13

The Fourier transform and multipliers

In this chapter, we complement the discussion of three major themes of Fourier analysis that we have studied in the previous Volumes. The first one is the Banach space valued Hausdorff–Young inequality

$$\|\widehat{f}\|_{L^{p'}(\mathbb{R}^d;X)} \leqslant C\|f\|_{L^p(\mathbb{R}^d;X)}. \tag{13.1}$$

As we recall from Section 2.4.b, this is a non-trivial condition, expressed by saying that the space X have *Fourier type* p. The basic theory around this notion was already developed in 2.4.b, but we now turn to the main result on this topic, Bourgain's Theorem 13.1.33, which says that (13.1) holds for some $p > 1$ if and only if X has some non-trivial type. Section 13.1 is dedicated to a detailed proof of this deep result.

The second theme is about connecting the Fourier multipliers $T_m : f \mapsto (m\widehat{f})^\vee$ from Chapter 5 and Section 8.3 with the Calderón–Zygmund theory of Chapter 11. In principle, we have

$$T_m f = (m\widehat{f})^\vee = \widehat{m} * f = k * f,$$

where the right-hand side has the formal structure of the operators studied in Chapter 11, but the question then becomes the correspondence of the conditions on the multiplier m and on the singular convolution kernel k. As we will see in Section 13.2.a, the function k will be a nice Calderón–Zygmund kernel, and hence $f \mapsto k * f$ will be in the scope of all results of Chapter 11 (notably, including those dealing with extrapolation of boundedness to the weighted $L^p(w;X)$ spaces), as soon as m satisfies assumptions like those in the Mihlin Multiplier Theorem 5.5.10 for sufficiently many derivatives $\partial^\alpha m$. Moreover, this result is very general in that it holds for multipliers taking values in arbitrary Banach spaces, and then in particular in $\mathscr{L}(X,Y)$ for any Banach spaces X and Y. However, the required number of derivatives on this level of generality is higher than that in the Mihlin Multiplier Theorem 5.5.10. Coping only with the same derivatives as in Mihlin's theorem turns out to be more delicate and require the use of a Banach space valued Hausdorff–Young inequality

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(13.1). It will be convenient to know, thanks to Bourgain's Theorem 13.1.33, that this estimate is always available in the UMD spaces that we so frequently deal with (recalling that every UMD space has non-trivial type by Proposition 7.3.15). As we have already seen in a number of occasions (notably, Bourgain's Theorem 5.2.10 on the Hilbert transform, and Guerre-Delabrière's Theorem 10.5.1 on the imaginary powers $(-\Delta)^{is}$ of the Laplacian), the UMD condition is often necessary for the theory that we develop.

As the third topic of this chapter, we complement these result by Theorem 13.3.5 of Geiss, Montgomery-Smith, and Saksman, which significantly extends the previous examples of Fourier multipliers whose $L^p(\mathbb{R}^d;X)$ boundedness implies the UMD condition. As one of its consequences, in Corollary 13.3.9, we are able to compete the characterisation of situations in which there is a continuous embedding $H^{k,p}(\mathbb{R}^d;X) \hookrightarrow W^{k,p}(\mathbb{R}^d;X)$ between two classes of classical function spaces studied in the previous Volumes. This also provides a link with the following Chapter 14, where we undertake a systematic development of the theory of function spaces of Banach space valued functions.

Despite the interconnected themes of the three sections of this chapter, any of them can be studied independently of the other two by a reader interested in a particular topic.

13.1 Bourgain's theorem on Fourier type

Already in Section 2.4.b, we discussed in some detail the notion of Fourier type, or the extent to which the Hausdorff–Young inequality $\|\widehat{f}\|_{p'} \leqslant C\|f\|_p$ remains valid for the Fourier transform of vector-valued functions. In the Notes of Chapter 2, we also mentioned without proof the main theorem on this topic, due to Bourgain, stating that non-trivial type implies non-trivial Fourier type (and hence is equivalent to it, the other direction being a rather easier Proposition 7.3.6). The aim of this section is to prove this fundamental result, which will also play a role in the subsequent parts of the book.

We recall from Proposition 2.4.20 that the Fourier type $p \in [1,2]$ of a Banach space X can be defined by any of the following equivalent conditions, where moreover any choice of $d \in \mathbb{Z}_+$ is equivalent by Proposition 2.4.11:

(1) The Fourier transform on $\mathbb{R}^d$, defined on $f \in L^1(\mathbb{R}^d;X)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\,\mathrm{d}\xi, \quad \xi \in \mathbb{R}^d,$$

extends to a bounded operator from $L^p(\mathbb{R}^d;X)$ to $L^{p'}(\mathbb{R}^d;X)$.

(2) The Fourier transform on $\mathbb{T}^d$, defined on $f \in L^1(\mathbb{T}^d;X)$ by

$$\widehat{f}(k) = \int_{\mathbb{T}^d} f(t) e^{-2\pi i t\cdot k}\,\mathrm{d}t, \quad k \in \mathbb{Z}^d,$$

restricts to a bounded operator from $L^p(\mathbb{T}^d)$ to $\ell^{p'}(\mathbb{Z}^d)$.

(3) The Fourier transform on $\mathbb{Z}^d$, defined on $x = (x_k)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d; X)$ by

$$\widehat{x}(t) = \sum_{k\in\mathbb{Z}} e^{-2\pi i k\cdot t} x_k, \quad t \in \mathbb{T}^d,$$

extends to a bounded operator from $\ell^p(\mathbb{Z}^d; X)$ to $L^{p'}(\mathbb{T}^d; X)$.

Denoting the norms of the respective extensions (or restrictions) by $\varphi_{p,X}(\mathbb{R}^d)$, $\varphi_{p,X}(\mathbb{T}^d)$ and $\varphi_{p,X}(\mathbb{Z}^d)$, we have:

Proposition 13.1.1. *Let X be a Banach space, $p \in (1, 2]$ and $d \in \mathbb{Z}_+$. Then*

$$\varphi_{p,\mathbb{C}}(\mathbb{R}^{d-1})\varphi_{p,X}(\mathbb{R}) \leqslant \varphi_{p,X}(\mathbb{R}^d) \leqslant \big(\varphi_{p,X}(\mathbb{R})\big)^d, \tag{13.2}$$

$$\varphi_{p,X}(\mathbb{R}^d) = \varphi_{p,X^*}(\mathbb{R}^d) \leqslant \begin{Bmatrix} \varphi_{p,X^*}(\mathbb{T}^d) = \varphi_{p,X}(\mathbb{Z}^d) \\ \varphi_{p,X}(\mathbb{T}^d) = \varphi_{p,X^*}(\mathbb{Z}^d) \end{Bmatrix} \leqslant \frac{\varphi_{p,X}(\mathbb{R}^d)}{\varphi_{p,\mathbb{C}}(\mathbb{R}^d)}. \tag{13.3}$$

It is actually known that $\varphi_{p,\mathbb{C}}(\mathbb{R}^d) = (p^{1/p}(p')^{-1/p'})^d$. For the purposes of deriving Proposition 13.1.1 with these explicit values, one only needs the easier lower bound $\varphi_{p,\mathbb{C}}(\mathbb{R}^d) \geqslant (p^{1/p}(p')^{-1/p'})^d$, which is readily deduced by computing the L^p norms of $\phi(x) = \widehat{\phi}(x) = e^{-\pi|x|^2}$.

As we shortly recall in more detail, most of the estimates of Proposition 13.1.1 have been proved in Section 2.4.b. To complete the picture with the final estimate in (13.3) (stated in Proposition 2.4.20 with a weaker constant), we begin with:

Lemma 13.1.2. *Let X be a Banach space and $p \in (1, \infty)$. Let $f \in L^p(\mathbb{T}^d; X)$ be a trigonometric polynomial, which we identify with its periodic extension to $\mathbb{R}^d$, and let $\phi \in \mathscr{S}(\mathbb{R}^d; X)$. Then*

$$\lim_{\varepsilon\downarrow 0} \|f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}\|_{L^p(\mathbb{R}^d;X)} = \|f\|_{L^p(\mathbb{T}^d;X)}\|\phi\|_{L^p(\mathbb{R}^d)},$$
$$\lim_{\varepsilon\downarrow 0} \|\mathscr{F}[f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}]\|_{L^{p'}(\mathbb{R}^d;X)} = \|\widehat{f}\|_{\ell^{p'}(\mathbb{Z}^d;X)}\|\widehat{\phi}\|_{L^{p'}(\mathbb{R}^d)}.$$

Proof. For the L^p norm we have

$$\begin{aligned} \|f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}\|^p_{L^p(\mathbb{R}^d;X)} &= \int_{\mathbb{R}^d} \|f(t)\phi(\varepsilon t)\|^p_X \varepsilon^d \,\mathrm{d}t \\ &= \int_{\mathbb{T}^d} \|f(t)\|^p_X \Big(\sum_{k\in\mathbb{Z}^d} |\phi(\varepsilon(t+k))|^p \varepsilon^d\Big)\,\mathrm{d}t, \end{aligned}$$

where in parentheses we have a Riemann sum of $\int_{\mathbb{R}^d} |\phi(t)|^p \,\mathrm{d}t$.

For the $L^{p'}$ norm, let us write $f(t) = \sum_{k\in\mathbb{Z}^d} x_k e_k(t)$. Then

$$\mathscr{F}[f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}](\xi) = \sum_{k\in\mathbb{Z}^d} x_k \int_{\mathbb{R}^d} \phi(\varepsilon t)\varepsilon^{d/p} e^{2\pi i k\cdot t} e^{-2\pi i \xi\cdot t}\,\mathrm{d}t$$

$$= \sum_{k\in\mathbb{Z}^d} x_k \widehat{\phi}(\varepsilon^{-1}(\xi - k))\varepsilon^{-d/p'}$$

Let us split this into two parts,

$$I := \sum_{k\in\mathbb{Z}^d} x_k \mathbf{1}_Q(\xi - k)\widehat{\phi}(\varepsilon^{-1}(\xi - k))\varepsilon^{-d/p'},$$
$$II := \sum_{k\in\mathbb{Z}^d} x_k \mathbf{1}_{\complement Q}(\xi - k)\widehat{\phi}(\varepsilon^{-1}(\xi - k))\varepsilon^{-d/p'},$$

where $Q = [-\frac{1}{2}, \frac{1}{2})^d$. The terms in I are disjointly supported, and hence

$$\begin{aligned}\|I\|_{L^{p'}(\mathbb{R}^d;X)} &= \Big(\sum_{k\in\mathbb{Z}^d} \|x_k\|^{p'} \|\mathbf{1}_Q(\cdot - k)\widehat{\phi}(\varepsilon^{-1}(\cdot - k))\varepsilon^{-d/p'}\|^{p'}_{L^{p'}(\mathbb{R}^d)}\Big)^{1/p'} \\ &= \Big(\sum_{k\in\mathbb{Z}^d} \|x_k\|^{p'} \|\mathbf{1}_Q(\varepsilon\cdot)\widehat{\phi}\|^{p'}_{L^{p'}(\mathbb{R}^d)}\Big)^{1/p'} \\ &\to \Big(\sum_{k\in\mathbb{Z}^d} \|x_k\|^{p'} \|\widehat{\phi}\|^{p'}_{L^{p'}(\mathbb{R}^d)}\Big)^{1/p'}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|II\|_{L^{p'}(\mathbb{R}^d;X)} &\leqslant \sum_{k\in\mathbb{Z}^d} \|x_k\| \|\mathbf{1}_{\complement Q}(\cdot - k)\widehat{\phi}(\varepsilon^{-1}(\cdot - k))\varepsilon^{-d/p'}\|_{L^{p'}(\mathbb{R}^d)} \\ &\leqslant \sum_{k\in\mathbb{Z}^d} \|x_k\| \|\mathbf{1}_{\complement Q}(\varepsilon\cdot)\widehat{\phi}\|_{L^{p'}(\mathbb{R}^d)} \to 0.\end{aligned}$$

Thus $\|I + II\|_{L^{p'}(\mathbb{R}^d;X)}$ indeed converges to the claimed limit. □

Proof of Proposition 13.1.1. The second bound in (13.2) is contained in Proposition 2.4.11. The first bound is also there, but in a slightly different form, and the present formulation is obtained by repeating the same proof: Given $f \in L^p(\mathbb{R};X)$ and $\phi \in L^p(\mathbb{R}^{d-1})$, we have

$$\begin{aligned}\|\widehat{f}\|_{L^{p'}(\mathbb{R};X)}\|\widehat{\phi}\|_{L^{p'}(\mathbb{R}^{d-1})} &= \|\mathscr{F}(f\otimes\phi)\|_{L^{p'}(\mathbb{R}^d;X)} \\ &\leqslant \varphi_{p,X}(\mathbb{R}^d)\|f\otimes\phi\|_{L^p(\mathbb{R}^d;X)} = \varphi_{p,X}(\mathbb{R}^d)\|f\|_{L^p(\mathbb{R};X)}\|\phi\|_{L^p(\mathbb{R}^{d-1})}.\end{aligned}$$

Choosing f and ϕ that (almost) achieve equality in the definition of the constants $\varphi_{p,X}(\mathbb{R})$ and $\varphi_{p,\mathbb{C}}(\mathbb{R}^{d-1})$, we obtain the first bound in (13.2).

The first equality in (13.3) is Proposition 2.4.16. The first pair of inequalities and the two equalities in the middle of in (13.3) are all contained in Proposition 2.4.20 (either as stated or substituting X^* in place of X).

Concerning the last pair of inequalities in (13.3), it suffices to prove that

$$\varphi_{p,X}(\mathbb{T}^d) \leqslant \frac{\varphi_{p,X}(\mathbb{R}^d)}{\varphi_{p,\mathbb{C}}(\mathbb{R}^d)}, \tag{13.4}$$

since the other bound follows with X^* in place of X and using the first equality in (13.3). To this end, it follows from Lemma 13.1.2 that

$$\begin{aligned}\|\widehat{f}\|_{\ell^{p'}(\mathbb{Z}^d;X)}\|\widehat{\phi}\|_{L^{p'}(\mathbb{R}^d)} &= \lim_{\varepsilon\downarrow 0}\|\mathscr{F}[f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}]\|_{L^{p'}(\mathbb{R}^d;X)}\\ &\leqslant \lim_{\varepsilon\downarrow 0}\varphi_{p,X}(\mathbb{R}^d)\|f(\cdot)\phi(\varepsilon\cdot)\varepsilon^{d/p}\|_{L^p(\mathbb{R}^d;X)}\\ &= \varphi_{p,X}(\mathbb{R}^d)\|f\|_{L^p(\mathbb{T}^d;X)}\|\phi\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Choosing, again, f and ϕ that (almost) achieve equality in the definition of the constants $\varphi_{p,X}(\mathbb{T}^d)$ and $\varphi_{p,\mathbb{C}}(\mathbb{R}^d)$, we complete the proof of (13.4), and hence the Proposition. □

Proposition 13.1.1 at hand, in order to prove that a given Banach space has Fourier type p, we can pick any of the equivalent conditions amenable to our analysis. We will eventually achieve our goal with the constant $\varphi_{p,X}(\mathbb{T})$, but a major part of the work will take place on the dual group $\mathbb{Z}$. This has the advantage of presenting a convenient finite formulation as follows:

Definition 13.1.3. *Let X be a Banach space, $p,q\in[1,\infty]$ and $n\in\mathbb{Z}_+$. Then $\varphi_{p,X}^{(q)}(n)$ is the smallest admissible constant such that the inequality*

$$\Big\|\sum_{k=1}^n e_k x_k\Big\|_{L^q(\mathbb{T};X)}\leqslant \varphi_{p,X}^{(q)}(n)\Big(\sum_{k=1}^n\|x_k\|^p\Big)^{1/p},\quad e_k(t):=e^{2\pi ikt}\quad(t\in\mathbb{T}),$$

holds for every choice of $x_1,\dots,x_n\in X$. We abbreviate $\varphi_{p,X}(n):=\varphi_{p,X}^{(p')}(n)$.

Although the case $q=p'$ is most directly linked with the Hausdorff–Young inequality on the infinite spaces $\mathbb{R}^d,\mathbb{T}^d$ and $\mathbb{Z}^d$, it turns out that our intermediate steps towards this final goal will also need to make use of the more general definition with "mismatched" exponents. Moreover, we will even need some further variations of this definition (e.g., involving other index sets F in place of $\{1,\dots,n\}$), but we postpone them until the point where they will be used. For the moment, we have the fairly obvious

Lemma 13.1.4. *Let X be a Banach space and $p,q\in[1,\infty]$. The sequence $(\varphi_{p,X}^{(q)}(n))_{n\geqslant 1}$ is increasing, and*

$$1\leqslant\varphi_{p,X}^{(q)}(n)\leqslant n^{1/p'},\qquad \varphi_{p,X}(\mathbb{Z})=\lim_{n\to\infty}\varphi_{p,X}(n)\in[1,\infty].$$

Proof. That the sequence is increasing follows simply by extending a shorter sequence by additional zeroes. This also shows the existence of a (possibly infinite) limit $\lim_{n\to\infty}\varphi_{p,X}(n)$. The lower bound follows by taking $x_1\neq 0=x_k$ for $k\geqslant 2$, and the upper bound is also simply the triangle and Hölder's inequality

$$\Big\|\sum_{k=1}^{n} e_k x_k\Big\|_{L^q(\mathbb{T};X)} \leqslant \sum_{k=1}^{n} \|x_k\| \leqslant n^{1/p'}\Big(\sum_{k=1}^{n}\|x_k\|^p\Big)^{1/p}.$$

Given $(x_k)_{k=1}^n$, let $x = (x_k)_{k\in\mathbb{Z}}$ be its zero extension. The upper bound $\varphi_{p,X}(n) \leqslant \varphi_{p,X}(\mathbb{Z})$ follows by observing that $\sum_{k=1}^n e_k(t)x_k$ is simply $\widehat{x}(-t)$.

It only remains to check that $\varphi_{p,X}(\mathbb{Z}) \leqslant \lim_{n\to\infty}\varphi_{p,X}(n)$. Let $x = (x_k)_{k\in\mathbb{Z}}$ be finitely supported, i.e., $x_k = 0$ if $|k| \geqslant N$ for some finite N. Now

$$\widehat{x} = \sum_{|k|\leqslant N-1} e_{-k}x_k = \sum_{j=1}^{2N-1} e_{-N+j}x_{N-j} = e_{-N}\sum_{j=1}^{2N-1} e_j x_{N-j},$$

hence

$$\begin{aligned}\|\widehat{x}\|_{L^{p'}(\mathbb{T};X)} &= \Big\|\sum_{j=1}^{2N-1} e_j x_{N-j}\Big\|_{L^{p'}(\mathbb{T};X)} \leqslant \varphi_{p,X}(2N-1)\Big(\sum_{j=1}^{2N-1}\|x_{N-j}\|^p\Big)^{1/p} \\ &= \varphi_{p,X}(2N-1)\|x\|_{\ell^p(\mathbb{Z};X)} \leqslant \lim_{n\to\infty}\varphi_{p,X}(n)\|x\|_{\ell^p(\mathbb{Z};X)}.\end{aligned}$$

By the density of finitely supported sequences in $\ell^p(\mathbb{Z};X)$, this shows that $\varphi_{p,X}(\mathbb{Z}) \leqslant \lim_{n\to\infty}\varphi_{p,X}(n)$, and completes the proof. □

The task of proving that a space X has non-trivial Fourier type (assuming non-trivial type) is hence reduced, in principle, to showing the boundedness of the sequence $(\varphi_{p,X}(n))_{n\geqslant 1}$ for some $p > 1$. Although the proof that we are about to give is eventually set up slightly differently, this idea serves as a good motivation for a major part of the subsequent analysis. The proof that we will present can be roughly divided into the following main steps, treated in the next four sections:

1. Using type bounds on Sidon sets that partition $\{1,\dots,n\}$ gives a first mild improvement $\varphi_{2,X}(n) = o(n^{1/2})$ over the trivial estimate $\varphi_{2,X}(n) \leqslant n^{1/2}$.
2. Comparison with the finite Fourier transform on $\mathbb{Z}_n$ gives sub-multiplicativity and leads to $\varphi_{2,X}(n) = O(n^{1/r-1/2})$ for some $r > 1$.
3. By a delicate Lemma 13.1.25 of Bourgain, this gives a first uniform bound $\varphi^{(2)}_{s;X}(n) = O(1)$, but with mismatched exponents $s \in (1,r)$ and $2 \neq s'$.
4. Standard duality and interpolation, combined with repeating the same key Lemma 13.1.25 on the dual side, allow us to conclude with $p \in (1,r)$.

A thorough reader may recognise some conceptual similarity with the considerations encountered in Section 7.3.b in the context of deducing non-trivial type (and cotype) from the non-containment of certain subspaces. There we defined the finite type constant $\tau_{2,X}(n)$ as the best constant in the estimate

$$\Big\|\sum_{k=1}^{n}\varepsilon_k x_k\Big\|_{L^2(\Omega;X)} \leqslant \tau_{2,X}(n)\Big(\sum_{k=1}^{n}\|x_k\|^2\Big)^{1/2} \quad \forall x_1,\dots,x_n\in X. \tag{13.5}$$

These numbers will play a role in the first proof step outlined above.

13.1.a Hinrichs's inequality: breaking the trivial bound

Recall that our goal is deriving non-trivial Fourier type from non-trivial type. Thus, from the knowledge that *random sums* $\sum_k \varepsilon_k x_k$ can be dominated by $\|(x_k)\|_{\ell^p}$, we would like to conclude that *trigonometric sums* $\sum_k e_k x_k$ can be similarly dominated (though possibly with a different p). An obvious idea that suggests itself is to try to dominate the trigonometric sum by the random sum. Indeed, we know from Section 6.5 that this can be done under particular circumstances if the trigonometric sum is restricted to a special set called a *Sidon set.* This leads to the following strategy: Given the initial sum over $k \in \{1, \dots, N\}$, we want to partition this into sums over Sidon sets on which we can make estimates, and this partitioning should be done sufficiently economically so that it allows us to beat the trivial estimate. To carry out this idea, we need to be able to

1. efficiently recognise Sidon sets, and
2. decompose arbitrary sets into as few as possible Sidon sets.

We now turn to these tasks. Recall from Section 6.5 that a subset $\Lambda \subseteq \mathbb{Z}$ is called a *Sidon set* if the following estimate holds uniformly over all finitely non-zero sequences $(c_\lambda)_{\lambda\in\Lambda}$ of complex numbers:

$$\sum_{\lambda\in\Lambda} |c_\lambda| \leqslant C \Big\| \sum_{\lambda\in\Lambda} c_\lambda e_\lambda \Big\|_\infty.$$

The smallest admissible constant C is called the *Sidon constant* of Λ and is denoted by $S(\Lambda)$. However, this definition in itself is hardly helpful in checking whether or not a particular set actually satisfies this property. A first sufficient condition for a set to be a Sidon set was achieved in Proposition 6.5.3, showing in particular that $S(\{2^k : k \in \mathbb{N}\}) \leqslant 4$. For the present purposes, we require a more robust criterion, which is provided in the following:

Definition 13.1.5 (Quasi-independent set). *A subset $F \subseteq \mathbb{Z} \setminus \{0\}$ is called* quasi-independent *if $\alpha_k \equiv 0$ is the only finitely non-zero sequence such that $\alpha_k \in \{-1, 0, +1\}$ for all $k \in F$ and*

$$\sum_{k\in F} \alpha_k \cdot k = 0.$$

Example 13.1.6. The sequence $\{2^k : k \in \mathbb{N}\}$ is quasi-independent. In fact, if $\sum_{k=0}^\infty \alpha_k 2^k = 0$ for a finitely non-zero sequence $(\alpha_k)_{k=1}^\infty$, then

$$\sum_{k:\alpha_k=+1} 2^k = \sum_{k:\alpha_k=-1} 2^k.$$

It follows from the uniqueness of the binary expansion that $\{k : \alpha_k = +1\} = \{k : \alpha_k = -1\}$, and this is possible only if both sets are empty. Hence $\alpha_k \equiv 0$.

Proposition 13.1.7 (Bourgain). *Every quasi-independent set $F \subseteq \mathbb{Z} \setminus \{0\}$ is a Sidon set with*

$$S(F) \leqslant 16.$$

By Example 13.1.6, this gives another proof of the fact that $\{2^k : k \in \mathbb{N}\}$ is a Sidon set, but with a slightly weaker constant than Proposition 6.5.3.

Proof. This is based on a variant of the Riesz product method also used in the proof of Proposition 6.5.3, but the details are somewhat different, and we will provide a self-contained argument. By considering every finite subset of the original F, we may assume without loss of generality that F is finite to begin with. Given parameters $\varrho \in (0,1]$ and $\xi = (\xi_k)_{k\in F} \in \mathbb{R}^F$, let then

$$\begin{aligned} R_\xi(t) &:= \prod_{k\in F} \big(1 + \varrho\cos(2\pi(kt+\xi_k))\big) \\ &= \prod_{k\in F} \big(1 + \frac{\varrho}{2}(e_k(t)e_1(\xi_k) + e_{-k}(t)e_{-1}(\xi_k))\big) \\ &= \sum_{\alpha\in\{-1,0,+1\}^F} 2^{-|\alpha|}\varrho^{|\alpha|} \exp\Big(2\pi i \sum_{k\in F}\alpha_k\cdot kt\Big)\exp\Big(2\pi i\sum_{k\in F}\alpha_k\xi_k\Big), \end{aligned}$$

where $|\alpha| := \sum_{k\in F}|\alpha_k|$ as usual for multi-indices,. (To relax the notation, we do not explicate the dependence of R_ξ on ϱ.)

From the assumption that F is quasi-independent, it follows that

$$\sum_{k\in F}\alpha_k\cdot k = 0 \quad \text{only if} \quad \alpha_k \equiv 0,$$

and hence $\widehat{R}_\xi(0) = 1$. It is also clear from the first line of the definition of $R_\xi(t)$ (recalling that $\varrho \in (0,1]$) that $R_\xi(t) \geqslant 0$, and hence

$$\|R_\xi\|_{L^1(\mathbb{T})} = \int_0^1 R_\xi(t)\,\mathrm{d}t = \widehat{R}_\xi(0) = 1.$$

Let us further write

$$R_\xi^{(m)}(t) := \sum_{\substack{\alpha\in\{-1,0,+1\}^F \\ |\alpha|=m}} 2^{-|\alpha|}\exp\Big(2\pi i\sum_{k\in F}\alpha_k\cdot kt\Big)\exp\Big(2\pi i\sum_{k\in F}\alpha_k\xi_k\Big),$$

so that

$$R_\xi(t) = \sum_{m=0}^{\#F}\varrho^m R_\xi^{(m)}(t), \qquad \text{where}$$

$$R_\xi^{(0)}(t) = 1, \qquad R_\xi^{(1)}(t) = \frac{1}{2}\sum_{k\in F}\big(e_k(t)e_1(\xi_k) + e_{-k}(t)e_{-1}(\xi_k)\big).$$

From the orthogonality of the exponentials, for each $j \in F$, we have

$$\int_0^1 R_\xi^{(1)}(t)e_j(t)\,\mathrm{d}t = \frac{1}{2}\sum_{k\in F}\big(\delta_{k,-j}e_1(\xi_k)+\delta_{k,j}e_{-1}(\xi_k)\big) = \frac{1}{2}e_{-1}(\xi_j),$$

where we observed that $k=-j$ is not possible when k,j belong to the same quasi-independent set F, since $1\cdot k+1\cdot j=0$ is a direct violation of the defining condition. It is also immediate that $\int R_\xi^{(0)}e_j = 0$ for all $j\in F\subseteq\mathbb{Z}\setminus\{0\}$.

