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MODEL REDUCTION BY PROPER ORTHOGONAL DECOMPOSITION FOR ESTIMATION OF SCALAR PARAMETERS IN ELLIPTIC PDES

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Abstract. *Proper orthogonal decomposition (POD) is a powerful technique for model reduction of linear and non-linear systems. It is based on a Galerkin type discretization with basis elements created from the system itself. In this work POD is applied to estimate scalar parameters in elliptic partial differential equations. The parameter estimation is formulated in terms of an optimal control problem that is solved by an augmented Lagrangian method combined with a sequential quadratic programming algorithm. Numerical examples illustrate the efficiency of the proposed approach.*

1 INTRODUCTION

Proper orthogonal decomposition is a method to derive low order models for systems of differential equations. It is based on projecting the system onto subspaces consisting of basis elements that contain characteristics of the expected solution. This is in contrast to, e.g., finite element techniques, where the elements of the subspaces are uncorrelated to the physical properties of the system that they approximate.

In this work we apply a POD Galerkin approximation to estimate scalar parameters in elliptic systems. The corresponding parameter identification problem is formulated as an optimal control problem with inequality constraints for the parameters. To solve this optimization problem with a scalar inequality constraint we apply an augmented

Lagrangian method [5, 6] combined with a globalized sequential quadratic programming (SQP) algorithm [12]. In [18] error estimates for POD Galerkin schemes for linear and certain semi-linear elliptic, parameter dependent systems were proved. The resulting error bounds depend on the number of POD basis functions and on the parameter grid that is used to generate the snapshots and to compute the POD basis.

Let us finally briefly comment on further literature containing applications of POD. It is successfully used in different fields including signal analysis and pattern recognition (see, e.g., [10]), fluid dynamics and coherent structures (see, e.g., [14, 30]) and more recently in control theory (see, e.g., [1, 3, 19, 22, 24]) and inverse problems [2]. Surprisingly good approximation properties are reported for POD based schemes in several articles, see [9, 31] for example. The relationship between POD and balancing is considered in [23, 29, 33]. Error analysis for nonlinear dynamical systems in finite dimensions are carried out in [15, 28]. In [20, 21] error estimates for POD Galerkin approximations are derived for non-linear parabolic differential equations. These results are extended to linear-quadratic optimal control problems in [13]. Reduced-basis element methods for parameter dependent elliptic are investigated in [4, 25, 26], for instance. Parameter identification problems are formulated in terms of optimal control problems and solved by SQP techniques in [7, 11, 16, 32], for instance.

The article is organized in the following manner: In Section 2 the parameter estimation is formulated interms of an optimal control problem. The POD Galerkin approximation is described in Section 3. In Section 4 we present numerical examples illustrating the efficient performance of the reduced-order approximation compared to a high-order approximation that utilizes a finite element discretization. Finally, we draw some conclusions and discuss future research in the last section.

2 PARAMETER ESTIMATION FOR ELLIPTIC EQUATIONS

Motivated by the good approximation properties [18] we apply the POD method to a parameter estimation problem, where we want to identify a scalar parameter in an elliptic equation from (perturbed) measurements of the solution on the boundary and/or in the domain. Let $q_a \in \mathbb{R}$ be a given lower bound for the unknown parameter q and

$$Q_{\text{ad}} = \big\{ q \in \mathbb{R} \,|\, q_a \leq q \big\} \subset \mathbb{R}$$

the (closed and convex) set of admissible coefficients. Suppose that Ω is a given bounded and open domain in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, with boundary $\Gamma = \partial\Omega$. Throughout, by $\nabla\varphi$ we denote the gradient of a function $\varphi : \Omega \to \mathbb{R}$. We suppose that for any $q \in Q_{\text{ad}}$ there exists a unique weak solution $u = u(q)$ to the elliptic problem

$$-c\Delta u + \beta \cdot \nabla u + qu = g \qquad \text{in } \Omega, \tag{1a}$$

$$c\,\frac{\partial u}{\partial n} + \sigma u = g_\Gamma \qquad \text{on } \Gamma, \tag{1b}$$

