**Delft University of Technology**
**Faculty of Electrical Engineering, Mathematics and Computer Science**
**Delft Institute of Applied Mathematics**

# Compound Options: Numerical valuation methods and a real option application

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# "Compound Options: Numerical valuation methods and a real option application"

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# Contents

# Frequently used notation

| | |
|---|---|
| $CV$ | Continuation value |
| $d$ | Downward factor in binomial tree |
| $M$ (binomial) | Number of time steps in binomial tree |
| $M$ (Monte Carlo) | Number of asset paths |
| $N$ (Monte Carlo) | Number of time steps |
| $p$ | Probability that the stock goes up |
| $r$ | Interest rate |
| $r_i^{daily}$ | Daily returns |
| $S_0$ | Value of the stock at $t = 0$ |
| $S$ | Price of the underlying stock |
| $t$ | Time variable |
| $T$ | Expiry time |
| $u$ | Upward factor in binomial tree |
| $V_C$ | European call option value |
| $V_P$ | European put option value |
| $V_C^{Am}$ | American call option value |
| $V_P^{Am}$ | American put option value |
| $V_{CoC}$ | Call on call option value |
| $X$ | Strike price |
| $\delta$t | Size of the time step |
| $\Lambda(S(T))$ | Payoff function at expiry |
| $\mu$ | Drift parameter |
| $\sigma$ | Volatility |

# 1 Introduction

For biotechnology companies that are aiming to produce and introduce new drugs, a lot of phases happen before a new drug can successfully enter the market. First, the drug has to be discovered: chemists and biologists synthesise molecular entities to create a successful medicine. After the discovery, the drug has to pass a pre-clinical phase and several clinical trial phases. In these phases, the drugs are tested on humans and animals. Finally, the drug has to pass a final stage where it has to be submitted to the EU Food and Drug Administration. All phases above have a certain cost. This process is called a *sequential R&D investment*. In this thesis, the goal is to value the sequential R&D investment using financial options.

Financial options are also called *derivatives* because they were derived from another financial product, for example an asset. Options give the holder the right, but not the obligation, to buy (call) or sell (put), a prescribed asset for a prescribed price at a prescribed time in the future. The most well-known options are European options. European options are an example of financial options. Several financial options will be introduced in Chapter 3.2. The underlying product of financial options is typically a financial product such as an asset.

In option valuation, one goal is to determine option prices. For some options, analytic solutions exist. In Chapter 3.2, analytic solutions will be provided and derived for European options and compound options. It will also be shown that a solution for American options has to be approximated numerically.

Chapter 4 shows valuation methods to value options that have no analytic solution. Methods discussed in this chapter are the binomial tree, the trinomial tree and Monte Carlo simulation. These methods should yield approximately the same results as the analytic solutions. Using the analytic solution together with the approximation of the numerical methods, the correctness of the numerical methods can be checked before they are applied further. This will be done in Chapter 5.

Compound options are options on options. The underlying product is an option instead of an asset. Therefore the payoff of a compound option involves the value of the underlying option. The goal of this thesis is to value different types of compound options and eventually apply this valuation to real options to value the sequential R&D investments of the biotechnology firm.

# 2 | Real Option: A Biotechnology Firm

In this chapter, a more detailed overview of the problem will be given. As mentioned in the introduction, the main focus of this thesis will be on the research of a real investment option, in particular the R&D investments of the biotechnology company. In this thesis, a link between real options and financial options will be established and the goal is to develop valuation methods for financial options and use these to value real options and the options of the biotechnology company. First, a brief introduction to real options will follow by stating its definition and introducing the most relevant types of real options. Then, several real options will be classified as being a put or call option to introduce simple, well-known payoff functions to these real options. Consequently, the *Option to Choose* will be introduced and finally a brief introduction to the bio-pharmaceutical company is made, which will be financially analyzed in this thesis.

## 2.1 Real options

The definition of a real option is the following:

**Definition 2.1.1.** A **real option** is a choice made available to the managers of a company with respect to business investment opportunities. [11]

The option is called *real* instead of *financial* because it mostly concerns projects that involve a physical asset instead of a financial instrument. Real options refer to choices or opportunities of which a business may take advantage or may realize. A real option typically gives the managers of a company the opportunity to *wait* with a certain business decision. The value of the real option represents the value of waiting with the decision. Several examples of real options will follow to clarify the matter.

**Example 2.1.1.** [12] Suppose market conditions decline severely resulting in a decrease in the value of the equipment and assets. The *option to abandon* gives management the opportunity to abandon its current operations and obtain the resale value of capital equipment and other assets in the second hand market.
Market conditions can also turn out to be more favorable than expected, the *option to expand* then gives management the opportunity to expand. When a firm is trading in a product that is highly volatile and changes in value often, the *option to switch* provides the firm with flexibility to change its output mix if prices or demand changes unexpectedly.
A firm holding valuable resources such as land, capital, specialized information or planning may benefit from the *option to wait*, which gives the opportunity to wait a certain time to see how uncertain conditions will develop.
The *option to invest* is present when an early investment opens up future growth opportunities.

All of the options just mentioned are examples of real options. They all involve tangible assets instead of only financial assets. The options however, can be described as financial options. The

*Net Present Value* of a project, i.e. the difference between present value of cash inflows and the present value of cash outflows over a period of time, will be defined as $S$ and the *Non Recoverable Costs* involved in the project represents the strike price denoted as $X$.
We will now classify the options above as call or put options: then we can use the payoff formulas from Chapter 3 in valuing the real options.

## 2.2 Financial Option Classification

The holder of the following options: *to abandon*, *to contract* or *to switch* has the right, but not the obligation, to sell a current investment. Therefore these options can be considered put options. These options are in-the-money when the value of selling the underlying real assets at current time is higher than the value of keeping the underlying real assets in the way it is currently deployed. Options *to wait, expand or make a new investment* all involve buying an investment. Therefore these options can be considered call options. These options go in-the-money when the value of the new investment exceeds the present value of its costs. An example of a more complicated real option, the *option to choose*, will be introduced in the following section. The option to choose is a combination of three of the options introduced earlier: the option to contract, option to expand and the option to abandon.

## 2.3 Option to Choose

An example of a real option is the option to choose. Suppose we have a large company that wants to hedge itself, i.e. protect itself from risk, through the use of strategic options. As discussed before, there exist a few types of real options to maximize business advantages [5]. The option to choose gives the holder the possibility, but not the obligation, at any time before and on expiry, to either:

- Expand current operations: increasing its value by 30% with 20 million euros of implementation costs;
- Contract current operations: contract 10% of its current operations, creating an additional 25 million euros in savings;
- Completely abandon its business: abandon all operations and selling its intellectual property for 100 million euros.

Suppose we are the holder of the option to choose. At any time before expiry, we have to determine whether it is optimal to: contract, expand, abandon or continue with the option. Different ways to determine the value of the option to choose will be discussed in Chapter 4.

## 2.4 The Biotechnology Company

This thesis will introduce valuation methods to value a special type of real option: R&D investments of a biotechnology company. Valuing investments like this is usually difficult, since the value is mostly determined by the ability to convert its current intellectual property to cash flow streams in the future, which is strongly dependent on approval processes and government regulations. Therefore these values are usually uncertain at the beginning. In our problem, all costs are in thousands of euros (K euros) and we will assume that the stages that the drug must pass in order to be released to the market are [10]:

**Discovery** - A trivial but necessary stage in the drug development is the discovery. We assume that the discovery takes 1 year and costs 2.200K euros. The costs in this stage mostly come from chemists and biologists synthesising new molecular entities.

**Pre-clinical phase and clinical trial phases** 1, 2 **and** 3 - The pre-clinical phase takes 3 years and the clinical trial phases 1, 2 and 3 respectively take 1, 2 and 3 years. The pre-clinical phase has a cost of 13.800K euros, and the three clinical trial phases respectively have costs of 2.800K, 64.000K and 18.100K. In these phases, the costs incur from testing the drugs on humans and animals.

**FDA filing and review** - The FDA filing and review takes 2 years and has a cost of 3.300K. The costs in this stage mostly come from submitting the drug to the EU Food and Drug Administration.

A graphical representation of this process can be found in Figure 2.1. This figure clearly shows the time it takes for each phase to pass. When the drug reaches $t = 12$, it enters the market. In the subsequent post-approval phase, the company receives revenues from selling the new drug while additional costs (e.g., marketing, product extensions) are incurred. Using the expected revenues, one can approximate the company's value using methods from Chapter 4. The details will be given in Chapter 6.

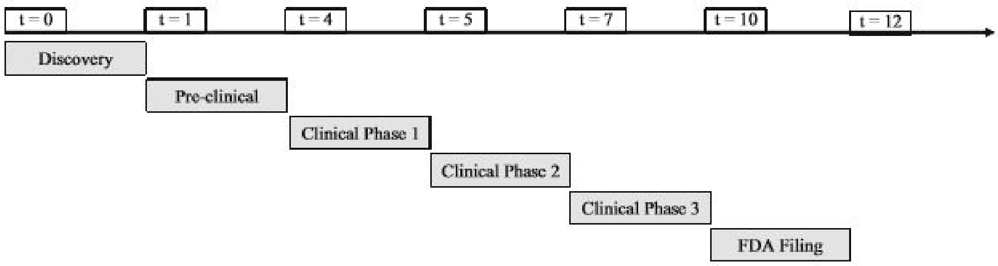


Figure 2.1: The 6 necessary phases with their duration [7].

As mentioned before, every phase in the approval process has a certain cost. Also, the phases in the approval process are related: a drug cannot fail the first clinical phase and continue to the second clinical phase. In order to pass the process, every step must be passed. Therefore, the drug approval process can be considered a (6-fold) compound option: once the first clinical trial phase is passed, the cost has to be paid and the drug can continue to the second clinical trial. The details of compound options will be discussed in Section 3.4.

At every phase, an amount has to be paid by the company. Therefore, the options can be considered call options. The value of the R&D investments of the company is equal to the value of the 6-fold compound option, where the compounded options are call options.

Using the cost of every phase $i$ as strike price $X_i$, and using the value of the company at time $i$ as $S$, we can combine the knowledge of financial options as elaborated on in Chapter 3.2 together with the numerical methods that will be introduced in Chapter 4 to value the R&D investments of a biotechnology company.

# 3 Option Constructions and Asset Dynamics

As mentioned in Chapter 2, knowledge of financial options is required to value the R&D investments. Financial options and real options are related: real options can be translated into financial options to make it easier to value the real options using valuation methods. The definition of real options has been introduced in Section 2.1, where two examples of real options have been introduced: the option to choose and the R&D investments of the biotechnology company. In this chapter, the asset dynamics and the basics of financial options and their constructions will be introduced, which will be used in the valuation of the real options.

## 3.1 Asset Price Model

In order to value an option, a mathematical description of how the underlying asset behaves should be developed. The derivation of this model is based on the work of Higham [8]. Because the price of an asset is highly dependent on various factors, we assume that the price today reflects all past information. This assumption is called the *Efficient Market Hypothesis*. Under this assumption, knowing the asset price $S$ today gives enough information to estimate an option price: knowing the complete history of an asset is not beneficial.

When researching the development of asset prices, usually the daily or weekly returns will be considered:

$$r_i^{daily} = \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}. \tag{3.1}$$

The daily returns can be normalized to obtain:

$$\hat{r}_i^{daily} = \frac{r_i^{daily} - \mu}{\sigma}. \tag{3.2}$$

Here, $\mu$ is the sample mean and $\sigma^2$ is the sample variance. If the daily return data looks like i.i.d. samples from the normal distribution, which is the case in practice, then $\hat{r}_i^{daily}$ will look like i.i.d. N(0, 1) samples.

Since the daily and weekly returns are small, the approximation $\log(1 + x) \approx x$ yields:

$$\log\Big(\frac{S(t_{i+1})}{S(t_i)}\Big) = \log\Big(1 + \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\Big) \approx \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}. \tag{3.3}$$

So the weekly or daily returns can be replaced by the log ratios. A mathematical description of the asset price movement will now be derived. Therefore some assumptions have to be made which will be assumed to be effective throughout the whole thesis when financial options are considered.

### 3.1.1 Market Assumptions

We agree upon the following assumptions for our further analysis [8]:

- The asset price may take any non-negative value;
- Buying and selling an asset may take place at any time $0 \leq t \leq T$;
- It is possible to buy and sell any amount of the asset;
- The bid-ask spread is zero - the price for buying equals the price for selling;
- There are no transaction costs;
- There are no dividends or stock splits;
- Short selling is allowed - it is possible to hold a negative amount of the asset;
- There is a single, constant, risk-free rate that applies to any amount of money deposited in a bank;

Furthermore, we will assume that there is no *arbitrage*: "arbitrage is the certainty of profiting from a price difference between a derivative and a portfolio of assets that replicates the derivative's cashflows" [17].

### 3.1.2 Mathematical Description of the Asset Price Model

We assume that the change in a risk-free investment with interest rate $r$ over a small time interval $\delta t$ can be modelled as:

$$D(t + \delta t) = D(t) + r\delta t D(t). \tag{3.4}$$

By the efficient market hypothesis, we assumed that the current asset price $S(t)$ contains all information of the asset price until today. This is added to our model by adding a random fluctuation increment $\sigma$ to the interest rate.
This yields the following equation for the discrete-time asset price model:

$$S(t_{i+1}) = S(t_i) + \mu\delta t S(t_i) + \sigma\sqrt{\delta t} Y_i S(t_i). \tag{3.5}$$

Here $\mu = r$, $\sigma \geq 0$ and $Y_0$, $Y_1$, $Y_2$, $\ldots$ are i.i.d. $\mathrm{N}(0, 1)$. Now that we arrived at a discrete asset price model, we would like to change it to a continuous asset price model.