For

$$f=\sum_{j\in F}c_je_j,$$

we then conclude that

$$\begin{aligned}\int_0^1 R_\xi f &= \int_0^1\Big(\sum_{m=0}^{\#F}\varrho^m R_\xi^{(m)}\Big)\Big(\sum_{j\in F}c_je_j\Big)\\ &= 0+\frac{\varrho}{2}\sum_{j\in F}c_je_{-1}(\xi_j)+\sum_{j\in F}c_j\sum_{m\geqslant 2}\varrho^m\int_0^1 R_\xi^{(m)}e_j.\end{aligned}\tag{13.6}$$

Using again the orthogonality of the exponentials, we have

$$\begin{aligned}\Big|\int_0^1 R_\xi^{(m)}e_j\Big| &= \Big|\sum_{\substack{\alpha\in\{-1,0,+1\}^F\\ |\alpha|=m\\ \sum_{k\in F}\alpha_k\cdot k=-j}} 2^{-m}\exp\Big(2\pi i\sum_{k\in F}\alpha_k\xi_k\Big)\Big|\\ &\leqslant \sum_{\substack{\alpha\in\{-1,0,+1\}^F\\ |\alpha|=m\\ \sum_{k\in F}\alpha_k\cdot k=-j}} 2^{-m} = \int_0^1 R_0^{(m)}e_j,\end{aligned}$$

where $R_0^{(m)}$ is simply $R_\xi^{(m)}$ with $\xi=0$. It follows that

$$\sum_{m=0}^{\#F}\Big|\int_0^1 R_\xi^{(m)}e_j\Big| \leqslant \sum_{m=0}^{\#F}\int_0^1 R_0^{(m)}e_j = \int_0^1 R_0e_j\leqslant \|R_0\|_{L^1(\mathbb{T})}=1.$$

The last term in (13.6) can now be estimated by

$$\Big|\sum_{j\in F}c_j\sum_{m\geqslant 2}\varrho^m\int_0^1 R_\xi^{(m)}e_j\Big|\leqslant\sum_{j\in F}|c_j|\sum_{m\geqslant 2}\varrho^2\Big|\int_0^1 R_\xi^{(m)}e_j\Big|\leqslant\sum_{j\in F}|c_j|\varrho^2.$$

If we now choose ξ_j so that $c_je_{-1}(\xi_j)=|c_j|$, then (13.6) gives

$$\begin{aligned}\frac{\varrho}{2}\sum_{j\in F}|c_j| &= \int_0^1 R_\xi f-\sum_{j\in F}c_j\sum_{m\geqslant 2}\varrho^m\int_0^1 R_\xi^{(m)}e_j\\ &\leqslant \|R_\xi\|_{L^1(\mathbb{T})}\|f\|_{L^\infty(\mathbb{T};X)}+\varrho^2\sum_{j\in F}|c_j|,\end{aligned}$$

and hence

$$\Big(\frac{\varrho}{2}-\varrho^2\Big)\sum_{j\in F}|c_j| \leqslant \|f\|_{L^\infty(\mathbb{T};X)} = \Big\|\sum_{k\in F} c_k e_k\Big\|_{L^\infty(\mathbb{T};X)}.$$

Choosing finally $\varrho = \frac{1}{4}$ completes the proof. □

By the previous result, our initial task of decomposing arbitrary sets into Sidon sets is reduced to decomposing into quasi-independent sets. A first step in this direction is to know that every set has a quasi-independent subset of somewhat substantial size.

Lemma 13.1.8. *Any finite subset $F \subseteq \mathbb{Z}\setminus\{0\}$ has a quasi-independent subset $F_0 \subseteq F$ of cardinality $\#F_0 \geqslant \lceil \log_3 \#F\rceil$.*

Proof. Let $F_0 \subseteq F$ be a quasi-independent subset of maximal cardinality, and let

$$F_1 := \Big\{\sum_{k\in F_0} \alpha_k \cdot k : \alpha_k \in \{-1,0,+1\}\Big\}.$$

Clearly $F_1 \supseteq F_0$, and we claim that in fact $F_1 \supseteq F$. If not, let $k_0 \in F\setminus F_1$. We will check that $F_0 \cup \{k_0\}$ is quasi-independent, contradicting the maximality of F_0. Namely, suppose that

$$\sum_{k\in F_0\cup\{k_0\}} \alpha_k \cdot k = 0,$$

where $\alpha_k \in \{-1,0,+1\}$. If $\alpha_{k_0} = \pm 1$, then

$$k_0 = \sum_{k\in F_0} (-\alpha_{k_0}\alpha_k)\cdot k \in F_1,$$

contradicting $k_0 \notin F_1$. Thus $\alpha_{k_0} = 0$, but then also $\alpha_k = 0$ for all $k \in F_0$, since F_0 is quasi-independent, and this proves that $F_0 \cup \{k_0\}$ is quasi-independent.

As explained above, this proves that $F_1 \supseteq F$, and hence

$$\#F \leqslant \#F_1 \leqslant 3^{\#F_0},$$

from which the proposition follows, since $\#F_0 \geqslant \log_3 \#F$ is necessarily an integer. □

By recursively removing big quasi-independent subsets, we arrive at the desired decomposition of the initial set:

Lemma 13.1.9. *For $N \in \mathbb{Z}_+$, let*

$$d(N) := \min\{k \in \mathbb{Z}_+ : \textit{any subset } F \subseteq \mathbb{Z}\setminus\{0\} \textit{ of size } \#F \leqslant N \textit{ can be divided into at most } k \textit{ quasi-independent subsets}\}.$$

Then $d(3^n) \leqslant \dfrac{2\cdot 3^n}{n+1}$ for all $n \in \mathbb{N}$. For all $n \geqslant 1$, each of the partitioning quasi-independent subsets can be chosen to have size at most n.

Proof. Since clearly $d(3^n) \leqslant 3^n$ (as each singleton is quasi-independent), the claim is obvious for $n \leqslant 1$. For $3 < \#F \leqslant 9$, Lemma 13.1.8 guarantees a quasi-independent subset of size 2. Starting from a set of size 9 and repeatedly extracting 3 quasi-independent subsets of size 2, we are left with a subset of size 3 that trivially splits into 3 quasi-independent subsets of size 1. Hence $d(3^2) \leqslant 3+3 = 6 = 2\cdot 3^2/(2+1)$. We then assume that, for some $n \geqslant 2$, any set of size 3^n can be divided into at most $2\cdot 3^n/(n+1)$ quasi-independent subsets of size at most n, and we prove the same for $n+1$.

If $\#F = 3^{n+1}$, Lemma 13.1.8 guarantees that we can repeatedly extract quasi-independent subsets F_i $(i = 1,\dots,j)$ of size $n+1$, until

$$3^{n+1} - j(n+1) \leqslant 3^n < 3^{n+1} - (j-1)(n+1),$$

thus

$$\frac{2\cdot 3^n}{n+1} \leqslant j < 1 + \frac{2\cdot 3^n}{n+1}.$$

The remaining set of size at most 3^n can then be divided into at most $d(3^n)$ quasi-independent subsets, and by the induction assumption we have

$$d(3^{n+1}) \leqslant j + d(3^n) < \Big(1 + \frac{2\cdot 3^n}{n+1}\Big) + \frac{2\cdot 3^n}{n+1} = 1 + \frac{4\cdot 3^n}{n+1}.$$

For $n \geqslant 2$, we have $1/(n+1) \leqslant \frac{4}{3}/(n+2)$ and $3^n/(n+2) \geqslant \frac{9}{4}$, and hence

$$d(3^{n+1}) \leqslant \Big(\frac{4}{9} + \frac{16}{3}\Big)\frac{3^n}{n+2} < 6\frac{3^n}{n+2} = \frac{2\cdot 3^{n+1}}{(n+1)+1},$$

and this completes the induction step. Note that all quasi-independent subsets that we constructed in the induction step had either size $n+1$, or they came from the induction assumption, in which case their size is at most n. $\square$

In the next remark, we indicate converses to the obtained bounds of Lemmas 13.1.8 and 13.1.9.

Remark 13.1.10. Let $F_0 \subseteq F := \{1,\dots,N\}$ be such that $N \geqslant 2$ and F_0 is quasi-independent. We claim that necessarily $\#F_0 \leqslant 2\log_2(N)$. Clearly, this implies $d(N) \geqslant \frac{N}{2\log_2(N)}$. Indeed, write $m = \#F_0$. It suffices to consider $m \geqslant 2$. Let $A \subseteq F_0$ be arbitrary. Then

$$0 \leqslant \sum_{a\in A} a \leqslant \sum_{a\in F_0} a < N^2.$$

Therefore, the number of different values can be estimated by

$$\#\Big\{\sum_{a\in A} a : A \subseteq F_0\Big\} \leqslant N^2.$$

One the other hand, if $A, B \subseteq F_0$ are such that $\sum_{a\in A} a = \sum_{b\in B} b$, then the quasi-independence of F_0 implies $A = B$. Therefore,

$$2^m = \#\{A \subseteq F_0\} \leqslant \#\Big\{ \sum_{a\in A} a : A \subseteq F_0 \Big\}.$$

We can conclude $2^m \leqslant N^2$ and thus the claim follows.

We now possess all the ingredients needed for the first estimate of Fourier type in terms of type, stated in terms of the finite versions of both properties. The reader may wish to compare the next proposition to Theorem 7.6.12 which gives a related inequality for the Walsh system.

Proposition 13.1.11 (Hinrichs's inequality). *For all $n \geqslant 1$ we have*

$$\frac{\varphi_{2,X}(3^n)}{\sqrt{3^n}} \leqslant 16\sqrt{2} \cdot \frac{\tau_{2,X}(n)}{\sqrt{n}}.$$

Proof. By Lemma 13.1.9, the set $\{1, \dots, 3^n\}$ can be divided into

$$A \leqslant 2 \cdot 3^n/(n+1)$$

quasi-independent subsets F_a of size $\#F_a \leqslant n$. By Proposition 13.1.7, each quasi-independent F_a is a Sidon set with $S(F_a) \leqslant 16$. By Pisier's Theorem 6.5.5, trigonometric series over a Sidon set is comparable in the L^p norm to the corresponding Rademacher series, up to the Sidon constant. Chaining these observations and using the definition of the type constants $\tau_{2,X}(n)$ and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\Big\|\sum_{k=1}^{3^n} e_k x_k\Big\|_{L^2(\mathbb{T};X)} &= \Big\|\sum_{a=1}^{A}\sum_{k\in F_a} e_k x_k\Big\|_{L^2(\mathbb{T};X)} \leqslant \sum_{a=1}^{A}\Big\|\sum_{k\in F_a} e_k x_k\Big\|_{L^2(\mathbb{T};X)} \\ &\leqslant \sum_{a=1}^{A} 16 \Big\|\sum_{k\in F_a} \varepsilon_k x_k\Big\|_{L^2(\Omega;X)} \\ &\leqslant \sum_{a=1}^{A} 16 \cdot \tau_{2,X}(\#F_a)\Big(\sum_{k\in F_a}\|x_k\|^2\Big)^{1/2} \\ &\leqslant 16 \cdot \max_{1\leqslant a\leqslant A} \tau_{2,X}(\#F_a)\sqrt{A}\Big(\sum_{a=1}^{A}\sum_{k\in F_a}\|x_k\|^2\Big)^{1/2} \\ &\leqslant 16 \cdot \tau_{2,X}(n)\sqrt{\frac{2\cdot 3^n}{n+1}}\Big(\sum_{k=1}^{3^n}\|x_k\|^2\Big)^{1/2},\end{aligned}$$

from which the proposition follows. □

The following corollary gives the promised improvement over the trivial bound $\varphi_{2,X}(3^n) \leqslant \sqrt{3^n}$ as soon as n is large enough.

Corollary 13.1.12. *Let X be a Banach space of type $p \in (1, 2]$. Then for all $n \geqslant 1$, we have*

$$\frac{\varphi_{2,X}(3^n)}{\sqrt{3^n}} \leqslant 16\sqrt{2} \cdot \tau_{p,X;2} \cdot n^{-1/p'}.$$

The type constant $\tau_{p,X;s}$ with a secondary parameter (above $s = 2$) was introduced right before Proposition 7.1.4 as the best constant in the inequality

$$\Big\| \sum_{k=1}^{K} \varepsilon_k x_k \Big\|_{L^s(\Omega;X)} \leqslant \tau_{p,X;s} \Big(\sum_{k=1}^{K} \|x_k\|^p \Big)^{1/p}, \tag{13.7}$$

where $x_1, \ldots, x_K \in X$ and $K \in \mathbb{Z}_+$ are arbitrary. Recall that $\tau_{p,X} := \tau_{p,X;p}$.

Proof. From the definition of the type constants and Hölder's inequality, it is immediate that

$$\frac{\tau_{2,X}(n)}{\sqrt{n}} \leqslant \frac{\tau_{p,X;2} \cdot n^{1/p-1/2}}{\sqrt{n}} = \tau_{p,X;2} \cdot n^{-1/p'}.$$

In combination with Proposition 13.1.11, this gives the result. □

13.1.b The finite Fourier transform and sub-multiplicativity

Note that the improvement of Corollary 13.1.12 over the trivial bound is only very slight. Our first goal in bootstrapping this initial estimate is to obtain a power-type bound of the form $\varphi_{2,X}(N) = O(N^{1/2-\delta})$. As the reader can easily verify (perhaps referring to Lemma 7.3.19), this would readily follow from the established bound, *if* in addition we had a sub-multiplicative estimate

$$\varphi_{2,X}(nm) \overset{?}{\leqslant} \varphi_{2,X}(n)\varphi_{2,X}(m).$$

As we do not know whether this is true, we take a detour by comparing the sequence $\varphi_{2,X}(n)$ with the following discretised variant:

Definition 13.1.13. *Let X be a Banach space and $n \in \mathbb{Z}_+$. Then $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ is the best constant in the following inequality with arbitrary $x_1, \ldots, x_n \in X$:*

$$\Big(\frac{1}{n} \sum_{h=1}^{n} \Big\| \sum_{k=1}^{n} e_k(h/n) x_k \Big\|^q \Big)^{1/q} \leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_n) \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}.$$

As the notation suggests, $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ has an interpretation as the norm of the Fourier transform (thus, a Fourier type constant) of functions on the finite group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, but there is no need to insist too much on this point here. The difference of the defining inequalities of $\varphi_{p,X}^{(q)}(n)$ and $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ is that the $L^p(\mathbb{T}; X)$ integral norm in the former is replaced by a finite Riemann sum approximation in the latter. We will next develop some tools for comparing the two kinds of norms. This will involve elements of some fairly classical Fourier analysis, and we begin with

Definition 13.1.14. *The* Dirichlet kernel *is defined by*

$$D_n(t) := \sum_{|k|\leqslant n} e_k(t), \quad t \in \mathbb{T},$$

the Fejér kernel *by*

$$F_n(t) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(t) = \sum_{|k|\leqslant n}\Big(1 - \frac{|k|}{n+1}\Big)e_k(t), \quad t \in \mathbb{T},$$

and the de la Vallée–Poussin kernel *by*

$$V_n(t) := \frac{1}{n}\sum_{k=n}^{2n-1} D_k(t) = \sum_{|j|\leqslant n} e_j(t) + \sum_{n<|j|<2n}\Big(2 - \frac{|j|}{n}\Big)e_j(t), \quad t \in \mathbb{T}.$$

Lemma 13.1.15. *These kernels satisfy the identities*

$$D_n(t) = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}, \qquad F_n(t) = \frac{1}{n+1}\frac{\sin^2(\pi(n+1)t)}{\sin^2(\pi t)} \geqslant 0,$$

$$V_n(t) = 2F_{2n-1}(t) - F_{n-1}(t).$$

Proof. The formula for D_n is the summation of a geometric series:

$$D_n(t) := \sum_{|k|\leqslant n} e^{2\pi i k t} = e^{-2\pi i n t}\frac{e^{2\pi i(2n+1)t} - 1}{e^{2\pi i t} - 1} = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}.$$

Since

$$\begin{aligned}\sum_{k=0}^{n}\sin(\pi(2k+1)t) &= \Im\sum_{k=0}^{n} e^{i\pi t}e^{i2\pi k t} = \Im\Big(e^{i\pi t}\frac{e^{i2\pi(n+1)t} - 1}{e^{i2\pi t} - 1}\Big)\\ &= \Im\Big(e^{i\pi(n+1)t}\frac{\sin(\pi(n+1)t)}{\sin(\pi t)}\Big) = \frac{\sin^2(\pi(n+1)t)}{\sin(\pi t)},\end{aligned}$$

we obtain the formula for F_n by summing over the formula for D_k. Finally,

$$V_n = \frac{1}{n}\sum_{k=n}^{2n-1} D_k = \frac{1}{n}\Big(\sum_{k=0}^{2n-1} D_k - \sum_{k=0}^{n-1} D_k\Big) = \frac{1}{n}\Big(2nF_{2n-1} - nF_{n-1}\Big).$$

□

Lemma 13.1.16. *If f is a trigonometric polynomial with* $\deg(f) < n$*, then for all $s \in \mathbb{R}$ we have*

$$\int_0^1 f(t)\,\mathrm{d}t = \frac{1}{n}\sum_{h=1}^{n} f(s + h/n),$$

i.e., f can be integrated exactly by uniform Riemann sums of order n.

Proof. It is enough to consider $f(t) = e_k(t)$, where $|k| < n$. We observe that

$$\sum_{h=1}^{n} e^{2\pi ikh/n} = \begin{cases} e^{2\pi ik/n} \frac{e^{2\pi ikn/n}-1}{e^{2\pi ik/n}-1} = 0, & 0 < |k| < n, \\ n, & k = 0, \end{cases}$$

and hence

$$\frac{1}{n}\sum_{h=1}^{n} f(s+h/n) = \frac{e_k(s)}{n}\sum_{h=1}^{n} e^{2\pi ikh/n} = e_k(s)\delta_{k,0} = \delta_{k,0} = \int_0^1 e_k(t)\,\mathrm{d}t.$$

□

On the level of L^p norms, this leads to the following comparison result:

Proposition 13.1.17 (Marcinkiewicz inequality). *Let X be a Banach space and $p \in [1,\infty)$. Then for all $n \in \mathbb{Z}_+$ and $x_1, \dots, x_n \in X$, we have*

$$\Big(\frac{1}{n}\sum_{h=1}^{n}\Big\|\sum_{k=1}^{n} e_k(h/n)x_k\Big\|^p\Big)^{1/p} \leqslant 3\Big\|\sum_{k=1}^{n} e_k x_k\Big\|_{L^p(\mathbb{T};X)}.$$

With the usual modification, the result is also true (and entirely trivial) for $p = \infty$: of course the supremum over $\{j/n : j = 1, \dots, n\}$ is dominated by the supremum over all of $\mathbb{T}$!

Proof. Let

$$f(t) := \sum_{k=1}^{n} e_k(t)x_k, \qquad m := \lfloor n/2 \rfloor.$$

Then $(n-1)/2 \leqslant m \leqslant n/2$ and the function $e_{-(m+1)}f$ is a linear combination of e_k with

$$-m = 1-(m+1) \leqslant k \leqslant n-(m+1) \leqslant (2m+1)-(m+1) = m,$$

so $e_{-(m+1)}f$ is a trigonometric polynomial of degree m.

Since the de la Vallée–Poussin kernel V_m from Definition 13.1.14 has Fourier coefficients $\widehat{V}_m(k) = 1$ for all values $|k| \leqslant m$ on which the Fourier coefficients of $e_{-(m+1)}f$ are supported, we conclude that

$$\widehat{V}_m[e_{-(m+1)}f]\widehat{\ } = [e_{-(m+1)}f]\widehat{\ },$$

hence $V_m * (e_{-(m+1)}f) = e_{-(m+1)}f$. Thus

$$\begin{aligned} \|f(t)\| &= \|e_{-(m+1)}(t)f(t)\| = \|V_m * (e_{-(m+1)}f)(t)\| \\ &\leqslant \int_{\mathbb{T}} |V_m(t-s)|\|f(s)\|\,\mathrm{d}s \\ &\leqslant \Big(\int_{\mathbb{T}} |V_m(t-s)|\,\mathrm{d}s\Big)^{1/p'}\Big(\int_{\mathbb{T}} |V_m(t-s)|\|f(s)\|^p\,\mathrm{d}s\Big)^{1/p} \end{aligned} \tag{13.8}$$

By Lemma 13.1.15, we have

$$|V_m| = |2F_{2m-1} - F_{m-1}| \leqslant 2F_{2m-1} + F_{m-1}, \tag{13.9}$$

where $\int_{\mathbb{T}} F_k(t)\,\mathrm{d}t = \widehat{F}_k(0) = 1$, and hence

$$\int_{\mathbb{T}} |V_m(t-s)|\,\mathrm{d}s \leqslant 3.$$

Substituting into (13.8) and summing, we have

$$\sum_{h=1}^{n} \|f(h/n)\|^p \leqslant 3^{p/p'} \int_{\mathbb{T}} \sum_{h=1}^{n} |V_m(h/n-s)| \|f(s)\|^p\,\mathrm{d}s.$$

Since the right-hand side of (13.9) is a trigonometric polynomial of degree $2m-1 \leqslant n-1$, Lemma 13.1.16 guarantees that

$$\begin{aligned}\sum_{h=1}^{n} |V_m(h/n-s)| &\leqslant \sum_{h=1}^{n} (2F_{2m-1} + F_{m-1})(h/n-s) \\ &= n\int_{\mathbb{T}} (2F_{2m-1} + F_{m-1})(u)\,\mathrm{d}u = 3n.\end{aligned}$$

Substituting back, we conclude that

$$\frac{1}{n}\sum_{h=1}^{n} \|f(h/n)\|^p \leqslant 3^{p/p'} \int_{\mathbb{T}} 3\|f(s)\|^p\,\mathrm{d}s = 3^p \|f\|^p_{L^p(\mathbb{T};X)}.$$

□

We now have the desired comparison of the two finite Fourier type constants:

Lemma 13.1.18. *For any Banach space X and $n \in \mathbb{Z}_+$, we have*

$$\varphi^{(q)}_{p,X}(n) \leqslant \varphi^{(q)}_{p,X}(\mathbb{Z}_n) \leqslant 3 \cdot \varphi^{(q)}_{p,X}(n).$$

Proof. Substituting $e_k(t)x_k$ in place of x_k in Definition 13.1.13, we find that

$$\frac{1}{n}\sum_{h=1}^{n} \Big\| \sum_{k=1}^{n} e_k(t+h/n)x_k \Big\|^q \leqslant \big(\varphi^{(q)}_{p,X}(\mathbb{Z}_n)\big)^q \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{q/p}$$

Integrating over $t \in \mathbb{T}$ and using the translation invariance

$$\|f(\cdot + h/n)\|_{L^q(\mathbb{T};X)} = \|f\|_{L^q(\mathbb{T};X)},$$

we obtain

$$\frac{1}{n}\sum_{h=1}^{n} \int_{\mathbb{T}} \Big\| \sum_{k=1}^{n} e_k(t)x_k \Big\|^q \,\mathrm{d}t \leqslant \big(\varphi^{(q)}_{p,X}(\mathbb{Z}_n)\big)^q \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{q/p},$$

and hence $\varphi_{p,X}^{(q)}(n) \leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_n)$.

