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Existence and uniqueness of stratified Antarctic flows

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ABSTRACT

We analyse a second-order boundary value problem arising from the motion of stratified ocean flows in Antarctica. Using a Lyapunov-type function and the monotonicity properties of the density function and of the oceanic vorticity, we prove results on the existence and uniqueness of solutions of the problem under consideration. Finally, we discuss possible extensions of the model that admit unique solutions for more general cases of density distribution.

1. Introduction

In oceanography, *gyres* are large-scale oceanic currents triggered by the wind stress and the combined forces of gravity and Coriolis due to Earth's rotation. It is well-known that there are *seven major gyres*, located in North Atlantic, South Atlantic, Indian, North Pacific, South Pacific, Arctic and Antarctic regions respectively, each of them featuring some particular flow patterns (see the discussions in [1–4]). However, the need of addressing the climate change issues has shifted the focus to two of these regions – *Arctic and Antarctic*. Ocean circulation here, being on the one hand attached to the melting ice-caps and on the other hand being the main mean of water exchange between multiple ocean basins, plays an indisputable role in the complex and vulnerable climate system.

In the core of our study is the mathematical model derived by Constantin and Johnson in [3] that describes the *general motion of ocean currents* as an elliptic boundary value problem. It is formulated in terms of a stream function in the spherical coordinate system, and was extensively used for, among others, derivation of the ODE setting of the Arctic gyres circulation [5], in which the first author obtained a nonlinear second-order ODE, and after combining it with an infinite-interval boundary conditions, the existence and uniqueness of solutions for different profiles of the oceanic vorticity function have been proved. Note that the equation from [5] can also describe the motion of Antarctic gyres if it is coupled with Dirichlet or Neumann boundary conditions on finite intervals [6,7]. For recent results in these directions, we refer to [4,8–14] on Arctic gyres and [15–21] on Antarctic gyres.

Despite the importance of the aforementioned studies, they neglected a very crucial feature of the ocean water — its *density variations* due to changes in salinity, temperature and pressure. This resulted in a series of papers devoted to revision of the existing mathematical models of oceanic gyres, where the *stratification effect* was initially not encountered (see discussions in [22,23]). In this work, we build upon these recent results to study the existence and uniqueness of solutions of the mathematical model of the *stratified flow in Antarctica*. Our paper is organised al follows. In Section 2 we present a PDE formulation of the mathematical model of the gyre circulation with its reduction to the ODE setting. In Section 3 we prove the results on the existence and uniqueness of solutions. Finally, in Section 4 we discuss possible extensions of the given model for incorporation of a broader class of density functions.

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2. Mathematical model of stratified ocean flows in Antarctica

In recent works [22,23], the following nonlinear elliptic equation was derived to describe the motion of *stratified ocean gyres*

$$\Delta\psi = F(\psi) - 2\omega\sqrt{\rho(\psi)}\sin\theta - \frac{1}{2}\omega^2\rho'(\psi)\sin^2\theta, \tag{1}$$

where $\omega > 0$ is the non-dimensional Coriolis parameter, the smooth function F is the oceanic vorticity and ρ is the density. By introducing a modified stereographic projection with the centre of projection being at the South Pole, equation 1 can be rewritten as the following semi-linear elliptic PDE

$$\Delta\psi + 8\omega\frac{\sqrt{\rho(\psi)}(1-x^2-y^2)}{\left(1+x^2+y^2\right)^3} + 2\omega^2\frac{\rho'(\psi)(1-x^2-y^2)^2}{(1+x^2+y^2)^4} - \frac{4F(\psi)}{\left(1+x^2+y^2\right)^2} = 0, \tag{2}$$

by regarding (x, y) as the Cartesian coordinates in the complex plane. Finally, using the change of variables $r = e^{-t}$ and $\psi(r) = u(t)$, equation 2 can be transformed into the second-order ODE that reads

$$u''(t) = \frac{F(u(t))}{\cosh^2(t)} - \frac{2\omega\sqrt{\rho(u(t))}\sinh(t)}{\cosh^3(t)} - \frac{\omega^2\rho'(u(t))\sinh^2(t)}{2\cosh^4(t)}. \tag{3}$$