In Equation 3.5 we see that over each $\delta t$ interval the asset price gets multiplied by $1+\mu\delta t+\sigma\sqrt{\delta t}$. Thus we can write the asset price at time $t$ as:

$$S(t) = S_0 \prod_{i=0}^{L-1} (1 + \mu\delta t + \sigma\sqrt{\delta t} Y_i). \tag{3.6}$$

We divide by $S_0$, take logs and use the approximation $\log(1 + \epsilon) \approx \epsilon - \epsilon^2/2 + \ldots$ to obtain:

$$\log\Big(\frac{S(t)}{S_0}\Big) \approx \sum_{i=0}^{L-1} (\mu\delta t + \sigma\sqrt{\delta t} Y_i - \frac{1}{2}\sigma^2 \delta t Y_i^2). \tag{3.7}$$

Now, we let $Z_i = \mu\delta t + \sigma\sqrt{\delta t} Y_i - \frac{1}{2}\sigma^2\delta t Y_i^2$. Then:

$$\mathbb{E}(Z_i) = \mu\delta t - \frac{1}{2}\sigma^2\delta t, \tag{3.8}$$

and

$$\text{var}(\mu\delta t + \sigma\sqrt{\delta t}Y_i - \frac{1}{2}\sigma^2\delta t Y_i^2) = \sigma^2\delta t + \text{H.O.T.} \tag{3.9}$$

The Central Limit Theorem states that the sum of a large number of i.i.d. random variables $Z_i$ with finite variance is approximately normally distributed. So:

$$\log(\frac{S(t)}{S_0}) \sim \text{N}((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t). \tag{3.10}$$

Now, we can write our continuous-time asset price as:

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}, \quad \text{where } Z \sim \text{N}(0,1). \tag{3.11}$$

If we subdivide the time in $N$ equidistant time intervals of length $\Delta t$, we can describe the evolution of the asset over a sequence of time points $0 = t_0 < t_0 < t_1 < \cdots < t_M$ by:

$$S(t_{i+1}) = S(t_i)e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}, \quad \text{for i.i.d. } Z_i \sim \text{N}(0,1) \tag{3.12}$$

We will use this mathematical description of the asset price later in Section 4. $S(t)$ as defined in Equation 3.11 follows a Geometric Brownian Motion. Moreover, $S(t)$ has a so-called lognormal distribution with the corresponding density function $f(u)$ [8]:

$$f(u) = \frac{\exp(\frac{-(\log(u/S_0) - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t})}{u\sigma\sqrt{2\pi t}} \quad \text{for } u > 0. \tag{3.13}$$

The probability measure used is the *risk neutral* probability measure, i.e. "the option value at asset price $S$ and time $t$ could be regarded as the suitably discounted, expectation of the payoff" [8]. Suppose $V(S,t)$ is the option value, then:

$$V(S,t) = e^{-r(T-t)}\mathbb{E}(\Lambda(S(T))). \tag{3.14}$$

Where $\Lambda(S(T))$ is the payoff of the option at expiry.

## 3.2 European Options

European options are the simplest of options in terms of their structure yet the most important since it lays the foundation to more structurally complicated options. The European option can be defined as follows:

**Definition 3.2.1.** A **European call option** gives its holder the right, but not the obligation, to purchase from the writer a prescribed asset for a prescribed price at a prescribed time in the future. [8]

**Definition 3.2.2.** A **European put option** gives its holder the right, but not the obligation, to sell to the writer a prescribed asset for a prescribed price at a prescribed time in the future. [8]

The prescribed purchase price is known as the exercise price or strike price and the prescribed time in the future is known as the expiry time.

Suppose we are the holder of European call option with strike price $X$. Suppose $S(t)$ is the stock price at time $t$. Then at maturity, we will have the following payoff $V_C(S,T)$:

- if $S(T) > X$: exercise the option: buy the asset for price $X$ and sell it for $S(T)$. This generates a payoff of $V_C(S,T) = S(T) - X$.
- if $S(T) \leq X$: do not exercise the option: the option generates no payoff so $V_C(S,T) = 0$.

If we were the holder of a European put option, the payoff $V_P(S,T)$ would be as follows:

- if $S(T) \geq X$: do not exercise the option: the option generates no payoff so $V_P(S,T) = 0$.
- if $S(T) < X$: exercise the option: buy the asset in the market for price $S(T)$ and use the option to sell it for $X$. This generates a payoff of $V_P(S,T) = X - S(T)$.

Now, we can summarize the payoffs of the European put and call options as follows:
The payoff of a European call option is given by [8]:

$$V_C(S,T) = \max(S(T) - X, 0). \tag{3.15}$$

The payoff of a European put option is given by [8]:

$$V_P(S,T) = \max(X - S(T), 0). \tag{3.16}$$

### 3.2.1 Analytic Solution of European Option

For European options, the partial differential equation (PDE) is given by equation 3.17 [8]. For European options, this equation is named the Black-Scholes PDE.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{3.17}$$

Now, the value $V(S,t)$ of a European option at stock price $S$ and time $t$ must satisfy this equation. To solve this partial differential equation, we need boundary conditions. As given in Chapter 3, we know that the boundary conditions for $t = T$ are the following:

$$\begin{aligned} V_C(S,T) &= \max(S(T) - X, 0), & (3.18)\\ V_P(S,T) &= \max(X - S(T), 0). & (3.19) \end{aligned}$$

If the asset price ever reaches zero, it will remain zero forever, so for a European call, the payoff at expiry will be zero. For a European put, if the asset price reaches 0, the payoff at expiry will be $X$, so the payoff at time $t$ is $Xe^{-r(T-t)}$. This leads to the following boundary conditions for $S = 0$ [8]:

$$\begin{aligned} V_C(0,t) &= 0, \text{ for all } 0 \leq t \leq T, & (3.20)\\ V_P(0,t) &= Xe^{-r(T-t)}, \text{ for all } 0 \leq T. & (3.21) \end{aligned}$$

Now we look at the behaviour of $V(S,t)$ when $S \to \infty$. When $S$ is extremely large, it will be very large compared to the strike price $X$, so: [8]

$$\begin{aligned} V_C(S,t) &\approx S, & (3.22)\\ V_P(S,t) &\approx 0. & (3.23) \end{aligned}$$

Now, the closed-form solution of a European call option will be derived. This derivation expands upon the work of Turner [21]. Because: $V_C(S,T) = \max(S(T) - X, 0)$, using density function

Equation 3.13 and $S_0 = S$, $V_C(S,t)$ can be rewritten to:

$$\begin{aligned} V_C(S,t) &= e^{-r(T-t)}\mathbb{E}(\max(S(T)-X,0)), \\ &= e^{-r(T-t)}\int_X^\infty (u-X)f(u)\mathrm{d}u, \\ &= e^{-r(T-t)}\Big(\int_X^\infty \frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\Big(\frac{-(\log(u/S)-(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}\Big)\mathrm{d}u \quad (3.24)\\ &- X\int_X^\infty \frac{1}{u\sigma\sqrt{2\pi(T-t)}}\exp\Big(\frac{-(\log(u/S)-(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}\Big)\mathrm{d}u\Big). \end{aligned}$$

Now, both integrals of equation 3.25 will be solved separately:

$$e^{-r(T-t)}\int_X^\infty \frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\Big(-\frac{1}{2}\Big(\frac{(\log(u/S)-(r-\sigma^2/2)(T-t))}{\sigma\sqrt{(T-t)}}\Big)^2\Big)\mathrm{d}u. \quad (3.25)$$

Now let $A = \frac{\log(\frac{X}{S})-(r-\sigma^2/2)(T-t)-\sigma^2(T-t)}{\sigma\sqrt{T-t}}$. Then, integral Equation 3.25 can be rewritten to:

$$\begin{aligned} e^{-r(T-t)}Se^{(r-\sigma^2/2)(T-t)+\sigma^2/2(T-t)}\int_A^\infty \frac{1}{\sqrt{2\pi}}e^{\frac{-y^2}{2}}\mathrm{d}y &= S(1-N(A)) = SN(-A), \\ &= SN\Big(\frac{\log(\frac{S}{X})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big), \\ &= SN(d_1). \quad (3.26) \end{aligned}$$

In this equation, $N$ is the $\mathcal{N}(0,1)$ distribution function, and:

$$d_1 = \frac{\log(S/X)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \quad (3.27)$$

Now the second integral of Equation 3.25 will be solved.

$$-e^{-r(T-t)}\int_X^\infty \frac{1}{\sigma u\sqrt{2\pi(T-t)}}\exp\Big(-\frac{1}{2}\Big(\frac{(\log(u/S)-(r-\sigma^2/2)(T-t))}{\sigma\sqrt{(T-t)}}\Big)^2\Big)\mathrm{d}u. \quad (3.28)$$

Let $z = \frac{\log(\frac{u}{S})-(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$. Then $\mathrm{d}z = \frac{\mathrm{d}u}{u\sigma\sqrt{T-t}}$ and integral 3.28 changes to:

$$\begin{aligned} -e^{-r(T-t)}X\int_{A+\sigma\sqrt{T-t}}^\infty \frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}\mathrm{d}z &= -e^{-r(T-t)}X(1-N(A+\sigma\sqrt{T-t})), \\ &= -e^{-r(T-t)}XN(-A-\sigma\sqrt{T-t}), \\ &= -e^{-r(T-t)}XN\Big(\frac{\log(\frac{S}{X})-(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big), \\ &= -e^{-r(T-t)}XN(d_2). \quad (3.29) \end{aligned}$$

Where again $N$ is the $\mathcal{N}(0,1)$ distribution function, and:

$$d_2 = \frac{\log(S/X)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \quad (3.30)$$

Now, substituting Equation 3.26 and Equation 3.29 into integral Equation 3.25 yields the closed-form solution for a European call option:

$$V_C(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2). \quad (3.31)$$

To find the solution of the European put option, we use the so-called put-call parity which is defined as follows [8]:

$$V_C(S,t) + Xe^{-r(T-t)} = V_P(S,t) + S. \tag{3.32}$$

Using this, we arrive at the Black-Scholes solution for a European put option: [8]

$$V_P(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{3.33}$$

Thus we found closed-form solutions for European options to determine the value at every time $t$.

## 3.3 American Options

Another type of financial options are American options. American options differ from European options by their exercise policy: an American option can be exercised at any time prior to expiry and on expiry. The definition of an American option is the following:

**Definition 3.3.1.** An **American call option** gives its holder the right, but not the obligation, to purchase from the writer a prescribed asset for a prescribed price at any time between the start date and a prescribed expiry date in the future. [8]

**Definition 3.3.2.** An **American put option** gives its holder the right, but not the obligation, to sell to the writer a prescribed asset for a prescribed price at any time between the start date and a prescribed expiry date in the future. [8]

The holder of the American option thus has to decide at every time prior to and on expiry whether it is optimal to exercise the option now or to hold the option and wait. This flexibility makes it harder to value American options than to value European options.

Suppose we are the holder of an American call option with strike price $X$. Suppose $S(t)$ is the stock price at time $t$, and suppose that the value of holding the option is equal to $CV(t)$. Then at any time before expiry we will have the following payoff $V_C^{Am}(S,t)$:

- if $S(t) - X > CV(t)$: exercise the option. This generates payoff $V_C^{Am}(S,t) = S(t) - X$ at time $t$, and the option stops to exist.
- if $S(t) - X \leq CV(t)$: do not exercise the option: $V_C^{Am}(S,t) = 0$ at time $t$, the option continues to exist.

When the option reaches expiry time, we will have:

$$V_C^{Am}(S,T) = \max(S(T) - X, 0). \tag{3.34}$$

The same can be done for an American put option with strike price $X$. At any time before expiry:

- if $X - S(t) > CV(t)$: exercise the option. This generates payoff $V_P^{Am}(S,t) = X - S(t)$ at time $t$, and the option stops to exist.
- if $X - S(t) \leq CV(t)$: do not exercise the option: $V_P^{Am}(S,t) = 0$ at time $t$, the option continues to exist.

When the option reaches expiry time, we will have:

$$V_P^{Am}(S,T) = \max(X - S(T), 0). \tag{3.35}$$

The details on how $CV(t)$ should be determined follow later in Section 4.3.3.