The other estimate follows at once from the Marcinkiewicz inequality (Proposition 13.1.17), which is the first step in

$$\begin{aligned}\Big(\frac{1}{n}\sum_{h=1}^{n}\Big\|\sum_{k=1}^{n} e_k(h/n)x_k\Big\|^q\Big)^{1/q} &\leqslant 3\Big\|\sum_{k=1}^{n} e_k x_k\Big\|_{L^q(\mathbb{T};X)} \\ &\leqslant 3\varphi_{p,X}^{(q)}(n)\Big(\sum_{k=1}^{n}\|x_k\|^p\Big)^{1/p}.\end{aligned}$$

□

The following lemma is our reason for considering the quantities $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$:

Lemma 13.1.19. *For any Banach space X and $m, n \in \mathbb{Z}_+$, we have the sub-multiplicative estimate*

$$\varphi_{p,X}^{(q)}(\mathbb{Z}_{mn}) \leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_m)\varphi_{p,X}^{(q)}(\mathbb{Z}_n), \qquad 1 \leqslant p \leqslant q \leqslant \infty;$$

in particular

$$\varphi_{p,X}(\mathbb{Z}_{mn}) \leqslant \varphi_{p,X}(\mathbb{Z}_m)\varphi_{p,X}(\mathbb{Z}_n) \qquad \forall p \in [1,2].$$

Proof. The second estimate is an obvious special case with $q = p' \geqslant 2 \geqslant p$. For the proof of the general estimate, it is convenient to observe that, by simple reindexing and modular arithmetic, the condition defining $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ is unchanged if instead of $\{1,\dots,n\}$ we take all sums over $\{0,\dots,n-1\}$. In the defining condition of the constant $\varphi_{p,X}^{(q)}(\mathbb{Z}_{mn})$, we should then sum over $\{0,\dots,mn-1\}$, and the key trick of the proof is to use a non-symmetric reindexing of this range for the h and k sums, namely

$$\begin{aligned} h &= an+b: \quad a = 0,\dots,m-1, \quad b = 0,\dots,n-1,\\ k &= um+v: \quad u = 0,\dots,n-1, \quad v = 0,\dots,m-1.\end{aligned}$$

Then

$$hk = (an+b)(um+v) = aumn + avn + bum + bv,$$

and hence, noting that $e^{2\pi i au} = 1$,

$$e_k(h/mn) = e_u(b/n)e_v(a/m)e_v(b/mn).$$

Thus

$$\begin{aligned}&\Big\{\frac{1}{mn}\sum_{h=0}^{mn-1}\Big\|\sum_{k=0}^{mn-1} e_k(h/mn)x_k\Big\|^q\Big\}^{1/q} \\ &= \Big\{\frac{1}{n}\sum_{b=0}^{n-1}\frac{1}{m}\sum_{a=0}^{m-1}\Big\|\sum_{v=0}^{m-1} e_v(a/m)y_v^{(b)}\Big\|^q\Big\}^{1/q},\end{aligned}$$

$$y_v^{(b)} := e_v(b/mn) \sum_{u=0}^{n-1} e_u(b/n) x_{um+v},$$

$$\leqslant \Big\{ \frac{1}{n} \sum_{b=0}^{n-1} \varphi_{p,X}^{(q)}(\mathbb{Z}_m)^q \Big(\sum_{v=0}^{m-1} \|y_v^{(b)}\|^p \Big)^{q/p} \Big\}^{1/q}$$

$$\leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_m) \Big\{ \sum_{v=0}^{m-1} \Big(\frac{1}{n} \sum_{b=0}^{n-1} \Big\| \sum_{u=0}^{n-1} e_u(b/n) x_{um+v} \Big\|^q \Big)^{p/q} \Big\}^{1/p}$$

by Minkowski's inequality for $p \leqslant q$,

$$\leqslant \varphi_{p,X}^{(q)}(\mathbb{Z}_m) \Big\{ \sum_{v=0}^{m-1} \varphi_{p,X}^{(q)}(\mathbb{Z}_n)^p \sum_{u=0}^{n-1} \|x_{um+v}\|^p \Big\}^{1/p}$$

$$= \varphi_{p,X}^{(q)}(\mathbb{Z}_m) \varphi_{p,X}^{(q)}(\mathbb{Z}_n) \Big\{ \sum_{k=0}^{mn-1} \|x_k\|^p \Big\}^{1/p},$$

where we used the defining condition for $\varphi_{p,X}^{(q)}(\mathbb{Z}_m)$ with the sequences $(y_v^{(b)})_{v=0}^{m-1}$ for each fixed $b = 0, \dots, n-1$, and that for $\varphi_{p,X}^{(q)}(\mathbb{Z}_n)$ with the sequences $(x_{mu+v})_{u=0}^{n-1}$ for each fixed $v = 0, \dots, m-1$. □

Combining the above results with Corollary 13.1.12 of Hinrichs's inequality, we achieve the desired power-type improvement over the trivial estimate $\varphi_{2,X}(N) \leqslant N^{1/2}$. One could try to deduce this from Lemma 7.3.19 applied to $\varphi_{2,X}(\mathbb{Z}_n)$. However, this time that does not work since we do not know whether $\varphi_{2,X}(\mathbb{Z}_n)$ is increasing in n. Therefore, we adapt the proof of the lemma and use the facts that $\varphi_{2,X}(n)$ is increasing and that $\varphi_{2,X}(\mathbb{Z}_n)$ is submultiplicative. Our choice of notation r' below is indicative of the fact that this is the Hölder conjugate of a (small) exponent $r > 1$.

Corollary 13.1.20. *Let X be a Banach space of type $p \in (1, 2]$. Then*

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/2-1/r'} = C \cdot N^{1/r-1/2},$$

where

$$r' := 3p'(68 \cdot \tau_{p,X;2})^{p'}, \qquad C := e^{\frac{r'}{2p'}}. \tag{13.10}$$

Proof. Given $N, n \in \mathbb{Z}_+$, let $k \in \mathbb{Z}_+$ satisfy

$$3^{n(k-1)} \leqslant N < 3^{nk}. \tag{13.11}$$

Then

$$\begin{aligned} \varphi_{2,X}(N) &\leqslant \varphi_{2,X}(3^{nk}) \quad \text{since } \varphi_{2,X} \text{ is increasing by Lemma 13.1.4,} \\ &\leqslant \varphi_{2,X}(\mathbb{Z}_{3^{nk}}) \quad \text{by the comparison in Lemma 13.1.18,} \end{aligned}$$

$$\leqslant \varphi_{2,X}(\mathbb{Z}_{3^n})^k \quad \text{by sub-multiplicativity (Lemma 13.1.19)},$$
$$\leqslant (3 \cdot \varphi_{2,X}(3^n))^k \quad \text{by the comparison in Lemma 13.1.18}.$$

Therefore, by (13.11) for any $s \in (1,2]$ we find

$$\begin{aligned} N^{1/2-1/s}\varphi_{2,X}(N) &\leqslant 3^{n(k-1)(\frac{1}{2}-\frac{1}{s})}(3 \cdot \varphi_{2,X}(3^n))^k \\ &= 3^{n(\frac{1}{s}-\frac{1}{2})}[3^{n(\frac{1}{2}-\frac{1}{s})} \cdot 3 \cdot \varphi_{2,X}(3^n)]^k \end{aligned}$$

For appropriate n and s, we will show that the term within brackets satisfies $[\dots] \leqslant 1$. By Corollary 13.1.12, we can estimate

$$[\dots] \leqslant 3^{n(\frac{1}{2}-\frac{1}{s})} \cdot 3 \cdot 16\sqrt{2} \cdot \tau_{p,X;2} \cdot 3^{n/2} \cdot n^{-1/p'} =: 3^{n/s'}Tn^{-1/p'},$$

where $T := 48\sqrt{2}\tau_{p,X;2}$. Therefore, setting $s' = (1+eT^{p'})p'\log(3)$ and taking $eT^{p'} \leqslant n < eT^{p'}+1$ we find that

$$3^{n/s'}Tn^{-1/p'} \leqslant e^{1/p'}T\frac{e^{-1/p'}}{T} = 1.$$

From the above we conclude that

$$N^{1/2-1/s}\varphi_{2,X}(N) \leqslant 3^{n(\frac{1}{s}-\frac{1}{2})} \leqslant 3^{n/2} = e^{n\log(3)/2} \leqslant e^{s'/(2p')}.$$

The above trivially holds true if we replace s' by any $r' > s'$. Since $50 \leqslant T \leqslant 68\tau_{p,X;2}$ and $p' \geqslant 2$, one can check that

$$s' = (1+eT^{p'})p'\log(3) = T^{p'}(T^{-p'}+e)p'\log(3) \leqslant (68\tau_{p,X;2})^{p'}3p' =: r'.$$

Thus the statement follows. □

Before turning to some of the sophisticated constructions and estimates for Bourgain's theorem, we discuss a much simpler situation where one can obtain $\varphi_{2,X}(n) = O(n^{1/r-1/2})$ with $r \in (1,2)$. It does not play a role in the proof of Bourgain's theorem.

Proposition 13.1.21. *If X has type p and cotype q, then for all $n \geqslant 1$,*

$$\varphi_{2,X}(n) \leqslant \tau_{2,X}(n)c_{2,X}(n) \leqslant \tau_{p,X;2}c_{q,X;2}n^{1/p-1/q}.$$

Of course the latter bound is nontrivial only if $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$.

Proof. Let $(\gamma_h)_{h\geqslant 1}$ be a complex Gaussian sequence (i.e., standard independent Gaussian random variables). Also let

$$\widetilde{\gamma}_k = \frac{1}{\sqrt{n}}\sum_{h=1}^{n}\gamma_h e_k(\frac{h}{n}), \quad k = 1,\dots,n.$$

Then $(\widetilde{\gamma}_k)_{k=1}^n$ are also independent standard Gaussian random variables (see Section E.2). Hence, using the natural Gaussian analogue of the finite type and cotype constants,

$$\begin{aligned}\frac{1}{n}\sum_{h=1}^{n}\Big\|\sum_{k=1}^{n}x_k e_k(\frac{h}{n})\Big\|^2 &\leqslant \frac{1}{n}c_{2,X}^{\gamma}(n)^2\mathbb{E}\Big\|\sum_{h=1}^{n}\gamma_h\sum_{k=1}^{n}x_k e_k(\frac{h}{n})\Big\|^2\\ &= c_{2,X}^{\gamma}(n)^2\mathbb{E}\Big\|\sum_{k=1}^{n}\widetilde{\gamma}_k x_k\Big\|^2\\ &\leqslant c_{2,X}^{\gamma}(n)^2\tau_{2,X}^{\gamma}(n)^2\sum_{k=1}^{n}\|x_k\|^2.\end{aligned}$$

Since $\|\gamma\|_2=1$, Proposition 7.1.18 informs us that $\tau_{2,X}^{\gamma}\leqslant\tau_{2,X}$ and $c_{2,X}^{\gamma}\leqslant c_{2,X}$, and the analogous result for the finite constants $\tau_{2,X}^{\gamma}(n)$ etc. follows by the same argument. Finally, Hölder's inequality implies $\tau_{2,X}(n)\leqslant\tau_{p,X;2}n^{\frac{1}{p}-\frac{1}{2}}$ and $c_{2,X}(n)\leqslant c_{q,X;2}n^{\frac{1}{2}-\frac{1}{q}}$. □

13.1.c Key lemmas for an initial uniform bound

The core of this section consists of two delicate lemmas of Bourgain that allow us to bootstrap the power-type improvement over the trivial bound on the growth of $\varphi_{2,X}(N)$, as given in Corollary 13.1.20, into a uniform estimate for the constants $\varphi_{s,X}^{(2)}(N)$ with some $s>1$. To streamline the presentation of the core arguments, we begin with the following classical identity:

Lemma 13.1.22. *Let $f=\sum_{j\in\mathbb{Z}}\widehat{f}(j)e_j$ with $(\widehat{f}(j))_{j\in\mathbb{Z}}\in\ell^1(\mathbb{Z})$. Then*

$$\sum_{j\equiv n \bmod N}\widehat{f}(j)=\frac{1}{N}\sum_{h=1}^{N}e_{-1}(nh/N)f(h/N).$$

Proof. We first observe that

$$\frac{1}{N}\sum_{h=1}^{N}f(h/N)=\sum_{j\in\mathbb{Z}}\widehat{f}(j)\frac{1}{N}\sum_{h=1}^{N}e_j(h/N)=\sum_{j\equiv 0 \bmod N}\widehat{f}(j),$$

which is case $n=0$ of the claim.

We apply this with f replaced by

$$e_{-n}f=\sum_{j\in\mathbb{Z}}\widehat{f}(j)e_{j-n}=\sum_{j\in\mathbb{Z}}\widehat{f}(j+n)e_j$$

to find that

$$\frac{1}{N}\sum_{h=1}^{N}(e_{-n}f)(h/N)=\sum_{j\equiv 0 \bmod N}\widehat{f}(j+n)=\sum_{j\equiv n \bmod N}\widehat{f}(j),$$

which is the general case. □

Lemma 13.1.23 (Bourgain). *Let $F \subseteq \mathbb{Z}$ be a finite subset with $\# F = N$. Then there exists $t_0 \in \mathbb{T}$ such that at least $\frac{1}{8}N$ of the pairwise disjoint intervals*

$$I_n := \frac{1}{N}[n - \frac{1}{2}, n + \frac{1}{2}), \qquad n = 1, \dots, N,$$

satisfy $t_0 k \in I_n + \mathbb{Z}$ for some $k \in F$.

Proof. We in fact show that this is true for the "average" choice of $t_0 \in \mathbb{T}$. For $t \in \mathbb{T}$ and $n = 1, \dots, N$, we denote

$$\begin{aligned} \nu_n(t) &:= \#\{k \in F : tk \in I_n + \mathbb{Z}\}, \\ N(t) &:= \#\{n = 1, \dots, N : \nu_n(t) > 0\}. \end{aligned}$$

The claim is then that $N(t_0) \geqslant \frac{1}{8}N$ for some $t_0 \in \mathbb{T}$, and we will prove that

$$\int_0^1 N(t)\,\mathrm{d}t \geqslant \frac{1}{8}N, \tag{13.12}$$

which clearly implies the existence of a desired t_0.

The strategy of the proof is as follows. Since each of the N different $k \in F$ satisfies $tk \in I_n + \mathbb{Z}$ for exactly one $n = 1, \dots, N$, we have

$$N = \sum_{n=1}^{N} \nu_n(t) = \sum_{\substack{1 \leqslant n \leqslant N \\ \nu_n(t) > 0}} \nu_n(t) \leqslant N(t)^{1/2} \Big(\sum_{n=1}^{N} \nu_n(t)^2 \Big)^{1/2}.$$

Integrating and using the Cauchy–Schwarz inequality, we obtain

$$N \leqslant \Big(\int_0^1 N(t)\,\mathrm{d}t \Big)^{1/2} \Big(\int_0^1 \sum_{n=1}^{N} \nu_n(t)^2\,\mathrm{d}t \Big)^{1/2},$$

and (13.12) follows if we can prove that

$$\int_0^1 \sum_{n=1}^{N} \nu_n(t)^2\,\mathrm{d}t \leqslant 8N. \tag{13.13}$$

Now

$$\nu_n(t) = \sum_{k \in F} \mathbf{1}_{I_n + \mathbb{Z}}(kt) = \sum_{k \in F} \mathbf{1}_{I_0 + \mathbb{Z}}(kt - n/N).$$

For the convenience of Fourier analysis, we replace the indicator

$$\mathbf{1}_{I_0 + \mathbb{Z}}(t) = \mathbf{1}_{[-\frac{1}{2N}, \frac{1}{2N})}(t), \qquad t \in [-\frac{1}{2}, \frac{1}{2}),$$

by the regularised version given by the 1-periodic extension of the "tent"

$$s(t) := (1 - N|t|)_+, \qquad t \in [-\frac{1}{2}, \frac{1}{2}).$$

An elementary computation of the Fourier coefficients shows that

$$\widehat{s}(j) = \frac{1}{N}\operatorname{sinc}^2(\pi j/N) = \begin{cases} 1/N, & j = 0, \\ 0, & 0 \neq j \equiv 0 \bmod N. \end{cases} \tag{13.14}$$

Note that the first equality above is valid for all $j \in \mathbb{Z}$, although in the second we only consider particular cases. Clearly $0 \leqslant \widehat{s}(j) = O(j^{-2})$, so that Lemma 13.1.22 applies to $f = s$. Since $s(h/N) = \mathbf{1}_{N\mathbb{Z}}(h)$ for $h \in \mathbb{Z}$, the conclusion of the lemma takes a particularly clean form, namely

$$\sum_{j \equiv n \bmod N} \widehat{s}(j) = \frac{1}{N} \qquad \forall n = 1, \dots, N. \tag{13.15}$$

We observe that $\mathbf{1}_{I_0+\mathbb{Z}}(t) \leqslant 2s(t)$, and hence

$$\begin{aligned} \nu_n(t) &\leqslant 2\sum_{k\in F} s(kt - n/N) = 2\sum_{k\in F}\sum_{j\in\mathbb{Z}} \widehat{s}(j) e_j(kt - n/N) \\ &= 2\sum_{h=1}^{N} e_h(-n/N) \sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \sum_{k\in F} e_j(kt). \end{aligned}$$

Substituting this into (13.13), we can now estimate

$$\begin{aligned} \int_0^1 \sum_{n=1}^{N} \nu_n^2 &\leqslant 4\int_0^1 \sum_{n=1}^{N} \Big| \sum_{h=1}^{N} e_h(-n/N) \sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \sum_{k\in F} e_j(kt) \Big|^2 \,\mathrm{d}t \\ &= 4\int_0^1 N \sum_{h=1}^{N} \Big| \sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \sum_{k\in F} e_j(kt) \Big|^2 \,\mathrm{d}t, \\ &\qquad \text{since the matrix } (N^{-1/2} e_h(-n/N))_{h,n=1}^{N} \text{ is unitary,} \\ &\leqslant 4N \sum_{h=1}^{N} \Big(\sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) \Big\| \sum_{k\in F} e_{jk} \Big\|_{L^2(\mathbb{T})} \Big)^2 \\ &\leqslant 4N \Big\{ \sum_{h=1}^{N-1} \Big(\sum_{\substack{j\equiv h \\ \bmod N}} \widehat{s}(j) N^{1/2} \Big)^2 + \Big(\widehat{s}(0) N + \sum_{\substack{0\neq j\equiv 0 \\ \bmod N}} \widehat{s}(j) N^{1/2} \Big)^2 \Big\}, \\ &\qquad \text{since } \Big\| \sum_{k\in F} e_{jk} \Big\|_{L^2(\mathbb{T})} = \begin{cases} N, & j = 0, \\ N^{1/2}, & \text{otherwise,} \end{cases} \\ &= 4N \Big\{ \sum_{h=1}^{N-1} \Big(\frac{1}{N} N^{1/2} \Big)^2 + \Big(\frac{1}{N} N + 0 \Big)^2 \Big\}, \end{aligned}$$

by (13.15) and (13.14),

$$= 4N\{(N-1)N^{-1} + 1\} = 4\{2N-1\} < 8N = \text{RHS}\,(13.13).$$

This confirms (13.13) and hence, as explained in the beginning of the proof, the assertion of the Lemma. ☐

From the comparison between $\ell^p_N(X)$ and $\ell^\infty_N(X)$, it is immediate that

$$\varphi^{(q)}_{\infty,X}(N) \leqslant \varphi^{(q)}_{p,X}(N) \cdot N^{1/p}.$$

This triviality admits a crucial improvement, where on the left we have a similar quantity associated to an *arbitrary* subset $F \subseteq \mathbb{Z}$ of size N.

Definition 13.1.24. *Given $F \subseteq \mathbb{Z}$, we denote by $\varphi^{(q)}_{\infty,X}(F)$ be the best constant in the estimate*

$$\Big\| \sum_{k\in F} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \varphi^{(q)}_{\infty,X}(F) \sup_{k\in F} \|x_k\|,$$

which is to holds for arbitrary families $(x_k)_{k\in F}$ in X.

Clearly the previously considered $\varphi^{(q)}_{\infty,X}(N)$ is the special case $\varphi^{(q)}_{\infty,X}(N) = \varphi^{(q)}_{\infty,X}(\{1,\ldots,N\})$ in this notation. In contrast to random sums with independent sequences of random variables, the particular choice of the indexing set F is very relevant here, since the joint distribution of $(e_k)_{k\in F}$ can be very different from that of $(e_k)_{k=1}^N$.

Lemma 13.1.25 (Bourgain). *For any Banach space X and exponents $p, q \in [1,\infty)$ we have*

$$\varphi^{(q)}_{\infty,X}(F) \leqslant A\varphi^{(q)}_{p,X}(N) \cdot N^{1/p}, \qquad A := \big(8p(\pi + 2^{1/q}\cdot 3)\big)^{1+1/q},$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$.

Remark 13.1.26. We only apply Lemma 13.1.25 with $p = 2 \leqslant q$. In this case

$$A \leqslant \big(16(\pi + \sqrt{2}\cdot 3)\big)^{3/2} < 1285.$$

Proof of Lemma 13.1.25. Since

$$\Big\| \sum_{k\in F} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \sum_{k\in F} \|x_k\| \leqslant N \max_{k\in F} \|x_k\|,$$

and $\varphi^{(q)}_{p,X}(N) \geqslant 1$, we have the trivial estimate

$$\varphi^{(q)}_{\infty,X}(F) \leqslant N \leqslant \varphi^{(q)}_{p,X}(N) N^{1/p'+1/p} \leqslant A\varphi^{(q)}_{p,X}(N) N^{1/p} \qquad \forall N \leqslant A^{p'}.$$

Suppose then, for induction, that $N > A^{p'}$, and moreover that the Lemma has been verified for all $N' < N$ in place of N. For $F \subseteq \mathbb{Z}_+$ of size N, we consider a splitting (with $\varnothing \neq F_0 \subsetneq F$ to be specified shortly)

$$\Big\| \sum_{k\in F} e_k x_k \Big\|_{L^q(\mathbb{T};X)} \leqslant \Big\| \sum_{k\in F_0} e_k x_k \Big\|_{L^q(\mathbb{T};X)} + \Big\| \sum_{k\in F\setminus F_0} e_k x_k \Big\|_{L^q(\mathbb{T};X)} =: I + II.$$

Since $F \setminus F_0 \subsetneq F$ is a strictly smaller set and $\varphi^{(q)}_{p,X}$ is clearly non-decreasing, the induction hypothesis applies to show that

$$\begin{aligned} II &\leqslant \varphi^{(q)}_{\infty,X}(F \setminus F_0) \max_{k\in F\setminus F_0} \|x_k\| \\ &\leqslant A\varphi^{(q)}_{p,X}(N)\#(F \setminus F_0)^{1/p} \max_{k\in F} \|x_k\|. \end{aligned} \tag{13.16}$$

Let us make a specific choice of $F_0 \subsetneq F$ as follows. By Lemma 13.1.23, there exist $t_0 \in T$ and $1 \leqslant n_1 < n_2 < \dots n_\ell \leqslant N$ with $\ell \geqslant \frac{1}{8}\#F$ such that each of the mutually disjoint sets

$$I_{n_j} + \mathbb{Z} = \frac{1}{N}[n - \frac{1}{2}, n + \frac{1}{2}) + \mathbb{Z}, \quad (j = 1, \dots, \ell),$$

intersects with the set $\{kt_0 : k \in F\}$. For each $j \in \{1, \dots, \ell\}$, we pick a $k_j \in F$ such that $k_j t_0 \in I_{n_j} + \mathbb{Z}$, and set $F_0 := \{k_j : j = 1, \dots, \ell\}$. Then $\#F_0 = \ell \geqslant \frac{1}{8}\#F$. The size bound on $\#F_0$ shows that (13.16) implies

$$II \leqslant A\varphi^{(q)}_{p,X}(N)\Big(\frac{7}{8}N\Big)^{1/p} \max_{k\in F} \|x_k\|. \tag{13.17}$$

Let $\psi : k_j \to n_j$ be the corresponding bijection from F_0 onto $\psi(F_0) \subseteq \{1, \dots, N\}$. Thus by definition that $k_j t_0 \in I_{\psi(k_j)} + \mathbb{Z} = I_{n_j} + \mathbb{Z}$ for all $j = 1, \dots, \ell$. For any $h \in \mathbb{Z}$, we then have

$$\begin{aligned} I &= \Big\| \sum_{k\in F_0} e_k(\cdot + ht_0)x_k \Big\|_{L^q(\mathbb{T};X)} \qquad \text{by translation invariance} \\ &\leqslant \Big\| \sum_{k\in F_0} [e_k(ht_0) - e_h(\frac{\psi(k)}{N})]e_k x_k \Big\|_{L^q(\mathbb{T};X)} + \Big\| \sum_{k\in F_0} e_h(\frac{\psi(k)}{N})e_k x_k \Big\|_{L^q(\mathbb{T};X)} \\ &=: I_1(h) + I_2(h), \end{aligned}$$

where (using again the induction hypothesis, now with the smaller set $F_0 \subsetneq F$)

$$\begin{aligned} I_1(h) &\leqslant \varphi^{(q)}_{\infty,X}(F_0) \max_{k\in F_0} \Big| \exp(2\pi i k h t_0) - \exp(i2\pi h\frac{\psi(k)}{N})\Big| \|x_k\| \\ &\leqslant A\varphi^{(q)}_{p,X}(\#F_0)^{1/p} \max_{k\in F_0} \Big(2\pi|h| \inf_{j\in\mathbb{Z}} \Big|kt_0 - \frac{\psi(k)}{N} - j\Big|\Big) \max_{k\in F_0} \|x_k\| \\ &\leqslant A\varphi^{(q)}_{p,X}(N)N^{1/p}\frac{\pi|h|}{N} \max_{k\in F} \|x_k\|, \end{aligned} \tag{13.18}$$

since $kt_0 \in I_{\psi(k)} + \mathbb{Z}$.

Having estimated both $I_1(h)$ and II in terms of the induction hypothesis, the serious work is left with $I_2(h)$, which we first average over a range $h = 1, \ldots, H \leqslant N$, where a favourable value of H is to be determined. We have

$$\begin{aligned}
\frac{1}{H}\sum_{h=1}^{H} I_2(h)^q &= \frac{N}{H}\int_{\mathbb{T}} \frac{1}{N}\sum_{h=1}^{H}\Big\| \sum_{j\in\psi(F_0)} e_h(j/N)e_{\psi^{-1}(j)}(t)x_{\psi^{-1}(j)}\Big\|^q \,\mathrm{d}t \\
&\leqslant \frac{N}{H}\int_{\mathbb{T}} \frac{1}{N}\sum_{h=1}^{N}\Big\| \sum_{j=1}^{N} e_h(j/N)y_j(t)\Big\|^q \,\mathrm{d}t \\
&\qquad y_j(t) := \begin{cases} e_{\psi^{-1}(j)}(t)x_{\psi^{-1}(j)}, & j\in\psi(F_0), \\ 0, & \text{else}, \end{cases} \\
&\leqslant \frac{N}{H}\int_{\mathbb{T}} \Big(\varphi_{p,X}^{(q)}(\mathbb{Z}_N)\Big[\sum_{j=1}^{N}\|y_j(t)\|^p\Big]^{1/p}\Big)^q \,\mathrm{d}t \\
&\qquad \text{by definition of } \varphi_{p,X}^{(q)}(\mathbb{Z}_N) \\
&\leqslant \frac{N}{H}\big(3\varphi_{p,X}^{(q)}(N)\big)^q\Big(\sum_{k\in F_0}\|x_k\|^p\Big)^{q/p} \qquad \text{by Lemma 13.1.18} \\
&\leqslant \frac{N}{H}\big(3\varphi_{p,X}^{(q)}(N)\big)^q(\#F_0)^{q/p}\max_{k\in F_0}\|x_k\|^q.
\end{aligned}$$

Combining the previous bound with (13.17) and (13.18), we have

$$\begin{aligned}
\Big\|\sum_{k\in F} e_k x_k\Big\|_{L^q(\mathbb{T};X)} &\leqslant I + II \leqslant \frac{1}{H}\sum_{h=1}^{H}(I_1(h)+I_2(h)) + II \\
&\leqslant \max_{1\leqslant h\leqslant H} I_1(h) + \Big(\frac{1}{H}\sum_{h=1}^{H} I_2(h)^q\Big)^{1/q} + II \\
&\leqslant \Big(A\frac{\pi H}{N} + 3\Big(\frac{N}{H}\Big)^{1/q} + A\Big(\frac{7}{8}\Big)^{1/p}\Big)N^{1/p}\varphi_{p,X}^{(q)}(N)\max_{k\in F}\|x_k\|,
\end{aligned} \tag{13.19}$$

where $N = \#F$, as we recall. We now choose H so as to essentially equate the first two terms:

$$H := \lfloor H' \rfloor, \qquad H' := A^{-q/(q+1)}N.$$

Since $A > 1$, we have $H \leqslant H' \leqslant N$. Recalling that $N > A^{p'}$, and noting that $p' > 1 > q/(q+1)$, we also observe that $H' \geqslant 1$, and hence $H \geqslant 1$. Thus this choice of H lies in the admissible range considered above. We also have $H' \leqslant H + 1 \leqslant 2H$, and thus

$$A\frac{\pi H}{N} + 3\Big(\frac{N}{H}\Big)^{1/q} \leqslant A\frac{\pi H'}{N} + 3\Big(\frac{2N}{H'}\Big)^{1/q} = (\pi + 2^{1/q}\cdot 3)A^{1/(q+1)}.$$

We also note that

$$\left(\frac{7}{8}\right)^{1/p} - 1 = \frac{1}{p}\xi^{1/p-1}\left(\frac{7}{8} - 1\right) \leqslant -\frac{1}{8p}, \quad \text{for some } \xi \in \left(\frac{7}{8}, 1\right).$$

Substituting into (13.19), we hence have

$$\varphi^{(q)}_{\infty,X}(F) \leqslant \left[(\pi + 2^{1/q} \cdot 3) \cdot A^{1/(q+1)} + \left(1 - \frac{1}{8p}\right)A\right]\varphi^{(q)}_{p,X}(N) \cdot N^{1/p}.$$

To complete the induction step, it remains to check that the quantity in brackets is at most A, which after easy simplification is the same as

$$(\pi + 2^{1/q} \cdot 3) \cdot A^{1/(q+1)} \leqslant \frac{1}{8p}A.$$

Clearly this is the case with the choice of A stated in the Lemma. □

We are now ready for a first uniform bound on the finite Fourier type constants:

Corollary 13.1.27. *Let X be a Banach space, $r \in (1, 2]$, and suppose that*

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r-1/2} \qquad \forall N \in \mathbb{Z}_+.$$

Then for all $s \in (1, r)$, we have

$$\varphi^{(2)}_{s,X}(N) \leqslant 3500\frac{Cr}{r-s} \qquad \forall N \in \mathbb{Z}_+.$$

Proof. By Lemma 13.1.25 and Remark 13.1.26 with $p = q = 2$, we have

$$\varphi^{(2)}_{\infty,X}(F) \leqslant 1285 \cdot \varphi_{2,X}(N) \cdot N^{1/2} \leqslant 1285 \cdot C \cdot N^{1/r} \tag{13.20}$$

whenever $F \subseteq \mathbb{Z}$ has size $\#F = N$.