The existence, uniqueness and stability of solutions of 3, complemented by the following the asymptotic conditions

$$\lim_{t\to\infty}\{u(t)\} = \psi_0 \quad \text{and} \quad \lim_{t\to\infty}\left\{u'(t)\cosh(t)\right\} = 0, \tag{4}$$

have been proved in [22], where the constant ψ_0 is the value of the stream function ψ at the North Pole, and conditions (4) express the fact that the flow is stagnant at the North Pole. Note that for constant density, equation 3 reduces to the following ODE

$$u''(t) = \frac{F(u(t))}{\cosh^2(t)} - 2\omega\frac{\sinh(t)}{\cosh^3(t)},$$

which together with (4) is precisely the model derived in [5].

When substituting (4) by Dirichlet or Neumann boundary conditions on a finite interval, equation 3 is also applicable to modelling of the *stratified Antarctic Circumpolar Current* (ACC), which flows eastward through the southern regions of the Atlantic, Indian and Pacific oceans [2]. In this paper, we focus on the qualitative analysis of 3 under Dirichlet boundary constraints:

$$u(0) = 0, \quad u(1) = 0,$$

which means that we regard the gyre as a region delimited by a streamline (a level set of ψ). We point out that the analysis in studying the Arctic gyres cannot be used to study the ACC because there are some essential differences between infinite-interval and finite-interval BVPs.

3. Main results

The proof of our main result relies on the following Lemma, which is a direct consequence of the main result from [24].

Lemma 1. *Let $f(t, y)$ be continuous on $[a, b+\varepsilon) \times \mathbb{R}$ with $\epsilon > 0$ and assume that all solutions of initial value problems for equation*

$$y'' + f(t, y) = 0 \tag{5}$$

exist on $[a, b+\varepsilon)$. Suppose further that there do not exist two solutions on $[a, c]$ to any of the problems of equation 5 with $y(a) = A, y(c) = C$, where A is fixed and $c \in (b-\varepsilon, b+\varepsilon), C \in \mathbb{R}$. Then there exists exactly one solution of equation 5 with boundary conditions $y(a) = A, y(b) = B$.

Let us rewrite our problem of modelling circulation of the stratified ACC in the following form

$$\begin{cases} u''(t) = a(t)F(u(t)) + \omega b(t)\sqrt{\rho(u(t))} + \omega^2 c(t)\rho'(u(t)), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{6}$$

where

$$a(t) = \frac{1}{\cosh^2(t)}, \quad b(t) = -\frac{2\sinh(t)}{\cosh^3(t)}, \quad c(t) = -\frac{\sinh^2(t)}{2\cosh^4(t)}.$$

From the physical perspective, we can assume that the oceanic vorticity $F(u)$ is continuous, and that the density function $\rho(u)$ is non-negative, bounded and has a bounded derivative, meaning that there exist positive constants $k, l > 0$ such that for all $t \in [0,1]$,

$$0 < \rho(u) \le l, \quad |\rho'(u)| \le k. \tag{7}$$

Theorem 1. *Assume that the inequalities 7 hold, and the continuous oceanic vorticity $F : \mathbb{R} \to \mathbb{R}$ satisfies the condition:*

$$\lim_{|u|\to\infty}\int_0^u [F(u)]\,ds = \infty. \tag{8}$$

Suppose further that there exists a constant $K > 0$ *and a strictly increasing continuous function* $W : [0,\infty) \to [0,\infty)$ *with* $W(0) = 0$, $W(\xi) > 0, \xi > 0$, *for which the conditions*

$$K + \int_0^u F(s)ds \geq W^{-1}(F^2(u)), \quad u \in \mathbb{R}, \tag{9}$$

and

$$\int_1^\infty \frac{du}{W(u)} = \infty \tag{10}$$

hold. Then for any given initial values u_0, u_0', *there is an* $\varepsilon > 0$ *such that the solution of the following initial-value problem*

$$\begin{cases} u''(t) = a(t)F(u(t)) + \omega b(t)\sqrt{\rho(u(t))} + \omega^2 c(t)\rho'(u(t)), \quad t \in (0,1), \\ u(0) = u_0, \quad u'(0) = u_0', \end{cases} \tag{11}$$

exists on $t \in [0, 1+\varepsilon)$ *and does not blow up for any finite* t.

