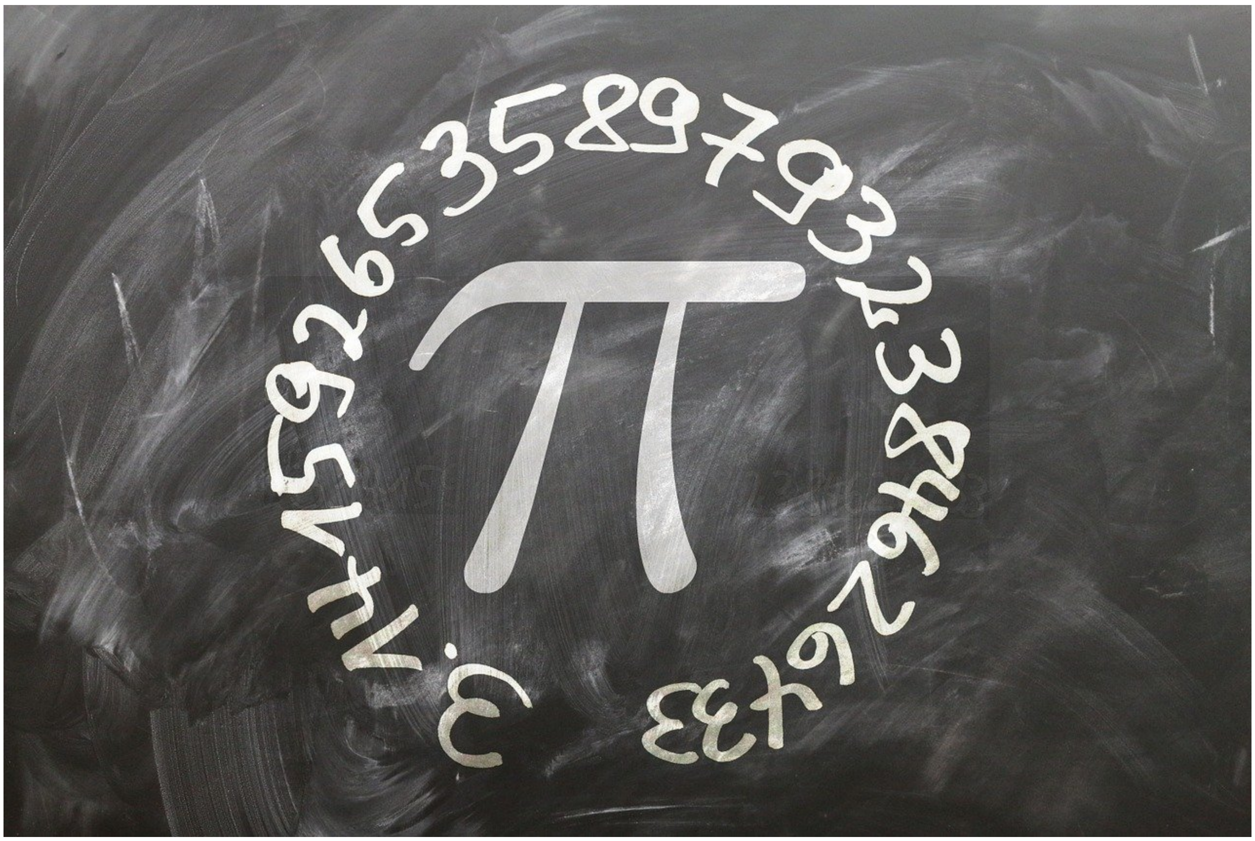
The Fourier Analysis behind Borwein Integrals

A Computer Bug or a Mathematical Phenomena?



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Abstract

Layman's Abstract

This paper studies apparent patterns in mathematics that break at a certain point and aims to provide a mathematical explanation for these breaks. We focus on two patterns. The first pattern, discovered by D. and J. Borwein, is as follows: π, π, π, $\dots$, π, $\pi - 0.000000000462....$ These numbers are the outcomes of integrals called the Borwein integrals. At first glance, it is not obvious why this pattern breaks. However, after performing a specific analysis called Fourier analysis, we will find the reason behind the apparent breakdown of the pattern.

Next, we study another very similar pattern: π, π, π, $\dots$, π, $\pi - 0.000000003589792\dots$. These numbers are the outcomes of different integrals, which we call the Nahin integrals. Once again, with the help of Fourier analysis we will find a reasonable explanation for why the pattern breaks and what the value of the following numbers will be.

In conclusion, this paper serves as a warning to those who assume a pattern exists based on a first glance. Furthermore, when an apparent pattern does not exist, this paper applies a methodology that can be more broadly used to determine the actual predictable behaviour.

Peer Abstract

This paper primarily studies the Borwein integrals B_n:

$$b_n(x) = \prod_{k=0}^{n} \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}$$
$$B_n = \int_{-\infty}^{\infty} b_n \, dx, \quad n = 0, 1, 2, \dots$$

These integrals are of interest because of their peculiar results, namely B_0 up to B_6 are all equal to π. However, B_7 is almost, but not quite, equal to π, equalling approximately $\pi - 0.000000000462$.