### 3.3.1 Analytic Solution of American Options

As explained, American options differ from European options in the sense that they can be exercised at any time before and on expiry. It can be shown that it is never optimal to exercise an American call before expiry. Therefore, an American call option has the same value as a European call option. For the American put option, the partial differential Equation 3.17 changes to the following partial differential inequality [8]:

$$\frac{\partial V_P^{Am}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_P^{Am}}{\partial S^2} + rS\frac{\partial V_P^{Am}}{\partial S} - rV_P^{Am} \leq 0. \tag{3.36}$$

To solve this, we need boundary conditions. At expiry, the value of the American option is equal to that of the European option [8]:

$$V_P^{Am}(S,T) = \max(X - S(T), 0), \text{ for all } S \geq 0. \tag{3.37}$$

If $S = 0$, the asset price is always zero, so we have a payoff of $X$. Therefore it is optimal to exercise immediately. This gives the following condition [8]:

$$V_P^{Am}(S,t) \to X, \text{ as } S \to 0, \text{ for all } 0 \leq t \leq T. \tag{3.38}$$

Now we look at the behaviour of the option when $S$ is large. In this case, the put will have no positive payoff [8]:

$$V_P^{Am}(S,t) \to 0, \text{ as } S \to \infty, \text{ for all } 0 \leq t \leq T. \tag{3.39}$$

This boundary problem is a lot harder to solve than that for the European variant. In general, we do not have a closed-form solution for an American put option. However, a solution for $V_P^{Am}(S,t)$ can be found using numerical methods.

## 3.4 Compound Options

A compound option is an *option on an option*. When a compound option is exercised, the holder pays the underlying option premium and receives the underlying option instead of the underlying asset. This premium is also called the *back fee*. The last underlying option of a compound option is related to the asset. Therefore, a compound option is a composition of options. Options like this are common to see in currency or fixed-income markets where there is an uncertainty regarding the option's risk protection capabilities. Compound options can be catagorized by the number of folds: i.e. the number of options on options. Thus an $n$-fold compound option consists of $n$ compounded options. Such an option has $n+1$ exercise dates and strike prices. Examples of 1-fold compound options are [13]:

- Call on a call - CoC;
- Call on a put - CoP;
- Put on a put - PoP;
- Put on a call - PoC.

Suppose we have a call on a call compound option. This option has strike prices $X_1$ and $X_2$ and expiration dates $T_1$ and $T_2$. The value of a call option will be denoted as: $V_C(S,\tau;X)$, where $S$ is the stock price, $\tau$ the time to expiry and $X$ the strike price.

On the first expiration date $T_1$, the holder has the right to buy the underlying call option using the strike price $X_1$. This new call has expiration date $T_2$ and strike price $X_2$. We introduce a new variable: $S^*$. $S^*$ is the value of $S$ such that $V_C(S, T_2 - T_1; X_2) = X_1$. The value of the call on the call on the first expiration date $T_1$ is as follows:

- if $S > S^*$: we have $V_C(S, T_2 - T_1; X_2) > X_1$, so the holder should exercise the call at $T_1$ and the value is $V_C(S, T_2 - T_1; X_2) - X_1$.
- if $S \leq S^*$: the call option should not be exercised at $T_1$ and thus the value is 0.

$S^*$ is the optimal exercise boundary: for all $S$ above $S^*$, one should exercise the call on call option and for all other values one should let the option expire. This yields:

$$V_{CoC}(S(T_1), T_1) = \max(V_C(S, T_2 - T_1; X_2) - X_1, 0). \tag{3.40}$$

We can argue in the same way to find the values of the other compound options. The option just discussed was an 1-fold compound option. We can also value $n$-fold compound options with $n > 1$, these options have $n + 1$ expiry dates and strike prices and $n$ values for $S^*$. Valuing these options works the same but one has to check at every expiry date if it is optimal to stop or to invest in the underlying option. Examples of multiple times compounded options will follow later in this thesis.

### 3.4.1 Analytic Solution of Compound Options

In Section 3.4 a few examples of compound options were given. These examples have closed-form solutions. In this subsection, the closed form solution for a call on call (CoC) compound option will be derived. This derivation will expand upon the work of Clewlow [3].

Let $T_2$ be the time to expiry of the underlying call option and $T_1$ be the time to expiry of the compound option. Furthermore, let $X_1$ be the strike price of the compound option and let $X_2$ be the strike price of the underlying option. $S(T_1)$ is the value of the asset after time $T_1$. At time $T_1$ which denotes the expiry time of the initial option, the value of the underlying call option can be found by filling in Equation 3.31 with the particular parameters for the CoC resulting in:

$$\begin{aligned} V_C(S(T_1), X_2, T_2 - T_1) &= S(T_1)N(z) - X_2 e^{-r(T_2 - T_1)} N(z - \sigma\sqrt{T_2 - T_1}), \\ z &= \frac{\ln(S(T_1)/X_2 e^{-r(T_2 - T_1)})}{\sigma\sqrt{T_2 - T_1}} + \frac{1}{2}\sigma\sqrt{T_2 - T_1}. \end{aligned}$$

The payoff of the compound option at $T_1$ can be summarised by:

$$\max(V_C(S(T_1), X_2, T_2 - T_1) - X_1, 0).$$

Therefore, the current value of the compound option is given by:

$$\begin{aligned} V_{CoC}(S_0, 0) &= e^{-rT_1}\mathbb{E}[\max(V_C(S(T_1), X_2, T_2 - T_1) - X_1, 0)], \\ V_{CoC}(S_0, 0) &= e^{-rT_1}\int_{-\infty}^{\infty} \max(V_C(e^u S_0, X_2, T_2 - T_1) - X_1, 0) f(u)\mathrm{d}u. \end{aligned} \tag{3.41}$$

Where:

$$f(u) = \frac{1}{\sigma\sqrt{2\pi T_1}} e^{-v^2/2},\ u = \ln(S(T_1)/S_0),\ v = \frac{u - \mu T_1}{\sigma\sqrt{T_1}} \text{ and } \mu = r - \frac{\sigma^2}{2}.$$

Here $S(T_1)$ has been transformed into $e^u S_0$. Let $S^*$ be the critical asset price such that:

$$V_C(S^*, X_2, T_2 - T_1) - X_1 = 0.$$

The initial option will be exercised at $T_1$ when $S(T_1) > S^*$. Then $V_C(S(T_1), X_2, T_2 - T_1) > X_1$. Therefore, $\max(V_C(e^u S_0, X_2, T_2 - T_1) - X_1, 0)$ will be nonzero for $S(T_1) > S^*$, which is equivalent to $u > ln(S^*/S_0)$. Therefore Equation 3.41 can be rewritten to:

$$V_{CoC}(S_0, 0) = e^{-rT_1} \int_{\ln(S^*/S_0)}^{\infty} S_0 e^u N(z) - X_2 e^{-r(T_2-T_1)} N(z - \sigma\sqrt{T_2 - T_1}) - X_1 \mathrm{d}u. \quad (3.42)$$

Integral 3.42 can be separated into three distinct integrals, which then can be solved resulting in:

$$e^{-rT_1} S_0 \int_{\ln(S^*/S_0)}^{\infty} e^u N(z) f(u) \mathrm{d}u = S_0 N_2(x, y; \rho), \quad (3.43)$$

$$X_2 e^{-rT_1} \int_{\ln(S^*/S_0)}^{\infty} N(z - \sigma\sqrt{T_2 - T_1}) f(u) \mathrm{d}u = X_2 e^{-rT_1} N_2(x - \sigma\sqrt{T_1}, y - \sigma\sqrt{T_2}; \rho), \quad (3.44)$$

$$X_1 e^{-rT_1} \int_{\ln(S^*/S_0)}^{\infty} f(u) \mathrm{d}u = X_1 e^{-rT_1} N(x - \sigma\sqrt{T_1}). \quad (3.45)$$

Where:

$$x = \frac{\ln(S_0/S^* e^{-rT_1})}{\sigma\sqrt{T_1}} + \frac{1}{2}\sigma\sqrt{T_1}, \quad y = \frac{\ln(S_0/k e^{-rT_2})}{\sigma\sqrt{T_2}} + \frac{1}{2}\sigma\sqrt{T_2}, \quad \rho = \sqrt{\frac{T_1}{T_2}}.$$

Here $N_2(x, y; \rho)$ denotes the bivariate normal distribution with interest rate $r$, dividend yield $d$, volatility $\sigma$ and $\rho$ the correlation coefficient. Substituting Equations 3.43, 3.44 and 3.45 into Equation 3.42 yields the closed-form solution for a CoC compound option [13]:

$$V_{CoC}(S_0, 0) = S_0 e^{-dT_2} N_2(x_+, y_+; \sqrt{T_1/T_2}) - X_2 e^{-rT_2} N_2(x_-, y_-; \sqrt{T_1/T_2}) - X_1 e^{-rT_1} N(x_-). \quad (3.46)$$

Here, the following substitutions were used for convenience:

$$\begin{aligned} x_+ &= \frac{\ln(S_0/S^*) + (r + \sigma^2/2)T_1}{\sigma\sqrt{T_1}}, \\ y_+ &= \frac{\ln(S_0/X_2) + (r + \sigma^2/2)T_2}{\sigma\sqrt{T_2}}, \\ x_- &= x_+ - \sigma\sqrt{T_1}, \\ y_- &= y_+ - \sigma\sqrt{T_2}. \end{aligned}$$

Hence, for certain options it is possible to determine a closed-form solution. Unfortunately, this is not possible for all options: for some options the equations cannot be solved analytically. The values of these options should be approximated numerically using numerical methods. The methods that can be used for this are the binomial- and trinomial methods and Monte-Carlo simulation. These methods will be explained in Chapter 4.

# 4 Numerical Valuation Methods

One goal in option valuation is to determine a *fair* value of an option at $t = 0$. The goal of this thesis is to determine the value of the R&D investments of a biotechnology company. As mentioned in Section 3.2, we can use valuation methods of financial options and adjust them to value real options. Therefore, our first focus will be to value several financial options and confirm the solutions. Then we will expand our methods to value the real option application. The valuation methods that will be explained in this chapter are: the binomial tree, the trinomial tree and the Monte Carlo method.

## 4.1 The Binomial Method

The first method we consider is the binomial tree. Suppose we have an option expiring at time $T$, and let $\delta t = T/M$ be the time step size. The key assumption made is that between two time levels, the asset price moves either up by a factor $u$, or down by a factor $d$. Such an upward movement occurs with probability $p$, and such a downward movement occurs with probability $1 - p$.

The asset price $S_0$ at time $t_0 = 0$ is known, so the asset price at time $t_1 = \delta t$ is either $uS_0$ or $dS_0$. Similarly, at time $t_2 = 2\delta t$, the asset price has either gone up twice, gone down twice, or gone up once and gone down once. This gives the following possible asset prices at time $t_2$: $u^2 S_0$, $udS_0$, $d^2 S_0$. Continuing this, we can construct a tree which gives the possible asset prices up to time $T$. At time $t = t_i = i\delta t$, there are $i + 1$ possible asset prices, which we label [8]:

$$S_n^i = d^{i-n} u^n S_0, \quad 0 \leq n \leq i. \tag{4.1}$$

So at time $M$ there are $M + 1$ possible asset prices. An example of such a tree with expiry $T = 2$ can be found in Figure 4.1.

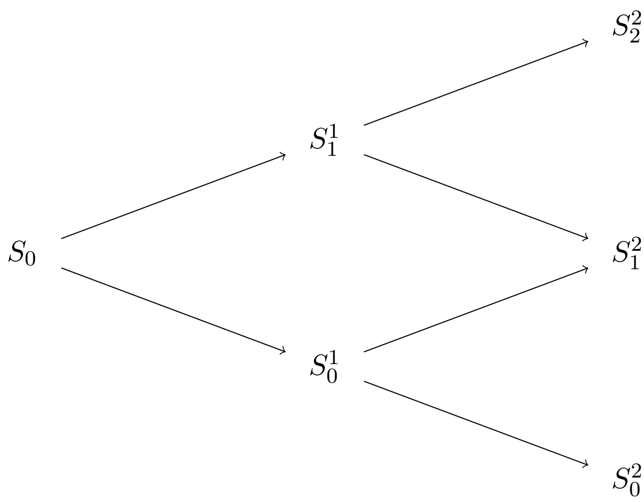


Figure 4.1: Binomial tree with $T = 2$.

Our goal here is to determine a *fair* value of the option at time 0. The binomial method can be used to value various types of options.

### 4.1.1 Binomial Method for European Options

Suppose we have a European option. Using the binomial tree, we know $M + 1$ possible asset prices at expiry. European options are only exercisable at expiry, and have payoff $\Lambda(S_n^M)$. So, the value of the option at expiry is [8]:

$$V_n^M = \Lambda(S_n^M), \quad 0 \leq n \leq i. \tag{4.2}$$

Now we want to find $V_0^0$. We do this by working backwards through the tree. We assume that the option values $V_n^{i+1}{}_{n=0}$ are known. Because of the assumption that the asset prices go either up or down, we know that asset price $S_n^i$ comes either from $S_{n+1}^{i+1}$ with probability $p$, or from $S_n^{i+1}$ with probability $1-p$. So the expected value of $S_n^i$ is: $pV_{n+1}^{i+1} + (1-p)V_n^{i+1}$. If we discount using the interest rate $r$, we get the following relation [8]:

$$V_n^i = e^{-r\delta t}(pV_{n+1}^{i+1} + (1-p)V_n^{i+1}), \quad 0 \leq n \leq i, \quad 0 \leq i \leq M-1. \tag{4.3}$$

If we know the parameters $u$, $d$, $p$ and $M$, we can use Equations 4.1, 4.2 and 4.3 to determine the option value. We are now going to derive expressions for $u$ and $d$.