Proof. Let us first rewrite problem (11) as the following equivalent system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = a(t)F(x(t)) + \omega b(t)\sqrt{\rho(x(t))} + \omega^2 c(t)\rho'(x(t)), \\ x(0) = u_0, \quad y(0) = u_0', \end{cases} \tag{12}$$

and construct a positive-defined Lyapunov-type function $V(x,y) : \mathbb{R}^2 \to [0,\infty)$ as

$$V(x,y) = y^2 + 2\int_0^x \left[F(s) + \sqrt{\rho(s)}\right] ds + 2\rho(x) + 2K.$$

Note that condition (8) ensures that $V(x,y) \to \infty$ for $|x| + |y| \to \infty$, whereas inequality 9 yields

$$K + \int_0^x F(s)ds \geq 0, \quad x \in \mathbb{R}.$$

By direct computations, the derivative of the functional $V(x,y)$ along the solution (x,y) of the system (12) is given by

$$\frac{dV}{dt} = \left[2F(x(t)) + 2\sqrt{\rho(x(t))} + 2\rho'(x(t))\right] y(t) + 2y(t)\left[a(t)F(x(t)) + \omega b(t)\sqrt{\rho(x(t))} + \omega^2 c(t)\rho'(x(t))\right]. \tag{13}$$

After setting

$$\gamma := \max_{t\in[0,1]} \left\{1 + a(t), 1 + \omega|b(t)|, 1 + \omega^2|c(t)|\right\} > 1,$$

we can show that $\frac{dV}{dt}$ from (13) satisfies the following inequality

$$\begin{aligned} \frac{dV}{dt} &\leq 2\left[|F(x(t))| + \sqrt{\rho(x(t))} + |\rho'(x(t))|\right]|y(t)| + 2(\gamma-1)\left[|F(x(t))| + \sqrt{\rho(x(t))} + |\rho'(x(t))|\right]|y(t)| \\ &= \gamma\left[2|F(x(t))| + 2\sqrt{\rho(x(t))} + 2|\rho'(x(t))|\right]|y(t)|, \end{aligned}$$

which under condition

$$\begin{aligned} F^2(x) &\leq W\left(K + \int_0^x F(s)ds\right) \\ &\leq W\left(y^2 + 2\int_0^x \left[F(s) + \sqrt{\rho(s)}\right]ds + 2\rho(x) + 2K\right) = W(V(x,y)) \end{aligned}$$

leads to

$$\begin{aligned} \frac{dV}{dt} &\leq \gamma\left[F^2(x) + y^2 + \rho(x) + y^2 + \rho'(x)^2 + y^2\right] \\ &\leq \gamma\left[1 + 3y^2 + F^2(x) + \rho(x) + \rho'^2(x)\right] \\ &\leq \gamma\left[1 + 3\left(y^2 + 2\int_0^x (F(s) + \sqrt{\rho(s)})ds + 2\rho(x) + 2K\right) + F^2(x) + k^2\right] \\ &= \gamma\left[3V(x,y) + W(V(x,y)) + k^2\right]. \end{aligned} \tag{14}$$

Let us now assume that $z(t)$ is the maximal solution of the differential equation

$$z'(t) = \gamma\left(3z(t) + W(z(t)) + k^2\right), \tag{15}$$

with the initial condition $z(0) = V(x(0), y(0))$. By the classical comparison principle [25], we know that

$$0 \leq V(x(t), y(t)) \leq z(t), \quad t \in [0, T), \tag{16}$$

where T is the maximal value such that $z(t)$ does not blow up on $[0, T)$. On the other hand, it follows from [26] that condition (10), applied to (14), can ensure that

$$\int_1^{\infty} \frac{d\xi}{3\xi + W(\xi) + k^2} = \infty,$$

which guarantees that all solutions of Eq. (15) are global. Hence all solutions of (11) are also global due to the obtained double inequality 16. □

Next we prove that the following boundary value problem

$$\begin{cases} u''(t) = a(t)F(u(t)) + \omega b(t)\sqrt{\rho(u(t))} + \omega^2 c(t)\rho'(u(t)), \\ u(0) = u_0, \quad u(\tau) = u_\tau, \end{cases} \tag{17}$$

cannot have two distinct solutions, where $\tau \in (1-\varepsilon, 1+\varepsilon)$ and $u_\tau \in \mathbb{R}$, with ε being the same value as in Theorem 1.