First, to try and observe a reason for the breaking of this apparent pattern, we will perform a graphical analysis on the integrands b_n. This will prove to be not very insightful, so another approach using Fourier analysis will be applied. To perform such an analysis, the Fourier transform of functions in L^2 must first be defined. We do this on the basis of functions in $L^1 \cap L^2$. Then, the Fourier transform of $\frac{\sin\left(\frac{x}{k}\right)}{\frac{x}{k}}, \quad k = 1$, is calculated and generalized to an arbitrary $k \in \mathbb{R}$. After this, the Fourier transform of the Borwein integrands is calculated, and their graphs are analyzed.

Interestingly, the Fourier transform of the first Borwein integrand is a Heaviside step function with a 'plateau' of width $\frac{1}{\pi}$, where the function is equal to π centered around 0. Each Fourier transform after this is a moving average of the one before, where the moving average window of the nth transform is determined by $\frac{1}{\pi(2n+1)}$. This means there is a very simple explanation for where the apparent pattern will break. Namely, if the difference between $\frac{1}{\pi}$ and $\frac{1}{\pi}(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1})$ becomes negative, then the plateau will vanish as the window becomes larger than the plateau, and even at the center point zero, the function value will become slightly less than π.

While the value of each Borwein integral decreases, there does exist a limit equal to approximately $\pi - 0.0000704$. Thus, while the pattern does 'break', it never breaks very badly and always remains quite close to π.

After studying the Borwein integrals and their behaviour at infinity, we will study another sequence of

functions. We call these functions the Nahin functions, defined as follows:

$$h_n(x) = \frac{\sin(4x)}{x} \prod_{k=0}^{n} \cos\left(\frac{x}{k+1}\right)$$
$$H_n = \int_{-\infty}^{\infty} h_n \, dx, \quad n = 0, 1, 2, \ldots$$

Note that $h_n(0) = H_n$. Using the same methodology as for the Borwein integrals, we will calculate the Fourier transform of the Nahin integrands and find a direct link between the Nahin integrands and the Borwein integrals. Namely, the Fourier transform of the Nahin integrands can be written as a factor times the sum of the dilated Fourier transform of B_0. The analysis will further result in an explanation for why the Nahin integrals' 'pattern' breaks at $n = 30$. When $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}$ is greater than four, the Nahin integrals will no longer be equal to but will be less than π. Finally, a closed expression will be found for the value of the Nahin integrals, and further research will be suggested to discover the value of this closed expression as n approaches infinity.

Contents

Preface

The bachelor's degree in Applied Mathematics at TU Delft concludes with a thesis on a mathematical topic, marking the end of four years of study for me. During the bachelor's program, I found myself interested in various fields of mathematics, including Discrete Mathematics, Statistics, and Analysis. As such, when faced with the choice of bachelor projects, these were the three fields in which I was most interested. After choosing a number of projects, I was allocated my top choice: The Fourier analysis behind Borwein integrals. This topic interested me because of the reversal of expectations and the mathematics behind the phenomena.

The first part of this report follows the layout of a YouTube video by G. Sanderson [1], which in turn was based on the results of a paper written by the father and son duo David and Jonathan Borwein [2]. David Borwein (1924-2021) had three children, one of whom was Jonathan (1951-2016), and all three became modern-day mathematicians. Many theorems and mathematical concepts that I was taught throughout the bachelor's program were developed by mathematicians who lived hundreds of years ago, with the exception of the elective Graph Theory. With this in mind, I found it interesting to be working on mathematics that had not been fully researched until recently.

I would like to thank my supervisor Emiel Lorist for his help and feedback over the past few months, as well as Cornelis Kraaikamp, who, after being my mentor at the start of the bachelor's program, agreed to be part of my thesis committee, marking the end of my bachelor's studies. Lastly, I would like to thank my family and friends for the support they have given me, not only during this project but throughout the entire process of studying mathematics.

Introduction

For $k = 0, 1, 2, \ldots$, the Borwein integrals are defined as follows:

$$B_n = \int_{-\infty}^{\infty} \prod_{k=0}^{n} \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}\, dx, \quad n = 0, 1, 2, \ldots$$

Interestingly, $B_0, B_1, \ldots, B_6 = \pi$, but $B_7 \approx \pi - 0.000000000462$. At first glance, one might assume that B_7 was calculated incorrectly, perhaps due to a rounding error by a computer. However, this is not the case. In fact, as n increases, B_n continues to deviate further from π. In Chapter 1.1, we perform a graphical analysis of the Borwein integrands to try and explain this behaviour. This proves largely unsuccessful, and thus in Chapter 1.2, we first define the Fourier transform of functions in L^2. Next, we calculate the Fourier transform of B_0 by hand in Section 1.3. In Section 1.4, we calculate the Fourier transform of the Borwein integrands, and in Section 1.5, we perform a graphical analysis on the Fourier transformations of the Borwein integrands. By doing so, we are able to explain the phenomena. Chapter 1 concludes with Section 1.6 where a closed form expression for the Borwein integrals is found, as well as their behaviour as n approaches infinity.