where $\beta \in \mathbb{R}^2$, c, $\sigma \in \mathbb{R}$ with $c > 0$. Further, $g : \Omega \to \mathbb{R}$ and $g_\Gamma : \Gamma \to \mathbb{R}$ are (essentially) bounded inhomogeinities. More precisely, u is a weak solution to (1) provided u belongs to the Sobolev space $H^1(\Omega) = \{\varphi : \Omega \to \mathbb{R} \mid \int_\Omega |\nabla\varphi|^2 + \varphi^2 \,\mathrm{d}\mathbf{x} < \infty\}$ and

$$\int_\Omega c\nabla u \cdot \nabla\varphi + \big(\beta \cdot \nabla u + qu\big)\varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma \sigma u\varphi \,\mathrm{d}\mathbf{s} = \int_\Omega g\varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma g_\Gamma\varphi \,\mathrm{d}\mathbf{s} \tag{2}$$

is satisfied for all $\varphi \in H^1(\Omega)$. For the definition and properties of Sobolev spaces we refer the reader to [8], for instance.

To determine the admissible coefficient $q \in Q_{\mathrm{ad}}$ from bounded measurements $u_d : \Gamma \to \mathbb{R}$ we minimize the quadratic cost functional J given by

$$J(q,u) = \int_\Gamma \frac{\alpha}{2} |u - u_d|^2 \,\mathrm{d}\mathbf{s} + \frac{\kappa}{2} |q - q_d|^2, \tag{3}$$

where α is a non-negative weight, $\kappa > 0$ denotes a regularization parameter and $q_d \geq q_a$ is a nominal/reference coefficient.

Now the constrained optimal control problem is given by

$$\min J(q,u) \quad \text{subject to (s.t.)} \quad (q,u) \in Q_{\mathrm{ad}} \times H^1(\Omega) \text{ satisfies (2).} \tag{\textbf{P}}$$

Note that (**P**) is an optimization problem in function spaces. It can be shown that there exists a solution (q^*, u^*) to (**P**). For more details we refer the reader to [17].

Since (**P**) is a non-convex programming problem, different local minima might occur. A numerical method will deliver a local minimum close to its starting value. Hence we focus on properties of local minima of (**P**). Suppose that (q^*, u^*) is a local optimal solution. This solution is characterized by first-order necessary optimality conditions [17, 27]: There exist unique Lagrange multipliers $p^* \in H^1(\Omega)$ and $\lambda^* \in \mathbb{R}$ with $\lambda^* \geq 0$ satisfying the dual/adjoint equation

$$\int_\Omega c\nabla p^* \cdot \nabla\varphi - \big(\beta \cdot \nabla p^* - q^*p^*\big)\varphi \,\mathrm{d}\mathbf{x} + \int_\Gamma \big(\sigma + \beta \cdot n\big)p^*\varphi \,\mathrm{d}\mathbf{s} = \int_\Gamma \alpha(u_d - u^*)\varphi \,\mathrm{d}\mathbf{s} \tag{4}$$

for all $\varphi \in H^1(\Omega)$ and the optimality condition

$$\lambda^* = \kappa(q^* - q_d) + \int_\Omega u^*p^* \,\mathrm{d}\mathbf{x} \quad \text{in } \mathbb{R}. \tag{5}$$

Moreover, the complementarity condition $\lambda^*(q_a - q^*) = 0 \in \mathbb{R}$ holds true. In (4) we denote by n the outward normal vector. Equation (4) is the weak (or variational) formulation of the elliptic boundary value problem

$$\begin{aligned} -c\Delta p^* - \beta \cdot \nabla p^* + q^*p^* &= 0 && \text{in } \Omega, \\ c\frac{\partial p^*}{\partial n} + \big(\sigma + \beta \cdot n\big)p^* &= \alpha(u_d - u^*) && \text{on } \Gamma. \end{aligned}$$

From $\lambda^* \geq 0$ and $q_a - q^* \leq 0$ we infer that λ^* satisfies

$$\lambda^* = \max\left\{0, \lambda^* + \varrho(q_a - q^*)\right\} \quad \text{for every } \varrho > 0. \tag{6}$$