We introduce a new variable $R_i$, which is 1 when the asset price goes up, and 0 when the asset price goes down between time $(i-1)\delta t$ and $i\delta t$. So $R_i$ is 1 with probability $p$, and 0 with probability $1-p$. $R_i$ is thus a Bernoulli random variable with parameter $p$. Therefore, $\mathbb{E}(R_i) = p$ and $\mathrm{Var}(R_i) = p(1-p)$. Suppose we look at $n$ time steps: $\sum_{i=1}^n R_i$ is now the number of upward movements, and $n - \sum_{i=1}^n R_i$ is the number of downward movements. We can now rewrite Equation 4.1 to [8]:

$$S(n\delta t) = S_0 u^{\sum_{i=1}^n R_i} d^{n-\sum_{i=1}^n R_i}. \tag{4.4}$$

Dividing both sides by $S_0$ and rewriting gives [8]:

$$\frac{S(n\delta t)}{S_0} = d^n (\frac{u}{d})^{\sum_{i=1}^n R_i}. \tag{4.5}$$

We can take the logarithm of this to get:

$$\frac{S(n\delta t)}{S_0} = \log(d^n(\frac{u}{d})^{\sum_{i=1}^n R_i}) = \log(d^n) + \log((\frac{u}{d})^{\sum_{i=1}^n R_i}) = n\log(d) + \log(\frac{u}{d})\sum_{i=1}^n R_i. \tag{4.6}$$

The Central Limit theorem states that the sum of $n$ (with $n$ big) equally distributed independent random variables with finite variance, is distributed as a normal random variable.

By the Central Limit theorem, $\sum_{i=1}^n R_i$ behaves like a normal random variable. So for $n$ large, $\log(S(n\delta t)/S_0)$ will be close to normal. Because of the assumptions about the continuous asset price model used in the Black-Scholes analysis, the mean of $\log(S(n\delta t)/S_0)$ must be equal to $(\mu - \frac{1}{2}\sigma^2)n\delta t$ and the variance must be equal to $\sigma^2 n\delta t$. Since the binomial method uses expected values, we assume that we work in a risk neutral world: $\mu = r$. Using the condition about the mean yield the following new condition:

$$\begin{aligned}
\mathbb{E}(\log(S(n\delta t)/S_0)) &= \mathbb{E}(n\log(d) + \log(\frac{u}{d})\sum_{i=1}^n R_i), \\
&= \mathbb{E}(n\log(d) - \log(d)\sum_{i=1}^n R_i + \log(u)\sum_{i=1}^n R_i), \\
&= \mathbb{E}(\log(d)(n - \sum_{i=1}^n R_i) + \log(u)\sum_{i=1}^n R_i), \\
&= \log(d)(n - \mathbb{E}(\sum_{i=1}^n R_i)) + \log(u)\mathbb{E}\sum_{i=1}^n R_i), \\
&= n(\log(d)(1-p) + p\log(u)). \qquad (4.7)
\end{aligned}$$

Setting this equal to $(r - \frac{1}{2}\sigma^2)n\delta t$ gives:

$$\log(d)(1-p) + p\log(u) = (r - \frac{1}{2}\sigma^2)\delta t. \tag{4.8}$$

Doing the same for the condition for the variance gives:

$$\begin{aligned}
\mathrm{Var}(\log(S(n\delta t)/S_0)) &= \mathrm{Var}(n\log(d) + \log(\frac{u}{d})\sum_{i=1}^n R_i), \\
&= \log(\frac{u}{d})^2 \mathrm{Var}(\sum_{i=1}^n R_i), \\
&= n\log(\frac{u}{d})p(1-p). \qquad (4.9)
\end{aligned}$$

Setting this equal to $\sigma^2 n\delta t$ gives:

$$\log(\frac{u}{d}) = \sigma\sqrt{\frac{\delta t}{p(1-p)}}. \tag{4.10}$$

We now have two equations, Equation 4.8 and Equation 4.10 and three unknowns: $u$, $d$ and $p$. We can fix one of these and solve for the other two. Suppose we let $p = \frac{1}{2}$, we get that:

$$u = e^{\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t}, \quad d = e^{-\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t}. \tag{4.11}$$

Using these expressions for $u$ and $d$, and Equations 4.1, 4.2 and 4.3 we can find a value for $V_0^0$.

### 4.1.2 Binomial Method for American Options

The binomial method can be adjusted to solve other options. Suppose we have an American option. The payoff of an American option at expiry is the same as the payoff of a European option, so Equation 4.1 can be used. Because an American option can be exercised at any time before expiry, we must check at every time step before expiry, whether it is optimal to exercise the option or to hold the option. Therefore, we compute the continuation value: $CV(t)$. In case of the binomial method, the continuation value is the expected value of $V_n^i$, discounted by the interest rate: $CV(t_i) = e^{-r\delta t}(pV_{n+1}^{i+1} + (1-p)V_{n-1}^{i+1})$. Thus Equation 4.2 changes to:

$$V_n^i = \max(\Lambda(S_n^i),\ CV(t_i)), \quad 0 \leq n \leq i, \quad 0 \leq i \leq M-1. \tag{4.12}$$

Now we can use the same expressions for $u$ and $d$, and Equations 4.1, 4.2 and 4.12 to find an expression for $V_0^0$.

## 4.2 Trinomial Method

An expansion of the binomial tree method is the trinomial method. The difference between the trinomial and binomial methods is that an extra factor $m$ is introduced, which we set equal to 1. Now the asset price can either go up, down or stay the same. We let the probabilities that the asset prices go up, down or stay the same be respectively: $p_u$, $p_d$ and $p_m$.

Suppose the asset price $S_0$ at time $t_0 = 0$ is known, so the asset price at time $t_1 = \delta t$ is either $uS_0$, $dS_0$ or $mS_0$. Similarly, at time $t_2 = 2\delta t$, we have the following possible asset prices: $u^2S_0$, $umS_0$, $m^2S_0$, $dmS_0$ and $d^2S_0$. Because $m = 1$, this is the same as the prices: $u^2S_0$, $uS_0$, $S_0$, $dS_0$ and $d^2S_0$.

We let $N_u$ be the times the asset price goes up, $N_d$ be the times that the asset price goes down and $N_m$ the times that the asset price remains the same. Now we can rewrite Equation 4.1 to [4]:

$$S_{i,j} = S_0 u^{N_u} d^{N_d}, \quad N_u + N_d + N_m = 1. \tag{4.13}$$

Here $j$ is the node, and $i$ is the time. So, at time $M$, there are $2M + 1$ possible asset prices. An example of such a tree with expiry $T = 2$ can be seen in Figure 4.2.

### 4.2.1 Trinomial Method for European Options

We can determine the value of the option at expiry using equation 4.2. Now we want to find $V_0^0$ by working backwards through the tree. Assume that the option values $(V_j^{i+1})_{j=0}^{i+1}$ are known. Because of the assumption that the asset prices can either go up, down or stay the same, we know that asset price $S_j^i$ comes from $S_{j+1}^{i+1}$ with probability $p_u$, from $S_j^{i+1}$ with probability $p_m$ or from $S_{j-1}^{i+1}$ with probability $p_d$. So the expected value of $S_j^i$ is: $p_u S_{j+1}^{i+1} + p_m S_j^{i+1} + p_d S_{j-1}^{i+1}$. If we discount this using the interest rate $r$, we get the following relation for the value of a European option:

$$V_j^i = e^{-r\delta t}(p_u S_{j+1}^{i+1} + p_m S_j^{i+1} + p_d S_{j-1}^{i+1}), \quad 0 \leq j \leq i, \quad 0 \leq i \leq M-1 \tag{4.14}$$

If we know te parameters $u$, $d$, $p_u$, $p_d$, $p_m$ and $M$, we can use Equation 4.14 and Equations 4.1 and 4.2 to find the value $V_0^0$. We are now going to derive expressions for $u$, $d$ and $p_u$, $p_d$ and $p_m$.

We assume that the asset price follows a geometric Brownian motion: $r$ is the rate of the risk-free

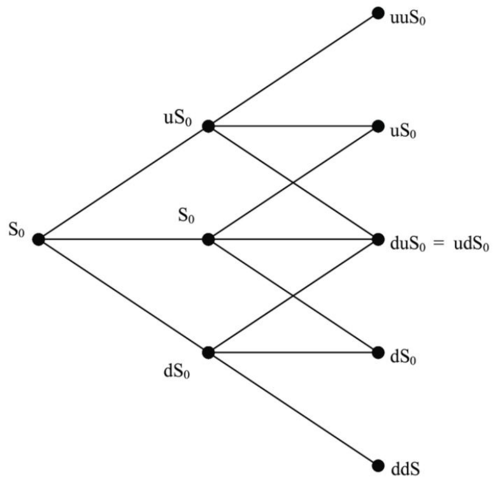


Figure 4.2: Trinomial tree with $T = 2$. [6]

investment, and we assume that the volatility $\sigma$ of the underlying asset is constant. This yields the following conditions [4]:

$$\mathbb{E}(S(t_{i+1})|S(t_i)) = e^{r\delta t}S(t_i), \tag{4.15}$$

$$\text{Var}(S(t_{i+1})|S(t_i)) = \delta t S(t_i)^2\sigma^2 + O(\delta t). \tag{4.16}$$

We can rewrite equation 4.15 to [4]:

$$\mathbb{E}(S(t_{i+1})|S(t_i)) = p_m m + p_u u + p_d d = 1 - p_u - p_d + p_u u + p_d d = e^{r\delta t}. \tag{4.17}$$

Here we used that $p_m = 1 - p_u - p_d$. We also add an extra constraint which requires that:

$$ud = 1. \tag{4.18}$$

This condition makes the numerical scheme way less complex: the number of nodes now grows polynomially instead of exponentially.
Using the three constaints 4.15, 4.16 and 4.18, we find a family of trinomial tree models. We choose to take [4]:

$$u = e^{\sigma\sqrt{2\delta t}}, \quad d = e^{-\sigma\sqrt{2\delta t}}. \tag{4.19}$$

Now $p_u$, $p_d$ and $p_m$ are defined as [4]:

$$p_u = \left(\frac{e^{\frac{r\delta t}{2}} - e^{-\sigma\sqrt{\frac{\delta t}{2}}}}{e^{\sigma\sqrt{\frac{\delta t}{2}}} - e^{-\sigma\sqrt{\frac{\delta t}{2}}}}\right)^2, \tag{4.20}$$

$$p_d = \left(\frac{-e^{\frac{r\delta t}{2}} + e^{-\sigma\sqrt{\frac{\delta t}{2}}}}{e^{\sigma\sqrt{\frac{\delta t}{2}}} - e^{-\sigma\sqrt{\frac{\delta t}{2}}}}\right)^2, \tag{4.21}$$

$$p_m = 1 - p_u - p_d. \tag{4.22}$$

We can now find $V_0^0$ of European options using Equations 4.1, 4.2 and 4.14.

### 4.2.2 Trinomial Method for American Options

For an American option, we need to determine the continuation value $CV(t_i)$. The continuation value of an American option using the trinomial tree is: $CV(t_i) = e^{-r\delta t}(p_u S_{j+1}^{i+1} + p_m S_j^{i+1} + p_d S_{j-1}^{i+1})$, so the value of the option is:

$$V_j^i = \max(\Lambda(S_j^i),\ V(t_i)), \quad 0 \leq j \leq i, \quad 0 \leq i \leq M-1 \tag{4.23}$$

We can use Equation 4.23 and Equations 4.1 and 4.2 to find the value $V_0^0$. Various other options can be priced using the trinomial tree, the payoff function and the formula for $V_n^i$ have to be changed depending on the type of option.

## 4.3 Monte Carlo Simulation

Monte Carlo simulation relies on repeated random sampling to obtain numerical results. In option pricing, Monte Carlo simulation is used to generate different stock paths. Suppose $M$ stock paths are generated using a psuedo random number generator. For each stock path, an estimation of the option price can be made. The *Law of Large Numbers* states that when an experiment is performed $M$ times, and let $M$ big, the mean of the experiments will converge to the theoretical expectation of the experiment. Creating a large number of stock paths is what makes Monte Carlo simulation computationally intensive and expensive.

The Monte Carlo method will now be further explained, based on the work of Higham [8]. Let $W$ be a general random variable with expectation $\mathbb{E}(W) = a$ and variance $\text{Var}(W) = b^2$ unknown. The aim is to give an approximation of $a$ and $b$. Therefore a psuedo random number generator is needed to take independent samples of $W$.

Let $W_1, W_2, ..., W_M$ be independent random samples of $W$ with the same distribution as $W$. Then, by the Law of Large Numbers one expects that:

$$a_M = \frac{1}{M}\sum_{i=1}^{M} W_i. \tag{4.24}$$

to be a good unbiased approximation of $a$. Unbiased means that $\mathbb{E}(a_M) = \mathbb{E}(W)$.
To estimate $b$, we use that:

$$\text{Var}(W) \quad = \quad \mathbb{E}((W - \mathbb{E}(W))^2). \tag{4.25}$$

Therefore, substituting $\mathbb{E}(W) = a_M$ yields that $\frac{1}{M}\sum_{i=1}^{M}(W_i - a_M)^2$ would be a good choice for $\text{Var}(W)$. This estimator should be rescaled to make it unbiased:

$$b_M^2 = \frac{1}{M-1}\sum_{i=1}^{M}(W_i - a_M)^2. \tag{4.26}$$

The Central Limit Theorem states that the sum of a large number of independently distributed variables with finite variance behaves like a normal distribution. So, by the Central Limit Theorem, $\sum_{i=1}^{M} W_i$ behaves like an $\text{N}(Ma,\ Mb^2)$ random variable, so:

$$a_M - a \text{ is approximately } \text{N}\left(0, \frac{b^2}{M}\right). \tag{4.27}$$

Using the expressions obtained for $a_M$ and $b_M$, a 95% confidence interval can be constructed for the option value $a$:

$$\left[a_M - \frac{1.96 b_M}{\sqrt{M}}, a_M + \frac{1.96 b_M}{\sqrt{M}}\right]. \tag{4.28}$$

Because the goal is to determine the option value at time 0, asset paths will be created until expiry time and the option value will be determined by working backwards until time 0. In the next sections, Monte Carlo algorithms will be given to estimate the option values of European and American options.

