body transitions between regions (e.g. scalings chosen for the capture region being applied to the near-Moon captured region), does not imply that the equations are now suddenly incorrect. The underlying equations have not changed, and simulate correctly in principle,¹¹ but in other regions of the problem other scalings may be considerably more appropriate for a proper understanding of the dynamics.

¹⁰For the near-Earth and very-near-Moon cases there is also a balance, but this is in the largest *perturbations*, rather than the dominant *forces*.

¹¹There may in practice, however, be large disadvantages to a bad scaling in connection with the roundoff error incurred by computations involving terms with large differences in magnitude during a numerical integration.

Chapter 5

Numerical Trajectory Integration

This chapter discusses 3 classes of integrators relevant to the work in Part III of this thesis:

- Single-step methods against which we will later benchmark, illustrated using the Runge-Kutta-Fehlberg 4th
 –order algorithm with step–size control;
- Multi-step methods, particularly in the context of their adaptability to conservative schemes, illustrated using the Adams–Bashforth–Moulton formulation of a 4th –order predictor-corrector algorithm with step–size control, and
- 3. Conservative methods (with which we will be primarily concerned), illustrated using the Bowman formulation of a 2^{nd} –order fixed step–size algorithm which conserves the Hamiltonian for the circular restricted 3–body problem exactly.

The key references for the algorithms are [Burden and Faires, 2001, sections 5.4–5.8] and [Kotovych and Bowman, 2002]. A discussion of symplectic integrators is omitted aside from some brief remarks in section 5.3, as they are not relevant to the approach taken in this thesis, but the interested reader is referred to the discussion in [Verzijl, 2006, section 6.4].

5.1 Preliminaries

The discussion here follows essentially the presentation of [Burden and Faires, 2001] with some customized notation which better suits present purposes. Note in particular that throughout, x always denotes an exact solution,¹ while w denotes the approximation at the corresponding time / output point.

We first introduce the discretization of time and corresponding notation for the solution and approximation. Introduce a constant step–size $h = (t_f - t_0)/N$ based on the difference between initial and final times t_i , t_f and the number of

¹In particular $x^{(i)} = x(t_i) = x(t_0 + i(t_f - t_0)/N)$ all refer to the same point along the exact solution.

output points N. This motivates the shorthand:

$$t_{i} = t_{0} + \frac{i(t_{f} - t_{0})}{N} \text{ for the } i^{th} \text{ discretized step , and} \\ x^{(i)} := x(t_{i}) = x\left(t_{0} + \frac{i(t_{f} - t_{0})}{N}\right) = x(t_{0} + ih) ,$$
$$w^{(i)} := w(t_{i}) = w\left(t_{0} + \frac{i(t_{f} - t_{0})}{N}\right) = w(t_{0} + ih) ,$$

for the exact solution x and its approximation w at the i^{th} step in the integration process.

In this chapter the assumed general form for the (vector) differential equation is $\dot{x} = f(x,t)$ and this will be further specified where appropriate. Commonly used trajectory integrators essentially use a time discretization which may be thought of as replacing the differential equation's formal solution:

$$x(t_{i+1}) = x(t_i) + \int_{t_i}^{t_{i+1}} f(x(t), t) \,\mathrm{d}t \;, \tag{5.1}$$

by the difference method:

$$x^{(i+1)} \approx w^{(i+1)} = w^{(i)} + hf(w^{(i)}, t_i) \quad \text{for single-step methods or}$$
(5.2)

$$x^{(i+1)} \approx w^{(i+1)} = \sum_{k=0}^{m-1} a_{i-k} w^{(i-k)} + h \sum_{k=0}^{m} b_{i-k} f(w^{(i-k)}, t_{i-k}) , \qquad (5.3)$$

for a general (explicit) *m*-step method, following the convention of [Burden and Faires, 2001].

With this two key concepts of integration-scheme error are introduced.

Definition 5.1.1 (Local Truncation Error)

The local truncation error of a method is defined as the difference at each step between the (true) solution to the differential equation and the approximation by the difference method, assuming the best-case scenario where all previous values are known.

For a one step difference method, again following the convention of [Burden and Faires, 2001]:

$$\tau^{(i+1)}(h) = \frac{x^{(i+1)} - \left(x^{(i)} + h f(x^{(i)}, t_i)\right)}{h} \quad \text{or for a multi-step method,}$$
(5.4a)

$$\tau^{(i+1)}(h) = \frac{x^{(i+1)} - \left(\sum_{k=0}^{m-1} a_{i-k} x^{(i-k)} + h \sum_{k=0}^{m} b_{i-k} f(x^{(i-k)}, t_{i-k})\right)}{h} .$$
(5.4b)

The division by *h* is by convention, so that the $\tau^{(i+1)}$ corresponds roughly to errors in the derivative and the truncation error in the solution is of the form $h\tau^{(i+1)}$. Note in particular that this difference is indeed local in the sense that it considers only the error induced by a single approximate step using perfect prior data.

Definition 5.1.2 (Global Truncation Error)

The global truncation error, by contrast, is defined as the difference between the true and approximated solution:

$$\bar{\tau}^{(i+1)}(h) = \frac{x^{(i+1)} - w^{(i+1)}}{h} \,. \tag{5.5}$$

Note here that, in contrast to the above, one compares only the exact solution with the (cumulative) approximation at a time–step. The problem with this definition is immediately obvious: if x(t) were available, one wouldn't need a numerical scheme in the first place.

Definition 5.1.3 (Consistency of an Integration Method)

A method is said to be consistent with the differential equation governing the trajectories if for the local truncation errors $\tau^{(i)}$ it holds that as the step-size $h \rightarrow 0$:

$$\lim_{h \to 0} \max_{1 \le i \le N} |\tau^{(i)}(h)| = 0.$$
(5.6)

Definition 5.1.4 (Convergence of an Integration Method)

A method is said to be convergent with respect to the differential equation if, similarly,

$$\lim_{h \to 0} \max_{1 \le i \le N} |w^{(i)} - x^{(i)}| = 0.$$
(5.7)

Note that comparing equation (5.5), it is clear that convergence implies global stability.

Summarizing, a local truncation error can always be defined, though the quality of a method is determined by the combination of its stability and its global truncation error. The latter is problematic by definition, though the two are linked intuitively. For single-step methods, one has the result:

Lemma 5.1.5 (Stability, Convergence and Error Bound for Single Step Methods) A Lipschitz condition on $\phi(w,t,h)$ as introduced above in the variable *w* on some domain $D \subset \mathbb{R}^2$ means

$$\exists L > 0 \quad \text{such that} \quad |\phi(w^{(i)}, t, h) - \phi(w^{(j)}, t, h)| \le L|w^{(i)} - w^{(j)}| \quad \forall (w^{(i)}, t), (w^{(j)}, t) \in D.$$
(5.8)

If $\exists h_0 > 0$ and $\phi(w,t,h)$ is continuous and satisfies a Lipschitz condition with constant L on the set

$$D = \{ (w,t,h) | a \le t \le b, -\infty < w < \infty, \ 0 \le h \le h_0 \} .$$
(5.9)

Then

- 1. The method is stable;
- 2. The method is convergent iff. it is consistent, which is equivalent to

$$\phi(x,t,0) = f(x,t) \ \forall \ a \le t \le b \ .$$

3. If a function $\tau(h)$ exists such that for each i = 1, 2, ..., N the local truncation error satisfies $|\tau^{(i)}(h)| \le \tau(h)$ whenever $0 \le h \le h_0$, then

$$|x^{(i)} - w^{(i)}| \le \frac{\tau(h)}{L} e^{L(t_i - a)}$$
.

Proof. See the references in section 5.10 of [Burden and Faires, 2001], or the discussion of the Gronwall inequality in chapter 1 of [Verhulst, 2000].

This result connects consistency and the local truncation error with convergence and the global truncation error for single-step methods, and is the foundation for strategies of local error control as a means of global error control.

Similar results for multi-step methods will not be discussed here; the reader is referred to section 5.10 of [Burden and Faires, 2001] and the references given therein. It suffices to say that the gist of the approach remains that small enough local errors lead to small global errors under certain conditions as above, though a key problem is typically that actual error estimates are difficult to come by.

5.2 Traditional Methods

With these preliminaries in mind, this section will discuss in somewhat more detail two classical algorithms in wide use as single-step and multi–step methods. Representative for the former will be the variable-step–size RKF4(5) method and for the latter a 4^{th} order Adams-Bashforth-Moulton variable-step–size method will be taken. We begin, however, with the simple predictor–corrector, which will be the prototype for the conservative schemes discussed in section 5.3.

5.2.1 Simple Predictor–Corrector

We begin with arguably the simplest conceivable predictor–corrector method. Let each step from $x(t_0 + ih)$ to $x(t_0 + (i + 1)h)$ be given by the integrator as:

$$w_p^{(i+1)} = w^{(i)} + h f(w^{(i)}), \qquad (5.10a)$$

$$w_c^{(i+1)} = w^{(i)} + h\left(f(w^{(i)}) + f(w_p^{(i+1)})\right).$$
(5.10b)

The Taylor expansion of the true solution is (we take $x(t_0) = x_0$ as usual):

$$x(t_0+h) = x_0 + hf(x_0) + \frac{h^2}{2}f'(x_0)f(x_0) + \frac{h^3}{6}f(x_0)\left(f''(x_0)f(x_0) + f'^2(x_0)\right) + \dots,$$
(5.11)

while we compare to find that the algorithm approximates this as:

$$x_{c}(t_{0}+h) = x_{0} + \frac{h}{2} \left(f(x_{0}) + f(x_{0}+hf(x_{0})) \right) \text{ which we Taylor-expand to find:}$$
$$= h f(x_{0}) + \frac{h^{2}}{2} f'(x_{0}) f(x_{0}) + \frac{h^{3}}{4} f''(x_{0}) f(x_{0})^{2} + \dots$$
(5.12)

Thus we see that it has local truncation error $\tau(h) = O(h^2)$ using the definition (5.4a).

5.2.2 Single–Step: RKF4(5)

The 4^{th} (5^{th}) order Runge-Kutta-Fehlberg method is the workhorse of integration engines, usually being chosen as a default due to its ease of implementation and relative accuracy. It is the integrator used in our previous internship work [Verzijl, 2005] which motivated present work and this thesis, and its choice was discussed there in detail.

Consider again the vector differential equation: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. The key step in Fehlberg's method [Fehlberg, 1968, 1969] is the use of two general expansions of the function about some x_0 to orders n and n + 1 to derive an estimate of the required step–size for stability of the $O(h^n)$ method.

Let each step from $x(t_0 + ih)$ to $x(t_0 + (i+1)h)$ be given by the integrator as:

$$w^{(i+1)} = w^{(i)} + h \sum_{j=0}^{4} c_j f_j + O\left(h^5\right)$$
(5.13a)

$$\hat{w}^{(i+1)} = \hat{w}^{(i)} + h \sum_{j=0}^{5} \hat{c}_j f_j + O\left(h^6\right) \,. \tag{5.13b}$$

Further, take the usual Runge-Kutta formulation for the functions f_i :

$$f_0 = f(w^{(i)}, t_i) , (5.14a)$$

$$f_j = f\left(\sum_{k=0}^{j-1} \beta_{jk} f_k, t_i + \alpha_j h\right).$$
(5.14b)

Now this formulation yields a total of 4 sets of coefficients (α_j , β_{jk} , c_j and \hat{c}_j) which must be determined such that the equations (5.13a) and (5.13b) represent 4th resp. 5th order Runge-Kutta formulas. The difference

$$\hat{w} - w \equiv \Delta = \sum_{j=0}^{5} (\hat{c}_j - c_j) f_j$$
(5.15)

then represents an approximation of the leading error term of the fourth order Runge-Kutta formula, which can be used to effect step–size control by adjusting the step–size such that the difference between the fourth and fifth order integrations (as an estimate for the local truncation error) never exceeds the user-specified error tolerance ε .

Specifically, taking a modified step-size qh and said error tolerance, choosing:

$$q < \left(\frac{h\varepsilon}{|w_{\rm rk5}^{(i+1)} - w_{\rm rk4}^{(i+1)}|}\right)^{1/4}$$

ensures that the local truncation error is bounded: $\tau^{(i+1)} < \epsilon$. See also [Burden and Faires, 2001, Fehlberg, 1969].

The above expression (5.14b) must be equivalent to the Taylor expansions of x, \hat{x} about x_0 . This requirement leads to the equations of condition for the coefficients, discussed in Fehlberg [1969]. In principle, on solving these one obtains an optimal set of coefficients, in the sense that they minimize the truncation error of the fourth order formula.²

With this approach, Fehlberg derived a combination of the coefficients α_j , β_{jk} , c_j and \hat{c}_j given in Fehlberg [1969]. However, there remain two degrees of freedom in the choice of parameters: α_2 , α_5 , and one implementation used in our code is that given in table 5.1, corresponding to the [Shampine et al., 1976] formulation implemented in [Burkardt,

 $^{^{2}}$ At higher orders there may be some further degrees of freedom in the choice of coefficients, leading to equivalent formulations at a given order, with slightly different properties and performance.

j	α_j				β_{jk}			\hat{c}_j	c_j
		k	0	1	2	3	4		
0	0		0					$\frac{16}{135}$	$\frac{25}{216}$
1	$\frac{1}{4}$		$\frac{1}{4}$					0	0
2	$\frac{3}{8}$		$\frac{3}{32}$	$\frac{9}{32}$				$\frac{6656}{12825}$	$\frac{1408}{2565}$
3	$\frac{12}{13}$		$\frac{1932}{2197}$	$-\frac{7200}{2197}$	$\frac{7296}{2197}$			$\frac{28561}{56430}$	$\frac{2197}{4104}$
4	1		$\frac{439}{216}$	-8	$\frac{3680}{513}$	$-\frac{845}{4104}$		$-\frac{9}{50}$	$-\frac{1}{5}$
5	$\frac{1}{2}$		$-\frac{8}{27}$	2	$-\frac{3544}{2565}$	$\frac{1859}{4104}$	$-\frac{11}{40}$	$\frac{2}{55}$	0

Table 5.1: RKF4(5) Coefficients for the Watts and Shampine Implementation of the RKF4(5) Method

It should be remarked that the industry-standard at the moment appears to be the code RKSuite introduced in [Brankin et al., 1993] and provided via Netlib. This suite is implemented parallel to the Burkardt integrator in our simulation code (*cf.* appendix C), and provides RKF2(3), 4(5) and 7(8) pairs, for integrations with $O(h^2, h^4, h^7)$ accuracy respectively.

5.2.3 Multi–Step: Adams–Bashforth–Moulton

We discuss multi-step predictor-corrector methods here primarily for what they offer as a building block towards higher-order conservative schemes. We will have more to say in this regard in Part III of this thesis, but in order to justify that discussion, introduce the methods here first together with a discussion of error control.

Following the notation of [Burden and Faires, 2001], Adams–Bashforth–Moulton methods take the following form (*cf.* equation 5.3):

$$x(t_{i+1}) \approx w^{(i+1)} = \sum_{k=0}^{m-1} a_k w^{(i-k)} + h \sum_{k=0}^m b_k f(w^{(i-k+1)}, t_{i-k+1}).$$

The first term in the second part of the approximation is $b_0w^{(i+1)}$, which is an implicit term, as one intends to solve for precisely this next $w^{(i+1)}$. The methods are essentially a concatenation of an explicit Adams-Bashforth method and an implicit Adams-Moulton method of the same order; the difference in the two schemes is what is done with the implicit term.

Explicit Adams–Bashforth

The Adams–Bashforth class of integrators are explicit multi–step algorithms, and set the coefficient b_m (the implicit term) 0. A typical scheme is the 4th order Adams–Bashforth:

$$w^{(0)} = \alpha, \ w^{(1)} = \alpha_1, \ w^{(2)} = \alpha_2, \ w^{(3)} = \alpha_3 \quad \text{initial data,}$$

$$w^{(i+1)} = w^{(i)} + \frac{h}{24} \left(55f(w^{(i)}, t_i) - 59f(w^{(i-1)}, t_{i-1}) + 37f(w^{(i-2)}, t_{i-2}) - 9f(w^{(i-3)}, t_{i-3}) \right) \quad \text{with } i = 3, \dots, N-1$$
(5.16a)

and local truncation error

$$\tau^{(i+1)}(h) = \frac{251}{720} \frac{d^5 x}{dt^5} \Big|_{\mu_i} h^4 \quad \text{for some } \mu_i \in (t_{i-3}, t_{i+1}) .$$
(5.16b)

Here the α terms are initial data, which if not available empirically may need to be estimated with a different method, *e.g.* Runge-Kutta-type algorithms (of at least the same order).

2004].

Implicit Adams-Moulton

The Adams–Moulton class of integrators are implicit multi–step algorithms, and take $b_m \neq 0$. For example, a typical three step 4th order scheme:

$$w^{(0)} = \alpha, w^{(1)} = \alpha_1, w^{(2)} = \alpha_2 \text{ initial data,}$$

$$w^{(i+1)} = w^{(i)} + \frac{h}{24} \left(9f(w^{(i+1)}, t_{i+1}) + 19f(w^{(i)}, t_i) - 5f(w^{(i-1)}, t_{i-1}) + f(w^{(i-2)}, t_{i-2}) \right) \text{ with } i = 2, \dots, N-1$$
(5.17a)

and local truncation error

$$\tau^{(i+1)}(h) = -\frac{19}{720} \frac{\mathrm{d}^5 x}{\mathrm{d}t^5} \Big|_{\mu'_i} h^4 \quad \text{for some } \mu'_i \in (t_{i-2}, t_{i+1}) .$$
(5.17b)

In general, with an implicit scheme, it depends rather critically on the form of the differential equation, $\dot{x} = f(x,t)$, whether the scheme can in fact be solved to yield $w^{(i+1)}$. The following section outlines how the combination of the two approaches can get around this caveat.

Predictor-Corrector Adams-Bashforth-Moulton Scheme

The schemes can efficiently be combined into a predictor-corrector scheme. The explicit Adams-Bashforth scheme predicts an approximation, which is then corrected by the implicit Adams-Moulton scheme, re-cast as an update equation:³

$$w_p^{(i+1)} = w^{(i)} + \frac{h}{24} \left(55f(w^{(i)}, t_i) - 59f(w^{(i-1)}, t_{i-1}) + 37f(w^{(i-2)}, t_{i-2}) - 9f(w^{(i-3)}, t_{i-3}) \right),$$
(5.18a)

$$w_c^{(i+1)} = w^{(i)} + \frac{h}{24} \left(9f(w_p^{(i+1)}, t_{i+1}) + 19f(w^{(i)}, t_i) - 5f(w^{(i-1)}, t_{i-1}) + f(w^{(i-2)}, t_{i-2}) \right).$$
(5.18b)

Combining them in this way gets us around the potential difficulty with solving for the implicit term.

Error Control

Moreover, the concept may be extended a step further by noting the analogy with RKF-type algorithms, where the relatively cheap availability of already evaluated function values from different order approximations was exploited to derive a step–size control.

In [Burden and Faires, 2001] it is shown that given the assumption $\frac{d^5x}{dt^5}\Big|_{\mu_i} \approx \frac{d^5x}{dt^5}\Big|_{\mu'_i}$, and assuming a step–size qh and error tolerance ε ,⁴ choosing:

$$q < \left(\frac{270}{19} \frac{h\varepsilon}{|w_c^{(i+1)} - w_p^{(i+1)}|}\right)^{1/4}$$

ensures that the local truncation error is bounded: $\tau^{(i+1)} < \epsilon$.

It should, however, be noted that modifying the step–size to meet such a bound is computationally more expensive than with *e.g.* RKF4(5), as the (equally spaced) points within the interval qh must be recalculated each time q is adjusted.⁵

³Here the subscripts p and c refer to predictor and corrector respectively.

⁴Strictly, one assumes that this error is much larger than the roundoff (machine) error.

⁵There are algorithmic workarounds discussed in the section 5.7 of [Burden and Faires, 2001] to minimize this, though these will

5.3 Conservative Integration

Before outlining the algorithmic approach, we begin by considering the motivation for conservative integration.

5.3.1 Motivation

Against the background of single– and multi–step methods which in principle attain $O(h^n)$ accuracy, and which are well-studied in the literature, the question might arise as to why other classes of integration methods are interesting to study. However, the posing of this question neglects the following issues:

- The measure of an integration scheme is not its accuracy in terms of the local truncation error, which is the accuracy to which is referred by the characterization $O(h^n)$ above; while this is a good first step, it says too little.
- In fact, the accuracy sought is more complex. What is in fact desired is a simulation which best approximates a given model, which is in turn only an approximation of the dynamics of a real-world system. Thus, two main sources of error with respect to the model must be taken into account:
 - 1. The global truncation error (rather than only the local truncation error), due to the finite approximation used in the integration algorithm;
 - 2. The cumulative numerical error, due to the buildup of roundoff error inherent to any numerical simulation.

However, even these skirt a deeper issue: how well does our simulation reproduce the features of the model itself? Precisely this consideration leads to both conservative and symplectic integrators, despite the fact that these may be considerably more complex (and possibly less efficient) than simple Runge-Kutta–type integrators.⁶ The reader will note that this connection is reprised below, in section 5.3.3.

Specifically, in this section, our attention turns to the question of preserving integrals of motion, as introduced in section 2.2. It stands to reason that a good numerical scheme should preserve the first integrals of a (Hamiltonian) system, *e.g.* the conservation of energy in the n-body problem.⁷ At worst, the actual value of energy should perhaps oscillate about the true value in some bounded manner.

But the reality is far more troubling, as it can be shown that for even Runge-Kutta schemes, the error in energy $\Delta \mathcal{E}$ may grow or decrease monotonically in time, [Yoshida, 1993], where it is shown that for just the simple 1-dimensional harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2) = T + V \quad \text{integrated a time-step } h \text{ further to } (p',q') \text{ becomes:}$$

$$(p'^2 + q'^2) = (1 + h^2)(p^2 + q^2) \quad \text{for an Euler scheme},$$

$$(p'^2 + q'^2) = (1 - \frac{1}{72}h^6 + \dots)(p^2 + q^2) \quad \text{for a RK4 scheme}.$$

not be discussed here.

⁶The real 'efficiency' issue is of course the amount and type of computations required relative to how accurately the integrator is reproducing the global trajectory, which is a fuzzy issue, and precisely the reason that traditional error analysis focuses on the minimization of local truncation error.

⁷Actually, the issue is a bit more complicated than simply using the integral as a direct measure, as its value may not be representative for the actual error in the integrated solution, as discussed in [Huang and Innanen, 1983]. This does not however, take away from the fundamental point that the energy integral *should* be conserved.

Since it is known from classical mechanics and the theory of Hamiltonian systems that analytical first integrals are key to understanding the structure of solutions and reducing the complexity of the dynamics, it seems a reasonable approach to try to build this knowledge into the integrator itself somehow, in order to obtain better results in this regard.

5.3.2 Conservative Integrators

This is precisely what Bowman et al. have attempted, and this section presents an extended version of their main theorem as a proposition, in addition to an example outlining their development of a conservative integration scheme for the circular restricted 3–body problem.

The development of conservative integrators begins in [Shadwick et al., 1999] with the idea of integrating the modified equation $\dot{x}^{(k)} = f(x^{(k)}) + s^{(k)}$, and then deriving conditions such that a desired integral is conserved, and where $\lim_{h\to 0} s^{(k)} = 0$ for consistency. This intuition is generalized in [Kotovych and Bowman, 2002] with the following simple but powerful proposition, which is here extended to the current formulation as well.⁸

Proposition 5.3.1 (Conservative Integration Theorem)

Given the usual (possibly non-autonomous) system of equations $\dot{x} = f(x,t)$, let $x, c \in \mathbb{R}^n$ be vectors.

If the left-hand side $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ has values orthogonal to *c*, *i.e.* $c \cdot f = 0$, and one considers a linear integral $I = c \cdot x$ of the ODE, then each stage of the explicit *m*-stage discretization (in the formulation of Bowman):

$$w^{(j)} = w^{(0)} + h \sum_{k=0}^{j-1} b_{jk} f(w^{(k)}, t_0 + a_j h), \qquad j = 1, \dots, m$$
(5.19)

also conserves I during each time-step h.

In the present formulation:

$$x^{(i+1)} \approx w^{(i+1)} = \sum_{k=0}^{m-1} a_{i-k} w^{(i-k)} + h \sum_{k=0}^{m} b_{i-k} f(w^{(i-k)}, t_{i-k}) .$$

Then setting $a_0 = 1$ and $a_{i-k} = 0 \forall k = 1, ..., m$, and letting t_{i-k} correspond to the partial steps $t_0 + a_j h$ in Bowman's formulation, one has the m-stage integrator:

$$w^{(i+1)} = w^{(i)} + h \sum_{k=0}^{m} b_{i-k} f(w^{(i-k)}, t_{i-k}), \qquad (5.20)$$

which also exactly conserves *I*, as will now be proven.

Proof. For each stage j = 1, ..., m one has, first for Bowman's formulation:

$$c \cdot w^{(j)} = c \cdot w^{(0)} + h \sum_{k=0}^{j-1} b_{jk} c \cdot f(w^{(k)}, t_0 + a_j h) = c \cdot w^{(0)} .$$
(5.21)

For the full *m*-stage step in present notation:

$$c \cdot w^{(i+1)} = c \cdot w^{(i)} + h \sum_{k=0}^{m} b_{i-k} c \cdot f(w^{(i-k)}, t_{i-k}) = c \cdot w^{(i)} , \qquad (5.22)$$

⁸We should remark that this idea has been around somewhat longer, and for example the work of Gear [?] already suggests some of the features brought together here.

which by induction preserves the integral value back to the exact value $c \cdot w^{(0)} = c \cdot x^{(0)} = I|_{t=0}$.

The novelty of Bowman's approach then, is to introduce a transformation $\xi = T(x)$ of the dependent variables *x* to new variables ξ , such that the the *generally nonlinear* integral *I* can be expressed as a linear function of the new variables: or $I' = c \cdot \xi$. Then by the above result, these transformed integrals I' are exactly conserved by an integrator of the form equation (5.19) or (5.20).⁹

For example, consider the second-order predictor-corrector algorithm introduced in [Shadwick et al., 1999]:

$$\hat{w}^{(i)} = w^{(i)} + h f(w^{(i)}, t) , \qquad (5.23a)$$

$$w^{(i+1)} = w^{(i)} + \frac{h}{2} \left(f(w^{(i)}, t) + f(\hat{w}^{(i)}, t+h) \right) .$$
(5.23b)

The first equation (predictor) is kept, and introducing the transformation to the corrector,

$$\xi(t+h) \equiv \xi^{(i+1)} \approx \xi^{(i)} + \frac{h}{2} \left(T'(w^{(i)}) f(w^{(i)}, t) + T'(\hat{w}^{(i)}) f(\hat{w}^{(i)}, t+h) \right)$$

= $\xi^{(i)} + \frac{h}{2} \left(\dot{\xi}^{(i)} + \dot{\xi}^{(i)} \right)$ (5.24)

where $\xi^{(i)} = T(w^{(i)})$, $\hat{\xi}^{(i)} = T(\hat{w}^{(i)})$ and T' is the derivative of the transformation with respect to $x, w.^{10}$ The new (estimated) value of the x is then obtained for the inverse transformation $w^{(i+1)} = T^{-1}(\xi^{(i+1)})$. This may in some cases involve technical issues which we will not delve into here, but postpone until the discussion in chapter 9. The conservation is implemented in the choice of the linearizing transformation T with regard to the integral.

In light of our earlier remarks in section 5.2.3, we draw the reader's attention as well to the amenability of Adams– Bashforth–Moulton schemes to this type of formulation, and note that only the corrector need be fully conservative.

Example 5.3.2 (Conservative Integrator for planar CR3BP)

As an illustration of the method, consider the planar (x, y) circular restricted 3–body problem, as in section 3.3.5 and [Kotovych and Bowman, 2002]. The Hamiltonian in rotating coordinates is given by (*cf.* section 3.3):

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(x^2 + y^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \quad \text{or in generalized coordinates } \mathbf{q}, \, \dot{\mathbf{q}}:$$
$$= \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_1^2 + q_2^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \,. \tag{5.25}$$

Here $\mu = m_2/(m_2 + m_1)$, while $r_1^2 = (x - \mu)^2 + y^2$ and $r_2^2 = (x + 1 - \mu)^2 + y^2$ are the scaled distances to the two primaries in rotating coordinates (see also figure 3.7). One proceeds in the usual way. Introducing canonical variables $q_1 = x$, $q_2 = y$, $p_1 = \dot{x} - y$ and $p_2 = \dot{y} + x$, the Hamiltonian becomes:

$$H = \frac{1}{2}(\dot{p}_1^2 + \dot{p}_2^2) + p_1 q_2 - p_2 q_1 - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}.$$
(5.26)

Now, as a predictor, take the following Euler step using the equations of motion (3.68) (subscripts refer to vector

⁹The approach is reminiscent of feedback linearization in control theory.

¹⁰Remark that the second formulation, advancing the corrector in transformed coordinates directly is simpler and less error-prone to code.

components):

1

$$\hat{q}_{i}^{(j)} = q_{i}^{(j)} + h\dot{q}_{i}^{(j)}, \qquad \hat{p}_{i}^{(j)} = p_{i}^{(j)} + h\dot{p}_{i}^{(j)} \qquad \text{for } i = 1, 2.$$
(5.27)

The (vector) transformation ξ for the corrector is taken as:

$$\xi_1 = \frac{1}{2}q_1^2, \tag{5.28a}$$

$$\xi_2 = \frac{1}{2}q_2^2, \tag{5.28b}$$

$$\xi_3 = \frac{1}{2}\dot{q}_1^2 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},\tag{5.28c}$$

$$\xi_4 = \frac{1}{2}\dot{q}_2^2 \,, \tag{5.28d}$$

such that the integral conserved becomes:

$$H(\xi) = -\xi_1 - \xi_2 + \xi_3 + \xi_4 . \tag{5.29}$$

Comparing equation (5.25), *H* is now a linear function of the new variables with vector $c = (-1, -1, +1, +1)^{\mathsf{T}}$. Differentiating the transformation gives:

$$\dot{\xi}_1 = q_1 \dot{q}_1, \quad \dot{\xi}_2 = q_2 \dot{q}_2, \quad \dot{\xi}_4 = \dot{q}_2 (\dot{p}_2 - \dot{q}_1), \quad \dot{\xi}_3 = \dot{\xi}_1 + \dot{\xi}_2 - \dot{\xi}_4 , \quad (5.30)$$

where the conservation of *H* has been used in obtaining $\dot{\xi}_3$ as a function of the others. The reader may check that in these new variables one indeed has that:

$$\mathbf{c}\cdot\mathbf{f}(\boldsymbol{\xi}) = \sum_{i=1}^{4} c_i \dot{\boldsymbol{\xi}}_i = 0 ;$$

i.e. orthogonality as required by the lemma above, and consequently each numerical step will preserve $H(\xi) = c \cdot \xi$. The corrector then, is as before the second-order relation:

$$\xi_i^{(j+1)} = \xi_i^{(j)} + \frac{h}{2} \left(\dot{\xi}_i + \dot{\xi}_i \right) \qquad \forall i = 1, \dots, 4,$$

which is inverted to obtain updates to the original coordinates as:

$$q_1 = \operatorname{sgn}(\hat{q}_1)\sqrt{2\xi_1}$$
, (5.31a)

$$q_2 = \operatorname{sgn}(\hat{q}_2)\sqrt{2\xi_2}$$
, (5.31b)

$$p_1 = -q_2 + \operatorname{sgn}(\hat{p}_1 + \hat{q}_2) \sqrt{2\left(\xi_3 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)},$$
(5.31c)

$$p_2 = q_1 + \operatorname{sgn}(\hat{p}_2 - \hat{q}_1)\sqrt{2\xi_4} .$$
(5.31d)

The scheme can, moreover, be shown to be $O(h^2)$ accurate. In the latter paper, this approach is extended to the full 3-body problem and the general *n*-body problem in Jacobi coordinates as well, and the 3-body case has been implemented in code, as discussed in appendix C. The corresponding algorithms and performance results form Part III of the present thesis, and can be found in chapters 9–11.

The power of this method then, is its approach of building conservation into existing algorithms in a relatively simple way. However, it should be remarked that in general two possible down-sides are:

- 1. the difficulty involved in finding the linearizing transformation, and
- 2. the technical issue of inverting the transformation to obtain updates in the original coordinates (for all possible cases).

In addition to these issues, the question of extending the method to incorporate multiple integrals is also open, and may be complicated by the above as well. Consequently, our discussion will return to all of the above in Part III of this thesis, where we treat them in some detail.

5.3.3 Conservative vs. Symplectic Methods

It is instructive, for purposes of context, to remark briefly on symplectic integration methods, despite the fact that this thesis will not concern itself further with them.

A parallel motivation to that given for conservative integrators applies to the development of symplectic integrators, in that one seeks to design an integrator which preserves (certain aspects of) the structure of the dynamical system for which it is developed.¹¹ The structure intended, in contrast to conservation of integrals of motion, is the symplectic structure of a Hamiltonian phase–space, introduced earlier in section 2.1.2. Symplectic integrators are, however, specific to Hamiltonian systems in contrast to conservative integrators, which can be developed for any dynamical system with known first integrals.¹²

Moreover, this preservation of the symplectic structure of the phase–space can, in fact, be shown to be equivalent to the conservation of a discrete Hamiltonian, which is related to, though not the same as the conservation of the actual Hamiltonian (*cf.* [Thijssen, 2007, section 8.4]).

Sadly, an integrator which is both symplectic and conservative, which is what one would ideally like, is out of reach in general. The following result, due to Ge and Marsden [Ge and Marsden, 1988], is cited to summarize the problem.

Proposition 5.3.3 (Non-conservation in Symplectic Algorithms)

Let \mathcal{H} be a Hamiltonian which has no conserved quantities other than functions of itself,¹³ or equivalently $\{I, \mathcal{H}\} = 0$ and $I(z) = I_0(\mathcal{H}(z))$ for any first integral I and coordinates z. Let further \mathcal{P} be an approximate solution algorithm which is defined for small step–size h in time and is smooth.

If this algorithm \mathcal{P} is symplectic, and conserves \mathcal{H} exactly, then it is the time advance map $g_t(x)$ for the Hamiltonian system (emphcf. section 2.3.1) up to a re–parameterization of time. This violates the assumptions of the proposition and so in other words, approximate symplectic algorithms cannot preserve energy for non–integrable systems.

Proof. The proof, outlined in [Ge and Marsden, 1988], hinges on the fact that if a system is non–integrable, there cannot be an approximate algorithm which is simultaneously the time advance map (up to a re–parameterization of time, in general), as this would simply be a time discretization of the true solution to the system. \Box

¹¹An interesting discussion in this regard, it should be noted, is that in [Shadwick et al., 2001], which considers a number of different approaches to structure preserving integration algorithms.

 $^{^{12}}$ In fact, as we shall argue in Parts II and III, we can extend this assertion to approximations of first integrals of the motion as well, and shall discuss the application of the method of integrating vectors (*cf.* chapter 4) for finding these.

 $^{^{13}}$ In a given class, *e.g.* the analytic functions.

Corollary 5.3.4

By the same argument, one cannot, for a non-integrable system, have a conservative (approximate) algorithm which is also symplectic.

Example 5.3.5 (Application to a 4–Body Choreography)

Figures 5.1 and 5.2, taken from [Kotovych and Bowman, 2002], illustrate the application of the three types of techniques as applied to the 4–body choreography recently studied by the group of Simó. The problem is very sensitive to initial conditions, and a demanding reference. The schemes implemented are low-order, and benchmarked not using the local truncation error, but relative to a 5th order Runge-Kutta scheme with very small fixed time–step.^{14,15}



Figure 5.1: Basic Predictor-Corrector, Conservative and Symplectic Integration results on Simó's 4–body choreography, [Kotovych and Bowman, 2002] — PC: naive predictor-corrector, C-PC: conservative predictor-corrector and SKP: a typical symplectic integrator; all are 2^{nd} order schemes with $h = 10^{-3}$ time–step, and integrated over 2 periods of the choreography: $t \in [0, 4\pi]$.

¹⁴Presumed by them to be more accurate due to its higher order, an issue to which we return in section 9.3.

¹⁵ The attentive reader will note also that no consideration is given by Bowman et al. to the CPU time required by the methods, and consequently little can be said about efficiency on this basis.



Figure 5.2: Predictor-Corrector, Conservative and Symplectic Integrator Performance, [Kotovych and Bowman, 2002] — Error for methods as in previous figure, computed relative to a 5^{th} order Runge-Kutta scheme with time-step $h = 10^{-5}$.

5.4 Error in Conservative Integration Schemes

We begin first with a discussion of the effect of conservative integration on the propagation of error.

5.4.1 Basic Error Analysis

Comparing the analysis of section 5.2.1, it is easily found that for a conservative scheme as outlined, a basic consideration of the error leads to the expression (5.32) for the approximation, in contrast to the earlier expression (5.12).

$$x(t_0 + h) = x_0 + h f(x_0) + \frac{h^2}{2} f'(x_0) f(x_0) + \frac{h^3}{4} \left(f''(x_0) f(x_0)^2 + \frac{T'''(x_0)}{3T'(x_0)} f(x_0)^3 \right) + \dots,$$
(5.32)

We remark that this shows that we may expect problems in the form of error blowup when $T'(x) \approx 0$, which is precisely when the inverse transformation would become singular with $T^{-1}(\xi)$, $T^{-1}(\xi) \rightarrow \infty$. We shall have more to say about this in section 10.2 further on.

5.4.2 Exact Integrals

Such schemes preserve an integral of motion exactly, but from the discussion of Hamiltonian dynamics, we also recognize that exact conservation of such an integral implies the removal of the associated degree of freedom from the dynamics being solved numerically.

In fact, this elimination of a degree of freedom was encountered earlier with *e.g.* the reduction of the 18×1^{st} order 3–body problem by means of coordinate transforms which eliminated the degrees of freedom associated with the conservation of linear momentum. An integrator which exactly conserves an integral performs precisely the same function, as the resulting numerical trajectory can have no error in precisely those same degrees of freedom.¹⁶

We propose then, that this leads to the physical interpretation illustrated in figure 5.3.

The removal of the *k* degrees of freedom associated with integrals being conserved, whether numerically (assume for the moment infinite machine precision) or analytically (by suitable coordinate transforms) reduces the dynamics from those on an *n*-dimensional manifold to those on an (n - k)-dimensional submanifold, by eliminating the in principle arbitrary $O(h^p)$ error of the integrator *normal* to the manifold.

¹⁶This remark and all others in the same vein should be understood as bearing the qualification "up to machine precision and roundoff error" throughout.



Figure 5.3: In- and Out-of-Manifold error for analytically conservative integration.

This would appear, then, to reduce the 2–norm local truncation error $\|\tau^{(m)}\|_2$ at iteration *m* as follows:

$$\|\boldsymbol{\tau}^{(m)}\|_{2} = \left(\sum_{i=1}^{n-k} |\boldsymbol{\tau}^{(m)}_{i}|^{2} + \sum_{i=n-k+1}^{n} |\boldsymbol{\tau}^{(m)}_{i}|^{2}\right)^{1/2}$$

$$= \left(\sum_{\substack{i=1\\in-\text{manifold}}}^{n-k} |c_{i}h^{p}|^{2} + \sum_{\substack{i=n-k+1\\normal-to-manifold}}^{n} |c_{i}h^{p}|^{2}\right)^{1/2} \text{ using the accuracy of the integrator, such that:}$$
(5.33)
$$\|\boldsymbol{\tau}^{(m)}_{\text{cons}}\|_{2} = \frac{\left(\sum_{i=1}^{n-k} |\boldsymbol{\tau}^{(m)}_{i}|^{2}\right)^{1/2}}{\left(\sum_{i=1}^{n-k} |\boldsymbol{\tau}^{(m)}_{i}|^{2} + \sum_{i=n-k+1}^{n} |\boldsymbol{\tau}^{(m)}_{i}|^{2}\right)^{1/2}} \cdot \|\boldsymbol{\tau}^{(m)}\|_{2}$$

$$= \sqrt{\frac{c_{1}^{2} + \dots + c_{n-k}^{2}}{c_{1}^{2} + \dots + c_{n}^{2}}} \cdot \|\boldsymbol{\tau}^{(m)}\|_{2},$$
(5.34)

Remark that this consideration may be slightly naive in practice, due to the issues raised below in section 5.4.4. Nonetheless, we may assert that in general terms, conservation of an integral reduces the numerically integrated motion to a lower–dimensional sub-manifold on which only the in–manifold error component still plays a role. It of course follows that the more integrals conserved by the numerical integration, the better our performance in this regard, up to the limit of a fully integrable system in which all errors are eliminated up to machine precision.

5.4.3 Approximate Integrals

Likewise, we may consider the question of what changes when our scheme conserves not an exact analytical integral of motion, but an approximation to one, such as those generated using the method of integrating vectors of chapter 4. Suppose that we have such an approximate integral, accurate to some $O(\varepsilon^d)$ in the local error due to the truncation of the approximation $\vec{\varsigma}^{(m)} = \vec{k}^{(m)} \varepsilon^d$ at order *m*.

1 /0

By the rationale of the preceding section, we expect that the local truncation error will behave as:

$$\|\tau_{approx}^{(m)}\|_{2} = \frac{\left(\sum_{i=1}^{n-k} |\tau_{i}^{(m)}|^{2} + \sum_{i=n-k+1}^{n} |\varsigma_{i}^{(m)}|^{2}\right)^{1/2} \cdot \|\tau^{(m)}\|_{2}}{\left(\sum_{i=1}^{n-k} |\tau_{i}^{(m)}|^{2} + \sum_{i=n-k+1}^{n} |\tau_{i}^{(m)}|^{2}\right)^{1/2} \cdot \|\tau^{(m)}\|_{2}}$$
$$= \frac{\left(h^{2p} \sum_{i=1}^{n-k} |c_{i}|^{2} + \epsilon^{2d} \sum_{i=n-k+1}^{n} |k_{i}|^{2}\right)^{1/2}}{\left(h^{2p} \sum_{i=1}^{n-k} |c_{i}|^{2} + h^{2p} \sum_{i=n-k+1}^{n} |c_{i}|^{2}\right)^{1/2} \cdot \|\tau^{(m)}\|_{2}}$$
$$= \sqrt{\left(\frac{c_{1}^{2} + \dots + c_{n-k}^{2}}{c_{1}^{2} + \dots + c_{n}^{2}} + \frac{\epsilon^{2d}}{h^{2p}} \frac{k_{n-k+1}^{2} + \dots + k_{n}^{2}}{c_{1}^{2} + \dots + c_{n}^{2}}\right)} \cdot \|\tau^{(m)}\|_{2}, \qquad (5.35)$$

The reader will remark also that the second term under the root in expression (5.35) is qualitatively different from the truncation error due to the integration scheme. This is because the ε -dependent part of the expression is bounded globally as well, since exact conservation of the *approximate* expression guarantees that all $\zeta^{(m)}$ are kept at this same order of magnitude, which precludes secular growth of the out-of-manifold error over long time-scales.

5.4.4 On Efficacy

Despite the promise that this approach suggests, in particular with regard to the maximal exploitation of the available knowledge about the dynamical system, there are some remarks to be made with respect to efficacy:

- First, it should be realized that the integrator formulations we present will be (sometimes considerably) computationally more intensive than their non-conservative cousins, and whether this increase in complexity is justifiable is an open (and problem-dependent) question.
- A second issue is potentially equally as important, and a likely objection to our approach. As Huang and Innanen discuss in [Huang and Innanen, 1983], the naive use of integral–conservation (typically conservation of energy) as a benchmark for integrator accuracy is potentially misleading. This is because integrators in general are naturally inclined to the preservation of integrals of motion, which will typically experience variations of a much smaller magnitude than the coordinates *x* and velocities *x*.

As is shown there, for problems amenable to Taylor-series expansions of the solution from a starting point (which is to say solutions which behave smoothly in a mathematical sense¹⁷), the local truncation error reflected

¹⁷In this specific case, an analysis of their derivation requires C^{p+2} smoothness in fact, which is an interesting issue that potentially undermines their arguments in the presence of stochastic regions of phase–space, which they raise as an example.

in an integral will behave as:

$$I^{(t+h)} = I^{(t)} + k h^{p+2} \quad \text{such that}$$
(5.36a)
$$\tau_I = O\left(h^{p+1}\right), \quad (5.36b)$$

where p is the order of the integrator.

This implies a slower variation than the coordinates which have an $O(h^p)$ local truncation error τ_x .¹⁸ As a result, the integral may be kept nearly constant over the time–span of the integration, while the coordinates deviate wildly from the true trajectory, and indeed the stochastic regions of phase–space characteristic of the breakdown of KAM tori for Hamiltonian systems are raised as an example where this could potentially be catastrophic.

However, the cautions raised by Huang and Innanen do not fundamentally change two key arguments in favor of the use of conservative schemes.

 First, from a purely theoretical point-of-view, conservative integration schemes remain interesting regardless of practical concerns due to the fact that they promise an inherently better numerical representation of the dynamics underlying the system by preserving more of its structure; this is essentially the same argument raised in the justification of symplectic integrators.

This is all the more relevant in a situation rarely discussed in the astrodynamics literature, when we have modeled a problem for which we have relatively little knowledge of the dynamics, and in particular lack the usual integrals *but* may be able to find approximations of first integrals instead.

- 2. More practically, the work of Bowman *et al.* suggests that there are certain types of problems where a unique combination of factors comes together:
 - on the one hand the overhead involved in a conservative scheme may be worth the investment in obtaining a better integration compared with similar integrators of the same order, and
 - on the other the accumulation of error in the degrees-of-freedom associated with the integral is a sufficiently serious issue in reproducing the true dynamics of the problem that it warrants the specific handling a conservative scheme provides.