Let $x = (x_k)_{k=1}^N \in \ell^s_N(X)$ have norm one. For $\alpha \in (0, 1)$ to be chosen, we denote

$$F_j := \{n \in \mathbb{Z} : \alpha^j < \|x_n\| \leqslant \alpha^{j-1}\}, \qquad x^{(j)} := (\mathbf{1}_{F_j}(k) \cdot x_k)_{k=1}^N.$$

Note that $F_j = \varnothing$ and $x^{(j)} = 0$ for $j \leqslant 0$, and

$$\#F_j \leqslant \#\{n \in \mathbb{Z} : \alpha^j < \|x_n\|\} \leqslant \alpha^{-js}\|x\|_{\ell^s_N(X)} = \alpha^{-js}, \qquad j \geqslant 1.$$

Thus

$$\Big\|\sum_{k=1}^N e_k x_k\Big\|_{L^2(\mathbb{T};X)} \leqslant \sum_{j=1}^\infty \Big\|\sum_{k\in F_j} e_k x_k^{(j)}\Big\|_{L^2(\mathbb{T};X)} \leqslant \sum_{j=1}^\infty \varphi^{(2)}_{\infty,X}(F_j)\max_{k\in F_j}\|x_k^{(j)}\|$$

$$\begin{aligned}
&\leqslant \sum_{j=1}^{\infty} 1285 \cdot \varphi_{2,X}^{(2)}(\#F_j) \cdot (\#F_j)^{1/2} \cdot \alpha^{j-1} \qquad \text{by (13.20)} \\
&\leqslant \sum_{j=1}^{\infty} 1285 \cdot C(\#F_j)^{1/r} \alpha^{j-1} \leqslant \sum_{j=1}^{\infty} 1285 \cdot C\alpha^{-js/r}\alpha^{j-1} \\
&= \frac{1285 \cdot C}{\alpha} \frac{\alpha^{1-s/r}}{1-\alpha^{1-s/r}}.
\end{aligned}$$

The choice $\alpha = (s/r)^{r/(r-s)}$ gives

$$\frac{1}{\alpha}\frac{\alpha^{1-s/r}}{1-\alpha^{1-s/r}} = \frac{\alpha^{-s/r}}{1-\alpha^{1-s/r}} = \frac{(r/s)^{s/(r-s)}}{1-s/r} \leqslant \frac{e}{1-s/r}$$

by an elementary optimisation in the last step. Substituting back, this gives

$$\Big\| \sum_{k=1}^{N} e_k x_k \Big\|_{L^2(\mathbb{T};X)} \leqslant 1285 \cdot C \cdot \frac{e}{1-s/r} = 1285 \cdot e \cdot \frac{Cr}{r-s} < 3500 \cdot \frac{Cr}{r-s}$$

for all $(x_k)_{k=1}^N \in \ell_N^s(X)$ of norm one, which is the claimed bound. □

13.1.d Conclusion via duality and interpolation

With the uniform bound of Corollary 13.1.27, we have already covered the core of the deep implication from non-trivial type to non-trivial Fourier type. The rest of the argument depends on the more routine techniques of duality and interpolation, but is still not entirely straightforward. We now turn our attention to giving these finishing touches to the proof. At the end of this section, a statement and proof of Bourgain's theorem will finally be given.

The first duality that we want to use is most elegantly expressed in terms of the Fourier type constants on the cyclic group $\mathbb{Z}_N$:

Lemma 13.1.28. *Let X be a Banach space, $N \in \mathbb{Z}_+$ and $p, q \in (1,\infty)$. Then*

$$N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N) = N^{1/p'}\varphi_{q',X^*}^{(p')}(\mathbb{Z}_N).$$

Proof. Since X is norming for X^*, Proposition 1.3.1 shows that $\ell_N^p(X)$ is norming for $\ell_N^{p'}(X^*)$, so that

$$\begin{aligned}
&\Big(\sum_{h=1}^{N} \Big\| \sum_{k=1}^{N} e_k(h/N) x_k^* \Big\|^{p'} \Big)^{1/p'} \\
&\qquad = \sup\Big\{ \sum_{h=1}^{N} \Big\langle x_h, \sum_{k=1}^{N} e_k(h/N) x_k^* \Big\rangle : \Big(\sum_{h=1}^{N} \|x_h\|^p \Big)^{1/p} \leqslant 1 \Big\},
\end{aligned}$$

where, observing the symmetry $e_k(h/N) = e^{2\pi i kh/N} = e_h(k/N)$,

$$\sum_{h=1}^{N}\Big\langle x_h, \sum_{k=1}^{N} e_k(h/N)x_k^*\Big\rangle = \sum_{k=1}^{N}\Big\langle \sum_{h=1}^{N} e_h(k/N)x_h, x_k^*\Big\rangle$$
$$\leqslant \Big(\sum_{k=1}^{N}\Big\|\sum_{h=1}^{N} e_h(k/N)x_h\Big\|^q\Big)^{1/q}\Big(\sum_{k=1}^{N}\|x_k^*\|^{q'}\Big)^{1/q'}$$
$$\leqslant N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N)\Big(\sum_{h=1}^{N}\|x_h\|^p\Big)^{1/p}\Big(\sum_{k=1}^{N}\|x_k^*\|^{q'}\Big)^{1/q'}.$$

Substituting back, this proves that

$$N^{1/p'}\varphi_{q',X^*}^{(p')}(\mathbb{Z}_N) \leqslant N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N).$$

Permuting the names of the exponents and using the isometric embedding of X into X^{**}, it also follows that

$$N^{1/q}\varphi_{p,X}^{(q)}(\mathbb{Z}_N) \leqslant N^{1/q}\varphi_{p,X^{**}}^{(q)}(\mathbb{Z}_N) \leqslant N^{1/p'}\varphi_{q',X^*}^{(p')}(\mathbb{Z}_N),$$

which proves the claimed equality. □

Corollary 13.1.29. *Let X be a Banach space, $r \in (1,2]$, and suppose that*

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r-1/2} \qquad \forall N \in \mathbb{Z}_+.$$

Then for all $s \in (1,r)$ we have

$$\varphi_{\infty,X^*}^{(s')}(F) \leqslant 1.35 \cdot 10^7 \frac{Cr}{r-s} N^{1/s} \qquad \forall s \in (1,r),$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$.

Recall from Corollary 13.1.20 that if X has type $p \in (1,2]$, then the assumption is satisfied with C and r as in (13.10).

Proof. By using both estimates of Lemma 13.1.18 with Lemma 13.1.28 in between, and finally Corollary 13.1.27, we have

$$N^{1/s'}\varphi_{2,X^*}^{(s')}(N) \leqslant N^{1/s'}\varphi_{2,X^*}^{(s')}(\mathbb{Z}_N) = N^{1/2}\varphi_{s,X}^{(2)}(\mathbb{Z}_N)$$
$$\leqslant N^{1/2} \cdot 3\varphi_{s,X}^{(2)}(N) \leqslant N^{1/2} \cdot 3 \cdot 3500\frac{Cr}{r-s}.$$

Then Lemma 13.1.25 and Remark 13.1.26 with $p = 2 \leqslant q = s'$ show that

$$\varphi_{\infty,X^*}^{(s')}(F) \leqslant 1285 \cdot \varphi_{2,X^*}^{(s')}(N) \cdot N^{1/2} < 1.35 \cdot 10^7 \frac{Cr}{r-s} \cdot N^{1/2+1/2-1/s'}.$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$. □

We now come to another form of duality, where we pass from the Fourier transform on $\mathbb{Z}$ to that on the circle $\mathbb{T}$, and it is in this latter setting that our argument will be completed.

Lemma 13.1.30. *Let X be a Banach space, $1 \leqslant s \leqslant \infty$, and suppose that*

$$\varphi^{(s')}_{\infty,X^*}(F) \leqslant K \cdot N^{1/s}$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$. Then the Fourier transform

$$\mathscr{F} : f \in L^1(\mathbb{T}; X) \mapsto (\widehat{f}(k))_{k \in \mathbb{Z}}, \qquad \widehat{f}(k) = \int_{\mathbb{T}} e_{-k}(t) f(t) \, \mathrm{d}t,$$

satisfies the weak-type estimate

$$\|\mathscr{F} f\|_{\ell^{s',\infty}(\mathbb{Z};X)} \leqslant K \|f\|_{L^s(\mathbb{T};X)}. \tag{13.21}$$

Proof. Let $f \in L^s(\mathbb{T}; X)$, let $\lambda > 0$, and let F be a finite subset of $\{k \in \mathbb{Z} : \|\widehat{f}(k)\| > \lambda\}$. (By a periodic analogue of the Riemann–Lebesgue Lemma 2.4.3, which has essentially the same proof, we could argue that this set is finite to begin with, but we do not need this here.) Then

$$\begin{aligned}
\#F &\leqslant \frac{1}{\lambda} \sum_{k \in F} \|\widehat{f}(k)\| = \frac{1}{\lambda} \sum_{k \in F} \langle \widehat{f}(k), x^*_{-k} \rangle \\
&\qquad \text{for suitable } x^*_{-k} \in X^* \text{ of norm one} \\
&= \frac{1}{\lambda} \int_{\mathbb{T}} f(t) \Big(\sum_{k \in F} e_{-k}(t) x^*_{-k} \Big) \, \mathrm{d}t \\
&\leqslant \frac{1}{\lambda} \|f\|_{L^s(\mathbb{T};X)} \Big\| \sum_{k \in -F} e_k x_k \Big\|_{L^{s'}(\mathbb{T};X^*)} \\
&\leqslant \frac{1}{\lambda} \|f\|_{L^s(\mathbb{T};X)} \varphi^{(s')}_{\infty,X^*}(-F) \leqslant \frac{1}{\lambda} \|f\|_{L^s(\mathbb{T};X)} K (\#F)^{1/s},
\end{aligned}$$

and hence

$$\lambda (\#F)^{1-1/s} \leqslant K \|f\|_{L^s(\mathbb{T};X)}.$$

Since this is true for any finite $F \subseteq \{k \in \mathbb{Z} : \|\widehat{f}(k)\| > \lambda\}$, it is also true for $F = \{k \in \mathbb{Z} : \|\widehat{f}(k)\| > \lambda\}$ (showing, *a posteriori*, the finiteness of this set). Then the supremum over $\lambda > 0$ of the left-hand side is precisely the $\ell^{s',\infty}(\mathbb{Z}; X)$ norm that we wanted to estimate. □

From (13.21) and the trivial fact that $\mathscr{F}$ is bounded from $L^1(\mathbb{T}; X) \to \ell^\infty(\mathbb{Z}; X)$, it seems apparent that we should conclude that $\mathscr{F}$ is bounded from $L^p(\mathbb{T}; X)$ to $\ell^{p'}(\mathbb{Z}; X)$ by interpolation. However, the version of the Marcinkiewicz Interpolation Theorem 2.2.3 covered in the text is not sufficient for this purpose, and we would need the generalisation stated in the Notes as Theorem 2.7.5. We will give a proof of a quantitative version of the special case relevant for the present application:

Lemma 13.1.31. *Let X be a Banach space such that (13.21) holds for some $s \in (1,2]$. Then*

$$\|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)} \leqslant \frac{3K}{(s-t)^{1/t'}} \|f\|_{L^t(\mathbb{T};X)} \qquad \forall t \in (1,s).$$

Proof. By homogeneity we may assume that $\|f\|_{L^t(\mathbb{T};X)} = 1$. We have

$$\begin{aligned} \|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)} &= \int_0^\infty t'\lambda^{t'-1} \#\{k : \|\widehat{f}(k)\| > \lambda\} \,\mathrm{d}\lambda \\ &\leqslant \int_0^\infty t'\lambda^{t'-1} \#\{k : \|\widehat{f_\lambda}(k)\| > \theta_0\lambda\} \,\mathrm{d}\lambda \\ &\qquad + \int_0^\infty t'\lambda^{t'-1} \#\{k : \|\widehat{f^\lambda}(k)\| > \theta_1\lambda\} \,\mathrm{d}\lambda, \end{aligned} \tag{13.22}$$

where $\theta_0 + \theta_1 = 1$ and, with parameters A and γ to be chosen shortly,

$$f_\lambda := f \cdot \mathbf{1}_{\{\|f\|_X \leqslant A\lambda^\gamma\}}, \qquad f^\lambda := f \cdot \mathbf{1}_{\{\|f\|_X > A\lambda^\gamma\}}.$$

Then

$$\|f^\lambda\|_{L^1(\mathbb{T};X)} = \int_{\{\|f\|_X > A\lambda^\gamma\}} \|f\|_X \leqslant (A\lambda^\gamma)^{1-t} \|f\|_{L^t(\mathbb{T};X)}^t = (A\lambda^\gamma)^{1-t}$$

and hence

$$\|\widehat{f^\lambda}\|_{\ell^\infty(\mathbb{Z};X)} \leqslant (A\lambda^\gamma)^{1-t} \leqslant \theta_1\lambda,$$

provided that we choose

$$\gamma = -1/(t-1), \qquad A = \theta_1^{-1/(t-1)}.$$

Then the second term on the right of (13.22) vanishes, and subsequently

$$\begin{aligned} \|\mathscr{F} f\|_{\ell^{t'}(\mathbb{Z};X)}^{t'} &\leqslant \int_0^\infty t'\lambda^{t'-1} \#\{k : \|\widehat{f_\lambda}(k)\| > \theta_0\lambda\} \,\mathrm{d}\lambda \\ &\leqslant \int_0^\infty t'\lambda^{t'-1} (\theta_0\lambda)^{-s'} K^{s'} \|f_\lambda\|_{L^s(\mathbb{T};X)}^{s'} \,\mathrm{d}\lambda \\ &\qquad \text{by Lemma 13.1.30} \\ &= t'\Big(\frac{K}{\theta_0}\Big)^{s'} \Big(\int_0^\infty \lambda^{t'-s'-1} \|f_\lambda\|_{L^s(\mathbb{T};X)}^{s'} \,\mathrm{d}\lambda\Big)^{s'/s'} \\ &\leqslant t'\Big(\frac{K}{\theta_0}\Big)^{s'} \Big\| \Big(\int_0^\infty \lambda^{t'-s'-1} \|f_\lambda\|_X^{s'} \,\mathrm{d}\lambda\Big)^{1/s'} \Big\|_{L^s(\mathbb{T})}^{s'} \\ &\qquad \text{by Minkowski's inequality with exponents } s \leqslant s' \\ &= t'\Big(\frac{K}{\theta_0}\Big)^{s'} \Big\| \Big(\int_0^{(A/\|f\|_X)^{t-1}} \lambda^{t'-s'-1} \|f\|_X^{s'} \,\mathrm{d}\lambda\Big)^{1/s'} \Big\|_{L^s(\mathbb{T})}^{s'} \end{aligned}$$

$$\text{keeping in mind the choice } \gamma = -1/(t-1)$$
$$= t'\Big(\frac{K}{\theta_0}\Big)^{s'}\Big\|\Big(\frac{1}{t'-s'}\Big[\frac{A}{\|f\|_X}\Big]^{(t-1)(t'-s')}\|f\|_X^{s'}\Big)^{1/s'}\Big\|_{L^s(\mathbb{T})}^{s'}$$
$$= t'\Big(\frac{K}{\theta_0}\Big)^{s'}\frac{1}{\theta_1^{t'-s'}}\frac{1}{t'-s'}\Big\|\Big(\|f\|_X^{s'-(t-1)(t'-s')}\Big)^{1/s'}\Big\|_{L^s(\mathbb{T})}^{s'},$$

where, observing that $tt' = t + t'$, we have

$$s' - (t-1)(t'-s') = s' - (t + t' - ts' - t' + s') = t(s'-1),$$

so that

$$\Big\|\Big(\|f\|_X^{s'-(t-1)(t'-s')}\Big)^{1/s'}\Big\|_{L^s(\mathbb{T})}^{s'} = \Big\|\|f\|_X^{t/s}\Big\|_{L^s(\mathbb{T})}^{s'} = \|f\|_{L^t(\mathbb{T};X)}^{ts'/s} = 1.$$

Taking $\theta_0 = \theta_1 = \frac{1}{2}$ and using $(t')^{1/t'} \leqslant e^{1/e} < \frac{3}{2}$, we obtain

$$\|\mathscr{F}f\|_{\ell^{t'}(\mathbb{Z};X)} \leqslant \frac{2(t')^{1/t'}K^{s'/t'}}{(t'-s')^{1/t'}} \leqslant \frac{3K^{s'/t'}}{(t'-s')^{1/t'}}.$$

Testing (13.21) with a constant function $f \equiv x$, with Fourier coefficients $\widehat{f}(k) = \delta_{k,0}x$, shows that $K \geqslant 1$ and hence $K^{s'/t'} \leqslant K$. Moreover,

$$t' - s' = \frac{t}{t-1} - \frac{s}{s-1} = \frac{t(s-1) - s(t-1)}{(s-1)(t-1)} = \frac{s-t}{(s-1)(t-1)} \geqslant s - t,$$

and hence

$$\|\mathscr{F}f\|_{\ell^{t'}(\mathbb{Z};X)} \leqslant \frac{3K}{(s-t)^{1/t'}}.$$

□

Lemma 13.1.32. *Let X be a Banach space, and suppose that there are constants C and $r \in (1,2]$ such that*

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r-1/2}$$

for all $N \in \mathbb{Z}_+$. Then for all $t \in (1,r)$, we have

$$\varphi_{t,X} \leqslant \frac{10^9 \cdot C}{(r-t)^{1+1/t'}}.$$

Proof. By Corollary 13.1.29, for all $s \in (1,r)$, we then have

$$\varphi_{\infty,X^*}^{(s')}(F) \leqslant 1.35 \cdot 10^7 \frac{Cr}{r-s} N^{1/s} =: K \cdot N^{1/s}$$

whenever $F \subseteq \mathbb{Z}$ is a subset of size $\#F = N$.

By Lemma 13.1.30, it follows that

$$\|\mathscr{F}\|_{\mathscr{L}(L^s(\mathbb{T};X),\ell^{s',\infty}(\mathbb{Z};X))} \leqslant K,$$

which by Lemma 13.1.31 implies

$$\varphi_{t,X}(\mathbb{T}) := \|\mathscr{F}\|_{\mathscr{L}(L^t(\mathbb{T};X),\ell^{t'}(\mathbb{Z};X))} \leqslant \frac{3K}{(s-t)^{1/t'}} \leqslant \frac{5\cdot 10^7 \cdot C \cdot r}{(r-s)(s-t)^{1/t'}}$$

for all $1 < t < s < r$. Optimising the bound with respect to s in this range, we choose

$$s = \frac{t^2 + (t-1)r}{2t-1}.$$

With this choice, a computation shows that

$$r - s = \frac{t(r-t)}{2t-1} \geqslant \frac{1}{3}(r-t), \quad s - t = \frac{(r-t)(t-1)}{2t-1} \geqslant \frac{1}{3}(r-t)(t-1).$$

Substituting back,

$$\varphi_{t,X}(\mathbb{T}) \leqslant 5\cdot 10^7 \cdot C \cdot r \frac{3^{1+1/t'}}{(r-t)^{1+1/t'}(t-1)^{1/t'}},$$

where $r \leqslant 2$ and $3^{1+1/t'} \leqslant 3^{3/2}$ and, for $t \in (1,2)$,

$$(t-1)^{1/t'} = [(t-1)^{t-1}]^{1/t} \geqslant [e^{-1/e}]^{1/t} \geqslant e^{-1/e}.$$

Thus

$$\varphi_{t,X}(\mathbb{T}) \leqslant 10^8 \cdot C \frac{3^{3/2}\cdot e^{1/e}}{(r-t)^{1+1/t'}} \leqslant \frac{10^9 \cdot C}{(r-t)^{1+1/t'}}.$$

□

We are finally ready for the main theorem:

Theorem 13.1.33 (Bourgain). *A Banach space X has non-trivial type if and only if it has non-trivial Fourier-type. Quantitatively:*

(1) *If X has Fourier-type $t \in (1,2]$, then it has type t with $\tau_{t,X} \leqslant \varphi_{t,X}(\mathbb{Z})$.*
(2) *If X has type $p \in (1,2]$ with related constant $\tau_{p,X;2}$ as defined in (13.7), then it has Fourier-type*

$$t = 1 + \frac{1}{6p'(68\cdot\tau_{p,X;2})^{p'}}$$

with constants

$$\varphi_{t,X}(\mathbb{R}) \leqslant \varphi_{t,X}(\mathbb{T}) \leqslant \exp\big(2(68\cdot\tau_{p,X;2})^{p'}\big).$$

Proof. (1): This is contained in Proposition 7.3.6.

(2): This is the main part of the proof, and depends on the results developed in the section. By Corollary 13.1.20, the assumptions imply that

$$\varphi_{2,X}(N) \leqslant C \cdot N^{1/r-1/2},$$

where, denoting $T := (68 \cdot \tau_{p,X;2})^{p'} \geqslant 68^2 > 4\,000$, we have

$$r' = 3p'T, \quad C = e^{\frac{r'}{2p'}} = e^{\frac{3}{2}T}.$$

Thus Lemma 13.1.32 shows that

$$\varphi_{t,X}(\mathbb{T}) \leqslant \frac{10^9 \cdot C}{(r-t)^{1+1/t'}}, \quad t \in (1,r),$$

where $r \geqslant 1 + (3p'T)^{-1}$. Hence, choosing $t := 1 + (6p'T)^{-1} \in (1,r)$, we have

$$r - t \geqslant (6p'T)^{-1}, \quad (r-t)^{1+1/t'} \leqslant (6p'T)^{\frac{3}{2}}.$$

Thus, noting that $p' \leqslant p' \log(68\tau_{p,X;2}) = \log T$, where $T \geqslant 68^2 > 4\,000$,

$$\begin{aligned} \varphi_{t,X}(\mathbb{T}) &\leqslant 10^9 \cdot e^{\frac{3}{2}T} \cdot (6p'T)^{\frac{3}{2}} \\ &= 10^9 \cdot 6^{\frac{3}{2}} \cdot (\log T)^{\frac{3}{2}} \cdot T^{\frac{3}{2}} \cdot e^{\frac{3}{2}T} \\ &\leqslant e^{\frac{1}{6}T} \cdot e^{\frac{1}{6}T} \cdot e^{\frac{1}{6}T} \cdot e^{\frac{3}{2}T} = e^{2T}. \end{aligned}$$

Finally, $\varphi_{t,X}(\mathbb{R}) \leqslant \varphi_{t,X}(\mathbb{T})$ is part of Propositions 13.1.1. □

Example 13.1.34. For each $r \in [2,\infty)$, the space $X = L^r(S)$ has type 2 with $\tau_{2,X;2} = \kappa_{r,2,\mathbb{K}}$ (the Kahane–Khintchine constant from the scalar-valued case of Theorem 6.2.4), but Fourier-type t if and only if $t \in [1,r']$. Hence, any estimate of the Fourier-type exponent in terms of the type of X must necessarily depend not only on the type exponent but also on the type constant of X.

Proof. The estimate $\tau_{2,X;2} \leqslant \kappa_{r,2,\mathbb{K}}$ follows from

$$\begin{aligned} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;L^r(S))} &\leqslant \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^r(\Omega;L^r(S))} = \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^r(S;L^r(\Omega))} \\ &\leqslant \kappa_{r,2,\mathbb{K}} \Big\| \{x_n\}_{n=1}^N \Big\|_{L^r(S;\ell^2_N)} \leqslant \kappa_{r,2,\mathbb{K}} \Big\| \{x_n\}_{n=1}^N \Big\|_{\ell^2_N(L^r(S))}. \end{aligned}$$

For the reverse estimate, it suffices to pick some non-zero $\phi \in L^r(S)$ and observe that the type inequality for $x_n = a_n\phi \in X$ implies the Kahane–Khintchine inequality for $a_n \in \mathbb{K}$.

The fact that X has Fourier-type t if $t \in [1,r']$ follows from the scalar-valued Hausdorff–Young inequality and Minkowski's inequality:

$$\|\widehat{f}\|_{L^{t'}(\mathbb{R};L^r(S))} \leqslant \|\widehat{f}\|_{L^r(S;L^{t'}(\mathbb{R}))} \leqslant \|f\|_{L^r(S;L^t(\mathbb{R}))} \leqslant \|f\|_{L^t(\mathbb{R};L^r(S))}$$

We indicate two alternative proofs of the "only if" part:

(1) In Example 2.1.15, it is verified directly that the Fourier transform is not bounded from $L^p(\mathbb{R};\ell^{r'})$ to $L^{p'}(\mathbb{R};\ell^{r'})$ for $p \in (r',2]$. By duality, it is also not bounded from $L^p(\mathbb{R};\ell^r)$ to $L^{p'}(\mathbb{R};\ell^r)$.
(2) Proposition 7.3.6 says that if X has Fourier type p, then it has cotype p'. But Corollary 7.1.6 says that $L^r(S)$ has cotype p' only for $p' \in [r,\infty]$.