In our approach we handle the scalar constraint $q_a - q \leq 0$ by an augmented Lagrangian penalization, where we make use of (6). See, e.g., [5, 6] for a detailed account of this technique. In fact, for $\varrho > 0$ and $\hat{\lambda} \geq 0$ we introduce the modified cost functional

$$J_{\hat{\lambda}}^{\varrho}(q,u) = J(q,u) + \frac{1}{2\varrho} \max\left\{0, \hat{\lambda} + \varrho\big(q_a - q\big)\right\}^2.$$

Then we consider the optimal control problem

$$\min J_{\hat{\lambda}}^{\varrho}(q,u) \quad \text{s.t.} \quad (q,u) \in \mathbb{R} \times H^1(\Omega) \text{ satisfies (2)} \tag{$\mathbf{P}_{\hat{\lambda}}^{\varrho}$}$$

for fixed $\varrho > 0$ and $\hat{\lambda} \geq 0$ instead of ($\mathbf{P}$). Using analogous arguments as for ($\mathbf{P}$) one can prove that ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$) has a solution for arbitrarily $\varrho > 0$ and $\hat{\lambda} \geq 0$. Notice that ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$) does not involve the inequality constraint explicitly. Rather this constraint is realized by adding an augmented Lagrangian-type penalty term to the original objective function. It is well-known that the augmented Lagrangian penalization is exact for sufficiently large $\varrho > 0$, i.e., a local solution (q^*, u^*) to ($\mathbf{P}$) is also a local solution to ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$).

Since ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$) has no inequality constraints, it can be solved by an SQP method with line search globalization. For more details we refer the reader to [12] for the globalized SQP method and to [17] for the application to our problem ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$). Let us review the augmented Lagrangian algorithm, which can be interpreted as a combination of penalty functions and local duality methods.

Algorithm 1 (Augmented Lagrangian method [5, 6])

(1) *Choose a starting value $\lambda^0 \geq 0$ for the Lagrange multiplier associated to the inequality constraint, the initial parameter $\varrho_0 > 0$ for the augmentation, a factor $\beta^{\varrho} > 1$ and a stopping criterion; set $k = 0$.*

(2) *Determine a (local) solution $x^{k+1} = (q^{k+1}, u^{k+1})$ of ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$) with $\varrho = \varrho_k$ and $\hat{\lambda} = \lambda^k$.*

(3) *Set $\lambda^{k+1} = \max\left\{0, \lambda^k + \varrho^k(q_a - q^k)\right\}$.*

(4) *Unless the stopping rule is satisfied, set $\varrho_{k+1} = \beta^{\varrho}\varrho_k$, $k = k + 1$, and continue with step (2).*

Remark 1

(a) Other augmentation rules for the parameter ϱ than the one realized in step (4) can be found, e.g., in [6, p. 405].

(b) In the process of solving ($\mathbf{P}$) the augmented Lagrangian algorithm acts as the outer iteration of our whole optimization method, whereas at each level of Algorithm 1 the solution of ($\mathbf{P}_{\hat{\lambda}}^{\varrho}$) is computed by the globalized SQP method that is described in detail in [17].

3 POD REDUCED-ORDER MODELLING

In this section we describe our POD Galerkin approximation of $(\mathbf{P}_{\hat{\lambda}}^{\varrho})$. For that purpose let $q_l \leq q_a$ and $q_u > q_a$ be chosen lower respectively upper estimates for the scalar coefficient q^*. We introduce the parameter grid by

$$\Delta q = \frac{q_u - q_l}{n-1} \quad \text{and} \quad q_j = q_l + (j-1)\Delta q \text{ for } j = 1, \ldots, n. \tag{7}$$

To simplify the presentation we choose an equidistant mesh in (7). Of course, the approach is analogous for non-equidistant parameter grids [18]. For $j \in \{1, \ldots, n\}$ let $u_j = u(q_j)$ denote the solution to (2) for the parameter q_j. Introducing the L^2 inner product

$$\langle \varphi, \psi \rangle = \int_\Omega \varphi \psi \, \mathrm{d}\mathbf{x} \tag{8}$$

and the associated induced norm $\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}$ we determine a POD basis of rank $\ell \in \{1, \ldots, n\}$ by solving

$$\min \sum_{j=1}^{n} \alpha_j \left\| u_j - \sum_{i=1}^{\ell} \langle u_j, \psi_i \rangle \, \psi_i \right\|^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}, \ 1 \leq i, j \leq \ell, \tag{9}$$

where the α_j's are chosen as the trapezoidal weights $\alpha_1 = \alpha_n = \Delta q/2$ and $\alpha_j = \Delta q$ for $2 \leq j \leq n-1$. In (9) we denote by δ_{ij} the Kronecker symbol with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise.