Next, Chapter 2 is centered around the analysis of another set of integrals, namely what we call the Nahin integrals, which take the following form:

$$H_n = \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \prod_{k=0}^{n} \cos\left(\frac{x}{k+1}\right) dx, \quad n = 0, 1, 2, \ldots$$

- Chapter 2 starts with the introduction of what we call the Nahin integrals and discusses the behaviour of their values.
- In Section 2.1, we discuss the graphical behaviour of the Nahin integrands.
- In Section 2.2, we calculate the explicit Fourier transform of the first few Nahin integrands, which we use in the next section.
- In Section 2.3, we graph the first six Nahin integrands and discuss their graphical behaviour.
- In Section 2.4, we perform a further analysis on the Nahin integrands, rewriting them in such a way that their Fourier transforms are linked to the Fourier transforms of the Borwein integrands. Using this information, we find a closed form for the value of the Nahin integrals and are able to explain the phenomena that arise with the Nahin integrals.

Finally, we conclude by giving a short summary of the conclusions made in the paper and a few suggestions for further study.

1 The Borwein Integrals

The Borwein integrals are a collection of integrals which David and Jonathan Borwein used as an example in their paper "Some Remarkable Properties of Sinc and Related Integrals" [2]. The Borwein integrals are defined as follows:

$$B_n = \int_{-\infty}^{\infty} \prod_{k=0}^{n} \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}\, dx, \quad n \in \mathbb{N}$$

Originally, D. and J. Borwein only integrated from 0 to ∞; however, to avoid unnecessary fractions, we will integrate over the entire real number line. The outcomes of the first few Borwein integrals are as follows:

$$\begin{aligned}
B_0 &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi \\
B_1 &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}\, dx = \pi \\
B_2 &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}} \frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}}\, dx = \pi \\
&\vdots \\
B_6 &= \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}} \frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}} \cdots \frac{\sin\left(\frac{x}{13}\right)}{\frac{x}{13}}\, dx = \pi \\
B_7 &\approx \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}} \frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}} \cdots \frac{\sin\left(\frac{x}{15}\right)}{\frac{x}{15}}\, dx = \pi - 0.0000000000462 \\
&\vdots
\end{aligned}$$

Noticeably, there is an apparent pattern of each integral having the value of π until we reach B_7. Here, the pattern breaks by the smallest of margins. In fact, the value of B_7 is so close to π that it initially seems to be a mistake. However, it isn't a mistake, and the value of each B_n after B_7 consistently decreases.

To understand this phenomenon, we will first evaluate the graphical nature of the integrands of the Borwein integrals.

Remark. The integrands of the Borwein integrals are defined at $x = 0$ by the limit of the function at that point, which exists. Similarly, throughout this paper, we will implicitly define functions at $x = 0$ by their limits at that point. In all cases used in this paper, these limits exist.

1.1 A Graphical Analysis of the Borwein Integrands

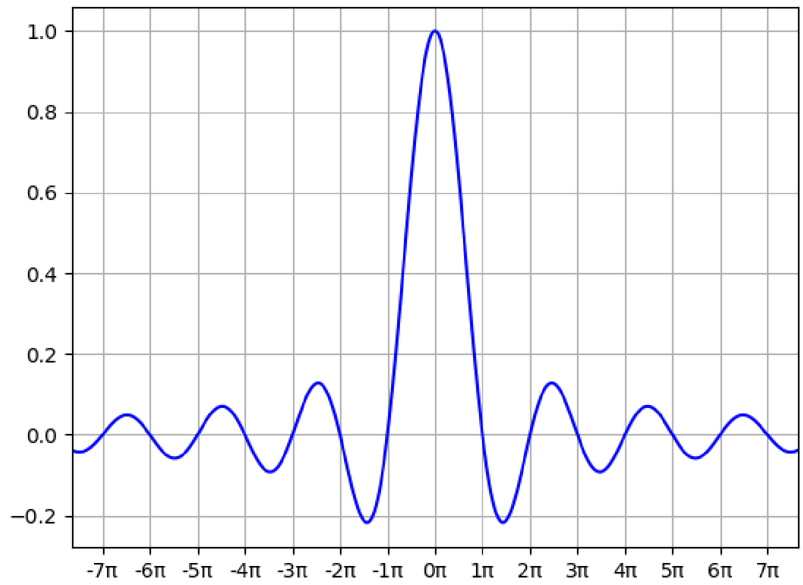


Figure 1: The graph of $\frac{\sin(x)}{x}$

Figure 1 shows the graph of $\frac{\sin(x)}{x}$. We know that the integral of this function is equal to π, and as such, this is our baseline for comparison with the following figures. The code for these figures can be found in Appendix 4.1.