This concludes the verification of the example. □

We also record the following simpler variant, which is nevertheless sufficient for many purposes:

Proposition 13.1.35. *Let X have type p and cotype q, where $\frac{1}{p}-\frac{1}{q}<\frac{1}{2}$. Let*

$$\frac{1}{r} := \frac{1}{2}+\frac{1}{p}-\frac{1}{q} \in \Big[\frac{1}{2},1\Big).$$

Then X has every Fourier-type $t \in (1,r)$, and

$$\varphi_{t,X}(\mathbb{R}) \leqslant \varphi_{t,X}(\mathbb{T}) \leqslant 10^9 \frac{\tau_{p,X;2}c_{q,X;2}}{(r-t)^{1+1/t'}}$$

Proof. By Proposition 13.1.21, we have

$$\varphi_{2,X}(N) \leqslant N^{\frac{1}{p}-\frac{1}{q}} = N^{\frac{1}{r}-\frac{1}{2}}, \quad C := \tau_{p,X;2}c_{q,X;2}$$

Thus Lemma 13.1.32 implies the bound for $\varphi_{t,X}(\mathbb{T})$, and Proposition 13.1.1 the bound for $\varphi_{t,X}(\mathbb{R})$. □

Remark 13.1.36. The assumptions of Proposition 13.1.35 are satisfied by many "common" spaces of nontrivial type (and hence finite cotype). Namely, such space often have type *or* cotype 2, and hence either $\frac{1}{p}-\frac{1}{q}=\frac{1}{2}-\frac{1}{q}<\frac{1}{2}$ or $\frac{1}{p}-\frac{1}{q}=\frac{1}{p}-\frac{1}{2}<1-\frac{1}{2}=\frac{1}{2}$.

13.2 Fourier multipliers as singular integrals

The goal of this section is to see how the results on singular integrals proved above can be applied to the theory Fourier multipliers developed in Sections 5.3 and 5.5. Given $m \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$, we recall that the operator T_m is a priori defined as $T_m : \check{L}^1(\mathbb{R}^d;X) \to \check{L}^1(\mathbb{R}^d;Y)$ by

$$T_m f(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi)e^{2\pi i\xi\cdot x}\,\mathrm{d}\xi.$$

The notation $\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ stands for the space of all $m \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ for which T_m extends to a bounded linear operator from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$. The connection of Fourier multipliers to integral operators is particularly simple in the following special case:

Proposition 13.2.1. *Let X, Y be Banach spaces and $m \in L^\infty_{\rm c}(\mathbb{R}^d; \mathscr{L}(X,Y))$. Then for all $f \in L^1 \cap \check{L}^1(\mathbb{R}^d; X)$, we have*

$$T_m f(x) = \int_{\mathbb{R}^d} k(x-y) f(y)\,\mathrm{d}y,$$

where $k = \check{m} \in \check{L}^1(\mathbb{R}^d; \mathscr{L}(X,Y))$.

Proof. Under these assumptions, we can make a direct computation

$$\begin{aligned} T_m f(x) &= \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi) e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} m(\xi)\Big(\int_{\mathbb{R}^d} f(y)e^{-2\pi i y\cdot\xi}\,\mathrm{d}y\Big)e^{2\pi i x\cdot\xi}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} m(\xi)e^{2\pi i(x-y)\cdot\xi}\,\mathrm{d}\xi\Big) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d}\check{m}(x-y)f(y)\,\mathrm{d}y, \end{aligned}$$

where the first step is the definition of T_m for $f \in \check{L}^1(\mathbb{R}^d; X)$, the second is the definition of $\widehat{f}(\xi)$ for $f \in L^1(\mathbb{R}^d; X)$, the third is Fubini's theorem that applies since both $m \in L^1(\mathbb{R}^d; \mathscr{L}(X,Y))$ and $f \in L^1(\mathbb{R}^d; X)$, and the fourth is the definition of the inverse Fourier transform of $m \in L^1(\mathbb{R}^d; \mathscr{L}(X,Y))$. □

The compact support assumption on m in Proposition 13.2.1 is not as restrictive as it may seem at first sight, as one can often reduce considerations to this situation by simple limiting arguments that we shortly explain. Recall from Definition 5.5.20 that $\psi \in \mathscr{S}(\mathbb{R}^d)$ is called a smooth Littlewood–Paley function if

(i) $\widehat{\psi}$ is smooth, non-negative, and supported in $\{\xi \in \mathbb{R}^d : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$;
(ii) $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Such functions exist by Lemma 5.5.21, whose proof also gives the identity $\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)$ and hence

$$\sum_{L<j\leqslant N} \widehat{\psi}(2^{-j}\xi) = \widehat{\varphi}(2^{-N}\xi) - \widehat{\varphi}(2^{-L}\xi)$$

for some $\widehat{\varphi} \in \mathscr{D}(\mathbb{R}^d)$ with $\widehat{\varphi}(0) = \int \varphi = 1$. Let

$$\begin{aligned} m_j(\xi) &:= \widehat{\psi}(2^{-j}\xi)m(\xi), \quad m^N(\xi) := \widehat{\varphi}(2^{-N}\xi)m(\xi), \\ m^N_L(\xi) &:= m^N(\xi) - m^L(\xi) = \sum_{L<j\leqslant N} m_j(\xi), \end{aligned} \tag{13.23}$$

and observe that $m^N \in L^\infty_{\rm c}(\mathbb{R}^d; \mathscr{L}(X,Y))$, whereas

$$m_j, m^N \in L^\infty_{\rm c}(\mathbb{R}^d \setminus \{0\}; \mathscr{L}(X,Y)),$$

i.e., these are supported away from both the origin and infinity. While the support away from zero is not required by Proposition 13.2.1, it is a convenience for forthcoming considerations due to the special role of the origin in various multiplier conditions. The next two lemmas describe a precise sense in which, for many purposes, it is "enough" to study the truncated multipliers m^N.

Lemma 13.2.2. *Let X, Y be Banach spaces and $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$. For $p \in (1,\infty)$, we have $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$, if and only if $m^N \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ uniformly in N, if and only if $m_L^N \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ uniformly in M and N.*

Proof. By the algebra of multipliers (Lemma 5.3.2), we have

$$T_{m^N} f = T_m(T_{\widehat{\varphi}(2^{-N}\cdot)} f) = T_m(\varphi_{2^{-N}} * f),$$

where $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$ and

$$\|\varphi_t * f\|_p \leqslant \|\varphi_t\|_1 \|f\|_p = \|\varphi\|_1 \|f\|_p,$$

so that $\|m^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \|\varphi\|_1 \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}$, and thus

$$\|m_L^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant 2\|\varphi\|_1 \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}.$$

On the other hand, it is evident from property (ii) of Littlewood–Paley functions that $m^N(\xi) \to m(\xi)$ as $N \to \infty$ for every $\xi \in \mathbb{R}^d$, and $m_L^N(\xi) \to m(\xi)$ as $N \to \infty$ and $L \to -\infty$ for every $\xi \in \mathbb{R}^d \setminus \{0\}$. In particular, both limits hold for almost every $\xi \in \mathbb{R}^d$. Then Proposition 5.3.16 implies that

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \liminf_{N\to\infty} \|m^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)},$$

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant \liminf_{\substack{N\to\infty \\ L\to-\infty}} \|m_L^N\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)}.$$

□

13.2.a Smooth multipliers have Calderón–Zygmund kernels

We will be mostly concerned with multipliers satisfying Mihlin-type conditions of the form

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \tag{13.24}$$

for some set of multi-indices $\alpha \in \mathbb{N}^d$. Recall that the Mihlin class, introduced and used in Definitions 5.3.17 and 5.5.9 and Theorems 5.3.18 and 5.5.10 (in one and several variables, respectively) to deduce that $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ for all $p \in (1,\infty)$ without any *a priori* boundedness assumptions on T_m, featured stronger R-boundedness versions of such conditions. The difference in the present context is that we are willing to assume that $m \in \mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)$ for some $p_0 \in (1,\infty)$ to begin with, and we wish to show that this *a priori* boundedness on one space can then be extrapolated to boundedness on other function spaces under conditions that are similar to those in Mihlin's theorems, but without the R-bounded aspects. As a matter of fact, these pointwise bounds can often be relaxed to weaker integrated versions, which is easily verified by inspecting the proofs, but for the clarity of the exposition we state the results under such pointwise assumptions. This is hardly a restriction for most applications.

The role of the multiplier conditions (13.24) for the kernel estimates is via careful use of the fundamental relation $\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \widehat{f}(\xi)$. So as to make most efficient use of the relation, and to unburden the formulae from inessential constants, we introduce the abbreviation

$$\not\partial := \partial / 2\pi i$$

so that

$$\widehat{\not\partial_j f}(\xi) = \xi_j \widehat{f}(\xi).$$

The deduction of the kernel estimates is easiest when sufficiently many derivatives are allowed in (13.24); as it turns out, this is somewhat more than the collection $\alpha \in \{0,1\}^d$ appearing in Mihlin's Theorem 5.5.10. We formulate several results for a generic Banach space Z instead of $\mathscr{L}(X,Y)$, as the operator structure plays no role here; this also makes the formulae slightly shorter. We say that a collection $\mathscr{A}$ of multi-indices is *convex*, if $\alpha \in \mathscr{A}$ implies $\beta \in \mathscr{A}$ whenever $0 \leqslant \beta \leqslant \alpha$.

Lemma 13.2.3. *If $m \in L^\infty(\mathbb{R}^d; Z)$ satisfies (13.24) for a convex set of multi-indices α, then each $m_j \in L^\infty_{\mathrm{c}}(B(0, 2^{j+1}); Z)$ satisfies*

$$\|\not\partial^\alpha m_j\|_\infty \leqslant M 2^{-j|\alpha|}$$

for the same set of multi-indices, where M is the constant of (13.24).

Proof. By the Leibniz rule, we have

$$\partial^\alpha m_j(\xi) = \partial^\alpha [\widehat{\psi}(2^{-j}\xi) m(\xi)] = \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{-j|\theta|} \partial^\theta \widehat{\psi}(2^{-j}\xi) \partial^{\alpha-\theta} m(\xi),$$

where each $\partial^{\alpha-\theta} m$ also satisfies (13.24) by convexity. Thus

$$\begin{aligned} \|\partial^\alpha m_j(\xi)\| &\leqslant \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{-j|\theta|} \mathbf{1}_{2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}} M |\xi|^{-|\alpha-\theta|} \\ &\leqslant \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{-j|\theta|} M (2^{j-1})^{-|\alpha|+|\theta|} \\ &= M 2^{-j|\alpha|} 2^{|\alpha|} \sum_{\theta \leqslant \alpha} \binom{\alpha}{\theta} 2^{|\alpha-\theta|} \cdot 1^{|\theta|} \end{aligned}$$

$$= M2^{-j|\alpha|}2^{|\alpha|}(2+1)^{|\alpha|} = M2^{-j|\alpha|}6^{|\alpha|},$$

where the binomial formula was used in the second to last step. The result follows after dividing both sides by $(2\pi)^{|\alpha|} \geqslant 6^{|\alpha|}$. □

Lemma 13.2.4. *Let Z be a Banach space and $f \in L^\infty_c(B(0,A);Z)$ have distributional derivatives that satisfy*

$$\|\partial^\alpha f\|_\infty \leqslant A^{-|\alpha|}$$

for some $A > 0$ and all multi-indices α in some convex set. Then

$$\|x \mapsto \partial^\alpha_x[(e^{2\pi i y\cdot x} - 1)f(x)]\|_\infty \leqslant (6 + 2^{|\alpha|})A|y| \cdot A^{-|\alpha|}$$

for all $y \in \mathbb{R}^d$ with $|y| \leqslant A^{-1}$, and for the same set of multi-indices.

Proof. The derivatives are given by

$$\partial^\alpha_x[(e^{2\pi i y\cdot x} - 1)f(x)] = (e^{2\pi i y\cdot x} - 1)\partial^\alpha f(x) + \sum_{0\neq\gamma\leqslant\alpha} y^\gamma e^{2\pi i y\cdot x}\partial^{\alpha-\gamma} f(x),$$

and hence

$$\begin{aligned}\|\partial^\alpha_x[(e^{2\pi i y\cdot x} - 1)f(x)]\| &\leqslant 2\pi|y|A \cdot A^{-|\alpha|} + \sum_{0\neq\gamma\leqslant\alpha} |y|^{|\gamma|}A^{-|\alpha|+|\gamma|} \\ &\leqslant |y|A \cdot A^{-|\alpha|}\Big(2\pi + \sum_{0\neq\gamma\leqslant\alpha}(A|y|)^{|\gamma|-1}\Big).\end{aligned}$$

If $A|y| \leqslant 1$, then $(A|y|)^{|\gamma|-1} \leqslant 1$ and $\sum_{0\neq\gamma\leqslant\alpha} 1 = 2^{|\alpha|} - 1$. □

Lemma 13.2.5. *Let Z be a Banach space and $f \in L^\infty_c(B(0,A);Z)$ have distributional derivatives that satisfy*

$$\|\partial^\alpha f\|_\infty \leqslant A^{-|\alpha|} \qquad \forall|\alpha| \leqslant d+1$$

for some $A > 0$. Then for almost all $x, y \in \mathbb{R}^d$ with $|y| \leqslant \frac{1}{2}|x|$, we have

$$|x|^n|\widehat{f}(x)| \leqslant c_d A^{d-n}, \tag{13.25}$$

$$|x|^n|\widehat{f}(x-y) - \widehat{f}(x)| \leqslant c_d A^{d-n}\min\{A|y|, 1\} \tag{13.26}$$

for all $n = 0, 1, \ldots, d+1$. In particular, $\widehat{f} \in L^1(\mathbb{R}^d;Z)$ and

$$\|\widehat{f}\|_1 \leqslant c_d.$$

Proof. For $x \in B(0,A)$, we have

$$\|x^\alpha \widehat{f}\|_\infty \leqslant \|\partial^\alpha f\|_1 \leqslant \|\partial^\alpha f\|_\infty\|\mathbf{1}_{B(0,A)}\|_1 \leqslant A^{-|\alpha|}\omega_d A^d,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. With $\alpha = ne_i$, this shows that $|x_i|^n|\widehat{f}(x)| \leqslant \omega_d A^{d-n}$ for $i = 1, \dots, d$, which readily gives (13.25).

We observe that $\widehat{f}(x-y) - \widehat{f}(x)$ is the Fourier transform of $(e^{2\pi i x\cdot y} - 1)f(x)$, which satisfies the same assumptions as f for $|y| \leqslant A^{-1}$, except for a multiplicative factor $(6+2^d)A|y|$, by Lemma 13.2.4. An application of (13.25) to this function in place of f hence gives

$$|x|^n|\widehat{f}(x-y) - \widehat{f}(x)| \leqslant c_d A^{d-n}A|y|$$

when $A|y| \leqslant 1$. On the other hand, if $A|y| > 1$, then we simply estimate $\widehat{f}(x-y)-\widehat{f}(x)$ by (13.25) and the triangle inequality, recalling the assumptions that $|y| \leqslant \frac{1}{2}|x|$ and $n \leqslant d+1$:

$$\begin{aligned}|\widehat{f}(x-y) - \widehat{f}(x)| &\leqslant |\widehat{f}(x-y)| + |\widehat{f}(x)| \leqslant c_d A^{d-n}(|x-y|^{-n} + |x|^{-n}) \\ &\leqslant c_d A^{d-n}(2^n+1)|x|^{-n} \leqslant c_d' A^{d-n}|x|^{-n}.\end{aligned}$$

The last two bounds are both seen to be dominated by the claimed bound in (13.26).

That $\widehat{f} \in L^1(\mathbb{R}^d; Z)$ is immediate from (13.25) by integrating the estimate

$$|\widehat{f}(x)| \leqslant c_d A^d \min\Big\{1, (A|x|)^{-d-1}\Big\}.$$

□

Proposition 13.2.6. *Let X, Y be Banach spaces and $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfy*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall |\alpha| \leqslant d+1.$$

Then each $K^N(x,y) = k^N(x-y) = \check{m}^N(x-y)$ is a Calderón–Zygmund kernel with the following bounds independent of the truncation N:

$$\|k^N(x)\| \leqslant \frac{c}{|x|^d}, \qquad \|k^N(x-y) - k^N(x)\| \leqslant \frac{1}{|x|^d}\omega\Big(\frac{|y|}{|x|}\Big),$$

for all $x, y \in \mathbb{R}^d$ with $|y| \leqslant \frac{1}{2}|x|$, where

$$c = c_d M, \qquad \omega(t) = c_d M \cdot t \cdot \Big(1 + \log\frac{1}{t}\Big).$$

Note that the modulus of continuity ω above is slightly "worse" (i.e., with slower decay as $t \to 0$) than the Lipschitz modulus $\omega_1(t) = t$, but "better" than any of the Hölder moduli $\omega_\delta(t) = t^\delta$ for $\delta \in (0,1)$.

Proof. By Lemma 13.2.3, the functions m_j satisfy the assumptions, and hence the conclusions, of Lemma 13.2.5 with $A = 2^{j+1}$ and a multiplicative factor $c_d M$. Thus,

$$\begin{aligned}|k^N(x)| &\leqslant \sum_{j\leqslant N}|k_j(x)| \leqslant \sum_{j\in\mathbb{Z}}\min_{0\leqslant h\leqslant d+1}\frac{c_d 2^{(j+1)(d-h)}}{|x|^h}M \\ &\leqslant \sum_{j:2^{j+1}\leqslant 1/|x|} c_d 2^{(j+1)d}M + \sum_{j:2^{j+1}\geqslant 1/|x|}\frac{c_d 2^{-(j+1)}}{|x|^{d+1}}M \leqslant c_d'|x|^{-d}M.\end{aligned}$$

Similarly, for $|y| \leqslant \frac{1}{2}|x|$,

$$\begin{aligned}&|k^N(x-y)-k^N(x)| \leqslant \sum_{j\in\mathbb{Z}}\min_{0\leqslant h\leqslant d+1}\frac{c_d 2^{(j+1)(d-h)}}{|x|^h}\min\{2^{j+1}|y|,1\}M \\ &\leqslant \sum_{j:2^{j+1}\leqslant 1/|x|} c_d 2^{(j+1)(d+1)}|y|M + \sum_{j:1/|x|\leqslant 2^{j+1}\leqslant 1/|y|}\frac{c_d}{|x|^{d+1}}|y|M \\ &\qquad + \sum_{j:2^{j+1}\geqslant 1/|y|}\frac{c_d 2^{-(j+1)}}{|x|^{d+1}}M \\ &\leqslant c_d'\frac{1}{|x|^{d+1}}|y|M + \frac{c_d'}{|x|^{d+1}}|y|\Big(1+\log\frac{|x|}{|y|}\Big)M + \frac{c_d'}{|x|^{d+1}}|y|M \\ &\leqslant \frac{c_d''}{|x|^{d+1}}|y|\Big(1+\log\frac{|x|}{|y|}\Big)M.\end{aligned}$$

This completes the proof. □

With the uniform pointwise bounds of Proposition 13.2.6 at hand, we can strengthen the sense in which the operator T_m with such bounds is associated with a Calderón–Zygmund kernel k:

Proposition 13.2.7. *Let X, Y be Banach spaces, $p \in [1,\infty)$, and $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$ satisfy*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall |\alpha| \leqslant d+1.$$

Then there is a kernel $k \in C(\mathbb{R}^d\setminus\{0\}; \mathscr{L}(X,Y))$ that satisfies the same bounds as k^N in Proposition 13.2.6 and such that

$$T_m f(x) = \int_{\mathbb{R}^d} k(x-y)f(y)\,\mathrm{d}y$$

for all $f \in L^p(\mathbb{R}^d; X)$ and almost all $x \in \mathbb{R}^d$ outside the support of f.

Proof. We split the proof into two cases:

Case $p \in (1,\infty)$: Let $f \in L^p(\mathbb{R}^d; X)$. Using the notation from the proof of Lemma 13.2.2 and the preceding discussion, we have

$$T_{m_L^N} f = T_m[(\varphi_{2^{-N}} * f) - (\varphi_{2^{-L}} * f)],$$

where $\varphi_{2^{-N}} * f \to f$ in $L^p(\mathbb{R}^d;X)$ as $N \to \infty$ by a standard mollifier result (e.g., Proposition 1.2.32). We also have $\|\varphi_R * f\|_p \leqslant \|\varphi_R\|_p \|f\|_1 = R^{-n/p'}\|f\|_1 \to 0$ as $R \to \infty$ if $f \in L^1(\mathbb{R}^d;X)$ and $\|\varphi_R * f\|_p \leqslant \|\varphi_R\|_1 \|f\|_p = \|f\|_p$ uniformly in R. Since $(L^1 \cap L^p)(\mathbb{R}^d;X)$ is dense in $L^p(\mathbb{R}^d;X)$, it follows that $\varphi_{2^{-L}} * f \to 0$ in $L^p(\mathbb{R}^d;X)$ as $L \to -\infty$ for all $f \in L^p(\mathbb{R}^d;X)$.

Summarising this discussion, it follows that, for all $f \in L^p(\mathbb{R}^d;X)$, we have the convergence $T_{m_L^N} f \to f$ in $L^p(\mathbb{R}^d;X)$ as $N \to \infty$ and $L \to -\infty$. By passing to a subsequence if needed, we may assume that this convergence also takes place almost everywhere.

If $f \in \check{L}^1(\mathbb{R}^d;X) \cap L^1(\mathbb{R}^d;X) \subseteq L^p(\mathbb{R}^d;X)$, then Proposition 13.2.1 shows that

$$T_{m_L^N} f = k_L^N * f,$$

where $T_{m_L^N}$ is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$ by Lemma 13.2.2. On the other hand, k_L^N is a finite sum of $k_j = \check{m}_j$, where the multipliers m_j are in the scope of Lemma 13.2.5, and hence $k_L^N \in L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$. But then also $f \mapsto k_N * f$ is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, and the previous display must remain valid for all $f \in L^p(\mathbb{R}^d;X)$ by continuity. Combining these pieces, we obtain

$$T_m f(x) = \lim_{\substack{N\to\infty \\ L\to-\infty}} T_{m_L^N} f(x) = \lim_{\substack{N\to\infty \\ L\to-\infty}} \int_{\mathbb{R}^d} k_L^N(x-y) f(y)\,\mathrm{d}y$$

for all $f \in L^p(\mathbb{R}^d;X)$ and almost every $x \in \mathbb{R}^d$.

Let us finally consider $x \in \complement \operatorname{supp} f$. Since this set is open, we can pick an $\varepsilon > 0$ such that $B(x,\varepsilon) \subseteq \complement \operatorname{supp} f$. For such x and any $y \in \operatorname{supp} f$, the series

$$\sum_{j\in\mathbb{Z}} k_j(x-y) = \lim_{\substack{N\to\infty \\ L\to-\infty}} k_L^N(x-y)$$

converges absolutely by the proof of Proposition 13.2.6. We denote by $k(x-y)$ the limit. Moreover, the same proposition shows that

$$\|k_L^N(x-y)f(y)\| \leqslant \frac{c_d M}{|x-y|^d} \|f(y)\|,$$

which is integrable over $y \in \mathbb{R}^d$ by Hölder's inequality, since $f \in L^p(\mathbb{R}^d;X)$ and $[y \mapsto |x-y|^{-d}] \in L^{p'}(\complement B(x,\varepsilon))$. Thus

$$T_m f(x) = \lim_{\substack{N\to\infty \\ L\to-\infty}} \int_{\mathbb{R}^d} k_L^N(x-y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} k(x-y) f(y)\,\mathrm{d}y$$

by dominated convergence. The pointwise estimates of k_L^N are clearly inherited by k by the pointwise convergence. This completes the proof for $p \in (1,\infty)$.

Case $p = 1$: We can still make use of large parts of the preceding considerations, but some details require a modification. The standard mollifier result (Proposition 1.2.32) still applies to show that $\varphi_{2^{-N}} * f \to f$, and hence $T_{m^N} f \to T_m f$, in $L^1(\mathbb{R}^d; X)$ as $N \to \infty$, but it no longer guaranteed that $\varphi_{2^{-L}} * f$ should converge to 0 as $L \to -\infty$. Hence, we will separately deal with $T_{m^L} f$.

For $f \in \check{L}^1(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$, we have

$$\begin{aligned}\|T_{m^L} f(x)\| &= \Big\| \int_{\mathbb{R}^d} \varphi(2^{-L}\xi) m(\xi) \widehat{f}(\xi) \, \mathrm{d}\xi \Big\| \\ &\leqslant \int_{|\xi| \leqslant 2^{L+1}} \|m\|_\infty \|\widehat{f}(\xi)\| \, \mathrm{d}\xi \leqslant \omega_d (2^{L+1})^d \|m\|_\infty \|f\|_1.\end{aligned}$$

Hence T_{m^L} extends to a bounded operator from $L^1(\mathbb{R}^d; X)$ to $L^\infty(\mathbb{R}^d; Y)$ of norm at most $\omega_d 2^{(L+1)d} \|m\|_\infty \to 0$ as $L \to -\infty$.