Remark 2 Instead of (8) one can also utilize the H^1 inner product

$$\langle \varphi, \psi \rangle_{H^1} = \int_\Omega \varphi \psi + \nabla \varphi \cdot \nabla \psi \, \mathrm{d}\mathbf{x}$$

with its associated induced norm $\| \cdot \|_{H^1}$; see [18, 20, 21].

The solution to (9) is given by the solution to the symmetric eigenvalue problem [14, 31]

$$\mathcal{R}\psi_i = \lambda_i \psi_i, \quad i = 1, \ldots, \ell$$

with $\mathcal{R}\psi = \sum_{j=1}^{n} \alpha_j \langle u_j, \psi \rangle \, u_j$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$.

Remark 3 (Snapshot POD [30]**)** Let us supply $\mathbb{R}^n$ with the weighted inner product

$$\langle v, w \rangle_{\mathbb{R}^n} = \sum_{i=1}^{n} \alpha_i v_i w_i \quad \text{for } v = (v_1, \ldots, v_n)^T, w = (w_1, \ldots, w_n)^T \in \mathbb{R}^n.$$

We define the symmetric non-negative matrix $\mathcal{K} \in \mathbb{R}^{n\times n}$ with the elements $\langle u_i, u_j\rangle$, $1 \le i, j \le n$, and consider the eigenvalue problem

$$\mathcal{K}v_i = \lambda_i v_i,\ 1 \le i \le \ell \quad \text{and} \quad \langle v_i, v_j\rangle_{\mathbb{R}^n} = \delta_{ij},\ 1 \le i, j \le \ell. \tag{10}$$

From singular value decomposition it follows that $\mathcal{K}$ has the same eigenvalues $\{\lambda_i\}_{i=1}^d$ as the operator $\mathcal{R}$. Furthermore, the POD basis functions are given by the formula

$$\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} \alpha_j (v_i)_j u_j \quad \text{for } i = 1, \dots, \ell, \tag{11}$$

where $(v_i)_j$ denotes the jth-component of the eigenvector $v_i \in \mathbb{R}^n$. In our numerical test examples (see Section 4) we compute the POD basis by solving (10) and using (11).

Remark 4 Note that (9) can be interpreted as the trapezoidal approximation of

$$\min \int_{q_l}^{q_u} \Big\| u(q) - \sum_{i=1}^{\ell} \langle u(q), \psi_i\rangle\, \psi_i \Big\|^2 \,\mathrm{d}q \quad \text{s.t.} \quad \langle \psi_i, \psi_j\rangle = \delta_{ij},\ 1 \le i, j \le \ell. \tag{12}$$

In (12) the function $u(q)$ is the weak solution to (1) for the parameter $q \in [q_l, q_u]$. This relationship is used in [21] to study asymptotic convergence properties of the eigenfunctions $\{\psi_i\}_{i=1}^{\ell}$ and the corresponding eigenvalues $\{\lambda_i\}_{i=1}^{\ell}$ as the mesh size Δq tends to zero or, equivalently, n tends to ∞.

When the POD basis functions $\{\psi_i\}_{i=1}^{\ell}$ have been computed we make the ansatz

$$u^\ell = \sum_{j=1}^{\ell} \gamma_j \psi_j, \quad \gamma_1, \dots, \gamma_\ell \in \mathbb{R}, \tag{13}$$

for the POD Galerkin approximation. Inserting (13) into (2) and choosing $\varphi = \psi_i$, $i = 1 \dots, \ell$, we get the finite-dimensional problem

$$\int_\Omega c\nabla u^\ell \cdot \nabla \psi_i + \big(\beta \cdot \nabla u^\ell + q u^\ell\big)\psi_i \,\mathrm{d}\mathbf{x} + \int_\Gamma \sigma u^\ell \psi_i \,\mathrm{d}\mathbf{s} = \int_\Omega g\psi_i \,\mathrm{d}\mathbf{x} + \int_\Gamma g_\Gamma \psi_i \,\mathrm{d}\mathbf{s} \tag{14}$$

for $i = 1, \dots, \ell$. Then we consider the following POD Galerkin approximation of $(\mathbf{P}_{\hat\lambda}^{\varrho})$:

$$\min J_{\hat\lambda}^{\varrho}(q, u^\ell) \quad \text{s.t.} \quad q \in \mathbb{R},\ u^\ell \text{ is given by (13) and } (q, u^\ell) \text{ satisfies (14)} \tag{15}$$

for fixed $\varrho > 0$ and $\hat\lambda \ge 0$.