An example in this regard is that of a multi–body choreography, such as the 4–body problem choreography discussed in [Kotovych and Bowman, 2002] and illustrated in figure 5.1. In this situation, moreover, we expect that despite the extreme sensitivity to initial conditions, the actual trajectory itself is safely in a regular region of phase–space, given the beautiful symmetry and regularity of the orbits, and indeed the comparison between the simple conservative scheme and the symplectic integrator is intriguing.

The reader should not come away with the idea, then, that conservative integration schemes are always preferable to *e.g.* Runge-Kutta–type integrators, but rather that there are certain classes of problems, typically those with regular solutions exhibiting sensitive dynamics, which may warrant the extra effort.

¹⁸This does not, of course, preclude a secular effect $t \cdot k \cdot h^{p+1}$.
It is this more holistic consideration which motivated this part of the present work, as ballistic capture trajectories may conceivably fall in just this class of problems. In the following chapters then, we shall try to (begin to) answer just that question, though we will build up to the goal by first laying a foundation using the standard problems and known integrals.

Part II

Approximations of First Integrals

Outline of Part II

The second part of this thesis (which comprises the second major part of the thesis for Applied Mathematics) is concerned with the application of the method of integrating vectors in the construction of the existing, and in the search for novel integrals of the motion in problems in astrodynamics (and in particular the capture problem introduced in section 3.5). We will proceed as follows:

- Building on the techniques introduced in chapter 4, chapter 6 treats the application to the planar Jacobi problem derived in chapter 3, which gave us insight into the dynamics cast as a perturbation of the 2-body problem. The method of integrating vectors will be shown to facilitate a very natural method for the *ab initio* construction of the known time-independent integrals, we will show how this may be exploited to easily reconstruct the known integrals of motion to any desired accuracy. Moving from this to the general time-dependent case, the system of PDE's for integrating vectors for the Jacobi problem are derived and discussed.
- Chapter 7 discusses the analogous application of the method to the circular restricted 3-body problem. The
 method of integrating vectors is then shown again to very naturally reproduce the Jacobi integral for the problem.
 Along the lines of the previous chapter, the general system of PDE's for integrating vectors for the restricted
 problem are then presented and discussed.
- 3. Finally, chapter 8 discusses the proposed model for the capture problem as a further (perturbative) extension of the circular restricted 3-body problem. It is shown in detail how, for this new problem, an analogue of the Jacobi integral may be constructed to $O(\varepsilon^4)$, and where the method, at present, breaks down due to the difficulty of dealing with the time-dependence. The results found up to that order have a particular interpretation related to the method of averaging and are linked, as well, to Andreu's quasi-bicircular problem, as remarked in section 3.5.2. As before, the general system of PDE's for integrating vectors for the capture problem are presented and discussed in closing, together with perspectives for their solution.

Chapter 6

The Jacobi 3–Body Problem

This chapter will treat the use of the method of integrating vectors in generating approximations of first integrals in the planar Jacobi 3–body problem(pJ3BP), formulated as a perturbation problem. Note that the computations summarized here are given in full in the MATHEMATICA notebook **IntegratingVectors02-P3BP-Jacobi.nb**, included on the CD provided with this thesis.

6.1 Approach

The approach of this chapter is best thought of as a road map for the following two as well, and so before beginning we outline the approach for each of the chapters in this part of the thesis. Beginning with a short review of the problem, and its casting as a perturbation problem, the method of integrating vectors will be applied to demonstrate the viability of an *ab initio* construction of the known integrals of motion. It is also useful to recall, at this point, the discussion in section 3.3.1 of where the Jacobi problem fits into the larger scheme of perturbations of the 2–body problem.

Following this, the approach to be followed in the construction of new integrals will be mapped out for each problem, by formulating the system of 1^{st} –order PDE's that must be solved. At the time of writing, it has not been possible to solve these yet, but we shall also detail the solution approaches attempted and their shortcomings.

We remark also that in this chapter we use mass–scales *only*, subject to further a simplification introduced below, to illustrate the main issues without introducing unnecessary complication to this already difficult problem. The mass–scales approach is, as we have shown in section 4.3, a straightforward though naive perturbation approach, and we will use somewhat more sophisticated approaches based around force–scales in the following two chapters after illustrating the main issues first, in the present chapter.¹

6.2 Review of the Problem

Beginning with the full 3–body problem, we recall equations (3.34):

$$\ddot{\mathbf{r}}_i = -\nabla_i U_i = \sum_{j=1,\neq i}^{n=3} \frac{\rho_j}{r_{ij}^3} \mathbf{r}_{ij}, \qquad i = 1, \dots, 3.$$

¹Remark that if it is desired to reformulate using force–scales, the resulting analysis is relatively simple using our MATHEMAT-ICA notebooks, and so we have focused our energies here on the conceptually simplest perturbation approach.

This system was reduced in chapter 3 to the equations of the planar Jacobi problem, equations (3.46):

$$\begin{split} \ddot{\mathbf{q}} &= -(\rho_1 + \rho_2) \frac{\mathbf{q}}{\|\mathbf{q}\|^3} + \rho_3 \left(\frac{\mathbf{Q} - (1 - \mu)\mathbf{q}}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} - \frac{\mathbf{Q} + \mu\mathbf{q}}{\|\mathbf{Q} + \mu\mathbf{q}\|\|^3} \right) \\ \ddot{\mathbf{Q}} &= -\frac{\rho_1 (1 + \upsilon)(\mathbf{Q} + \mu\mathbf{q})}{\|\mathbf{Q} + \mu\mathbf{q}\|^3} - \frac{\rho_2 (1 + \upsilon)(\mathbf{Q} - (1 - \mu)\mathbf{q})}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} \,. \end{split}$$

It is these that will be considered in the following sections.

6.3 The 10 Known Integrals

In the following subsections, it will be shown how all 10 known integrals of the 3–body problem may be constructed *ab initio* (*i.e.* specifically without prior knowledge of the dynamics and solution techniques for the system) using the method of integrating vectors from a perturbation perspective.

6.3.1 Simplified Expansions of Terms in ε

To this end, recall first the governing equations (3.46) of this problem. It is clear that a certain number of terms depend on the mass ratios and can be seen as perturbations; thus we require the following expansions of terms in what follows, and present them first for that reason.

• Concerning the parameters in the equations of motion, remark that it will be assumed for simplicity that $m_0 := m_1, m_2 = \varepsilon^{k_2} m_0$ and $m_3 = \varepsilon^{k_3} m_0$ with the k_i corresponding to the m_i and thus fixed by the physical system.

Typically, for the Sun–Earth–Moon system with $\varepsilon = 10^{-2}$ we normalize by the Solar mass, and so take for the mass ratios $k_1 = 0$, $k_2 = 3$, $k_3 = 4$ to yield:

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{\epsilon^3 m_0}{m_0 (1 + \epsilon^3)} = \epsilon^3 - \epsilon^6 + \epsilon^9 + O\left(\epsilon^{12}\right) , \qquad (6.1a)$$

$$\upsilon = \frac{m_3}{m_1 + m_2} = \frac{\varepsilon^4 m_0}{m_0 (1 + \varepsilon^3)} = \varepsilon^4 - \varepsilon^7 + \varepsilon^{10} + \mathcal{O}\left(\varepsilon^{12}\right).$$
(6.1b)

• Likewise, for the parameters used in the reconstruction of the original coordinates, (and which are needed in constructing the conservation laws in this coordinate system):

$$\frac{\upsilon}{1+\upsilon} = \varepsilon^4 - \varepsilon^7 - \varepsilon^8 + \varepsilon^{10} + 2\varepsilon^{11} + \mathcal{O}\left(\varepsilon^{12}\right) , \qquad (6.2a)$$

$$\frac{1}{1+\upsilon} = 1 - \varepsilon^4 + \varepsilon^7 + \varepsilon^8 - \varepsilon^{10} - 2\varepsilon^{11} + \mathcal{O}\left(\varepsilon^{12}\right) = 1 - \frac{\upsilon}{1+\upsilon}.$$
(6.2b)

• Next, we consider the consequences of these ε -dependencies for the denominator terms which contain μ resp.

 $1 - \mu$. First the potential terms:

$$\frac{1}{\|\mathbf{r}_{12}\|} = \frac{1}{\|\mathbf{q}\|} \quad \text{unchanged}; \tag{6.3a}$$

$$\frac{1}{\|\mathbf{r}_{13}\|} = \frac{1}{\|\mathbf{Q} + \mu\mathbf{q}\|} = \frac{1}{\|\mathbf{Q}\|} + \frac{q_1 Q_1 + q_2 Q_2}{\|\mathbf{Q}\|^3} \varepsilon^3 + \mathcal{O}\left(\varepsilon^6\right),$$
(6.3b)

$$\frac{1}{\|\mathbf{r}_{23}\|} = \frac{1}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|}$$
$$= \frac{1}{\|\mathbf{Q} - \mathbf{q}\|} + \frac{q_1(Q_1 - q_1) + q_2(Q_2 - q_1)}{\|\mathbf{Q} - \mathbf{q}\|^3} \,\varepsilon^3 + \mathcal{O}\left(\varepsilon^6\right) \,, \tag{6.3c}$$

and then also the force terms:

$$\frac{1}{\|\mathbf{r}_{12}\|^3} = \frac{1}{\|\mathbf{q}\|^3} \quad \text{unchanged;}$$
(6.4a)

$$\frac{1}{\|\mathbf{r}_{13}\|^{3}} = \frac{1}{\|\mathbf{Q} + \mu\mathbf{q}\|^{3}}$$
$$= \frac{1}{\|\mathbf{Q}\|^{3}} - \frac{3(q_{1}Q_{1} + q_{2}Q_{2})}{\|\mathbf{Q}\|^{5}} \epsilon^{3} + O\left(\epsilon^{6}\right), \qquad (6.4b)$$

$$\frac{1}{\|\mathbf{r}_{23}\|^3} = \frac{1}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3},$$

$$= \frac{1}{\|\mathbf{Q} - \mathbf{q}\|^3} + \frac{3(q_1(Q_1 - q_1) + q_2(Q_2 - q_1))}{\|\mathbf{Q} - \mathbf{q}\|^5} \,\varepsilon^3 + \mathcal{O}\left(\varepsilon^6\right).$$
(6.4c)

• Lastly, note that the details of the expansion of the expression $f(x; \varepsilon)$ will not be worked out in detail because it will not be used in this explicit form.

6.3.2 On More Accurate Expansions in ε

In reality, as always, we need to strike a balance between model simplicity and model accuracy, and as we shall see further on, the simplified expansions given above are not accurate enough for simulation purposes.

It was assumed that $m_i = \varepsilon^{k_i} m_0$ where $m_0 := m_1$ the normalizing mass and k_i the exponent corresponding to body *i*'s mass ratio. In practice it will generally not be possible to find an ε such that this holds exactly, and we must return to the use of ε for order and dimensionless constants \bar{m}_i for correction to the exact values (as in section 4.3). Thus, taking the ratios used above, the correct expressions are in fact:

 $m_2 = \varepsilon^3 \bar{m}_2 m_1$, $m_3 = \varepsilon^4 \bar{m}_3 m_1$ and so, for example:

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{m_1 \varepsilon^3 \bar{m}_2}{m_1 (1 + \bar{m}_2 \varepsilon^3)} = \bar{m}_2 \varepsilon^3 - \bar{m}_2^2 \varepsilon^6 + \bar{m}_2^3 \varepsilon^9 + O\left(\varepsilon^{12}\right)$$
(6.5a)

$$\upsilon = \frac{m_3}{m_1 + m_2} = \frac{m_1 \varepsilon^4 \bar{m}_3}{m_1 (1 + \bar{m}_2 \varepsilon^3)} = \bar{m}_3 \varepsilon^4 - \bar{m}_2 \bar{m}_3 \varepsilon^7 + \bar{m}_2^2 \bar{m}_3 \varepsilon^{10} + O\left(\varepsilon^{12}\right).$$
(6.5b)

The expressions for the distances and force terms change in the same way.

However, these \bar{m}_2 , \bar{m}_3 are just O(1) multiplicative constants, and so it is not unreasonable to neglect them in the initial analysis for purposes of clarity in this and the next two chapters. We must, however, keep in mind that when we want to use the results for simulation the expressions must be corrected to their 'real-world' counterparts, which we will in fact do in chapters 10 and 11 in Part III of this thesis.

6.3.3 Equations of Condition

Next (continuing to work with the simplified expansions) we derive the actual "equations of condition" which the integral approximation must satisfy at each order by specification of the equations (4.22) - (4.23) for the current case, in terms of the expansions of the ODE's in orders *k* of ε :

$$\nabla I_k = \mathbf{v}_k \quad \text{and}$$

$$\frac{\partial I_k}{\partial t} = - [\mathbf{v} \cdot \mathbf{f}]_k \quad k = 0, 1, 2, \dots \quad \text{implying for the integrating vector that:}$$

$$\frac{\partial \mathbf{v}_{i,k}}{\partial x_j} = \frac{\partial \mathbf{v}_{j,k}}{\partial x_i} \quad \text{and}$$

$$\frac{\partial \mathbf{v}_k}{\partial t} = -\nabla [\mathbf{v} \cdot \mathbf{f}]_k \quad k = 0, 1, 2, \dots$$

In these equations the key expression is $v \cdot f$. Its gradients determine the time–dependent part of the equation for the integrating vectors at the "low–level" expression (4.23), and the *a priori* existence of such a part at the "high–level" expression (4.22) of the integral term I_k .²

This latter point is particularly relevant in constructing first the time-independent integrals of the problem, which have been derived before by problem-specific methods in chapter 3. In general finding an integrating factor will prove decidedly complicated, but even there the "high-level" equations sometimes allow for considerable limitation of the search-space for candidates.

In general however, the approach is two-pronged:

1. First, solve the equations (4.23) for the integrating factor at a given order of the expansion of

$$\mathbf{v}(\mathbf{x},t;\mathbf{\varepsilon}) = \sum_{i=0}^{\infty} \mathbf{v}_i(\mathbf{x},t) \mathbf{\varepsilon}^i \quad \text{where}$$

$$\mathbf{v}_i(\mathbf{x},t) = \left(\mathbf{v}_{1,i}(\mathbf{x},t), \dots, \mathbf{v}_{2n,i}(\mathbf{x},t)\right)^\mathsf{T},$$
(6.7)

with 2n the number of components of **x**, and so also the number of first-order ordinary differential equations that we obtain when we reduce the second–order system.

2. Then use this result to solve the equations (4.22) for the integral at a given order of the expansion

$$I(\mathbf{x},t;\varepsilon) = \sum_{i=0}^{\infty} I_i(\mathbf{x},t)\varepsilon^i, \qquad (6.8)$$

²Recall that the terms high–level and low–level are here chosen for easy reference, motivated by the difference between the integrating factor v_k (which is determined by the so-called "low–level equations") and used as a building block for the actual integral approximation I_k (which must satisfy the so-called "high–level equations").

which gives, order–by–order, an approximation to a first integral of the system (which, recalling the discussion in previous sections, has been made exact by means of the integrating vector v).

With regard to $v \cdot f$ then, we expand in the current problem as:

$$\mathbf{v} \cdot \mathbf{f} = [\mathbf{v} \cdot \mathbf{f}]_{0} + [\mathbf{v} \cdot \mathbf{f}]_{1} \varepsilon + [\mathbf{v} \cdot \mathbf{f}]_{2} \varepsilon^{2} + \dots, \text{ where}$$
$$[\mathbf{v} \cdot \mathbf{f}]_{k=0,1,2} = -\frac{\rho_{0} \mathbf{v}_{5,k} \, q_{1}}{\|\mathbf{q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{6,k} \, q_{2}}{\|\mathbf{q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{7,k} \, Q_{1}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{8,k} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + \mathbf{v}_{1,k} \, \dot{q}_{1} + \mathbf{v}_{2,k} \, \dot{q}_{2} + \mathbf{v}_{3,k} \, \dot{Q}_{1} + \mathbf{v}_{4,k} \, \dot{Q}_{2} \,, \tag{6.9}$$

$$- [\mathbf{v} \cdot \mathbf{f}]_{3} = \frac{\rho_{0} \mathbf{v}_{5,0} \, q_{1}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{5,3} \, q_{1}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{6,0} \, q_{2}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{6,3} \, q_{2}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{7,3} \, Q_{1}}{\|\mathbf{Q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{8,3} \, Q_{2}}{\|\mathbf{Q}\|^{3}}$$

$$+ \mathbf{v}_{7,0} \left(-\frac{3\rho_{0} \, q_{1} Q_{1}^{2}}{\|\mathbf{Q}\|^{5}} - \frac{3\rho_{0} \, q_{2} Q_{1} Q_{2}}{\|\mathbf{Q}\|^{5}} + \frac{\rho_{0} \, q_{1}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \, q_{1}}{\|\mathbf{Q}-\mathbf{q}\|^{3}} + \frac{\rho_{0} \, Q_{1}}{\|\mathbf{Q}-\mathbf{q}\|^{3}} \right)$$

$$+ \mathbf{v}_{8,0} \left(-\frac{3\rho_{0} \, q_{1} Q_{1} Q_{2}}{\|\mathbf{Q}\|^{5}} - \frac{3\rho_{0} \, q_{2} Q_{2}^{2}}{\|\mathbf{Q}\|^{5}} + \frac{\rho_{0} \, q_{2}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \, q_{2}}{\|\mathbf{Q}-\mathbf{q}\|^{3}} + \frac{\rho_{0} \, Q_{2}}{\|\mathbf{Q}-\mathbf{q}\|^{3}} \right)$$

$$- \mathbf{v}_{1,3} \, \dot{q}_{1} - \mathbf{v}_{2,3} \, \dot{q}_{2} - \mathbf{v}_{3,3} \dot{Q}_{1} - \mathbf{v}_{4,3} \, \dot{Q}_{2} \, ,$$

$$- [\mathbf{v} \cdot \mathbf{f}]_{4} = \frac{\rho_{0} \mathbf{v}_{5,1} \, q_{1}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{5,4} \, q_{1}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{6,1} \, q_{2}}{\|\mathbf{q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{7,0} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{7,4} \, Q_{1}}{\|\mathbf{Q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{8,0} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + \frac{\rho_{0} \mathbf{v}_{8,4} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + \mathbf{v}_{5,0} \left(\frac{\rho_{0} \, Q_{1}}{\|\mathbf{Q}\|^{3}} + \frac{\rho_{0} \, q_{1}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} - \frac{\rho_{0} \, Q_{1}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} \right) + \mathbf{v}_{7,1} \left(-\frac{3\rho_{0} \, q_{1} Q_{1}^{2}}{\|\mathbf{Q}\|^{5}} - \frac{3\rho_{0} \, q_{2} Q_{1} Q_{2}}{\|\mathbf{Q}\|^{5}} + \frac{\rho_{0} \, q_{1}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \, q_{1}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} + \frac{\rho_{0} \, Q_{1}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} \right) + \mathbf{v}_{6,0} \left(\frac{\rho_{0} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + \frac{\rho_{0} \, q_{2}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} - \frac{\rho_{0} \, Q_{2}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} \right) + \mathbf{v}_{8,1} \left(-\frac{3\rho_{0} \, q_{1} Q_{1} Q_{2}}{\|\mathbf{Q}\|^{5}} - \frac{3\rho_{0} \, q_{2} Q_{2}^{2}}{\|\mathbf{Q}\|^{5}} + \frac{\rho_{0} \, q_{2}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \, q_{2}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} + \frac{\rho_{0} \, Q_{2}}{\|\mathbf{Q} - \mathbf{q}\|^{3}} \right) - \mathbf{v}_{1,4} \, \dot{q}_{1} - \mathbf{v}_{2,4} \, \dot{q}_{2} - \mathbf{v}_{3,4} \dot{Q}_{1} - \mathbf{v}_{4,4} \, \dot{Q}_{2} , \qquad (6.11)$$

(6.10)

and so on for higher orders in ε .

These are only the first 5 orders, but they already show the clear effect of the explicit scaling of magnitudes in mass. The first 3 equations are the same, and so are their low–level counterparts, since no perturbation enters the picture until $O(\varepsilon^3)$ in mass. At that order the equations couple $O(\varepsilon^{\{3,0\}})$ –terms.³ At the next order, both perturbations are active for the first time, and now couple the $O(\varepsilon^{\{4,1,0\}})$ –terms.

This qualitative relation remains valid on increasing the order. The $O(\varepsilon^5)$ equation, for example, is identical in form to the $O(\varepsilon^4)$ equation since there are no 5th-order terms in **f**, and so there the previous coupling is the same, though again 'shifted' such that $O(\varepsilon^{\{4,1,0\}}) \rightarrow O(\varepsilon^{\{5,2,1\}})$. We might instead characterize this as preserving (unaltered) the coupling $O(\varepsilon^{\{k,k-3,k-4\}})$ as a general scheme for $0 \le k \le 5$ in the present notation.⁴

³The reader will note that by this notation we intend a coupling between $O(\varepsilon^3)$ and $O(\varepsilon^0) = O(1)$ terms.

⁴Two comments are in order for this extension of the notation: first, this scheme is understood only to contain terms of order $k \ge 0$, and second, the scheme is extended at higher orders as new ε -couplings enter the picture of our expansions.

This pattern continues at each order where there are no explicit new contributions to the expression. However, when this is not the case we find terms shifted up but *also* coupled to an additional order, as with the $O(\varepsilon^6)$ terms: $O(\varepsilon^{\{6,3,2,0\}}) = O(\varepsilon^{\{k,k-3,k-4,k-6\}})$ for $0 \le k \le 6$.

As indicated above, at each order k in ε we must first solve the $\frac{1}{2}n(n+1)$ system (4.23). As a first indication of what these equations imply, let us derive the equations of condition for time. These are presumably critical for integrals *other* than the 10 known time–independent algebraic integrals of section 3.1.1, as it is here that the approach becomes interesting; indeed this is the main point where the explicit form of the differential equation, $\mathbf{f}(\mathbf{x};\varepsilon)$ enters the picture.

$$\frac{\partial \mathbf{v}_{0}}{\partial t} = \begin{pmatrix} \partial_{t} \mathbf{v}_{1,0} \\ \partial_{t} \mathbf{v}_{2,0} \\ \partial_{t} \mathbf{v}_{3,0} \\ \partial_{t} \mathbf{v}_{3,0} \\ \partial_{t} \mathbf{v}_{4,0} \\ \partial_{t} \mathbf{v}_{5,0} \\ \partial_{t} \mathbf$$

Any integral of the equations of motion will satisfy this set of 8 equations and the 36 - 8 = 28 spatial PDE's, and their correspondingly more complex counterparts at higher orders in ε .

However, it is immediately apparent that even at $O(\varepsilon^0)$ we lack any good means, or even an idea of where to start in considering the general setting. To rectify that, the following sections will spend some time on first developing solutions for the integrating vectors of the known algebraic integrals, which are time-independent.

While these integrals' integrating factors do in principle follow from the above low-level equations (4.23), that approach itself turns out to be difficult for lack of a good idea of where to start. Instead, they are best approached using the high-level equations which allow us to derive, if not immediately the correct integrating factor, at least a limitation of the search space for valid factors. This is accomplished by setting the high-level expression $\frac{\partial I_k}{\partial t} = -[\mathbf{v} \cdot \mathbf{f}]_k \equiv 0$, which is a scalar rather than vector equation.⁵

6.3.4 Conservation of Linear Momentum

The first integrals we will briefly consider are those associated with conservation of linear momentum. However, on considering equations (3.46) for the planar Jacobi formulation of the 3-body problem, it is intuitively clear that linear momentum is automatically conserved, as the origin of the system is co-moving with the system barycenter $\mathbf{r}_0 = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_1 + m_2 + m_3}$ (cf. section 3.3.3 and in particular equation (3.35)).

Nonetheless it is instructive to consider the way in which this intuitive understanding manifests itself in the current framework. If we consider the high–level equations (4.22) and the factors (6.9), it is clear that in order to obtain a time–independent integral of the form $k \dot{\mathbf{q}}$ or $k \dot{\mathbf{Q}}$ where the constants correspond to the masses in $\sum_{i} m_i \dot{\mathbf{r}}_i$, we must take at order 0:

⁵Equivalently, one chooses to deal with the determining equation rather than its gradient.

 $v_{1...4;0} = 0$ corresponding to variables $q_{1...2}, Q_{1...2}$, (6.13a)

$$v_{5...8;0} = c_{1...4}$$
 constants corresponding to variables $\dot{q}_{1...2}$, $\dot{Q}_{1...2}$. (6.13b)

However, matching the intuition that $\sum_i m_i \dot{r}_i = \mathbf{a} = 0$, it is clear that using the reconstruction equations (3.43):

$$m_{1}\dot{\mathbf{r}}_{1} = m_{1}\left(-\frac{m_{3}}{m_{1}+m_{2}+m_{3}}\dot{\mathbf{Q}} - \frac{m_{2}}{m_{1}+m_{2}}\dot{\mathbf{q}}\right),$$

$$m_{2}\dot{\mathbf{r}}_{2} = m_{2}\left(-\frac{m_{3}}{m_{1}+m_{2}+m_{3}}\dot{\mathbf{Q}} + \frac{m_{1}}{m_{1}+m_{2}}\dot{\mathbf{q}}\right),$$

$$m_{3}\dot{\mathbf{r}}_{3} = m_{3}\left(\frac{m_{1}+m_{2}}{m_{1}+m_{2}+m_{3}}\dot{\mathbf{Q}}\right),$$

such that

$$\sum_{i} m_i \dot{\mathbf{r}}_i = 0 \dot{\mathbf{q}} + 0 \dot{\mathbf{Q}} \quad \text{as expected.}$$
(6.14)

Consequently all 4 constants $c_{1...4} \equiv 0$, which is indeed seen to be precisely the trivial solution of equations (6.9), and by extension of the high–level equation at each order, as the lower orders are substituted.

6.3.5 Conservation of Angular Momentum

The next step is to consider the conservation of angular momentum, again based on the simplified expressions in ε . For reference, the 'true' expansion of the known integral is given first.

Expansion of the Angular Momentum in ϵ

First, for reference, we use the expansions given earlier to derive perturbation-expansion expressions for the angular momentum. Given that the bodies move in a single plane, the relevant expression is scalar (the z-component of the 3-vector $\mathbf{h} = \sum_{i} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i$):

$$h = m_1(x_1\dot{y}_1 - y_1\dot{x}_1) + m_2(x_2\dot{y}_2 - y_2\dot{x}_2) + m_3(x_3\dot{y}_3 - y_3\dot{x}_3)$$
(6.15)

in inertial coordinates. Transformed to Jacobi coordinates using the same expressions (3.43) and again substituting the simplified choice $m_1 = m_0$, $m_2 = \varepsilon^3 m_0$, $m_3 = \varepsilon^4 m_0$ yields:

$$h = \epsilon^{3} m_{0}(q_{1}\dot{q}_{2} - q_{2}\dot{q}_{1}) + \epsilon^{4} m_{0}(Q_{1}\dot{Q}_{2} - Q_{2}\dot{Q}_{1}) - \epsilon^{6} m_{0}(q_{1}\dot{q}_{2} - q_{2}\dot{q}_{1}) + 2\epsilon^{7} m_{0}(Q_{1}\dot{q}_{2} - Q_{2}\dot{q}_{1} - q_{2}\dot{Q}_{1} + q_{1}\dot{Q}_{2}) - \epsilon^{8} m_{0}(Q_{1}\dot{Q}_{2} - Q_{2}\dot{Q}_{1}) + \epsilon^{9} m_{0}(q_{1}\dot{q}_{2} - q_{2}\dot{q}_{1}) + \mathcal{O}\left(\epsilon^{10}\right).$$

$$(6.16)$$

Note in particular the lack of contributions below $O(\varepsilon^3)$ and the correspondence of that and the $O(\varepsilon^4)$ terms to what would naively be expected using $\mathbf{Q} + \mu \mathbf{q} \approx \mathbf{Q}$ and $\mathbf{Q} - (1 - \mu)\mathbf{q} \approx \mathbf{Q} - \mathbf{q}$. Interesting also is the cross-coupling of terms at $O(\varepsilon^7)$.

Construction of the Angular Momentum in $\boldsymbol{\epsilon}$

With the above expression for comparison, we next consider the equivalent *ab initio* construction using the method of integrating vectors. Beginning again with equation (6.9), it must be null (zero) for the time–independent angular momentum integral:

$$[\mathbf{v} \cdot \mathbf{f}]_{0} = 0 = -\frac{\rho_{0} \mathbf{v}_{5,0} \, q_{1}}{\|\mathbf{q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{6,0} \, q_{2}}{\|\mathbf{q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{7,0} \, Q_{1}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{8,0} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + \mathbf{v}_{1,0} \, \dot{q}_{1} + \mathbf{v}_{2,0} \, \dot{q}_{2} + \mathbf{v}_{3,0} \, \dot{Q}_{1} + \mathbf{v}_{4,0} \, \dot{Q}_{2} ,$$

and there are a number of different ways to accomplish this. Two simple ones stand out, and in fact it is precisely these two that lead (up to a certain intrinsic lack of uniqueness discussed further on) to the angular momentum and energy integrals. The first, used in this section, is that of ignoring the factors related to the gravitational terms⁶ in equation (6.9).

In that case, remark that the above expression (indeed, for k = 0, 1, 2) will be null if we:

- Disregard the gravitational factors by setting $v_{5...8;k} = 0$;
- Use the remaining factors $v_{1...4;k}$ to cancel the expression internally, by setting:

$$\mathbf{v}_{1,k} = \dot{q}_2 \,, \tag{6.17a}$$

$$W_{2,k} = -\dot{q}_1 , \qquad (6.17b)$$

$$\mathbf{v}_{3,k} = \dot{Q}_2 \,, \tag{6.17c}$$

$$v_{4,k} = -Q_1$$
. (6.17d)

The first pair leads (after multiplication with the factor m_0), to the expansion term $(q_1\dot{q}_2 - q_2\dot{q}_1)m_0\varepsilon^k$, while the second pair leads to $(Q_1\dot{Q}_2 - Q_2\dot{Q}_1)m_0\varepsilon^k$, on integration of the high–level equations.

• We remark that these pairs may be chosen independently, e.g. $v_{1,2;k}$ as above and $v_{3,4;k} = 0$ or *vice-versa*. These choices, however, are not the only way to cancel the remaining terms in the expression. There is also a way in which the same 4 factors may be used, and again these may or may not be necessary independent:

$$\mathbf{v}_{1,7} = \dot{Q}_2 \,, \tag{6.18a}$$

$$v_{2,7} = \dot{Q}_1 ,$$
 (6.18b)

$$\mathbf{v}_{3,7} = -\dot{q}_2 \,, \tag{6.18c}$$

$$\mathbf{v}_{4,7} = -\dot{q}_1 \ . \tag{6.18d}$$

This leads to the expansion term $(Q_1\dot{q}_2 - Q_2\dot{q}_1 - q_2\dot{Q}_1 + q_1\dot{Q}_2)m_0\varepsilon^k$, and yields, unlike the previous possibility, a coupling of the q, Q terms. The justification for this, however, can here only be physical, and is discussed after the construction of the integral.

These observations, then, are the basis of the construction of the angular momentum integral in this perturbation formulation. We now walk through the construction, resolving the issues of uniqueness on the basis of physical con-

 $^{^{6}}$ Gravitational factors are not essential to the angular momentum; though this would be a bad choice for *e.g.* the energy, it is a valid approach here.

siderations.

1. First, note that the first term in any correct expansion must be third-order in ε , as this is the lowest order at which we may expect a contribution from the perturbed system; this is due to the fact that the O(1) effect of the primary would be null, as an unperturbed primary would remain motionless. It is only when the perturbing $O(\varepsilon^3)$ effect of a secondary (corresponding to the combined **q** in Jacobi coordinates) is added that the system begins to move.

On these grounds, we choose the vector $v_k = 0$ for k = 0, 1, 2, and with this choice the spatial equations of (4.23) are now satisfied trivially, and it follows that $\nabla I_{0,1,2} = \overline{0}$ and $I_{0,1,2} = c$ which may be taken null so that the integral $I(\mathbf{x}; \varepsilon)$ is null up to third order.

- 2. The factors for k = 3 are now chosen using the first cancelation scheme worked out. Remark here that the expression $[\mathbf{v} \cdot \mathbf{f}]_3$ is of the same form due to the fact that the extra terms are null, as can be seen in equation (6.10).
- 3. The same process can be repeated at $O(\varepsilon^4)$ where the first perturbation due to **q** plays a role. Together with the 3^{rd} -order terms in step 2, this produces the first two parts of the expansion (6.16), $\varepsilon^3 m_0(q_1\dot{q}_2 q_2\dot{q}_1)$ and $\varepsilon^4 m_0(Q_1\dot{Q}_2 Q_2\dot{Q}_1)$, in a quite natural way.
- 4. The next order to play a role is $O(\varepsilon^6)$ as it is the next order reflected in the force vector $\mathbf{f}(\mathbf{x}; \varepsilon)$,⁷ and the first to couple to a lower order which has not been fully set null: $O(\varepsilon^{\{6,3,2,0\}}) = O(\varepsilon^{\{6,3\}})$. However, the terms which couple are in fact $v_{5...8;3}$ and since those *were* set null, they do not influence the $O(\varepsilon^6)$ term, and here choosing $v_{1,2;6} = -v_{1,2;3}$ gives the expansion term $-\varepsilon^6 m_0(q_1\dot{q}_2 q_2\dot{q}_1)$ sought.
- 5. Lastly, the furthest order discussed here is $O(\varepsilon^7)$, due to its cross-coupling of terms. In terms of the coupling pattern derived from $\mathbf{f}(\mathbf{x}; \varepsilon)$, the pattern here is $O(\{7,4,3,1,0\} = \{7,4,3\})$, reflecting a mixing of effects due to the interaction between the secondary and the tertiary masses. However, on substituting the results from lower orders, it is clear that the equation here (and indeed through $O(\varepsilon^9)$) is the same as before.

Nonetheless, in order to obtain the correct result here, it is appropriate to change the choice of the factor to the second possibility for cancelation:

$$\begin{split} \mathbf{v}_{1,7} &= \dot{Q}_2 \;, \\ \mathbf{v}_{2,7} &= \dot{Q}_1 \;, \\ \mathbf{v}_{3,7} &= -\dot{q}_2 \;, \\ \mathbf{v}_{4,7} &= -\dot{q}_1 \;. \end{split}$$

This reverses the usual pattern, but it is easily checked that:

$$\begin{bmatrix} \mathbf{v} \cdot \mathbf{f} \end{bmatrix}_7 = \mathbf{v}_{1,7} \, \dot{q}_1 + \mathbf{v}_{2,7} \, \dot{q}_2 + \mathbf{v}_{3,7} \, \dot{Q}_1 + \mathbf{v}_{4,7} \, \dot{Q}_2$$

= $\dot{Q}_2 \, \dot{q}_1 + \dot{Q}_1 \, \dot{q}_2 - \dot{q}_2 \, \dot{Q}_1 - \dot{q}_1 \, \dot{Q}_2$
= 0.

⁷One may think of this as arising from the expansion $\varepsilon^3(1-\varepsilon^3+\varepsilon^6+...)$, and on these grounds we would also expect a similar contribution at $O(\varepsilon^9)$, which is indeed the case as reflected in the reference expansion.

Multiplying each integrating factor by the same constant 2 does not alter the result, and thus the integral approximation (*cf.* equation 6.16):

$$h^{*} = \varepsilon^{3} m_{0}(q_{1}\dot{q}_{2} - q_{2}\dot{q}_{1}) + \varepsilon^{4} m_{0}(Q_{1}\dot{Q}_{2} - Q_{2}\dot{Q}_{1}) - \varepsilon^{6} m_{0}(q_{1}\dot{q}_{2} - q_{2}\dot{q}_{1}) + 2\varepsilon^{7} m_{0}(Q_{1}\dot{q}_{2} - Q_{2}\dot{q}_{1} - q_{2}\dot{Q}_{1} + q_{1}\dot{Q}_{2}) + \mathcal{O}\left(\varepsilon^{8}\right)$$
(6.19)

is constructed after integrating out the factors $v_{i,7}$.

There is a question of uniqueness which needs to be addressed however, which we postpone until section 6.3.7 below; however, in practice some physical considerations should already help to reduce the seeming arbitrariness somewhat.⁸ The situation is, however, slightly different for the expansion of v. Here, the first term is of course due to the tertiary only, and so the form is $Q_1\dot{Q}_2 - Q_2\dot{Q}_1$, but at the next order, even in the expansion of v, which is properly

$$\upsilon = \frac{m_3}{m_1 + m_2} = \bar{m}_3 \varepsilon^4 - \bar{m}_2 \bar{m}_3 \varepsilon^7 + \bar{m}_2^2 \bar{m}_3 \varepsilon^{10} + \mathcal{O}\left(\varepsilon^{12}\right) \,.$$

It is clear that the $O(\varepsilon^7)$ effect is due to the interaction of secondary and tertiary. Letting this guide our choice of the integrating vector, the second cancelation possibility is more sensible, as it is a relatively direct coupling:

$$v_{1,2} \leftrightarrow Q_{1,2}$$
 recalling that $v_{1,2} = \frac{\partial I}{\partial q_{1,2}}$ and
 $v_{3,4} \leftrightarrow -q_{1,2}$ recalling that $v_{3,4} = \frac{\partial I}{\partial Q_{1,2}}$.

The need for the factor 2, however, is not immediately apparent, and is considered further on.

6.3.6 Conservation of Energy

Finally, in this section we consider the conservation of energy, again based on the simplified expressions in ε . For reference, the 'true' expansion of the known integral is given first.

Expansion of the Energy Integral in ϵ

First, we use the expansions given earlier to derive perturbation-expansion expressions for the energy (which is necessary as it doesn't take a very simple form in ε in Jacobi-coordinates). This is done by starting from the expressions in inertial coordinates and substituting the reconstruction expressions (3.43) with their respective expansion-expressions given above. Thus:

$$T = \frac{1}{2}m_{1}\dot{\mathbf{r}}_{1} \cdot \dot{\mathbf{r}}_{1} + \frac{1}{2}m_{2}\dot{\mathbf{r}}_{2} \cdot \dot{\mathbf{r}}_{2} + \frac{1}{2}m_{3}\dot{\mathbf{r}}_{3} \cdot \dot{\mathbf{r}}_{3}$$

$$= \frac{1}{2}m_{0}\left(\left\|-\frac{\upsilon\dot{\mathbf{Q}}}{1+\upsilon}-\mu\dot{\mathbf{q}}\right\|^{2} + \varepsilon^{3}\left\|-\frac{\upsilon\dot{\mathbf{Q}}}{1+\upsilon}+(1-\mu)\dot{\mathbf{q}}\right\|^{2} + \varepsilon^{4}\left\|\frac{\dot{\mathbf{Q}}}{1+\upsilon}\right\|^{2}\right)$$
(6.20a)

$$= \frac{1}{2}m_0 \left(\epsilon^3 (\dot{q}_1^2 + \dot{q}_2^2) + \epsilon^4 (\dot{Q}_1^2 + \dot{Q}_2^2) - \epsilon^6 (\dot{q}_1^2 + \dot{q}_2^2) - \epsilon^8 (\dot{Q}_1^2 + \dot{Q}_2^2) + \epsilon^9 (\dot{q}_1^2 + \dot{q}_2^2) \right) + \mathcal{O}\left(\epsilon^{10}\right), \quad (6.20b)$$

⁸It should be realized that an expansion of the force vector is made in ε , of the form $\varepsilon^3(1-\varepsilon^3+\varepsilon^6+...)$ resp. $\varepsilon^4(1-\varepsilon^3+\varepsilon^6+...)$. In consequence, on this physical basis alone, the integrating vectors which should play a role at all can be deduced to a large extent. The validity of this comment should be seen as limited to the present discussion, and is certainly subject to the discussion of uniqueness in section 6.3.7; in general it would be incorrect to suppose that orders not reflected in the force vector may always be ignored in generating an integral approximation, as orders of v and **f** mix in $[v \cdot f]_k$.

and

$$U = -\frac{Gm_1m_2}{\|\mathbf{r}_{12}\|} - \frac{Gm_1m_3}{\|\mathbf{r}_{13}\|} - \frac{Gm_2m_3}{\|\mathbf{r}_{23}\|}$$
$$= -\frac{Gm_0^2\varepsilon^3}{\|\mathbf{q}\|} - \frac{Gm_0^2\varepsilon^4}{\|\mathbf{Q} + \mu\mathbf{q}\|} - \frac{Gm_0^2\varepsilon^7}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|}$$
(6.21a)

$$= -\frac{Gm_0^2 \varepsilon^3}{\|\mathbf{q}\|} - \frac{Gm_0^2 \varepsilon^4}{\|\mathbf{Q}\|} - \frac{Gm_0^2 \varepsilon^7}{\|\mathbf{Q} - \mathbf{q}\|} + \frac{Gm_0^2 \varepsilon^7 (q_1 Q_1 + q_2 Q_2)}{\|\mathbf{Q}\|^3} + O\left(\varepsilon^{10}\right),$$
(6.21b)

such that the energy,⁹ is $\mathcal{E} = T + U$.

Note in particular that there are no contributions to the energy below $O(\varepsilon^3)$, and that to $O(\varepsilon^4)$ the terms are essentially what would naively be expected using the mass ratios and assuming that $\mathbf{Q} + \mu \mathbf{q} \approx \mathbf{Q}$ and $\mathbf{Q} - (1-\mu)\mathbf{q} \approx \mathbf{Q} - \mathbf{q}$, while at $O(\varepsilon^7)$ new terms appear.

Construction of the Energy Integral in ϵ

The above is again given for reference, using the known integral. Let us now proceed instead to the construction of the energy integral by the method of integrating vectors. As noted above, the integral is time–independent, and this is the first requirement at the "high–level," which will in turn help to find a useful approach to satisfy the "low–level" equations for the integrating factor.

Note first that at the $0^{th} - 2^{nd}$ orders in ε , the equations (6.9) hold, which can be trivially set null (and thus $\frac{\partial I_{0,1,2}}{\partial t} = 0$) by setting $v_{i;0,1,2} = 0 \quad \forall \quad i = 0, \dots, 8$. The motivation for this is as before, *i.e.* that at these orders there are no physical effects represented in the force vector.

With this choice the spatial equations of (4.23) are now satisfied trivially, and it follows that $\nabla I_{0,1,2} = I_{0,1,2} = 0$ and the integral $I(\mathbf{x}; \varepsilon)$ is null up to third order. At the third order in ε things get interesting. Recalling equation (6.10),

$$- [\mathbf{v} \cdot \mathbf{f}]_3 = \frac{\rho_0 \mathbf{v}_{5,3} \, q_1}{\|\mathbf{q}\|^3} + \frac{\rho_0 \mathbf{v}_{6,3} \, q_2}{\|\mathbf{q}\|^3} + \frac{\rho_0 \mathbf{v}_{7,3} \, Q_1}{\|\mathbf{Q}\|^3} + \frac{\rho_0 \mathbf{v}_{8,3} \, Q_2}{\|\mathbf{Q}\|^3} - \mathbf{v}_{1,3} \, \dot{q}_1 - \mathbf{v}_{2,3} \, \dot{q}_2 - \mathbf{v}_{3,3} \dot{Q}_1 - \mathbf{v}_{4,3} \, \dot{Q}_2 \,,$$

where the lower order $v_{i,j}$'s have been substituted.

The null-solution which essentially "jumps right out" is of course:

$$\mathbf{v} = \left(\begin{array}{cccc} 0, & 0, & \frac{m_0 \rho_0 q_1}{\|\mathbf{q}\|^3}, & \frac{m_0 \rho_0 q_2}{\|\mathbf{q}\|^3}, & 0, & 0, & m_0 \dot{q}_1, & m_0 \dot{q}_2 \end{array}\right)^{\mathsf{T}} .$$
(6.22)

While this candidate-vector ensures that the integral will not depend explicitly on time, does it also satisfy the spatial equations of (4.23)? To this end observe that:

$$\frac{\partial \mathbf{v}_{1,3}}{\partial q_2} = -\frac{3m_0 \rho_0 q_1 q_2}{\|\mathbf{q}\|^5} = \frac{\partial \mathbf{v}_{2,3}}{\partial q_1} ,$$

⁹Equal to the Hamiltonian, though not formulated explicitly in those terms here.

while all other relations are satisfied trivially. This is clearly a straightforward cancelation of terms, but we can now exploit it to reconstruct the integral from the relation $\nabla I_3 = v_3$, obtaining:

$$I_3 = \int \mathbf{v}_3 \cdot d\mathbf{x} = -\frac{Gm_0^2}{\|\mathbf{q}\|} + \frac{1}{2}m_0(\dot{q}_1^2 + \dot{q}_2^2) , \qquad (6.23)$$

which is readily confirmed to be the $O(\epsilon^3)$ term of the energy \mathcal{E} .