Theorem 2. *Assume that all conditions in Theorem 1 are satisfied. Suppose further that the function F is non-decreasing, the density function ρ and its derivative ρ' are non-increasing. Then the solution of problem (17) is unique.*

Proof. Suppose that there exist two solutions u_1, u_2 of problem (17) with $u_1(t) \not\equiv u_2(t)$. Let $t^* \in (0, \tau)$ be such that $u_1(t^*) \neq u_2(t^*)$. Without loss of generality, we assume that $u_1(t^*) > u_2(t^*)$. Let us now introduce a difference $u = u_1 - u_2$. Then by continuity and in view of the fact that $u(0) = u(\tau) = 0$, there must be a non-empty interval $(a, b) \subset (0, \tau)$ with $t^* \in (a, b)$ such that $u(t) > 0$ for $t \in (a, b)$ and $u(a) = u(b) = 0$. We know that $u(t)$ satisfies the differential equation

$$u''(t) = a(t)[F(u_1) - F(u_2)] + \omega b(t)[\sqrt{\rho(u_1)} - \sqrt{\rho(u_2)}] + \omega^2 c(t)[\rho'(u_1) - \rho'(u_2)].$$

By using integration by parts, we obtain the inequality

$$\begin{aligned} 0 &\geq -\int_a^b [u'(t)]^2 dt = \int_a^b u''(t)u(t)dt \\ &= \int_a^b a(t)[F(u_1) - F(u_2)]u(t)dt + \omega \int_a^b b(t)[\sqrt{\rho(u_1)} - \sqrt{\rho(u_2)}]u(t)dt + \omega^2 \int_a^b c(t)[\rho'(u_1) - \rho'(u_2)]u(t)dt. \end{aligned}$$

By using the monotonicity of the functions F, ρ, ρ' and the fact $u(t) > 0$ for $t \in (a, b)$, it is easy to see that

$$\int_a^b a(t)[F(u_1) - F(u_2)]u(t)dt + \omega \int_a^b b(t)[\sqrt{\rho(u_1)} - \sqrt{\rho(u_2)}]u(t)dt + \omega^2 \int_a^b c(t)[\rho'(u_1) - \rho'(u_2)]u(t)dt > 0, \tag{18}$$

which is a contradiction. Thus there is no such point $t^* \in (0, \tau)$, for which $u_1(t^*) \neq u_2(t^*)$. Therefore the solution is unique. □

By a direct application of 3, we have proved the following result for the problem (6) based on Theorem 1 and Theorem 2.

Theorem 3. *Assume that all conditions in Theorems 1 and 2 are satisfied. Then the problem (6) admits a unique solution.*

Remark 1. There are large classes of the vorticity function F and the density function ρ such that all conditions hold. For example, we can take the vorticity function as the linear form $F(u) = \alpha u$ or $F(u) = u\ln(1+|u|)$, and the density function as the form $\rho(u) = \rho_0 - \beta u^2$ or $\rho(u) = \rho_0(1 - \beta u)$, where ρ_0, α, β are some positive number.

4. Discussions

In this paper we extended the existing study of the ACC circulation to the stratified flow setting. Even though the ODE formulation of the physical problem is restrictive in the sense that it does not consider the most general cases of the oceanic vorticity function and of the density distribution, its analysis gives valuable insights into the ocean circulation in the region. To prove the existence and uniqueness of solutions to the BVP under consideration, we made restrictive, but still physically relevant, assumptions on the density function requiring it to be non-increasing. In the follow-up works we would like to extend this setting to admit discontinuous stratification. This regime can arise, for example, from fresh water influx due to the ice melting, and would fit very well into the shallow-water formulation of the ACC flow.

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Data availability

No data was used for the research described in the article.