### 4.3.1 Monte Carlo Simulation for European Options

Algorithm 1 shows how to determine $V_0^0$ of a European option using Monte Carlo simulation.

**Result:** Value of a European Option
Create $M$ random paths for the stock price $S$
For each path, calculate the payoff at maturity using $\max(S(T) - X, 0)$ for a call option and $\max(X - S(T), 0)$ for a put option
Start at maturity $i = N$;
**for** $i : -1 : 1$ **do**
  for each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
**end**
For each path $j$, discount the price $V_j(t_0) = e^{-\int_{t_0}^{t_1} r(s)ds} V_j(t_1)$
The price of the option at time zero is the mean of the vector $V_j(t_0)$

**Algorithm 1:** Monte Carlo algorithm for European options.

### 4.3.2 Monte Carlo simulation for American options

Monte Carlo simulation can also be used to determine $V_0^0$ and $S^*$ for an American put option. Because an American option can be exercised at any time prior to and on expiry, one has to determine at every time step whether the value of continuation (Section 4.3.3) is higher than exercising at that time. The algorithm to value an American option using Monte Carlo simulation can be found in Algorithm 2.

**Result:** Value of a American option
Create $M$ random paths for the stock price $S$
For each path, calculate the payoff at maturity using $\max(S(T) - X, 0)$ for a call option and $\max(X - S(T), 0)$ for a put option
Start at maturity $i = N$;
**for** $i = N : -1 : 1$ **do**
  for each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
  For each path $j$, compute the continuation value $c_j$ using the first three Laguerre polynomials
  For each path $j$ with $h_j > c_j$, exercise the option, so let $V_j(t_i) = \Lambda(S_n^i)$
**end**
For each path $j$, discount the price $V_j(t_0) = e^{-\int_{t_0}^{t_1} r(s)ds} V_j(t_1)$
The price of the option at time zero is the mean of the vector $V_j(t_0)$

**Algorithm 2:** Monte Carlo algorithm for American options. [18]

The steps to compute the continuation value can be found in Section 4.3.3. Depending on the exercise policy of the option, the algorithms above can be adjusted to value various other types of options.

### 4.3.3 Continuation Value for Monte Carlo Method

This section shows how to compute the continuation value for the Monte Carlo method. The work in this section is based on the work of Longstaff and Schwarz [14].

Suppose $(\Omega,\ \mathcal{F},\ P)$ is a probability space and $[0,T]$ is a finite time horizon. $\Omega$ is the set of all possible outcomes of the asset price between time 0 and $T$, and $\omega$ represents a sample path. Let $\mathcal{F}$ be a *sigma algebra* of all different events at time $T$, and $P$ is the probability measure defined on the elements of $\mathcal{F}$. Let $F = \mathcal{F}_t; t \in [0,T]$ be the filtration generated by the asset process, and assume that $\mathcal{F}_T = \mathcal{F}$.

**Definition 4.3.1.** Given a probability space $(\Omega,\ \mathcal{F},\ P)$ a filtration $\mathcal{F}_t$ is a collection of sub-sigma-algebras of $\mathcal{F}$ satisfying $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$. [15]

By the no-arbitrage assumption, an assumption made is that there exists a martingale measure $Q$.

**Definition 4.3.2.** A martingale is a sequence of random variables $X_0$, $X_1$, $X_2$,...,$X_n$ such that for any time n: [16]

- $\mathbb{E}(|X_n|) < \infty$ ,
- $\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = X_n$.

Now let $C(\omega, s; t, T)$ be the path of cash flows generated by the option, conditional on not being exercised at or before time $t$. The goal is to find the optimal stopping rule which maximizes the value of the American option.

An assumption made to compute the continuation value is that an American option can be can be exercised at $K$ discrete points between time 0 and $T$. In practice, an American option can be exercised continuously, continuity can be approximated by taking $K$ large. Since the cash flow of immediate exercise at $t_k$ is known, we want to know the cash flows from continuation at time $t_k$. The value of continuation $CV(\omega; t_k)$ is given by:

$$CV(\omega, t_k) = \mathbb{E}_Q\Big[\sum_{j=k+1}^{K} e^{-\int_{t_k}^{t_j} r(\omega,s)ds} C(\omega, t_j; t_k, T)|\mathcal{CV}_{t_k}\Big]. \tag{4.29}$$

We will now use least squares to approximate the conditional expectation function at $t_{K-1}$, $t_{K-2}$, ...,$t_1$. We work backwards since the path of cash flows $C(\omega, s; t, T)$ is defined recursively so $C(\omega, s; t_k, T)$ can differ from $C(\omega, s; t_{k+1}, T)$, so it may be optimal to stop at time $t_{k+1}$. At time $t_{K-1}$ we assume that the unknown functional form of $CV(\omega; t_{K-1})$ can be represented as a linear combination of a countable set of $\mathcal{F}_{t_{K-1}}$-measurable basis function. Assuming that $X$ is the value of the underlying asset and that $X$ follows a *Markov* process, i.e. "X is a random process whose future probabilities are determined by its most recent values" [22]. The following set of functions can be chosen as basis functions:

$$L_n(X) = e^{-X/2}\frac{e^X}{n!}\frac{d^n}{dX^n}\Big(X^n e^{-X}\Big). \tag{4.30}$$

These basis functions are the weighted Laguerre polynomials. Other possible choices for basis functions are Hermite, Chebyshev, Gegenbauer and Jacobi polynomials. The choice of the polynomials does not influence the numerical results significantly [14]. Equation 4.29 can now be rewritten to:

$$CV(\omega; t_{K-1}) = \sum_{j=0}^{\infty} a_j L_j(X). \tag{4.31}$$

Here, the $a_j$ coefficients are constants, which need to be determined by using least squares regression.

### 4.3.4 Least Squares Regression for Continuation Value Coefficients

A short explanation on how to determine the coefficients $a_j$ in Equation 4.31 will follow. The coefficients will be determined by using least squares regression (LS). The coefficients $a_j$ can be written as a vector:

$$\hat{\mathbf{a}} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{y}. \tag{4.32}$$

In this equation, $\mathbf{B}$ is a matrix which looks as:

$$\mathbf{B} = \begin{bmatrix} 1 & L_0(X_1) & L_1(X_1) & \dots & L_\infty(X_1) \\ 1 & L_0(X_2) & L_1(X_2) & \dots & L_\infty(X_2) \\ 1 & L_0(X_3) & L_1(X_3) & \dots & L_\infty(X_3) \\ 1 & L_0(X_4) & L_1(X_4) & \dots & L_\infty(X_4) \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & L_0(X_M) & L_2(X_M) & \dots & L_\infty(X_M) \end{bmatrix}. \tag{4.33}$$

In Equation 4.33, the vector $\mathbf{X}$ is the stock price at time $t_{K-1}$. The vector $\mathbf{y}$ is the discounted stock price at $t_K$, so:

$$\mathbf{y} = e^{-r\delta t} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ \vdots \\ X_M \end{bmatrix}. \tag{4.34}$$

Substituting Equations 4.34 and 4.33 into Equation 4.32 will yield a vector of coefficients which can be substituted into Equation 4.31. After this substitution the continuation value can be determined and used in Monte Carlo simulation.

# 5 Academical Tests and Validation of Algorithms

This chapter will elaborate on the validation of the algorithms for the binomial, trinomial and Monte Carlo methods for the American option and compound option, which is possible since the prices of these latter options are also known analytically. The valuation techniques will also be applied to the option to choose. For validation, the American option is considered as it serves as an introduction to compound options. This is logical since an $n$-fold compound option with $n$ large resembles the American option: if $n \to \infty$, the $n$-fold compound option is an American option where the underlying products are options instead of assets. Furthermore, there is a lot more known about American option values and their optimal exercise boundary than there is about compound options, so these options serve as a first check of the correctness of the algorithms.

## 5.1 American Put Option

The first option to be considered is the American put option. The option has payoff function $\Lambda(X) = \max(X - S, 0)$, and the option can be exercised at any time prior to and on expiry. At every time step $t$, there exists a critical asset price $S^*(t)$. For an American put option, the holder should exercise when $S(t) < S^*(t)$. When $S^*(t)$ is computed for all $t$, an optimal exercise boundary can be constructed. For all $S(t)$ above the boundary, the option should be exercised. $S^*$ can be determined by using the binomial and trinomial methods and using Monte Carlo simulation. We define two methods to establish $S^*$:

1. Let $S^*(t_i)$ the smallest $S_n^i$ for which the binomial method indicates that it is not beneficial to exercise the option;
2. Let $S^*(t_i)$ the average of: the biggest $S_n^i$ for which the binomial method indicates that it is beneficial to exercise the option and the smallest $S_n^i$ for which the binomial indicates that it is not beneficial to exercise the option.

### 5.1.1 Critical Asset Price Approximation using Binomial Tree

Figure 5.1 shows a plot of $S^*$ for both methods using the binomial tree. The plots approximately give the same result hence it can be concluded that both methods perform similarly. Therefore it is arbitrarily decided that from now on method 2 will be used for further computations.

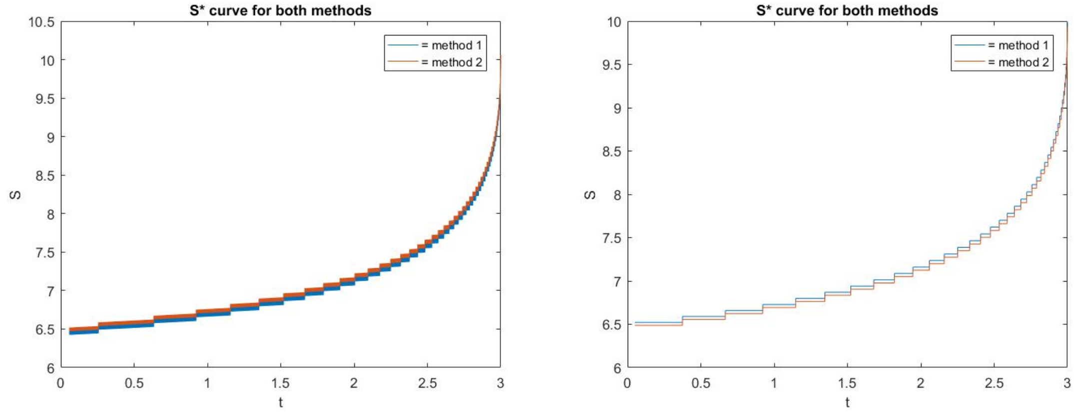


Figure 5.1: Comparison of respectively binomial and trinomial method for $M = 5000$, $S_0 = 9$, $X = 10$, $T = 3$, $r = 0.06$ and $\sigma = 0.3$.

### 5.1.2 Critical Asset Price Approximation using Trinomial Tree

Another way to determine the optimal exercise boundary is using the trinomial tree. The comparison of the optimal exercise boundaries of the binomial and trinomial methods for methods 1 and 2 and for $M = 100$ and $M = 5000$ can be found in Figure 5.2. This figure shows that the optimal exercise boundary of the binomial methods oscillates, whereas the optimal exercise boundary of the trinomial methods does not. Apparently, the trinomial methods converges quicker for smaller $M$. A comparison of the trinomial and binomial methods for $M = 5000$ can be found in Figure 5.1. For $M = 100$, Figure 5.2 shows that the binomial method produces more possible outcomes per time step. When $M = 5000$, the methods give almost the exact same results and both the trinomial and the binomial methods converge to the strike price $X = 9$.

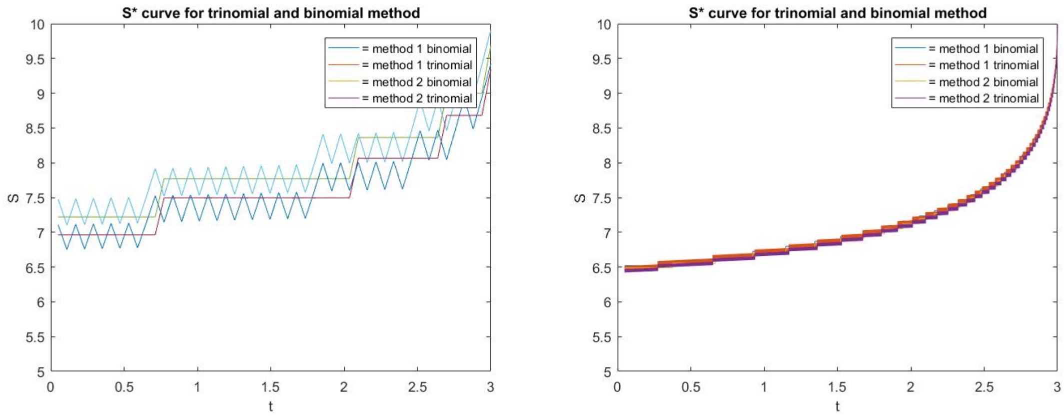


Figure 5.2: Comparison of binomial and trinomial method for $M = 100$ and $M = 5000$.