At the fourth order the process may be repeated (recalling that the $O(\varepsilon^1)$ terms of v are likewise null) with:

$$- [\mathbf{v} \cdot \mathbf{f}]_4 = \frac{\rho_0 \mathbf{v}_{5,4} \, q_1}{\|\mathbf{q}\|^3} + \frac{\rho_0 \mathbf{v}_{6,4} \, q_2}{\|\mathbf{q}\|^3} + \frac{\rho_0 \mathbf{v}_{7,4} \, Q_1}{\|\mathbf{Q}\|^3} + \frac{\rho_0 \mathbf{v}_{8,4} \, Q_2}{\|\mathbf{Q}\|^3} - \mathbf{v}_{1,4} \, \dot{q}_1 - \mathbf{v}_{2,4} \, \dot{q}_2 - \mathbf{v}_{3,4} \dot{Q}_1 - \mathbf{v}_{4,4} \, \dot{Q}_2 + \mathbf{v}_{4,4} \,$$

Now, consider the following, noting that this approach is possible because the equations of condition are in a certain sense 'decoupled,' which was *not* the case before the expansion in ε :

$$\mathbf{v} = \begin{pmatrix} 0, & 0, & \frac{m_0 \rho_0 Q_1}{\|\mathbf{q}\|^3}, & \frac{m_0 \rho_0 Q_2}{\|\mathbf{q}\|^3}, & 0, & 0, & m_0 \dot{Q}_1, & m_0 \dot{Q}_2 \end{pmatrix}^{\mathsf{T}} .$$
(6.24)

It is easily seen that this satisfies the spatial equations of (4.23):

$$\frac{\partial \mathbf{v}_{3,3}}{\partial Q_1} = -\frac{3m_0\rho_0 Q_1 Q_2}{\|\mathbf{Q}\|^5} = \frac{\partial \mathbf{v}_{4,3}}{\partial Q_2}$$

and again all other relations are satisfied trivially, such that:

$$I_4 = \int \mathbf{v}_4 \cdot d\mathbf{x} = -\frac{Gm_0^2}{\|\mathbf{Q}\|} + \frac{1}{2}m_0(\dot{Q}_1^2 + \dot{Q}_2^2) , \qquad (6.25)$$

which is confirmed to be the $O(\varepsilon^4)$ term of the energy \mathcal{E} . The process is readily continued beyond this point, following the approach of section 6.3.3 to recover the integral we gave as the sum of expressions (6.20) and (6.21).

6.3.7 On Uniqueness

In the above discussion of the different integrals, it is clear that there is some room for choices of the integrating vectors which while different, would still lead to roughly the same integrals, *e.g.* the same up to an arbitrary function of the integral.¹⁰ This raises the question of uniqueness, which this section will briefly address.

A first remark is in order with respect to the uniqueness of any first integral of motion. It is easily shown that any integral is unique not only up to an additive or multiplicative constant, but up to any smooth function of the integral. To see this, suppose that $I(\mathbf{x},t)$ is an integral, *i.e.* $L_t(I(\mathbf{x},t)) = 0$. Then, for an arbitrary function $\phi(I)$ which depends *only* on the integral (though implicitly on \mathbf{x} , *t* through $I(\mathbf{x},t)$):

¹⁰Including the special cases of an additive or multiplicative constant.

$$L_{t} (\phi(I)) = \partial_{t} \phi(I) + \nabla \phi(I) \cdot \partial_{t} I \quad \text{which of course is simply:}$$

$$= \frac{\partial \phi(I)}{\partial I} \frac{\partial I(\mathbf{x}, t)}{\partial t} + \frac{\partial \phi(I)}{\partial I} \nabla I(x, t) \cdot \dot{\mathbf{x}}$$

$$= \frac{\partial \phi(I)}{\partial I} \left(\frac{\partial I(\mathbf{x}, t)}{\partial t} + \nabla I(x, t) \cdot \dot{\mathbf{x}} \right)$$

$$= \frac{\partial \phi(I)}{\partial I} L_{t} (I(\mathbf{x}, t)) = 0. \qquad (6.26)$$

Consequently it is not entirely unexpected that the process in the method of integrating vectors leaves some "wiggleroom," of which we briefly discuss three examples:

- 1. When, for example, a given choice (*e.g.* a choice of sign, or equivalently a constant ± 1 in the vector) is made, the fact that higher orders an expansion couple to lower orders forces the equations to incorporate that choice further on, and themselves enforce a basic consistency.
- 2. A related question we encounter is what to do when at subsequent orders of an expansion, the expressions $[\mathbf{v} \cdot \mathbf{f}]_k$ are essentially the same because they are only shifted in order (*e.g.* the $O(\epsilon^{\{6,0\}})$ –coupling at k = 6 becomes a $O(\epsilon^{\{7,1\}})$ –coupling at k = 7.

Intuitively, this means that the expansion has added no new information, and can probably be ignored; indeed, this is typical of a model incorporating disparate effects at different orders in ε . Thus if we do not ignore these orders as done in the preceding sections, they add only multiplicative constants, which is illustrated for the energy integral in example 6.3.1.

Example 6.3.1 (Non–Uniqueness of the Constructed Energy Integral)

Recalling the construction of the energy integral and the integrating vectors chosen there, one could choose analogously for the terms corresponding to \mathbf{Q} at $\mathcal{O}(\varepsilon^3)$ and \mathbf{q} at $\mathcal{O}(\varepsilon^4)$ respectively. As an example, why not take:

$$\mathbf{v} = \left(\frac{m_0 \rho_0 q_1}{\|\mathbf{q}\|^3}, \frac{m_0 \rho_0 q_2}{\|\mathbf{q}\|^3}, \frac{m_0 \rho_0 Q_1}{\|\mathbf{q}\|^3}, \frac{m_0 \rho_0 Q_2}{\|\mathbf{q}\|^3}, m_0 \dot{q}_1, m_0 \dot{q}_2, m_0 \dot{Q}_1, m_0 \dot{Q}_2\right)^\mathsf{T},$$

at one or both orders?

Taken at both, it follows by collecting the terms, that (accurate to $O(\varepsilon^6)$):

$$\tilde{I}(\mathbf{x}; \mathbf{\epsilon}) = \left(\frac{1}{2}m_1(\dot{q}_1^2 + \dot{q}_2^2) - \frac{Gm_1^2}{\|\mathbf{q}\|}\right)(1+\mathbf{\epsilon})\,\mathbf{\epsilon}^3 + \left(\frac{1}{2}m_1(\dot{Q}_1^2 + \dot{Q}_2^2) - \frac{Gm_1^2}{\|\mathbf{Q}\|}\right)(1+\frac{1}{\mathbf{\epsilon}})\,\mathbf{\epsilon}^4 + \mathcal{O}\left(\mathbf{\epsilon}^6\right)$$

It is easily recognized that the terms obtained above are simply multiplied at each order k by a constant $(1 + c_k(\varepsilon) (c_k(\varepsilon)))$ at order k depending on the details of the expansion), the latter part of which is independent of both **x** and t, and so contributes only a constant to the integral and so does not change the orbital derivative $L_t(F(\mathbf{x}; \varepsilon))$.

This is consistent with the general non–uniqueness of integrals, in the sense that functions of integrals $\phi(I)$ are themselves integrals, and in fact it is for this reason that one does not test the uniqueness but rather the functional

independence of integrals of a problem.

However, while as a consequence the above modification would be a valid integral–approximation, the question of whether it has value in terms of physical interpretation is dubious. The essential point here is simply that while the integral isn't unique, it is non–unique in a benign way: the additional terms that could be constructed simply change the integral by a constant, and can be modified for a specific purpose simply by choosing an appropriate combination of (functions of) factors such that the resulting $\phi(I)$ is the integral desired.

3. Likewise, we may consider the situation encountered in the construction of the conservation of angular momentum, where we needed a multiplicative factor of 2. This factor is relevant for its physical significance, which might not be guessed from first principles *without* knowledge of the dynamics being considered. Nonetheless, given sufficient background knowledge to motivate it, this factor can always be safely added to obtain an equally valid integral–approximation term at $O(\varepsilon^7)$, as we have just discussed under point 2.

It should in fact be clear to the reader, that all 3 points are in fact slightly different examples of a single issue (*i.e.* whether the constant is ± 1 , $1 + c_k(\varepsilon)$ or another number).

In practice this means that integrating vectors differing in such small details lead to integrals which may differ slightly from the traditional formulations of the integrals of motion (though certainly not functionally independent of them). This does not, however, change the fact that integrals thus constructed remain novel in their construction from first principles,¹¹ and just as useful, in principle, as the traditional formulations because functions of an integral are themselves integrals, allowing us to reformulate them to suit our needs (*e.g.* to the traditional formulations).

6.3.8 On Demonstration vs. Proof

A further remark is in order with respect to the constructions of the previous sections.

The attentive reader has no doubt noticed that this thesis has asserted the sufficiency of the constructive approaches to the known integrals of motion without proof. It should be intuitively clear, particularly in light of the previous discussion, that there is no reason to doubt that the expansions may be continued with the above strategy to construct the desired integral up to any desired degree of accuracy, up to the issues of uniqueness just raised.

Nonetheless, the reader is advised that this thesis will *prove* the asserted validity for the simpler case of the restricted 3–body problem in the next chapter (*cf.* proposition 7.2.1), by way of illustration. Analogous proofs for the present chapter's results are more involved, and so will be omitted in order not to excessively burden the reader with extensive calculations.

6.4 Approximations of New Integrals

The previous sections considered the integrating vectors for the 10 known time-dependent integrals first, in order to develop an idea of what we may expect in the general case, in hopes of suggesting some solution approaches.

A pertinent question raised by the earlier discussion is of course whether there is any other way to cancel the terms in each $[\mathbf{v} \cdot \mathbf{f}]_k$ (leading to different time–independent integrals), or alternately to satisfy the entire system of equations

¹¹And so with no knowledge of the system beyond the equations of motion.

which an arbitrary time-dependent integrating vector must satisfy. In discussing this, we begin by giving the full equations of condition.

6.4.1 Full Equations of Condition

The foundation of these equations is a slightly more general version of expressions (6.9)–(6.11):

$$[\mathbf{v} \cdot \mathbf{f}]_{k} = -\frac{\rho_{0} \mathbf{v}_{5,k} \, q_{1}}{\|\mathbf{q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{6,k} \, q_{2}}{\|\mathbf{q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{7,k} \, Q_{1}}{\|\mathbf{Q}\|^{3}} - \frac{\rho_{0} \mathbf{v}_{8,k} \, Q_{2}}{\|\mathbf{Q}\|^{3}} + g_{k}(\mathbf{q}, \mathbf{Q}, \mathbf{v}_{5,0 \le l < k}) + h_{k}(\mathbf{q}, \mathbf{Q}, \mathbf{v}_{6,0 \le l < k}) + m_{k}(\mathbf{q}, \mathbf{Q}, \mathbf{v}_{7,0 \le l < k}) + n_{k}(\mathbf{q}, \mathbf{Q}, \mathbf{v}_{8,0 \le l < k}) + \mathbf{v}_{1,k} \, \dot{q}_{1} + \mathbf{v}_{2,k} \, \dot{q}_{2} + \mathbf{v}_{3,k} \, \dot{Q}_{1} + \mathbf{v}_{4,k} \, \dot{Q}_{2} , \qquad (6.27)$$

where what might be referred to as the inhomogeneous terms of the expression (*i.e.* the terms in addition to the basic terms of expression (6.9), which were shown to recur at each order) have been split into the 4 terms g_k , h_k , m_k , n_k which collect the terms at order k that couple to lower orders $0 \le l < k$.

The equations of condition then become:

$$\begin{aligned} \partial_{t} \mathbf{v}_{j,k} &= -\rho_{0} \left(\mathbf{v}_{5,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{5,k} \right) - \rho_{0} \left(\mathbf{v}_{6,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{6,k} \right) \\ &- \rho_{0} \left(\mathbf{v}_{7,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{7,k} \right) - \rho_{0} \left(\mathbf{v}_{8,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{8,k} \right) \\ &+ \partial_{j} \mathbf{v}_{1,k} \, \dot{q}_{1} + \partial_{j} \mathbf{v}_{2,k} \, \dot{q}_{2} + \partial_{j} \mathbf{v}_{3,k} \, \dot{Q}_{1} + \partial_{j} \mathbf{v}_{4,k} \, \dot{Q}_{2} \\ &+ \partial_{j} g_{k} (\cdots) + \partial_{j} h_{k} (\cdots) + \partial_{j} m_{k} (\cdots) + \partial_{j} n_{k} (\cdots) \\ &\text{for } j = 1, \dots, 4 \,, \end{aligned}$$

$$\partial_{t} \mathbf{v}_{j,k} &= -\rho_{0} \left(\mathbf{v}_{5,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{5,k} \right) - \rho_{0} \left(\mathbf{v}_{6,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{6,k} \right) \\ &- \rho_{0} \left(\mathbf{v}_{7,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{7,k} \right) - \rho_{0} \left(\mathbf{v}_{8,k} \, \partial_{j} \, \frac{q_{1}}{\|\mathbf{q}\|^{3}} + \frac{q_{1}}{\|\mathbf{q}\|^{3}} \, \partial_{j} \mathbf{v}_{8,k} \right) \\ &+ \partial_{j} \mathbf{v}_{1,k} \, \dot{q}_{1} + \partial_{j} \mathbf{v}_{2,k} \, \dot{q}_{2} + \partial_{j} \mathbf{v}_{3,k} \, \dot{Q}_{1} + \partial_{j} \mathbf{v}_{4,k} \, \dot{Q}_{2} + \frac{1}{1} \cdot \mathbf{v}_{j,k} \end{matrix}$$

$$+ \partial_j g_k(\dots) + \partial_j h_k(\dots) + \partial_j m_k(\dots) + \partial_j n_k(\dots)$$
for $j = 5, \dots, 8$, and:
$$(6.28b)$$

extra term

 $\partial_i v_{j,k} = \partial_j v_{i,k}$ for an additional 28 equations. (6.28c)

Thus, if an integrating vector v_k can be found which satisfies these equations, it may be integrated out via the relations (4.22) to obtain I_k , the $O(\varepsilon^k)$ term in the integral approximation.

6.4.2 Solution Perspectives

We will now briefly discuss the approaches attempted to the full equations of condition just presented, and conclude with some remarks on further perspectives for solution.

Approaches Attempted

We remark that the primary difficulty in solving the system (6.28) lies in the fact that instead of a single linear 1^{st} –order partial differential equation, we are confronted with a $\frac{1}{2}n(n+1)$ system of coupled such equations, for which there is no well–developed solution strategy. In place of this, a number of approaches were tried:

• The system, being a linear system of 1st –order partial differential equations, was approached using the method of characteristics (see *e.g.* [Evans, 1997, section 3.2]), as in section 4.1, extended to the system 6.28. However, this method leads to the original Jacobi–coordinate equations of motion (3.46).

In retrospect this should not surprise us: as the method of characteristics transforms a PDE system into a characteristic ODE system, we find that applying the method to the above system recovers the "characteristic" ODE with which we began: system (3.46). As these are the equations we invoked the method of integrating vectors in solving originally, this is not a fruitful approach.

- The equations of the system were also checked against the extensive listing in the Handbook of Nonlinear Partial Differential Equations [Polyanin and Zaitsev, 2003], in search of a (perhaps more complicated) equation or system of equations with some form of usable similarity to our problem. This was done in hopes of finding a nonlinear problem containing ours as a special case, and which might provide a viable solution strategy.
- We have also attempted to find a solution using the relatively straightforward, if naive vector extension of the method of separation of variables, taking:

$$\mathbf{v} = (\hat{\mathbf{v}}_1(x)\bar{\mathbf{v}}_1(t), \dots, \hat{\mathbf{v}}_8(x)\bar{\mathbf{v}}_8(t))^\mathsf{T}$$
(6.29)

as integrating vector and substituting into the usual "low-level" expressions above. This approach did not yield a viable approach due to its still excessive generality for our problem.

Less naively, on the hypothesis that the forms of the integrating vectors found using the existing integrals might serve as a building block towards new integrals, we have also tried a number of approaches using combinations of the integrating vectors with the factors $v_{\mathcal{E}}$, v_H found above for energy and angular momentum, as in the method of integrating vectors, *e.g.*:

$$\mathbf{v} = (\,\hat{\mathbf{v}}_1(x,t)\mathbf{v}_{\mathcal{E},1}(x),\,\ldots,\,\hat{\mathbf{v}}_8(x,t)\mathbf{v}_{\mathcal{E},8}(x)\,)^{\mathsf{T}}\,. \tag{6.30}$$

However, this approach has met with little success within the time-constraints of work on this thesis.

Further Perspectives

The fact that these approaches to a difficult class of equations from a 300–year old problem did not immediately meet with much success should perhaps not surprise us, and it is also in that light that we will, in the next two chapters, focus on the considerably simpler restricted 3 and 4–body problems introduced in sections 3.3.5 and 3.5.

Nonetheless, we would like to comment on a somewhat different approach which strikes us as promising, and in that regard suggestible as a further avenue of research: seeking an analogue of the Runge–Lenz vector for the 3–body problem.

The reader will recall that this integral of the 2-body problem, $\dot{\mathbf{r}} \times \mathbf{H} - \rho \frac{\mathbf{r}}{r}$, allowed us to fully reduce that problem to the traditional solution in terms of Kepler's conic sections. Its generalization to the full 3-body problem is a complicated and open problem, and we would draw the reader's attention specifically to [Dahl, 1997], which makes a link between this integral and the question of a generating symmetry and invariance principle, as guaranteed by Noether's theorem.

It is known that the Runge–Lenz vector arises from a dynamical symmetry of the Kepler problem, which has been studied using primarily the methods of group theory. What has been missing is the connection to a physical invariance principle, which Dahl asserts arises from the generator of Lorentz transformations in the relativistic 2–body problem.

We would tentatively suggest the extension of his work to a 3-body problem, cast in a perturbed 2-body problem formulation as in this thesis, and it would be our hope that by shedding light on the form sought, this might lead to more fruitful approaches to integrating vectors and subsequent integral approximations than the "blind" methods outlined above. Such work would, however, go considerably beyond the scope and level of this thesis.

Chapter 7

The Circular Restricted 3–Body Problem

This chapter treats the use of the method of integrating vectors in generating approximations of first integrals in the planar circular restricted 3–body problem (pCR3BP), formulated in terms of perturbations. Before beginning, a brief review of the problem's equations is given. Note that the computations summarized here are given in full in the MATHEMATICA notebook **IntegratingVectors03-CR3BP.nb**, included on the CD provided with this thesis.

7.1 Review of the Problem

The equations of motion are derived from the limiting case of the 3-body problem where the third body influences neither the primary nor the secondary, which orbit as solutions of the 2-body problem, while its motion is determined by them both (*cf.* case C in table 3.2). The equations are formulated in rotating coordinates, and are normalized for purposes of analysis, *cf.* expression (3.59), repeated here:

$$\begin{split} \ddot{x} - 2\dot{y} &= \frac{\partial\Omega}{\partial x} = x - \frac{1 - \mu}{r_1^3} (x + \mu) - \frac{\mu}{r_2^3} (x - (1 - \mu)) ,\\ \ddot{y} + 2\dot{x} &= \frac{\partial\Omega}{\partial y} = y - \frac{1 - \mu}{r_1^3} y - \frac{\mu}{r_2^3} y , \quad \text{where:} \\ \mu &= \frac{m_2}{m_1 + m_2} ,\\ r_1 &= |z + \mu| = \sqrt{(x + \mu)^2 + y^2} ,\\ r_2 &= |z - (1 - \mu)| = \sqrt{(x - (1 - \mu))^2 + y^2} . \end{split}$$

One takes for **x** the 4-element vector $\mathbf{x} = (x, y, \dot{x}, \dot{y})^{\mathsf{T}}$,¹ and so the integrating vector sought is likewise $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)^{\mathsf{T}}$, where each element will be expanded in a small parameter.

With regard to the scaling approach used in this problem, we recall the reader's attention to sections 3.3.5 and 4.3.3. On consideration, it is readily apparent that when we performed the textbook normalization of the circular restricted

¹In the notebooks the state-vector $\mathbf{x} = (y_1, y_2, y_3, y_4)^{\mathsf{T}}$ is used instead, pursuant to the standard reduction of a *n*-dimensional 2^{*nd*} order system to a 2*n*-dimensional 1^{*st*} order system, but we have chosen for the real meanings of these here for notational clarity.

3-body problem, we in fact implicitly introduced a force-scale mixed with a time-scale. We recall that we effectively chose there, following [Szebehely, 1967]:

$$\begin{split} M &= m_1 + m_2 \quad \text{as a mass-scale, such that:} \\ \frac{m_1}{m_1 + m_2} &= 1 - \mu \quad \text{and} \quad \frac{m_2}{m_1 + m_2} = \mu \quad \text{which sum to 1;} \\ L &= r_0 = \langle \|\mathbf{r}_{12}\| \rangle \quad \text{as a distance-scale, and} \\ T &= \frac{1}{\omega_{12}} \quad \text{as a time-scale,} \end{split}$$

where the subscripts $_{12}$ refer to the primaries orbiting in a 2-body problem-solution. This means that we have implicity introduced the force-scale $\frac{G(m_1+m_2)}{r_{12}^2}$, in addition to making the time dimensionless using the scale *T*. Thus our analysis here is in fact already a force-scales approach, save that the literature conventions have chosen the scale for us.

The problems that will concern us in Part III of this thesis are the Sun–Earth–Moon(–Satellite) and Earth–Moon– Satellite cases. In principle, when considering the motion of the Moon in the Sun–Earth–Moon (SEM) model or of the Satellite in the Earth–Moon–Sat (EMSat), the choice of scaling by the primaries is, if naive, not necessarily a bad first choice, and for that reason we shall treat it without modification in the analysis of this chapter.

Nonetheless, the reader is asked to bear in mind the comments of section 4.3.3, as in particular for the motion of the satellite in the Sun–Earth–Moon–Satellite (SEMSat) model we use for the Capture Problem, it is precisely the changing force–scale regimes along a trajectory that make ballistic capture possible. We shall have more to say about the issue later, as we consider the drawbacks to the naive scale used here in the next chapter and in Part III in particular.

7.2 The Jacobi Integral

The Jacobi integral is the only known integral for the circular restricted 3-body problem, and the reader will recall that it was derived in section 3.3.5 as equation (3.62); it is given here in slightly modified form:

$$\begin{split} \mathcal{I} &= \frac{1}{2}V^2 - \Omega \\ &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}\left(x^2 + y^2\right) - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} \end{split}$$

7.2.1 Expansions in ε

The small parameter in the dimensionless formulation of the CR3BP is the small $\mu = \frac{m_2}{m_1 + m_2}$, and as before the "true" perturbation description is found by setting:

$$m_0 = m_1$$

$$m_1 = \bar{m}_1 \,\varepsilon^0 m_0 \quad \rightarrow \bar{m}_1 = \frac{m_1}{\varepsilon^0 \cdot m_0} = 1$$

$$m_2 = \bar{m}_2 \,\varepsilon^1 m_0 \quad \rightarrow \bar{m}_2 = \frac{m_2}{\varepsilon^1 \cdot m_0} > 1$$

where (since there's only a single small parameter) ε is taken to reflect the order of the perturbation. This is typically 10^{-6} for the Sun–Earth 2–body problem and 10^{-2} for the Earth–Moon 2–body problem (the two cases of the CR3BP which will concern us). In this section we will use the latter choice when relevant, but the reader will note that the case $\varepsilon = 10^{-6} = (10^{-2})^3$ may be obtained from it by the substitution $\varepsilon \longrightarrow \varepsilon^3$ throughout.

Consequently, the small parameter becomes:

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{m_0 \varepsilon \bar{m}_2}{m_0 (1 + \bar{m}_2 \varepsilon)} = \bar{m}_2 \varepsilon - \bar{m}_2^2 \varepsilon^2 + \bar{m}_2^3 \varepsilon^3 + O\left(\varepsilon^4\right)$$

though for simplicity, not much is lost by setting:

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{\varepsilon m_0}{m_0(1 + \varepsilon)} = \varepsilon - \varepsilon^2 + \varepsilon^3 + O\left(\varepsilon^4\right) \quad \text{instead},$$

as long as $\bar{m}_2 = \mathcal{O}(1)$.

Alternatively, we can also simply redefine $\varepsilon := \bar{m}_2 \varepsilon$ of course, though that approach would not scale to additional small parameters with different constants \bar{m}_i , such as we encounter in the next chapter.

For purposes of clarity we will instead continue with the latter, simpler expression as in the previous chapter. Corrected expressions are given in section 11.1 where we will discuss the convergence of the approximations constructed here for the classes of trajectories of interest to us. Taking the simpler expressions for the moment, remark that the terms r_1 , r_2 have expansions in the small parameter, and in particular, the terms needed for the equations of motion are:

$$\frac{1}{r_1^3} = \frac{1}{\left((x+\mu)^2 + y^2\right)^{3/2}} \quad \text{which becomes:}$$
$$= \frac{1}{\left(x^2 + y^2\right)^{3/2}} - \frac{3x}{\left(x^2 + y^2\right)^{5/2}} \varepsilon + \frac{3(4x^2 + 2x^3 - y^2 + 2xy^2)}{2\left(x^2 + y^2\right)^{7/2}} \varepsilon^2 + O\left(\varepsilon^3\right)$$
(7.1)

and

$$\frac{1}{r_2^3} = \frac{1}{\left(\left(x - (1 - \mu)\right)^2 + y^2\right)^{3/2}} \quad \text{which becomes:}$$

$$= \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} - \frac{3(x - 1)}{\left(\left(x - 1\right)^2 + y^2\right)^{5/2}} \varepsilon + \frac{3(2 - 2x^2 + 2x^3 - 3y^2 + 2x(y^2 - 1))}{2\left(x^2 + y^2\right)^{7/2}} \varepsilon^2 + O\left(\varepsilon^3\right) . \tag{7.2}$$

7.2.2 Equations of Condition

The equations of condition which the integrating vector must satisfy at each order k may be derived by specifying equations (4.23) in terms of the expansions of the terms above and the force function **f** in orders k of ε . The equations (4.23) are based, in turn, on the term $\mathbf{v} \cdot \mathbf{f}$, which is again expanded as:

$$\mathbf{v} \cdot \mathbf{f} = [\mathbf{v} \cdot \mathbf{f}]_0 + [\mathbf{v} \cdot \mathbf{f}]_1 \mathbf{\varepsilon} + [\mathbf{v} \cdot \mathbf{f}]_2 \mathbf{\varepsilon}^2 + \dots,$$

and it is straightforward to show that for the CR3BP the first few terms are:

$$[\mathbf{v} \cdot \mathbf{f}]_0 = \dot{x} \,\mathbf{v}_{1,0} + \dot{y} \,\mathbf{v}_{2,0} + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,0} + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,0} \,, \tag{7.3}$$

$$[\mathbf{v} \cdot \mathbf{f}]_{1} = \dot{x} \,\mathbf{v}_{1,1} + \dot{y} \,\mathbf{v}_{2,1} + \left(x - \frac{x}{\left(x^{2} + y^{2}\right)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,1} + \left(y - \frac{y}{\left(x^{2} + y^{2}\right)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,1} \\ + \left(-\frac{-1 + x}{\left(\left(-1 + x\right)^{2} + y^{2}\right)^{3/2}} + \frac{3x^{2}}{\left(x^{2} + y^{2}\right)^{5/2}} - \frac{1 - x}{\left(x^{2} + y^{2}\right)^{3/2}}\right) \mathbf{v}_{3,0} \\ + \left(-\frac{y}{\left(\left(-1 + x\right)^{2} + y^{2}\right)^{3/2}} - y\left(-\frac{3x}{\left(x^{2} + y^{2}\right)^{5/2}} - \frac{1}{\left(x^{2} + y^{2}\right)^{3/2}}\right)\right) \mathbf{v}_{4,0},$$
(7.4)

$$\begin{bmatrix} \mathbf{v} \cdot \mathbf{f} \end{bmatrix}_{2} = \dot{x} \mathbf{v}_{1,2} + \dot{y} \mathbf{v}_{2,2} + \left(x - \frac{x}{\left(x^{2} + y^{2}\right)^{3/2}} + 2\dot{y} \right) \mathbf{v}_{3,2} + \left(y - \frac{y}{\left(x^{2} + y^{2}\right)^{3/2}} - 2\dot{x} \right) \mathbf{v}_{4,2} \\ + \left(-\frac{x - 1}{\left(\left(x - 1\right)^{2} + y^{2}\right)^{3/2}} + \frac{3x^{2}}{\left(x^{2} + y^{2}\right)^{5/2}} - \frac{1 - x}{\left(x^{2} + y^{2}\right)^{3/2}} \right) \mathbf{v}_{3,1} \\ + \left(-\frac{y}{\left(\left(x - 1\right)^{2} + y^{2}\right)^{3/2}} - y \left(-\frac{3x}{\left(x^{2} + y^{2}\right)^{5/2}} - \frac{1}{\left(x^{2} + y^{2}\right)^{3/2}} \right) \right) \mathbf{v}_{4,1} \\ + \left(-\frac{2 - x}{\left(\left(x - 1\right)^{2} + y^{2}\right)^{3/2}} + \frac{3\left(x - 1\right)x}{\left(x^{2} + y^{2}\right)^{5/2}} - \frac{-2 + x}{\left(x^{2} + y^{2}\right)^{3/2}} + \frac{3\left(x - 1\right)^{2}}{\left(\left(x - 1\right)^{2} + y^{2}\right)^{5/2}} \\ - \frac{3x\left(4x^{2} + 2x^{3} - y^{2} + 2xy^{2}\right)}{2\left(x^{2} + y^{2}\right)^{7/2}} \right) \mathbf{v}_{3,0} + \left(-y\left(-\frac{1}{\left(\left(x - 1\right)^{2} + y^{2}\right)^{3/2}} - \frac{3\left(x - 1\right)}{\left(\left(x - 1\right)^{2} + y^{2}\right)^{5/2}} \right) \\ -y\left(\frac{3x}{\left(x^{2} + y^{2}\right)^{5/2}} + \frac{1}{\left(x^{2} + y^{2}\right)^{3/2}} + \frac{3\left(4x^{2} + 2x^{3} - y^{2} + 2xy^{2}\right)}{2\left(x^{2} + y^{2}\right)^{7/2}} \right) \right) \mathbf{v}_{4,0} .$$
(7.5)

Higher orders can be computed analogously, and these computations to $O(\varepsilon^4)$ are produced in the MATHEMAT-ICA notebook IntegratingVectors03-CR3BP.nb, which is easily extended further. The orders shown here are sufficient, however, to demonstrate the principles used in the construction of the Jacobi integral in the following section.

Before this, however, the following remarks should be made:

- First, we note that the orders are *all* coupled to each other: due to the expansion in ε no orders are skipped. In consequence, each order influences the next, carrying the previous influences as well in the pattern described in the previous chapter (instead of coupling as *e.g.* O (ε^{6,3,2,0}) as in the Jacobi 3–body problem, we obtain O (ε^{6,5,4,3,2,1,0}), with 0 carrying the newest terms and the rest each shifted one order).
- This yields an interesting pattern: at each order k of the expansion, the terms

$$\dot{x} \mathbf{v}_{1,k} + \dot{y} \mathbf{v}_{2,k} + \left(x - \frac{x}{\left(x^2 + y^2\right)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,k} + \left(y - \frac{y}{\left(x^2 + y^2\right)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,k}$$

form the basis, and then each of the previous k-1 terms recur, each at $v_{3,4;i}$ an order higher, followed by a final

set of new terms multiplied by $v_{3,4;0}$.

• It also bears mentioning that the factors $v_{1,2;k}$ only occur as $\dot{x} v_{1,k} + \dot{y} v_{2,k}$, while all the other components of the expression multiply $v_{3,4;k}$; this effectively gives us less freedom in choosing the components to cancel each other for a time-independent integrating vector.

7.2.3 Construction of the Integral

As in the previous chapter, the expansion of the Jacobi integral is briefly given for reference.

Expansion of the Jacobi Integral in ϵ

The integral may be expanded, using only the above expressions for the terms involving the small parameter μ , as follows (to third order in ε):

$$\mathcal{I} = \frac{1}{2} \left(x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2}{\left(x^2 + y^2\right)^{1/2}} \right) + \left(\frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{1/2}} - \frac{x}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(x^2 + y^2\right)^{1/2}} \right) \varepsilon$$

$$+ \left(\frac{3x^2}{2\left(x^2 + y^2\right)^{5/2}} - \frac{1}{2\left(x^2 + y^2\right)^{3/2}} + \frac{1}{\left(x^2 + y^2\right)^{1/2}} + \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} - \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{1/2}} \right) \varepsilon$$

$$+ x \left(\frac{2}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} \right) \varepsilon$$

$$(7.6)$$

Construction of the Jacobi Integral in ϵ

In order to construct the integral (presuming that it were unknown and that the expansion of the previous section were not available to us), the key again lies in observing that the goal is to find a time–independent integral, and for this it is sufficient to require that

$$[\mathbf{v} \cdot \mathbf{f}]_k = 0 \quad \forall k = 0, 1, 2, \dots$$

Unlike the Jacobi 3–body problem's energy integral in the previous chapter, here we cannot ignore the low orders in ε , and must begin with the null-order equation. With a bit of trial and error, it is found that the simplest way to cancel the terms consistent with the spatial equations (4.23) is the following:

$$\nu_{0} = \begin{pmatrix} -\left(x - \frac{x}{(x^{2} + y^{2})^{3/2}}\right) \\ -\left(y - \frac{y}{(x^{2} + y^{2})^{3/2}}\right) \\ \frac{\dot{x}}{\dot{y}} \end{pmatrix}.$$
(7.7)

Remark that this is almost a straightforward pairing, except that the terms $+2\dot{y}$, $-2\dot{x}$ are omitted, as they naturally cancel when multiplied by $v_{3:0}$ resp. $v_{4:0}$ and subsequent summation, due to their opposed signs. Then, on integrating

each factor with respect to the coordinates $\mathbf{x} = (x, y, \dot{x}, \dot{y})^{\mathsf{T}}$ pursuant to equation (4.22), we find that:

$$I_0 = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 - x^2 - y^2 - \frac{2}{\left(x^2 + y^2\right)^{1/2}} \right),$$
(7.8)

up to a constant. This matches up exactly with the expansion given above, and recalling the discussion of uniqueness in the previous chapter, this is roughly what we expect, given that there is only one known time-independent integral for this problem. At the next order in ε , with $[\nu \cdot \mathbf{f}]_1$ as above, the equations may be canceled in the same way by choosing the factors:

$$\nu_{1} = \begin{pmatrix} -\left(-\frac{-1+x}{\left((-1+x)^{2}+y^{2}\right)^{3/2}} + \frac{3x^{2}}{\left(x^{2}+y^{2}\right)^{5/2}} - \frac{1-x}{\left(x^{2}+y^{2}\right)^{3/2}}\right) \\ -\left(-\frac{y}{\left((-1+x)^{2}+y^{2}\right)^{3/2}} - y\left(-\frac{3x}{\left(x^{2}+y^{2}\right)^{5/2}} - \frac{1}{\left(x^{2}+y^{2}\right)^{3/2}}\right)\right) \\ 0 \\ 0 \end{pmatrix} \right).$$
(7.9)

This choice is a strategic one: simply ignore the gravitational and coriolis terms by setting $v_{3,4;k>0} = 0$ and cancel the terms arising from order 0 using $v_{1,2;k>0}$. This approach turns out to not only cancel terms, but also to yield an integrating vector satisfying (4.23) such that the integral approximation constructed on integrating (4.22) is *precisely* the Jacobi integral. This observation is formalized in the following proposition.

Proposition 7.2.1 (Pattern Cancelation for Jacobi Integral)

The structure of the expansions (7.3)–(7.5) and their counterparts $[\mathbf{v} \cdot \mathbf{f}]_k$ for k > 0 follows a specific pattern which allows for the easy cancelation using only the factors $v_{1,2;k}$. It is precisely this naive choice which when combined with the factor chosen above for k = 0 which leads to the approximation, by order ε , of the Jacobi integral.

Proof.

Recall the expression for $v \cdot f$, which is generalized for arbitrary order k > 0 as:

$$[\mathbf{v} \cdot \mathbf{f}]_{k} = \dot{x} \mathbf{v}_{1,k} + \dot{y} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \mathbf{v}_{4,k} + \dot{x} g_{k}(x, y) + \dot{y} h_{k}(x, y) .$$

The functions g_k , h_k , in particular, depend only on the first two coordinates, and contain terms coming from the lowerorder expressions for $v_{j < k}$. Consequently, in searching for stationary solutions ($\partial_t I_k = 0$), the following choice serves to cancel the factor and obtain a candidate for the next integral approximation:

 $\mathbf{v}_{1,k} = -g_k(x, y) ,$ (7.10a)

$$v_{2,k} = -h_k(x, y),$$
 (7.10b)

$$v_{3,k} = 0$$
, (7.10c)

$$v_{4,k} = 0$$
. (7.10d)

That this approach indeed yields not just a valid integral but precisely the Jacobi integral is demonstrated as follows.

The expression determining the time-dependence of the integral via $\partial_t I$ is constructed as follows:

$$\mathbf{v} \cdot \mathbf{f} = [\mathbf{v} \cdot \mathbf{f}]_0 + [\mathbf{v} \cdot \mathbf{f}]_1 \mathbf{\varepsilon} + [\mathbf{v} \cdot \mathbf{f}]_2 \mathbf{\varepsilon}^2 + \dots$$

= $\mathbf{v}_0 \cdot \mathbf{f}_0 + (\mathbf{v}_0 \cdot \mathbf{f}_1 + \mathbf{v}_1 \cdot \mathbf{f}_0) \mathbf{\varepsilon} + (\mathbf{v}_0 \cdot \mathbf{f}_2 + \mathbf{v}_1 \cdot \mathbf{f}_1 + \mathbf{v}_2 \cdot \mathbf{f}_0) \mathbf{\varepsilon}^2 + \dots$ (7.11)

Recall also that in this formulation, the equations of the circular restricted 3-body problem become:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{pmatrix} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \dot{x} \\ \dot{y} \\ x - \frac{1-\mu}{r_1^3} (x+\mu) - \frac{\mu}{r_2^3} (x-(1-\mu)) + 2\dot{y} \\ y - \frac{1-\mu}{r_1^3} (x+\mu) - \frac{\mu}{r_2^3} (x-(1-\mu)) - 2\dot{x} \end{pmatrix}.$$

In this, the reader will of course recognize that in the expansion $\mathbf{f} = \mathbf{f}_0 + \mathbf{f}_1 \boldsymbol{\epsilon} + \mathbf{f}_2 \boldsymbol{\epsilon}^2 + \dots$ there are contributions to the higher orders only from the components involving μ in the equations of motion, and these occur only in the 3^{th} resp. 4^{th} vector components.

Thus given that $v_{3,4;k>0} = 0$,

$$\mathbf{v}_{i} \cdot \mathbf{f}_{j} = \begin{cases} \mathbf{v}_{3,0} \, f_{3,j} + \mathbf{v}_{4,0} \, f_{4,j} & \text{if } i = 0\\ 0 & \text{if } i \neq 0 \end{cases}$$
(7.12)

and so recalling that $v_{3,0} = \dot{x}$ and $v_{4,0} = \dot{y}$, the above unknown functions are $g_k(x, y) = f_{3,k}$ and $h_k(x, y) = f_{4,k}$, which on integration with respect to the coordinates yields:

$$I_{k} = \int \mathbf{v}_{k} \cdot \mathbf{d}\mathbf{x} = \int \left(\begin{array}{c} \left[-\frac{1-\mu}{r_{1}^{2}} (x+\mu) - \frac{\mu}{r_{2}^{2}} (x-(1-\mu)) \right]_{k} \\ \left[-\frac{1-\mu}{r_{1}^{2}} y - \frac{\mu}{r_{2}^{2}} y \right]_{k} \end{array} \right) \cdot \mathbf{d}\mathbf{x}$$
$$= \int \nabla \left[\frac{1-\mu}{r_{1}} + \frac{\mu}{r_{2}} \right]_{k} \cdot \mathbf{d}\mathbf{x}$$
$$= \left[\frac{1-\mu}{r_{1}} + \frac{\mu}{r_{2}} \right]_{k} \quad \text{up to a constant,}$$
(7.13)

as the integrand is of course simply the derivative with respect to *x*, *y* of the result. This in turn is the $O(\varepsilon^k)$ part of the potential term: precisely the next contribution to the expansion of \mathcal{I} .

Thus it is established that precisely this naive approach (7.10) provides the reconstruction of the Jacobi integral (7.6).² Furthermore, the approach can be used to construct the integral to arbitrary order, and so further expressions will be omitted here.

7.3 Approximations of New Integrals

A pertinent question is of course whether there is any other way to cancel the terms in each $[\mathbf{v} \cdot \mathbf{f}]_k$ (leading to different time–independent integrals), or perhaps to satisfy the entire system of equations which an arbitrary time–dependent integrating vector must satisfy. In discussing this, we begin by giving the full equations of condition.

²Subject to the usual understanding of limited uniqueness, cf. section 6.3.7.

7.3.1 Full Equations of Condition

Before the brief discussion below, however, let us state the full set of equations (4.23) as applied to the circular restricted 3-body problem, for the case when the scope of the approach is not limited to the reconstruction of the Jacobi integral. The foundation of these is a slightly more general version of what was used above:

$$[\mathbf{v} \cdot \mathbf{f}]_{k} = g_{k}(x, y, \mathbf{v}_{3,0 \le l < k}) + h_{k}(x, y, \mathbf{v}_{4,0 \le l < k}) + +\dot{x}\mathbf{v}_{1,k} + \dot{y}\mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right)\mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right)\mathbf{v}_{4,k},$$
(7.14)

where the inhomogeneous terms have been split into 2 terms which act as more general versions of the g_k , h_k introduced earlier in proposition 7.2.1. Instead of always multiplying $v_{3,0} \rightarrow \dot{x}$ resp. $v_{4,0} \rightarrow \dot{y}$, these now collect the terms multiplying $v_{3,0\leq l < k}$ and $v_{4,0\leq l < k}$, representing the couplings to all earlier orders (l < k). The equations of condition then become:

$$\partial_{t} \mathbf{v}_{1,k} = \dot{x} \,\partial_{1} \mathbf{v}_{1,k} + \dot{y} \,\partial_{1} \mathbf{v}_{2,k} + \left(1 - \partial_{1} \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \mathbf{v}_{3,k} + \left(x + 2\dot{y} - \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \,\partial_{1} \mathbf{v}_{3,k} \\ - \,\partial_{1} \left(\frac{y}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \,\partial_{1} \mathbf{v}_{4,k} + \,\partial_{1} g_{k} + \,\partial_{1} h_{k} \,, \tag{7.15a}$$

$$\partial_{t} \mathbf{v}_{2,k} = \dot{x} \,\partial_{2} \mathbf{v}_{1,k} + \dot{y} \,\partial_{2} \mathbf{v}_{2,k} - \partial_{2} \left(\frac{x}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \mathbf{v}_{3,k} + \left(x + 2\dot{y} - \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \,\partial_{2} \mathbf{v}_{3,k} \\ + \left(1 - \partial_{2} \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \,\partial_{2} \mathbf{v}_{4,k} + \partial_{2} g_{k} + \partial_{2} h_{k} \,, \tag{7.15b}$$

$$\partial_{t} \mathbf{v}_{3,k} = \mathbf{v}_{1,k} + \dot{x} \,\partial_{3} \mathbf{v}_{1,k} + \dot{y} \,\partial_{3} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{3} \mathbf{v}_{3,k} \\ - 2\mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{3} \mathbf{v}_{4,k} + \,\partial_{3} g_{k} + \,\partial_{3} h_{k} \,,$$
(7.15c)

$$\partial_{t} \mathbf{v}_{4,k} = \dot{x} \,\partial_{4} \mathbf{v}_{1,k} + \mathbf{v}_{2,k} + \dot{y} \,\partial_{4} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{4} \mathbf{v}_{3,k} \\ + 2\mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{4} \mathbf{v}_{4,k} + \,\partial_{4} g_{k} + \,\partial_{4} h_{k} \quad \text{and}$$
(7.15d)

 $\partial_i v_{j,k} = \partial_j v_{i,k}$ for an additional 6 equations. (7.15e)

As usual, if an integrating vector v_k can be found which satisfies these equations, it may be integrated out via the relations (4.22) to obtain I_k , the $O(\varepsilon^k)$ term in the integral approximation.

7.3.2 Solution Perspectives

As with the results of the previous chapter, we will now briefly discuss the approaches attempted to the full equations of condition just presented, and conclude with some remarks on further perspectives for solution.

Approaches Attempted

The same general remarks apply as in the case of the Jacobi problem: our difficulties stem from the necessity of solving a $\frac{1}{2}n(n+1)$ -dimensional system of coupled linear 1st –order PDE equations, for which there is no well–developed solution strategy. In place of this, a number of approaches were tried:

- Here too, an approach using the method of characteristics is fruitless, as the characteristic ODE system that the method yields is simply the original ODE system of the CR3BP, which is too difficult to solve except in a few special cases which are not our present concern.
- The equations of the system were again checked for similarity against [Polyanin and Zaitsev, 2003] for the existence a (perhaps more complicated) equation-type which might contain our problem as a special case, and provide a viable solution strategy; this was again without fruit.
- As in the previous chapter, we note as hopeful the attempt to find a solution using the straightforward extension of the method of separation of variables, taking:

$$\mathbf{v} = (\,\hat{\mathbf{v}}_1(x)\bar{\mathbf{v}}_1(t), \, \dots, \, \hat{\mathbf{v}}_4(x)\bar{\mathbf{v}}_4(t)\,)^{\mathsf{T}} \,, \tag{7.16}$$

or using the existing vectors as building blocks for new integrals, using combinations of the integrating vectors with the factors $v_{\mathcal{I}}$ found above for the Jacobi integral as:

$$\mathbf{v} = (\hat{\mathbf{v}}_1(x,t)\mathbf{v}_{\mathcal{I},1}(x), \dots, \hat{\mathbf{v}}_4(x,t)\mathbf{v}_{\mathcal{I},4}(x))^{\mathsf{T}}.$$
(7.17)

However, this approach has likewise met with little success within the time-constraints of work on this thesis.