For $f \in \check{L}^1(\mathbb{R}^d; X) \cap L^1(\mathbb{R}^d; X)$, we can now write

$$T_{m^N} f = T_{m_L^N} f + T_{m^L} f = k_{m_L^N} * f + T_{m^L} f.$$

Since all of the operators acting on f above are bounded from $L^1(\mathbb{R}^d; X)$ to $L^1(\mathbb{R}^d; Y) + L^\infty(\mathbb{R}^d; Y)$, the identity continues to hold for all $f \in L^1(\mathbb{R}^d; X)$. Taking the limits $N \to \infty$ and $L \to -\infty$, we have $T_{m^N} f \to T_m f$ in $L^1(\mathbb{R}^d; Y)$ and $T_{m^L} f \to 0$ in $L^\infty(\mathbb{R}^d; Y)$. Along suitable subsequences, we have both limits almost everywhere, and hence we arrive at the same pointwise limit

$$T_m f(x) = \lim_{\substack{N \to \infty \\ L \to -\infty}} \int_{\mathbb{R}^d} k_L^N (x - y) f(y) \, \mathrm{d}y$$

as in the case $p \in (1, \infty)$. The rest of the proof can then be concluded in the same way as before. Specifically, let us note that the final application of dominated convergence is justified simple because the product of $[y \mapsto |x - y|^{-d}] \in L^\infty(\complement B(x, \varepsilon))$ and $f \in L^1(\mathbb{R}^d; X)$ is integrable. □

Corollary 13.2.8. *Let X, Y be Banach spaces and $p_0 \in [1, \infty)$. Suppose that $m \in \mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M |\xi|^{-|\alpha|} \qquad \forall |\alpha| \leqslant d + 1.$$

Then T_m extends to a bounded operator from $L^p(w; X)$ to $L^p(w; Y)$ for every $p \in (1, \infty)$ and every Muckenhoupt weight $w \in A_p$. Moreover,

$$\|T_m\|_{\mathscr{L}(L^p(w;X), L^p(w;Y))} \leqslant c_{d,p} (\|m\|_{\mathfrak{M}L^{p_0}(\mathbb{R}^d; X,Y)} + M) [w]_{A_p}^{\max\{1, \frac{1}{p-1}\}}.$$

Proof. By Proposition 13.2.7, the A_2 Theorem 11.3.26 applies to such an operator T_m, and this gives precisely the stated conclusions. □

Corollary 13.2.9. *Let X, Y be UMD spaces. Suppose that $m \in L^\infty(\mathbb{R}^d; X, Y)$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall|\alpha| \leqslant d+1,$$

and in addition

$$\mathscr{R}\big(\big\{|\xi|^{|\alpha|}\partial^\alpha m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}\big\}\big) \leqslant \widetilde{M} \qquad \forall \alpha \in \{0,1\}^d,$$

Then T_m extends to a bounded operator from $L^p(w;X)$ to $L^p(w;Y)$ for every $p \in (1,\infty)$ and every Muckenhoupt weight $w \in A_p$. Moreover,

$$\|T_m\|_{\mathscr{L}(L^p(w;X),L^p(w;Y))} \leqslant c_{d,p}(\min(\hbar^d_{p,X}, \hbar^d_{p,Y})\beta_{p,X}\beta_{p,Y}\widetilde{M}+M)[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}.$$

Proof. By Mihlin's Multiplier Theorem 5.5.10, the assumptions imply that

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X,Y)} \leqslant c_d \min(\hbar^d_{p,X}, \hbar^d_{p,Y})\beta_{p,X}\beta_{p,Y}\widetilde{M}.$$

We then conclude with an application of Corollary 13.2.8. □

This proof displays a certain dichotomy between the multiplier conditions needed to get the boundedness of T_m to begin with, and the conditions needed to extrapolate this boundedness to other spaces. The former one needs the stronger R-boundedness assumptions, but only for a smaller number of derivatives, while the latter only needs usual pointwise bounds, but for a larger set of derivatives. This dichotomy disappears from sight in the following important special case:

Corollary 13.2.10. *Let X be a UMD space. Suppose that a scalar-valued $m \in L^\infty(\mathbb{R}^d)$ satisfies*

$$|\partial^\alpha m(\xi)| \leqslant M|\xi|^{-|\alpha|} \qquad \forall|\alpha| \leqslant d+1.$$

Then T_m extends to a bounded operator on $L^p(w;X)$ for every $p \in (1,\infty)$ and every Muckenhoupt weight $w \in A_p$. Moreover,

$$\|T_m\|_{\mathscr{L}(L^p(w;X)))} \leqslant c_{d,p}\hbar^d_{p,X}\beta^2_{p,X}M[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}}.$$

Proof. The assumed pointwise bounds coincide with the R-bounds required by Corollary 13.2.10 in the case of a scalar-valued multiplier m. □

13.2.b Mihlin multipliers have Hörmander kernels

We now turn to the question of kernel estimates assuming only the multiplier conditions appearing in Mihlin's Theorem 5.5.10. It turns out that the maximal order of d derivatives is just on the border of what we need to make useful estimates, and in order to cope with this condition, we need to impose an additional assumption on the underlying Banach space X in terms of the notion of Fourier type discussed in Section 13.1.

The analogue of Lemma 13.2.5 in the present context is the following rather more complicated assertion.

Lemma 13.2.11. *Let X be a Banach space with Fourier type $p \in (1,2]$. Let $f \in L^\infty_c((-A,A)^d;X)$ satisfy*

$$\|\partial\!\!\!/^{\alpha} f\|_\infty \leqslant A^{-|\alpha|} \qquad \forall \alpha \in \{0,1\}^d$$

for some $A > 0$. Then $\widehat{f} \in L^1(\mathbb{R}^d;X)$ and, denoting

$$\Phi_{p,X} := 4p'(4 + \log_2^+ \varphi_{p,X}),$$

we have the estimates

$$\|\widehat{f}\|_1 \leqslant \Phi^d_{p,X}, \tag{13.27}$$

$$\|\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1 \leqslant \Phi^d_{p,X}\frac{4d\varphi_{p,X}}{(AR)^{1/p'}} \quad \forall R > 0, \tag{13.28}$$

$$\|\widehat{f}(\cdot - y) - \widehat{f}(\cdot)\|_1 \leqslant \Phi^d_{p,X} \cdot 4 \cdot 2^d A|y| \quad \forall y \in \mathbb{R}^d, \tag{13.29}$$

$$\|\mathbf{1}_{\complement B(0,3|y|)}[\widehat{f}(\cdot - y) - \widehat{f}(\cdot)]\|_1 \leqslant \Phi^d_{p,X} \min\Big\{2, \frac{8d^2\varphi_{p,X}}{(Ar)^{1/p'}}, 4\cdot 2^d Ar\Big\}. \tag{13.30}$$

Remark 13.2.12. Thanks to Bourgain's Theorem 13.1.33, the assumption on the Banach space X in Lemma 13.2.11 is simply that X has some non-trivial type $r \in (1,2]$. Namely, Theorem 13.1.33 guarantees that we can then take

$$p' = 1 + 6r'T, \quad \varphi_{p,X} \leqslant e^{2T}, \quad T := (68\tau_{r,X;2})^{r'} \geqslant 68^2 > 4\,000,$$

and hence

$$\begin{aligned}\Phi_{p,X} &\leqslant 4\big(1+6r'T\big)\Big(4 + \frac{2}{\log 2}T\Big) = \frac{48}{\log 2}\big(\frac{1}{6r'T}+1\big)\big(\frac{2\log 2}{T}+1\big)r'T^2 \\ &\leqslant 70 \cdot r'T^2 = 70r'(68\tau_{r,X;2})^{r'}.\end{aligned}$$

Proof of (13.27). For $k \in \mathbb{Z}^d$, let

$$D_k = \{x \in \mathbb{R}^d : x_i \in [2^{k_i}, 2^{k_i+1}) \ \forall i = 1,\ldots,d\}$$

so that obviously

$$\|\widehat{f}\|_1 = \sum_{k\in\mathbb{Z}^d} \|\mathbf{1}_{D_k} f\|_1.$$

For each $k \in \mathbb{Z}^d$, we partition $\mathbf{1} = \alpha + \beta + \gamma$ for some $\alpha, \beta, \gamma \in \{0,1\}^d$ yet to be chosen. Then

$$D_k = D_k^\alpha \times D_k^\beta \times D_k^\gamma, \qquad D_k^\theta = \{(x_i)_{i:\theta_i=1} : x_i \in [2^{k_i}, 2^{k_i+1})\}.$$

Similarly,

$$\mathbb{R}^d = \mathbb{R}^\alpha \times \mathbb{R}^\beta \times \mathbb{R}^\gamma, \qquad \mathbb{R}^\theta = \{(x_i)_{i:\theta_i=1} : x_i \in \mathbb{R}\},$$

and we abbreviate $L^s L^t_\gamma := L^s(\mathbb{R}^\alpha \times \mathbb{R}^\beta; L^t(\mathbb{R}^\gamma;X))$.

For $x \in D_k$, we have $|x_i| \geqslant 2^{k_i}$, and hence $|x^{\beta+\gamma}| \geqslant 2^{k\cdot(\beta+\gamma)}$. We can now make the following estimate. At a critical point, passing from a norm of the Fourier transform $\widehat{f}$ to a norm of f itself, we apply the Fourier type assumption to $\mathscr{F} : L^p(\mathbb{R}^\gamma; X) \to L^{p'}(\mathbb{R}^\gamma; X)$, producing the constant $\varphi_{p,X}(\mathbb{R}^\gamma) \leqslant \varphi_{p,X}^{|\gamma|}$, and the trivial boundedness of the Fourier transform $\mathscr{F} : L^1(\mathbb{R}^{\alpha+\beta}; Z) \to L^\infty(\mathbb{R}^{\alpha+\beta}; Z)$, with $Z = L^q(\mathbb{R}^\gamma; X)$ for either $q = p$ or $q = p'$, depending on the (irrelevant) order in which we perform these two steps:

$$\begin{aligned}
\|\mathbf{1}_{D_k}\widehat{f}\|_{L^1} &\leqslant 2^{-k\cdot(\beta+\gamma)}\|\mathbf{1}_{D_k}x^{\beta+\gamma}\widehat{f}\|_{L^1} \\
&\leqslant 2^{-k\cdot(\beta+\gamma)}\|\mathbf{1}_{D_k}\|_{L^1L^p_\gamma}\|x^{\beta+\gamma}\widehat{f}\|_{L^\infty L^{p'}_\gamma} \\
&\leqslant 2^{-k\cdot(\beta+\gamma)} \cdot 2^d 2^{k\cdot(\alpha+\beta+\gamma/p)} \cdot \varphi_{p,X}^{|\gamma|}\|\partial^{\beta+\gamma} f\|_{L^1L^p_\gamma} \\
&\leqslant 2^d 2^{k\cdot(\alpha-\gamma/p')} \cdot \varphi_{p,X}^{|\gamma|}A^{-|\beta|-|\gamma|}2^d A^{|\alpha|+|\beta|+|\gamma|/p} \\
&\leqslant 4^d 2^{k\cdot(\alpha-\gamma/p')} \cdot \varphi_{p,X}^{|\gamma|}A^{|\alpha|-|\gamma|/p'} \\
&= 4^d \times \prod_{i:\alpha_i=1}(A2^{k_i}) \times \prod_{i:\beta_i=1} 1 \times \prod_{i:\gamma_i=1}(\varphi_{p,X}(2^{k_i}A)^{-1/p'}).
\end{aligned}$$

Since the splitting $\mathbf{1} = \alpha + \beta + \gamma$ is free for us to choose, it is obvious that, for each i, we choose it to be in the first, second or third category according to which of the three numbers

$$A2^{k_i}, \qquad 1, \qquad \varphi_{p,X}(2^{k_i}A)^{-1/p'}$$

is the smallest. This gives us the estimate

$$\begin{aligned}
\|\widehat{f}\|_1 &= \sum_{k\in\mathbb{Z}^d}\|\mathbf{1}_{D_k}f\|_1 \\
&\leqslant 4^d \sum_{k\in\mathbb{Z}^d}\prod_{i=1}^d \min\{A2^{k_i}, 1, \varphi_{p,X}(2^{k_i}A)^{-1/p'}\} \\
&= 4^d\Big(\sum_{k\in\mathbb{Z}}\min\{A2^k, 1, \varphi_{p,X}(2^kA)^{-1/p'}\}\Big)^d \\
&\leqslant 4^d\Big(\sum_{k:A2^k\leqslant 1} A2^k + \sum_{k:1\leqslant A2^k\leqslant\varphi_{p,X}^{p'}} 1 + \sum_{k:A2^k\geqslant\varphi_{p,X}^{p'}}\varphi_{p,X}(2^kA)^{-1/p'}\Big)^d \\
&\leqslant 4^d\Big(2 + (1+\log_2^+\varphi_{p,X}^{p'}) + \frac{\varphi_{p,X}(\varphi_{p,X}^{p'})^{-1/p'}}{1-2^{-1/p'}}\Big)^d \\
&\leqslant 4^d(3 + p'\log_2^+\varphi_{p,X} + 2p')^d \leqslant (4p')^d(4+\log_2^+\varphi_{p,X})^d
\end{aligned}$$

where we observed that $1 - 2^{-1/p'} \geqslant 1/(2p')$, since the function $g(u) = u/2 + 2^{-u}$ satisfies $g(u) \leqslant 1$ for $u = 1/p' \in [0, \frac{1}{2}]$, being convex with $g(0) = 1$ and $g(\frac{1}{2}) = 1/4 + 2^{-1/2} < 1$. $\square$

Proof of (13.28). Making the same decomposition

$$\|\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1 = \sum_{k\in\mathbb{Z}^d}\|\mathbf{1}_{D_k}\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1$$

as in the proof of (13.27), we observe hat $\mathbf{1}_{D_k}\mathbf{1}_{\complement[-R,R]^d}$ is non-zero only if at least one k_i satisfies $2^{k_i+1} > R$. Thus

$$\begin{aligned}\|\mathbf{1}_{\complement[-R,R]^d}\widehat{f}\|_1 &\leqslant \sum_{i=1}^{d}\sum_{\substack{k\in\mathbb{Z}^d\\ 2^{k_i}>R/2}}\|\mathbf{1}_{D_k}\widehat{f}\|_1\\ &\leqslant d\cdot 4^d\Big(\sum_{k\in\mathbb{Z}}\min\{A2^k,1,\varphi_{p,X}(2^kA)^{-1/p'}\}\Big)^{d-1}\\ &\qquad\times\Big(\sum_{k:2^k>R/2}\min\{A2^k,1,\varphi_{p,X}(2^kA)^{-1/p'}\}\Big),\end{aligned}$$

by inspection of the proof of (13.27). The factor raised to power $d-1$ is estimated as in the proof of (13.27) by

$$\Big(\sum_{k\in\mathbb{Z}}\min\{A2^k,1,\varphi_{p,X}(2^kA)^{-1/p'}\}\Big)^{d-1}\leqslant (p')^{d-1}\Big(4+\log_2^+\varphi_{p,X}\Big)^{d-1}.$$

On the other hand, we have

$$\begin{aligned}\sum_{k:2^k>R/2}&\min\{A2^k,1,\varphi_{p,X}(2^kA)^{-1/p'}\}\\ &\leqslant\sum_{k:2^k>R/2}\varphi_{p,X}(2^kA)^{-1/p'}\\ &\leqslant\frac{\varphi_{p,X}(AR/2)^{-1/p'}}{1-2^{-1/p'}}\leqslant 4p'\varphi_{p,X}(AR)^{-1/p'},\end{aligned}$$

again by recycling some estimates from the proof of (13.27). Collecting the bounds, the proof of (13.28) is complete. □

Proof of (13.29). We observe that $\widehat{f}(x-y)-\widehat{f}(x)$ is the Fourier transform of $f(x)e^{2\pi i x\cdot y}$, which verifies the same assumptions as f by Lemma 13.2.4, aside from the multiplicative factor $(6+2^d)A|y|$, provided that $A|y|\leqslant 1$. Applying (13.27) to this function gives (13.29) for $A|y|\leqslant 1$. But for $A|y|>1$, (13.29) is an immediate consequence of (13.27) by the triangle inequality. □

Proof of (13.30). This final bound is a certain synthesis of the other bounds. The first and third bounds in the minimum are obtained from (13.27) (with the triangle inequality) and from (13.29), respectively, ignoring the restriction to $\complement B$, which only increases the norm.

For the second bound, we also use the triangle inequality, but keeping the restriction to $\complement B$. Then

$$\begin{aligned}\|\mathbf{1}_{\complement B(0,3|y|)}\widehat{f}(\cdot - y)\|_1 &= \|\mathbf{1}_{\complement B(-y,3|y|)}\widehat{f}\|_1 \\ &\leqslant \|\mathbf{1}_{\complement B(0,2|y|)}\widehat{f}\|_1 \leqslant \|\mathbf{1}_{\complement(-2r/\sqrt{d},2r/\sqrt{d})^d)}\widehat{f}\|_1,\end{aligned}$$

and the same bound is obvious for $\widehat{f}$ in place of $\widehat{f}(\cdot - y)$. Applying (13.28) with $R = 2r/\sqrt{d}$ produces the required bound. □

Proposition 13.2.13. *Let X, Y be Banach spaces, and suppose that $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall \alpha \in \{0,1\}^d. \tag{13.31}$$

Let $K^N(t,s) = k^N(t-s) = \check{m}^N(x-y)$ be the kernels related to the Littlewood–Paley truncations m^N of m as in (13.23).

(1) *If the space Y has Fourier type $p \in (1,2]$, then the kernels K^N satisfy the Hörmander condition uniformly in N, and quantitatively*

$$\int_{|t|>3|s|} \|(k^N(t-s) - k^N(t))x\|_Y \,\mathrm{d}t \leqslant (2\Phi_{p,Y})^{d+1} M\|x\|_X \quad \forall x \in X,$$

where $\Phi_{p,Y} = 4p'(4 + \log_2^+ \varphi_{p,Y})$.

(2) *If the space X has Fourier type $p \in (1,2]$, then the kernels K^N satisfy the dual Hörmander condition uniformly in N, and quantitatively*

$$\int_{|t|>3|s|} \|(k^N(t-s)^* - k^N(t)^*)y^*\| \,\mathrm{d}t \leqslant (2\Phi_{p,X})^{d+1} M\|y^*\|_{Y^*} \quad \forall y^* \in Y^*,$$

where $\Phi_{p,X} = 4p'(4 + \log_2^+ \varphi_{p,X})$.

Proof of (1). From Lemma 13.2.3 it follows that each Littlewood–Paley truncation $m_j \in L^\infty_{\mathrm{c}}(B(0,2^{j+1}); \mathscr{L}(X,Y))$ satisfies

$$\|\eth^\alpha m_j\|_\infty \leqslant 2^d M 2^{-(j+1)|\alpha|},$$

which is like the condition of Lemma 13.2.11 with $A = 2^{j+1}$ and an additional multiplicative constant $2^d M$.

Moreover, for $x \in X$, the function $m_j(\cdot)x \in L^\infty_{\mathrm{c}}(B(0,2^{j+1}); Y)$ satisfies the same assumption with constant $2^d M\|x\|$, and now the range Y also has Fourier type $p \in (1,2]$, as required to apply Lemma 13.2.11. In particular, from (13.30), we conclude that

$$\begin{aligned}&\int_{|t|>3|s|} \|(k_j(t-s) - k_j(t))x\|_Y \,\mathrm{d}t \\ &\qquad \leqslant \Phi_{p,Y}^d 2^d M\|x\| \min\Big\{2, \frac{8d^2\varphi_{p,Y}}{(2^{j+1}r)^{1/p'}}, 8\cdot 2^d 2^{j+1} r\Big\}.\end{aligned}$$

Since $m^N \in L^\infty_{\rm c}(\mathbb{R}^d;\mathscr{L}(X,Y)) \subseteq L^1(\mathbb{R}^d;\mathscr{L}(X,Y))$, the kernels $k^N = \check{m}^N \in C_0(\mathbb{R}^d;\mathscr{L}(X,Y))$ are well defined, and we can estimates

$$\begin{aligned}
\int_{|t|>3|s|} &\|(k^N(t-s)-k^N(t))x\|_Y \,\mathrm{d}t \\
&\leqslant \sum_{j\leqslant N}\int_{\complement B}\|(k_j(t-s)-k_j(t))x\|_Y\,\mathrm{d}t \\
&\leqslant \varPhi^d_{p,Y}2^dM\|x\|\Big(\sum_{j:8\cdot 2^d2^{j+1}r\leqslant 2^{-d-5}} 8\cdot 2^d2^{j+1}r \\
&\qquad\qquad + \sum_{j:2^{-d-3}\leqslant 2^{j+1}r\leqslant (8d^2\varphi_{p,Y})^{p'}} 1 \\
&\qquad\qquad + \sum_{2^{j+1}r\geqslant (8d^2\varphi_{p,Y})^{p'}} \frac{8d^2\varphi_{p,Y}}{(2^{j+1}r)^{1/p'}}\Big) \\
&\leqslant \varPhi^d_{p,Y}2^dM\|x\|\Big(2+(\log_2^+(8d^2\varphi_{p,Y})^{p'}+d+4)+\frac{1}{1-2^{-1/p'}}\Big) \\
&\leqslant \varPhi^d_{p,Y}2^dM\|x\|\Big(6+3d+\log_2^+\varphi_{p,Y}\Big)p' \\
&\leqslant \varPhi^d_{p,Y}2^dM\|x\|\cdot d\cdot\varPhi_{p,Y}\leqslant (2\varPhi_{p,Y})^{d+1}M\|x\|.
\end{aligned}$$

□

Proof of (2). We note that (13.31) implies a similar bound for the pointwise adjoint function $m^* = m(\cdot)^* \in L^\infty(\mathbb{R}^d;\mathscr{L}(Y^*,X^*))$, while the assumption that X has Fourier type $p\in(1,2]$ implies that X^* has the same Fourier type with $\varphi_{p,X^*}=\varphi_{p,X}$ (Proposition 2.4.16). Thus case (2) follows from the already proven case (1) applied to (m^*,Y^*,X^*) in place of (m,X,Y). □

Corollary 13.2.14. *Let X,Y be Banach spaces with non-trivial Fourier type, let $p_0\in[1,\infty)$, and suppose that $m\in\mathfrak{M}L^{p_0}(\mathbb{R}^d;X,Y)$ satisfies*

$$\|\partial^\alpha m(\xi)\|\leqslant M|\xi|^{-|\alpha|}\qquad \forall\alpha\in\{0,1\}^d.$$

Then $m\in\mathfrak{M}L^p(\mathbb{R}^d;X,Y)$ for all $p\in(1,\infty)$.

Proof. By Lemma 13.2.2, the Littlewood–Paley truncations of m satisfy $m^N\in\mathfrak{M}L^{p_0}(\mathbb{R}^d;X,Y)$ uniformly in $N\in\mathbb{Z}$. By Proposition 13.2.13, the kernels $k^N=\check{m}^N$ satisfy both Hörmander and dual Hörmander conditions uniformly in $N\in\mathbb{Z}$. On the other hand, by Lemma 13.2.11, the kernel $k_j=\check{m}_j$ satisfy $k_j(\cdot)x\in L^1(\mathbb{R}^d;Y)$ for all $x\in X$, uniformly in $\|x\|\leqslant 1$, and hence $k_j\in L^1_{\rm so}(\mathbb{R}^d;\mathscr{L}(X,Y))$.

It follows that the kernels k^N_L satisfy both Hörmander and dual Hörmander conditions uniformly in $L,N\in\mathbb{Z}$, and they belong to $L^1_{\rm so}(\mathbb{R}^d;\mathscr{L}(X,Y))$ (but in general *not* uniformly). Thus the convolution with k^N_L defines a bounded operator from $L^{p_0}(\mathbb{R}^d;X)$ to $L^{p_0}(\mathbb{R}^d;Y)$. So does $T_{m^N_L}$, and hence the identity

$$T_{m_L^N} f = k_L^N * f,$$

initially guaranteed by Proposition 13.2.1 for all $f \in L^1 \cap \check{L}^1(\mathbb{R}^d; X)$, extends by continuity and density to all $f \in L^{p_0}(\mathbb{R}^d; X)$. Since the operators are uniformly bounded on this space, and their kernels satisfy both Hörmander and dual Hörmander conditions uniformly, it follows from the Calderón–Zygmund Theorem 11.2.5 that they extend boundedly from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ for all $p \in (1, \infty)$, again uniformly in $L, N \in \mathbb{Z}$. This is the same as $m_L^N \in \mathfrak{M}L^{p_0}(\mathbb{R}^d; X, Y)$ uniformly in $L, N \in \mathbb{Z}$, which, by Lemma 13.2.2, implies that $m \in \mathfrak{M}L^p(\mathbb{R}^d; X, Y)$. □

The following corollary is just the operator-valued Mihlin Multiplier Theorem 5.5.10 in the special case of Hilbert spaces (in contrast to general UMD spaces covered by Theorem 5.5.10); we state it here for the sake of pointing out the alternative approach to this special case via the Calderón–Zygmund extrapolation theory developed in this chapter.

Corollary 13.2.15. *Let H_1, H_2 be Hilbert spaces and suppose that $m \in L^\infty(\mathbb{R}^d; \mathscr{L}(H_1, H_2))$ satisfies*

$$\|\partial^\alpha m(\xi)\| \leqslant M|\xi|^{-|\alpha|} \qquad \forall \alpha \in \{0,1\}^d.$$

Then $m \in \mathfrak{M}L^p(\mathbb{R}^d; H_1, H_2)$ for all $p \in (1, \infty)$.

Proof. By Plancherel's theorem in both Hilbert spaces, we have

$$\|T_m f\|_{L^2(\mathbb{R}^d; H_2)} = \|m\widehat{f}\|_{L^2(\mathbb{R}^d; H_2)} \leqslant M\|\widehat{f}\|_{L^2(\mathbb{R}^d; H_1)} = M\|f\|_{L^2(\mathbb{R}^d; H_1)},$$

and thus $\|m\|_{\mathfrak{M}L^2(\mathbb{R}^d; H_1, H_2)} \leqslant M$. Since both H_i have Fourier type 2, Corollary 13.2.14 applies to give that $m \in \mathfrak{M}L^p(\mathbb{R}^d; H_1, H_2)$ for all $p \in (1, \infty)$. □

13.3 Necessity of UMD for multiplier theorems

In the previous sections, we have seen Fourier multiplier theorems of roughly two types:

1. If we already know the boundedness of such an operator on one $L^{p_0}(\mathbb{R}^d; X)$, then this boundedness can be extrapolated to other $L^p(\mathbb{R}^d; X)$ spaces under relatively mild (or even no) assumptions on the space X.
2. If we need to prove the boundedness "from scratch", then the required assumptions on X tend to be much stronger, and in particular involve the UMD property.

Let us also recall from the previous volumes that the need of the UMD property is not only imposed by the chosen proof strategies, but by the very nature of things: for prominent examples of multipliers like $-i\,\mathrm{sgn}(\xi)$ corresponding to the Hilbert transform (Theorem 5.2.10), or $|\xi|^{is}$ corresponding to imaginary powers of the Laplacian (Corollary 10.5.2), the UMD property is indeed necessary. The goal of this section is to continue this list by yet another class of Fourier multipliers whose boundedness requires UMD, and thereby close the circle of implications in a number of useful characterisations of UMD spaces. We start by discussing the types of multipliers that we are going to consider:

Definition 13.3.1. *We say that m is constant in the direction of $x \in \mathbb{R}^d \setminus \{0\}$ if $m(tx) = m(x)$ for all $t > 0$. We say that m is* stably constant *in the direction of $x \in \mathbb{R}^d \setminus \{0\}$ if, in addition, we have*

$$\lim_{t\to\infty} m(y+tx) = m(x) \qquad \forall y \in \mathbb{R}^d.$$

Note that if m is stably constant in the direction of x, then for every $s > 0$,

$$\lim_{t\to\infty} m(y+tsx) = \lim_{t\to\infty} m(y+tx) = m(x) = m(sx),$$

where the last step follows from the assumption (included in the definition of stably constant) that m is in particular constant in the direction of x.

Example 13.3.2. Suppose that $m \in C(\mathbb{R}^d \setminus \{0\})$ is homogeneous, $m(tx) = m(x)$ for all $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$. Then m is stably constant in every direction. Indeed

$$\lim_{t\to\infty} m(y+tx) = \lim_{t\to\infty} m(t^{-1}y+x) = m(x)$$

simply by the continuity of m at x.