Remark 5 Introducing the matrices

$$A^\ell = ((a_{ij})) \in \mathbb{R}^{\ell\times\ell} \text{ with } a_{ij} = \int_\Omega c\nabla\psi_j \cdot \nabla\psi_i + (\beta\cdot\nabla\psi_j)\psi_i \,\mathrm{d}\mathbf{x} + \int_\Gamma \sigma\psi_j\psi_i \,\mathrm{d}\mathbf{s},$$
$$B^\ell = ((b_{ij})) \in \mathbb{R}^{\ell\times\ell} \text{ with } b_{ij} = \int_\Omega \psi_j\psi_i \,\mathrm{d}\mathbf{x},$$

and the vectors

$$\gamma^\ell = (\gamma_i) \in \mathbb{R}^\ell \quad \text{and} \quad b^\ell = (b_i) \in \mathbb{R}^\ell \text{ with } b_i = \int_\Omega g\psi_i \,\mathrm{d}\mathbf{x} + \int_\Gamma g_\Gamma\psi_i \,\mathrm{d}\mathbf{s},$$

we derive from (13) that (14) yields the linear system in $\mathbb{R}^\ell$:

$$\left(A^\ell + qB^\ell\right)\gamma^\ell = b^\ell.$$

4 NUMERICAL EXPERIMENTS

This section is devoted to present two numerical test examples. All coding is done in MATLAB using routines from the FEMLAB package concerning finite element implementation. The programs are executed on a standard 1.7 Ghz desktop PC.

Run 1. The domain Ω is the unit square $(0,1)\times(0,1)$ and we choose $c = 2$, $\beta = (0,1)^T$, $g = 1$, $\sigma = 1.5$, $g_\Gamma = -1$ in (1). To compute the snapshots for the POD reduced-order modelling we apply a finite element discritization with a rectangular, equidistant mesh with mesh size $h = 1/50$. As ansatz function we utilize piecewise linear finite elements. In (7) we take $q_l = 0$ and $q_u = 50$ and compute the finite element solution to (2) for $n = 51$ different coefficients. Then we determine $\ell = 7$ POD basis functions as described in Section 3. The decay of the first eigenvalues is shown in the left plot of Figure 1. The needed CPU time for the computation of the POD basis is 40.19 seconds. Our aim

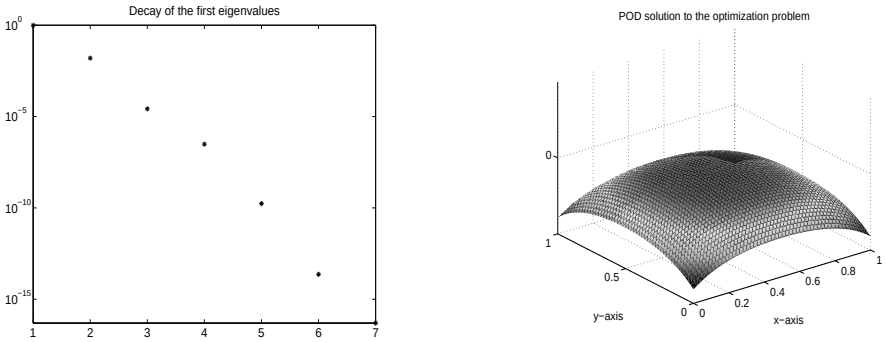

Figure 1: Decay of the first eigenvalues (left) and optimal POD state (right) for Run 1.

is to estimate the ideal coefficient $q_{\text{ideal}} = 40$ from a measurement u_d that is given by

the boundary values of the finite element solution $u_{\text{ideal}} = u(q_{\text{ideal}})$ to (2) for the ideal coefficient q_{ideal}. Moreover, let $\alpha = 10000$ and $\kappa = 0.001$ in (3). For the inequality constraints we take the lower bound $q_a = 20$. Since we assume that we do not have an a-priori estimation for q^*, we set $q_d = 0$.