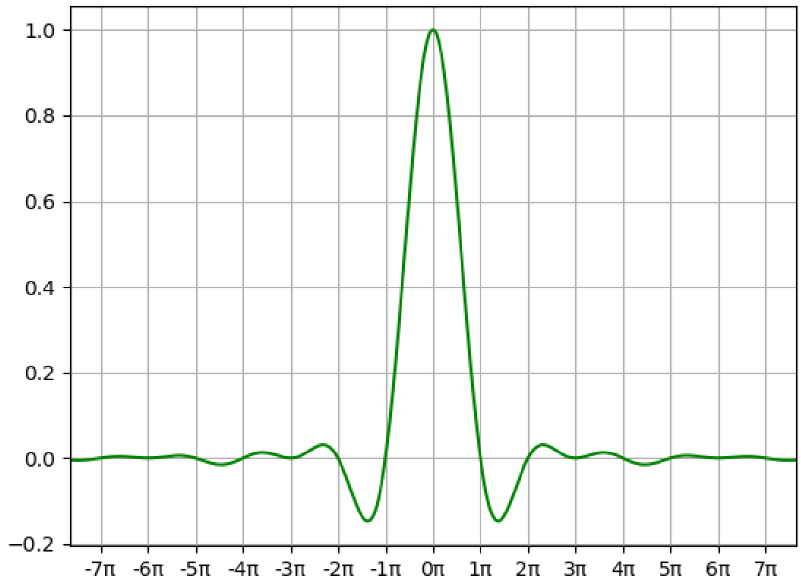


Figure 2: The graph of $\frac{\sin(x)}{x}\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}$

In Figure 2, we have the graph of $\frac{\sin(x)}{x}\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}$. In the vicinity of 0, this graph is very similar to Figure 1, but as it leaves the direct vicinity of 0, the function becomes a damped version of Figure 1 in the sense that local maxima and minima are closer to the x-axis the further you get away from zero. Nevertheless, we know that the integral of this function is also π.

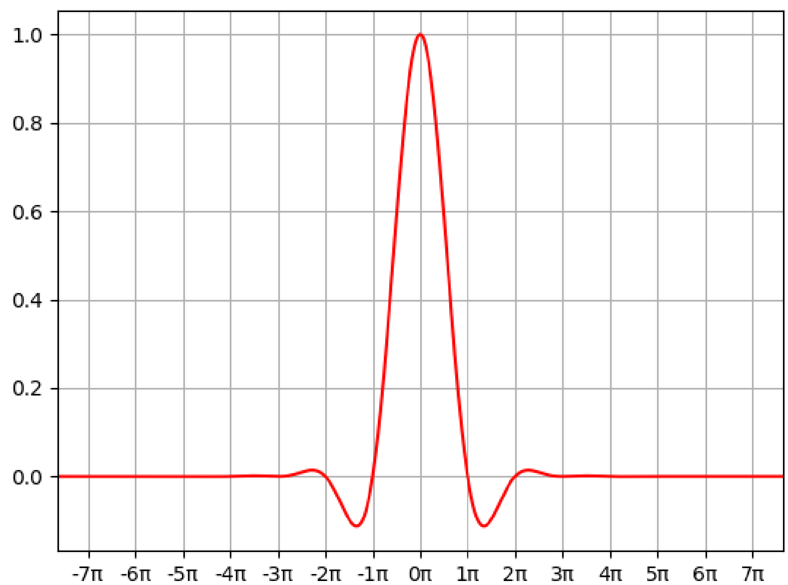


Figure 3: The graph of $\frac{\sin(x)}{x}\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}\frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}}\cdots\frac{\sin\left(\frac{x}{13}\right)}{\frac{x}{13}}$

Next, we skip a few steps and regard the integrand of B_6. Figure 3 shows the graph of $\frac{\sin(x)}{x}\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}\frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}}\cdots\frac{\sin\left(\frac{x}{13}\right)}{\frac{x}{13}}$. Notably, this function is considerably more damped, with significant values occurring only within the range $[-2\pi, 2\pi]$. Nevertheless, we know that the integral is still equal to π.

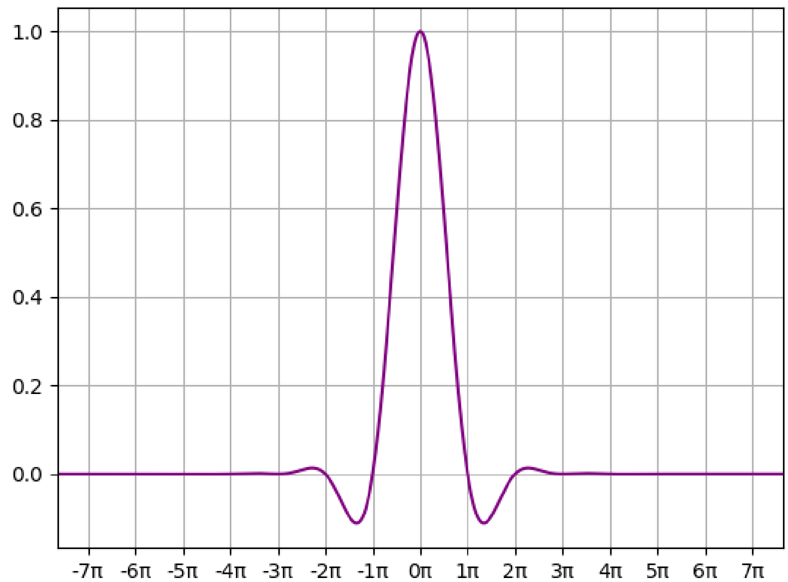


Figure 4: The graph of $\frac{\sin(x)}{x}\frac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}}\frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}}\cdots\frac{\sin\left(\frac{x}{15}\right)}{\frac{x}{15}}$

Lastly, in Figure 4, we know that the integral is just barely less than π. However, we notice no remarkable change in the function in comparison to Figure 3, and therefore, we have no clear-cut reason for why the pattern breaks. Since we cannot come to a definite conclusion based upon the current form of our Borwein integrals, in the next section, we will attempt another method. Namely, by introducing and then calculating the Fourier transform of the integrands.