Further Perspectives

The main further approach which suggests itself for the CR3BP is one based on Fourier analysis, motivated by the roughly periodic motion of *e.g.* the Moon about the Earth and the two about the Sun. While we will not go into this further here, we discuss it in somewhat more detail in the next chapter for the Capture Problem, which is in essence only a small perturbation of the present problem, making the roughly periodic motion explicit as simple harmonic motion, *cf.* section 3.5.

An analogue of the idea proposed for the full Jacobi 3-body problem under the present restrictions is sufficiently speculative that we will not remark further.

Chapter 8

The Capture Problem

This chapter treats the use of the method of integrating vectors in generating approximations of first integrals in the Capture Problem (CP) formulated in section 3.5, formulated in terms of perturbations. Note that the computations summarized here are given in full in the MATHEMATICA notebook **IntegratingsVectors04-CaptureProblem-mod.nb**, included on the CD provided with this thesis.

8.1 Review of the Problem

The capture problem discussed and derived in section 3.5, stated in rotating (synodic) dimensionless coordinates (and normalized by the Earth–Sun distance and mass ratios) is given by equation (3.83):

$$\begin{split} \ddot{x} - 2\dot{y} &= x - \frac{1 - \mu}{r_1^3} (x + \mu) - \frac{\mu}{r_2^3} \left(x - (1 - \mu) \right) - \frac{\upsilon}{r_3^3} \left(x - (1 - \mu + \eta \cos \omega' t) \right) ,\\ \ddot{y} + 2\dot{x} &= y - \frac{1 - \mu}{r_1^3} y - \frac{\mu}{r_2^3} y - \frac{\upsilon}{r_3^3} (y - \eta \sin \omega' t) , \quad \text{where:} \end{split}$$
$$\begin{aligned} \mu &= \frac{m_2}{m_1 + m_2} ,\\ \upsilon &= \frac{m_3}{m_1 + m_2} ,\\ r_1 &= |z + \mu| = \sqrt{(x + \mu)^2 + y^2} ,\\ r_2 &= |z - (1 - \mu)| = \sqrt{(x - (1 - \mu))^2 + y^2} ,\\ r_3 &= |z - (1 - \mu + \eta e^{i\omega' t})| = \sqrt{(x - (1 - \mu + \eta \cos \omega' t))^2 + (y - \eta \sin \omega' t)^2} \end{split}$$

Remark in particular that the parameter η is a scaled distance in this model, and is in fact given by $\eta = \frac{r_{0.EM}}{r_{0.ES}}$, which in practice becomes $\eta \approx 2.5 \cdot 10^{-3} = 0.25 \epsilon$ in magnitude for the conventional choice of $\epsilon = 10^{-2}$ for Sun–Earth–Moon system.¹

With regard to the scaling approach used in this problem, we recall the reader's attention to sections 3.5 and 4.3.3. On consideration, it is readily apparent that as with the normalization of the circular restricted 3-body problem, we have in fact implicitly introduced a force-scale mixed and a time-scale in the Capture Problem as well. We have effectively

¹In consequence, it is both $O(\varepsilon^2)$ and $O(\varepsilon^3)$ though we shall use the latter in keeping with the convention of letting the constant $1 \le k \le 10$.

chosen:

$$M = m_1 + m_2 \text{ as a mass-scale, such that:}$$

$$\frac{m_1}{m_1 + m_2} = 1 - \mu \text{ and } \frac{m_2}{m_1 + m_2} = \mu ,$$
and additionally $\upsilon = \frac{m_3}{m_1 + m_2}$ derives from *M* analogously;
$$L = r_0 = \langle ||\mathbf{r}_{12}|| \rangle \text{ as a distance-scale, and}$$

$$T = \frac{1}{\omega_{12}} \text{ as a time-scale,}$$

where the subscripts $_{12}$ refer to the Sun and Earth as primaries orbiting in a 2-body problem-solution. This means that we have implicitly introduced the force-scale $\frac{G(m_1+m_2)}{r_{12}^2}$, in addition to making the time dimensionless using the scale *T*. Thus our analysis here is in fact already a force-scales approach, save that the literature conventions of the CR3BP have chosen the scale for us.

Thus, since the parameter η here refers to a distance–scale rather than a mass–scale, it makes sense to incorporate it as in equation 8.1. Note that in doing this, we now take $\varepsilon = 10^{-1}$, with -1 as the greatest common divisor of the magnitudes 10^{-3} , 10^{-6} and 10^{-8} .

$$\eta = \frac{r_{EM}}{r_0} = \frac{r_{EM}}{r_{ES}} \approx 2.57 \cdot 10^{-3} = \bar{\eta}\varepsilon^3 = \mathcal{O}\left(\varepsilon^3\right) \,. \tag{8.1}$$

With this in mind, we will for the moment use the scales introduced implicitly in our derivation in section 3.5. However, we must first lay some groundwork for the discussion to follow, which will consider the possibilities for perturbations of the Jacobi integral in some detail (as the only known integral for the CR3BP which the present CP is a further perturbation of).

8.2 The Analogue of the Jacobi Integral

Recall that the Jacobi integral of the circular restricted 3-body problem was given by:

$$\begin{split} \mathcal{I} &= \frac{1}{2}V^2 - \Omega \\ &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 \;\right) - \frac{1}{2}\left(x^2 + y^2 \;\right) - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} \;, \end{split}$$

and had the expansion of equation (7.6). It was further proven that this was precisely the integral–approximation constructed using the method of integrating vectors naively (in the search for a time–dependent integral) in the previous chapter, *cf.* proposition 7.2.1.

The fact that the Capture Problem proposed in this thesis is simply a small further perturbation of the circular restricted 3-body problem from the perturbation-perspective, motivates trying first a similar approach in deriving an integral for the problem. We remark though, that the problem is not Hamiltonian, otherwise, as we showed in chapter 2, the hypothetical Hamiltonian \mathcal{H}_{CP} would itself be an integral.

8.2.1 Analytical Considerations

The Jacobi integral was derived using an approach traditionally motivated by conservation–of–energy considerations, multiplying (3.59) by the velocities \dot{x}_i and summing to obtain (in the planar case) :

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\frac{\partial\Omega}{\partial x} + \dot{y}\frac{\partial\Omega}{\partial y} \quad \text{or}$$
$$\frac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2\right) = L_t(\Omega) \equiv \nabla\Omega \cdot \dot{\mathbf{x}} ,$$

and on integrating with respect to time:

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)=\Omega+k\;.$$

This is possible specifically *because* we can write:

$$L_t(\Omega) = \nabla \Omega \cdot \dot{\mathbf{x}}$$
 and
 $\int L_t(\Omega) \, dt = \Omega + k$, with *k* some constant of integration.

For the capture problem, on considering the equations of motion (3.83), it is easy to see that an analogous potential can be formulated in normalized rotating coordinates:

$$\bar{\Omega}(t) = \frac{1}{2} \left(x^2 + y^2 \right) - \frac{(1-\mu)}{r_1} - \frac{\mu}{r_2} - \frac{\upsilon}{r_3(t)} , \qquad (8.2)$$

which fulfills exactly the same role here as Ω in the CR3BP. The key difference is the introduction of time dependence to $\overline{\Omega}(t)$, as a result of which it now holds that:

$$L_t\left(\bar{\Omega}(t)\right) = \nabla \bar{\Omega}(t) \cdot \dot{\mathbf{x}} + \partial_t \bar{\Omega}(t) .$$

Naively proceeding as before, on multiplying by \dot{x}_i and summing, we now obtain:

$$\begin{split} \dot{x}\ddot{x} + \dot{y}\ddot{y} &= \dot{x}\frac{\partial\bar{\Omega}(t)}{\partial x} + \dot{y}\frac{\partial\bar{\Omega}(t)}{\partial y} , \quad \text{which leads to:} \\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2 + \dot{y}^2\right) &= \nabla\bar{\Omega}(t) \cdot \dot{\mathbf{x}} \equiv \underbrace{L_t\left(\bar{\Omega}(t)\right) - \partial_t\bar{\Omega}(t)}_{\text{now explicitly }\bar{\Omega}(t)} . \end{split}$$

Integrating with respect to time, this yields:

$$\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \bar{\Omega}(t) - k - \int \partial_t \bar{\Omega}(t) \, \mathrm{d}t \quad \text{where:}$$
(8.3)

$$\int \partial_t \bar{\Omega}(t) dt = \int \frac{\partial}{\partial t} \left(\frac{\upsilon}{r_3(t)} \right) dt$$

= $\eta \upsilon \int \frac{y \omega' \cos \omega' t - (x - (1 - \mu)) \omega' \sin \omega' t}{\left((x - (1 - \mu + \eta \cos \omega' t))^2 + (y - \eta \sin \omega' t)^2 \right)^{3/2}} dt$, (8.4)

where the integral is understood to also take x = x(t) and y = y(t) into account, for which no analytically closed expression is known. It is pertinent to remark here that due to the multiplication by the product ηv , this can in fact be

seen as an $O(\eta(\varepsilon) \times \upsilon(\varepsilon) = \varepsilon^{k_{\upsilon}+k_{\eta}})$ correction to the Jacobi integral:

$$\mathcal{J}' = \frac{1}{2}V^2 - \bar{\Omega}(t) + \varepsilon^{k_0 + k_\eta} \cdot \int (\cdots) dt , \qquad (8.5)$$

where k_{υ} , k_{η} correspond to the order in ε of the parameters υ , η respectively.

While this is a useful illustration, it does little for our practical problem of finding a new integral for the capture problem. It does, however, suggest the possibility of approaching integral approximations by superposition of the Jacobi integral results of the previous chapter with new integrating vectors corresponding to only the time-dependent part of the integral.

8.2.2 Expansions in ε

In this section the approach will focus on extending that of section 7.2.1. However, in this problem, the distance parameter cast as $\eta = \eta(\epsilon)$ is quite relevant, and it makes sense to incorporate it into our force–scales, though we do so naively for the moment, continuing with the conventional scaling based on the primaries.

Our goal in this section is mainly to discuss the departures from the derivations already given in section 7.2.1, and the reader will note that we do so only for the simplified expansion-expressions (*i.e.* neglecting the scaling constants \bar{m}_i), to keep the discussion clear. Corrected expressions are given in section 11.1 where we will discuss the convergence of the approximations constructed here for the classes of trajectories of interest to us.

The definitions of the parameters μ , υ is as before, and we take $\varepsilon = 10^{-1}$. However, the masses m_i , i = 1, 2, 3, 4 now correspond to the Sun, Earth, Moon and Satellite respectively, and we take $m_0 := m_1$.² Thus the Earth enters with mass of order $O(\varepsilon^3 m_0)$ while the Moon is $O(\varepsilon^8 m_0)$, such that:

$$\mu = \frac{m_2}{m_1 + m_2} = \frac{\epsilon^6 m_0}{m_0 (1 + \epsilon^6)} = \epsilon^6 - \epsilon^{12} + \epsilon^{18} + \mathcal{O}\left(\epsilon^{24}\right),\tag{8.6}$$

$$\upsilon = \frac{m_3}{m_1 + m_2} = \frac{\varepsilon^8 m_0}{m_0 (1 + \varepsilon^6)} = \varepsilon^8 - \varepsilon^{14} + \varepsilon^{20} + O\left(\varepsilon^{24}\right) \,. \tag{8.7}$$

Likewise, the expressions for r_1 , r_2 change accordingly to:

$$\frac{1}{r_1^3} = \frac{1}{\left((x+\mu)^2 + y^2\right)^{3/2}} \quad \text{which becomes:}$$
$$= \frac{1}{\left(x^2 + y^2\right)^{3/2}} - \frac{3x}{\left(x^2 + y^2\right)^{5/2}} \varepsilon^6 + \frac{3(4x^2 + 2x^3 - y^2 + 2xy^2)}{2\left(x^2 + y^2\right)^{7/2}} \varepsilon^{12} + O\left(\varepsilon^{18}\right)$$
(8.8)

and

²We might take $m_1 + m_2$ but this is the same up to the sixth decimal place due to the huge mass of the Sun.
$$\frac{1}{r_2^3} = \frac{1}{\left((x - (1 - \mu))^2 + y^2\right)^{3/2}} \quad \text{which becomes:}$$

$$= \frac{1}{\left((x - 1)^2 + y^2\right)^{3/2}} - \frac{3(x - 1)}{\left((x - 1)^2 + y^2\right)^{5/2}} \varepsilon^6 + \frac{3(2 - 2x^2 + 2x^3 - 3y^2 + 2x(y^2 - 1))}{2\left(x^2 + y^2\right)^{7/2}} \varepsilon^{12} + \mathcal{O}\left(\varepsilon^{18}\right) . \quad (8.9)$$

The above cases do not depend directly on how η is handled, but the $r_3(t)$ term does. We remark that on introducing it into a force–scales framework, there are two possible approaches:

- 1. We can make a complete switch to force–scales from first principles, and consequently completely re–derive the equations of motion for the capture problem along the lines of the discussion in section 4.3.3. We then seek a new scaling in which the question of an appropriate m_0 , r_0 must again be faced; which is coupled to the issue that the Kepler-relation used to derive the CR3BP normalization is only strictly applicable to the 2–body problem being perturbed (here: Earth–Sun). Choosing *e.g.* $\mathbf{r}_0 = \mathbf{r}_{EM} \neq \mathbf{r}_{ES}$ to make another distance scale explicit, as suggested by section 4.3.3, leads to highly nontrivial modifications of the equations of motion which we prefer to avoid for the moment, though we return to it below in section ??.
- 2. Alternately, keeping the primary goal of studying the equations as a perturbation problem, we may accept the scaling as before based on the CR3BP formulation with the Sun and Earth as primaries, and choose to modify only η to make its nature as a distance-scale in this context explicit. This choice involves only expression (8.1).³ It is justified to make "just" this substitution rather than a full re-derivation of the equations of motion *only* because the normalized equations were made dimensionless, introducing distance-scales implicitly.

Preferring the latter approach for both its simplicity and the advantage of comparability with the previous chapter, we proceed by substitution, though omitting the factor $\bar{\eta}$ to simplify notation (as we did with the \bar{m}_i terms). Thus:

$$\frac{1}{r_3^3} = \frac{1}{\left(\left(x - (1 - \mu + \eta \cos \omega' t)\right)^2 + (y - \eta \sin \omega' t)^2\right)^{3/2}} \quad \text{which now becomes:}
= \frac{1}{\left((x - 1)^2 + y^2\right)^{3/2}} + \frac{3\left((x - 1)\cos \omega' t + y\sin \omega' t\right)}{\left((x - 1)^2 + y^2\right)^{5/2}} \varepsilon^3
+ \frac{3\left(7 + 5\cos 2\omega' t + 5\left(3 + \cos 2\omega' t\right)x^2 - 4x^3 - 10y\sin 2\omega' t\right)}{4\left((x - 1)^2 + y^2\right)^{7/2}} \varepsilon^6
+ \frac{3\left((7 - 5\cos 2\omega' t\right)y^2 - 2x\left(9 + 5\cos 2\omega' t - 5y\sin 2\omega' t + 2y^2\right)\right)}{4\left((x - 1)^2 + y^2\right)^{7/2}} \varepsilon^6 + O\left(\varepsilon^9\right).$$
(8.10)

Note that on taking this into account, the time-dependencies now occur only in the numerator at each order, due to the fact that we are expanding the terms around a time-independent position in the denominator.

³With $\varepsilon = 10^{-1}$ as remarked above.

8.2.3 Equations of Condition

Using the above expansions, the equations of condition may be derived which the integral approximation must satisfy at each order, by specifying equations (4.22) – (4.23) in terms of the expansions of the ODE's in orders *k* of ε . In order to discuss this in more detail, however, we must first derive the key terms in those equations: $[\mathbf{v} \cdot \mathbf{f}]_k$, and that will be treated first, before discussing the equations of condition proper.

In line with the approach taken above, let us here take $\eta = \eta(\epsilon)$ into account as well. As before we naively set $\eta = 10^{-3}$, and as the parameters η, μ, υ are now $O(10^{-3}, 10^{-6}, 10^{-8})$, and take $\epsilon = 10^{-1}$ to obtain integer powers in the expressions to follow.

The above expansion of $\boldsymbol{\nu}\cdot\boldsymbol{f}$ then takes the following form:

$$[\mathbf{v} \cdot \mathbf{f}]_{k=0,\dots,5} = \dot{x} \,\mathbf{v}_{1,k} + \dot{y} \,\mathbf{v}_{2,k} + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,k} + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,k} , \qquad (8.11)$$

$$[\mathbf{v} \cdot \mathbf{f}]_{k=6,7} = \dot{x} \,\mathbf{v}_{1,k} + \dot{y} \,\mathbf{v}_{2,k} + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,k} + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,k} + \left(-\frac{-1 + x}{\left((-1 + x)^2 + y^2\right)^{3/2}} + \frac{3x^2}{\left(x^2 + y^2\right)^{5/2}} - \frac{1 - x}{\left(x^2 + y^2\right)^{3/2}}\right) \mathbf{v}_{3,k-6} + \left(-\frac{y}{\left((-1 + x)^2 + y^2\right)^{3/2}} - y\left(-\frac{3x}{\left(x^2 + y^2\right)^{5/2}} - \frac{1}{\left(x^2 + y^2\right)^{3/2}}\right)\right) \mathbf{v}_{4,k-6},$$
(8.12)

$$[\mathbf{v} \cdot \mathbf{f}]_{k=8,9,10} = \dot{x} \, \mathbf{v}_{1,k} + \dot{y} \, \mathbf{v}_{2,k} + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,k} + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,k}$$
$$- \frac{(x-1) \, \mathbf{v}_{3,k-8}}{\left((-1+x)^2 + y^2\right)^{3/2}} + \left(-\frac{-1+x}{\left((-1+x)^2 + y^2\right)^{3/2}} + \frac{3x^2}{(x^2 + y^2)^{5/2}} - \frac{1-x}{(x^2 + y^2)^{3/2}}\right) \mathbf{v}_{3,k-6}$$
$$- \frac{y \, \mathbf{v}_{4,k-8}}{\left((-1+x)^2 + y^2\right)^{3/2}} + \left(-\frac{y}{\left((-1+x)^2 + y^2\right)^{3/2}} - y \left(-\frac{3x}{(x^2 + y^2)^{5/2}} - \frac{1}{(x^2 + y^2)^{3/2}}\right)\right) \mathbf{v}_{4,k-6},$$
(8.13)

$$\mathbf{v} \cdot \mathbf{f}]_{k=11} = \dot{x} \, \mathbf{v}_{1,k} + \dot{y} \, \mathbf{v}_{2,k} + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y}\right) \mathbf{v}_{3,k} + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x}\right) \mathbf{v}_{4,k}$$

$$- \frac{(x-1) \, \mathbf{v}_{3,k-8}}{\left((-1+x)^2 + y^2\right)^{3/2}} + \left(-\frac{-1+x}{\left((-1+x)^2 + y^2\right)^{3/2}} + \frac{3x^2}{(x^2 + y^2)^{5/2}} - \frac{1-x}{(x^2 + y^2)^{3/2}}\right) \mathbf{v}_{3,k-6}$$

$$- \frac{y \, \mathbf{v}_{4,k-8}}{\left((-1+x)^2 + y^2\right)^{3/2}} + \left(-\frac{y}{\left((-1+x)^2 + y^2\right)^{3/2}} - y\left(-\frac{3x}{(x^2 + y^2)^{5/2}} - \frac{1}{(x^2 + y^2)^{3/2}}\right)\right) \mathbf{v}_{4,k-6}$$

$$+ \left(\frac{3(-1+x) \left(-2\cos \omega' t \left(-1+x\right) - 2y \sin \omega' t\right)}{2\left((-1+x)^2 + y^2\right)^{5/2}} + \frac{\cos \omega' t}{\left((-1+x)^2 + y^2\right)^{3/2}}\right) \mathbf{v}_{3,k-11}$$

$$+ \left(\frac{3y(-2\cos \omega' t \left(-1+x\right) - 2y \sin \omega' t\right)}{2\left((-1+x)^2 + y^2\right)^{5/2}} + \frac{\sin \omega' t}{\left((-1+x)^2 + y^2\right)^{3/2}}\right) \mathbf{v}_{4,k-11}.$$
(8.14)

Higher orders can be computed analogously, and these computations to $O(\epsilon^{11})$ are produced in the Mathematica notebook IntegratingVectors04-CaptureProblem-mod.nb. The orders shown here are sufficient, however, to illustrate the pattern of the couplings between orders in the expansion, which has been emphasized in the above notation.

A few observations may be made:

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- The equations are the same for k = 1, ..., 5, corresponding to the influence of the 2^{nd} primary not entering until $O(\varepsilon^6)$. Thus the first perturbing terms enter at $O(\varepsilon^6)$ due to the influence of mass m_2 (Earth) at $x = 1 \mu$.
- The next order of the expansion adds no perturbations and so just follows the pattern, while the second perturbing term now enters at $O(\varepsilon^8)$. This term represents only the O(1) contribution of the expansion of the m_3 terms in ε , and introduces no time-dependence.
- The pattern again stays the same through orders 9 and 10, whereupon the first *time-dependent* correction by the $O(\epsilon^3)$ term from the time-dependent part of the equations of motion is observed at $O(\epsilon^{11})$.
- Beyond this the expressions of course continue to increase in complexity as different parts of the expansions of each mass m_i 's contribution interact with themselves and each other. Note that the patterns do, of course, continue to build on the basis of equations (8.11)–(8.14).

Common Parts of Equations of Condition

The actual equations of condition are of course simply equations (4.22) and (4.23); the key to finding valid integrating vectors lies in satisfying the latter, and that system of $\frac{1}{2}4(4+1) = 10$ equations can be compactly stated as follows, at

each order k:

$$\frac{\partial \mathbf{v}_k}{\partial t} = -\nabla [\mathbf{v} \cdot \mathbf{f}]_k \quad 4 \text{ equations,}$$
$$\frac{\partial \mathbf{v}_{j,k}}{\partial x_i} = \frac{\partial \mathbf{v}_{i,k}}{\partial x_j} \quad 1 \le i < j \le 4 \quad \text{a further 6 equations}$$

This is in general a time-dependent system, though in the case of *e.g.* the Jacobi integral of chapter 7 we took precisely a stationary (time-independent) solution of these equations at each order k in the construction (and likewise for the 10 integrals of the 3-body problem treated in chapter 6).

8.2.4 On the Construction of the Integral Approximation

With the above tools in hand, it remains to consider the construction of an integral-approximation parallel to the Jacobi integral of the previous chapter. Recall that the form any such integral must take (if it exists) was given by equation (8.5). On this basis we find that the equations of condition are separable into two parts, one reproducing \mathcal{I} and only the latter contributing to the corrections:

$$\begin{aligned} \mathcal{I}' &= \frac{1}{2}V^2 - \bar{\Omega}(t) + \varepsilon^{k_0 + k_\eta} \cdot \int (\cdots) dt \\ &= \mathcal{I} - \frac{\upsilon}{r_3(t)} + \varepsilon^{k_0 + k_\eta} \cdot \int (\cdots) dt , \\ &= \mathcal{I} - \underbrace{\varepsilon^8 \left(1 - \varepsilon^6 + \mathcal{O}\left(\varepsilon^{12}\right) \right) \cdot \left(\frac{1}{r_3(t;\varepsilon)}\right) + \varepsilon^{11} \left(1 - \varepsilon^3 + \mathcal{O}\left(\varepsilon^9\right) \right) \cdot \int (\cdots) dt }_{\text{Capture Problem Correction}}, \end{aligned}$$
(8.15)

which has been obtained using the ϵ -dependencies as in the discussion thus far, such that:

$$\varepsilon^{k_{v}+k_{\eta}} \cdot \int (\cdots) dt = \varepsilon^{3} \left(\varepsilon^{8} - \varepsilon^{14} + O\left(\varepsilon^{20}\right) \right) \cdot \int (\cdots) dt$$

Remark also that in the above $1/r_3(t)$ depends explicitly on ε as well, though this term has leading O(1), and so does not change the leading order of the expression. This is seen in expression (8.15) to be $O(\varepsilon^8)$ for a single time-independent term which is easy to find, and a common $O(\varepsilon^{11})$ in all time-dependent terms, which pose considerably more difficulty, as will become clear.

The discussion that follows is split into two parts, each playing a role at *every* order *k* of the expansion: the discussion of the homogeneous part which is reflected in terms multiplying the $v_{1,...,4;k}$ -term in expressions (8.11)–(8.14), and the inhomogeneous part reflecting the choices of integrating vector components at previous orders of the expansion.

Homogeneous Part

At every order *k*, on substituting the earlier terms $v_{i,j}$, j = 0, ..., k - 1, we obtain the PDE equations for $v_{i,k}$, and find them composed of a common homogeneous part and inhomogeneous terms resulting from the substitution of earlier $v_{i,j}$. The homogenous part of the PDE's in time at every order derives from:

$$\frac{\partial \mathbf{v}_k}{\partial t} = -\nabla [\mathbf{v} \cdot \mathbf{f}]_k ,$$

where:

$$\mathbf{v} \cdot \mathbf{f} = \dot{x} \mathbf{v}_{1,k} + \dot{y} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{\left(x^2 + y^2\right)^{1/2}} \right) \mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{\left(x^2 + y^2\right)^{1/2}} \right) \mathbf{v}_{4,k} \,.$$
(8.16)

Taking the gradient one obtains the homogeneous PDE system for $v(\mathbf{x}, t)$:

$$\partial_{t} \mathbf{v}_{1,k} = \dot{x} \,\partial_{1} \mathbf{v}_{1,k} + \dot{y} \,\partial_{1} \mathbf{v}_{2,k} + \left(1 - \partial_{1} \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \mathbf{v}_{3,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \partial_{1} \mathbf{v}_{3,k} \\ -\rho \,\partial_{1} \left(\frac{y}{(x^{2} + y^{2})^{1/2}}\right) \mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \partial_{1} \mathbf{v}_{4,k} , \qquad (8.17a)$$

$$\partial_{t} \mathbf{v}_{2,k} = \dot{x} \,\partial_{2} \mathbf{v}_{1,k} + \dot{y} \,\partial_{2} \mathbf{v}_{2,k} - \partial_{2} \left(\frac{x}{(x^{2} + y^{2})^{1/2}} \right) \mathbf{v}_{3,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}} \right) \partial_{2} \mathbf{v}_{3,k} + \left(1 - \partial_{2} \frac{y}{(x^{2} + y^{2})^{1/2}} \right) \mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}} \right) \partial_{2} \mathbf{v}_{4,k} ,$$
(8.17b)

$$\partial_{t} \mathbf{v}_{3,k} = \mathbf{v}_{1,k} + \dot{x} \,\partial_{3} \mathbf{v}_{1,k} + \dot{y} \,\partial_{3} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{3} \mathbf{v}_{3,k} \\ - 2\mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{3} \mathbf{v}_{4,k} , \qquad (8.17c)$$

$$\partial_{t} \mathbf{v}_{4,k} = \dot{x} \,\partial_{4} \mathbf{v}_{1,k} + \mathbf{v}_{2,k} + \dot{y} \,\partial_{4} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{4} \mathbf{v}_{3,k} \\ + 2\mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{4} \mathbf{v}_{4,k} \quad \text{and}$$
(8.17d)

 $\partial_i \mathbf{v}_j = \partial_j \mathbf{v}_i$ for an additional 6 equations. (8.17e)

While we showed previously that the Jacobi integral may be constructed precisely a time-independent solution of this system, at present, there is to the best of the author's knowledge no general (time-dependent) solution for this coupled system of PDE's in the literature, nor have the author's best efforts yielded one within the limited time-frame of work on this thesis (cf. section 8.3).

However, these equations are the same in both the CR3BP and the Capture Problem, and form the basis of the expansion which was proven to lead to the construction of \mathcal{I} in the former problem. As noted in the previous section, that would necessarily be a part of an analogous integral in the current problem, and so it stands to reason that the correct choice in this approach is the same as it was there, choosing again:

$$\mathbf{v}_{0} = \begin{pmatrix} -\left(x - \frac{x}{(x^{2} + y^{2})^{3/2}}\right) \\ -\left(y - \frac{y}{(x^{2} + y^{2})^{3/2}}\right) \\ \frac{\dot{x}}{\dot{y}} \end{pmatrix},$$
(8.18)

which is easily checked to satisfy the equations of condition, thus leading to the O(1) integral approximation

,

$$I_0 = \int \mathbf{v}_0 \cdot d\mathbf{x} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} (x^2 + y^2) - \frac{1}{(x^2 + y^2)^{1/2}}$$
(8.19)

as before. Likewise, the usual remarks on uniqueness apply, as only the homogeneous part is relevant for order $k = 0, \dots, 5$; at each order we could of course pick analogous factors, but these only contribute a constant, rather than any new information.

Inhomogeneous Part

In both the CR3BP and in the Capture Problem, at higher orders of ε , these homogenous equations are extended in the following way (which was also exploited in proposition 7.2.1):

$$[\mathbf{v} \cdot \mathbf{f}]_{k} = \dot{x} g_{k}(x, y, t) + \dot{y} h_{k}(x, y, t) + \dot{x} \mathbf{v}_{1,k} + y_{4} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \mathbf{v}_{4,k} .$$
(8.20)

The functions g_k , h_k , notably, now depend on the first two coordinates *and* time in general, and contain terms coming from the lower-order expressions for $v_{j \le k}$. When searching for the stationary parts of the solution, we may take:

$$\mathbf{v}_{1,k} = -g_k(x, y),$$
 (8.21a)

$$v_{2,k} = -h_k(x, y),$$
 (8.21b)

$$v_{3,k} = 0,$$
 (8.21c)

$$\mathsf{v}_{4,k} = \mathbf{0},\tag{8.21d}$$

for those parts which do not contain a time-dependence, which gives approximations which in the circular restricted 3-body problem yielded the Jacobi integral (with $\varepsilon \longrightarrow \varepsilon^6$ for the effects due to m_2).

In the capture problem, however, there are *new* time-independent terms due to m_3 starting at $O(\varepsilon^8)$ which can be found in exactly the same way, but also new terms which are periodic and time-dependent, starting at $O(\varepsilon^{11})$, which are numerous.

Taking for example equation (8.14):

$$\begin{split} [\mathbf{v} \cdot \mathbf{f}]_{11} &= \dot{x} \, \mathbf{v}_{1,11} + \dot{y} \, \mathbf{v}_{2,11} + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y} \right) \mathbf{v}_{3,11} + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x} \right) \mathbf{v}_{4,11} \\ &- \frac{(x-1) \, \mathbf{v}_{3,3}}{\left((-1+x)^2 + y^2 \right)^{3/2}} + \left(- \frac{-1+x}{\left((-1+x)^2 + y^2 \right)^{3/2}} + \frac{3x^2}{(x^2 + y^2)^{5/2}} - \frac{1-x}{(x^2 + y^2)^{3/2}} \right) \mathbf{v}_{3,5} \\ &- \frac{y \, \mathbf{v}_{4,3}}{\left((-1+x)^2 + y^2 \right)^{3/2}} + \left(- \frac{y}{\left((-1+x)^2 + y^2 \right)^{3/2}} - y \left(- \frac{3x}{(x^2 + y^2)^{5/2}} - \frac{1}{(x^2 + y^2)^{3/2}} \right) \right) \mathbf{v}_{4,5} \\ &+ \left(\frac{3(-1+x) \left(-2\cos \omega' t \left(-1+x \right) - 2y \sin \omega' t \right)}{2 \left((-1+x)^2 + y^2 \right)^{5/2}} + \frac{\cos \omega' t}{\left((-1+x)^2 + y^2 \right)^{3/2}} \right) \mathbf{v}_{3,0} \\ &+ \left(\frac{3y(-2\cos \omega' t \left(-1+x \right) - 2y \sin \omega' t \right)}{2 \left((-1+x)^2 + y^2 \right)^{5/2}} + \frac{\sin \omega' t}{\left((-1+x)^2 + y^2 \right)^{3/2}} \right) \mathbf{v}_{4,0} \,, \end{split}$$

it is clear that these time-dependent terms complicate the homogenous equations considerably. One can construct a partial solution in the way outlined above (using pattern cancellation as with the Jacobi integral in the previous chapter),

which then eliminates all terms but the time-dependent ones, leaving:⁴

$$\begin{split} [\mathbf{v} \cdot \mathbf{f}]_{11}^* &= \dot{x} \, \mathbf{v}_{1,11}^* + \dot{y} \, \mathbf{v}_{2,11}^* + \left(x - \frac{x}{(x^2 + y^2)^{3/2}} + 2\dot{y} \right) \mathbf{v}_{3,11}^* + \left(y - \frac{y}{(x^2 + y^2)^{3/2}} - 2\dot{x} \right) \mathbf{v}_{4,11}^* \\ &+ \left(\frac{3\left(-1 + x\right) \left(-2\cos\omega' t \left(-1 + x\right) - 2y\sin\omega' t \right)}{2\left(\left(-1 + x\right)^2 + y^2 \right)^{5/2}} + \frac{\cos\omega' t}{\left(\left(-1 + x\right)^2 + y^2 \right)^{3/2}} \right) \dot{x} \\ &+ \left(\frac{3y\left(-2\cos\omega' t \left(-1 + x\right) - 2y\sin\omega' t \right)}{2\left(\left(-1 + x\right)^2 + y^2 \right)^{5/2}} + \frac{\sin\omega' t}{\left(\left(-1 + x\right)^2 + y^2 \right)^{3/2}} \right) \dot{y} \,. \end{split}$$

With regard to possible solutions of these remaining equations, the following may be remarked:

- 1. It is important to recall that the equation given is essentially a stand-in for $\partial_t I_{11} = -[\mathbf{v} \cdot \mathbf{f}]_{11}$, and so realize that we cannot in general choose $\mathbf{v}_{1,11}(t)$, $\mathbf{v}_{2,11}(t)$ in such a way that they simply cancel the terms above to yield $\partial_t I_{11} \equiv 0$. Since the calculation proceeds by effectively setting $I_{11} = \cdots + \int \frac{\partial I_{11}}{\partial t} dt$, this would, barring some unlikely serendipitous cancelation at integration time, lead to $I_{11}(t) = \text{const}$ and thus contradiction.
- 2. A different approach would involve the method of averaging (*cf.* section 2.3.2), but without going into detail it is immediately clear that:

$$\int_{T=\frac{2\pi}{\omega'}} \left(g_1(x,y) \cos \omega' t + g_2(x,y) \sin \omega' t \right) \mathrm{d}t = 0$$

due to the inherent periodicity of the terms in $[\mathbf{v} \cdot \mathbf{f}]_{11}^*$. This also holds for higher harmonics (*e.g.* $\sin(m\omega' t)$ for integer m > 1), and so this approach yields no new information, as what remains after averaging is simply the homogeneous equations.

This result can be extended somewhat, as an observation regarding the project of averaging itself. It tells us, in effect, that while we can average the equations, we remain with only the time–independent terms due to the Moon since the averages of all harmonics are in principle 0. These terms have an interesting geometric interpretation, illustrated in figure 8.1.

Essentially, they represent the addition of the mass of the Moon (as an $O(\epsilon^8(1+\cdots))$ term) to the influence of the Earth, which corresponds geometrically to the simple harmonic motion of the Moon around the average (*i.e.* fixed) position of the Earth in the rotating coordinate system of our model.

3. A third approach rests on remarking that as the time-derivative of the equations contains periodic terms, it is reasonable to expect that any correct integrating vector must contain the same periodicity, albeit via a combination of higher harmonics of the base frequency ω' .

In light of the above, would be to try Fourier series instead of simply straightforward matching of the lowest harmonic, which does not work. This idea is analogous to the method used in [Andreu, 1998], noted earlier

⁴The * superscript here denotes that these terms are to be added in superposition to the partial solution obtained by the procedure outlined above.



Figure 8.1: Averaging of Moon's simple harmonic motion about Earth in Capture Problem rotating coordinate system

in section 3.5, and we discuss this idea and the link with Andreu's approach in more detail below in section 8.3.2.

Thus it appears that at least for the moment the question remains open, and the strongest result that can be stated incorporates the time-independent m_3 -term at $O(\varepsilon^8)$ in addition to the $m_{1,2}$ results of the Jacobi problem:

$$I(x,y,\varepsilon) = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) - \frac{1}{2} \left(x^2 + y^2 \right) - \frac{1}{\left(x^2 + y^2 \right)^{1/2}} - \left(\frac{1}{\left(x^2 + y^2 \right)^{1/2}} - \frac{x}{\left(x^2 + y^2 \right)^{3/2}} - \frac{1}{\left(x^2 + y^2 \right)^{1/2}} \right) \varepsilon^6 - \frac{1}{\left((x-1)^2 + y^2 \right)^{1/2}} \varepsilon^8 + O\left(\varepsilon^{11} \right) ,$$

$$(8.22)$$

where time–dependent and time–independent terms mix starting at $\mathcal{O}(\epsilon^{11})$.

We remark, in that light, that it does not make sense to pursue the analysis further (*i.e.* isolating the time–independent terms, as we did with expression (8.14)) without being able to solve the time–dependent parts of the integral. Given that those terms may in general become influential to the same magnitude as any time–independent terms we *could* solve for, further work would not necessarily increase the accuracy of the approximation.

8.3 Approximations of New Integrals

In this final section, we briefly consider the issue of finding approximations of new integrals using the method of integrating vectors. The reader will remark that the negative results of the previous section give cause for a measure of pessimism, but there is reason for some optimism as well.

8.3.1 Full Equations of Condition

Before the brief discussion below, however, let us state the full set of equations (4.23) as applied to the capture problem, for the case when the scope of the approach is not limited to extensions of the Jacobi integral. The foundation of these is a slightly more general version of expression (8.20):

$$[\mathbf{v} \cdot \mathbf{f}]_{k} = g_{k}(x, y, t, \mathbf{v}_{3,0 \le l < k}) + h_{k}(x, y, t, \mathbf{v}_{4,0 \le l < k}) + +\dot{x}\mathbf{v}_{1,k} + \dot{y}\mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right)\mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right)\mathbf{v}_{4,k},$$
(8.23)

where the inhomogeneous terms have been split into 2 terms which act as more general versions of the g_k , h_k introduced earlier. Instead of always multiplying $v_{3,0}$ resp. $v_{4,0}$, these now collect the terms multiplying $v_{3,0 \le l < k}$ and $v_{4,0 \le l < k}$, representing the couplings to earlier orders (l < k), including possible time–dependence. The equations of condition then become:

$$\partial_{t} \mathbf{v}_{1,k} = \dot{x} \,\partial_{1} \mathbf{v}_{1,k} + \dot{y} \,\partial_{1} \mathbf{v}_{2,k} + \left(1 - \partial_{1} \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \mathbf{v}_{3,k} + \left(x + 2\dot{y} - \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \,\partial_{1} \mathbf{v}_{3,k} \\ - \,\partial_{1} \left(\frac{y}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}}\right) \,\partial_{1} \mathbf{v}_{4,k} + \,\partial_{1} g_{k} + \,\partial_{1} h_{k} \,, \tag{8.24a}$$

$$\partial_{t} \mathbf{v}_{2,k} = \dot{x} \,\partial_{2} \mathbf{v}_{1,k} + \dot{y} \,\partial_{2} \mathbf{v}_{2,k} - \partial_{2} \left(\frac{x}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \mathbf{v}_{3,k} + \left(x + 2\dot{y} - \frac{x}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \,\partial_{2} \mathbf{v}_{3,k} \\ + \left(1 - \partial_{2} \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{\left(x^{2} + y^{2}\right)^{1/2}} \right) \,\partial_{2} \mathbf{v}_{4,k} + \partial_{2} g_{k} + \partial_{2} h_{k} \,, \tag{8.24b}$$

$$\partial_{t} \mathbf{v}_{3,k} = \mathbf{v}_{1,k} + \dot{x} \,\partial_{3} \mathbf{v}_{1,k} + \dot{y} \,\partial_{3} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{3} \mathbf{v}_{3,k} \\ - 2\mathbf{v}_{4,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{3} \mathbf{v}_{4,k} + \,\partial_{3} g_{k} + \,\partial_{3} h_{k} , \qquad (8.24c)$$

$$\partial_{t} \mathbf{v}_{4,k} = \dot{x} \,\partial_{4} \mathbf{v}_{1,k} + \mathbf{v}_{2,k} + \dot{y} \,\partial_{4} \mathbf{v}_{2,k} + \left(x + 2\dot{y} - \frac{x}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{4} \mathbf{v}_{3,k} \\ + 2\mathbf{v}_{3,k} + \left(y - 2\dot{x} - \frac{y}{(x^{2} + y^{2})^{1/2}}\right) \,\partial_{4} \mathbf{v}_{4,k} + \,\partial_{4} g_{k} + \,\partial_{4} h_{k} \quad \text{and}$$
(8.24d)

 $\partial_i \mathbf{v}_{j,k} = \partial_j \mathbf{v}_{i,k}$ for an additional 6 equations. (8.24e)

As usual, if an integrating vector v_k can be found which satisfies these equations, it may be integrated out via the relations (4.22) to obtain I_k , the $O(\varepsilon^k)$ term in the integral approximation.

8.3.2 Solution Perspectives

As with the results of the previous chapters, we will now briefly discuss the approaches attempted to the full equations of condition just presented, and conclude with some remarks on further perspectives for solution.

Approaches Attempted

The same general remarks apply as in the case of the Jacobi and circular restricted 3-body problems: our difficulties stem from the necessity of solving a $\frac{1}{2}n(n+1)$ -dimensional system of coupled linear 1st -order PDE equations, for which there is no well-developed solution strategy. In place of this, the usual approaches were tried, with the following results.

- Here too, an approach using the method of characteristics was shown to be fruitless, as the characteristic ODE system that the method yields is simply the original ODE system.
- The equations of the system were again checked for similarity against [Polyanin and Zaitsev, 2003] for the existence a (perhaps more complicated) equation-type which might contain our problem as a special case, and provide a viable solution strategy; this was again without fruit.
- As in the previous chapter, we note as hopeful the attempt to find a solution using the straightforward extension of the method of separation of variables, taking *e.g.*:

$$\mathbf{v} = (\,\hat{\mathbf{v}}_1(x)\bar{\mathbf{v}}_1(t),\,\dots,\,\hat{\mathbf{v}}_4(x)\bar{\mathbf{v}}_4(t)\,)^{\mathsf{T}} \,. \tag{8.25}$$

This has likewise met with little success within the time-constraints of work on this thesis.

• Lastly, of course, the approach of section 8.2 was tried, and we have described in detail the difficulties involved, due to the time-dependence of the modified potential $\bar{\Omega}(t)$.

Further Perspectives

In light of the discussion of the problems arising in the already restricted approach of section 8.2, where we sought an analogue of the Jacobi integral, the strongest option for finding a time-dependent solution to the equations for the integrating vectors would seem to lie with the final approach suggested there: the use of Fourier techniques expanding in higher harmonics of the frequency ω' of the Moon's motion.

The motivation for this is that while the problem is perturbed from the simpler and already difficult circular restricted 3-body problem, the perturbation is of a particularly simple periodic nature. While in the present work we found that simple sine/cosine terms cannot satisfy the equations (4.23), Andreu's work in [Andreu, 1998] suggests that superposition of such terms may be adequate to the task, recalling that he considered a similar problem in seeking near-circular, near-oscillatory motion of the Moon about the Earth as a true (rather than approximate) solution of the 3-body problem.

Thus, though his approach is sufficiently involved that it must be seen as beyond the scope of this thesis, we highly recommend its investigation in any future work.

Part III

Integrator Performance

Outline of Part III

The final part of this thesis (which comprises the second major part of the thesis for Aerospace Engineering) is concerned with the implementation of approximations of astrodynamic first integrals in conservative integration schemes, and their performance relative to baseline Runge–Kutta–(Fehlberg) schemes.

- 1. In chapter 9, general design considerations will be discussed against the background of the motivation for conservative integrators. In particular, we introduce the design approach, the rationale of the integral–benchmarks used, and give a summary of the coding issues involved, which are treated in more detail in appendix C.
- 2. In chapter 10, the design of integrators which are conservative with regard to the exact (known) first integrals is discussed for the Jacobi 3-body problem and the circular restricted 3-body problem. Our approach is based on Bowman et al.'s earlier work on integrators conservative of the exact energy integral for the 3-body problem [Shadwick et al., 1999, Kotovych and Bowman, 2002].

We present results for the Sun–Earth–Moon and Earth–Moon–Satellite systems and give considerably more detailed performance data than Bowman *et al.*, as well as considering three different formulations of the integrators and their relative merits. We discuss observed instabilities in a particular class of problems. Further, we also discuss the possibilities for integrators which are multiply conservative within Bowman *et al.*'s framework.

3. In the final chapter 11, the approach of the previous chapter is extended to the use of approximations of integrals, rather than the actual integrals, motivated by the capture problem for which we have only such an approximation (from Part II of this thesis). We illustrate the approach using the circular restricted 3–body problem first (as we have the full integral available), and then present the results for the capture problem.

Chapter 9

Conservative Integrator Design Notes

This chapter sets the stage for the results presented in the next two chapters by briefly recapping the background raised earlier in chapter 5. We then present two related sections, the first summarizing issues related to the design of the code used to obtain the results of the following chapters, and the second giving a brief discussion of the approach taken in evaluating integrator performance.

9.1 Background

The general issues and contrasts between classical and conservative methods have already been summarized in chapter 5, and the reader will recall that conservative methods are relatively flexible, in that they can be combined with both single-step and multi-step methods, provided the integrator prototype can be cast in the form of proposition 5.3.1.