Algorithm 1 is initialized by $\lambda^0 = 1$ and $\varrho^0 = 2$. Further, we choose the factor $\beta^\varrho = 2$. Due to the optimality condition (5) we take

$$\left| \kappa\big(q^{k+1} - q_d\big) + \int_\Omega u^{k+1} p^{k+1} \, \mathrm{d}\mathbf{x} - \lambda^{k+1} \right| < 10^{-8}$$

and $k \leq k_{\max} = 100$ as a stopping rule. The augmented Lagrangian method stops after two iterations and needs 0.15 seconds. As the POD solution we obtain $q^\ell \approx 39.50$ for the estimated coefficient and $\lambda^\ell = 0$ for the Lagrange multiplier associated with the inequality constraint. From

$$\frac{|q_{\text{ideal}} - q^\ell|}{|q_{\text{ideal}}|} \cdot 100\% \approx 1.25\%$$

we conclude that the POD Galerkin approximation yields a satisfactory result. The POD optimal state is presented in the right plot of Figure 1. In the left plot of Figure 2 you can see the difference between the POD optimal state and the measurement u_d which is very small relative to the optimal state. The performance of the algorithm and of the

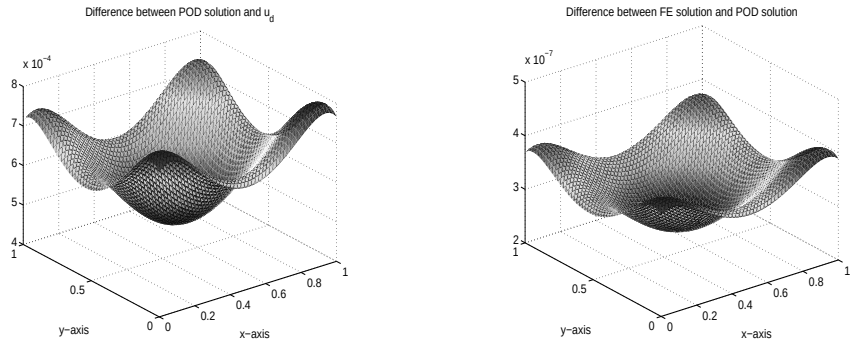

Figure 2: Difference between optimal POD state and measurements (left) and Difference between optimal POD and finite element state (right) for Run 1.

SQP method as the inner iteration method is shown in Table 1. When we consider the

Alg. 1	SQP method	q^k
$k = 1$	6 iterations	39.504506
$k = 2$	2 iterations	39.504519

Table 1: Performance of Algorithm 1 with the globalized SQP method as the inner iteration method for the POD discretized problem (Run 1).

same algorithm, except that we use the finite element discretization of $(\mathbf{P}^\varrho_{\hat{\lambda}})$ instead of the

POD approximation (15), we find the solution $q^h \approx 39.50$ for the estimated coefficient and $\lambda^h = 0$ for the Lagrange multiplier. The obtained result nearly coincides with (q^ℓ, λ^ℓ), yet the computing time is 96.73 seconds that is much larger than the computing time for the POD-based SQP solver. The difference between the optimal POD and optimal finite element state is shown in Figure 2 (right plot). The performance of the algorithm and of the SQP method as the inner iteration method is shown in Table 2. In Table 3

Alg. 1	SQP method	q^k
$k = 1$	6 iterations	39.504754
$k = 2$	1 iteration	39.504775
$k = 3$	1 iteration	39.504771

Table 2: Performance of Algorithm 1 with the globalized SQP method as the inner iteration method for the finite element discretization (Run 1).

we compare the optimal POD state and optimal POD coefficient with the corresponding finite element solution for different numbers ℓ of POD ansatz functions. We observe from

ℓ	$\frac{\|u^\ell - u^h\|}{\|u^h\|}$	$\frac{\|u^\ell - u^h\|_{H^1}}{\|u^h\|_{H^1}}$	$\frac{\|q^\ell - q^h\|}{\|q^h\|}$
4	5.6e-09	1.7e-08	7.9e-05
5	1.0e-10	1.5e-11	1.3e-05
6	5.7e-11	5.6e-12	9.8e-06
7	1.9e-12	1.9e-13	1.8e-06

Table 3: Relative errors for the optimal POD state u^ℓ and optimal POD coefficient q^ℓ compared to the corresponding finite element solutions (u^h, q^h) for different numbers ℓ of POD ansatz functions (Run 1).