### 5.1.3 Critical Asset Price Approximation using Monte Carlo method

In Figure 5.3, $S^*$ is plotted for the Monte Carlo method, where $S^*$ is defined using method 2.

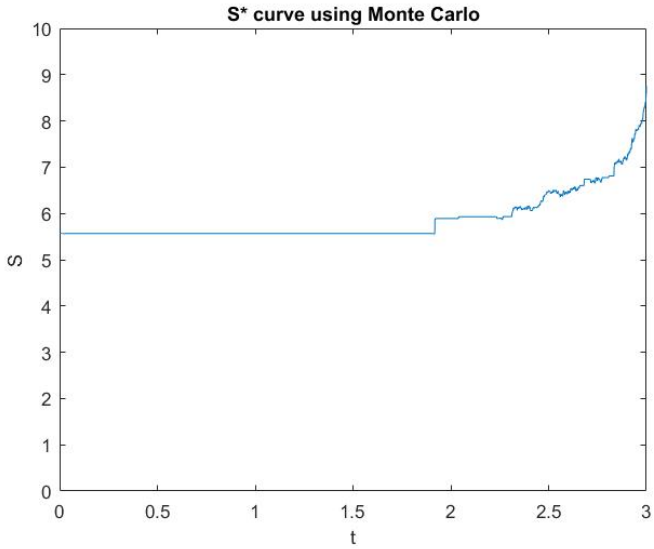


Figure 5.3: Optimal exercise boundary for an American put option for method 2 using Monte Carlo simulation and $S_0 = 9$, $X = 10$, $T = 3$, $r = 0.06$, $\sigma = 0.3$, $M = 5000$ and $dt = 1e - 3$.

Figure 5.1 has shown that for $t \to T$, $S^*(t)$ converges to the strike price $X = 9$. This happens because at expiry there is no continuation value to consider: one only has to decide whether it is optimal to exercise or to let the option expire, so one should exercise the option when $S(T) < X = S^*(T)$. Figure 5.3 however shows that the convergence of the Monte Carlo method is slower, since for the same $M$ it has not converged to the strike price yet.

### 5.1.4 Convergence of the Valuation Methods for American Put

The optimal exercise boundary for the American put option has just been determined. Using the methods described in Chapter 4, the value of the option can also be determined. As mentioned in Chapter 3.2, there is no closed-form solution for American options. The differential equation however can be numerically approximated using various techniques. To validate the outcomes of the algorithm, results from Higham [8] will be used. Table 5.1 shows the results of the binomial and trinomial methods for various values of $M$. The methods appear to converge non-monotonically to the value 1.7958, which is the value as found in Higham [8]. Hence, the algorithm for the binomial and trinomial methods give the desired results.

| $M$ | Binomial | Trinomial |
|---|---|---|
| 100 | 1.7970 | 1.7946 |
| 200 | 1.7957 | 1.7947 |
| 400 | 1.7958 | 1.7954 |
| 1000 | 1.7958 | 1.7957 |
| 5000 | 1.7958 | 1.7958 |

Table 5.1: American put option values for the binomial and trinomial method with $S_0 = 9$, $X = 10$, $T = 3$, $\sigma = 0.3$ and $r = 0.06$.

When the Monte Carlo method is considered, $M$ represents the number of simulated stock paths. As explained before, the Monte Carlo method gives a 95% confidence interval for the option value. The valuation is considered appropriate if the interval covers the real option value. Table 5.3

shows the confidence intervals for different values of $M$. This table shows that taking a larger $M$ yields a smaller confidence interval thus a more accurate approximation. All of the intervals cover the true option value but taking $M$ bigger gives a more precise solution.

| M | 95% confidence interval |
|---|---|
| 500 | [1.7052, 1.9781] |
| 1000 | [1.7034, 1.8684] |
| 5000 | [1.7328, 1.8041] |
| 10000 | [1.7600, 1.8135] |

Table 5.2: American put option values for the Monte Carlo method with $\sigma = 0.3$, $r = 0.06$, $S_0 = 9$, $dt = 1e-3$ and $X = 10$.

## 5.2 Single Compound Option

As mentioned in Chapter 4, there exist closed-form solutions for a few single compound options. In this section, the call on call option will be valued using the binomial and trinomial methods as well as the Monte Carlo method. Afterwards, these values will be checked using the closed form solution for a call-on-call as stated in Equation 3.46. When a multiple compounded option will be researched further in this thesis, there will not be a closed form solution to check with. Therefore, this section verifies that the algorithms used to valuate the compounded options yield the desired results.

### 5.2.1 Binomial and Trinomial Method for Single Compound Option

In Chapter 4, algorithms have been introduced for the valuation of certain options. However, the algorithms introduced in that chapter are not capable of valueing a call-on-call option yet, hence modifications to these algorithms are necessary. Algorithm 3 shows the modified algorithm that will be used to value the CoC option.

**Result:** Value of a CoC option
Create a binomial or trinomial tree with $M$ steps until expiry ($T_2$)
For all possible outcomes for $S$ at maturity, calculate the payoff at maturity, $T_2$, is given by $\max(S(T_2) - X_2, 0)$
Start at maturity $i = M$;
**for** $i = N : -1 : \frac{T_1}{T_2}M$ **do**
| Work backwards through the tree using the method from chapter 4.1 or 4.2.
**end**
The value of the call on call option at $T_1$ is $\max(V_{CoC}(S, T_2 - T_1; X_2) - X_1, 0)$
**for** $i = \frac{T_1}{T_2}M - 1 : -1 : 1$ **do**
| Work backwards through the tree using method from chapter 4.1 or 4.2.
**end**

**Algorithm 3:** Binomial and trinomial algorithms for CoC option.

Using Algorithm 3, the results in Table 5.3 are obtained. The result of the closed form solution with the same parameters was $V_0^0 = 0.4798$, so both methods converge to the exact solution.

| M | Binomial | Trinomial |
|---|---|---|
| 100 | 0.4753 | 0.4767 |
| 200 | 0.4785 | 0.4777 |
| 400 | 0.4793 | 0.4792 |
| 1000 | 0.4796 | 0.4795 |
| 5000 | 0.4798 | 0.4798 |

Table 5.3: Call on call option values for binomial and trinomial method with $\sigma = 0.3$, $r = 0.05$, $S_0 = 3$, $X_1 = 1$ and $X_2 = 2$.

### 5.2.2 Monte Carlo method for Single Compound Option

The algorithm to value the single compound option using the Monte Carlo method is as follows:

**Result:** Value of a CoC option
Create $M$ random paths for the stock price $S$ until expiry ($T_2$)
For each path, calculate the payoff at maturity $T_2$ is given by: $\max(S(T_2) - X_2, 0)$.
Start at maturity $i = N$;
**for** $i = N : -1 : \frac{T_1}{T_2}N$ **do**
  For each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
  For each path $j$, compute the continuation value $c_j$ using the first three weighted Laguerre polynomials and set $V_j(t_i) = c_j$
**end**
$V_j(T_1) = \max(V(T_1) - X_1, 0)$
**for** $i = \frac{T_1}{T_2}N - 1 : -1 : 1$ **do**
  For each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
**end**
for each path $j$, discount the price $V_j(t_0) = e^{-\int_{t_0}^{t_1} r(s)ds} V_j(t_1)$
The price of the option at time zero is the mean of the vector $V_j(t_0)$

**Algorithm 4:** Monte Carlo algorithm for CoC option.

| M | 95% confidence interval |
|---|---|
| 300 | [0.3823, 0.5439] |
| 500 | [0.4431, 0.5710] |
| 1000 | [0.4621, 0.5570] |
| 3000 | [0.4465, 0.4964] |
| 5000 | [0.4384, 0.4772] |

Table 5.4: Call on call option values for Monte Carlo method with $\sigma = 0.3$, $r = 0.05$, $S_0 = 3$, $X_1 = 1$ and $X_2 = 2$.

Algorithm 4 yields the results of Table 5.4. The exact value is given by 0.4798. The 95% confidence interval covers this value for almost all values of $M$, so the Monte Carlo simulation gives the desired results. When the results from table 5.4 are compared with the results of the binomial and trinomial methods in Table 5.3, one can conclude that both the binomial tree and the trinomial tree converge quicker compared to the Monte Carlo method.

## 5.3 Multiple Compounded Call Option

As an expansion on the CoC option, the multiple compounded call option can be constructed. The phases introduced in Chapter 2 of the biotechnology firm, form a 6-fold compound option. In this section, option values and the optimal exercise boundary of an $n$-fold compound option are obtained via the binomial method and Monte Carlo method, where $1 \leq n \leq 10$. Using the algorithms designed in this section, the biotechnology company can be valued in Chapter 6.

### 5.3.1 Binomial Method for Multiple Compound Option

Algorithm 5 shows how to value an $n$-fold compound option.

**Result:** Value of $n$-fold compound option
Create a binomial tree with $M$ steps until expiry ($T_n$)
For all possible outcomes for $S$ at maturity, the payoff at maturity is calculated using $\max(S(T_n) - X_n, 0)$.
Start at maturity $i = M$;
**for** $i = M : -1 : \frac{T_{n-1}}{T_n} M$ **do**
| Work backwards through the tree using the method described in Chapter 4.1
**end**
The value of the call on call option at $T_{n-1}$ is $\max(V_C(S, T_n - T_{n-1}; X_n) - X_{n-1}, 0)$
**for** $i = \frac{T_{n-1}}{T_n} M - 1 : -1 : \frac{T_{n-2}}{T_{n-1}} M$ **do**
| Work backwards through the tree using the method described in Chapter 4.1.
**end**
The value of the call on call option at $T_{n-2}$ is $\max(V_C(S, T_{n-1} - T_{n-2}; X_{n-1}) - X_{n-2}, 0)$
Continue with this process until $T_1$ is reached
From $T_1$, work backwards through the tree using the method from chapter 4.1 to obtain the option value $V_0$.

**Algorithm 5:** Binomial algorithm for $n$-fold compound option

| Number of folds → S* values ↓ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S^*_{T_1}$ | 4.34 | 5.91 | 7.76 | 9.17 | 11.08 | 12.30 | 13.96 | 15.84 | 17.24 | 18.76 |
| $S^*_{T_2}$ | 2.40 | 4.47 | 6.35 | 8.15 | 9.84 | 11.15 | 12.91 | 14.64 | 15.92 | 18.06 |
| $S^*_{T_3}$ | | 2.40 | 4.58 | 6.39 | 8.19 | 9.88 | 11.93 | 13.24 | 15.33 | 16.32 |
| $S^*_{T_4}$ | | | 2.40 | 4.59 | 6.54 | 8.40 | 10.13 | 12.23 | 13.57 | 15.08 |
| $S^*_{T_5}$ | | | | 2.40 | 4.70 | 6.84 | 8.60 | 10.38 | 12.00 | 13.90 |
| $S^*_{T_6}$ | | | | | 2.40 | 4.81 | 6.71 | 8.81 | 10.62 | 12.29 |
| $S^*_{T_7}$ | | | | | | 2.40 | 4.81 | 7.01 | 9.01 | 10.87 |
| $S^*_{T_8}$ | | | | | | | 2.40 | 4.81 | 7.02 | 8.83 |
| $S^*_{T_9}$ | | | | | | | | 2.40 | 4.82 | 7.03 |
| $S^*_{T_{10}}$ | | | | | | | | | 2.40 | 4.82 |
| $S^*_{T_{11}}$ | | | | | | | | | | 2.40 |
| Option value $V_0^0$ | 8.51 | 5.24 | 3.07 | 1.53 | 6.65e-1 | 2.43e-1 | 7.67e-2 | 2.05e-2 | 4.70e-3 | 9.09e-4 |

Table 5.5: Table of $S^*$ values and option values for $n$ times compounded call option, with $1 \leq n \leq 10$. The $S^*$ value in every column is the value at expiry ($T = 10$). The following parameters were used: $S_0 = 15$, $T = 10$, $X_n = 5$, $r = 0.05$, $\sigma = 0.3$ and $M = 5000$.

In Table 5.5, $V_0^0$ and $S^*$ values can be found for an $n$-fold compound option. Here, a strike price

$X = 5$ is used for all options. The price of a European call option (a 0-fold option) with these parameters is $V_0^0 = 12.0787$. The option values for $n = 3$, $n = 1$ and $n = 0$ have been confirmed to be correct [20], so based on this and the previous tests, we will conclude that all the obtained values are approximately correct.

For an $n$-fold compound option, $S^*_{T_{n+1}}$ is the critical asset price at expiry. Hence, for an 2-fold compound option, $S^*_3$ represents the critical asset price at expiry. The option values $V_0^0$ and $S^*$ of an $n$-fold compound option with $1 \leq n \leq 10$ can be found in Table 5.5. In Table 5.5, one can see that the price of a compound option decreases when $n$ increases. When $n$ increases, the number of options on options increases and therefore the number of option premiums increases. Therefore, the total premium paid with a 10-fold compound option is higher than with a 5-fold compound option.