In the following chapters we will focus on conservative formulations based on a simple prototype 2^{nd} –order predictor– corrector algorithm, following [Kotovych and Bowman, 2002]. The remarks of section 5.3 remain applicable to higher– order and multi–step methods, but these are beyond the scope of present work, as our goal is to establish the feasibility and possible desirability of the conservative–integration approach even at low orders in accuracy.

Note that for the conservative integration algorithms, we reference the discussion in [Shadwick et al., 1999, 2001], [Kotovych and Bowman, 2002] and we have coded a number of implementations of the algorithms for our few–body astrodynamics problems. For comparison we take primarily reference Runge–Kutta–Fehlberg integrators which are standard implementations of the Netlib package RKSuite, documented in [Brankin et al., 1993], and based on the theory discussed in [Fehlberg, 1968, 1969, Shampine et al., 1976, Burden and Faires, 2001].

9.2 General Design Notes

In the consideration of exactly and approximately conservative integration schemes, we have coded an extensive suite of integrators for the Jacobi 3-body problem, the circular restricted 3-body problem and the capture problem introduced in chapter 3. This IntegrationMethods code is in a sense a 'deliverable' for this thesis project as a whole, and this section will discuss a number of design issues relevant to the development.

A more detailed discussion of the simulation suite itself is given in appendix C.

Remark also that when discussing the integration schemes, that by "implicitly–formulated" and "explicitly–formulated", we intend here the design choice of whether to formulate the coordinate transformations of the conservative scheme with explicit coordinates (making one of the coordinates $\xi_i = x_i + U$), or whether to treat one of the coordinates implicitly, sacrificing it for increased control over the potential (taking $\xi_i = U$ and $x_i = g(x_{j\neq i}, U(x_{j\neq i}, x_i))$ implicitly, which must be reconstructed using *e.g.* Newton-Raphson nonlinear root–finding).

9.2.1 Models and Methods

The models and methods implemented for IntegrationMethods are the following:

- 1. For the planar Jacobi 3-body problem model, the methods:
 - (a) The Runge-Kutta-Fehlberg pairs 2(3), 4(5) & 7(8) from RKSuite.
 - (b) An explicitly formulated implementation of Bowman's pJacobi-integrator in natural Jacobi coordinates.
 - (c) An implicitly formulated implementation of Bowman's pJacobi-integrator in natural Jacobi coordinates.
 - (d) An implicitly formulated implementation of Bowman's pJacobi–integrator in polar Jacobi coordinates (which should be conservative of both the energy and angular momentum).

Models were implemented for both Sun-Earth-Moon and Earth-Moon-Satellite test problems.

- 2. For the circular restricted 3-body problem model, the methods:
 - (a) The Runge-Kutta-Fehlberg pairs 2(3), 4(5) & 7(8) from RKSuite.
 - (b) An explicitly formulated implementation of Bowman's CR3BP–integrator in natural dimensionless model coordinates (rather than canonical variables corresponding to the Hamiltonian formulation).
 - (c) An approximately conservative integrator developed specifically for this model, with approximation order $O(\varepsilon^k)$, k = 0, ..., 3.

Models were implemented for both Sun-Earth-Moon and Earth-Moon-Satellite test problems.

- 3. For the capture problem model, the methods:
 - (a) The Runge-Kutta-Fehlberg pairs 2(3), 4(5) & 7(8) from RKSuite.
 - (b) An approximately conservative integrator developed specifically for this model.
- 4. For the ephemeride model coded for and used in the Internship report [Verzijl, 2005]:
 - (a) Burkardt's [Burkardt, 2004] 4(5) Runge-Kutta-Fehlberg integrator, for reference.
 - (b) The Runge-Kutta-Fehlberg pairs 2(3), 4(5) & 7(8) from RKSuite.

9.2.2 Conceptual Design Issues

Next, we discuss a number of conceptual issues arising in the design of the integrators:

• An important first remark is reserved for the lack of actual implementations in the Bowman *et al.* papers,¹ and as will become clear in the next chapter, this leaves considerable ambiguity with regard to the design choices not determined by the integration scheme. We discuss 3 different implementations there, and will give some suggestions as to best practice.

¹The author has had brief communication with dr. Bowman, but the issues encountered have not yet been fully resolved.

- It should also be remarked in light of the above that the implementations have been coded primarily for accuracy at this point, and considerable time was necessary for debugging. As a result, they have explicitly *not* been performance-tuned yet, and may be somewhat slower in execution than is strictly necessary. Note, however, that we will gauge performance mostly by function evaluations and RMS-error estimates rather than CPU time, and so while this would be an important step in the future development of the code, it is highly unlikely to qualitatively change our present conclusions.
- The rationale for the algorithms introduced in the previous section was the following:
 - 1. We have made 3 different implementations of Bowman's algorithms for the planar Jacobi 3–body problem in order to develop both experience and intuition for the differences between them, as well as to get a better picture of the relative differences in performance between the three approaches.
 - 2. We have chosen to work towards our interest in the capture problem, and for that reason developed approximate implementations only for it and the circular restricted 3–body problem; this in the expectation that the behavior on the planar Jacobi problem would not be qualitatively different.
 - 3. We have based our test problems on actual ephemeride data for initial conditions where applicable, and have included reference ephemeride–model code for a 3–body problem and a 4–body problem based on the "internship code" described in [Verzijl, 2005].
- There is an open issue regarding the choice of appropriate benchmarks. The ideal case for any numerical method is to benchmark against a known solution, which is unavailable to us. In the case of the RKF integrators, one can presumably test the accuracy against Kepler orbits, though we must immediately raise the objection that the results on such stable 2-body problem solutions are by no means necessarily representative for the 3-body problem-dynamics, let alone the dynamically sensitive trajectories of the 4-body capture problem that motivated this work. We return to this issue below in section 9.3.
- Lastly we should remark that with our approach to benchmarking and the choice of algorithms, we are essentially making an "unfair" comparison: we compare a 2nd –order *fixed* step–size scheme with *variable* step–size Runge–Kutta integrators that implement error–control, and within this category 2 out of 3 of the integrators we compare with are also higher-order methods in the step–size. Our goal with this is to provide what might be thought of as worst–case performance data, in that changing either aspect of this unfair comparison is likely to result in significant improvements in performance, while the current formulations should demonstrate the key potential drawbacks to the conservative schemes as a class of integrators.

Remark that to somewhat balance this unfair comparison, we have chosen for a relative error tolerance of 10^{-6} for the RKF–methods, corresponding to roughly 1 mm in physical units, in order to give our "error–naive" conservative schemes a "fighting chance."

• The reader is nonetheless advised to recall that a conservative scheme can easily be formulated for higher–order methods, and in particular that we can formulate such methods using error–control as well, by choosing multi-step predictor–corrector methods as our prototype, *cf.* section 5.2.3.

9.2.3 Coding and Validation Issues

The following are more practical issues concerning the actual coding and validation of the IntegrationMethods suite, which also bear remarking:

- The front-end to the IntegrationMethods suite of implementations is the FORTRAN code IntegrationMethods.f90, and the suite and its structure are described in more detail in appendix C.
- With regard to the validation of the code, we have followed and built on the approach taken in [Verzijl, 2005]:
 - Testing RKF2(3), 4(5) & 7(8) implementations against test-cases from the Ephemeride problems in the above document, which were obtained using Burkardt's RKF 4(5) and validated there.
 - Testing new integrators against test-cases based on two standard sets of initial conditions with trajectories verified with the RKF7(8) codes from the Netlib RKSuite.
 - Testing for scaling with time-step, and due to the formulation of the conservative schemes on the basis of an explicit predictor-corrector method, it should be remarked that there is a model-dependent threshold for the time-step to ensure the stability of the integration; see also section 9.3 below.
 - Testing scaling with satellite mass (using masses below a threshold of $O(10^{20} \text{ kg})$ at which the "satellite" would be expected to noticeably perturb the primaries).
 - Testing convergence of the Newton–Raphson process for the implicitly–formulated conservative integration schemes.
 - We have also coded coordinate-transform functions based on work in the above document, and tested different methods with the same initial conditions for comparability of results in multiple coordinate systems, all validated qualitatively against comparable ephemeride-model results.
- As remarked in the previous section, the code has not yet been performance tweaked, and so despite a large measure of standardization across the modules and procedures, it is to be expected that there remains considerable room for improvement in terms of CPU time.

Particularly relevant in this regard is the potential speed–up resulting from shifting the code back to double– precision for actual orbit analysis. Before doing this, however, due consideration should be given to the question of whether terms are appropriately balanced in the integrator to avoid loss of significance at double precision (particularly when treating the potential term U, cf. the discussion of algorithms 10.4.1 and 10.4.2).

• Lastly, we remark that the astrodynamic constants used in the new integrators have been taken from [Montenbruck and Gill, 2001], and are standardized across all models and methods.

9.3 On Performance Evaluation

Lastly, before moving on to the results, we will briefly discuss the methods available for the evaluation of the performance of the integrators for our problems. This is a surprisingly thorny issue, and Berry & Healy give an excellent summary, [Berry and Healy, 2003], of the methods available for the accuracy assessment of numerical trajectory integration.

9.3.1 Approaches to Performance Evaluation

We discuss first the primary approaches in common use,² in relation both to our validation process and the performance discussion of the next chapters. These are:

- 1. Testing against a problem with a known analytical solution, typically a Kepler 2–body problem. The difficulty with this in our present work is that the integrators we consider are model–specific, tailored algorithms, and so cannot usefully be tested in this way.
- 2. Comparison with integral invariants of the motion, which while we do not make use of in the traditional manner, is of course intimately connected to the present work. The key difference is that we are trying to make the conservation of the integral internal to the method as opposed to functioning as an external benchmark.
- 3. Time–reversal of the trajectory, a method which is not universally accepted, due to its susceptibility to (*i.e.* failure to identify) time–reversible integration errors. This approach has not played a role in the present work.
- 4. Step-size halving, as a test of the convergence of the solution, which is based on the fact that the local truncation error is step-size dependent (by convention $\tau = O(h^p)$ for a *p*-order integrator); as noted in section 9.2.3, we have indeed used this method to validate trajectory integration in the validation phase of both present and previous work.
- 5. Comparison with a higher–order integrator; as remarked in the same section 9.2.3, we have chosen to make extensive use of this technique in the present work, as will be clear from the results presented in the next chapters.

9.3.2 On Comparison with Higher–Order Integrators

However, it is incumbent upon us to point out a critical weakness in this last approach, which we must consistently bear in mind:

- First of all, it assumes that the higher–order integrator is *better* than the integrator being tested, and even in present work, while it is tempting to consider *e.g.* RKF7(8) as a "gold–standard" against which to test conservative schemes, it is far wiser to consider correspondence between the results of the two methods, validated independently,³ as confirmation that RKF7(8) is giving correct results, and vice–versa.
- Secondly, and more subtly, if both the integrator in testing or validation and the higher–order integrator share a common fundamental approach, *e.g.* both based on expansions in terms of Taylor series, presuming smoothness of the solutions near each starting point, then both will be susceptible to pathological trajectories which cannot be handled on the basis of those fundamentals.

 $^{^{2}}$ An interesting method also discussed in [Berry and Healy, 2003] is Zadunaisky's method, which involves the construction of an analytic function near an integrated solution, and then the construction of corresponding differential equations which together with the function form a quasi–exact solution in the neighborhood of interest, which can be used in the same way as testing against a known solution such as the Kepler 2–body problem solution. Its application is involved however, and will not be discussed here.

 $^{^{3}}$ In particular also ideally with a conservative scheme of the same order, which is, however, somewhat beyond the scope of this thesis.

Moreover, while tempting to reach for exotic regions in complicated dynamical systems, a very telling and *much* simpler example of this type of pathology is already clearly visible in the following system.

Example 9.3.1 (Exponential Decay and Singular Perturbations) Consider the system:

$$\varepsilon \frac{dx}{dt} = -x$$
 with initial condition:
 $x(0) = x_0$
(9.1)

It is easy to find, separating as $\frac{dx}{x} = -\frac{dt}{\varepsilon}$, that the solution is given by:

$$\mathbf{x}(t) = \mathbf{x}_0 \ e^{-\frac{t}{\varepsilon}} \ . \tag{9.2}$$

However, let us now consider the effect of discretization on the solution. We reprise the notation of chapter 5. *Any* correct discretized solution *must* behave (with time–step h at iteration n) as:

$$w^{(n)} = w^{(0)} e^{-\frac{n\hbar}{\varepsilon}} . (9.3)$$

Suppose first that we integrate using Heun's implicit discretization (analogous to the Trapezoidal rule, *cf.* [Burden and Faires, 2001, chapter 4]), with time–step *h*:

$$\varepsilon \frac{w^{(n+1)} - w^{(n)}}{h} = -\frac{1}{2} \left(w^{(n)} + w^{(n+1)} \right) \quad \text{such that:}$$
$$w^{(n+1)} = \left(\frac{\varepsilon - \frac{h}{2}}{\varepsilon + \frac{h}{2}} \right) w^{(n)} .$$

This is a difference equation with solution:

$$w^{(n)} = \left(\frac{\varepsilon - \frac{h}{2}}{\varepsilon + \frac{h}{2}}\right)^n w^{(0)} = e^{n \ln\left(\frac{\varepsilon - \frac{h}{2}}{\varepsilon + \frac{h}{2}}\right)} w^{(0)}.$$

We remark two things. First, this solution is stable iff. $2\varepsilon > h$, otherwise we get power–law decay modulated by oscillatory behavior about 0, which is certainly not present in the solution (9.2). Second, it is clear that this is a qualitatively different solution than what we expected. In fact we remark that for any *h*, as $\varepsilon \downarrow 0$:

$$\begin{split} \lim_{\epsilon \downarrow 0} \frac{nh}{\epsilon} &= \mathcal{O}\left(\epsilon^{-1}\right) \quad \text{while} \\ \lim_{\epsilon \downarrow 0} n \ln \left| \frac{\epsilon - \frac{h}{2}}{\epsilon + \frac{h}{2}} \right| &= n \ln \left| 1 + \mathcal{O}\left(\epsilon^{1}\right) \right| = \mathcal{O}\left(\epsilon^{1}\right) \,. \end{split}$$

So the behavior exhibits completely different asymptotics, which means that naive application gives an incorrect solution. Perhaps the issue is the method, after all it is not A–stable. We use the following implicit method, which *is* A–stable, and discretize instead as:

$$\varepsilon \frac{w^{(n+1)} - w^{(n)}}{h} = -w^{(n+1)} \quad \text{such that:}$$
$$w^{(n+1)} = \left(\frac{\varepsilon}{\varepsilon + h}\right) w^{(n)} .$$

This is again a difference equation, with solution:

$$w^{(n)} = \left(\frac{\varepsilon}{\varepsilon + h}\right)^n w^{(0)} = e^{n \ln\left(\frac{\varepsilon}{\varepsilon + h}\right)} w^{(0)} \; .$$

The asymptotic behavior here, in turn, is:

$$\lim_{\epsilon \downarrow 0} n \ln \left(\frac{\varepsilon}{\varepsilon + h} \right) = \mathcal{O}(\ln \varepsilon) \; .$$

This is arguably closer to correct as $\lim \varepsilon \downarrow 0$, and not subject to un-physical oscillations, but still not the true $O\left(\frac{1}{\varepsilon}\right)$ behavior of the analytical solution.

The reason that even this ostensibly "better" method still gets the solution wrong, is that this is a classic example of singularly perturbed behavior. The correct solution combines two qualitatively different regimes: a constant null-solution for almost all time corresponding to the omission of the ε -terms in equation (9.1), and an exponential fall-off from x_0 to 0 over a very short time-space immediately preceding it due to the singular perturbation by the $O(\varepsilon)$ term. The reader will note that this behavior is in fact typical for what are known as stiff equations, and the discussion in *e.g.* [Press et al., 1992] is illustrative of the phenomena.

Equation (9.1) is a pathological system for the method because no matter how small the time–step h is chosen, there will always be cases as $\varepsilon \downarrow 0$ that the method returns errors beyond whatever error bounds are set, simply because the solution it is returning is close but not equal to the true solution, with the discrepancy determined by the value of ε .

The solution though, is intuitive, particularly in light of our discussion in section 4.3: one makes the problematic distance scale explicit. Typically one introduces a new variable *y* such that $y := e^{-\frac{t}{\varepsilon}} x(t)$, making the scale associated with the singular perturbation explicit, and then integrates the new $\varepsilon \frac{d(e^{-\frac{t}{\varepsilon}}x)}{dt} = \cdots$ system using a normal integrator.

• The point we wish to make with the preceding example is that it is not always enough to rely on the known behavior of an integrator in terms of its local truncation error $\tau(h)$. In the case of singularly perturbed problems, one should, in principle, investigate whether the discretization actually provides the correct asymptotic behavior in the singular–perturbation regime.

The problem, in practice, is that the investigation of the asymptotic behavior of the discretization may be a very difficult process when the form of the equations is not as simple as in example 9.3.1, and moreover, an analytical solution is not available. For our present problems, it might be suspected that the introduction of a small mass moving through multiple force–scale regions at high velocities, typical of ballistic lunar capture trajectories, might give rise to singular perturbation issues. Indeed, as we discuss in the next chapters, and in appendix C,⁴ we find some indications of this in our numerical investigations. However, the difficulty of the analysis for our much more complicated systems of equations places an explicit singular perturbations investigation beyond the scope of this thesis.

⁴The reader may wish to peruse this, and in particular section C.2.5 in connection with the present discussion, and that of the results to follow in the next 2 chapters.

9.3.3 Present Approach

Against the background of the above discussion, the reader is further advised that the numerical aspects of the present work rely primarily on:

- Time-step halving and comparison with alternate integrators (both same-order and higher-order Runge-Kutta-Fehlberg types) for *validation* on comparatively well-known trajectories, for lack of exact solutions along the lines of the Kepler problem (*e.g.* circular and near-Earth orbit solutions of the 3-body problem which should be near Kepler orbits, but also comparison with ephemeride-based solutions of actual planetary motion for the Sun-Earth-Moon case).
- 2. Comparison with the RKF7(8) integrations for *performance* analysis, where in the next chapter we will focus on the accuracy of coordinates and velocities, the required number of function evaluations to produce the trajectory and on the conservation of integrals of the motion.

In light of the above discussion then, we stress again that strictly taken, our results are by no means "absolute" in the sense of certainty with regard to the true solution of a system; instead, they merely compare performance between our conservative integrators and the Runge–Kutta schemes, although this is justified somewhat by the considerable evidence that good performance may be expected from higher–order RKF–type integrators based on *e.g.* the issues raised in [Berry and Healy, 2003].

Chapter 10

Exactly Conservative Integrators

In this chapter we discuss a number of exactly conservative integrators and their performance both against simple 3-body problem test-cases, as well as against Ballistic-Lunar-Capture (BLC)-like departure trajectories from the near-Earth region.¹ The integrators considered are implementations of those introduced by Bowman *et al.* [Shadwick et al., 1999, Kotovych and Bowman, 2002, Shadwick et al., 2001], but to the author's best knowledge this is the first detailed consideration of their performance,² and particularly so with regard to BLC-type trajectories, which involve a small mass transiting regions with markedly differing force-scales, which we introduced in section 4.3.3.

Note that we structure the discussion slightly differently here, in order of increasing complexity of the integrator for variants of the 3-body problem, starting from the circular restricted 3-body problem and continuing through the different approaches to the Jacobi 3-body problem, and finally discussing a doubly-conservative integrator for the same. After discussing the approach and the integrators themselves, we will also give some consideration to the general problem of constructing multiply-conservative integrators for astrodynamics problems, which is a more troublesome issue than it might seem at first glance.

10.1 Design for Exact Conservation of the Hamiltonian

The approach to the design of each integrator is based on conservation of the Hamiltonian (effectively the energy), and we will see that for the Jacobi 3–body problem this also allows us to incorporate the conservation of angular momentum "for free." The building block for the integrators discussed is the conservation of energy, in principle, but remark that we have formulated all integrators in classical *x*, *y* and Jacobi **q**, **Q** coordinates in keeping with the theory introduced in chapter 3, and so omit the formulation in Hamiltonian canonical variables that *e.g.* Bowman uses for the circular restricted 3–body problem.

Nonetheless, as the classical energy here coincides with the Hamiltonian in either set of coordinates, we are essentially using the Hamiltonian as a building block even when we don't use a Hamiltonian formulation. In section 10.5 we exploit this further in polar coordinates to obtain a doubly conservative integral, but will consider the limits of extending this building block in section 10.6 when we then discuss the problem of incorporating an integral that isn't amenable

¹Full BLC–trajectories such as the one indicated in figure 3.9 do not occur in the 3–body problem, and the reader will recall that their study was the motivation for the derivation of the capture problem in section 3.5. Consequently true BLC–trajectories will not occur in this chapter, but we discuss them in the next when we turn to approximately conservative integrators for the Capture Problem.

²Despite the fact that our discussion here and in the next chapter will be driven by our concerns with BLC-type trajectories and so is by no means comprehensive.

to that same extension due to the form of the quadratic terms and inverse transforms.

10.2 Core Conservative Integration Scheme

Remark that throughout the discussion, vectors will be purposely emphasized as **x** to avoid confusion with the scalar coordinate *x*. Also, we occasionally reference elements of a vector in computer memory using block parentheses as $\mathbf{x}[1] = x$, $\mathbf{x}[2] = y$, *etc.* With this in mind, the first two algorithms we present take a simple predictor–corrector step respectively a "naive" conservative predictor–corrector step as follows:

Algorithm 10.2.1 (Simple Predictor Corrector)

for $t = t_0 : h : t_f$ do $\dot{\mathbf{x}} \leftarrow f(\mathbf{x})$ \triangleright Evaluate current derivative at current state \mathbf{x} $\mathbf{x}_p \leftarrow \mathbf{x} + h \cdot \dot{\mathbf{x}}$ \triangleright Generate predictor $\dot{\mathbf{x}}_p \leftarrow f(\mathbf{x}_p)$ \triangleright Evaluate predictor derivative $\mathbf{x}_c \leftarrow \mathbf{x} + \frac{h}{2} (\dot{\mathbf{x}} + \dot{\mathbf{x}}_p)$ \triangleright Update original state with corrector $\mathbf{x} \leftarrow \mathbf{x}_c$ end for

Algorithm 10.2.2 (Naive Core Conservative Integrator)

for $t = t_0$: h : t_f do	
$\dot{\mathbf{x}} \leftarrow f(\mathbf{x})$	\triangleright Evaluate current derivative at current state x
$\mathbf{x}_p \leftarrow \mathbf{x} + h \cdot \dot{\mathbf{x}}$	⊳ Generate predictor
$\dot{\mathbf{x}}_p \leftarrow f(\mathbf{x}_p)$	▷ Evaluate predictor derivative
$\xi \leftarrow T(\mathbf{x}) \ \xi_p \leftarrow T(\mathbf{x}_p)$	▷ Transform variables
$\dot{\boldsymbol{\xi}} \leftarrow f_{\boldsymbol{\xi}}(\mathbf{x}) \equiv T'(\mathbf{x}) \cdot \dot{\mathbf{x}}$ $\dot{\boldsymbol{\xi}}_{p} \leftarrow f_{\boldsymbol{\xi}}(\mathbf{x}_{p}) \equiv T'(\mathbf{x}_{p}) \cdot \dot{\mathbf{x}}_{p}$	▷ Evaluate derivatives in transformed coordinates
${{\xi }_{c}} \leftarrow {{\xi }+rac{h}{2}}\left({{{\dot {\xi }}+{{\dot {\xi }}_{p}}} ight)$	▷ Update transformed state
$ \begin{aligned} \mathbf{x}_c \leftarrow T^{-1}\left(\boldsymbol{\xi}_c\right) \\ \mathbf{x} \leftarrow \mathbf{x}_c \end{aligned} $	▷ Update original state with corrector

end for

A few comments are in order:

- First, note that this latter integrator is, in our terminology, *explicitly–formulated* with regard to the coordinates. For the CR3BP we have taken this approach in algorithm 10.3.1 below with the third transformed variable ξ_3 containing the sum of a transformed coordinate and the CR3BP potential. This is in contrast to algorithm 10.4.2 where the transformed coordinates use *U* in place of a coordinate, such that the inverted result \mathbf{x}_c must be passed through a Newton–Raphson solver in order to reconstruct the actual coordinate value; we refer to this approach as *implicitly–formulated*.
- Note in the above that we have given a formal expression for the derivative of the transform. In practice, however, we form the transform derivatives $\vec{f}_{\xi}(\mathbf{x}, \dot{\mathbf{x}})$ from the original coordinates rather than using the equivalent matrix vector expression $T'(\mathbf{x}) \cdot \dot{\mathbf{x}}$ encountered in the theoretical setting; this for reasons of efficiency and numerical stability. The above expression on the other hand, is useful in the error-analysis of the methods, which is discussed in [Kotovych and Bowman, 2002, appendix A].
- There is however another issue, raised in the above reference and also in [Shadwick et al., 1999] which is quite important in practice: that of turning points of the transformation (*i.e.* where T'(x) = 0). When inverting the transformation back from original variables, we need $T^{-1}(\xi)$, which is related to T'(x) as follows. Implicitly differentiating $T^{-1}(T(x)) = x$:

$$\frac{d(T^{-1}(T(x)))}{dx} = \frac{dx}{dx} \text{ we find that:} T^{-1'}(\xi) = \frac{1}{T'(x)}.$$
 (10.1)

This is a relation between the derivatives of the forward and inverse transformations, and we note that at turning points, we expect that $T^{-1}(x) \to \infty$ and then likewise $T^{-1}(x) \to \infty$, implying a singularity. This is more than just a hypothetical issue; if we recall the expression (5.32) introduced earlier in chapter 5:

$$\begin{aligned} x(t_0+h) - x(t_0) &= h f(x_0) + \frac{h^2}{2} f'(x_0) f(x_0) \\ &+ \frac{h^3}{4} \left(f''(x_0) f(x_0)^2 + \frac{T'''(x_0)}{3T'(x_0)} f(x_0)^3 \right) + \dots , \end{aligned}$$

we see that we may indeed expect problems in the form of error blowup when $T'(x) \approx 0$, which is precisely when the inverse transformation becomes singular. This has, furthermore, been our experience in practice as well, though its occurrence is difficult to predict.³

Bowman *et al.* suggest two alternative approaches to this situation: either reducing the time-step to just prior to the turning point, such that steps may be safely taken, or switching to a non-conservative scheme such as the

³It appears, however, to be related to the "balance" of scales in a problem, occurring more frequently in the Earth–Moon–Satellite problems than in the Sun–Earth–Moon problems, possibly indicating issues stemming from a singular perturbation scenario, *cf.* section 9.3. Also, we remark that similar problems manifest when the time–step of the integrator is chosen too large, though this may be due simply to the instabilities of the explicitly formulated algorithm.

simple predictor-corrector for a single point in the integration.⁴

Our experience has been somewhat different however: we have found the first solution sometimes unworkable in practice, and have opted for the latter, with the caveat that in order to preserve integration accuracy, we combine it with time-step reduction by a factor rf, and we use this to integrate over the singular point of the transformation (effectively a $2h = 2 \times rf \times \frac{h}{rf}$ step) before continuing with the conservative scheme. We include data on this in our performance benchmarks below.⁵

Incorporating these changes, we arrive at the following core integration scheme, for an explicitly-formulated problem:

Algorithm 10.2.3 (Core Conservative Integrator)

for $t = t_0$: h : t_f do	
$\dot{\mathbf{x}} \leftarrow f(\mathbf{x})$	\triangleright Evaluate current derivative at current state x
$\mathbf{x}_p \leftarrow \mathbf{x} + h \cdot \dot{\mathbf{x}}$	▷ Generate predictor
$\dot{\mathbf{x}}_p \leftarrow f(\mathbf{x}_p)$	▷ Evaluate predictor derivative
$\begin{aligned} \boldsymbol{\xi} &\leftarrow \boldsymbol{T}(\mathbf{x}) \\ \boldsymbol{\xi}_p &\leftarrow \boldsymbol{T}(\mathbf{x}_p) \end{aligned}$	▷ Transform variables
$\dot{\boldsymbol{\xi}} \leftarrow f_{\boldsymbol{\xi}}(\mathbf{x}) \equiv T'(\mathbf{x}) \cdot \dot{\mathbf{x}}$ $\dot{\boldsymbol{\xi}}_p \leftarrow f_{\boldsymbol{\xi}}(\mathbf{x}_p) \equiv T'(\mathbf{x}_p) \cdot \dot{\mathbf{x}}_p$	▷ Evaluate transform-derivatives
${{\xi }_{c}} \leftarrow {{\xi }+\frac{h}{2}}\left({{{\dot {\xi }}+{{\dot {\xi }}_{p}}}} ight)$	▷ Update transformed transformed state
if <i>T</i> nonsingular then $\mathbf{x}_c \leftarrow T^{-1}(\boldsymbol{\xi}_c)$	\triangleright Accept conservative step – singularity test by elements of vector $T^{-1}(\xi_c)$
else $h \leftarrow h/rf$	\triangleright Reject, reduce step-size by rf and take non-conservative step
for $t = t : h/rf : t + 2h$ do Execute $2 \times rf$ predictor– end for	corrector steps from current state \mathbf{x} > Algorithm 10.2.1
$h \leftarrow h \cdot rf$	▷ Restore original time–step
end if	
$\mathbf{x} \leftarrow \mathbf{x}_c$	

end for

⁴Private communication with Bowman suggests that it may be enough to use this approach for only the coordinate experiencing blowup, rather than the entire state vector, based on numerical experiments.

⁵We remark that in general when the time-step is chosen appropriately for a problem, we typically need little to no nonconservative steps, and in practice, an excess of these steps is more likely an indicator of an unsuitable choice of h.

10.3 An Integrator for the Circular Restricted 3–Body Problem

We now turn to the design for the circular restricted 3-body problem specifically, and the reader will recall our derivation of the Jacobi integral \mathcal{I} in section 3.3.5 as the only known integral, which we also constructed via the method of integrating vectors in section 7.2.

10.3.1 Integrator Design

This first algorithm is also the simplest: it is a relatively straightforward implementation of Bowman's conservative integrator for the circular restricted 3-body problem of section 3.3.5. The vector **x** is used here to denote the extended state vector $\mathbf{x} = (x, y, \dot{x}, \dot{y})^{\mathsf{T}}$, and consequently $\dot{\mathbf{x}} = (\dot{x}, \dot{y}, \ddot{x}, \ddot{y})^{\mathsf{T}}$.

The core algorithm is as given above as algorithm 10.2.3. This is then extended by the specification of the necessary procedures in the following algorithm 10.3.1:

Algorithm 10.3.1 (CR3BP Conservative Integrator Implementation)

 $t := t_0$ ▷ Initial time ⊳ Initial state $\mathbf{x} := (x_0, y_0, \dot{x}_0, \dot{y}_0)^{\mathsf{T}}$ k := 0▷ Output counter $X_{\text{out}}[k] \leftarrow \mathbf{x}$ $h := \frac{t_f - t_0}{N}$ ▷ First entry in 2D output array ▷ Determine step size **for** $t = t_0 : h : t_f$ **do** Take Conservative Integration Step ▷ Algorithm 10.2.3 if (Intermediate Output Point k) then $X_{\text{out}}[k] \leftarrow \mathbf{x}$ ▷ Next entry in output array $k \leftarrow k+1$ ▷ Increment counter end if end for **Procedures:** function $f(\mathbf{x})$ $f \leftarrow \begin{pmatrix} \dot{x} = \mathbf{x}[3] \\ \dot{y} = \mathbf{x}[4] \\ 2\dot{y} + x - \frac{1-\mu}{r_1^3}(x+\mu) - \frac{\mu}{r_2^3}(x-(1-\mu)) \\ -2\dot{x} + y - \frac{1-\mu}{r^3}y - \frac{\mu}{r^3}y \end{pmatrix}$ \triangleright 1st derivatives already available!

end function

function $T(\mathbf{x})$

$$T \leftarrow \begin{pmatrix} \frac{1}{2}x^{2} \\ \frac{1}{2}y^{2} \\ \frac{1}{2}\dot{x}^{2} - \frac{1-\mu}{r_{1}} - \frac{\mu}{r_{2}} \\ \frac{1}{2}\dot{y}^{2} \end{pmatrix}$$
d function

end function

function $f_{\xi}(\mathbf{x})$

end

$$f_{\xi} \leftarrow \begin{pmatrix} x\dot{x} = \mathbf{x}[1]\dot{\mathbf{x}}[1] \\ y\dot{y} = \mathbf{x}[2]\dot{\mathbf{x}}[2] \\ f_{\xi}[1] + f_{\xi}[2] - f_{\xi}[4] \\ \dot{y}\ddot{y} = \mathbf{x}[4]\dot{\mathbf{x}}[4] \end{pmatrix}$$

function
$$T^{-1}(\mathbf{x}_p, \boldsymbol{\xi})$$

$$\begin{pmatrix} \operatorname{sgn}(x_p) \\ & \end{array}$$

$$T^{-1} \leftarrow \begin{pmatrix} \operatorname{sgn}(y_p)\sqrt{2\xi_2} \\ \operatorname{sgn}(\dot{x}_p)\sqrt{2\left(\xi_3 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)} \\ \operatorname{sgn}(\dot{y}_p)\sqrt{2\xi_4} \end{pmatrix}$$

end function

 $\sqrt{2\xi_1}$

 \triangleright Potential terms absorbed into 3^{rd} coordinate

▷ 3rd element using Hamiltonian conservation

End Procedures

Note the following in algorithm 10.3.1:

- r_1 and r_2 are of course as in section 3.3.5, and the reader will note that the terms are evaluated sequentially, guaranteeing that r_1 , r_2 can be formed before they are needed in the reconstruction of the third element of \mathbf{x}_c .
- The terms $sgn(x_p)$ are used to pick the correct sign for the root based on the predicted value. This can usually be done with sufficient accuracy, but can in principle form a problem near a zero–crossing of a coordinate; the appropriate solution would then be to reduce the time–step so that the signs of predictor and corrector again agree.⁶
- As remarked above, $T(\mathbf{x})$ is somewhat different from the form given in [Kotovych and Bowman, 2002, p. 253], where the canonical⁷ state vector $\mathbf{x} = (q_1 = x, q_2 = y, p_1 = \dot{x} - y, p_2 = \dot{y} + x)^T$ was used. This choice makes essentially no difference to the normalized equations of motion,⁸ and we prefer the formulation above as it lends itself somewhat more easily to the analysis of chapter 7.

Note also that comparing with example 5.3.2, the integral being conserved here can be cast in the appropriate form $\mathbf{c} \cdot \boldsymbol{\xi}$ using the vector $\mathbf{c} = (-1, -1, +1, +1)^{\mathsf{T}}$ (while of course $\boldsymbol{\xi} = T(\mathbf{x})$, as given in the algorithm).

• As remarked above, the derivative of the transformed variables, given as $T'(x) \cdot \dot{x}$, is not evaluated in this form for reasons of numerical stability (*i.e.* numerical experiments showed the matrix T'(x) to be a significant source

⁶For a small enough time-step, the predictor and corrector will both indicate a point just before the zero-crossing for the next iterate, and on the subsequent step both indicate a point just after the zero-crossing.

⁷*i.e.* corresponding to the CR3BP Hamiltonian.

⁸The two may be exchanged by simple coordinate substitution.

of error, likely due to ill-conditioning). Instead, in this problem and the others treated in this thesis, the derivative denoted $f_{\xi}(\mathbf{x})$ is worked out in the original coordinates, which are of course readily available during the computation.

Moreover, the reader will note that we have side–stepped the calculation of the derivative of the potential in $\mathbf{f}_{\xi}[3]$ by exploiting the conservation of the Hamiltonian to formulate it from the other terms, and consequently this step is performed last in actual code.⁹

10.3.2 Integrator Performance

With this algorithm in mind, and implemented as the routine bowman_cr3bp (cf. appendix C), we can consider its performance on our two test problems.

Sun-Earth-Moon

Our first test–case is the Sun–Earth–Moon circular restricted 3–body problem, integrated over a period of 3 months (roughly 3 full revolutions). The initial coordinates are taken from the DE405 ephemerides relative position of the Moon on 01/01/2007, and in dimensional (km resp. km/s) coordinates become:

$$\mathbf{x}_{0} = \begin{pmatrix} (1-\mu)r_{ES} + 3.136850 \cdot 10^{5} \\ -2.019042 \cdot 10^{5} \\ +5.483922 \cdot 10^{-1} \\ +8.062581 \cdot 10^{-1} \end{pmatrix}$$

The resulting orbit in the rotating system is near-circular as expected (coming from a more complicated model, the initial condition does not correspond exactly to a circular orbit, compare also figure 10.15), shifting slightly relative with respect to the Earth–Sun line, and is illustrated in figure 10.1.

We consider the performance of algorithm 10.3.1 with initial time-steps 100s and 1000s against the same-order RKF2(3) pair and the reference higher-order RKF7(8) pair in order to gain an idea of the performance of our algorithm. Table 10.1 gives the function evaluations, counts for the number of non-conservative steps, and the RMS-deviation from the RKF7(8) solution (as an estimate of the true error).

Integrator:	RKF7(8)	RKF2(3)	Conservative $h = 1000 s$	Conservative $h = 100 s$
fevals	1.24E+5	1.21E+4	4.42E+4	3.39E+5
NC-steps	-	_	0	0
Moon				
$\Delta x_{\rm RMS}$	_	1.49E+1	2.06E+1	1.81E-1
Δy_{RMS}	_	1.56E+1	2.13E+1	2.13E-1
$\Delta \dot{x}_{RMS}$	_	3.68E-5	5.07E-5	4.66E-7
Δý _{RMS}	-	3.75E-5	5.20E-5	4.83E-7

Table 10.1: Comparison of performance metrics for Sun-Earth-Moon pCR3BP

We see in the table that with a time-step of 1000 seconds we have the same order of function evaluations as the RKF2(3) pair, and similar performance with respect to the position and velocity differences. When we proceed to 100

⁹The actual potential-derivatives were calculated with MATHEMATICA and then implemented for comparison; agreement was found to be good, though not to machine precision.



Figure 10.1: Circular restricted 3-body problem motion of the Moon relative to the Earth in Earth–Sun–rotating coordinates.

second steps, the number of function evaluations goes up by slightly less than a factor 10, and is now comparable to the RKF7(8) pair, and the errors in the state vector have gone down by approximately 2 orders of magnitude, as we expect from a 2^{nd} –order integrator. Remark also that no non-conservative steps were necessary.

Figures 10.2 - 10.4 give the actual state vector differences (our estimate of the error, *cf.* section 9.3) plotted on a logarithmic scale, and we note that the RMS difference is indeed a good measure of the actual performance, gauged by the average level of the "error" with respect to RKF7(8). We note in these that on a logarithmic scale we have large negative dips corresponding to the difference between the integrator and the RKF7(8) benchmark becoming ≈ 0 . This corresponds to the periodic oscillation of the RKF2(3) and algorithm 10.3.1–solutions about the reference RKF7(8) solution, with a period of roughly 1 month (2 zero–crossings of the difference per month for a total of 6 dips over a 3 month simulation).

We note also that it appears, on comparing the similar-performance figures 10.2 and 10.3 that the growth of the error is slower for the conservative scheme, though it would be premature to draw general conclusions based on this.

In order to investigate slightly further, we have also simulated a 3–year span of orbits, and find the very interesting results of figures 10.5 and 10.6. The estimated error in the RKF2(3) solution grows as expected, but the conservative scheme first experiences a jump in the error at a single non–conservative step,¹⁰ and then demonstrates a subsequent tendency toward correction of the error back to a lower level over time.

We remark that this suggests that while, as we shall see, the conservation of the integral is not representative for the position- and velocity-errors over short time-spans, it can have a significant positive effect on error control over a long

¹⁰This occurs beyond the first 3 months, and so was not apparent in the earlier figures.

time-span.



Figure 10.2: State difference magnitude for Moon: Algorithm 10.3.1 (h = 1000 s) vs. RKF7(8)



Figure 10.3: State difference magnitude for Moon: RKF2(3) vs. RKF7(8)



Figure 10.4: State difference magnitude for Moon: Algorithm 10.3.1 (h = 100 s) vs. RKF7(8)



Figure 10.5: State difference magnitude for Moon over a longer 3 year simulation: Algorithm 10.3.1 (h = 1000 s) vs. RKF7(8) – Note the jump in the error at a non–conservative step, and the subsequent tendency toward correction over time.



Figure 10.6: State difference magnitude for Moon over a longer 3 year simulation: RKF2(3) vs. RKF7(8) – Note the continuing growth of the error.

Next, we consider the performance of the conservative scheme against the RKF2(3) pair specifically with regard to the conservation of the Jacobi integral, and find the results of figures 10.7 and 10.8, where we see that the conservative scheme, as promised, indeed conserves the Jacobi integral up to essentially machine precision (as does RKF7(8)), while the same–order RKF2(3) code induces a clear secular drift. This, while it may not be significant over the short term, is cause for concern in longer time–scale simulations.¹¹



Figure 10.7: Integral–Conservation Performance: Algorithm 10.3.1 (red, h = 100s) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral.

 $^{^{11}}$ Unlike the figure given at the beginning of this section, these values are of course calculated in the natural *i.e.* rotating CR3BP coordinate system, in order to obtain the correct values of the Jacobi integral.



Figure 10.8: Integral–Conservation Performance: RKF2(3) (red) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral.

Earth-Moon-Satellite

Our second test–case is the Earth–Moon–Satellite circular restricted 3–body problem, likewise integrated over a period of 3 months. The initial coordinates are taken from the DE405 ephemerides relative position of the Moon on 01/01/2007, and in dimensional (km resp. km/s) coordinates become:

 $\mathbf{x}_{0} = \begin{pmatrix} -\mu r_{EM} - 6.999993 \cdot 10^{3} \\ -9.893934 \\ +9.685109 \cdot 10^{-1} \\ +1.058167 \cdot 10^{1} \end{pmatrix}$

The resulting orbit is considerably more complex than the previous one, and includes a near-Earth swingby encounter; the orbit is illustrated in figure 10.9 in EM–rotating coordinates.





We consider the performance of algorithm 10.3.1 with initial time–steps 100s and 10s against the same–order RKF2(3) pair and the reference higher–order RKF7(8) pair, in order to gain an idea of the performance of our algorithm. We note that the integration fails (*i.e.* becomes unstable) for O(1000s) time–steps. Table 10.2 gives the function evaluations, counts for the number of non–conservative steps, and the RMS–deviation from the RKF7(8) solution (as an estimate of the true error).

We see in the table that in contrast to the previous test-case, the step-size necessary to get RKF-comparable results with the conservative scheme is now no longer on the order of 1000s, but roughly 2 orders of magnitude smaller than before. As a result, we now need 2 orders of magnitude more function evaluations in order to match the estimated accuracy in position of the RKF2(3) pair, though as we shall see our performance with respect to the integral is a
Integrator:	RKF7(8)	RKF2(3)	Conservative $h = 100 s$	Conservative $h = 10s$
fevals	1.24E+5	1.60E+4	3.48E+5	3.21E+6
NC-steps	-	_	1	0
Satellite				
$\Delta x_{\rm RMS}$	-	1.34E+2	2.51E+4	1.32E+2
Δy_{RMS}	-	1.40E+2	2.59E+4	1.30E+2
$\Delta \dot{x}_{RMS}$	-	1.74E-2	3.65E-1	1.25E-2
$\Delta \dot{y}_{RMS}$	-	2.16E-2	4.10E-1	1.34E-2

Table 10.2: Comparison of performance metrics for Earth-Moon-Satellite pCR3BP

slightly different story.

Figures 10.10 and 10.11 give the actual state vector differences plotted on a logarithmic scale. We note also that it appears, on comparing the similar-performance between the two with respect to the error magnitude, that the error grows slowly for the conservative scheme, staying at roughly the same magnitude throughout the orbit. This is in line with our previous test-case, but here the RKF2(3) solution appears to be considerably more location-dependent with regard to the magnitude of the error, confirming our suspicion that the apparent advantage of the conservative scheme in the SEM-case may not be a universal property, though it would be useful when it occurs.

We note in particular the sudden spike in estimated error occurring in all the plots, at step 6060. This is due to a close–approach swing–by of Earth, which can be seen in figure 10.9. We note on comparing the two integrators that it appears that the RKF2(3) integrator is considerably more sensitive to this maneuver. Moreover, the error grows noticeably at both transits (departure near step 0 and return near step 6060) of the region of the primaries, in contrast to the conservative algorithm, which exhibits only the spike.

Next, we consider the performance of the conservative scheme against the RKF2(3) pair specifically with regard to the conservation of the Jacobi integral, and find the results of figures 10.12 - 10.14, where we observe, in addition to the drift of the RKF2(3) pair two very different phenomena:

- 1. First, we notice for the first time an offset in the value of \mathcal{I} even in the RKF7(8) pair. This is due to an initial jump at departure from the Earth–Moon region, and we see at step 6060 that the integrator manifests a second discontinuity in the integral at this second close–approach to the primaries. Both times, this is reflected in a permanent shift for the RKF–type integrators.
- 2. With regard to the conservative scheme we also see a very interesting effect at the departure from Earth near step 0: using h = 100 s the algorithm no longer conserves the integral, and this is here manifests again as a permanent shift due to a single non-conservative step. Numerical investigation confirms that the single non-conservative step taken is in fact the same 1st step, and by slightly altering the step-size downwards, this may be avoided, as is evident from figure 10.13.