Table 3 that the relative errors for the states and coefficients decrease for increasing ℓ.

Run 2 For the same domain Ω as in Run 1 we now regard other values for the parameters of the elliptic differential equation (1): $c = 0.75$, $\beta = (1, 1)^T$, $g(x) = x$, $\sigma = 1.5$, $g_\Gamma = -1$. In (3) the weight and the regularization parameter are $\alpha = 10000$ and $\kappa = 0.001$, respectively. Further, we choose $q_{\text{ideal}} = 25$ and analogous to Run 1 we take u_d as the boundary values of the finite element solution $u_{\text{ideal}} = u(q_{\text{ideal}})$ to (2) for $q = q_{\text{ideal}}$. The snapshots for the computation of the POD basis are computed by the same finite element discretization and same parameter grid as in the previous run. Then we determine $\ell = 7$ POD basis functions and perform the reduced-order model (15). Note that for the computation of the POD basis functions we use the H^1 inner product (see Remark 2) in Run 2 opposite to Run 1 where we used the mass matrix. The decay of the first eigenvalues is shown in the left plot in Figure 3. The lower bound q_a is set equal to 10. Algorithm 1 is initialized and ed as in Run 1. After 3 augmented Lagrangian iterations we obtain the POD solution $q^\ell \approx 24.95$ for the estimated coefficient and $\lambda^\ell = 0$ for the multiplier. Again, the relative error

$$\frac{|q_{\text{ideal}} - q^\ell|}{|q_{\text{ideal}}|} \cdot 100\% \approx 0.2\%$$

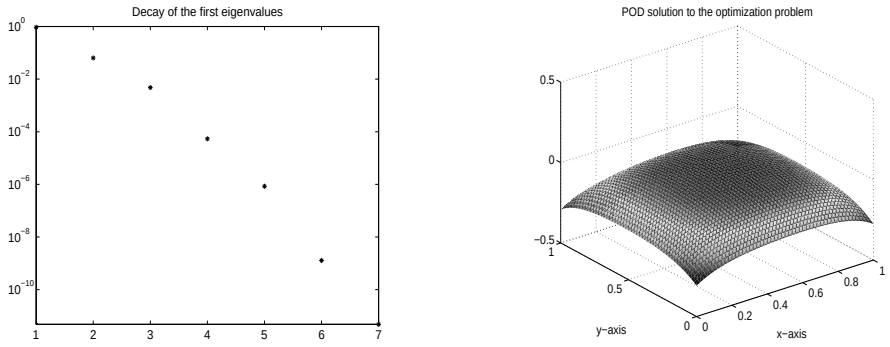

Figure 3: Decay of the first eigenvalues (left) and optimal POD state (right) for Run 2.

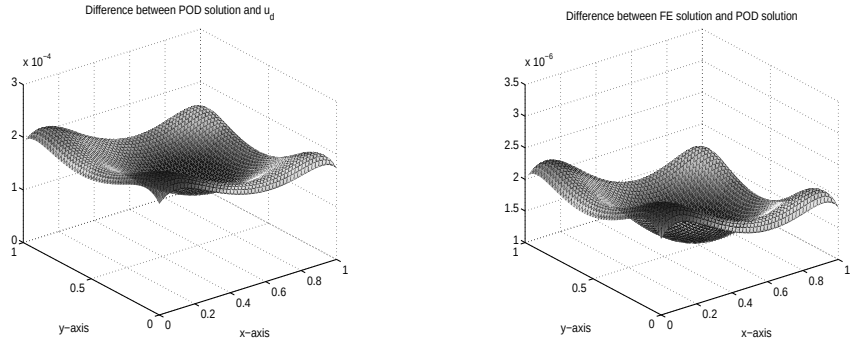

Figure 4: Difference between optimal POD state and measurements (left) and Difference between optimal POD and finite element state (right) for Run 2.