The lower price can therefore be seen as a compensation of the higher total premium paid. Furthermore, the critical asset price $S^*$ increases when $n$ increases. This is also related to the option premiums that have to be paid: the owner of a 10-fold compound option still has to pay the option premium of 5 for 10 times, so therefore $S^*$ is higher.

### 5.3.2 Monte Carlo Method for Multiple Compound Option

The goal of this section is to expand Algorithm 4 to an algorithm that values an $n$-fold compound option and yields the same results as Table 5.5. The following algorithm was used:

**Result:** Value of $n$-fold compound option
Create $M$ random paths for the stock price $S$ until expiry $(T_n)$
For each path, calculate the payoff at maturity $T_n$ is given by: $\max(S(T_n) - X_n, 0)$.
Start at maturity $i = N$;
**for** $i = N : -1 : \frac{T_{n-1}}{T_n} N$ **do**
  For each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
  For each path $j$, compute the continuation value $c_j$, and set $V_j(t_i) = c_j$
**end**
The value of the option at $T_{n-1}$ is $\max(V_C(S, T_n - T_{n-1}; X_n) - X_{n-1}, 0)$
**for** $i = \frac{T_{n-1}}{T_n} N - 1 : -1 : \frac{T_{n-2}}{T_{n-1}} N$ **do**
  For each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
  For each path $j$, compute the continuation value $c_j$, and set $V_j(t_i) = c_j$
**end**
The value of the option at $T_{n-1}$ is $\max(V_C(S, T_{n-1} - T_{n-2}; X_{n-1}) - X_{n-2}, 0)$
Repeat this process until $\frac{T_1}{T_n}$ is reached.
**for** $i = \frac{T_2}{T_n} N - 1 : -1 : \frac{T_1}{T_n}$ **do**
  For each path $j$, discount the price $V_j(t_i) = e^{-\int_{t_i}^{t_{i+1}} r(s)ds} V_j(t_{i+1})$
**end**
for each path $j$, discount the price $V_j(t_0) = e^{-\int_{t_0}^{t_1} r(s)ds} V_j(t_1)$
The price of the option at time zero is the mean of the vector $V_j(t_0)$

**Algorithm 6:** Monte Carlo algorithm for $n$-fold compound option

For the computations of the continuation value (CV), the weighted Laguerre polynomials were used as basis functions: for an $n$-fold option, $n+1$ weighted Laguerre polynomials were used. The results can be found in Table 5.6. When these results are compared to the results in Table 5.5, the Monte Carlo method appears to give different results. Therefore, the Monte Carlo method will not be used to value the real option.

| Number of folds | 95 % confidence interval of the option value |
|---|---|
| 0 | [11.7038, 12.6821] |
| 1 | [8.6238, 8.8545] |
| 2 | [5.0922, 5.2638] |
| 3 | [2.5754, 2.7282] |
| 4 | [0.8058, 0.8841] |
| 5 | [0.0392, 0.0494] |

Table 5.6: Monte Carlo results for an $n$-fold compound option with $S_0 = 15$, $T = 10$, $X_n = 5$, $r = 0.05$, $\sigma = 0.3$, $dt = 1e-2$ and $M = 5000$.

In the following sections, the aim is to understand the solutions of the multiple compounded call options, by varying specific parameters and other choices.

### 5.3.3 Effect of the Volatility

The effect of $\sigma$ on the optimal exercise boundary, $S^*$, can be investigated. The impact of the volatility on the optimal exercise boundary for an 10-fold compound option can be found in Figure 5.4. Figure 5.4 shows that if $\sigma$ is smaller, the optimal exercise boundary will be higher. Since a compounded call option is considered, the optimal exercise boundary is a lower bound for the exercise area. For all $S$ above the curve, the option should be exercised. Therefore, from Figure 5.4, a compound option with a higher volatility is exercised quicker than a compound option with a lower volatility.

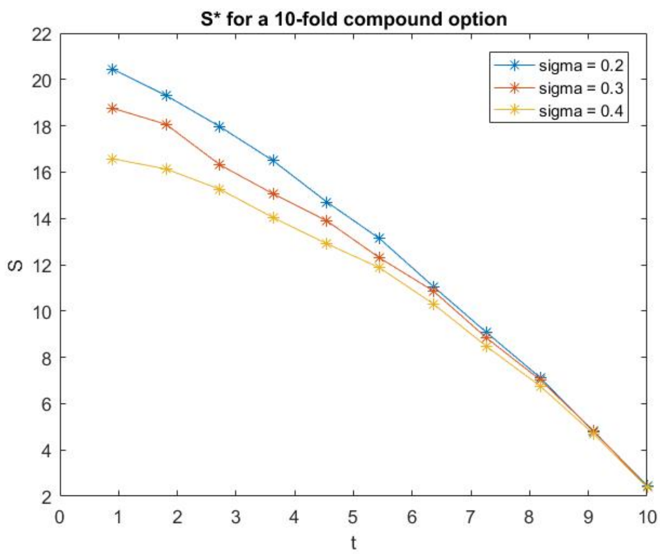


Figure 5.4: Optimal exercise boundary for a 10-fold compound option, $S_0 = 15$, $T = 10$, $X_n = 5$, $r = 0.05$, $\sigma = 0.3$ and $M = 5000$.

The impact of the volatility on the option price $V_0^0$ can be found in Table 5.7. Table 5.7 shows that the value of the compound call option is always higher when the volatility is higher. This happens because higher volatility increases both the up potential and down potential. Higher up and down potential can be considered an uncertainty with a certain cost. Uncertainties make prices increase, so an increase in volatility will result in an increase in option value. The same explanation holds for the results in Figure 5.4. Since an increase in the volatility increases the uncertainty, the optimal exercise boundary will be lower.

| | $V_0^0$ | | |
|---|---|---|---|
| Number of exercise dates ↓ | $\sigma = 0.2$ | $\sigma = 0.3$ | $\sigma = 0.4$ |
| 2 | 4.6444 | 5.2414 | 6.3181 |
| 4 | 0.6256 | 1.5262 | 2.5749 |
| 6 | 0.0172 | 0.2425 | 0.8139 |
| 8 | 7.2609e-5 | 0.0205 | 0.1959 |
| 10 | 4.4836e-8 | 9.0889e-04 | 0.0356 |

Table 5.7: Option prices for different values of $\sigma$ and 2, 4, 6, 8 or 10 exercise dates with $S_0 = 15$, $T = 10$, $X_n = 5$, $r = 0.05$, $\sigma = 0.3$ and $M = 5000$.

### 5.3.4 Effect of the Strike Prices

The effect of taking increasing or decreasing strike prices can also be analyzed. Suppose we have the biotechnology company introduced earlier. If a type of medicine is very new, the company manager might prefer to start with small investments and make bigger investments when the medicine turns out to be successful. In this situation we have increasing strike prices. Another possible scenario is that the company manager invests a lot in the beginning: money could be needed for for example facilities, computers and the development of drugs. In this case the strike prices will be decreasing: once the company has invested in a factory and equipment, it only needs a certain monthly investments to pay employees and keep the production going.

We will now research the difference these investment scenarios make on the optimal exercise boundary, $S^*$. We assume that the company has a 4-fold option for the investments and we assume that the time zero value is $S_0 = 15$. The strike prices we use will be: $X_1 = 2$, $X_2 = 5$, $X_3 = 9$, $X_4 = 11$ and $X_T = 13$ for the increasing case, and in the decreasing case we will have: $X_1 = 13$, $X_2 = 11$, $X_3 = 9$, $X_4 = 4$ and $X_T = 2$.
The effect of both strike prices can be found in Figure 5.5.

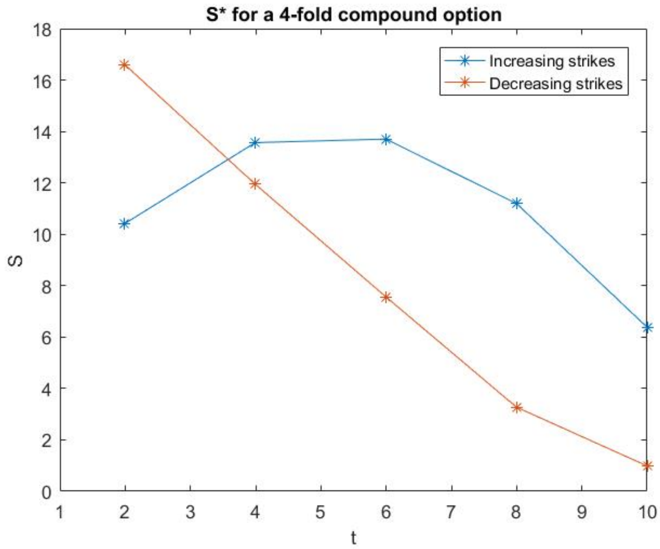


Figure 5.5: Comparison of $S^*$ of 4-fold option with increasing and decreasing strike prices with $S_0 = 15$, $r = 0.05$, $\sigma = 0.3$ and $M = 2000$.

The result looks interesting: the orange line is decreasing whereas the blue line is increasing until time $t = 6$ and decreasing after this point. Because we would like to conclude if this result is case specific or not, we will do another experiment, with different strike prices. We will now choose: $S_0 = 70$ and strike prices $X_1 = 10$, $X_2 = 20$, $X_3 = 30$, $X_4 = 40$ and $X_T = 50$ in the increasing case and $X_1 = 50$, $X_2 = 40$, $X_3 = 30$, $X_4 = 20$ and $X_T = 10$ as decreasing strike prices. The results in Figure 5.6 confirm that Figure 5.5 was not case specific. We have seen that the optimal exercise boundary for a compound option with increasing strike prices is not monotonic. Therefore the biotechnology company, the holder of such a compound option, should carefully consider at every time step whether the investments should be continued.

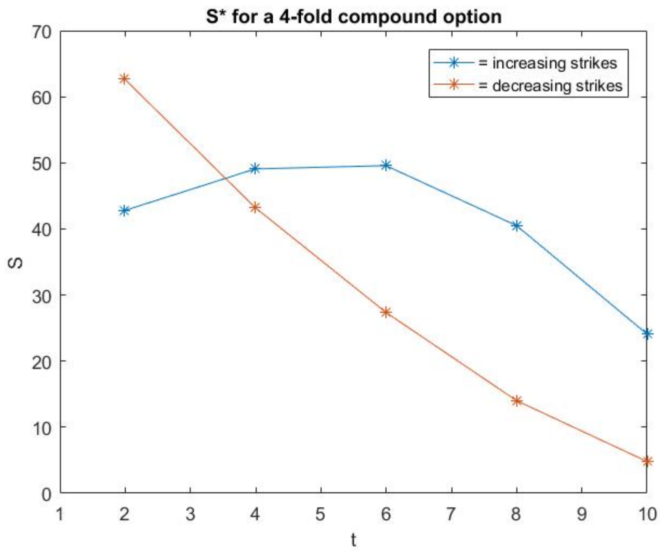


Figure 5.6: Comparison of $S^*$ of 4-fold option with increasing and decreasing strike prices with $S_0 = 70$, $r = 0.05$, $\sigma = 0.3$ and $M = 2000$

$V_0^0$ for the first option with increasing strike prices is $V_0^0 = 0.7369$ and for the option with decreasing strike prices we get $V_0^0 = 0.1618$. This can be explained as follows: because this option has decreasing strike prices, we have $X_1 = 13$. In Figure 5.5 we see that we should exercise the option at time $t = 2$ when $S > 16.61$. So when $S < 16.61$, our option is out of the money, so it generates no payoff. Since $S_0 = 15$, the probability of $S$ reaching 16.61 is relatively small, making the probability of the option finishing out of the money relatively big. This explains the lower price of the option with decreasing strike prices.

## 5.4 Option to Choose

The first example of a real option is the option to choose. The option to choose has been described in Chapter 2.3: the holder has the right to choose at every time before and on expiry whether he wants to: expand current operations, contract current operations or completely abandon its business. The exact options are the following [5]:

- Option to contract: contract 10% of its current operations, creating an additional 25 million euro in savings after this contraction.
- Option to expand: expanding its current operations, increasing its value by 30% with 20 million euro of implementation costs.
- Option to abandon: abandoning its operations, selling its intellectual property for 100 million euro.

The holder of the option to choose has to determine at any time before expiry whether it is optimal to: contract, expand, abandon or continue with the option. The value of the option to choose can be determined by the binomial and trinomial method and Monte Carlo simulation. At every time step in these methods, the following has to be computed:

$$\max(0.1 * V(t) + 25, 1.3 * V(t) - 20, 100). \tag{5.1}$$

Here $V(t)$ is the value of the current operations. Because the option to choose can be exercised at any time, the valuation methods of American options as explained in Chapter 4 can be used, using Equation 5.1 as payoff in every time step. In the calculations, a large company is considered with a value of 100 million euro. This will be $S_0$. Furthermore, a volatility of 15% and a risk free rate of 5% are assumed.

### 5.4.1 Binomial and Trinomial Method for Option to Choose

First the binomial and trinomial methods are considered. As mentioned above, the binomial and trinomial algorithm of American options can be used to value the option to choose. Table 5.8 shows the estimated option values for different values of $M$. Using the computations by Dumrauf [5] as a reference, Table 5.8 shows that both methods converge to the correct solution.

| M | Binomial | Trinomial |
|---|---|---|
| 100 | 119.3350 | 119.3390 |
| 200 | 119.3474 | 119.3454 |
| 400 | 119.3480 | 119.3463 |
| 1000 | 119.3501 | 119.3492 |
| 5000 | 119.3502 | 119.3504 |

Table 5.8: Option to choose values for binomial and trinomial methods and different values of $M$.