Coupled with the qualitative difference in estimated RMS–error, this behavior on the part of the conservative scheme is indicative of a heightened susceptibility to subsequent error of the method if problems occur at–or–near initial conditions.¹² We note, however, that when we take an appropriately small time–step, the integral is conserved properly, and the method does not exhibit any discontinuities in the integral, despite the rapid changes at the swing–by. It is readily

 $^{^{12}}$ It also suggests that it may in such cases be worthwhile to implement the more complex corrective approach suggested in [Shadwick et al., 1999] instead of taking a non-conservative step.



Figure 10.10: State difference magnitude for Satellite: Algorithm 10.3.1 (h = 10s) vs. RKF7(8)

checked that the swing-by is correctly simulated by the conservative integrator, and in combination with the behavior of the RKF pairs on integral conservation, this should make us suspicious of *their* behavior at the swing-by, and more generally at close approaches.

In principle the simplest remedy for both problems is related: for the RKF integrator pairs, we must decrease the error tolerance (forcing a smaller time-step), as it appears that in these close approaches the integrators do not track the dynamics accurately enough. For the conservative scheme, we must reduce the time-step sufficiently in the current formulation, but given the excessive computational cost over an entire trajectory, this strongly motivates the introduction of step-size adjustment for error control, *e.g.* via the use of Adams-Bashforth-Moulton-based conservative schemes, as suggested in section 5.2.3.



Figure 10.11: State difference magnitude for Satellite: RKF2(3) vs. RKF7(8)



Figure 10.12: Integral–Conservation Performance: Algorithm 10.3.1 (red, h = 100 s) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral.



Figure 10.13: Integral–Conservation Performance: Algorithm 10.3.1 (red, h = 10s) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral.



Figure 10.14: Integral–Conservation Performance: RKF2(3) (red) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral.

10.4 Energy Conservative Integrators for the 3–Body Problem

In this section we discuss 2 approaches to the design of conservative integrators for the planar Jacobi 3-body problem. In both cases the vector **x** is used here to denote the extended state vector $\mathbf{x} = (q_1, q_2, Q_1, Q_2, \dot{q}_1, \dot{q}_2, \dot{Q}_1, \dot{Q}_2)^{\mathsf{T}}$. The reader will note that we will focus more on the performance metrics and in contrast to the previous section include only those figures that give insight into specific aspects of the discussion.

10.4.1 Explicit Integrator Design

The first algorithm is an implementation of Bowman's conservative integrator which integrates all coordinates explicitly. (As remarked in the previous chapter, we use the terms *explicit* and *implicit* to refer to the treatment of coordinates here, and not to the method itself, *cf.* section 5.2.3.)

Algorithm 10.4.1 (Explicit Implementation of Bowman's 3BP Conservative Integrator)

$t := t_0$	⊳ Initial time
$\mathbf{x} := (x_0, y_0, \dot{x}_0, \dot{y}_0)^T$	▷ Initial state
k := 0	▷ Output counter
$X_{\text{out}}(k) \leftarrow \mathbf{x}$	▷ First entry in 2D output array
$h := rac{t_f - t_0}{N}$	▷ Determine step size
$g_1 := \frac{m_1 m_2}{m_1 + m_2 + m_3}$	
$g_2 := \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_2}$	
$m_1 + m_2 + m_3$	
for $t = t_0$: h : t_f do	
Take Conservative Integration Step	⊳ Algorithm 10.2.3
if (Intermediate Output Point k) then	
$X_{ ext{out}}[k] \leftarrow \mathbf{x}$	▷ Next entry in output array
$k \leftarrow k+1$	▷ Increment counter
end if	
end for	
Procedures:	
function $f(\mathbf{x})$	

$$f \leftarrow \begin{pmatrix} \dot{q}_1 = \mathbf{x}[5] \\ \dot{q}_2 = \mathbf{x}[6] \\ \dot{Q}_1 = \mathbf{x}[7] \\ \dot{Q}_2 = \mathbf{x}[8] \\ -(\rho_1 + \rho_2) \frac{q_1}{\|\mathbf{q}\|^3} + \rho_3 \left(\frac{Q_1 - (1-\mu)q_1}{\|\mathbf{Q} - (1-\mu)q\|^3} - \frac{Q_1 + \mu q_1}{\|\mathbf{Q} + \mu q\|\|^3} \right) \\ -(\rho_1 + \rho_2) \frac{q_2}{\|\mathbf{q}\|^3} + \rho_3 \left(\frac{Q_2 - (1-\mu)q_2}{\|\mathbf{Q} - (1-\mu)q\|^3} - \frac{Q_2 + \mu q_2}{\|\mathbf{Q} + \mu q\|\|^3} \right) \\ -\rho_1 (1 + \upsilon) \frac{(Q_1 + \mu q_1)}{\|\mathbf{Q} + \mu q\|^3} - \rho_2 (1 + \upsilon) \frac{(Q_2 - (1-\mu)q_1)}{\|\mathbf{Q} - (1-\mu)q\|^3} \\ -\rho_1 (1 + \upsilon) \frac{(Q_2 + \mu q_2)}{\|\mathbf{Q} + \mu q\|^3} - \rho_2 (1 + \upsilon) \frac{(Q_2 - (1-\mu)q_2)}{\|\mathbf{Q} - (1-\mu)q\|^3} \end{pmatrix}$$

▷ 1st derivatives already available!

end function

function
$$T(\mathbf{x})$$

$$\begin{pmatrix}
q_1 \\
q_2 \\
Q_1 \\
Q_2 \\
\frac{1}{2}g_1 \dot{q}_1^2 \\
\frac{1}{2}g_2 \dot{Q}_1^2 \\
\frac{1}{2}g_2 \dot{Q}_2^2 + U \\
\frac{1}{2}g_2 \dot{Q}_2^2 + U
\end{pmatrix}$$
end function

 \triangleright Potential absorbed into 8th element; spatial coordinates unmodified

function $f_{\xi}(\mathbf{x})$

$$f_{\xi} \leftarrow \begin{pmatrix} \dot{q}_1 = \mathbf{x}[5] \\ \dot{q}_2 = \mathbf{x}[6] \\ \dot{Q}_1 = \mathbf{x}[7] \\ \dot{Q}_2 = \mathbf{x}[8] \\ g_1 \dot{q}_1 \ddot{q}_1 = g_1 \mathbf{x}[5] \dot{\mathbf{x}}[5] \\ g_1 \dot{q}_2 \ddot{q}_2 = g_1 \mathbf{x}[6] \dot{\mathbf{x}}[6] \\ g_2 \dot{Q}_1 \ddot{Q}_1 = g_2 \mathbf{x}[7] \dot{\mathbf{x}}[7] \\ - \left(f\xi[5] + f_{\xi}[6] + f_{\xi}[7]\right) \end{pmatrix}$$

end function

$$\begin{aligned} \textbf{function} \ T^{-1}(\textbf{x}_{p}, \boldsymbol{\xi}) \\ T^{-1} \leftarrow \begin{pmatrix} \xi_{1} \\ \xi_{2} \\ \xi_{3} \\ \xi_{4} \\ sgn(\dot{q}_{1,p})\sqrt{2g_{1}\xi_{5}} \\ sgn(\dot{q}_{2,p})\sqrt{2g_{1}\xi_{6}} \\ sgn(\dot{Q}_{1,p})\sqrt{2g_{2}\xi_{7}} \\ sgn(\dot{Q}_{2,p})\sqrt{2g_{2}(\xi_{8}-U)} \end{aligned}$$

 $> 8^{th}$ element last using Hamiltonian conservation

 $\triangleright U$ reconstructed from first 4 elements

end function

End Procedures

We make a number of observations with regard to algorithm 10.4.1:

- Note first that again comparing example 5.3.2, the integral being conserved here can be cast in the appropriate form $\mathbf{c} \cdot \boldsymbol{\xi}$ using the vector $\mathbf{c} = \left(\mathbf{0}_{[1 \times 4]}, +1, +1, +1, +1 \right)^{\mathsf{T}}$ (and of course $\boldsymbol{\xi} = T(\mathbf{x})$ as given in the algorithm).
- We have chosen here to incorporate the potential U into the energy term $\xi[8]$ corresponding to \dot{Q}_2 ; in light of the remarks here and in the next section on the issue of balancing contributions such that the magnitude of the potential doesn't dwarf the other contributions, this may be an unlucky choice for problems involving a small parameter (such as our 2nd test–case for a small satellite).
- A related issue is that when the contributions are unbalanced, as we expect for a small-mass satellite, this has consequences for the choice of machine representation. If *e.g.* $U = O(10^{22})$ while the kinetic energy associated with even a large ($m \sim 10^4$ kg) satellite leaving the Earth near escape-velocity, some $T = O(10^5)$, then in double precision the energy $\mathcal{E} = T + U \equiv U$. The contribution of the satellite is completely lost due to insufficient machine precision, and this may be expected to cause issues with the inverse transformation.¹³

In practice, we have avoided this issue by using quadruple precision (which can track approximately 32 digits), but this incurs a noticeable performance penalty in code–execution times. An alternative would be to at least couple U to the coordinate with the largest kinetic energy in a given problem, though whether this would be a viable solution would still depend on the specifics of the problem.

• We also remark briefly that instead of the simple expression for the 8^{th} element using the conservation of the Hamiltonian, we might instead use the full expression for $\frac{d\xi_8}{dt}$:

$$-\left(f_{\xi}[5]+f_{\xi}[6]+f_{\xi}[7]\right) \equiv g_2 \dot{Q}_2 \dot{Q}_2 + \sum_{i=1,2} \left(\frac{\partial U}{\partial q_i} \dot{q}_i + \frac{\partial U}{\partial Q_i} \dot{Q}_i\right).$$
(10.2)

As we noted earlier, we have chosen not to do this mainly because it is computationally more intensive, but as we shall see in the next chapter, when we conserve an integral only approximately, we shall have no choice, and use a formulation closely linked to the above expression.

10.4.2 Implicit Integrator Design

Given the above issues, an even better work–around with respect to the balancing of magnitudes is to integrate U separately and recover one of the coordinates implicitly. Our considerations above, moreover, suggest that this should still be done with the coordinate which would have the largest "leverage" on U in terms of the derivative $\frac{\partial U}{\partial x_i}$, and so we suggest using a coordinate from one of the primaries.

Thus, consider the following variation on the above where the coordinate q_2 is sacrificed and U is integrated instead, followed by using a Newton's algorithm as a nonlinear root finder for the reconstruction of q_2 from $\{U, q_1, Q_1, Q_2\}$ (with $q_{2,p}$ as initial guess):

¹³Issues which would conceivably mimic the "turning point" behavior remarked on previously, even in the absence of such points.

Algorithm 10.4.2 (Implicit Implementation of Bowman's 3BP Conservative Integrator)

 $t := t_0$ $\mathbf{x} := (x_0, y_0, \dot{x}_0, \dot{y}_0)^{\mathsf{T}}$ k := 0 $X_{\text{out}}(k) \leftarrow \mathbf{x}$ $h := \frac{t_f - t_0}{N}$ $g_1 := \frac{m_1 m_2}{m_1 + m_2 + m_3}$ $g_2 := \frac{m_3(m_1 + m_2)}{m_1 + m_2 + m_3}$

for $t = t_0 : h : t_f$ **do**

Take Conservative Integration Step $\mathbf{x}_c \leftarrow T^{-1}(\boldsymbol{\xi}_c)$ $\mathbf{x}_c \leftarrow NR(\mathbf{x}_c, q_{2,p}, U = \boldsymbol{\xi}[2])$ $\mathbf{x} \leftarrow \mathbf{x}_c$

if (Intermediate Output Point k) then $X_{out}[k] \leftarrow \mathbf{x}$ $k \leftarrow k + 1$ end if ▷ Initial time
 ▷ Initial state
 ▷ Output counter
 ▷ First entry in 2D output array
 ▷ Determine step size

▷ modified from Algorithm 10.2.3 as follows: ▷ Update original state with corrector ▷ Newton-Raphson Solver for $q_{2,c}$

Next entry in output arrayIncrement counter

end for

Procedures:

$$\begin{aligned} \text{function } f(\mathbf{x}) & \dot{q}_1 = \mathbf{x}[5] \\ \dot{q}_2 = \mathbf{x}[6] \\ \dot{Q}_1 = \mathbf{x}[7] \\ \dot{Q}_2 = \mathbf{x}[8] \\ & -(\rho_1 + \rho_2) \frac{q_1}{\|\mathbf{q}\|^3} + \rho_3 \left(\frac{Q_1 - (1 - \mu)q_1}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} - \frac{Q_1 + \mu q_1}{\|\mathbf{Q} + \mu\mathbf{q}\|\|^3} \right) \\ & -(\rho_1 + \rho_2) \frac{q_2}{\|\mathbf{q}\|^3} + \rho_3 \left(\frac{Q_2 - (1 - \mu)q_2}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} - \frac{Q_2 + \mu q_2}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} \right) \\ & -\rho_1(1 + \upsilon) \frac{(Q_1 + \mu q_1)}{\|\mathbf{Q} + \mu\mathbf{q}\|^3} - \rho_2(1 + \upsilon) \frac{(Q_2 - (1 - \mu)q_1)}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} \\ & -\rho_1(1 + \upsilon) \frac{(Q_2 + \mu q_2)}{\|\mathbf{Q} + \mu\mathbf{q}\|^3} - \rho_2(1 + \upsilon) \frac{(Q_2 - (1 - \mu)q_2)}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3} \end{aligned}$$
end function

 \triangleright 1st derivatives already available!

function *T*(**x**)

$$T \leftarrow \begin{pmatrix} q_1 \\ U \\ Q_1 \\ Q_2 \\ \frac{1}{2}g_1 \dot{q}_1^2 \\ \frac{1}{2}g_2 \dot{q}_2^2 \\ \frac{1}{2}g_2 \dot{Q}_2^2 \end{pmatrix}$$

1

end function

$$f_{\text{function } f_{\xi}(\mathbf{x})} = \begin{pmatrix} \dot{q}_1 = \mathbf{x}[5] & \\ -\left(f_{\xi}[5] + f_{\xi}[6] + f_{\xi}[7] + f_{\xi}[8]\right) & \\ \dot{Q}_1 = \mathbf{x}[7] & \\ \dot{Q}_2 = \mathbf{x}[6] & \\ g_1 \dot{q}_1 \ddot{q}_1 = g_1 \mathbf{x}[5] \dot{\mathbf{x}}[5] & \\ g_1 \dot{q}_2 \ddot{q}_2 = g_1 \mathbf{x}[6] \dot{\mathbf{x}}[6] & \\ g_2 \dot{Q}_1 \ddot{Q}_1 = g_2 \mathbf{x}[7] \dot{\mathbf{x}}[7] & \\ g_2 \dot{Q}_2 \ddot{Q}_2 = g_2 \mathbf{x}[8] \dot{\mathbf{x}}[8] & \end{pmatrix}$$

end function

function $T^{-1}(\mathbf{x}_p, \xi)$

$$T^{-1} \leftarrow \begin{pmatrix} \zeta_{1} \\ \xi_{2} \\ \xi_{3} \\ \xi_{4} \\ sgn(\dot{q}_{1,p})\sqrt{2g_{1}\xi_{5}} \\ sgn(\dot{q}_{2,p})\sqrt{2g_{1}\xi_{6}} \\ sgn(\dot{Q}_{1,p})\sqrt{2g_{2}\xi_{7}} \\ sgn(\dot{Q}_{2,p})\sqrt{2g_{2}\xi_{8}} \end{pmatrix}$$

end function

function NR($\mathbf{x}_c, q_{2,p}, U_{ref} = \xi_{2,c}$) maxit := 20 $tol := 10^{-6}$ $q_2 \leftarrow q_{2,p}$ while $i < \max$ it do

Form:
$$\|\mathbf{q}\|$$
, $\|\mathbf{Q} - (1-\mu)\mathbf{q}\|$, $\|\mathbf{Q} + \mu\mathbf{q}\|$
 $U = -G\left(\frac{m_1m_2}{\|\mathbf{q}\|} + \frac{m_1m_3}{\|\mathbf{Q} + \mu\mathbf{q}\|} + \frac{m_2m_3}{\|\mathbf{Q} - (1-\mu)\mathbf{q}\|}\right)$
 $dU = G\left(\frac{m_1m_2}{\|\mathbf{q}\|^3} + \frac{m_1m_3\mu(Q_2 + \mu q_2)}{\|\mathbf{Q} + \mu\mathbf{q}\|^3} - \frac{m_2m_3(1-\mu)(Q_2 - (1-\mu)q_2)}{\|\mathbf{Q} - (1-\mu)\mathbf{q}\|^3}\right)$
 $\delta q_2 = \frac{U - U_{\text{ref}}}{dU}$

 \triangleright Potential as separate 2^{nd} coordinate

 $\triangleright 2^{nd}$ element last using Hamiltonian conservation

 $\triangleright U$ passed as element 4

▷ Maximum iterations to perform
 ▷ Tolerance for q₂ convergence
 ▷ Initial guess based on q_{2,p}

 \triangleright Newton-Raphson Correction to q_2

```
if |\delta q_2| < \text{tol then}

NR \leftarrow Q_2

Exit While Loop \triangleright q_{2,c} acceptable

else

q_2 \leftarrow q_2 + \delta q_2

i \leftarrow i + 1 \triangleright Increment counter towards maxit

end if

end while

end function
```

End Procedures

Note that the integral being conserved by the algorithm is cast in the appropriate form $\mathbf{c} \cdot \boldsymbol{\xi}$ using the vector $\mathbf{c} = (0, +1, 0, 0, +1, +1, +1, +1)^{\mathsf{T}}$ and the usual $\boldsymbol{\xi} = T(\mathbf{x})$.

With regard to the Newton-Raphson procedure $NR(\mathbf{x}_c, q_{2,p}, U_{ref} = \xi_{2,c})$, a few issues require comment:

- 1. First, the algorithm as implemented performs a maximum of *e.g.* 20 iterations, based on experience both by the author and in [Shadwick et al., 1999, Kotovych and Bowman, 2002] that the method usually converges in very few iterations (typically 3–5), but may be relaxed to *e.g.* 100 to allow for slower convergence.
- 2. Second, convergence is based on a tolerance which applies to q_2 and so carries a dimension of km, thus the cited criterium requires convergence to within 10^{-6} km = 1mm, which is expected to be accurate enough to prevent the accumulation of significant numerical error.
- 3. Typically in the literature [*e.g.* Press et al., 1992], it is suggested that a "safe Newton–Raphson"–algorithm be used preferentially, which brackets the root and forces the iterative process to remain bounded by the brackets. We have omitted this here due to the fact that in our practice, we are confronted with the following either/or situation.

Either the routine converges very quickly due to a sufficiently good initial guess, or it doesn't converge at all, which appears (on investigation with exactly such a bracketing approach) to be due to the absence of a root near our guess, which would seem to indicate that either the step–size is too large near a turning point, or that we have lost the root due to difficulties with balancing and rounding error in the components of our transformation.

10.4.3 Integrator Performance

We now turn to the performance of the algorithms introduced in the preceding section.

Sun-Earth-Moon

As with the CR3BP in section 10.3, we again begin the discussion of the results for the planar Jacobi 3–body problem with the Sun–Earth–Moon test case integrated over a period of 3 months. The initial coordinates are now taken from a

hypothetical circular Moon orbit (based on the Kepler expressions for velocity)¹⁴ and as we now treat a full 3–body problem, the initial conditions are thus given in Jacobi variables $\mathbf{x} = (\mathbf{q}, \mathbf{Q}, \dot{\mathbf{q}}, \dot{\mathbf{Q}})^{\mathsf{T}}$ as:¹⁵

$$\mathbf{x}_{0} = \begin{pmatrix} r_{ES} \\ 0 \\ (1+\upsilon)(1-\mu)r_{ES} - r_{EM} \\ 0 \\ 0 \\ \omega_{ES}r_{ES} \\ 0 \\ -\omega_{EM}r_{EM} + (1-\mu)\omega_{ES}r_{ES} \end{pmatrix}$$

The resulting orbit is illustrated in the Earth-centered Earth-Sun-rotating frame in figure 10.15.



Figure 10.15: Motion of the Moon relative to the Earth in the Earth–centered, Earth–Sun–rotating coordinate system, solution from planar Jacobi 3–body problem.

We begin our discussion with a summary of the estimated RMS–errors, that is to say the RMS–differences with the RKF7(8) results calculated for this test case, given in table 10.3. The features which stand out are that:

1. For a step-size of about 100s, which we also took in section 10.3, we still have estimated errors roughly 2 orders of magnitude smaller than the RKF2(3) integrator pair, at the cost 1 order of magnitude greater number

¹⁴This test-case was introduced for validation purposes, and is included here as a slight contrast to the case used in the discussion of the CR3BP.

¹⁵The numerical values of the constants can, as always, be found either in [Montenbruck and Gill, 2001] or in the IntegrationMethods code.

of function evaluations;

- 2. In some cases this advantage is increased to 3 orders of magnitude smaller error, where we refer particularly to the results for this problem's small body, the Moon;
- 3. We also note that there is essentially no difference between the performance of the explicit and implicit methods. This is not surprising given our discussion of the algorithms, as the major gains favoring the implicit method would come from precision–limited calculations, which we have purposely taken out of the picture for the moment.

Integrator:	RKF7(8)	RKF2(3)	Explicit $h = 100 s$	Implicit $h = 100 s$
fevals	1.24E+5	9.88E+3	3.62E+5	3.62E+5
NC-steps	-	-	0	0
Sun				
$\Delta x_{\rm RMS}$	-	7.32E-06	4.99E-08	4.99E-08
Δy_{RMS}	-	3.33E-06	5.64E-08	5.65E-08
$\Delta \dot{x}_{RMS}$	-	1.01E-12	1.99E-14	1.99E-14
Δý _{RMS}	-	5.82E-13	1.98E-14	1.98E-14
Earth				
$\Delta x_{\rm RMS}$	-	5.98E+00	1.76E-02	1.76E-02
Δy_{RMS}	-	4.34E+00	1.84E-02	1.84E-02
$\Delta \dot{x}_{RMS}$	-	1.17E-05	1.27E-08	1.27E-08
Δý _{RMS}	-	1.30E-05	1.39E-08	1.39E-08
Moon				
$\Delta x_{\rm RMS}$	-	4.01E+02	3.26E-01	3.26E-01
Δy_{RMS}	-	3.54E+02	3.11E-01	3.11E-01
$\Delta \dot{x}_{RMS}$	-	9.46E-04	8.28E-07	8.28E-07
Δý _{RMS}	-	1.05E-03	8.76E-07	8.78E-07

Table 10.3: Comparison of performance metrics for Sun–Earth–Moon planar Jacobi Problem – RKF2(3), RKF7(8) and Algorithms 10.4.1 and 10.4.2.

We present a few illustrations of this performance in figures 10.16 - 10.17, in line with the earlier discussion. We note here also the difference in magnitude between the estimated errors in the 3 bodies, corresponding prominently to their relative motion in the problem, figures 10.16 and 10.17, which appears to confirm our expectation that smaller, faster moving bodies are prone to a larger magnitude of error.

Next, we consider the performance of the conservative scheme against the RKF2(3) pair specifically with regard to the conservation of the energy integral,¹⁶ and find the results of figures 10.18 and 10.19, where we see that the conservative scheme again conserves the integral up to essentially machine precision (as does RKF7(8)), while the same–order RKF2(3) code again induces clear drift.

¹⁶Calculations made in natural (*i.e.* planar Jacobi) coordinate system with integral expression in Jacobi coordinates.



Figure 10.16: State difference magnitude: Algorithm 10.4.1 (h = 100 s) vs. RKF7(8) – Sun in magenta, Earth in blue and Moon in green. Figure also representative for Algorithm 10.4.2.



Figure 10.17: State difference magnitude: RKF2(3) vs. RKF7(8) – Sun in magenta, Earth in blue and Moon in green.



Figure 10.18: Integral–Conservation Performance: Algorithm 10.4.1 (red, h = 100 s) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral. Figure also representative for Algorithm 10.4.2.



Figure 10.19: Integral–Conservation Performance: RKF2(3) (red) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral.

Earth-Moon-Satellite

We continue with the analogous presentation of results for the Earth–Moon–Satellite case. The results are along the lines of the 3–body problem but somewhat more troublesome, in that particularly small time–steps are necessary to get a solution comparable to the RKF integrator pairs, where comparability is judged by the observed convergence towards RKF7(8) when we reduce the step–size. As we shall see, this is largely due to repeated close encounters with the Earth, followed by an orbit–altering swing–by of the Moon which is difficult to track correctly.

The initial conditions are given in Jacobi variables $\mathbf{x} = (\mathbf{q}, \mathbf{Q}, \dot{\mathbf{q}}, \dot{\mathbf{Q}})^{\mathsf{T}}$ as:¹⁷

$$\mathbf{x}_{0} = \begin{pmatrix} r_{EM} \\ 0 \\ -\mu(1+\upsilon)r_{EM} - 6.999993 \cdot 10^{3} \\ -9.893934 \\ 0 \\ \omega_{EM} \\ -9.685109 \cdot 10^{-1} \\ -1.058167 \cdot 10^{1} \end{pmatrix}$$

The resulting orbit is illustrated in the Earth-centered psuedo-inertial frame in figure 10.20.



Figure 10.20: Planar Jacobi 3BP motion of the Earth, Moon and Satellite, in the Earth–centered psuedo–inertial frame – RKF7(8) integrator pair solution.

In figure 10.20 we see the solution found using the RKF7(8) integrator pair. Figures 10.21 and 10.22 by contrast, give

¹⁷The numerical values of the constants can, as always, be found either in [Montenbruck and Gill, 2001] or in the IntegrationMethods code.



Figure 10.21: Planar Jacobi 3BP motion of the Earth, Moon and Satellite, in the Earth–centered psuedo–inertial frame – Algorithm 10.4.2 solution with h = 10s.



Figure 10.22: Planar Jacobi 3BP motion of the Earth, Moon and Satellite, in the Earth–centered psuedo–inertial frame – Algorithm 10.4.2 solution with h = 5s.

the solutions found using the implicit conservative scheme of algorithm 10.4.2, where we see that even at 10s steps (*cf.* tables 10.4 - 10.5 for an impression of the relative computational effort), the solution returned by the integrator is bad. Only on taking steps below 5s is the swing–by of the Moon on the return leg modeled correctly, and results for the explicit algorithm 10.4.1 are similar.

Integrator:	RKF7(8)	RKF2(3)	Explicit $h = 100 s$	Implicit $h = 100 s$
fevals	1.24E+5	2.04E+4	8.66E+4	3.62E+4
NC-steps	-	_	100%	0
Earth				
$\Delta x_{\rm RMS}$	-	4.58E-02	2.47E-04	3.94E-03
Δy_{RMS}	-	4.07E-02	2.31E-04	3.69E-03
$\Delta \dot{x}_{RMS}$	-	1.04E-07	6.15E-10	9.84E-09
Δý _{RMS}	-	1.19E-07	6.59E-10	1.05E-08
Moon				
$\Delta x_{\rm RMS}$	-	3.79E+00	2.01E-02	3.21E-01
$\Delta y_{\rm RMS}$	-	3.30E+00	1.88E-02	3.00E-01
$\Delta \dot{x}_{RMS}$	-	8.48E-06	5.00E-08	8.00E-07
Δý _{RMS}	-	9.73E-06	5.36E-08	8.57E-07
Satellite				
$\Delta x_{\rm RMS}$	-	8.49E+03	2.44E+05	3.90E+05
Δy_{RMS}	-	1.56E+04	5.23E+05	5.10E+05
$\Delta \dot{x}_{RMS}$	-	2.88E-02	1.07E+00	1.13E+00
Δÿ _{RMS}	_	2.93E-02	6.67E-01	6.69E-01

Table 10.4: Comparison of performance metrics for Earth–Moon–Satellite planar Jacobi 3–body problem – RKF2(3), RKF7(8) and Algorithms 10.4.1 and 10.4.2 – Note that the explicit algorithm 10.4.1 needs 100% non–conservative steps for this integration, indicating step–size issues.

We note that while in table 10.4 the required work in terms of function evaluations is comparable with the RKF7(8) integrator pair, the estimated error is worse for both algorithms. The explicit integrator in particular needs 100% non–conservative steps. Moreover, this difficulty persists down to a 10s step–size, and indicates that the implicit formulation is indeed somewhat more robust in this case.

Table 10.5 next compares the effect of time–step reduction for the implicit integrator at steps of 50, 10 and 5 seconds. We note there that the position differences remain of magnitude $1 \cdot 10^5$ km for the satellite, and are not reduced as the primaries. This we ascribe to the different source of error for the primaries, which are not subject to the high–velocity dynamics the satellite experiences (relative to their much larger mass), as we discuss next.

Comparing figures 10.23 and 10.24, we may draw the conclusion that in this respect this problem presents a difficult test case for the algorithms. It comprises a high velocity departure from the near–Earth area, multiple close–approaches corresponding to the perigee of a high–eccentricity Kepler orbit, and lastly an orbit–altering swing–by which determines the behavior beyond roughly step 6000 (*cf.* figure 10.20).

The near–Earth apogees account for the series of peaks in the estimated error observed in the figures, while the last swing–by where the satellite encounters the Moon accounts for the sudden jump near step 6000. The difference in performance illustrated shown between the RKF integrator pairs and the conservative schemes is in this case, however, indicative of an issue with the latter, because cumulative error up to the encounter of the Moon prevents correct timing of the swing–by. This conclusion is also borne out by the the convergence of the conservative algorithms to the correct trajectory with decreasing time–step.

Integrator:	RKF7(8)	Implicit $h = 50 s$	Implicit $h = 10s$	Implicit $h = 5 s$
fevals	1.24E+5	6.80E+5	3.22E+6	6.40E+6
NC-steps	-	0	0	0
Earth				
$\Delta x_{\rm RMS}$	-	9.86E-04	3.94E-05	9.86E-06
Δy_{RMS}	-	9.23E-04	3.69E-05	9.23E-06
$\Delta \dot{x}_{RMS}$	-	2.46E-09	9.84E-11	2.46E-11
$\Delta \dot{y}_{RMS}$	-	2.63E-09	1.05E-10	2.63E-11
Moon				
$\Delta x_{\rm RMS}$	-	8.02E-02	3.21E-03	8.02E-04
Δy_{RMS}	-	7.50E-02	3.00E-03	7.50E-04
$\Delta \dot{x}_{RMS}$	-	2.00E-07	8.00E-09	2.00E-09
$\Delta \dot{y}_{RMS}$	-	2.14E-07	8.57E-09	2.14E-09
Satellite				
$\Delta x_{\rm RMS}$	-	3.07E+05	1.54E+05	5.16E+04
Δy_{RMS}	-	5.29E+05	3.99E+05	1.27E+05
$\Delta \dot{x}_{RMS}$	-	1.37E+00	5.54E-01	2.67E-01
$\Delta \dot{y}_{RMS}$	-	8.60E-01	4.06E-01	2.20E-01

Table 10.5: Comparison of performance metrics for Earth–Moon–Satellite planar Jacobi 3–body problem – RKF2(3), RKF7(8) and Algorithms 10.4.2 for $h_0 = 50$, 10, 5s.

We remark further that In using the explicit integrator, we see in fact that the performance is considerably worse than in the implicit formulation, in that even down to 10 second steps, a few hundred loops are still made non–conservatively. These ensure that the integral is not conserved, whereas for the implicit formulation, it is conserved to machine accuracy, as we saw earlier in *e.g.* figure 10.18.

With regard to conservation of the energy integral, we obtain the same results as in earlier sections, with conservation to machine precision as long as there are no non–conservative steps. For the explicit algorithm, this means that we in fact see numerical drift of the value of the integral throughout our numerical experiments, in stark contrast to the implicit formulation. Remark, however, that the drift over the three month integration period accumulates to $O(10^{-12})$ relative to the true value, while for the RKF2(3) pair it accumulates to $O(10^{-6})$.



Figure 10.23: State difference magnitude: Algorithm 10.4.2 (h = 5s) vs. RKF7(8) – Earth in blue, Moon in green and Satellite in black. Figure also essentially representative for Algorithm 10.4.1, and the results of both at higher time–steps.



Figure 10.24: State difference magnitude: RKF2(3) vs. RKF7(8) – Earth in blue, Moon in green and Satellite in black.

10.5 Doubly Conservative Integrator for the 3–Body Problem

Finally, we will consider of Bowman's doubly conservative integrator for the (Jacobi) 3–body problem. We will first introduce the coordinate transformation, and then give the algorithm. A few details regarding the force function are left to the comments after algorithm 10.5.1.

10.5.1 Integrator Design

As we discussed earlier, this algorithm is based on a change of coordinates from the usual Jacobi variables $\{q_1, q_2, Q_1, Q_2, \dot{q}_1, \dot{q}_2, \dot{Q}_1, \dot{Q}_2\}$ to a new (double) set of polar coordinates $\{q, \theta, p, l, Q, \Theta, P, L\}$, where the linear and angular momenta take the place of the velocities.

This is effected via the coordinate transformation:

$$q = \sqrt{q_1^2 + q_2^2} \,, \tag{10.3a}$$

$$Q = \sqrt{Q_1^2 + Q_2^2}, \qquad (10.3b)$$

$$\theta = \arctan \frac{q_2}{2}, \qquad (10.3c)$$

$$\Theta = \arctan \frac{q_1}{Q_1}, \qquad (10.3d)$$

$$p = g_1(\dot{q}_1 \cos \theta + \dot{q}_2 \sin \theta), \qquad (10.3e)$$

$$P = g_2(\dot{Q}_1 \cos \Theta + \dot{Q}_2 \sin \Theta), \qquad (10.3f)$$

$$l = g_1 q(\dot{q}_2 \cos \theta - \dot{q}_1 \sin \theta), \qquad (10.3g)$$

$$L = g_2 Q(\dot{Q}_2 \cos \Theta - \dot{Q}_1 \sin \Theta), \qquad (10.3h)$$

and so from this point on, let $\mathbf{x} = (q, \theta, p, l, Q, \Theta, P, L)^T$. Associated with the new variables are of course transformed equations of motion, as well as a new choice of variables, which we summarize in the procedures of the following algorithm:

Algorithm 10.5.1 (Implementation of Bowman's 3BP Conservative Integrator in Polar–Coordinates)

$t := t_0$	⊳ Initial time
$\mathbf{x} := (x_0, y_0, \dot{x}_0, \dot{y}_0)^{T}$	⊳ Initial state
k := 0	▷ Output counter
$X_{\text{out}}(k) \leftarrow \mathbf{x}$	▷ First entry in 2D output array
$h := \frac{t_f - t_0}{N}$	▷ Determine step size
$g_1 := \frac{m_1 m_2}{m_1 + m_2 + m_3}$	
$g_2 := \frac{m_3(m_1+m_2)}{m_1+m_2+m_3}$	

for $t = t_0 : h : t_f$ **do**

Take Conservative Integration Step $\mathbf{x}_c \leftarrow T^{-1}(\boldsymbol{\xi}_c)$ $\mathbf{x}_c \leftarrow NR(\mathbf{x}_c, q_p, U = \boldsymbol{\xi}[1])$ $\mathbf{x} \leftarrow \mathbf{x}_c$ ▷ modified from Algorithm 10.2.3 as follows: ▷ Update original state with corrector ▷ Newton-Raphson Solver for $q_{2,c}$ if (Intermediate Output Point k) then $X_{\text{out}}[k] \leftarrow \mathbf{x}$ $k \leftarrow k + 1$

end if



Procedures:

function $f(\mathbf{x})$ $f \leftarrow \begin{pmatrix} \frac{p}{g_1} \\ \frac{l}{g_1q^2} \\ \frac{l^2}{g_1q^3} - \frac{\partial U}{\partial q} \\ -\frac{\partial U}{\partial \theta} \\ \frac{p}{g_2} \\ \frac{L}{g_2Q^2} - \frac{\partial U}{\partial Q} \\ -\frac{\partial U}{g_2Q^3} - \frac{\partial U}{\partial Q} \end{pmatrix}$

end function

function $T(\mathbf{x})$ $T \leftarrow \begin{pmatrix} \frac{p^2}{2g_1} + \frac{l^2}{2g_1q^2} \\ \frac{p^2}{2g_2} + \frac{L^2}{2g_2Q^2} \\ U \\ Q \\ l \\ L \\ \theta \end{pmatrix}$

▷ Next entry in output array

Increment counter

> All quantities available from state vector

▷ Potential a separate coordinate as in algorithm 10.4.2

end function

$$f_{\text{function } f_{\xi}(\mathbf{x})} \begin{pmatrix} \frac{p\dot{p}}{g_1} + \frac{lq^2\dot{l}-ql^2\dot{q}}{g_1q^4} \\ \frac{p\dot{p}}{g_2} + \frac{LQ^2\dot{L}-QL^2\dot{Q}}{g_2Q^4} \\ \frac{\partial U}{\partial q} + \frac{\partial U}{\partial \theta} + \frac{\partial U}{\partial Q} + \frac{\partial U}{\partial \Theta} = -\left(f_{\xi}[1] + f_{\xi}[2]\right) \\ \dot{Q} = \dot{\mathbf{x}}[5] \\ \dot{l} = \dot{\mathbf{x}}[4] \\ \dot{L} = \dot{\mathbf{x}}[8] \\ \dot{\Theta} = \dot{\mathbf{x}}[2] \\ \dot{\Theta} = \dot{\mathbf{x}}[6] \\ end \text{ function } \end{cases}$$

▷ 3^{*rd*} element last using Hamiltonian conservation

function $T^{-1}(\mathbf{x}_p, \boldsymbol{\xi})$

$$T^{-1} \leftarrow \begin{pmatrix} \xi_{3} \\ \xi_{7} \\ sgn(\dot{p},p)\sqrt{2g_{1}\left(\xi_{1}-\frac{l^{2}}{2g_{1}q^{2}}\right)} \\ \xi_{5} \\ \xi_{4} \\ \xi_{8} \\ sgn(\dot{P},p)\sqrt{2g_{2}\left(\xi_{2}-\frac{L^{2}}{2g_{2}Q^{2}}\right)} \\ \xi_{6} \end{pmatrix}$$

end function

maxit := 20

 $tol:=10^{-6}$

while *i* < maxit do

 $q \leftarrow q_p$

function NR($\mathbf{x}_c, q_p, U_{\text{ref}} = \xi_{3,c}$)

 $\triangleright U$ passed as 1^{st} element

▷ Maximum iterations to perform
 ▷ Tolerance for q₂ convergence

▷ Initial guess based on $q_{2,p}$

Form: $\|\mathbf{q}\|$, $\|\mathbf{Q} - (1-\mu)\mathbf{q}\|$, $\|\mathbf{Q} + \mu\mathbf{q}\|$ $U = -G\left(\frac{m_1m_2}{\|\mathbf{q}\|} + \frac{m_1m_3}{\|\mathbf{Q} + \mu\mathbf{q}\|} + \frac{m_2m_3}{\|\mathbf{Q} - (1-\mu)\mathbf{q}\|}\right)$ $dU = \frac{\partial U}{\partial q}$ \triangleright See notes below for $\frac{\partial U}{\partial q}$ expression $\delta q = \frac{U - U_{\text{ref}}}{dU}$ \triangleright Newton-Raphson Correction to q_2 if $|\delta q| < \text{tol then}$ $NR \leftarrow Q_2$ Exit While Loop $\triangleright q_c$ acceptable else $q \leftarrow q + \delta q$ $i \leftarrow i + 1$ Increment counter towards maxit end if end while end function **End Procedures**

There are now 2 integrals being conserved by the algorithm, which we may cast in the appropriate form $\mathbf{c} \cdot \boldsymbol{\xi}$ using the vector $\mathbf{c} = (+1, +1, +1, 0_{[1 \times 5]})^{\mathsf{T}}$ and the usual $\boldsymbol{\xi} = T(\mathbf{x})$, such that:

$$\mathcal{E} = T + U = \frac{p^2}{2g_1} + \frac{p^2}{2g_2} + U \quad \text{and}$$
(10.4a)

$$H = \frac{l^2}{2g_1 q^2} + \frac{L^2}{2g_2 Q^2} \tag{10.4b}$$

(where we intend by *H* the angular momentum, rather than the Hamiltonian) are summed together to form the integral:

$$I = \mathcal{E} + H = \frac{p^2}{2g_1} + \frac{P^2}{2g_2} + U + \frac{l^2}{2g_1 q^2} + \frac{L^2}{2g_2 Q^2} , \qquad (10.5)$$

which is now exactly conserved by the integrator.

Lastly, as we mentioned in the beginning of the section, we also need the following expressions for the evaluation of the force model in polar coordinates:

$$\frac{\partial U}{\partial q} = G\left(\frac{m_1m_2}{q^2} + \frac{m_1m_3\mu(\mu q + Q\cos(\theta - \Theta))}{\|\mathbf{Q} + \mu\mathbf{q}\|^3} - \frac{m_2m_3(1 - \mu)\left((\mu - 1)q + Q\cos(\theta - \Theta)\right)}{\|\mathbf{Q} - (1 - \mu)\mathbf{q}\|^3}\right),$$
(10.6a)

$$\frac{\partial U}{\partial \theta} = G\left(-\frac{m_1 m_3 (\mu q Q \sin(\theta - \Theta))}{\|\mathbf{Q} + \mu \mathbf{q}\|^3} + \frac{m_2 m_3 ((1 - \mu) q Q \sin(\theta - \Theta))}{\|\mathbf{Q} - (1 - \mu) \mathbf{q}\|^3}\right),\tag{10.6b}$$

$$\frac{\partial U}{\partial Q} = G\left(\frac{m_1 m_3 (Q + \mu q \cos(\theta - \Theta))}{\|\mathbf{Q} + \mu \mathbf{q}\|^3} + \frac{m_2 m_3 (Q - (1 - \mu) q \cos(\theta - \Theta))}{\|\mathbf{Q} - (1 - \mu) \mathbf{q}\|^3}\right),$$
(10.6c)

$$\frac{\partial U}{\partial \Theta} = G\left(\frac{m_1 m_3 \mu q Q \sin(\theta - \Theta)}{\|\mathbf{Q} + \mu \mathbf{q}\|^3} - \frac{m_2 m_3 (1 - \mu) q Q \sin(\theta - \Theta)}{\|\mathbf{Q} - (1 - \mu) \mathbf{q}\|^3}\right),$$
(10.6d)

and with this we may consider the presentation of this third conservative integration scheme complete.

10.5.2 Integrator Performance

Now, using algorithm 10.5.1, we reprise our discussion of conservative schemes for the planar Jacobi 3–body problem, though our discussion will be more brief than in the previous sections.

Sun-Earth-Moon

For both this and the following test–case, we use the same initial conditions as for the explicit and implicit integrators discussed previously. We begin our discussion with a summary of the estimated RMS–errors, which is to say the RMS–differences with the RKF7(8) results calculated for this test case, given in table 10.6. The features which stand out are that:

- 1. For a step-size of about 300s, comparable in terms of function evaluations to the the RKF7(8) pair, we find that the error is still more or less an order of magnitude better than the RKF2(3) pair, at the cost of 2 orders of magnitude greater number of function evaluations, in addition to more complicated function evaluations for the transformation $T(\mathbf{x})$ and the overhead of the Newton–Raphson solver;
- 2. In some cases this advantage is increased to 2 orders of magnitude smaller error, where we refer particularly to the results for this problem's small body, the Moon;¹⁸
- 3. We also note that there is essentially no difference between the performance of the polar algorithm compare with the explicit and implicit methods. This is somewhat surprising, though it will be less so in light of our remarks on the conservation of the second integral further on. We may summarize them as follows, however: conservation of the angular momentum appears to be much less important to the current problems than the conservation

¹⁸The reader will note further on in this chapter that for the Earth-Moon-Satellite configuration it is typically the satellite which causes problems!

of energy, and consequently shows much less deviation from constant for all algorithms except the RKF2(3) pair.

Integrator:	RKF7(8)	RKF2(3)	Polar $h = 300 s$	Polar $h = 100 s$
fevals	1.24E+5	9.88E+3	1.50E+5	3.62E+5
NC-steps	-	-	0	1
Sun				
$\Delta x_{\rm RMS}$	-	7.32E-06	1.00E-06	4.62E-08
Δy_{RMS}	-	3.33E-06	5.56E-07	2.64E-08
$\Delta \dot{x}_{RMS}$	-	1.01E-12	2.80E-13	1.37E-14
$\Delta \dot{y}_{RMS}$	-	5.82E-13	2.22E-13	1.04E-14
Earth				
$\Delta x_{\rm RMS}$	-	5.98E+00	3.27E-01	1.50E-02
Δy_{RMS}	-	4.34E+00	1.89E-01	9.65E-03
$\Delta \dot{x}_{RMS}$	-	1.17E-05	1.08E-07	8.94E-09
Δÿ _{RMS}	-	1.30E-05	9.00E-08	8.47E-09
Moon				
$\Delta x_{\rm RMS}$	_	4.01E+02	2.17E+00	2.63E-01
Δy_{RMS}	-	3.54E+02	1.96E+00	2.43E-01
$\Delta \dot{x}_{RMS}$		9.46E-04	5.33E-06	6.57E-07
$\Delta \dot{y}_{RMS}$	-	1.05E-03	5.63E-06	6.96E-07

Table 10.6: Comparison of performance metrics for Sun–Earth–Moon planar Jacobi Problem – RKF2(3) ,RKF7(8) and Algorithm 10.5.1 with h = 300s and h = 100s respectively.

We present illustrations of this performance in figures 11.1 and 10.26, in line with the earlier discussion. We note here also the difference in magnitude between the estimated errors in the 3 bodies, corresponding prominently to their relative motion in the problem.