Example 13.3.3. Suppose that $m \in C^1(\mathbb{R}^d \setminus \{0\})$ satisfies the first order Mihlin condition $|\nabla m(x)| \leqslant M|x|^{-1}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. If m is constant in the direction of some x, then m is stably constant in this direction. Indeed

$$\begin{aligned}|m(y+tx)-m(x)| = |m(y+tx)-m(tx)| &= \Big|\int_0^1 y\cdot\nabla m(ys+tx)\,\mathrm{d}s\Big| \\ &\leqslant |y|\int_0^1 \frac{M\,\mathrm{d}s}{|ys+tx|} \leqslant \frac{M|y|}{t|x|-|y|},\end{aligned}$$

and clearly this converges to 0 as $t \to \infty$.

Proposition 13.3.4 (Transference from $\mathbb{T}^d$ to $\mathbb{T}^{rd}$). *Let*

$$m \in C(\mathbb{R}^d \setminus \{0\}; \mathscr{L}(X)),$$

and suppose that it induces a periodic Fourier multiplier

$$T := \widetilde{T}_{(m(j))_{j\in\mathbb{Z}^n\setminus\{0\}}} \in \mathscr{L}(L^p_0(\mathbb{T}^d; X)).$$

If T_k is the extension of T to $L^p_0(\mathbb{T}^d; L^p(\mathbb{T}^{(k-1)d}; X))$ ($L^p(\mathbb{T}^0; X) := X$), then

$$\Big\| \sum_{k=1}^{r} T_k f_k \Big\|_{L^p(\mathbb{T}^{rd},\, \mathrm{d}t_1 \ldots \mathrm{d}t_r; X)} \leqslant \|T\|_{\mathscr{L}(L^p_0(\mathbb{T}^d;X))} \Big\| \sum_{k=1}^{r} f_k \Big\|_{L^p(\mathbb{T}^{rd},\, \mathrm{d}t_1 \ldots \mathrm{d}t_r; X)}$$

for all $f_k = f_k(t_1, \ldots, t_k) \in L^p_0(\mathbb{T}^d, \mathrm{d}t_k; L^p(\mathbb{T}^{(k-1)d}, \mathrm{d}t_1 \ldots \mathrm{d}t_{k-1}; X))$ *that have non-zero Fourier coefficients with respect to* t_k *only in the directions where* m *is stably constant.*

Proof. By the density of trigonometric polynomials in L^p, we may assume that

$$f_k(t_1, \ldots, t_{k-1}, t_k) = f_k(\bar{t}_{k-1}, t_k) = \sum_{\substack{\ell \in \mathbb{Z}^n \\ 0 < |\ell| \leqslant B}} \sum_{\substack{j \in \mathbb{Z}^{(k-1)n} \\ |j| \leqslant B}} a^{(k)}_{j,\ell} e_j(\bar{t}_{k-1}) e_\ell(t_k),$$

where

$$\bar{t}_{k-1} = (t_1, \ldots, t_{k-1}) \in (\mathbb{T}^d)^{k-1}, \quad t_k \in \mathbb{T}^d,$$

$$e_j(\bar{t}_{k-1}) := \exp(2\pi i j \cdot \bar{t}_{k-1}), \quad e_\ell(t_k) := \exp(2\pi i \ell \cdot t_k),$$

and we may choose the same B for all the f_k, since there are only finitely many of them. Then $T_k f_k$ has a similar expansion with the (j, ℓ) term multiplied by $m(\ell)$.

Let us fix some $\bar{t}_k := (\bar{t}_{k-1}, t_k) = (t_1, \ldots, t_k) \in \mathbb{T}^{kd}$ for the moment, and

$$\bar{N}_k := (N_1, \ldots, N_{k-1}, N_k) = (\bar{N}_{k-1}, N_k) \in \mathbb{Z}^k_+$$

to be chosen below.

We will shortly define an auxiliary function of the new variable $t \in \mathbb{T}^d$. For this we need to introduce a couple of product-like operations between vectors of different lengths. We set

$$\bar{N}_k \otimes t := (\bar{N}_{k-1} \otimes t, N_k t) = (N_1 t, \ldots, N_k t) \in (\mathbb{T}^d)^k, \quad \bar{N}_k \in \mathbb{Z}^k,$$

$$\bar{N}_{k-1} \odot j := N_1 j_1 + \ldots + N_{k-1} j_{k-1} \in \mathbb{Z}^d, \quad j = (j_1, \ldots, j_{k-1}) \in (\mathbb{Z}^d)^{k-1}.$$

These operations satisfy the identity

$$j \cdot (\bar{N}_{k-1} \otimes t) = (\bar{N}_{k-1} \odot j) \cdot t, \quad \text{hence} \quad e_j(\bar{N}_{k-1} \otimes t) = e_{\bar{N}_{k-1} \odot j}(t),$$

where $\cdot$ stands for the usual Euclidean scalar product.

The new function is then defined by

$$\begin{aligned} \tilde{f}_k(t) &:= f_k(\bar{t}_k + \bar{N}_k \otimes t) \\ &= \sum_{\substack{\ell \in \mathbb{Z}^n \\ 0 < |\ell| \leqslant B}} \sum_{\substack{j \in \mathbb{Z}^{(k-1)n} \\ |j| \leqslant B}} a^{(k)}_{j,\ell} e_j(\bar{t}_{k-1}) e_\ell(t_k) e_{\bar{N}_{k-1} \odot j + N_k \ell}(t), \end{aligned} \qquad (13.32)$$

The function $\widetilde{T_k f_k} : \mathbb{T}^n \to X$ is defined analogously.

We now want to compare $\widetilde{T_k f_k}$ with $T\tilde{f}_k$. They are both multiplier transforms of $\tilde{f}_k$, where in the first one the exponential $e_{\bar{N}_{k-1}\odot j+N_k\ell}$ is multiplied by $m(\bar{N}_{k-1}\odot j+N_k\ell)$, and in the second one by $m(\ell)$.

By the assumption on f_k, we know that m is stably constant in the direction of ℓ whenever $a_{j,\ell}^{(k)}\neq 0$, and therefore

$$\lim_{N_k\to\infty} m(\bar{N}_{k-1}\odot j+N_k\ell)=m(\ell).$$

Hence, assuming that $\bar{N}_{k-1}$ was already chosen, and recalling that $j\in\mathbb{Z}^{(k-1)d}$ and $\ell\in\mathbb{Z}^d$ with $|j|,|\ell|\leqslant B$ take only finitely many different values, we can choose N_k large enough so that

$$|m(\bar{N}_{k-1}\odot j+N_k\ell)-m(\ell)|\leqslant\varepsilon$$

for any preassigned $\varepsilon>0$ and all relevant values of j and ℓ.

In conclusion, denoting by $\|g\|_A$ the sum of the norms of the Fourier coefficients of a trigonometric polynomial g (on a torus of any dimension), we have

$$\|\widetilde{T_k f_k}-T\tilde{f}_k\|_p\leqslant\|\widetilde{T_k f_k}-T\tilde{f}_k\|_A\leqslant\varepsilon\|\tilde{f}_k\|_A\leqslant\varepsilon\|f_k\|_A.$$

Of course the $\|\cdot\|_A$ norms are finite since the functions above are all trigonometric polynomials.

Summing up, it follows that

$$\Big\|\sum_{k=1}^{r}\widetilde{T_k f_k}\Big\|_p\leqslant\Big\|T\sum_{k=1}^{r}\tilde{f}_k\Big\|_p+\varepsilon\sum_{k=1}^{r}\|f_k\|_A. \tag{13.33}$$

Here the L^p norms are taken with respect to the variable $t\in\mathbb{T}^d$, and we recall that the variables $t_1,\ldots,t_r\in\mathbb{T}^d$ were kept fixed until now. We now take the L^p norms of (13.33) with respect to $\bar{t}_r=(t_1,\ldots,t_r)\in\mathbb{T}^{rd}$ and use the triangle inequality to get

$$\begin{aligned}
&\Big(\int_{\mathbb{T}^{rn}}\int_{\mathbb{T}^n}\Big\|\sum_{k=1}^{r}(T_k f_k)(\bar{t}_r+\bar{N}_r\otimes t)\Big\|_X^p\,\mathrm{d}t\,\mathrm{d}\bar{t}_r\Big)^{1/p}\\
&\leqslant\|T\|_{\mathscr{L}(L_0^p(\mathbb{T}^n;X))}\Big(\int_{\mathbb{T}^{rn}}\int_{\mathbb{T}^n}\Big\|\sum_{k=1}^{r}f_k(\bar{t}_r+\bar{N}_r\otimes t)\Big\|_X^p\,\mathrm{d}t\,\mathrm{d}\bar{t}_r\Big)^{1/p}\\
&\qquad+\varepsilon\sum_{k=1}^{r}\|f_k\|_A.
\end{aligned}$$

Exchanging the order of the integrations on $\mathbb{T}^{rd}$ and $\mathbb{T}^d$, we find by translation invariance that the dependence on t and $\bar{N}_r$ disappears and we are left with

$$\Big\|\sum_{k=1}^{r}T_k f_k\Big\|_{L_0^p(\mathbb{T}^{rd};X)}\leqslant\|T\|_{\mathscr{L}(L_0^p(\mathbb{T}^d;X))}\Big\|\sum_{k=1}^{r}f_k\Big\|_{L_0^p(\mathbb{T}^{rd};X)}+\varepsilon\sum_{k=1}^{r}\|f_k\|_A.$$

Since there is no more explicit $\bar{N}_r$ dependence, we may take $\varepsilon \to 0$, and this gives the assertion. □

Theorem 13.3.5 (Geiss–Montgomery-Smith–Saksman). *Let $d \geqslant 2$ and $m \in C(\mathbb{R}^d \setminus \{0\})$ be a multiplier that is stably constant in the directions of four vectors $\pm u_i$, $i = 1, 2$, where moreover*

$$m(-u_1) = m(u_1) \neq m(u_2) = m(-u_2).$$

If $m \in \mathfrak{M}L^p(\mathbb{R}^d; X)$, then X is a UMD space and

$$\beta^{\mathbb{R}}_{p,X} \leqslant \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u_1) - m(u_2)|}. \tag{13.34}$$

To streamline the proof, we recall a transference result that we already observed and used in the proof of Corollary 10.5.2:

Lemma 13.3.6. *If $m \in C(\mathbb{R}^d \setminus \{0\}) \cap \mathfrak{M}L^p(\mathbb{R}^d; X)$, then $(m(k))_{k\in\mathbb{Z}^d\setminus\{0\}} \in \mathfrak{M}L^p_0(\mathbb{T}^d; X)$ and*

$$\|(m(k))_{k\in\mathbb{Z}^d\setminus\{0\}}\|_{\mathfrak{M}L^p_0(\mathbb{T}^d;X)} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}.$$

Proof. This is a slight variant of Proposition 5.7.1, which says that if every $k \in \mathbb{Z}^d$ is a Lebesgue point of $m \in L^\infty(\mathbb{R}^d)$, then $(m(k))_{k\in\mathbb{Z}^d}$ is a Fourier multiplier on $L^p(\mathbb{T}^d; X)$ of at most the norm of the Fourier multiplier m on $L^p(\mathbb{R}^d; X)$. A slight obstacle is that 0 may fail to be a Lebesgue point of our $m(\xi)$, no matter how we define $m(0)$. But, if we only consider the action of these operators on $L^p_0(\mathbb{T}^d; X)$, the 0th frequency never shows up, and one can check that the proof of Proposition 5.7.1 also applies, with trivial modifications, to the case that each $k \in \mathbb{Z}^d \setminus \{0\}$ is a Lebesgue point, giving exactly what we claimed. □

Proof of Theorem 13.3.5. We begin by essentially the same reduction as in the proofs of both Theorems 5.2.10 and 10.5.1 (the necessity of UMD for the boundedness of the Hilbert transform and the imaginary powers of the Laplacian, respectively); but we repeat this short step for the reader's convenience: By Theorem 4.2.5 it suffices to estimate the dyadic UMD constant. In order to most conveniently connect this with Fourier analysis, we choose a model of the Rademacher system $(r_k)_{k=1}^n$, where the probability space is $\mathbb{T}^{dn} = \mathbb{T}^d_1 \times \cdots \times \mathbb{T}^d_n$ (each $\mathbb{T}^d_k$ is simply an indexed copy of $\mathbb{T}^d$), and $r_k = r_k(t_k)$ is a function of the kth coordinate $t_k \in \mathbb{T}^d_k$ only. Moreover, we are free to choose any instance of such function, as long as it takes both values ± 1 on subsets of $\mathbb{T}^d$ of measure $\frac{1}{2}$. Then it is sufficient to prove that

$$\Big\| \sum_{k=1}^n \epsilon_k f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant K \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)},$$

where K is the constant on the right of (13.34), for all signs $\epsilon_k = \pm 1$, for all f_k of the form $f_k = \phi_k(r_1, \ldots, r_{k-1})r_k$; these are precisely the martingale differences of Paley–Walsh martingales (see Proposition 3.1.10). We use the convention that $L^p(\mathbb{T}^0; X) := X$.

Let us then observe that, with suitable choice of the invertible matrices A_j, $j = 1, 2$, the multipliers $m_j(\xi) = m(A_j\xi)$ (of the same multiplier norm as the original m) are stably constant in the directions of $\pm e_k$, $k = 1, 2$, and moreover $m_j(\pm e_k) = m(u_1)$ if $j = k$ and $m_j(\pm e_k) = m(u_2)$ if $j \neq k$. Defining yet another multiplier $m' = \frac{1}{2}(m_1 - m_2)$ (of at most the same multiplier norm as m), we find that m' is also stably constant in the directions of $\pm e_k$, $k = 1, 2$, and moreover $m'(\pm e_1) = \frac{1}{2}(m(u_1) - m(u_2)) =: a$ and $m'(e_2) = -a$. If we can prove the claim with m', e_1, e_2 in place of the original m, u_1, u_2, then the original claim also follows from

$$\beta_{p,X} \leqslant \frac{2\|m'\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m'(e_1) - m'(e_2)|} \leqslant \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u_1) - m(u_2)|}.$$

Dropping the primes, we assume without loss of generality that $m(\pm e_1) = a = -m(\pm e_2)$, and m is stably constant in the directions of $\pm e_j$, $j = 1, 2$.

From Proposition 13.3.4 and Lemma 13.3.6 we know that, for suitable functions f_k,

$$\begin{aligned}\Big\| \sum_{k=1}^n \widetilde{T}_k f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} &\leqslant \|m\|_{\mathfrak{M}L^p_0(\mathbb{T}^d;X)} \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)},\end{aligned}$$

where T_k is a copy $\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}}$ acting in the kth $\mathbb{T}^d_k$, thus

$$T_k f_k = \phi_k(r_1, \ldots, r_{k-1})\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} r_k.$$

The required condition on f_k above is that its Fourier coefficients with respect to the variable t_k should be non-zero only in the directions, where m is stably constant, i.e., only in the directions $\pm e_1$ and $\pm e_2$. Given the product form of f_k, this means more simply that r_k should have non-zero Fourier coefficients only in these directions, which holds in particular if r_k is a function of only the first or only the second coordinate. Note that this gives still (more than) enough flexibility to make r_k equidistributed with a Rademacher variable.

Now, given a sequence $(\varepsilon_k)_{k=1}^r$, we choose r_k to be a function of the first coordinate if $\varepsilon_k = +1$, and of the second coordinate if $\varepsilon_k = -1$. It then follows that in either case $\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} r_k = a\varepsilon_k r_k$, and we conclude that

$$\Big\|\sum_{k=1}^{n}\epsilon_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} = \frac{1}{|a|}\Big\|\sum_{k=1}^{n} T_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)}$$
$$\leqslant \frac{2}{|m(e_1)-m(e_2)|}\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\Big\|\sum_{k=1}^{n} f_k\Big\|_{L^p(\mathbb{T}^{dn};X)},$$

which is what we claimed. □

For the sake of precise quantitative conclusions, we also record the following variant of Theorem 13.3.5. The assumptions of the next result are much stronger than those of Theorem 13.3.5, so that the qualitative conclusion that X is a UMD space is immediate from the previous theorem. The point of this variant is that under the stronger assumption we can directly estimate the complex UMD constant $\beta^{\mathbb{C}}_{p,X}$ of X. The result is not strictly a corollary of Theorem 13.3.5 itself, but follows by a modification of its proof, as we are about to see.

Corollary 13.3.7. *Let $d \geqslant 2$ and $m \in C(\mathbb{R}^d \setminus \{0\})$ be an even, homogeneous multiplier whose range contains the complex unit circle. If $m \in \mathfrak{M}L^p(\mathbb{R}^d;X)$, then X is a UMD space and*

$$\beta^{\mathbb{C}}_{p,X} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}.$$

Proof. By the same reductions and notation as in the proof of Theorem 13.3.5, we now need to check that

$$\Big\|\sum_{k=1}^{n}\sigma_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}\Big\|\sum_{k=1}^{n} f_k\Big\|_{L^p(\mathbb{T}^{dn};X)},$$

for any $\sigma_k \in \mathbb{C}$ with $|\sigma_k| = 1$. By the assumption about the range of m, we can further write $\sigma_k = m(u_k)$ for some $u_k \in \mathbb{C}$ with $|u_k| = 1$.

Consider a large number $R > 0$. For each k, we can find an integer vector $n_k \in \mathbb{Z}^d$ such that $\|n_k - Ru_k\|_{\ell^\infty} \leqslant \frac{1}{2}$. Thus $\|u_k - R^{-1}n_k\|_{\ell^\infty} \leqslant \frac{1}{2R}$. Since m is continuous, by choosing R large enough we ensure that $|m(u_k)-m(R^{-1}n_k)| \leqslant \delta$ for each $k = 1,\dots,n$ and any given $\delta > 0$. Thus

$$\Big\|\sum_{k=1}^{n}\sigma_k f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} = \Big\|\sum_{k=1}^{n} m(u_k) f_k\Big\|_{L^p(\mathbb{T}^{dn};X)}$$
$$\leqslant \Big\|\sum_{k=1}^{n} m(n_k) f_k\Big\|_{L^p(\mathbb{T}^{dn};X)} + \sum_{k=1}^{n}\delta\|f_k\|_{L^p(\mathbb{T}^{dn};X)},$$

where we also used the homogeneity $m(R^{-1}n_k) = m(n_k)$.

We now come to our choice of the Rademachers functions r_k appearing in the martingale differences $f_k = \phi_k(r_1,\dots,r_{k-1})r_k$. Fixing any Rademacher function r on $\mathbb{T}$, we take $r_k(t) := r(n_k \cdot t)$ for $t \in \mathbb{T}^d$. Substituting $n_k \cdot t$ into the Fourier series of r, we find that

$$r_k(t) = \sum_{j\in\mathbb{Z}} \widehat{r}(j) e^{2\pi i j n_k\cdot t}$$

has a Fourier series involving only frequencies that are multiples of the vector n_k. By the homogeneity of m again, this means that

$$\widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} r_k = m(n_k) r_k,$$

and thus

$$\begin{aligned}\Big\| \sum_{k=1}^n m(n_k) f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} &= \Big\| \sum_{k=1}^n \widetilde{T}_{(m(j))_{j\in\mathbb{Z}^d\setminus\{0\}}} f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)}.\end{aligned}$$

Collecting the estimates, we have checked that

$$\begin{aligned}\Big\| \sum_{k=1}^n \sigma_k f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &\Big\| \sum_{k=1}^n f_k \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &+ \delta \sum_{k=1}^n \|f_k\|_{L^p(\mathbb{T}^{dn};X)},\end{aligned}$$

or in other words

$$\begin{aligned}&\Big\| \sum_{k=1}^n \sigma_k r_k \phi_k(r_1,\dots,r_{k-1}) \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\qquad \leqslant \|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \Big\| \sum_{k=1}^n r_k \phi_k(r_1,\dots,r_{k-1}) \Big\|_{L^p(\mathbb{T}^{dn};X)} \\ &\qquad\quad + \delta \sum_{k=1}^n \|r_k \phi_k(r_1,\dots,r_{k-1})\|_{L^p(\mathbb{T}^{dn};X)}.\end{aligned}$$

While the specific choice of the Rademacher functions r_k depended on the numbers n_k, which in turn depended on δ, it is clear that this last bound is true for any Rademacher sequence $(r_k)_{k=1}^n$, as soon as it is true for one. Once this observation is made, we see that everything is independent of δ, and taking the limit $\delta \to 0$, we obtain the required bound. □

Corollary 13.3.8. *Let X be a Banach space, $d \geqslant 2$ and $p \in (1,\infty)$. If any of the following operators is bounded on $L^p(\mathbb{R}^d; X)$, then X is a UMD space:*

(1) *a second-order Riesz transform $R_j R_k$, $1 \leqslant j, k \leqslant d$,*
(2) *their non-zero difference $R_j^2 - R_k^2$, $1 \leqslant j \neq k \leqslant d$,*
(3) *the Beurling transform $B = (R_2^2 - R_1^2) + i2R_1R_2$ $(d = 2)$.*

Moreover, we have the following estimates:

(1) $\beta^{\mathbb{R}}_{p,X} \leqslant 2\|R_j R_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$,
(2) $\beta^{\mathbb{R}}_{p,X} \leqslant \|R_j^2 - R_k^2\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$,
(3) $\beta^{\mathbb{C}}_{p,X} \leqslant \|B\|_{\mathscr{L}(L^p(\mathbb{R}^2;X))}$.

Proof. These operators correspond to the multipliers

$$m_{R_jR_k}(\xi) = -\frac{\xi_j\xi_k}{|\xi|^2}, \quad m_{R_j^2-R_k^2}(\xi) = -\frac{\xi_j^2-\xi_k^2}{|\xi|^2}, \quad m_B(\xi) = -\frac{\xi_1 - i\xi_2}{\xi_1 + i\xi_2},$$

each of which is even and homogeneous, in particular stably constant in all directions.

Writing $\xi_1 + i\xi_2$ in the polar coordinates as $re^{i\theta}$, it is clear that $m_B(\xi) = m_B(re^{i\theta}) = -e^{-i2\theta}$ takes all values in the complex unit circle. Hence the claims concerning B are immediate from Corollary 13.3.7.

For R_jR_k, we observe that $m_{R_j^2}(\xi) = -\xi_j^2/|\xi|^2$ is -1 for $\xi = e_j$ and 0 for $\xi = e_k$, $k \neq j$, whereas $m_{R_jR_k}(\xi) = -\frac{1}{2}$ for $\xi = (e_j+e_k)$ and $\frac{1}{2}$ for $\xi = (e_j-e_k)$ when $k \neq j$; in each case we have $|m(u_1) - m(u_2)| = 1$ for suitable vectors u_i. For $R_j^2 - R_k^2$, the multiplier is -1 for $\xi = e_j$ and $+1$ for $\xi = e_k$, so that $|m(e_j) - m(e_k)| = 2$. In each case, the claimed conclusion is immediate from Theorem 13.3.5. □

Corollary 13.3.8 allows us to complete a characterisation of a function space embedding that we studied in Section 5.6:

Corollary 13.3.9. *Let X be a Banach space, let $d, k \geqslant 1$ and $p \in (1,\infty)$. Then there is a constant C such that*

$$\|f\|_{W^{k,p}(\mathbb{R}^d;X)} \leqslant \|f\|_{H^{k,p}(\mathbb{R}^d;X)} \qquad \forall f \in \mathscr{S}(\mathbb{R}^d;X)$$

if and only if at least one of the following holds:

(1) *$d = 1$ and k is even, or*
(2) *X is a UMD space.*

Proof. The sufficiency of (1) has been established in Proposition 5.6.10 and the sufficiency of (2) in Theorem 5.6.11. Moreover, in Theorem 5.6.12, it has been shown that the UMD property is necessary when k is odd, and that the boundedness of the second-order Riesz transform R_1^2 is necessary when k is even and $d \geqslant 2$. By Corollary 13.3.8, the UMD property follows from this, and hence it is necessary in all cases except (1). □

In our final corollary to Theorem 13.3.5, we dispense with the evenness condition.

Corollary 13.3.10. *Let $d \geqslant 1$ and $m \in C(\mathbb{R}^d \setminus \{0\})$ be any positively homogeneous multiplier (i.e., $m(\lambda\xi) = m(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\lambda > 0$) that is not identically constant. If $m \in \mathfrak{M}L^p(\mathbb{R}^d; X)$, then X is a UMD space and*

$$\beta^{\mathbb{R}}_{p,X} \leqslant \min_{u_1,u_2 \in S^{d-1}} \frac{4\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u_1) + m(-u_1) - m(u_2) - m(-u_2)|},$$

$$\beta^{\mathbb{R}}_{p,X} \leqslant (\hbar_{p,X})^2 \leqslant \Big(\min_{u \in S^{d-1}} \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u) - m(-u)|} \Big)^2,$$

where at least one of the right-hand sides is finite.

The assumption that m is not identically constant, rather than the perhaps expected "not identically zero", is necessary: the Fourier multiplier T_m with $m \equiv c$ coincides with the scalar multiplication $f \mapsto c \cdot f$, whose boundedness certainly needs no UMD.

Proof. As pointed out right before Proposition 5.3.7, the assumption that $m \in \mathfrak{M}L^p(\mathbb{R}^d; X)$ implies the same property for the reflected function $\widetilde{m}(\xi) := m(-\xi)$. Then, by the triangle inequality, the even and odd parts $m_{\text{even}} := \frac{1}{2}(m + \widetilde{m})$ and $m_{\text{odd}} := \frac{1}{2}(m - \widetilde{m})$ are also positively homogeneous multipliers of at most the same multiplier norm as m. Since m is not identically constant, and $m = m_{\text{even}} + m_{\text{odd}}$, at least one of m_{even} or m_{odd} is not identically constant.

If m_{even} is not identically constant, there are two directions $u_1, u_2 \in S^{d-1}$ such that $m_{\text{even}}(u_1) \neq m_{\text{even}}(u_2)$ and hence, by evenness,

$$m_{\text{even}}(-u_1) = m_{\text{even}}(u_1) \neq m_{\text{even}}(u_2) = m_{\text{even}}(-u_2).$$

By Example 13.3.2, the homogeneous $m_{\text{even}} \in C(\mathbb{R}^d \setminus \{0\})$ is stably constant in every directions. Hence m_{even} satisfies the assumptions of the Geiss–Montgomery-Smith–Saksman Theorem 13.3.5, and the said theorem guarantees that, for any such $u_1, u_2 \in S^{d-1}$,

$$\begin{aligned}\beta^{\mathbb{R}}_{p,X} &\leqslant \frac{2\|m_{\text{even}}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{m_{\text{even}}(u_1) - m_{\text{even}}(u_2)} \\ &\leqslant \frac{4\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{m(u_1) + m(-u_1) - m(u_2) - m(-u_2)}.\end{aligned}$$

(Note that the condition that $m_{\text{even}}(u_1) \neq m_{\text{even}}(u_2)$ is precisely the requirement that the denominator is non-zero, and hence can extend the previous display to all pairs of $u_1, u_2 \in S^{d-1}$; interpreting $1/0 = \infty$, as usual, this only amounts to adding the triviality $\beta^{\mathbb{R}}_{p,X} \leqslant \infty$.)