References

[1] A. Constantin, R.S. Johnson, An exact, steady, purely azimuthal equatorial flow with a free surface, J. Phys. Oceanogr. 46 (2016) 1935–1945.
[2] A. Constantin, R.S. Johnson, An exact, steady, purely azimuthal flow as a model for the antarctic circumpolar current, J. Phys. Oceanogr. 46 (2016) 3585–3594.
[3] A. Constantin, R.S. Johnson, Large gyres as a shallow-water asymptotic solution of Euler's equation in spherical coordinates, Proc. Roy. Soc. Lond. A 473 (2017) 20170063.
[4] A. Constantin, R.S. Johnson, On the dynamics of the near-surface currents in the arctic ocean, Nonlinear Anal. Real World Appl. 73 (103894) (2023) 43.
[5] J. Chu, On a differential equation arising in geophysics, Monatsh. Math. 187 (2018) 499–508.
[6] J. Chu, K. Marynets, Nonlinear differential equations modeling the antarctic circumpolar current, J. Math. Fluid Mech. 23 (92) (2021) 9.
[7] J. Chu, K. Marynets, Z. Wang, Existence and approximate solutions of a nonlinear model for the antarctic circumpolar current, Differential Integral Equations 36 (2023) 537–558.
[8] J. Chu, On a nonlinear integral equation for the ocean flow in arctic gyres, Quart. Appl. Math. 76 (2018) 489–498.
[9] J. Chu, On a nonlinear model for arctic gyres, Ann. Mat. Pura Appl. 197 (4) (2018) 651–659.
[10] J. Chu, Monotone solutions of a nonlinear differential equation for geophysical fluid flows, Nonlinear Anal. 166 (2018) 144–153.
[11] J. Chu, On an infinite-interval boundary-value problem in geophysics, Monatsh. Math. 188 (2019) 621–628.
[12] S.V. Haziot, Explicit two-dimensional solutions for the ocean flow in arctic gyres, Monatsh. Math. 189 (2019) 429–440.
[13] S.V. Haziot, Study of an elliptic partial differential equation modeling the ocean flow in arctic gyres, J. Math. Fluid Mech. 23 (48) (2021) 9.
[14] K. Marynets, Study of a nonlinear boundary-value problem of geophysical relevance, Discrete Contin. Dyn. Syst. 39 (2019) 4771–4781.
[15] A. Abrashkin, A. Constantin, A steady azimuthal stratified flow modelling the antarctic circumpolar current, J. Differential Equations 374 (2023) 632–641.
[16] J. Chu, Z. Wang, Nagumo-type uniqueness and stability for nonlinear differential equations on semi-infinite intervals, J. Differential Equations 367 (2023) 229–245.
[17] S.V. Haziot, Study of an elliptic partial differential equation modeling the antarctic circumpolar current, Discrete Contin. Dyn. Syst. 39 (2019) 4415–4427.
[18] K. Marynets, Two-point boundary problem for modeling the jet flow of the antarctic circumpolar current, Electron. J. Differential Equations (56) (2018) 12.
[19] K. Marynets, On the modeling of the flow of the antarctic circumpolar current, Monatsh. Math. 188 (2019) 561–565.
[20] K. Marynets, Stuart-type vortices modeling the antarctic circumpolar current, Monatsh. Math. 191 (2020) 749–759.
[21] R. Quirchmayr, A steady, purely azimuthal flow model for the Antarctic Circumpolar Current, Monatsh Math. 187 (2018) 565–572.
[22] Q. Ding, F. Liao, S. Wang, Existence, uniqueness and stability for a nonlinear problem arising from stratified arctic gyres, Ann. Mat. Pura Appl. 203 (4) (2024) 1513–1527.
[23] Q. Ding, L. Roberti, Stratified ocean gyres with stuart-type vortices, Ann. Mat. Pura Appl. 203 (4) (2024) 2847–2862.
[24] K. Schrader, P. Waltman, An existence theorem for nonlinear boundary value problems, Proc. Amer. Math. Soc. 21 (1969) 653–656.
[25] W.A. Coppel, Stability and asymptotic behavior of differential equations, D. C. Heath and Company, Boston, 1965.
[26] A. Constantin, Global existence of solutions for perturbed differential equations, Ann. Mat. Pura Appl. 168 (4) (1995) 237–299.