is small. The performance of the algorithm is shown in Table 4. Using the finite element

Alg. 1	SQP method	q^k
$k = 1$	7 iterations	24.946926
$k = 2$	3 iterations	24.947143
$k = 3$	2 iterations	24.947143

Table 4: Performance of Algorithm 1 with the globalized SQP method as the inner iteration method for the POD discretized problem (Run 2).

discretization (see Table 5) we find the solution $q^h \approx 24.95$ and $\lambda^h = 0$. However, the finite element based SQP solver needs 147.01 seconds, whereas the POD-based one only requires 0.19 seconds for the optimization plus 42.49 seconds for the computation of the POD basis functions. Again we compare the relative errors between the optimal POD and the optimal finite element state and the corresponding optimal coefficients for several numbers of POD basis functions (Table 6).

Alg. 1	SQP method	q^k
$k=1$	8 iterations	24.947724
$k=2$	1 iteration	24.947702
$k=3$	1 iteration	24.947696
$k=4$	1 iteration	24.947696
$k=5$	1 iteration	24.947697

Table 5: Performance of Algorithm 1 with the globalized SQP method as the inner iteration method for the finite element discretization (Run 2).

ℓ	$\frac{\|u^\ell - u^h\|}{\|u^h\|}$	$\frac{\|u^\ell - u^h\|_{H^1}}{\|u^h\|_{H^1}}$	$\frac{\|q^\ell - q^h\|}{\|q^h\|}$
4	2.3e-07	4.9e-07	5.7e-04
5	1.3e-10	7.8e-10	8.0e-06
6	2.1e-10	2.7e-11	2.0e-05
7	2.5e-10	2.8e-11	2.2e-05

Table 6: Relative errors for the optimal POD state u^ℓ and optimal POD coefficient q^ℓ compared to the corresponding finite element solutions (u^h, q^h) for different numbers ℓ of POD ansatz functions (Run 2).

Finally, we add noise to our measurement:

$$u_d^h = (1 + \varepsilon\delta) u^h(q_{\text{ideal}})\Big|_\Gamma.$$

where $\varepsilon \in [-1, 1]$ is a random variable and $\delta = 5\%$. Algorithm 1 stops after three iterations and the (typical) performance of the algorithm for one choice of the random variable ε is presented in Table 7. The optimal POD coefficient is $q^\ell \approx 24.85$ and the

Alg. 1	SQP method	q^k
$k=1$	7 iterations	24.847891
$k=2$	3 iterations	24.848113
$k=3$	2 iterations	24.848112

Table 7: Performance of Algorithm 1 with the globalized SQP method as the inner iteration method for the POD discretization in case of noisy data (Run 2).

Lagrange multiplier λ^ℓ is zero. The relative errors

$$\frac{\|u^\ell - u^h(q_{\text{ideal}})\|}{\|u^h(q_{\text{ideal}})\|} \approx 1.87 \cdot 10^{-5} \quad \text{and} \quad \frac{\|u^\ell - u^h(q_{\text{ideal}})\|_{H^1(\Omega)}}{\|u^h(q_{\text{ideal}})\|_{H^1(\Omega)}} \approx 2.091.87 \cdot 10^{-6}$$

are rather small. Moreover, we have

$$\frac{\|u^\ell - u_d^h\|}{\|u_d^h\|} \approx 4.35 \cdot 10^{-4}.$$

Of course, due to the non-smooth measurement data u_d^h we find

$$\frac{\|u^\ell - u_d^h\|_{H^1(\Omega)}}{\|u_d^h\|_{H^1(\Omega)}} \approx 0.26.$$

5 CONCLUSIONS

In this article we consider estimation problems for scalar parameters in elliptic partial differential equations. These problems can be formulated as parameter identification problems so that we are in the position to apply optimization methods. In many applications a repeated, reliable and real-time estimation is essential. To obtain an efficient solver we apply a POD-based discretization of the elliptic equation and make use of an augmented Lagrangian algorithm combined with a globalized SQP method. It turns out that in our numerical examples we get satisfactory results. Motivated by this observations we plan to utilize POD approximation for vibroacoustic applications in future research.

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