1.2 The Fourier Transform $L^2(\mathbb{R})$ Functions

We saw at the start of the Chapter that the Borwein integrals seem to exhibit a pattern that "breaks" at B_7, as each integral thereafter distances itself further from π. After failing to explain this behaviour through the graphs of the integrands, we will now try another method using Fourier analysis. The Fourier transform of an arbitrary function $f \in L^1(\mathbb{R})$ is defined in [3] as follows:

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx, x \in \mathbb{R} \tag{1}$$

This is a useful formulation since $\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\, dx$ is equal to B_0, and we can find a similar formulation for the following Borwein integrals. However, this definition of the Fourier transform assumes that $f \in L^1(\mathbb{R})$. Our function $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{\sin(x)}{x}$ is an element of $L^2(\mathbb{R})$ but not $L^1(\mathbb{R})$; however, we would like to have a similar definition. Therefore, we will first define the Fourier transform for L^2 functions, then calculate the Fourier transform of f, and then a dilated f, ending the chapter by calculating and analyzing the Fourier transforms of the Borwein integrals.

The Fourier transform for functions on $L^1(\mathbb{R})$ is already neatly defined in Grafakos [3], and we will now define it as follows:

Definition 1.1 (The Fourier Transform in $L^1(\mathbb{R})$). *Given $f \in L^1(\mathbb{R})$, we define the Fourier transform $\widehat{f}(\xi)$ by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}\,dx, \quad \xi \in \mathbb{R}.$$

Here, $\widehat{f}$ denotes the Fourier transform of f.

Remark. While our definitions, theorems, and propositions are formulated within $\mathbb{R}$, it's important to note that extensions to $\mathbb{R}^n$ exist. However, for clarity, we restrict our focus in this paper exclusively to functions defined within $\mathbb{R}$.

The Borwein integrands are functions in $L^2(\mathbb{R})$, and thus, this section will provide a proper definition of the Fourier transform for an arbitrary function $g \in L^2(\mathbb{R})$. To do this, we first require the following theorem [4] and proposition:

Theorem 1.2 (Plancherel). *Let f be a function such that the Fourier transform of f is defined. Then*

$$||\widehat{f}||_2^2 = ||f||_2^2.$$

Proposition 1.3. *Given $f, g \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$, $b \in \mathbf{C}$, and $a > 0$, and $(\delta^a f)(x) = f(ax)$, we have*

$$\begin{aligned}\widehat{f+g} &= \widehat{f} + \widehat{g},\\ \widehat{bf} &= b\widehat{f},\\ \widehat{\delta^a f} &= a^{-n}\delta^{a^{-1}}\widehat{f} = (\widehat{f})_a, \quad \textit{where } a > 0.\end{aligned}$$

For functions in $L^1(\mathbb{R})$ a similar proposition can be found in [3]. We will prove the first of the three statements for functions in $L^2(\mathbb{R})$ after Proposition 1.7, but the proofs of the other two are omitted to avoid unnecessary repetition as they use the same technique. To prove this first statement, we need the following theorems and proposition from [5]:

Proposition 1.4 (Hölder's inequality). *Let $S \subseteq \mathbb{R}, p, q \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(S)$ and $g \in L^q(S)$, then $fg \in L^1(S)$ and*

$$||fg||_1 \leq ||f||_p ||g||_q.$$

Theorem 1.5 (Dominated Convergence theorem). *Let $f_n : \mathbb{R} \to \mathbb{R}$ be measurable functions for $n \geq 1$ such that $f_n \to f$ pointwise. If there exists an integrable function $g : \mathbb{R} \to [0,\infty)$ such that $f_n \leq g$ for all $n \geq 1$, then f_n and f are integrable and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n dx = \int_{\mathbb{R}} f dx$$

Theorem 1.6 (Density of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ in $L^2(\mathbb{R})$). *$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, i.e., given $g \in L^2(\mathbb{R})$, there exists $(g_n)_{n\geq 1} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g_n \to g$ in $L^2(\mathbb{R})$.*

Proof. Let $g \in L^2(\mathbb{R})$, and define $g_n = g \cdot \mathbb{1}_{[-n,n]}$.