We also indicate the performance with respect to the energy integral in figure 10.27. We note that here the conservative integrator does not actually conserve the integral up to machine precision; it is not far off, though still differing roughly two orders of magnitude (as indicated by the red curve). We remark that the curve appears to be relatively constant in magnitude, as opposed to a secular growth, though it had an interesting 'stepped' profile corresponding to the periodicity of the Moon's motion (6 steps over 3 lunar cycles, *cf.* the dips in figures 11.1 and 10.26 as well).

In light of this discussion and the results presented, it appears that while the algorithm shows good performance relative to the RKF2(3) pair (traded against a higher RKF7(8)–like number of function evaluations), there is no clear reason to prefer this particular implementation. This seems particularly so given that it has the highest overhead of the three variants.



Figure 10.25: State difference magnitude: Algorithm 10.5.1 (h = 300 s) vs. RKF7(8) – Sun in magenta, Earth in blue and Moon in green.



Figure 10.26: State difference magnitude: Algorithm 10.5.1 (h = 100 s) vs. RKF7(8) – Sun in magenta, Earth in blue and Moon in green.



Figure 10.27: Energy Integral–Conservation Performance: Algorithm 10.4.1 (red, h = 300 s) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral. Figure also representative for Algorithm 10.4.2 and the smaller h = 100 s time–step.

Earth-Moon-Satellite

For the Earth–Moon–Satellite case, we find results for algorithm 10.5.1 in polar Jacobi coordinates roughly in line with the explicit and implicit algorithms 10.4.1–10.4.2 discussed in section 10.4. We would expect, however, on the basis of the double conservation, and the previous test–case, to find slightly better results, and that is borne out by table 10.7 and figure 10.29 below.



Figure 10.28: Planar Jacobi 3BP motion of the Earth, Moon and Satellite, in the Earth–centered psuedo–inertial frame – Algorithm 10.5.1 solution with h = 5s – Note the subsequent trajectory after swing–by, where a slightly different approach of the near–Earth region leads to a second swing–by and subsequent escape from the system.

We note that here the RMS-measure is not a very good one, as the error behaves very differently during different parts of the trajectory. In fact the trajectory is a very good match (significantly better than the explicit and implicit algorithms considered in the previous section) *up to* the swing-by encounter with the Moon, as we see on comparing figures 10.20 and 10.28. The subsequent trajectory after swing-by, however, is very different, in that a slightly different approach of the near-Earth region leads to a second swing-by and subsequent escape from the system.

Further, the energy integral is not properly conserved for this problem, as we see in figure 10.30, which is representative for the range of time–steps $2.5s \le h \le 50s$. This indicates the need for a yet–smaller time–step, which we have not investigated further. Lastly, figure 10.31 gives an indication of the performance with respect to the conservation of the angular momentum integral. In contrast to the energy integral, we find that this is conserved very well, which is in fact typical of all three algorithms on both test–cases.

Again, it seems that we may tentatively conclude that there is no clear advantage to this formulation of the algorithm, despite our expectations in light of its doubly–conservative nature. In light of the picture that emerges from evaluating the integral, as in figure 10.31, this is perhaps understandable.



Figure 10.29: State difference magnitude: Algorithm 10.5.1 (h = 2.5s) vs. RKF7(8) – Earth in blue, Moon in green and Satellite in black. Compare figure 10.23.

Integrator:	RKF7(8)	Polar $h = 50 s$	Polar $h = 5 s$	Polar $h = 2.5 s$
fevals	1.24E+5	6.80E+5	6.40E+6	1.28E+7
NC-steps	-	0.015%	0.014%	0.014%
Earth				
$\Delta x_{\rm RMS}$	-	3.92E+00	3.65E+00	1.83E+00
Δy_{RMS}	-	3.89E+00	3.21E+00	1.61E+00
$\Delta \dot{x}_{RMS}$	-	9.94E-06	8.58E-06	4.31E-06
$\Delta \dot{y}_{RMS}$	-	1.02E-05	9.78E-06	4.90E-06
Moon				
$\Delta x_{\rm RMS}$	-	3.19E+02	2.97E+02	1.49E+02
Δy_{RMS}	-	3.16E+02	2.61E+02	1.31E+02
$\Delta \dot{x}_{RMS}$	-	8.08E-04	6.98E-04	3.50E-04
$\Delta \dot{y}_{RMS}$	-	8.30E-04	7.95E-04	3.99E-04
Satellite				
$\Delta x_{\rm RMS}$	-	1.63E+06	8.01E+06	1.68E+06
Δy_{RMS}	-	7.80E+05	2.23E+07	1.81E+06
$\Delta \dot{x}_{RMS}$	-	8.33E-01	4.89E+00	9.89E-01
Δÿ _{RMS}	-	5.48E-01	1.26E+01	9.30E-01

Table 10.7: Comparison of performance metrics for Earth–Moon–Satellite planar Jacobi 3–body problem – RKF2(3), RKF7(8) and Algorithms 10.5.1 for $h_0 = 100$, 10, 5*s*.



Figure 10.30: Energy Integral–Conservation Performance: Algorithm 10.5.1 (red, h = 10s) vs. RKF7(8) (blue); also included is a dashed black line for the true value of the integral. Performance representative for range $2.5s \le h \le 50s$


Figure 10.31: Angular Momentum Integral–Conservation Performance: Algorithm 10.5.1 (red, h = 300 s, blue, h = 100 s) vs. RKF7(8) (green); also included is a dashed black line for the true value of the integral.

10.6 On Conservation of Multiple Integrals in General

Let us make some closing remarks on the conservation of multiple integrals in general.

Essentially, the above case of double conservation is a "lucky" side–effect of the formulation in polar coordinates, which makes the conservation of the angular momentum easy to implement. It is important to note, however, that expression (10.5) is a *single* function of two integrals. This is in principle always possible *if* the linearizing transformation for the two integrals is the same, for then:

$$I_1 = c_1 \cdot \xi = c_1 \cdot T(x) \quad \text{and}$$

$$I_2 = c_2 \cdot \xi = c_2 \cdot T(x) \quad \text{can be combined as:}$$

$$I_3 = (c_1 + c_2) \cdot T(x) . \quad (10.7)$$

At the very least, the transformations must be compatible in the sense we outline below, and this is the benefit of polar coordinates. In general however, the quadratic forms involved of the transformation for conservation of the Hamiltonian makes it difficult to envision further compatible integrals.

Some possibilities and difficulties do present themselves:

• The exception to this difficulty which Bowman *et al.* exploit is that, much as with our explicit formulation of a singly–conservative integrator, we can always add a term to one of the transformed variables, as long as it does not obstruct the reconstruction sequence in the inverse transformation (this is the sense of compatibility we intended above).

We remark that it does appear, however, that this may have some consequences for the conservation of both, as the effects of machine error become coupled in a non-trivial way. Indeed this may explain the stepping behavior observed in figure 10.27.

• Further, one way in which such an obstruction could occur also happens to set practical limits on what we can do with the other coordinates. A reason that we are able to formulate an implicit algorithm along the above lines, is that we can reconstruct the coordinate using the potential and the remaining coordinates using an efficient algorithm: Newton–Raphson.

This is no longer possible (at least, not efficiently in a robust sense) if we remove a further coordinate from the set on which U depends, such that we must consider instead an *e.g.* 2–dimensional root–finding problem. In that case there are no known "good" algorithms, and we would be reliant on 2– or multi–dimensional Newton–Raphson type codes which are very strongly dependent on the initial guess, and in the limit of this dependence we would hypothetically no longer need a correction–step in our algorithm, *i.e.* we undermine the assumptions of the prototype on which our framework is built.

• That leaves the possibility of conservation of an integral which can be part of the transformation T(x) analogous to the potential, but which leads to a root-finding problem in different variables than those determining U(x), such that the root-finding routines can be run if not in parallel, then at least sequentially, we may have a way out of the above dilemma. While this is certainly possible, it is not to say that nature will oblige by providing us with such a useful integral of course.

Despite the above, rather abstract discussion, it is prudent to bear the following in mind. A doubly–conservative integrator as formulated in algorithm 10.5.1 applies to the Jacobi problem, which was in a sense already maximally reduced with the other 8 integrals of motion, effectively exploiting all the known integrals. There is a caveat in that this is only true for the planar restriction of the problem, but the extension of the above integrator by the addition of a z–coordinate seems a straightforward next step. Barring unforeseen difficulties with the formulation of the angular momentum in these coordinates, it should be possible to extend to the 3–dimensional case.

That this does not change when we extend the model itself beyond 3 bodies implies that the "low–hanging fruit" has *already* been picked, and that for the foreseeable future, the best candidate for similar conservative approaches would lie with new integral–approximations of the kind discussed in Part II of this thesis.

Supposing of course that headway *can* be made in that area, it further implies that any additional integrals being incorporated into a multiply–conservative integrator will likely be of a relatively complex form, and consequently, it is unlikely that true multiply–(approximately)–conservative algorithms can be formulated that are not so computationally complex that their implementation poses a problem of diminishing returns.

Chapter 11

Approximately–Conservative Integrators

In this chapter we discuss the design and performance of 2^{nd} –order integrators which conserve an approximation (to a given order¹) to an integral of motion. We present results for the circular restricted 3–body problem and the capture problem only; the process of design for the Jacobi 3–body problem is exactly analogous, with its discussion differing mainly in details of implementation, which are of lesser interest.

11.1 Convergence of Integral Approximations

Before formulating the two approximately conservative integrators we discuss in this chapter, let us briefly consider the convergence of the corresponding integral approximations constructed in Part II of this thesis (sections 7.2 and 8.2).

It is not possible, nor is it useful, for present purposes, to give a general consideration of the convergence of integral approximations (even for analytically known integrals) for an *arbitrary* orbit of the 3–body problem, whether full or restricted. However, we can predict the performance of integral approximations for our test–cases, using *e.g.* our RKF7(8) integrator pair solutions as input. We have done this for the CR3BP in particular.

The approximations we consider are (in their naive formulation):

$$\begin{split} \mathcal{I} &= \frac{1}{2} \left(x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2}{\left(x^2 + y^2\right)^{1/2}} \right) - \left(\frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{1/2}} - \frac{x}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(x^2 + y^2\right)^{1/2}} \right) \varepsilon \\ &- \left(\frac{3x^2}{2\left(x^2 + y^2\right)^{5/2}} - \frac{1}{2\left(x^2 + y^2\right)^{3/2}} + \frac{1}{\left(x^2 + y^2\right)^{1/2}} + \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} - \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{1/2}} \right) \varepsilon \\ &+ x \left(\frac{2}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} \right) \right) \varepsilon^2 + \mathcal{O} \left(\varepsilon^3 \right) \,, \end{split}$$

with $\varepsilon = 10^{-2}$ for the CR3BP. For the capture problem, with $\varepsilon = 10^{-1}$:

¹In a small parameter ε resulting from a perturbation formulation of the system, as discussed extensively in Part II of this thesis.

$$\begin{split} I &= -\frac{1}{2} \left(x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2}{\left(x^2 + y^2\right)^{1/2}} \right) - \left(\frac{1}{\left(\left(x - 1\right)^2 + y^2 \right)^{1/2}} - \frac{x}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(x^2 + y^2\right)^{1/2}} \right) \, \epsilon^6 \\ &- \frac{1}{\left(\left(x - 1\right)^2 + y^2 \right)^{1/2}} \, \epsilon^8 + \mathcal{O}\left(\epsilon^{11} \right) \, . \end{split}$$

However, as remarked earlier in chapters 6 - 8 these expressions were derived under simplifying assumptions for purposes of clarity. When comparing with real simulations based on real initial conditions, we must of course correct them to the true expressions, which is easily done in MATHEMATICA. The calculations are provided on the CD included with this thesis as the notebook **IntegralApproxExpressions.nb**, for reference.

We find there that the correct expressions are:

$$\begin{aligned} \mathcal{I} &= -\frac{1}{2} \left(x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2}{\left(x^2 + y^2\right)^{1/2}} \right) - \left(\frac{1}{\left(\left(x - 1\right)^2 + y^2 \right)^{1/2}} - \frac{x}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(x^2 + y^2\right)^{1/2}} \right) \bar{m}_2 \varepsilon \\ &- \left(\frac{3x^2}{2 \left(x^2 + y^2\right)^{5/2}} - \frac{1}{2 \left(x^2 + y^2\right)^{3/2}} + \frac{1}{\left(x^2 + y^2\right)^{1/2}} + \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} - \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{1/2}} \right) \\ &+ x \left(\frac{2}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(\left(x - 1\right)^2 + y^2\right)^{3/2}} \right) \right) (\bar{m}_2 \varepsilon)^2 + \mathcal{O}\left((\bar{m}_2 \varepsilon)^3 \right), \end{aligned}$$
(11.1)

and

$$I = -\frac{1}{2} \left(x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2}{\left(x^2 + y^2\right)^{1/2}} \right) - \left(\frac{1}{\left((x-1)^2 + y^2\right)^{1/2}} - \frac{x}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\left(x^2 + y^2\right)^{1/2}} \right) \bar{m}_2 \varepsilon^6 - \frac{1}{\left((x-1)^2 + y^2\right)^{1/2}} \bar{m}_3 \varepsilon^8 + O\left(\varepsilon^{11}\right) .$$
(11.2)

Here, as earlier in the work of Part II of this thesis:

1

 $m_0 = m_1$ for each problem, and so: (11.3)

$$\bar{m}_2 = \frac{m_2}{m_0 \epsilon}$$
 for the CR3BP, while (11.4)

$$\begin{cases} \bar{m}_2 &= \frac{m_2}{m_0 e^6} \\ \bar{m}_3 &= \frac{m_3}{m_0 e^8} \end{cases} \text{ for the CP.}$$
(11.5)

These approximate integral expressions (11.1)–(11.2) have been used in the approximately conservative integrators discussed in this chapter, but we also use them to obtain the qualitative impression of the speed of convergence for the integrators.

1. The results for the Sun–Earth–Moon circular restricted 3–body problem are given in figure 11.1.



Figure 11.1: Estimation of the convergence of the integral by $O(\epsilon)$ using the Sun–Earth–Moon CR3BP test–case.

We see that the convergence is quite quick, and reaches 10 digits of precision by the third term in the approximation for these types of trajectories. We also note that not all terms of the approximation contribute in the same way, with the expansion's RMS–deviations appearing to stagnate at alternating orders, which may of course be useful computationally.

- 2. The situation for the Earth–Moon–Satellite configuration is illustrated in figure 11.2, and we will postpone the discussion it until later. However, we note briefly that the convergence appears to be considerably slower, which is in line with the much smaller magnitudes of the ε^k terms at each step in the approximation, since the corrections by powers of ε are much larger for this problem than in a Sun–Earth–Moon configuration.
- 3. The convergence in the capture problem in turn cannot strictly be estimated, as we do not have a limit to which it converges. However, qualitatively, we remark that it is based on the Jacobi integral \mathcal{I} for the Sun-Earth-Moon CR3BP, and we might naively expect the convergence to be comparable. The caveat lies in the fact that the

 $\mathcal{O}(\varepsilon^8)$ term corresponds precisely to the $\mathcal{O}(\varepsilon^0)$ term of the Moon, and has a relatively small impact, as we shall see, so we effectively have only 1 contributing term. This makes convergence slower than we might have hoped.

Approximately Conservative Integrators for the CR3BP 11.2

We first consider the algorithm we have developed for the circular restricted 3-body problem.

11.2.1 **Integrator Design**

In order to formulate the approximately conservative integrator for the circular restricted 3-body problem, we base ourselves on the core algorithm 10.2.3 and algorithm 10.3.1 for the CR3BP specifically. This is then modified to algorithm 11.2.1.

Algorithm 11.2.1 (CR3BP-based Approximately Conservative Integrator)

▷ Initial time $t := t_0$ $\mathbf{x} := (x_0, y_0, \dot{x}_0, \dot{y}_0)^{\mathsf{T}}$ ⊳ Initial state k := 0▷ Output counter $X_{\text{out}}[k] \leftarrow \mathbf{x}$ $h := \frac{t_f - t_0}{N}$ ▷ First entry in 2D output array ▷ Determine step size **for** $t = t_0 : h : t_f$ **do** Take Conservative Integration Step ▷ Algorithm 10.2.3 if (Intermediate Output Point k) then $X_{\text{out}}[k] \leftarrow \mathbf{x}$ ▷ Next entry in output array $k \leftarrow k + 1$ ▷ Increment counter end if end for **Procedures:** function $f(\mathbf{x})$ $f \leftarrow \begin{pmatrix} \dot{x} = \mathbf{x}[3] \\ \dot{y} = \mathbf{x}[4] \\ 2\dot{y} + x - \frac{1 - \mu}{r_1^3} (x + \mu) - \frac{\mu}{r_2^3} (x - (1 - \mu)) \\ -2\dot{x} + y - \frac{1 - \mu}{r_2^3} y - \frac{\mu}{r_2^3} y \end{pmatrix}$

end function

function $T(\mathbf{x})$

 \triangleright 1st derivatives already available!

$$T \leftarrow \begin{pmatrix} \frac{1}{2}x^2 \\ \frac{1}{2}y^2 \\ \frac{1}{2}\dot{x}^2 + U_{app} \\ \frac{1}{2}\dot{y}^2 \end{pmatrix}$$

 \triangleright **Approximate** potential terms absorbed into 3^{rd} coordinate

end function

function $f_{\xi}(\mathbf{x})$

$$f_{\xi} \leftarrow \begin{pmatrix} x\dot{x} \\ y\dot{y} \\ \dot{x}\ddot{x} + \frac{\partial U_{app}(x, y, \varepsilon)}{\partial x} + \frac{\partial U_{app}(x, y, \varepsilon)}{\partial y} \\ \dot{y}\ddot{y} \end{pmatrix}$$
end function

 $\triangleright \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$ must now be formed explicitly

function
$$T^{-1}(\mathbf{x}_p, \xi)$$

$$T^{-1} \leftarrow \begin{pmatrix} \operatorname{sgn}(x_p)\sqrt{2\xi_1} \\ \operatorname{sgn}(y_p)\sqrt{2\xi_2} \\ \operatorname{sgn}(\dot{x}_p)\sqrt{2(\xi_3 - U_{app})} \\ \operatorname{sgn}(\dot{y}_p)\sqrt{2\xi_4} \end{pmatrix}$$
end function

End Procedures

Note the following in algorithm 10.3.1:

- The reader will note that in being based on the CR3BP-integrator of algorithm 10.3.1 this integrator is explicitly formulated with respect to the coordinates, but this is not an essential point.
- As noted above, for the Earth–Moon–Satellite problem we take $\varepsilon = 10^{-2}$. This too is not essential, in that the same algorithm works for the Sun–Earth–Moon problem if we adjust ε to that problem, and we remark that for purposes of the algorithm, we may there directly take $\varepsilon = 10^{-6}$.
- In formulating conservation of the approximate integral, we note that the approximate nature comes from the Jacobi integral's potential terms, which we have denoted U_{app} above. In contrast to the earlier cases, we can no longer enforce the conservation of this using the other coordinates, since conservation is only approximate. Instead, we must form the terms $\frac{\partial U_{app}}{\partial x}$ and $\frac{\partial U_{app}}{\partial y}$ explicitly. We give the expressions below in section 11.2.2.
- In forming these terms, we note that $U_{app}(x, y, \varepsilon)$ and its derivatives are formed by the order, which is passed to the algorithm as a variable of the same name by the IntegrationMethods driver routine. Algorithm 11.2.2 outlines the construction.
- Lastly, we note that our approximate integral is determined by the linearizing transformation ξ = T(x) and the vector c = (-1, -1, +1, +1)^T, which the reader will note is indeed in the appropriate form c ⋅ ξ of section 5.3.2, if we take U_{app} as discussed in the next section.

11.2.2 Approximation Construction

,

The necessary approximations are given by the following expressions:

$$[U_{app}]_0 = -\frac{1}{\left(x^2 + y^2\right)^{1/2}} \tag{11.6a}$$

at $\mathcal{O}(\epsilon^0)$, which we denote order=0 in terms of the variable passed,

$$[U_{app}]_{1} = -\left(\frac{1}{\left((x-1)^{2}+y^{2}\right)^{1/2}} - \frac{x}{\left(x^{2}+y^{2}\right)^{3/2}} - \frac{1}{\left(x^{2}+y^{2}\right)^{1/2}}\right)\bar{m}_{2}\varepsilon$$
(11.6b)

at $\mathcal{O}(\epsilon^1)$, which we denote order=1 in terms of the variable passed,

$$[U_{app}]_{2} = -\left(\frac{3x^{2}}{2(x^{2}+y^{2})^{5/2}} - \frac{1}{2(x^{2}+y^{2})^{3/2}} + \frac{1}{(x^{2}+y^{2})^{1/2}} + \frac{1}{((x-1)^{2}+y^{2})^{3/2}} - \frac{1}{((x-1)^{2}+y^{2})^{1/2}} + x\left(\frac{2}{(x^{2}+y^{2})^{3/2}} - \frac{1}{((x-1)^{2}+y^{2})^{3/2}}\right)\right)(\bar{m}_{2}\varepsilon)^{2}$$

$$(11.6c)$$

at $O(\epsilon^2)$, which we denote order=2 in terms of the variable passed.

Consequently for the approximation derivatives, we find that:

$$\left[\frac{\partial U_{app}}{\partial x}\right]_0 = \frac{x}{\left(x^2 + y^2\right)^{3/2}} \tag{11.7a}$$

at $\mathcal{O}(\epsilon^0)$, which we denote order=0 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial x}\right]_{1} = \bar{m}_{2}\varepsilon \left(\frac{(-1+x)}{\left((-1+x)^{2}+y^{2}\right)^{3/2}} - \frac{3x^{2}}{\left(x^{2}+y^{2}\right)^{5/2}} + \frac{1}{\left(x^{2}+y^{2}\right)^{3/2}} - \frac{x}{\left(x^{2}+y^{2}\right)^{3/2}}\right)$$
(11.7b)

at $\mathcal{O}\left(\epsilon^{1}\right)$, which we denote order=1 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial x}\right]_{2} = \frac{1}{2}\bar{m}_{2}^{2}\varepsilon^{2}\left(\frac{15x^{3}}{\left(x^{2}+y^{2}\right)^{7/2}} - \frac{9x}{\left(x^{2}+y^{2}\right)^{5/2}} - \frac{4}{\left(x^{2}+y^{2}\right)^{3/2}} + \frac{2x}{\left(x^{2}+y^{2}\right)^{3/2}} + \frac{3\left(-2+2x\right)}{\left(1-2x+x^{2}+y^{2}\right)^{5/2}} + \frac{2}{\left(1-2x+x^{2}+y^{2}\right)^{3/2}} - \frac{-2+2x}{\left(1-2x+x^{2}+y^{2}\right)^{3/2}} + x\left(\frac{12x}{\left(x^{2}+y^{2}\right)^{5/2}} - \frac{3\left(-2+2x\right)}{\left(1-2x+x^{2}+y^{2}\right)^{5/2}}\right)\right)$$
(11.7c)

at $O(\varepsilon^2)$, which we denote order=2 in terms of the variable passed,

with respect to x and:

$$\left[\frac{\partial U_{app}}{\partial y}\right]_{0} = \frac{y}{\left(x^{2} + y^{2}\right)^{3/2}}$$
(11.8a)

at $\mathcal{O}\left(\epsilon^{0}\right)$, which we denote order=0 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial y}\right]_{1} = \bar{m}_{2} \varepsilon \left(\frac{y}{\left((-1+x)^{2}+y^{2}\right)^{3/2}} - \frac{3xy}{\left(x^{2}+y^{2}\right)^{5/2}} - \frac{y}{\left(x^{2}+y^{2}\right)^{3/2}}\right)$$
(11.8b)

at $\mathcal{O}(\epsilon^1)$, which we denote order=1 in terms of the variable passed,

$$\begin{bmatrix} \frac{\partial U_{app}}{\partial y} \end{bmatrix}_{2} = \frac{1}{2} \bar{m}_{2}^{2} \epsilon^{2} \left(\frac{15x^{2}y}{\left(x^{2} + y^{2}\right)^{7/2}} - \frac{3y}{\left(x^{2} + y^{2}\right)^{5/2}} + \frac{2y}{\left(x^{2} + y^{2}\right)^{3/2}} + \frac{6y}{\left(1 - 2x + x^{2} + y^{2}\right)^{5/2}} - \frac{2y}{\left(1 - 2x + x^{2} + y^{2}\right)^{3/2}} + x \left(\frac{12y}{\left(x^{2} + y^{2}\right)^{5/2}} - \frac{6y}{\left(1 - 2x + x^{2} + y^{2}\right)^{5/2}} \right) \right)$$

at $O(\varepsilon^2)$, which we denote order=2 in terms of the variable passed,

with respect to y.

The actual construction is then performed stepwise by the following algorithm:

Algorithm 11.2.2 (Approximation Constructor)

$$\begin{array}{l} U_{app} \leftarrow 0 \\ \frac{\partial U_{app}}{\partial x} \leftarrow 0 \\ \frac{\partial U_{app}}{\partial y} \leftarrow 0 \end{array}$$

for i = 0 : 1 : order do

$$U_{app} \leftarrow U_{app} + \left[U_{app} \right]_{i} \qquad \qquad \triangleright \text{ Add order } i \text{ term}$$

$$\frac{\partial U_{app}}{\partial x} \leftarrow \frac{\partial U_{app}}{\partial x} + \left[\frac{\partial U_{app}}{\partial x} \right]_{i} \qquad \qquad \triangleright \text{ Add order } i \text{ term}$$

$$\frac{\partial U_{app}}{\partial y} \leftarrow \frac{\partial U_{app}}{\partial y} + \left[\frac{\partial U_{app}}{\partial y} \right]_{i} \qquad \qquad \triangleright \text{ Add order } i \text{ term}$$

end for

In the above algorithm terms of the form $\left[\frac{\partial U_{app}}{\partial y}\right]_i$ refer to the *i*th term in the approximation of the potential, or the derivative thereof, *cf.* equations (11.6)–(11.8).

11.2.3 Integrator Performance

We next turn to a consideration of integrator performance.

Sun-Earth-Moon

Given our interest in the phenomenon of capture, our design for the CR3BP is best seen as a stepping stone to the capture problem, for which the reader will recall that we have only an approximate integral to work with. As a result, in the performance considerations below, we will focus on the performance on satellite trajectories in the Earth–Moon region, and so a omit discussion of the results of the approximate conservation scheme for the Sun–Earth–Moon circular restricted 3–body problem.

We do remark, however, that given the dominance of the main masses, the simulation results confirm that with $\varepsilon = 10^{-6}$ in this case (*cf.* section 7.2.1), the approximate integrals converge quickly, as we suspected in section 11.1. The corrections of the $O(\varepsilon^3)$ terms are already beyond the double–precision level of accuracy available to us in the MAT-LAB environment, and at $O(\varepsilon^2)$ we reach machine precision in our deviations from ideal.²

The interested reader will find simulation output in the directory model3-approx on the CD included with this thesis, which can be visualized using the included MATLAB scripts (for details, see appendix C).

Earth-Moon-Satellite

We thus focus in this section on the same Earth–Moon–Satellite test–case as in section 10.3.2. The trajectory in the (natural) Earth–Moon–rotating system has already been given in figure 10.9, corresponding to initial condition:

$$\mathbf{x}_{0} = \begin{pmatrix} -\mu r_{EM} - 6.999993 \cdot 10^{3} \\ -9.893934 \\ +9.685109 \cdot 10^{-1} \\ +1.058167 \cdot 10^{1} \end{pmatrix}$$

Here, we compare the performance of the algorithm 11.2.1 for 4 different approximations of the Jacobi integral \mathcal{J} , constructed as explained in section 11.2.2. Let us begin by considering the predicted convergence based on the RKF7(8) solution of the CR3BP, given in figure 11.2. We see, on the basis of both the RKF7(8) pair and the solution via algorithm 10.3.1, that we may expect convergence to be considerably slower than that of the above test–case. This is understandable as now $\varepsilon = 10^{-2}$ rather than³ $\varepsilon = 10^{-6}$ (again *cf.* section 7.2.1), and as such the deviation from the initial value goes down more slowly.

Remark though, that in figure 11.2 we give the RMS-deviation from the initial value by order, and this means that the results are worse than might be expected for general trajectories in the CR3BP, since this test-case involves phenomena such as the swing-by near step 6060 which are difficult for the integrator to handle, increasing the error considerably, as discussed earlier in section 10.3.2.

We note also, considering this figure, that there is a divergence between the predictions of convergence due to the conservative scheme and the RKF integrator pair. Comparing the actual reduction by order in table 11.1, it appears that the reduction estimated by algorithm 10.3.1 was too optimistic, and that the actual performance is closer to that predicted by the RKF7(8) data.

 $^{^{2}}$ This could of course, already be estimated using expressions (11.6)–(11.8).

³In the Sun–Earth–Moon configuration.



Figure 11.2: Predicted convergence of integral approximations based on Algorithm 10.3.1 and RKF7(8) solutions of the CR3BP – First plot shows error at departure which converges to a stable value, second plot shows disturbance from this stable value by swing–by of Moon (Ordering is $O(\varepsilon^0)$ –red, $O(\varepsilon^1)$ –green, $O(\varepsilon^2)$ –blue, $O(\varepsilon^3)$ –magenta). Third plot shows expected convergence behavior of the RMS–deviation from the Jacobi integral's true value, by order of the approximation (red is algorithm 10.3.1, blue is the RKF7(8) pair).

When we consider the usual performance metrics for the approximately conservative scheme, we find the results of table 11.1, and give the corresponding state magnitude difference plots (as an estimate of the error) for the different orders in figure 11.3. The reader may compare these with figures 10.10 and 10.11 for the fully conservative and RKF2(3) integrator pair solutions respectively.

We note, on comparing with the earlier fully–conservative algorithm 10.3.1 that the estimated error is roughly an order of magnitude higher than previously, which is in line with what we expect for relatively slow convergence of the approximate integral.

It is, however, incumbent upon us to point out that using the integral approximations rather than the true integral in closed analytical form adds considerably to the computational effort involved. This is particularly so in terms of floating point operations per (transformed) force function evaluation $f_{\xi} = T'(\mathbf{x}) \cdot \dot{\mathbf{x}}$, which is not yet reflected in the 'fevals' count in table 11.1. In consequence, it would probably not be worthwhile to proceed along these lines in practice, at least in this type of problem involving a small satellite.

Integrator:	RKF7(8)	Approx $O(\epsilon^0)$	Approx $O(\varepsilon^1)$	Approx $O(\varepsilon^2)$	Approx $O(\varepsilon^3)$
fevals	1.24E+5	3.21E+6	3.21E+6	3.21E+6	3.21E+6
NC-steps	-	2	0	0	0
Satellite					
$\Delta x_{\rm RMS}$	-	1.56E+4	1.03E+4	6.23E+3	3.30E+3
Δy_{RMS}	-	1.63E+4	1.08E+4	6.61E+3	3.52E+3
$\Delta \dot{x}_{RMS}$	_	3.32E-1	3.04E-1	2.64E-1	2.14E-1
Δý _{RMS}	-	3.83E-1	3.49E-1	3.00E-1	2.37E-1

 Table 11.1: Comparison of performance metrics for Earth–Moon–Satellite pCR3BP with approximately conservative algorithms of varying order for a 10 second time–step.



Figure 11.3: State difference magnitude for Satellite: Algorithm 11.2.1 at varying orders for a 10 second time–step vs. RKF7(8) – Ordering of integral approximations conserved is $\mathcal{O}(\varepsilon^0)$ –red, $\mathcal{O}(\varepsilon^1)$ –green, $\mathcal{O}(\varepsilon^2)$ –blue, $\mathcal{O}(\varepsilon^3)$ –black.

For comparison of what these plots mean in practice, we give for comparison figures 11.4 and 11.5, which show the RKF7(8) trajectories and an approximately conservative scheme together in one plot, with the order=0 approximation in 11.4 and the highest order=3 approximation in 11.5.



Figure 11.4: Comparison of trajectories in Earth–Moon–rotating system – Algorithm 11.2.1 with order=0 (*i.e.* $O(\varepsilon^0)$) approximation in black, reference RKF7(8) integrator pair in blue.

In these we see that while the function evaluations necessary are relatively high, the rough form of the trajectory is already present in the $O(\epsilon^0)$ approximate integrator. However, this will typically not track the details of the swing-by accurately enough, forcing us to use a higher-order approximation anyway (assuming we were to persist in this approach for this particular problem).



Figure 11.5: Comparison of trajectories in Earth–Moon–rotating system – Algorithm 11.2.1 with order=3 (*i.e.* $O(\epsilon^3)$) approximation in black, reference RKF7(8) integrator pair in blue.

11.3 Approximately Conservative Integrators for Capture

Finally, we shall consider the design and performance of the approximately conservative algorithm we have developed for the Capture Problem.

11.3.1 Integrator Design

In order to formulate the approximately conservative integrator for the Capture Problem, we base ourselves on algorithm 11.2.1 in the previous section. The key change, remarking the structural similarity between expressions (11.1) and (11.2), is that now the terms we denoted as an "approximate potential" change form, and so algorithm 11.2.1 is unchanged, as is the approximation constructor 11.2.2.

What does change, however, are the expressions used in the algorithms to form the approximations:

$$[U_{app}]_0 = -\frac{1}{\left(x^2 + y^2\right)^{1/2}}$$
(11.9a)

at $\mathcal{O}\left(\epsilon^{0}\right)$, which we denote order=0 in terms of the variable passed,

$$[U_{app}]_1 = -\bar{m}_2 \varepsilon^6 \left(\frac{1}{\sqrt{(-1+x)^2 + y^2}} - \frac{x}{\left(x^2 + y^2\right)^{3/2}} - \frac{1}{\sqrt{x^2 + y^2}} \right)$$
(11.9b)

at $\mathcal{O}(\epsilon^6)$, which we denote order=1 in terms of the variable passed,

$$[U_{app}]_2 = -\frac{\bar{m}_3 \varepsilon^8}{\sqrt{(-1+x)^2 + y^2}}$$
(11.9c)

at $O(\epsilon^8)$, which we denote order=2 in terms of the variable passed.

Consequently for the approximation derivatives, we find that:

$$\left[\frac{\partial U_{app}}{\partial x}\right]_{0} = \frac{x}{\left(x^{2} + y^{2}\right)^{3/2}}$$
(11.10a)

at $O(\varepsilon^0)$, which we denote order=0 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial x}\right]_{1} = -\bar{m}_{2}\varepsilon^{6}\left(-\frac{-1+x}{\left((-1+x)^{2}+y^{2}\right)^{3/2}} + \frac{3x^{2}}{\left(x^{2}+y^{2}\right)^{5/2}} - \frac{1}{\left(x^{2}+y^{2}\right)^{3/2}} + \frac{x}{\left(x^{2}+y^{2}\right)^{3/2}}\right)$$
(11.10b)

at $O(\epsilon^6)$, which we denote order=1 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial x}\right]_{2} = \bar{m}_{3} \varepsilon^{8} \frac{(-1+x)}{\left((-1+x)^{2}+y^{2}\right)^{3/2}}$$
(11.10c)

at $\mathcal{O}(\epsilon^8)$, which we denote order=2 in terms of the variable passed,

with respect to x and:

$$\left[\frac{\partial U_{app}}{\partial y}\right]_{0} = \frac{y}{\left(x^{2} + y^{2}\right)^{3/2}}$$
(11.11a)

at $O(\varepsilon^0)$, which we denote order=0 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial y}\right]_{1} = -\bar{m}_{2}\varepsilon \left(-\frac{y}{\left((-1+x)^{2}+y^{2}\right)^{3/2}} + \frac{3xy}{\left(x^{2}+y^{2}\right)^{5/2}} + \frac{y}{\left(x^{2}+y^{2}\right)^{3/2}}\right)$$
(11.11b)

at $\mathcal{O}(\epsilon^6)$, which we denote order=1 in terms of the variable passed,

$$\left[\frac{\partial U_{app}}{\partial y}\right]_{2} = \bar{m}_{3}\varepsilon^{8} \frac{y}{\left((-1+x)^{2}+y^{2}\right)^{3/2}}$$
(11.11c)

at $\mathcal{O}(\epsilon^8)$, which we denote order=2 in terms of the variable passed,

with respect to y. The actual construction is then performed stepwise by the algorithm 11.2.2 given above.

11.3.2 Integrator Performance

In order to evaluate the performance of the approximate integrator in the capture problem, we of course first need a ballistic lunar capture trajectory. While the initial conditions given in the previous chapter are close (being based on initial conditions used in [Verzijl, 2005]), they do not lead to capture in the CP model. This is due to the fact that the initial conditions used there were based on the full DE405 ephemeride model [Standish, 1997], in contrast to the simplified Capture Problem model we introduced in section 3.5 and treat here.

However, as the model we introduced indeed reproduces all the essential features necessary for ballistic capture trajectories, it is not that difficult to find a suitable set of initial conditions in CP–model coordinate system as:

$$\mathbf{x}_{0} = \begin{pmatrix} (1-\mu) r_{ES} + 6.999993 \cdot 10^{3} \\ -9.893934000 \\ +9.719510960 \cdot 10^{-1} \\ +1.059463426 \cdot 10^{1} \end{pmatrix}$$

The resulting orbit has the geometric form typical of exterior ballistic capture trajectories (*cf.* figure 3.9), is illustrated in figure 11.6 in Sun–Earth–rotating coordinates. Figure 11.7 gives some further detail of the capture, and we see that the radial distance goes down below 10000 km during the capture phase. Likewise this capture is unstable, and after a period of some days the satellite leaves the vicinity of the Moon, its Moon–relative energy \mathcal{E}_2 again becoming positive (*cf.* section 3.4).

Let us begin by again considering the predicted convergence based on the RKF7(8) solution of the CR3BP, given in figure 11.8. We see there that while the integral is near constant throughout, it diverges considerably at departure and in the capture region, where if we assume that it is converging back to the constant solution, it is doing so quite slowly. Thus, on the basis of the RKF7(8) pair data, we may expect convergence to be slower, as in the previous CR3BP test–case (though for different reasons, as outlined in section 11.1: while the basis of the capture problem is the Sun–Earth–Moon CR3BP, this slow behavior is due to the availability of so few terms in our approximation.)

When we consider the actual performance of the approximately conservative scheme, we find the results of tables 11.2



Figure 11.6: Capture Problem motion of the Satellite relative to the Earth and Moon in Earth–Sun–rotating coordinates – trajectory demonstrates unstable ballistic lunar capture effected by a direct impulse transfer from a 600 km parking orbit, as in [Verzijl, 2005] (Earth in blue, Moon in green, Satellite in black).

and 11.3, and give the corresponding state magnitude difference plots (as an estimate of the error) for the different orders of approximation in figures 11.9–11.10 for h = 10s resp. h = 1s integration steps.

Integrator:	RKF7(8)	Approx $O(\varepsilon^0)$	Approx $O(\varepsilon^1)$	Approx $O(\epsilon^2)$	Approx $O(\varepsilon^2)$ $(h = 10s)$
fevals	1.56E+5	4.01E+7	4.01E+7	4.01E+7	4.04E+6
NC-steps	-	2	0	0	0
Satellite					
$\Delta x_{\rm RMS}$	-	8.58E+3	5.42E+3	5.70E+3	1.15E+5
Δy_{RMS}	-	1.02E+4	5.91E+3	6.21E+3	1.63E+5
$\Delta \dot{x}_{RMS}$	-	8.89E-2	5.47E-2	5.74E-2	3.89E-1
$\Delta \dot{y}_{RMS}$	-	9.07E-2	5.55E-2	5.84E-2	3.83E-1

Table 11.2: Comparison of performance metrics for Capture Problem with approximately conservative algorithms of varying order for a 1 second time–step ($O(\epsilon^2)$) term included with both 1 and 10 second time–steps for reference).



Figure 11.7: Capture Details – illustrated are negative 2–body energy \mathcal{E}_2 , very close approach in terms of radial distance, and an extended period of close coupling of velocities (relative velocity near 0).

Integrator:	RKF7(8)	KF7(8) RKF2(3) Simple Predictor Correcto		Approx $O(\epsilon^2)$
fevals	1.24E+5	1.76E+4	2.00E+7	4.01E+7
NC-steps	-	0	0	0
Satellite				
$\Delta x_{\rm RMS}$	-	7.60E+3	8.58E+3	5.70E+3
Δy_{RMS}	-	8.25E+3	1.01E+4	6.21E+3
$\Delta \dot{x}_{RMS}$	-	7.66E-2	8.90E-2	5.74E-2
$\Delta \dot{y}_{RMS}$	-	7.81E-2	9.07E-2	5.84E-2

Table 11.3: Comparison of performance metrics for Capture Problem Algorithm 11.2.1 at order 2 (*i.e.* $O(\varepsilon^8)$) and h = 1s with RKF2(3) pair and Simple Predictor–Corrector, also h = 1s.



Figure 11.8: Predicted behavior of integral approximations based on Algorithm 10.3.1 and RKF7(8) solutions of the Capture Problem – First plot shows error at departure which converges to a stable value followed by disturbance from this stable value during capture by Moon. Second plot is scaled to emphasize the features of the first (Ordering is $O(\epsilon^0)$ –red, $O(\epsilon^1)$ –green, $O(\epsilon^2)$ –blue).



Figure 11.9: State difference magnitude for Satellite: Algorithm 11.2.1 at varying orders for h = 10s time-step vs. RKF7(8) – Ordering is $O(\varepsilon^0)$ –red, $O(\varepsilon^6)$ –green, $O(\varepsilon^8)$ –blue.



Figure 11.10: State difference magnitude for Satellite: Algorithm 11.2.1 at varying orders for h = 1s time–step vs. RKF7(8) – Ordering is $O(\varepsilon^0)$ –red, $O(\varepsilon^6)$ –green, $O(\varepsilon^8)$ –blue.

We see in these plots that the conservative scheme performs roughly the same as the simple predictor–corrector for the $O(\varepsilon^0)$ term, though it becomes considerably better with the $O(\varepsilon^6)$ –correction. What is interesting is that it appears to stagnate there (an issue we raised earlier), and there is no noticeable improvement on adding the $O(\varepsilon^8)$ –correction, at least for this capture trajectory. Noting that the computational effort scales with the order of the approximation, we find in this good reason to neglect the latter term.

For comparison of what these plots mean in practice, we give for comparison figures 11.11 and 11.12, which show the RKF7(8) trajectories and an approximately conservative scheme together in one plot, with the order=0 (*i.e.* $O(\varepsilon^0)$) approximation in figure 11.11 and the highest order=2 (*i.e.* $O(\varepsilon^8)$) approximation in figure 11.12.



Figure 11.11: Comparison of trajectories in Earth–Moon–rotating system – Algorithm 11.2.1 with order=0 (*i.e.* $O(\epsilon^0)$) approximation in black, reference RKF7(8) integrator pair in blue.

We also give a comparison of the error of the best approximately conservative scheme (second order or $O(\varepsilon^8)$ accurate, and taking 1 second steps) with the simple predictor corrector of algorithm 10.2.1 (likewise 1 second time–step) and the RKF2(3) integrator pair. This is illustrated in figure 11.13.

We also consider, along the lines of the above discussion, the qualitative difference in the trajectory found using these three methods in figures 11.14–11.15. It is clear that the extra computational effort makes a difference relative to the simple–predictor corrector, which finds capture, but not the moment at which it occurs, which might be important for the timing of maneuvers.

We thus develop, by the above considerations, a picture of relatively slow convergence due to the rather basic integral approximation available to us, though it does appear that the conservative formulation improves the results relative to the simple predictor–corrector. However, in light of the fact that the integrals converge to the same solution found with the Runge–Kutta-Fehlberg pair for much higher computational effort, the advantages of conservative integration likewise seem a moot point in the capture problem as well.



Figure 11.12: Comparison of trajectories in Earth–Moon–rotating system – Algorithm 11.2.1 with h = 10s and order=2 (*i.e.* $O(\varepsilon^8)$) approximation in black, reference RKF7(8) integrator pair in blue.



Figure 11.13: State difference magnitude for Satellite: Algorithm 11.2.1 order 2 (h = 1 s, black) vs. Simple Predictor–Corrector (h = 1 s, red) vs. RKF7(8) (blue).



Figure 11.14: Trajectory Difference: Algorithm 11.2.1 order 2 (h = 1 s, black) vs. Simple Predictor–Corrector (h = 1 s, red) vs. RKF7(8) (blue).



Figure 11.15: Trajectory Difference zoomed in on capture region: Algorithm 11.2.1 order 2 (h = 1 s, black) vs. Simple Predictor–Corrector (h = 1 s, red) vs. RKF7(8) (blue).

Part IV

Conclusions

Chapter 12

Conclusions

In the preceding chapters, we have presented a body of interrelated work on open problems in astrodynamics and celestial mechanics. We have developed the 3–body problem, the circular restricted 3–body problem and a simple model for 4–body ballistic lunar capture as a series of successive perturbation problems. To these, we have taken a new approach to the problem of finding integrals of the motion on the one hand, and considered the possibilities offered by a new approach to the integral–conservative numerical solution of the equations of motion of each problem.

The sum of this work constitutes the author's thesis work at the faculties of Applied Mathematics and Aerospace Engineering. Consequently, in this chapter, we summarize the conclusions of the work performed, and will give a number of recommendations for future work.

12.1 Conclusions

We summarize our conclusions by subject area, roughly coinciding with the division of this thesis into parts submitted for Applied Mathematics and for Aerospace Engineering.