### 5.4.2 Monte Carlo method for Option to Choose

For the Monte Carlo simulation, again the algorithm of the American option can be used. At every time step, the value of continuation has to be compared with the value of immediate exercise, which is given by Equation 5.1. Table 5.9 shows the results. Using reference [5] and the solutions of the binomial and trinomial methods, the Monte Carlo method shows to cover the real option value for all values of $M$. So the Monte Carlo method gives a good approximation of the option to choose value.

| M | 95% confidence interval |
|---|---|
| 200 | [113.1003, 124.1748] |
| 500 | [116.1089, 123.5266] |
| 1000 | [115.9359, 120.9910] |
| 5000 | [117.4156, 119.6682] |

Table 5.9: Option to choose values for Monte Carlo simulation and different values of $M$

# 6 | Valuation of the R&D investments of a Biotechnology Firm

One of the goals of this thesis is to value R&D investments of the biotechnology company. As mentioned in Chapter 2, several algorithms had to be designed and checked before they could be used. Chapter 5 showed that the algorithms designed yield the correct results. In this chapter, the binomial method will be used as valuation method. In Chapter 2, it was mentioned that there exist several trials the drug has to pass before it can enter the market. Each of these trials has a specific cost and duration, these were given in Chapter 2. Table 6.1 shows an overview of the different R&D stages with their duration and total costs.

| **R&D stage** | **Total costs (000s euro)** | **Years in stage** |
|---|---|---|
| Discovery | 2,200 | 1 |
| Pre Clinical | 13,800 | 3 |
| Phase I | 2,800 | 1 |
| Phase II | 6,400 | 2 |
| Phase III | 18,100 | 3 |
| FDA Filing | 3,300 | 3 |

Table 6.1: Table of different R&D stages with their duration and costs. [10]

After the drug has successfully entered the market, the drug can differ in quality. The quality of the drug will determine the payoff of the drug. To value the biotechnology company, it is necessary to come up with a set of expected revenues, where the revenues vary, with the highest revenue coming from the drug being considered break through and the lowest revenue coming from the drug being considered bad quality.

The expected revenues will be computed using the binomial tree. To use the binomial tree, a set of parameters is needed. The paper of Kellogg [10] shows how to compute the value of the drugs at $t = 0$. Considering this is not the focus of this thesis, this value will be assumed and not further derived. Therefore, it is assumed that $S_0 = 123.000\text{K}$, $u = e^{\sigma\sqrt{dt}}$ and $d = e^{-\sigma\sqrt{dt}}$ where $dt = T/M$ and $T = 12$, $M = 5000$. Using these parameters and $\sigma = 0.26$ and $r = 7.09\%$, a binomial tree can be constructed. From the binomial tree, we can extract the possible values of the drug at expiry. For the binomial tree with $M = 5000$, there will be 5001 possible prices at expiry. To compute the expected revenues using the possible prices for $S(T)$, the following equation from Kellog will be used [10]:

$$V(S,T) = \max[0.75S(T)] - 1.619\text{K}, 0] \tag{6.1}$$

These values for $V(S,T)$ will be used to work backwards through the tree. As mentioned in Chapter 2, this real option is a $6-$fold compound option, with strike prices being the total costs in Table 6.1. Therefore Algorithm 5 and the parameters specified above can be used to obtain the value of R&D investments of the biotechnology firm:

$$V(S, 0) = 46.2097\text{K} \tag{6.2}$$

Considering that the option premium, the strike price at $t = 0$, is only 2.200K, investing in the R&D project can be beneficial for the biotechnology firm. Managers of the biotechnology company will have to evaluate when they should continue with the sequential investments and when they should stop the investments. As explained in Chapter 5, it has been shown that an optimal exercise boundary shows for which values of $S$ one should exercise the option. The optimal exercise boundary for the R&D investments of the biotechnology firm can be found in Figure 6.1. For all $S$ values above the curve, the option should be exercised. Thus, the optimal exercise boundary also confirms that at $t = 0$, the managers of the firm should invest in the R&D investments.

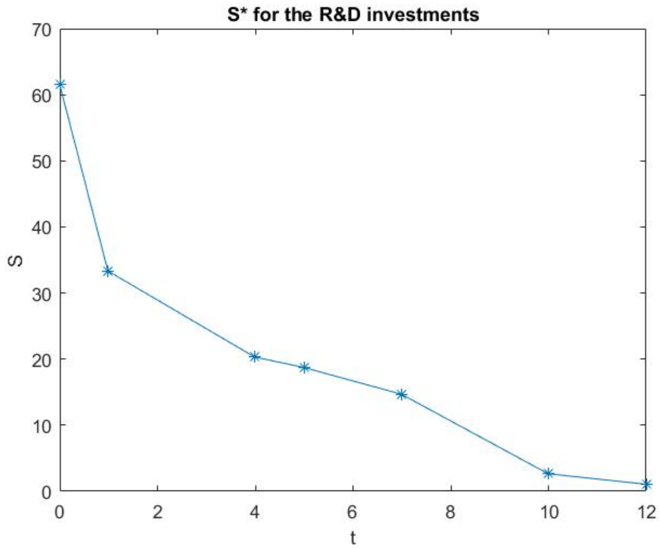


Figure 6.1: Optimal exercise boundary of the R&D investments of the biotechnology firm.

# 7 | Conclusion

In this thesis, the goal was to determine the value of R&D investments of a biotechnology firm. In Chapter 2, the concept of real options was explained and the option to choose was introduced. Furthermore, the steps in the R&D investments of a biotechnology company were explained.

In Chapter 3, financial options were introduced. European options, American options, several compound options and real options have been described extensively. The asset price model following a Geometric Brownian motion has been described. Furthermore, possible closed form solutions for the option prices for the financial options named above were given in this chapter.

Chapter 4 described three numerical methods used to implement the described models: the binomial and trinomial tree construct a tree of possible asset prices until $t = T$ and compute the option value at $t = 0$ by working backwards through the tree using expectations. The Monte Carlo method uses the asset price at $t = 0$ to simulate $M$ different asset paths until $t = T$ and computes the option value at $t = 0$ by discouting the option values at $t = T$ to $t = 0$ and taking the expectation.

Chapter 5 was used to check the designed algorithms on their correctness. First the option value and critical asset boundary of an American put option were examined. Then the single compounded call option was investigated because it has a closed form solution. Then this algorithm was expanded to value an $n$-fold compound option. For the 10-fold compound option, the effect of the volatility was tested, and for a 4-fold compound option, the effect of the order of input of exercise dates and the effect of the strike prices was investigated. The last part of Chapter 5 was the valuation of the option to choose which served as a bridge between the financial options and the target real option.

Chapter 6 finally shows the valuation of the target option: the R&D investments of a biotechnology firm. In this chapter, the information and techniques from previous chapters were combined to obtain the desired option value.

# 8 | Discussion

Various factors in this thesis can be discussed. To begin with, the validity of the algorithms could be discussed. To convince the reader of the correctness of the algorithms, they were first shown to yield the desired results for different options with closed form solutions. Assuming that the modifications made to value other options were done correctly, it can be assumed that the algorithms approximate the right results.

In Chapter 3, various market assumptions were introduced which were made throughout this thesis. An assumption that might violate the correctness of the results is absence of transaction costs. For a compounded option with multiple exercise times, a transaction cost might need to be paid every exercise time. The total sum of the transaction costs may not be negligible and the estimated option value might be higher than the actual option value.

For the Monte Carlo algorithm, continuation values are computed using methods described in Chapter 4. To compute the continuation value, the previous cash flows are regressed on a set of basis functions. Suppose the underlying asset increases or decreases more than expected, the predictions based on the regression will not be correct, and therefore the option value will be approximated incorrectly.

To compute the value of the R&D investments of the biotechnology firm, the binomial method was used and a formula from Kellogg and Charnes [10] was used to compute expected revenues. Because the focus of this thesis was the numerical implementation of the valuation methods, this formula has not been checked or adapted. In a further research, a more extensive method could be developed to compute the expected revenues. This might yield a better approximation.

Finally, further research could be done to improve the correctness of the Monte Carlo algorithms. Algorithm 6 yields approximately the right results up to 3-fold compound option, when the continuation value is computed using $n + 1$ basis functions. A robust algorithm should converge to the correct solution when the number of asset paths and basis functions are taken large. This did not happen in Algorithm 6. To approximate the regression coefficients, a Matlab function called 'bicgstab' was used. The matrices could not be inverted directly because they were almost singular. Other functions might approximate the regression coefficients better and quicker, resulting in a better approximation of the option value. Furthermore, to reduce the variance, Antithetic or control variates could be used.

# Bibliography


[1] Areal, N., Rodrigues, A., & Armada, M.J.R. (z.d.). Improvements to the Least Squares Monte Carlo Option Valuation Method, 4-4. Source:
https://pdfs.semanticscholar.org/bc69/f35deb3b976eca1779bd5b13c44373b04fba.pdf

[2] Chen, J. (2019, 25th January). *Compound Option.* Accessed on: 21-07-2019, source:
https://www.investopedia.com/terms/c/compoundoption.asp

[3] Clewlow, Llanos, J., & Strickland, C. (z.d.). Pricing Exotic Options in a Black-Scholes World, 3-7. Source:
https://warwick.ac.uk/fac/soc/wbs/subjects/finance/research
/wpaperseries/1994/94-54.pdf

[4] Clifford, P., Wang, Y., Zaboronski, O., & Zhang, K. (2010, 29th November). *Pricing Options Using Trinomial Trees* [Slides]. Accessed on: 26-05-2019, source:
https://warwick.ac.uk/fac/sci/maths/people/staff/oleg_zaboronski
/fm/trinomial_tree_2010_kevin.pdf

[5] Dumrauf, G.L. (z.d.). *Real options models* [Slides]. Accessed on: 01-06-2019, source:
https://ucema.edu.ar/ gl24/Slides/Real_options_models_ejercicios.pdf

[6] Fusari, N. (z.d.). *Three essays in option pricing* [Figure]. Source:
https://www.researchgate.net/figure/Evolution-of-the-underlying-trinomial
-tree_fig4_40754739

[7] Hauschild, B., & Reimsbach, D. (2016). Erratum to: Modeling sequential R&D investments: a binomial compound option approach. *Business Research*, 10(1), 157-157.
https://doi.org/10.1007/s40685-016-0038-3

[8] Higham, D. J. (2004). *An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation.* Cambridge, Great-Britain: Cambridge University Press.

[9] *How Interest Rates and Volatility Affect Option Prices?* (2013, 12th January). Accessed on: 15-07-2019, source:
https://financetrain.com/how-interest-rates-and-volatility-affect-option-prices/

[10] Kellogg, D., & Charnes, J. M. (2000). Using Real-Options Valuation Methods for a Biotechnology firm. *Financial Analysis Journal.* Source:
https://pdfs.semanticscholar.org/b69e/5d967eec08f1d58a5054f16ebf0fae68c271.pdf

[11] Kenton, W. (2019, 15th June). *Real Option.* Accessed on: 17-06-2019, source:
https://www.investopedia.com/terms/r/realoption.asp

[12] Knudsen, O. K., & Scandizzo, P. L. (z.d.). Real Options and Project Evaluation: a Primer. Source:
http://realoptionsinternational.com/Real_options_and_project_evaluation_a_primer.pdf

[13] Kwok, Y. K. (z.d.). Compound options. *Encyclopedia of Financial Engineering and Risk Managament.* Source:
https://www.math.ust.hk/ maykwok/piblications/Articles/comp%20option.pdf

[14] Longstaff, F. A., & Schwarz, E. S. (2001). Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies,* 14(1), 113-147. Source:
https://people.math.ethz.ch/ hjfurrer/teaching/LongstaffSchwartz
AmericanOptionsLeastSquareMonteCarlo.pdf

[15] Lowther, G. (2016, 1st November). Filtrations and Adapted Processes. Accessed on 19-05-2019, source:
https://almostsure.wordpress.com/2009/11/08/filtrations-and-adapted-processes/

[16] *Martingale: Definition - Statistics How To.* (2018, 2nd May). Accessed on: 05-05-2019, source:
https://www.statisticshowto.datasciencecentral.com/martingale-definition/

[17] No-arbitrage pricing definition - Risk.net. (2019, 13th August). Accessed on: 23-07-2019, source:
https://www.risk.net/definition/no-arbitrage-pricing

[18] Pricing American Options with Monte Carlo Methods. (2018), 12-14. Source:
https://www.maths.ox.ac.uk/system/files/attachments/TT18_dissertation_1000246.pdf

[19] Shreve, S. (2004). *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model.* Springer New York.

[20] *the Financial Math Calculator.* (z.d.). Accessed on: 05-06-2019, source:
http://quantcalc.net/CompoundFFT.html

[21] Turner, E. (2010). The Black-Scholes Model and Extensions, 1-6. Source:
http://www.math.uchicago.edu/ may/VIGRE/VIGRE2010/REUPapers/Turner.pdf

[22] Weisstein, Eric W. "Markov Process." From MathWorld–A Wolfram Web Resource.
http://mathworld.wolfram.com/MarkovProcess.html