We first prove that $g_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. For $g_n \in L^1(\mathbb{R})$ we use Hölder's inequality (1.4) to find the following:

$$\int_{\mathbb{R}} |g_n|\,dx = \int_{\mathbb{R}} |g|\mathbb{1}_{[-n,n]}\,dx \leq ||g||_2 \cdot ||\mathbb{1}_{[-n,n]}||_2$$

Since $g \in L^2(\mathbb{R})$, $||g||_2 < \infty$. Furthermore, $||\mathbb{1}_{[-n,n]}||_2 = \left(\int_{\mathbb{R}} \mathbb{1}_{[-n,n]}\,dx\right)^{\frac{1}{2}} = (2n)^{\frac{1}{2}} < \infty$. Thus, $\int_{\mathbb{R}} |g_n|\,dx \leq ||g||_2 \cdot ||\mathbb{1}_{[-n,n]}||_2 < \infty$, so $g_n \in L^1(\mathbb{R})$.

Next, we show that $g_n \in L^2(\mathbb{R})$:

$$|g_n| \leq |g| \implies \left(\int_{\mathbb{R}} |g_n|^2 \, dx\right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}} |g|^2 \, dx\right)^{\frac{1}{2}} < \infty$$

This holds since $g \in L^2(\mathbb{R})$. Thus, $g_n \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Now, we show that $g_n \to g$ in $L^2(\mathbb{R})$. Trivially, $g_n \to g$ pointwise. Furthermore, $|g_n - g|^2 \leq (2|g|)^2 = 4|g|^2$, which is integrable since $g \in L^2(\mathbb{R})$. Therefore, by Theorem 1.5, $g_n \to g$ in $L^2(\mathbb{R})$. Thus, $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. □

Now we can define the Fourier transform for functions in L^2 as follows:

Proposition 1.7. *Given $g \in L^2(\mathbb{R})$, then $\widehat{g} = \lim_{n\to\infty} \widehat{g_n}$, is the well-defined Fourier transform of g, where $(g_n)_{n\geq 1} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g_n \to g$ in $L^2(\mathbb{R})$ norm.*

Proof. We will prove this claim in three steps.

(i) $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, i.e., for all $g \in L^2(\mathbb{R})$, there exists $(g_n)_{n\geq 1} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g_n \to g$ in $L^2(\mathbb{R})$.

(ii) Given g and $(g_n)_{n\geq 1}$, then $\widehat{g_n} \to h$ in $L^2(\mathbb{R})$ norm for some $h \in L^2(\mathbb{R})$.

(iii) h is unique.

(i) Note that this is proven in Theorem 1.6.

(ii) If $g \in L^2(\mathbb{R})$, then by (i) there exists $(g_n)_{n\geq 1}$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $g_n \to g$ in $L^2(\mathbb{R})$ norm. Therefore $||g - g_n||_2 \to 0$. Note that $(g_n)_{n\geq 1}$ must be Cauchy since $(g_n)_{n\geq 1}$ is convergent. We will prove that $(\widehat{g_n})_{n\geq 1}$ is also Cauchy. Namely, since $(g_n)_{n\geq 1}$ is Cauchy, the following holds: $\forall \epsilon > 0, \exists N \geq 1$ such that $\forall n, m \geq N; ||g_n - g_m||_2 < \epsilon$. Therefore by Theorem 1.2, $||\widehat{g_n} - \widehat{g_m}||_2 = ||g_n - g_m||_2 < \epsilon$. This implies that $(\widehat{g_n})_{n\geq 1}$ is Cauchy in $L^2(\mathbb{R})$. By the completeness of L^2, a Cauchy sequence in L^2 is also convergent. Therefore $(\widehat{g_n})_{n\geq 1}$ is convergent in $L^2(\mathbb{R})$ with limit $h \in L^2(\mathbb{R})$.

(iii) Take $(g_n)_{n\geq 1}$ and $(G_n)_{n\geq 1}$ such that $g_n \to g$ in $L^2(\mathbb{R})$ and $G_n \to g$ in $L^2(\mathbb{R})$. Let $h = \lim_{n\to\infty} \widehat{g_n}$, $H = \lim_{n\to\infty} \widehat{G_n}$, with $h, H \in L^2(\mathbb{R})$. We must prove that $h = H$ almost everywhere (a.e.) $\iff h - H = 0$ a.e.. We know by the properties of norms that $h - H = 0$ a.e. $\iff ||h - H||_2 = 0$. Therefore we use this to find the following:

$$||h - H||_2 = ||\lim_{n\to\infty} \widehat{g_n} - \lim_{n\to\infty} \widehat{G_n}||_2 = \lim_{n\to\infty} ||\widehat{g_n} - \widehat{G_n}||_2$$

which using Proposition 1.3:

$$\lim_{n\to\infty} ||\widehat{g_n} - \widehat{G_n}||_2 = \lim_{n\to\infty} ||\widehat{g_n - G_n}||_2$$

which by Theorem 1.2:

$$\lim_{n\to\infty} ||\widehat{g_n - G_n}||_2 = \lim_{n\to\infty} ||g_n - G_n||_2 = \lim_{n\to\infty} ||g_n - g + g - G_n||_2 \leq \lim_{n\to\infty} ||g_n - g||_2 + \lim_{n\to\infty} ||g - G_n||_2$$

using the triangle inequality for norms. Furthermore,

$$\lim_{n\to\infty} ||g_n - g||_2 + \lim_{n\to\infty} ||g - G_n||_2 = 0$$

since $g_n \to g$ in $L^2(\mathbb{R})$ and $G_n \to g$ in $L^2(\mathbb{R})$. Therefore $||h - H||_2 = 0$, and thus $h = H$ in $L^2(\mathbb{R})$ as desired. With these three elements we have a well-defined Fourier transform for functions in $L^2(\mathbb{R})$, namely for $g \in L^2(\mathbb{R}), \widehat{g} = \lim_{n\to\infty} \widehat{g_n}$. □