12.1.1 Modeling Aspects

With regard to the modeling aspect of the thesis problem posed in the introduction:

- We have provided a detailed analysis of the links between the three key problems (planar Jacobi 3–body problem, the circular restricted 3–body problem and the capture problem), and have demonstrated that they can be conceived as a set of successive perturbations and restrictions starting from the known solution of the 2–body problem.
- We have derived and given an analysis of a simplified capture problem (3.83), which functions as a perturbation of the circular restricted 3-body problem, and shown that it is a limiting case of the more realistic quasi-bicircular problem treated by Andreu (specifically, our periodic term is the lowest, dominant harmonic in that model). In particular we note that in light of the results presented in chapter 11 for the capture problem, we may conclude that our model is indeed suited to the task of studying ballistic lunar capture for which it was developed, preserving the essential features which make such trajectories possible.

• We have introduced a framework for treating problems in astrodynamics in terms of multiple force–scales and considered the implications of its application to the capture problem. In this we have shown that a key to understanding the dynamics of the capture problem lies in the fact that an exterior ballistic capture trajectory passes through at least 3 distinct regions of physical space, which each involve different force–scales.

However, due to the time–constraints of this thesis work, our approach to the subsequent problems has been largely in terms of mass scales for the planar Jacobi 3–body problem, and the naive force–scales of the CR3BP and CP. Thus this conclusion has served us more for its explanatory power than for its application in the present work.

12.1.2 On Approximations of First Integrals

With regard to the application of the method of integrating vectors in the search for approximations of integrals of the motion:¹

- We have demonstrated the *ab initio* construction of all the known integrals for the 3-body problem and circular restricted 3-body problems, and it is hoped that the types of integrating factors found in chapters 6-8 may serve as a guide in future work.
- We have constructed the approximation (8.22) of a first integral in the specific case of the capture problem, using the integrating vectors found for the CR3BP's Jacobi integral as a foundation. The approximation is valid to O(ε⁸), formulated in mass-scales based on the Sun-Earth 2- and 3-body problems, and we have shown that the O(ε⁸) term, which is the O(1) contribution of the Moon, corresponds to the time-averaged influence of the Moon in our capture problem.

This approach is, however, limited by the time-dependence of the equations for the integrating vectors, but remains promising as a subject of further study, as we discuss in the recommendations below.

• We have also given the derivation of the systems (6.28), (7.15) and (8.24) of linear 1st –order partial differential equations which any novel approximations of new integrals must satisfy for each of our three key problems, and have pointed out the difficulties in solving them, as well as discussing possible approaches for further work.

12.1.3 On Integral–Conservative Numerical Simulation

With regard to the integral-conservative integration of the equations of motion for our three key problems:²

• We have given algorithms for the implementation of (three) 2nd –order energy–conservative integrators for the planar Jacobi 3–body problem (algorithms 10.4.1–10.5.1) and a fourth for the circular restricted 3–body problem(algorithm 10.3.1). These have been implemented in FORTRAN and benchmarked against Runge–Kutta–Fehlberg integrator pairs, where we have discussed the strengths & weaknesses of the conservative integrators.

¹Together with section 12.1.1: the Applied Mathematics work of this thesis project.

²Together with section 12.1.1: the Aerospace Engineering work of this thesis project.

In particular, we have noted the robustness of the implicit formulation in practice,³

- We further conclude that somewhat counter to our intuition⁴ the algorithms do not perform very well on problems involving a small satellite:
 - They require considerably larger⁵ time-steps than RKF-type methods to achieve similar accuracy, by our estimation. This is caused primarily by their difficulty in dealing with close-approaches to the primary bodies by the small satellite, which require a small time-step to be resolved properly. Our algorithms are formulated without step-size adjustment, and so must integrate the entire trajectory with a fixed small time-step. This causes a large disadvantage with respect to the step-size adaptive RKF algorithms.
 - Further, we observe not only the numerical instabilities due to a too large choice of time-step typical of an explicit integration scheme, but also observe evidence for singularly perturbed behavior in test cases involving a small satellite and large primaries.

Specifically, we observe that the stability of the integrator appears to be dependent on the mass of the satellite (improving with increasing mass) and the velocities during close encounters with the primaries during *e.g.* high–velocity departure and/or later swing–bys. This appears to emphasize the need for making the force–scales in a problem explicit in its numerical solution, though for the capture problem this would imply coupling solutions obtained separately in different force–regimes.

• We contrast this behavior, however, with the Sun–Earth–Moon simulations, which we might consider more typical of larger scale planetary systems. Here we find comparatively good performance, in line with the earlier findings of Bowman *et al.*. In particular, we find that the algorithms are better than the prototype on which they are based, and can hold their own against an RKF2(3) pair by trading function evaluations against increased accuracy.

This, in line with the remarks above on the potential for long time–scale error–constraint suggests the conclusion that a more promising field of application for conservative integration schemes may indeed be precisely long–timescale planetary simulations.

• We have also implemented integrators conservative of *approximate* integrals for the circular restricted 3–body problem and the capture problem, (algorithm 11.2.1 together with problem–specific variants of algorithm 11.2.2).

We note that with regard to performance, the approximately conservative integrators are strongly dependent on the speed of convergence, which is high for the Sun–Earth–Moon 3–body problem, but low for the Earth– Moon–Satellite 3–body problem. The capture problem forms an exception in that it is based on the Sun–Earth– Moon configuration, but performs badly due to the availability of only a few low–order terms.

We note moreover that the convergence appears to plateau in some cases, and in particular we observe that for the capture problem, the additional computational effort of taking the $O(\varepsilon^8)$ term into account does not appear

³In comparison with the explicit and polar formulations presented in chapter 10, in addition to its advantages with respect to numerical roundoff error.

⁴And our motivation of this approach.

⁵Order of magnitude larger than the reference RKF7(8) integrator, and often multiple orders of magnitude larger than the RKF2(3) pair.

worthwhile. We expect however, that the corrections at $O(\varepsilon^{11})$ which follow *would* make a large contribution, based on our experience with the Sun–Earth–Moon CR3BP (*cf.* figure 11.1).

• For the planar Jacobi 3-body problem in particular we have also implemented an integrator in polar coordinates, which is conservative of both the energy and the angular momentum. It appears that this integrator performs on par with the other implementations in the Sun-Earth-Moon configuration, and can perform somewhat worse in the Earth-Moon-Satellite configuration. This is likely due to a discrepancy between the role that angular momentum plays in the two problems (the satellite having little to no influence in the latter).

In consequence, for the problems studied it does not appear to be worth the additional computational effort, though the opposite may be the case in our revised conception of its application to long–scale planetary simulations.

• Finally, we remark that these implementations are themselves part of a software-development effort which has resulted in the IntegrationMethods suite of simulation and visualization code for the methods of this thesis, and which we discuss briefly in appendix C.

12.1.4 Minor Results

Finally, with regard to some minor new results developed in this thesis:

- We have clarified and slightly extended the proof of Bowman *et al.*'s conservative integration theorem slightly, which result is our proposition 5.3.1.
- We have *proved* the construction of the Jacobi integral of the circular restricted 3–body problem by the method of integrating vectors (*cf.* our proposition 7.2.1), in addition to *demonstrating* the construction of the ten 3–body problem integrals in the planar Jacobi 3–body problem.
- We have shown explicitly that any integral playing a role analogous to that of the Jacobi integral in the capture problem must necessarily be time-dependent, and contain as its time-independent part the Jacobi integral proper. This knowledge cannot, however, be used effectively at present, due to the fact that the time-dependent corrections to the integral are in principle of the same order of magnitude as the Jacobi-like terms.
- We have also discussed the relationship between local truncation error and integral approximation error in exactly and approximately conservative algorithms in section 5.4. An exactly conservative algorithm essentially forces the solution onto a lower–dimensional manifold in phase–space, since the conservation of the integral eliminates normal–to–manifold errors in the degrees of freedom corresponding to the integral.

In practice this is only true up to machine precision, but unlike the local truncation error, this normal-tomanifold error is not free to grow, but in principle remains constrained to the same order of magnitude throughout the course of the integration. We have observed (*cf.* figure 10.5) that this fact may enable the integrators to constrain the error over long time-span integrations more effectively than Runge-Kutta-based methods, though further work is necessary to establish this and the conditions under which it might hold true.

12.2 Recommendations

Against the background of the above conclusions of our work, we present the following six recommendations for future work.

12.2.1 Applied Mathematics Part

We make two key and one minor recommendation in light of our work for Applied Mathematics:

• First, with regard to the capture problem, we would like to emphasize that while we have not succeeded in solving the time-dependent PDE system for the integrating vectors within the time-constraints of work on this thesis, we envision this as being very much a tractable problem.

In particular, as we have eliminated simple lowest-harmonic time-dependence of integrating vectors, we highly recommend the application of Fourier techniques (harmonic analysis) in order to study the possibilities for a series solution which may itself be expressible in terms of our perturbation-formulation.

• Second, and more tentatively, we recommend the further investigation of 3-body problem-extensions of the Runge-Lenz vector theory, for possible approaches to developing new integral approximations in the 3-body problem. This issue, which we raised in section 6.4, is motivated by our modeling perspective in that we assume to have only slightly perturbed the 2-body problem.

We have seen the effects of this perturbation on all the integrals of the 2-body problem except the Runge-Lenz vector, suggesting that we can push the idea a bit further. We recall our earlier discussion however, and note that this may require a different approach than the techniques of this thesis, particularly recommending a study based on the perturbation of the underlying dynamical symmetry which gives rise to the integral in the 2-body problem.

• Lastly, we also recommend, in particular for our restricted CR3BP and capture problems, their extension to 3–dimensional problems in future work, for better correspondence of the model to the physical reality, in so far as the problems remain tractable.

12.2.2 Aerospace Engineering Part

We make two key and one tentative recommendation in light of our work for Aerospace Engineering:

- In light of the difficulties experienced with all the lower–order integrators (and in particular the conservative schemes) in handling of close approaches by a small satellite to large primaries, we strongly recommend two approaches to dealing with this type of trajectory, which are not necessarily mutually exclusive:
 - First, the development of (implicitly formulated⁶) conservative algorithms which incorporate the estimation of local truncation error and subsequent step-size adjustment. It is nearly certain that this would eliminate the largest performance roadblock observed, in that the time-step is determined by the most

⁶That is with an implicit coordinate and explicit potential to minimize rounding issues; coupled to this, we recommend taking the coordinate with the greatest influence on the potential U implicit, to improve the performance of the nonlinear solver, *cf.* section sec:energyconservative.

critical parts of a trajectory, while it could safely be higher elsewhere.

We suggest, further, that this may be fruitfully implemented on the basis of a Adams–Bashforth–Moulton type multi–step predictor–corrector pair. As remarked in section 5.2.3, this also provides a clear path to higher–order integration schemes, and so would remove the second major drawback to the methods of this thesis (as being only $O(h^2)$ methods in time).

 Second, the reformulation of problems involving a satellite, in particular our capture problem, in terms of explicit force-scales for different regions of a trajectory for small satellites (in particular for any future work on the ballistic lunar capture problem).

This is motivated by the analysis of section 4.3.3 together with the remarks on singularly perturbed behavior, in that it may be that certain types of instabilities persist even in higher–order methods, even with step–size control, due to inappropriate modeling of the local dynamics. The best known approach to such a situation is to make the local dynamics explicit, and we have shown that in the present class of problems, a force–scales approach is a suitable way to handle the problem.

The potential down–side to this, however, is the need to couple solutions of different formulations of the same problem in different physical regions, though this should not be an insurmountable problem.

• Lastly, we make a tentative recommendation concerning a hybrid approach to the numerical simulation of Hamiltonian systems. It was remarked in our brief comparison of conservative and symplectic methods in section 5.3.3 that both are motivated as structure preserving algorithms for Hamiltonian systems. It is understood that they are, strictly speaking, mutually exclusive, but we would tentatively suggest that *fully* symplectic, *approximately* conservative algorithms are not ruled out by theorem 5.3.3, suggesting the possibility of a best of both worlds approach. We suggest this very carefully however, and note that such an approach would be very complicated, and quite possibly not worth the return on investment in terms of computational effort.
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Part V

Appendices

Appendix A

Notational Conventions

A few notes on the notational conventions in this thesis.

- We use the common convention 'iff.' as shorthand for the phrase 'if and only if,' and 'resp.' as shorthand for respectively. Latin shorthand is typeset in italics.
- We also draw the reader's attention to the distinction between lemmas, theorems and propositions. Following the convention in mathematics, the term *lemma* has been reserved for theorems of lesser importance, while *theorem* refers to a principal result.

However, to make a distinction between results in the literature and new results presented in this thesis, we have reserved use of the term *proposition* for theorems either given here for the first time, or significantly extended from the literature.

- We use the counting convention $a \times b^{th}$ to denote the combination of the number of equations *a* and order of each equation *b* in a system of differential equations. This is used in order to facilitate the analysis of transformations of the order of a system, *e.g.* from a 9×2^{nd} -order to a 18×1^{st} -order system, where the product *must* be conserved.
- We also use the shorthand $O(\varepsilon^{\{k,k-2,k-4\}})$ to denote a coupling between orders of an expansion in ε ; thus, the above is intended as shorthand for the statement "a coupling between $O(\varepsilon^k)$, $O(\varepsilon^{k-2})$ and $O(\varepsilon^{k-4})$ terms in the expansion."

Two comments are in order for this extension of our notation: first, this scheme is understood only to contain terms of order $k \ge 0$ (*i.e.* we neglect any terms that are < 0), and second, it is understood that the scheme is extended at higher orders as new ε -couplings enter the picture in our expansions (*cf.* chapter 4 and Part II of this thesis).

• We use boldface vectors such as **x** only occasionally to emphasize the vector character of the variable being discussed, preferring in general *x* with the vector character implied, which will usually imply an extended state vector $\mathbf{x} = (x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n)^{\mathsf{T}}$. When *x* refers only to the coordinate *x*, this will be clear from the context.

- The exception to the above rule is the convention in the discussion of astrodynamics in chapter 3 where $r := ||\mathbf{r}||$ is always scalar and in a slight departure from convention elsewhere, vector quantities are emphasized in bold-face to distinguish them clearly from their scalar magnitudes.
- We use $|| \cdot ||$ to denote the Euclidean norm and $| \cdot |$ to denote the complex norm or magnitude. The latter is also used to denote the absolute value when applied to reals, a distinction which should be clear from context.
- We use the angled braces $\langle x \rangle$ to denote the average of a quantity, where the type of average taken will be clear from context (typically the time-average).
- We use subscripts *x_i* to denote the scalar components *i* of a vector; where on occasion **x**_{*i*} is used, the subscript denotes the order of an expansion of a vector quantity.
- We use superscripts $w^{(i)}$ to denote iteration, whether in a numerical integration scheme or an iteration of a mapping. In particular, we will use $w^{(i)} \approx x^{(i)} \equiv x(t_i)$ for the approximation to the solution given by a numerical integrator.
- We also draw the reader's attention to the distinction between the symbols v and v. Throughout the thesis, v will be reserved for integrating factors and integrating vectors, *cf.* chapter 4, while $v = \frac{m_3}{m_1+m_2}$ a dimensionless parameter corresponding to a small secondary or tertiary mass.
- We will *not* use f⁽³⁾(x) for d³f/dx³ but write the derivative explicitly, while 'clean' superscripts x⁵ are reserved for powers. We *do*, however, use the shorthand ∂_i for component derivatives, e.g. the derivative of the scalar f(x) with respect to the third vector component of x: ∂₃ f(x) := df(x)/∂x₃.
- We lastly draw the reader's attention to the distinction between Q, the rational numbers and Q, the generalized coordinate space in the context of Lagrangian and Hamiltonian mechanics.

Appendix B

Hamiltonian Formulations

We give, for reference, the Hamiltonians for the problems discussed in this thesis.

2-Body Problem Lagrangian and Hamiltonian **B.1**

For the 2-body problem the inertial formulation is simplest; the n-body equations (3.3a) are specified for the case n = 2, resulting in:

$$\ddot{\mathbf{r}}_1 = \frac{\rho_2}{r_{12}^3} \mathbf{r}_{12} \tag{B.1a}$$

$$\ddot{\mathbf{r}}_2 = \frac{p_1}{r_{21}^3} \mathbf{r}_{21} \tag{B.1b}$$

Lagrangian Formulation

Introducing:

$$M := m_1 + m_2$$
 and $\bar{\mu} := \frac{m_1 m_2}{m_1 + m_2}$,

the following generalized coordinates may be introduced:1,2

$$\mathbf{r} := \mathbf{r}_2 - \mathbf{r}_1 , \qquad (B.2a)$$
$$\mathbf{r}_b := \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{\sum_{i=1}^{N=2} m_i \mathbf{r}_i}{\sum_{i=1}^{N=2} m_i} . \qquad (B.2b)$$

¹The subscript on the center-of-mass coordinate \mathbf{r}_b carries the meaning binary, which will be used in the 3-body problem and 4-body problems to distinguish it from the true center of mass \mathbf{r}_0 in those cases. In the 2-body problem the two coincide. ²Note also that the use of $\bar{\mu}$ is a convention which is used only here, and is not to be confused with the parameter $\mu = \frac{m_2}{m_1+m_2}$ will

be a dimensionless parameter in the 3-body problem, cf. section 3.3.5.

With these it is found that:

$$T = \frac{1}{2}M\dot{r}_{b}^{2} + \frac{1}{2}\bar{\mu}\dot{r}^{2} \quad \text{and}$$
(B.3)
$$U = -G\frac{m_{1}m_{2}}{r}, \quad (B.4)$$

and so it follows that the Lagrangian is given by:

$$\mathcal{L} = T - U = \frac{1}{2}M\dot{r}_b^2 + \frac{1}{2}\bar{\mu}\dot{r}^2 + G\frac{m_1m_2}{r} .$$
(B.5)

Hamiltonian Formulation

The Hamiltonian formulation approaches the problem as in the previous section, but introduces a slightly different set of generalized coordinates. We introduce the same mass conventions and kinetic and potential energy as in the previous section to obtain (B.5).

Note that $M\bar{\mu} \equiv m_1m_2$. Now, intuitively the Hamiltonian is simply T + U (and expressed in generalized coordinates and momenta respectively), but this is demonstrated explicitly using the Legendre transform:

$$\mathcal{H} = \sum_{j} p_{j} \dot{q}_{j} - \mathcal{L}(q, p) . \tag{B.6}$$

With the same \mathbf{r} , \mathbf{r}_b as generalized coordinates, the generalized momenta become:

$$q_1 = \mathbf{r}_b , \tag{B.7a}$$

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = M \dot{\mathbf{r}}_b , \qquad (B.7b)$$

$$q_2 = \mathbf{r} , \qquad (B.7c)$$

$$p_2 = \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \mu \dot{\mathbf{r}} , \qquad (B.7d)$$

by which:

$$\begin{aligned} \mathcal{H} &= M\dot{r}_{b}\dot{r}_{b} + \bar{\mu}\dot{r}\dot{r} - \left(\frac{1}{2}M\dot{r}_{b}^{2} + \frac{1}{2}\bar{\mu}\dot{r}^{2} + G\frac{m_{1}m_{2}}{r}\right) \\ &= \frac{1}{2}M\dot{r}_{b}^{2} + \frac{1}{2}\bar{\mu}\dot{r}^{2} - G\frac{m_{1}m_{2}}{r} \\ &= \frac{p_{1}^{2}}{2M} + \frac{p_{2}^{2}}{2\bar{\mu}} - G\frac{M\bar{\mu}}{\|q_{2}\|} . \end{aligned}$$
(B.8)

The equations of motion are derived from the relations $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$, $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$, which give:

$$\dot{q}_1 = \frac{p_1}{M} \longrightarrow \dot{\mathbf{r}}_b = \dot{\mathbf{r}}_b , \qquad (B.9a)$$

$$\dot{q}_2 = \frac{p_2}{\bar{\mu}} \longrightarrow \dot{\mathbf{r}} = \dot{\mathbf{r}},$$
 (B.9b)

$$\dot{p}_1 = 0 \longrightarrow M\ddot{\mathbf{r}}_b = 0,$$
 (B.9c)

$$\dot{p}_2 = -\frac{GM\mu}{\|q_2\|^3} \longrightarrow \bar{\mu}\ddot{\mathbf{r}} = -\frac{GM\mu}{r^3}\mathbf{r}.$$
(B.9d)

B.2 Full 3–Body Problem Hamiltonian

The full 3–body problem Hamiltonian is derived as follows. In the coordinates of figure 3.1, the reader will recall the formulation of the equations of motion as:

$$\ddot{\mathbf{r}}_{i} = -\nabla_{i} U_{i} = \sum_{j=1,\neq i}^{n=3} \frac{\rho_{j}}{r_{ij}^{3}} \mathbf{r}_{ij}, \qquad i = 1, \dots, 3.$$
(B.10)

We take as generalized coordinates \mathbf{q}_i simply the position vectors \mathbf{r}_1 , \mathbf{r}_2 , \mathbf{r}_3 , and with the usual convention $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$, we obtain the Lagrangian $\mathcal{L} = T - U$ as:

$$\mathcal{L} = \frac{1}{2} \left(m_1 \mathbf{r}_1 \cdot \mathbf{r}_1 + m_2 \mathbf{r}_2 \cdot \mathbf{r}_2 + m_3 \mathbf{r}_3 \cdot \mathbf{r}_3 \right) + \frac{Gm_1 m_2}{\|\mathbf{r}_{12}\|} + \frac{Gm_1 m_3}{\|\mathbf{r}_{13}\|} + \frac{Gm_2 m_3}{\|\mathbf{r}_{23}\|}$$
(B.11)

It is easily verified that the momenta corresponding to these coordinates via $p_i = \frac{d\mathcal{L}}{d\dot{q}_i}$ are simply

$$\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$$
.

Consequently, the kinetic energy terms of the Lagrangian \mathcal{L} are reformulated as:

$$\frac{1}{2}m_i \mathbf{r}_i \cdot \mathbf{r}_i = \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{2m_i} = \frac{p_{1,i}^2 + p_{2,i}^2 + p_{3,i}^2}{2m_i}$$

such that the Hamiltonian for the full 3-body problem (cf. equation B.6) is given by:

$$\mathcal{H} = \frac{\mathbf{p}_1 \cdot \mathbf{p}_1}{2m_1} + \frac{\mathbf{p}_2 \cdot \mathbf{p}_2}{2m_2} + \frac{\mathbf{p}_3 \cdot \mathbf{p}_3}{2m_3} - \frac{Gm_1m_2}{\|\mathbf{r}_{12}\|} - \frac{Gm_1m_3}{\|\mathbf{r}_{13}\|} - \frac{Gm_2m_3}{\|\mathbf{r}_{23}\|} .$$
(B.12)

B.3 Circular Restricted 3–Body Problem Hamiltonian

The derivation of the (planar) CR3BP Hamiltonian follows the same outline as the above, save for the derivation of the Lagrangian. We use the coordinates of figure 3.7 In dimensionless inertial coordinates, it is given by:

$$\mathcal{L} = \frac{1}{2} \left(\dot{X}^2 + \dot{Y}^2 \right) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu (1 - \mu)$$
(B.13)

(B.14)

(B.16)

where:

$$\mu = \frac{m_2}{m_1 + m_2}$$

$$r_1 = \sqrt{(X + \mu \cos \omega t)^2 + (Y + \mu \sin \omega t)^2}$$

$$r_2 = \sqrt{(X - (1 - \mu) \cos \omega t)^2 + (Y - (1 - \mu) \sin \omega t)^2}$$

We remark that this of course implies that the potential U = U(X, Y, t) in inertial coordinates. On changing to the rotating frame (which we will not derive, though the interested reader may consult *e.g.* [José and Saletan, 1998]):

$$\mathcal{L} = \frac{1}{2} \left((\dot{x} - y)^2 + (\dot{y} + x)^2 \right) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}$$
(B.15)

where:

$$r_1 = \sqrt{(x+\mu)^2 + y^2}$$

$$r_2 = \sqrt{(x-(1-\mu))^2 + y^2}.$$

Here, the potential U = U(x, y) only, as we expect given that the change of coordinates to the rotating system makes the gravity field time-independent. Taking as canonical coordinates $q_1 = x$, $q_2 = y$, we find via $p_i = \frac{dL}{dq_i}$ that:

$$p_1 = \dot{x} - y \,, \tag{B.17a}$$

$$p_2 = \dot{y} + x \,. \tag{B.17b}$$

Consequently, the Lagrangian may be transformed to the Hamiltonian using B.6 to find:

$$\mathcal{H} = \frac{1}{2} \left(p_1^2 + p_2^2 \right) + p_1 q_2 - p_2 q_1 - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \,. \tag{B.18}$$

B.4 Angular–Momentum Reduced 3–Body Problem Hamiltonian

As remarked in section 3.3.3, one can try to find a coordinate transform analogous to the conservation of linear momentum approach used in chapter 3, such that the reduction is implicit in the transformed system. Historically, this task has proven considerably harder than one might initially imagine. No coordinate transform has been found to effect such a reduction for the general n-body problem, as detailed in [Wintner, 1947].

However, in the specific case of the 3-body problem the fact that the 3 bodies at each point in time span a plane can, however, be exploited to find a coordinate transformation which Wintner gives for non-collinear configurations of the 3-body problem, *i.e.* general triangle configurations $\triangle(t)$.

Specifically, letting m_1, m_2, m_3 be at the vertices of the triangle, which has an area $|\triangle(t)|$; and their corresponding exterior angles $\theta_i(t)$, he introduces new coordinate ρ_i corresponding to the length of the vertex opposite m_i (and the variables i, j, k run through the cyclic permutations of 1,2,3.)

Then,

$$\sin \theta_i = \frac{2|\Delta|}{\rho_i \rho_k} \tag{B.19a}$$

$$\cos\theta_i = \frac{\rho_i^2 + \rho_j^2 + \rho_k^2}{2\rho_j \rho_k} \tag{B.19b}$$

$$|\triangle| = \frac{\prod(\rho_j + \rho_k - \rho_i)^{1/2}}{4/\left(\Sigma\rho_i\right)^{1/2}} > 0$$
(B.19c)

and it is observed that the motion of the triangle, which instantaneously defines a plane, is fully determined by these three vectors ρ plus the inclination of said plane w.r.t. the barycentric coordinate system through the center of mass (as above), which is introduced as the fourth coordinate 1.

This leads to the Hamiltonian formulation:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}(\iota, \rho_1, \rho_2, \rho_3, I, P_1, P_2, P_3) \\ &= \frac{\|\mathbf{H}_0\|^2 \sin^2 \iota}{4|\triangle|} \sum \frac{\rho_i^2}{m_i} \sin^2 \left(\frac{I}{\|\mathbf{H}_0\| \sin \iota} + \frac{\theta_j - \theta_k}{3} \right) + \sum \frac{P_j^2 + P_k^2 - 2P_j P_k \cos \theta_i}{2m_i} \\ &+ \|\mathbf{H}_0\| \cos \iota \sum \left(\frac{P_j}{\rho_k} - \frac{P_k}{\rho_j} \right) \frac{\sin \theta_i}{3m_i} + \|\mathbf{H}_0\|^2 \cos^2 \iota \sum \frac{\rho_j^2 + \rho_k^2 - \frac{1}{2}\rho_i^2}{36m_i \rho_j^2 \rho_k^2} - \sum \frac{m_j m_k}{\rho_i} \,. \end{aligned}$$
(B.20)

And the usual Hamiltonian equations hold:

$$I' = -H_{1}, \, \iota' = H_{I}, \, \rho_{i}' = -H_{P_{i}}, \, P_{i}' = H_{\rho_{i}}.$$
(B.21)

On consideration, while this is interesting from a historical point of view, it is pretty clear that due to the explicit triangle formulation, this does not scale to multi-body n > 3 problems, which means that it is not a useful basis for our extensions to 4-body problems later on. Moreover, its value in light of explicit complexity is perhaps dubious, and so has been included only for completeness.

Noteworthy and discussed in section 3.3.4, however, is that this triangle's motion always determines an instantaneous plane, though its orientation w.r.t. inertial space is changing; hence $\iota = \iota(t)$. However, there is a particular invariant case ($\iota = \iota_0$ for all time), where the plane is perpendicular to the constant vector \mathbf{H}_0 . Now while this plane can always be defined (specifically, by the relation $\mathbf{H}_0 \cdot \mathbf{x} = 0 \longrightarrow h_1 x + h_2 y + h_3 z = 0$), the motion need not necessarily be restricted to it.

We remark, then, that you also have explicit conservation of angular momentum if we *restrict* motion to such a plane, and by appropriate orientation of the coordinate system, we may choose the invariant z = 0 plane. It is easily verified that this is an invariant plane, and that the angular momentum vector becomes a single *z*-component perpendicular to it. While this is a special case, unlike the triangle-configuration approach we have just described, it does scale beyond 3 bodies.

B.5 4–Body Capture Problem

We remark that the Capture Problem is not Hamiltonian; otherwise, as we showed in chapter 2, the hypothetical Hamiltonian \mathcal{H}_{CP} would itself be an integral.

Appendix C

Simulation Suite Manual

This appendix presents the simulation suite used in obtaining the results pertaining to numerical integration methods, as presented in the third part of this thesis. It begins with a section on the usage of the main simulation code, and then discusses the structure of the FORTRAN simulation, MATLAB visualization and MATHEMATICA analysis codes in turn, closing with a section on code validation.

C.1 Usage

We first discuss the basic parameters controlling the IntegrationMethods FORTRAN code, and then give a brief discussion of the options for fine tuning available to the user.

C.1.1 Getting Started

The suite's entry point is the driver program IntegrationMethods.f90 in which the user specifies the force model, integration method, the input and output coordinate systems, initial conditions, simulation time-frame and output frequency.

The parameters to be set by the user are the following:

- **t_start** is the simulation start time given as a Julian date; the function JD (yyyy, mm, dd, hh, mm, ss) is available for the conversion of normal date+time-stamps to Julian dates.
- **t_stop** is the simulation stop time given as a Julian date.
- **n_step** is the number of steps at which the simulation will provide output.
- **model** is the force model to be used in the integration:
 - 1. planar Jacobi 3BP [SEM]
 - 2. planar Jacobi 3BP [EMSat]
 - 3. planar CR3BP [SEM]
 - 4. planar CR3BP [EMSat]
 - 5. planar Capture Problem [SEMSat]
 - 6. 3BP with ephemerides in 3D [EMSat]
 - 7. 4BP with ephemerides in 3D [SEMSat]

- **method** is the integration method to be used with the above force model; note that there are 3 different exact approaches coded to the Jacobi 3–body problem, but the others take only 2 (exact and approximate), while the RKF–pairs and the simple predictor–corrector can be used for any problem.
 - 1. Burkardt's RKF4(5) Integrator pair
 - 2. Netlib RKF2(3),4(5),7(8) Integrator pairs
 - 3. Bowman's planar Jacobi 3BP Conservative Integrator (explicit variant)
 - 4. Bowman's planar Jacobi 3BP Conservative Integrator (implicit variant)
 - 5. Bowman's planar Jacobi 3BP Conservative Integrator (implicit polar variant)
 - 6. Bowman's planar CR3BP Conservative Integrator
 - 7. Simple Predictor-Corrector
 - 8. Our planar CR3BP Approximately Conservative Integrator
 - 9. Our planar Capture Problem Approximately Conservative Integrator
- **rk** is the setting 'l', 'm', 'h' which determines the (low-, medium- or high-order) pair chosen for the Netlib RKF integrator pairs; for other choices this is a dummy variable. Note also that the error tolerances for the RKF pairs can be set at the beginning of the Traj subroutine, via the variables abserr and relerr.
- order is the order of the integral approximation to be conserved for the approximate integrators. Valid settings are 0–3, referring to the lowest orders of approximation (though depending on the problem, not necessarily the $O(\varepsilon^0) O(\varepsilon^3)$ terms specifically).
- **csys0** is the coordinate system chosen for the input initial conditions:
 - 0 selects the model-based coordinate systems, which is valid only for models 1-5
 - 1 selects the Earth co-moving coordinate system using the coordinate orientations of the inertial system.
 - 2 selects the Earth–Moon rotating coordinate system.
 - 3 selects the Earth–Sun rotating coordinate system.
- **csys** is the coordinate system chosen for the simulation output. Note that for problems not including the Sun in the simulation, csys=3 is not a valid choice.
- x[1:neqn] finally, is the initial state at t_start; neqn is the number of the variables in the state vector (4 for planar CR3BP, 8 for planar Jacobi 3BP's, set automatically once model is chosen by the user).

The driver then sets the appropriate constants, and calls the subroutine Traj, which in turns sets up the integration and makes calls to the appropriate integrator, force model and output routines. While running, the program displays the current and total output points, and a successful simulation run ends with the display of the total number of force function evaluations performed during the integration. The program pauses to display unrecoverable errors for traceability purposes, and then aborts execution.

C.1.2 Fine–Tuning and Troubleshooting

In order to go beyond the basics outlined above, the user will find it necessary to make additions and alterations to the code. For this purpose an understanding of the code structure outlined in the next section is essential, but we note first a few fine-tuning parameters which are not too complicated.

- h_0 is the time-step used internally by the exactly and approximately conservative integrators. This is found in the variable declarations of each integrator subroutine, and declared as a parameter, which the user must tweak at runtime. A good initial setting for both problems has been found to be h_0=1D2 or 100 seconds.
- **rfac** is the reduction factor used when a non-conservative step must be made due to the inverse transformation returning a bad state vector. This is typically due to a turning point, though if it occurs repeatedly during an integration, it is often an indicator of a too-large time-step h_0 , leading to integrator instability.

Beyond this, the user will note that the most likely cause of bad integration results are, in order of expected occurrence:

- 1. A bad choice of time-step for the problem (h_0 too large),
- 2. A badly formulated transformed force function f_{ξ} ,
- 3. A badly formulated state transformation T(x),
- 4. A badly formulated inverse state transformation $T^{-1}(x)$.

Consequently these should always be checked first when unexpected output is encountered.

C.2 Code Structure

This section outlines the code structure for the simulation, visualization and analysis portions of the code provided on the accompanying CD.

C.2.1 FORTRAN Simulation Code

The IntegrationMethods code suite has been developed following FORTRAN 90/95 conventions, and is written using routines either wrapped in a module for global accessibility, or included with the driver program for its local accessibility. All code was compiled in quadruple precision on the Intel[®] FORTRAN Compiler Version 9.1.3192.2005, integrated into Microsoft Visual Studio 2005.¹

The main components of the IntegrationMethods suite are as follows.

- The IntegrationMethods driver program **IntegrationMethods.f90** in which the user specifies the force model, integration method, the input and output coordinate systems, initial conditions, simulation time-frame and output frequency. The driver then sets the appropriate constants, and calls the subroutine Traj.
- The subroutine Traj is in turn the driver for the integration. The settings outlined above are either local and passed explicitly (start & stop-times, initial conditions, output frequency), or global modular. Traj uses them, in particular the force model and integration method to call the appropriate integrator and return the trajectory.

Each force model is integrated in its natural coordinates, and Traj performs the necessary scaling of variables from conventional positions and velocities to those used by the models, as well as calling the appropriate coordinate transforms to first transform the initial conditions to the model input, and subsequently at each output point to transform from the (rescaled) model output frame to the appropriate user defined output frame.

¹For comparison, the trajectory integration codes were also validated in double precision on the G95 open-source FOR-TRAN compiler.

- A number of subroutines are local to the IntegrationMethods.f90 driver, and provided as local includes.
 - **rkf45.f90** The integrator code which provides Burkardt's implementation of the Runge-Kutta-Fehlberg 4(5) integrator pair, used in [Verzijl, 2005].
 - **rksuite.f90** The integrator suite which provides the standard Netlib implementation of Runge-Kutta-Fehlberg 2(3), 4(5) and 7(8) integrator pairs.
 - **BowmanPC.f90** The integrator suite which provides four implementations (designed by the author) of Bowman *et al.*'s 2nd order conservative predictor corrector algorithms. Specifically, it provides their planar CR3BP integrator, and three implementations of their general 3–body problem integrator: two using normal Jacobi coordinates which conserves energy and one using polar Jacobi coordinates which conserves both energy and angular momentum.
 - **ApproxPC.f90** The integrator suite ApproxPC.f90 which provides two 2^{nd} –order predictor-corrector integrators. The first is conservative of an approximate Jacobi integral of varying orders, and the second is conservative of an approximate Jacobi–like integral for the Capture Problem, as discussed in Part II of this thesis.
 - **JPLSub.f90** The subroutine collection provided by Standish *et al.* at JPL [Standish, 1997] as a driver for the retrieval and interpolation of their (binary) DE405 ephemeride data; this is necessary for the ephemeride–based integrations.
- A number of subroutines, by contrast, are more usefully defined global, and provided as module includes.

ForceModel.f90 The collection of the 7 force models used in the simulations:

- f_Jacobi: the three body problem force model in Jacobi coordinates for both the Sun-Earth-Moon and Earth-Moon-Satellite problems;
- f_Jacobi_p: the same as the above in polar Jacobi coordinates;
- f_cr3bp: the CR3BP force model in scaled (dimensionless) coordinates for both the Sun-Earth-Moon and Earth-Moon-Satellite planar CR3BPs;
- f_Capture: the Capture Problem force model in scaled (dimensionless) coordinates, and
- two ephemeride-based implementations: f_3bp_eph of the Earth-Moon-Satellite restricted 3-body problem and the Sun-Earth-Moon-Satellite restricted 4-body problemused in [Verzijl, 2005], f_4bp_eph respectively.
- **Coordinates.f90** The subroutine collection which contains functions and subroutines for coordinate transformations and the generation of the necessary rotation matrices, as well as for handling time (*e.g.* conversions to and from Julian dates).
- **Output.f90** The subroutine which takes the output array generated by Traj and writes to a file on disk. This in turn is loaded by a separate set of scripts implemented in MATLAB for visualization, discussed next.

C.2.2 Output File Format

The output file, by default IntegrationMethods .out is a plain ASCII text file with the following structure:

- 1. First, the number of output steps n_steps,
- 2. the step-size h_0 (only for the fixed time-step integrators) and the

- 3. Julian start date t_start are indicated.
- 4. The next fields record the four vector-elements x, y, \dot{x} , \dot{y} of the output for the Earth, the Moon, the Sun and the Satellite, in that order. When a body does not play a role in the simulation, the output is a $4 \times n_{\text{steps}}$ zero matrix.
- 5. Following this, the driver outputs the Julian end date t_stop,
- 6. the model: model,
- 7. the method: method,
- 8. the number of force evaluations, tracked by the variable counter,
- 9. the number of internal loops (only for conservative integrators), tracked by the variable loop1,
- 10. the number of these loops which were used for non-conservative steps, tracked by the variable loop2,
- 11. and finally the mass of the satellite m[4] (where applicable).

This file is then post-processed in MATLAB .

C.2.3 MATLAB Visualization Code

MATLAB Version 7.0.1.24704 (R14) Service Pack 1 was used for the visualization of the simulation output, as presented in Part III of this thesis. Note that the visualizations of data are based on double–precision data, as MAT-LAB appears unable to handle quadruple precision, and so the routines round the raw results as they are loaded, before post–processing. The data may be loaded in MATLAB in three ways:

- The simulation output file is loaded using subroutines QuickLoadEMSat.m, QuickLoadSEM.m or Quick-LoadEph.m, which generate a simple *x*, *y*-plot for the Earth–Moon–Satellite, Sun–Earth–Moon(–Satellite) or ephemeride–model geometries, centered at the Earth.
- 2. The simulation output file can be processed into an MPEG movie using the subroutine **Animation2D.m**, which generates an animation from the simulation data.
- 3. Most importantly, for analysis purposes, the function **postprocess.m** is the driver for a set of post–processing routines which loads 2 input files, given as arguments, and then generates the following plots:
 - Two *x*, *y*–plots of the trajectories as integrated;
 - A four-subfigure plot of the along-track absolute difference between the two integrations, as a function of the integration step (roughly corresponding to the time);
 - A three-subfigure plot consisting of the value of the integral as compared with the reference initial value for the two integrations, a plot of the absolute difference between integral values, and a plot of the relative variations about the initial value for both integrators;
 - A three-subfigure plot consisting of the estimated values of approximations of the integral to different orders, plotted separately for the two integrators, and a third subfigure plotting the reduction of the difference between these approximations and the true value of the integral.

postprocess.m needs the subroutines:

- loaddata.m the routine which actually loads the data, as in QuickLoadEMSat.m above;
- trimspikes.m a routine which (optionally) trims upward spikes from the trajectory differences;²
- **makeplots.m** the routine which processes the data into the forms required for plotting and which generates the resulting plots, which are then displayed on–screen.

Additionally, **postprocess.m** outputs a single cell structure containing the simulation data and data labels, a matrix of the RMS trajectory differences, and 4 large matrices containing the actual simulation data. All except these latter 4 are also output to the screen.

C.2.4 MATHEMATICA Analysis Code

MATHEMATICA 5.2 was used for purposes of analysis, in particular for the calculations of Part II of this thesis.

• The key MATHEMATICA notebooks used are:

IntegratingVectors01-3BP.nb – notebook briefly treating the traditional 3–body problem; IntegratingVectors02-P3BP-Jacobi.nb – notebook treating the planar Jacobi 3–body problem; IntegratingVectors03-CR3BP.nb – notebook treating the planar circular restricted 3–body problem; IntegratingVectors04-CaptureProblem-mod.nb – notebook treating the Capture Problem; IntegratingVectors05-2BP.nb – notebook treating the 2–body problem.

IntegralApproxExpressions.nb – notebook treating the corrected expressions for the approximate integrals (*cf.* chapters 10–11.

• Further, we have included for completeness a further number of utility-notebooks which give some useful subcalculations supporting the approach in the above five. We mention in particular the notebooks **ReductionAngularMomentum.nb** treating the reduction of the angular momentum, **Averaging the Potential.nb** treating the effects of averaging in the Capture Problem as discussed in section 8.2.4 and **Waluya-2.3.nb** treating some of the analysis in [Waluya, 2003] for reference.

C.2.5 Code Validation

The reader will note that for the purposes of validation, we have built on the approach taken earlier in developing simulation code for ballistic lunar capture trajectories. That code has been used as a validation reference, and for a discussion of its validation, the reader is referred to [Verzijl, 2005, appendix B]. Beyond this basis, code validation in the present work was done against two primary references:

- 1. qualitative analytical considerations for closed (Kepler) orbits (e.g. for the Moon in the Earth-Sun system) and
- 2. validation against RKSuite results and ephemeride-based code for other cases.

We remark also in this context that the astrodynamic constants used in the new integrators have been taken from [Montenbruck and Gill, 2001], and are standardized across all models and methods. Validation tests were roughly performed successively as follows:

²These have been found to be numerical artifacts due to a sub-optimal tolerance choice for the RKF7(8) pair (the user would encounter this as a series of alternating error messages in IntegrationMethods –output indicating that either too much or too little computational effort was required over a series of output steps).

- Validation of the simulation with 3BP and 4BP ephemeride force models using new the RKF pairs against Internship code to verify the correct implementation of the RKSuite package.
- Using these RKF integrator pairs in turn as a reference, we validated the conservative schemes against the RKFpair results for the same initial conditions. In this we used a number of test-cases:
 - First, test-cases of simple circular orbits for the Moon resp. satellite were checked to integrate as expected based on the initial conditions.
 - We next coded coordinate-transform functions based on earlier work, and tested different methods with
 same initial conditions for comparability of results across multiple coordinate systems, all validated qualitatively against comparable ephemeride-model results.
 - The new integrators were then used with test-cases based on two standard sets of initial conditions for the Sun-Earth-Moon and Earth-Moon-Satellite configurations (so total 4 sets of initial conditions), with trajectories verified against the RKF7(8) codes.
- We further tested the methods for scaling with time-step, and noted a qualitative picture of convergence to the RKF7(8) solutions. Due to the formulation of the conservative schemes on the basis of an explicit predictor-corrector method, it should be remarked that there is a model-dependent stability threshold for the time-step; see also section 9.3.
- We also tested scaling with satellite mass for the Earth–Moon–Satellite configurations in particular (using masses below the threshold of $O(10^{20} \text{ kg})$ at which the satellite would be expected to noticeably perturb the primaries), as well as with changing the magnitude of the initial velocity.

From this test, we found behavior in the RKF integrator pairs indicating that the system had become stiff: the RKSuite routines gave repeated warnings that alternated between on the one hand warning that the work required to achieve the $O(10^{-6})$ relative tolerance was too high, and on the other that too many output points were desired, indicating that too much work was being done. Both issues are "resolvable" by increasing the error tolerance to *e.g.* $O(10^{-3})$, which would significantly degrade the quality of the solution.

As we remarked earlier in section 9.3.2, stiff behavior is expected to be linked to singular perturbation issues, and indeed on physical grounds it might be suspected that the introduction of a small mass moving through multiple force–scale regions at high velocities, typical of ballistic lunar capture trajectories, might give rise to such problems.³ This is indeed what we observe, in that the problems encountered are more pronounced for models involving the Satellite than those concerning only the Sun–Earth–Moon system, and conversely, the problems were noticeably less on taking a higher satellite mass m_{Sat} or lower initial velocity $V_{Sat,0}$.

By themselves bad results would raise the question of the stability of the integrator, and indeed we found that reducing the step–size (often by a factor 100 or more) resolved the problem; however, the fact that the problems were m_{Sat} – and $V_{\text{Sat},0}$ –dependent does seem to indicate a mismatch in the scales of the dynamics of the primaries and the small satellite, as we expect from a singular perturbation scenario.

³This might also account for the alternating nature of the warnings, which we have not yet investigated in full.

• Lastly. as remarked above, the code has not yet been performance tweaked, and so despite a large measure of standardization across the modules and procedures, it is to be expected that there remains considerable room for improvement in terms of performance. Particularly relevant in this regard is the potential speed–up resulting from shifting the code back to double–precision for actual orbit analysis, though we recall the remarks of section 9.2.3 on the need for ensuring by proper formulation of the force model that this does not lead to loss of significance at machine precision.