For the odd part m_{odd}, being not identically constant is equivalent to being not identically zero. If this is the case, there is some direction $u \in S^{d-1}$ such that $m(-u) = -m(u) \neq 0$. Writing $\xi \in \mathbb{R}^d$ as $\xi = (\xi \cdot u)u + [\xi - (\xi \cdot u)u]$, we consider the invertible linear transformations $A_\lambda \xi = (\xi \cdot u)u + \lambda[\xi - (\xi \cdot u)u]$, where $\lambda > 0$. By Proposition 5.3.8, each $m_{\rm odd} \circ A_\lambda$ has the same multiplier norm as $m_{\rm odd}$. As $\lambda \to 0$, it is clear that $A_\lambda \xi \to (\xi \cdot u)u$ for all $\xi \in \mathbb{R}^d$ and thus, by the continuity of m and hence $m_{\rm odd}$,

$$m_{\rm odd} \circ A_\lambda(\xi) \to m_{\rm odd}((\xi \cdot u)u) = \operatorname{sgn}(\xi \cdot u) m_{\rm odd}(u).$$

A convergence result for multipliers, Proposition 5.3.16, then implies that

$$\begin{aligned} |m_{\rm odd}(u)| \|\xi \mapsto \operatorname{sgn}(\xi \cdot u)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &\leqslant \liminf_{\lambda \to 0} \|m_{\rm odd} \circ A_\lambda\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\ &= \|m_{\rm odd}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}. \end{aligned}$$

By another application of Proposition 5.3.8 with a rotation that sends u to e_1, it follows that

$$\begin{aligned} \|\xi \mapsto \operatorname{sgn}(\xi_1)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} &= \|\xi \mapsto \operatorname{sgn}(\xi \cdot u)\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \\ &\leqslant \frac{\|m_{\rm odd}\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m_{\rm odd}(u)|} \leqslant \frac{2\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)}}{|m(u) - m(-u)|}. \end{aligned}$$

(The bound remains valid for all $u \in S^{d-1}$, reducing to a triviality if $m(u) = m(-u)$.) By Fubini's theorem, we find that

$$\hbar_{p,X} := \|\xi \mapsto \operatorname{sgn}(\xi)\|_{\mathfrak{M}L^p(\mathbb{R};X)} \|\xi \mapsto \operatorname{sgn}(\xi_1)\|_{\mathfrak{M}L^p(\mathbb{R};X)}.$$

The bound between $\beta^{\mathbb{R}}_{p,X} \leqslant (\hbar_{p,X})^2$ is contained in Corollary 5.2.11. □

13.4 Notes

Section 13.1

The precise quantitative form of the final bound in the comparison of various Fourier-type constants in Proposition 13.1.1 seems to be new; we were not aware of this estimate at the time of completing Volume II, where a weaker version was given. The identity $\varphi_{p,\mathbb{C}}(\mathbb{R}^d) = (p^{1/p}(p')^{-1/p'})^d$ mentioned below the said proposition is due to Babenko [1961] in the special case that p' is an even integer, and due to Beckner [1975] in full generality.

The main result of this section, Theorem 13.1.33 is from Bourgain [1988a], with preliminary versions going back to Bourgain [1981, 1982]. The main theorem of Bourgain [1982] reads as follows: If X is a B-convex Banach space (which is equivalent to non-trivial type by Proposition 7.6.8), then there are $u, v \in (1,\infty)$ and $\delta, M \in (0,\infty)$ such that

$$\delta\Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^u\Big)^{1/u} \leqslant \Big\|\sum_{\gamma\in\Gamma} \gamma x_\gamma\Big\|_{L^2(G;X)} \leqslant M\Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^v\Big)^{1/v}, \qquad (13.35)$$

whenever $\{x_\gamma\}_{\gamma\in\Gamma}$ is a finitely non-zero sequence of elements of X and Γ is the spectrum of the compact abelian group G. This is a Hausdorff–Young inequality with mismatched exponents; our Corollary 13.1.27 is the special case of the right-hand inequality with $G = \mathbb{T}$ and $\Gamma = \{e_k\}_{k\in\mathbb{Z}}$. For these particular G and Γ, and under the stronger assumption that X be super-reflexive, (13.35) was proved in Bourgain [1981]. A further predecessor of such results is due to James [1972], who proved a bound like (13.35) with a super-reflexive Z in place of both X and $L^2(G;X)$, and $z_k \in Z$ in place of both x_γ and γx_γ, under the assumption that $(z_k)_{k=1}^\infty$ is a *basic sequence* in Z, i.e.,

$$\Big\| \sum_{k=1}^{K} a_k z_k \Big\|_Z \leqslant C \Big\| \sum_{k=1}^{L} a_k z_k \Big\|_Z \tag{13.36}$$

for all scalars a_k and integers $K \leqslant L$. Requiring (13.36) for $z_k = e_k x_k \in Z = L^2(\mathbb{T};X)$, uniformly in $x_k \in X$, is equivalent to the still stronger property that X be a UMD space, which is why additional work was required by Bourgain [1981] to obtain his result for trigonometric series in super-reflexive spaces. (The estimate (13.36) in the said special case is equivalent to the $L^2(\mathbb{T};X)$-boundedness of the periodic Hilbert transform by Proposition 5.2.7, and this is equivalent to the UMD property by Corollary 5.2.11. UMD spaces are super-reflexive by Corollary 4.3.8, but the converse is false. Various examples showing the last point are due to Pisier [1975], Bourgain [1983], Garling [1990], Geiss [1999], and Qiu [2012]. The example of Qiu [2012] is an infinitely iterated $L^p(L^q)$ space, which has been presented in Theorem 4.3.17, but the super-reflexivity of this space is not treated there.)

As in our treatment in the section under discussion, getting from estimate (13.35) with mismatched exponents to dual pairs requires further ideas. This was achieved by Bourgain [1988b], who proved that, for some $u_1, v_1 \in (1,\infty)$ and $\delta_1, M_1 \in (0,\infty)$, there further holds

$$\begin{aligned}\delta_1 \Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^{u_1'} \Big)^{1/u_1'} &\leqslant \Big\| \sum_{\gamma\in\Gamma} \gamma x_\gamma \Big\|_{L^{u_1}(G;X)} \\ &\leqslant \Big\| \sum_{\gamma\in\Gamma} \gamma x_\gamma \Big\|_{L^{v_1'}(G;X)} \leqslant M_1 \Big(\sum_{\gamma\in\Gamma} \|x_\gamma\|_X^{v_1} \Big)^{1/v_1},\end{aligned} \tag{13.37}$$

when G is either $\mathbb{T}$ or the Cantor group $\{-1,1\}^{\mathbb{N}}$. For $G = \mathbb{T}$, the leftmost and rightmost estimates correspond, in our notation, to $\varphi_{u_1,X}(\mathbb{T}) \leqslant 1/\delta_1$ and $\varphi_{v_1,X}(\mathbb{Z}) \leqslant M_1$, respectively. The easy estimate $\varphi_{p,X}(\mathbb{R}) \leqslant \varphi_{p,X}(\mathbb{T})$ was also observed by Bourgain [1988b]. In contrast to the case of $\mathbb{T}$, a scaling argument (substituting $f(\lambda\cdot)$ in place of f and considering the limit $\lambda \to 0$ or $\lambda \to \infty$) shows that an estimate of the from $\|\widehat{f}\|_{L^q(\mathbb{R};X)} \leqslant C\|f\|_{L^p(\mathbb{R};X)}$ can only hold for $q' = p$; thus, in order to deduce any Hausdorff–Young inequality on $\mathbb{R}$ at all, the additional steps from the mismatched exponents of Bourgain [1982] to the dual exponents of Bourgain [1988b] seem to be necessary.

The second half of the argument leading to Bourgain's Theorem 13.1.33, as presented in Sections 13.1.c and 13.1.d, is close to the treatment of Bourgain

[1988b], although we have also benefited from the exposition of these steps by Pietsch and Wenzel [1998]. On the other hand, the first half of our treatment, in Sections 13.1.a and 13.1.b, is also based on Pietsch and Wenzel [1998] but deviates from the original approach of Bourgain [1982]. The beginning of the argument, leading to Proposition 13.1.11 on "breaking the trivial bound" is due to Hinrichs [1996], but it also uses a result of Bourgain [1985], Proposition 13.1.7, on the Sidon property of quasi-independent sets.

We have chosen this approach of Hinrichs [1996] and Pietsch and Wenzel [1998] due to an independent interest, in our opinion, of some of its intermediate steps, despite the fact that the original argument of Bourgain [1982, 1988b] seems slightly more efficient in terms of the final quantitative conclusions. In any case, the main result says that every Banach space of type $p \in (1, 2]$ will have Fourier-type $r = 1 + (c\tau_{p,X;2})^{-p'}$, for some absolute constant c. (The additional factor $6p'$ in our formulation of Theorem 13.1.33 could obviously be absorbed by choosing a larger constant c.) The difference is in the numerical value of c, which is 68 in our formulation (up to the lower order factor just mentioned) and 17 in Bourgain [1982, 1988b].

In our approach, this constant comes from the proof of Corollary 13.1.20, where the estimate $48\sqrt{2}$ (≈ 67.88) $\leqslant 68$ is made. (Since we are clearly off Bourgain's constant at this point already, it would seem pointless to insist in the decimals here.) The constant $48\sqrt{2}$, in turn, is produced as

$$48\sqrt{2} = 16 \cdot \sqrt{2} \cdot 3, \qquad \text{where}$$

(i) 16 is the upper bound of the Sidon constants of quasi-independent sets from Proposition 13.1.7;
(ii) $\sqrt{2}$ comes from the factor in front of the upper bound of the number of quasi-independent sets required to partition a given set in Lemma 13.1.9; the root is due to the use of this number count after an application of the Cauchy–Schwarz inequality in the proof of Proposition 13.1.11;
(iii) 3 is the constant from the Marcinkiewicz inequality (Proposition 13.1.17), which enters into the estimate through an application of the Comparison Lemma 13.1.18 in the proof of Corollary 13.1.20.

One may speculate that the constant 16 (just below the 17 of Bourgain [1982]) is the heart of the matter, and the other two factors are only produced by secondary details that should be avoidable by more careful reasoning.

The approach of Bourgain [1982] is based on two abstract results (avoided in the present treatment) about the collection of tuples of functions

$$\mathscr{O} := \{\boldsymbol{\xi} = (\xi_i)_{i=1}^n \in L^2(\Omega;\mathbb{R})^n : \|\xi_i\|_\infty \leqslant 1, \int \xi_i = \int \xi_i\xi_j = 0 \quad \text{for all } 1 \leqslant i \neq j \leqslant n\}$$

on a probability space Ω; namely:

(1) The set $\mathscr{E}$ of extreme points of $\mathscr{O}$ consists of tuples of ±1-valued functions.

(2) For each $\boldsymbol{\xi} \in \mathscr{O}$, there is a Borel probability measure μ on $\mathscr{E}$ such that

$$\xi_i = \int_{\mathscr{E}} \eta_i \, \mathrm{d}\mu(\boldsymbol{\eta}), \quad \text{for every } i = 1, \dots, n.$$

According to Bourgain [1982], the proof of (1) is "essentially contained in" Dor [1975], while (2) can be derived from a generalisation of Choquet's integral representation theorem due to Edgar [1976]. Combining these abstract tools with delicate hard analysis, Bourgain [1982] eventually arrives at his key technical estimate, which in our notation (and exchanging the roles of X and X^* compared to Bourgain [1982]) may be stated as

$$\varphi^{(2)}_{\infty,X^*}(F) \leqslant K \cdot N^{1/t}, \quad t' = (17 \cdot \tau_{p,X;2})^{p'}. \tag{13.38}$$

This is recognised as a close relative of Corollary 13.1.29, where the bound

$$\varphi^{(s')}_{\infty,X^*}(F) \leqslant K \cdot N^{1/s}, \quad s' > r' = 3p'(68 \cdot \tau_{p,X;2})^{p'}$$

is obtained. While the left-hand sides are not identical, (13.38) allows Bourgain [1982] to deduce the Hausdorff–Young inequality with mismatched exponents as in (13.35) (with X^* in place of X) for any $v \in (1,t)$, and finally, in Bourgain [1988b], also the classical Hausdorff–Young inequality (13.37) (again with X^* in place of X) with any $u_1 \in (1,v)$. Since $v \in (1,t)$ is arbitrary, one can reach any $u_1 \in (1,t)$, and thus in particular the r determined by

$$r' = (18 \cdot \tau_{p,X;2})^{p'} \tag{13.39}$$

is a Fourier type of X^*, and hence of X.

Remark 13.4.1 (A typo in the statement of Bourgain's theorem in König [1991]). It seems to be claimed by König [1991] that every space of type $p > 1$ would have Fourier-type r with $r' = c \cdot \tau^{p'}_{p,X;2}$ and $c = 18$ (forgetting brackets from (13.39)). As written, this is absurd for any absolute constant c:

It is straightforward to verify that, for every $p \in (1,2]$, the space $X = \ell^p$ has type p with constant $\tau_{p,X;2} = 1$:

$$\begin{aligned} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;\ell^p)} &\leqslant \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{\ell^p(L^2(\Omega))} = \Big\| \{x_n\}_{n=1}^N \Big\|_{\ell^p(\ell^2_N)} \\ &\leqslant \Big\| \{x_n\}_{n=1}^N \Big\|_{\ell^p(\ell^p_N)} = \Big(\sum_{n=1}^N \|x_n\|_X^p \Big)^{1/p} \end{aligned}$$

Thus, were the claim in the beginning of the remark true, all these spaces would have the Fourier-type $r = \frac{c}{c-1} > 1$, which is impossible for $p \in (1,r)$ by Example 2.1.15.

Since the numerical constant in (13.38) may be affected by an equivalent choice of the type constant, we note that Bourgain [1982] is not explicit about the precise definition of the constant that he denotes by C, but one can see in the proof of the first step of his Proposition 4 that $C = \tau_{p,X^*;2}$; recall that we exchanged the roles of X and X^* compared to Bourgain [1982].

More details on quasi-independent sets can be found in the monograph of Graham and Hare [2013]. Sometimes quasi-independent sets are called *dissociate sets*, but it seems that in more recent works this terminology is reserved for the slightly stronger property where one allows $\alpha_k \in \{-2, -1, 0, 1, 2\}$ in Definition 13.1.5. In particular, one can find there that quasi-independent are Sidon sets with constant $6\sqrt{6} \approx 14.70$, which is slightly better than the constant 16 in Proposition 13.1.7. The converse bounds of Remark 13.1.10 have been shown to us by Dion Gijswijt. If one replaces the group $\mathbb{Z}$ by another group it was shown on page 203 in Pietsch and Wenzel [1998] that the bound of Lemma 13.1.8 is sharp.

The result of Proposition 13.1.21 states that type p and cotype q with $1/p - 1/q < 1/r - 1/2$ with $r \in (1, 2)$ implies Fourier type r. In the limiting case of equality it is unknown what happens. However, the result is sharp in the sense that for every $r \in (1, 2)$ and for every $p \in (r, 2)$ there exists a Banach space X such that X has type p, cotype q, and Fourier type r with $\frac{1}{p} - \frac{1}{q} = \frac{1}{r} - \frac{1}{2}$, and none of the exponents (p, q, r) can be improved (see Bourgain [1988a] and García-Cuerva, Torrea, and Kazarian [1996]). This example was also used to show that the dependence on the type constant is necessary in Theorem 13.1.33. The following improvement was observed in García-Cuerva, Torrea, and Kazarian [1996] for Banach lattices X:

$$\begin{aligned}\sup\{p \in (1, 2] :& X \text{ has Fourier type } p\} \\ &= \sup\{p \in (1, 2] : X \text{ has type } p \text{ and cotype } p'\}.\end{aligned}$$

Section 13.2

In the scalar-valued case, considerations of the kind that we have presented in this section go back to Hörmander [1960] who used similar methods to rederive (a variant of) the multiplier theorem of Mihlin [1956, 1957] by transforming it into a form where the theory of Calderón and Zygmund [1952] could be applied. The methods of Hörmander [1960] are already very close to the ones in the Section 13.2.b, the key difference being that he can make use of the Plancherel theorem to pass between L^2 estimates in the space and frequency variables. For functions taking values in a general Banach space, the only available substitute is the elementary $L^1(\mathbb{R}^d; X)$-to-$L^\infty(\mathbb{R}^d; X)$ boundedness of the Fourier transform. This still allows essentially similar conclusions, at the cost of requiring estimates for a higher number of derivatives as input. On the other hand, as soon as we start imposing such stronger assumptions, we can also obtain stronger conclusions, namely, standard Calderón–Zygmund kernels rather than just Hörmander kernels, as in Section 13.2.a. Scalar-valued versions of such results are again well known; for example, a version of Proposition 13.2.7 with $d+2$ derivatives (instead of $d+1$ in the said proposition) appears in the book of Stein [1993]. Under this stronger assumption, Stein [1993] deduces that $k \in C^1(\mathbb{R}^d \setminus \{0\})$, while Proposition 13.2.6 gives the slightly weaker conclusion that k is just barely below Lipschitz, with a modulus of continuity $\omega(t) = O(t \cdot \log(1+1/t))$. This is still quite enough to derive like Corollaries 13.2.8, 13.2.9, and 13.2.10 on the boundedness of Fourier multipliers on weighted $L^p(w;X)$ spaces. Using the result from Stein [1993] in place of Proposition 13.2.7, a version of Corollary 13.2.10 assuming $d+2$ derivatives was formulated by Meyries and Veraar [2015]. In principle, variants of Propositions 13.2.6 and 13.2.7 sufficient for Corollaries 13.2.8 through 13.2.10 would only require smoothness of order $d+\varepsilon$, but such statements and proofs are bound to have additional technicalities due to the very formulation of fractional order smoothness conditions. Various results in this direction, involving kernel bounds for Fourier multipliers with close-to-critical fractional smoothness, were explored by Hytönen [2004].

To get rid of the $\varepsilon > 0$ altogether, i.e., to deduce useful (in view of Calderón–Zygmund extrapolation) kernel estimates for $k = \check{m}$ from just d derivatives of m, one needs to impose assumptions on the Fourier-type of the underlying spaces. While we have only dealt with the sufficiency of the Fourier-type assumption in Section 13.2.b, an early result involving both directions, in dimension $d = 1$, is the following:

Theorem 13.4.2 (König [1991]). *A Banach space X is K-convex if and only if every $f \in C^1(\mathbb{T}, X)$ has Fourier coefficients $(\widehat{f}(n))_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z}; X)$.*

Recall that K-convexity is equivalent to non-trivial type by Pisier's Theorem 7.4.23, and non-trivial type is equivalent to non-trivial Fourier-type by Bourgain's Theorem 13.1.33. The proof of "⇒" in Theorem 13.4.2 is then straightforward from non-trivial Fourier type. For the converse, König [1991] starts with a concrete counterexample when $X = L^1(\mathbb{T})$, and approximates this finite versions that can be represented in ℓ^1_N, with blow-up in the limit $N \to \infty$. By the Maurey–Pisier Theorem 7.3.8, if X does not have non-trivial type, then it contains subspaces isomorphic to ℓ^1_N uniformly, and hence the said finite examples can also be represented in X. Finally, the closed graph theorem guarantees that a sequence of examples with blow-up also guarantees the existence of a single $f \in C^1(\mathbb{T}, X)$ with $(\widehat{f}(n))_{n\in\mathbb{Z}} \notin \ell^1(\mathbb{Z}; X)$.

In our formulation of Proposition 13.2.13, the assumed Fourier-type $p \in (1, 2]$ only affects the constant in the estimate. However, by more careful reasoning, one could show that also the number of the required derivative $\partial^\alpha m$ could be reduced as a function of p; roughly speaking, one needs only derivatives up to order $\lfloor d/p \rfloor + 1$, or more generally fractional smoothness of order $d/p + \varepsilon$, to obtain the same conclusions. Such results can be found in Hytönen [2004]. In the more general context of various function spaces, this phenomenon will be explored further in Chapter 14; see Proposition 14.5.3 and take $q = \infty$ there.

Our focus in the section under discussion has been exploring conditions that one needs to assume on a multiplier m in order that their associated kernel $k = \check{m}$ satisfies the assumptions of one of the extrapolation theorems of Chapter 11 (so that the *a priori* boundedness of T_m on one $L^{p_0}(\mathbb{R}^d; X)$ extends to other spaces), but similar considerations can also be used to reduce the required smoothness, as a function of the Fourier-type of the underlying spaces, in results like Mihlin's Multiplier Theorem 5.5.10 (where the boundedness of T_m on $L^p(\mathbb{R}^d; X)$ is deduced "from scratch"). Such results were pioneered by Girardi and Weis [2003b] and further elaborated by Hytönen [2004]. If m is scalar-valued, it is also possible to replace Fourier-type by quantitatively weaker assumptions on type or cotype; see Hytönen [2010].

Section 13.3

The main results of this section, notably Proposition 13.3.4, Theorem 13.3.5, and Corollary 13.3.8, are essentially from Geiss, Montgomery-Smith, and Saksman [2010], but we have incorporated some improvements, partially inspired by unpublished observations of Alex Amenta that he kindly shared with us.

These results may be seen as successors, in terms of both statement and proof, of Theorem 5.2.10 of Bourgain [1983] and Theorem 10.5.1 of Guerre-Delabrière [1991], which deal with the necessity of UMD for the boundedness of the Hilbert transform and the imaginary powers $(-\Delta)^{is}$ of the Laplacian, respectively. However, none of these three results contains any of the other two.

Certain elaborations of Corollary 13.3.8 are due to Castro and Hytönen [2016]. Namely, the identity $\partial_j \partial_k u = -R_j R_k \Delta u$ implies that

$$\|\partial_j \partial_k u\|_{L^p(\mathbb{R}^d;X)} \leqslant C \sum_{i=1}^{d} \|R_i^2 u\|_{L^p(\mathbb{R}^d;X)}, \tag{13.40}$$

where $C \leqslant \|R_j R_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$, but C could *a priori* be much smaller. However, Castro and Hytönen [2016] show that the seemingly weaker inequality (13.40) still implies the UMD property with the same control

$$\beta_{p,X} \leqslant 2C_{(13.40)} \tag{13.41}$$

as in Corollary 13.3.8 for $\|R_j R_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}$. More generally, the same paper proves the necessity of UMD for any member of a family of inequalities of the form

$$\|\partial^\beta u\|_{L^p(\mathbb{R}^d;X)} \leqslant C \sum_{\alpha \in \mathscr{A}} \|\partial^\alpha u\|_{L^p(\mathbb{R}^d;X)},$$

but the relation between the constants is particularly clean in the example just mentioned.

It could be of interest to identify more general criteria (subsuming previous related results) for inequalities of classical/harmonic analysis to

(1) imply the UMD property of X (as in all mentioned results), or
(2) control the UMD constant $\beta_{p,X}$ linearly by the constant in the inequality (as in Theorems 10.5.1 and 13.3.5, but not Theorem 5.2.10).

While we have concentrated, in this section, on lower bounds of multiplier norms by the UMD constants, Geiss, Montgomery-Smith, and Saksman [2010] also treat the other direction. In particular, they show that the first two bounds of Corollary 13.3.8 are actually identities:

$$\|2R_jR_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \|R_j^2 - R_k^2\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \beta_{p,X}^{\mathbb{R}} \tag{13.42}$$

for all $1 \leqslant j \neq k \leqslant d$. The upper bounds for the norms are proved by representing and estimating the operators by means of stochastic integrals. Yaroslavstev [2018] obtained further variants of these estimates for related operators. We plan to detail this in a forthcoming Volume. By (13.41), a trivial bound, and (13.42), it follows that

$$\beta_{p,X} \leqslant 2C_{(13.40)} \leqslant 2\|R_jR_k\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \beta_{p,X},$$

and hence all these quantities must be equal. In particular, as observed by Castro and Hytönen [2016], it follows that

$$C_{(13.40)}^{X=\mathbb{R}} = \frac{1}{2}\beta_{p,\mathbb{R}} = \frac{1}{2}(\max(p,p')-1),$$

using Burkholder's Theorem 4.5.7 for the last equality. We are not aware of another method than that of Geiss, Montgomery-Smith, and Saksman [2010] to determine the exact norms (13.42) or the sharp constant in (13.40), which highlights the benefits of martingale techniques even for questions of classical analysis.

In the third case of Corollary 13.3.8 concerning the Beurling–Ahlfors transform, the matching upper bound is an outstanding open problem even for $X = \mathbb{C}$ (see Problems O.1 and O.2).

More generally, Geiss, Montgomery-Smith, and Saksman [2010] prove that all real, even, and homogeneous (i.e., $m(t\xi) = m(\xi) \in \mathbb{R}$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and $t \in \mathbb{R} \setminus \{0\}$) multipliers $m \in C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfy the estimate

$$\|m\|_{\mathfrak{M}L^p(\mathbb{R}^d;X)} \leqslant C_m \cdot \beta_{p,X},$$

where C_m depends only on m. Note in particular that the estimate is linear in $\beta_{p,X}$, improving on the quadratic estimate provided by $T(1)$ Theorem 12.4.21, or the still higher order dependence in the Mihlin Multiplier Theorem 5.5.10. By elaborations of the $T(1)$ technology, linear dependence has also been obtained for a class of even non-convolution operators on $L^p(\mathbb{R};X)$ (but only in dimension $d = 1$, as written) by Pott and Stoica [2014], but beyond that the availability of linear bounds in terms of $\beta_{p,X}$ remains open. In particular, a possible linear estimate between $\beta_{p,X}$ and the norm of the Hilbert transform $\hbar_{p,X} = \|H\|_{\mathscr{L}(L^p(\mathbb{R};X))}$, in either direction, is unknown (see Problem O.6).

Certain substitute results related to the latter are due to Domelevo and Petermichl [2023c,d]. They construct a new dyadic operator and show that its boundedness is equivalent to that of the Hilbert transform, with linear dependence between the respective norms in both directions. Analogous results for the Riesz transforms are obtained in Domelevo and Petermichl [2023a,b].

Further estimates between the Hilbert transform (and variants) and decoupling constants related to the UMD constant can be found in Osękowski and Yaroslavtsev [2021].

Corollary 13.3.9 characterises situations in which there is a continuous embedding $H^{k,p}(\mathbb{R}^d; X) \hookrightarrow W^{k,p}(\mathbb{R}^d; X)$. Several related results, including versions on domains $\mathcal{O} \subseteq \mathbb{R}^d$, are due to Arendt, Bernhard, and Kreuter [2020].