Now we have the correct tools, we finish with the proof of the first statement of Proposition 1.3:

Proof. Let $f, g \in L^2(\mathbb{R})$. By Theorem 1.6, there exist sequences $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ in $L^1(\mathbb{R})$ such that

$f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R})$. Therefore, we have:

$$\widehat{f+g} = \widehat{\lim_{n\to\infty} f_n + \lim_{n\to\infty} g_n} = \lim_{n\to\infty} \widehat{f_n + g_n}.$$

Since the statement holds for functions in $L^1(\mathbb{R})$, we get:

$$\lim_{n\to\infty} \widehat{f_n + g_n} = \lim_{n\to\infty} (\widehat{f_n} + \widehat{g_n}) = \widehat{\lim_{n\to\infty} f_n} + \widehat{\lim_{n\to\infty} g_n} = \hat{f} + \hat{g}.$$

Hence, we conclude that:

$$\widehat{f+g} = \hat{f} + \hat{g}.$$

□

1.3 Calculating the Fourier Transform of the First Borwein Integrand

The objective of this section is to compute the Fourier transform of $b_0(x) = \frac{\sin(x)}{x}$. We will accomplish this using techniques from complex analysis. From the preceding section, it is known that the Fourier transform of $\frac{\sin(x)}{x}$ is defined in $L^2(\mathbb{R})$; specifically, for $g \in L^2(\mathbb{R})$,

$$\hat{g} = \lim_{n\to\infty} \widehat{g_n}.$$

Given $b_R(x) = \frac{\sin(x)}{x} e^{-2\pi i\xi x} \mathbb{1}_{[-R,R]} \in L^1$, we ascertain that $\lim_{R\to\infty} b_R(x) = \frac{\sin(x)}{x}$. Consequently, the Fourier transform of $\frac{\sin(x)}{x}$ is expressed as follows:

$$\lim_{R\to\infty} \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-2\pi i\xi x} \mathbb{1}_{[-R,R]} \, dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(x)}{x} e^{-2\pi i\xi x} \, dx.$$

Therefore, we proceed by analyzing the function $\int_{C_i} \frac{\sin(z)}{z} e^{-2\pi i\xi z} \, dz$ along contours C_1 and C_2 to evaluate the original integral. Initially, we rewrite it as:

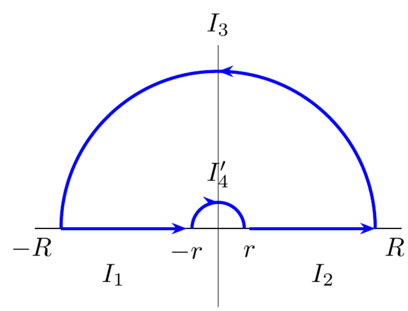
$$\int_{C_i} \frac{\sin(z)}{z} e^{-2\pi i\xi z} \, dz = \int_{C_i} \frac{1}{2iz}(e^{iz} - e^{-iz}) e^{-2\pi i\xi z} \, dz = \int_{C_i} \frac{1}{2iz} e^{iz} e^{-2\pi i\xi z} \, dz - \int_{C_i} \frac{1}{2iz} e^{-iz} e^{-2\pi i\xi z} \, dz$$

$$= \text{(i)} - \text{(ii)}.$$

We first analyze (i):

$$\int_{C_i} \frac{1}{2iz} e^{iz(1-2\pi\xi)} \, dz$$

If $1 - 2\pi\xi > 0$, we use the following contour $C_1 = I_1 + I_2 + I_3 + I_4'$:



Remark. The code for contours C_1 and C_2 can be found in Appendix 4.2.

I_1, I_2, I_3, I_4, and I_4' are then parameterized as follows:

I_1 is parameterized by $\gamma_1 : [-R, -r] \to \mathbb{C}, \gamma_1(z) = z$
I_2 is parameterized by $\gamma_2 : [r, R] \to \mathbb{C}, \gamma_2(z) = z$
I_3 is parameterized by the positively orientated circular arc σ_R at angle π consisting of all $z = Re^{it}, t \in [0, \pi]$
I_4 is parameterized by the positively orientated circular arc σ_r at angle π consisting of all $z = re^{it}, t \in [0, \pi]$
I_4' is parameterized by the negatively orientated circular arc σ_r' at angle π consisting of all $z = re^{it}, t \in [0, \pi]$

We introduce the following necessary theorems from [6]:

Theorem 1.8 (Cauchy's theorem for Simply Connected Regions)**.** *Let f be an analytic function on a simply connected region Ω. If γ is a closed path in Ω, then*

$$\int_\gamma f(z)\, dz = 0$$

Theorem 1.9 (Shrinking Path Lemma)**.** *Suppose that f is a continuous complex-valued function on a closed disk $B_{r_0}(z_0)$ with centre at z_0 and radius r_0. For $0 < r \le r_0$, let σ_r denote the positively oriented circular arc at angle α consisting of all $z = z_0 + re^{i\theta}$, where $\theta_0 \le \theta \le \theta_0 + \alpha, \theta_0$ and α are fixed, and $\alpha \neq 0$. Then*

$$\lim_{r \to 0^+} \frac{1}{i\alpha} \int_{\sigma_r} \frac{f(z)}{z - z_0}\, dz = f(z_0).$$

Theorem 1.10 (Jordan's Generalized Lemma)**.** *Let $R_0 > 0$ and $0 \le \theta_1 < \theta_2 \le \pi$. For $R \ge R_0$, let σ_R be the circular arc of all $z = Re^{i\theta}$ with $0 \le \theta_1 \le \theta \le \theta_2 \le \pi$. Let f be a continuous complex-valued function defined on all arcs σ_R and let $M(R)$ denote the maximum value of $|f|$ on σ_R. If $\lim_{R\to\infty} M(R) = 0$, then for all $s > 0$,*

$$\lim_{R\to\infty} \int_{\sigma_R} e^{isz} f(z)\, dz = 0.$$

Since we integrate over a simply connected region we can use Theorem 1.8 giving us the following:

$$\oint_{C_1} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \int_{\sigma_R} f(z)\, dz + \int_{\sigma_r'} f(z)\, dz = 0.$$

Note that

$$\begin{aligned} \lim_{r\to 0^+, R\to\infty} \left(\int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz \right) &= \lim_{r\to 0^+, R\to\infty} \left(\int_{-R}^{-r} \frac{1}{2iz} e^{iz} e^{-2\pi i \xi z}\, dz + \int_r^R \frac{1}{2iz} e^{iz} e^{-2\pi i \xi z}\, dz \right) \\ &= \int_{\mathbb{R}} \frac{1}{2iz} e^{iz} e^{-2\pi i \xi z}\, dz. \end{aligned}$$

Therefore, we only need to calculate $\lim_{R\to\infty} \int_{\sigma_R} f(z)\, dz$ and $\lim_{r\to 0^+} \int_{\sigma_r'} f(z)\, dz$ to find the value of (i).

We start by calculating the integral over I_3:

$$\int_{\sigma_R} \frac{1}{2iz} e^{iz} e^{-2\pi i \xi z}\, dz.$$

Call $h(z) = \frac{1}{2iz}$. Then

$$\max_{z\in\sigma_R} \left| \frac{1}{2iz} \right| = \frac{1}{2R}.$$

Since $\lim_{R\to\infty} \frac{1}{2R} = 0$, we find by Jordan's General Lemma 1.10 that

$$\lim_{R\to\infty} \int_{\sigma_R} \frac{1}{2iz} e^{iz(1-2\pi\xi)}\, dz = 0.$$

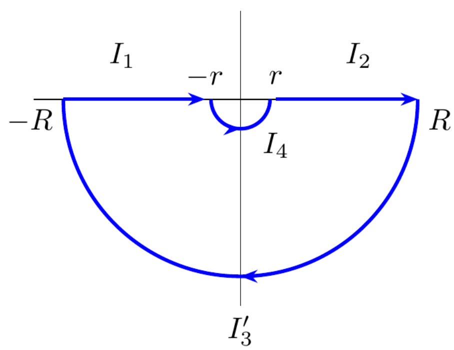
Next, we calculate the integral over I_4'. Since σ_r' has negative orientation we first rewrite it to an integral over σ_r which runs over I_4:

$$\int_{\sigma_r'} \frac{1}{2iz} e^{iz(1-2\pi\xi)}\, dz = -\frac{1}{i\pi} \int_{\sigma_r} \frac{\pi}{2} \frac{e^{iz(1-2\pi\xi)}}{z}\, dz.$$

which, by the Shrinking Path Lemma 1.9 when taking the limit as $r \to 0^+$, is equal to $-\frac{\pi}{2}$.
Therefore

$$\int_{\mathbb{R}} \frac{1}{2iz} e^{iz} e^{-2\pi i\xi z}\, dz = \lim_{r\to 0^+, R\to\infty} \left(-\int_{\sigma_R} f(z)\, dz - \int_{\sigma_r'} f(z)\, dz \right) = \frac{\pi}{2}$$

If $1 - 2\pi\xi \leq 0$, we use the following contour C_2:



I_1, I_2, I_3', and I_4 are then reparametrized as follows:

I_1 is parameterized by $\gamma_1 : [-R, -r] \to \mathbb{C}, \gamma_1(z) = z$
I_2 is parameterized by $\gamma_2 : [r, R] \to \mathbb{C}, \gamma_2(z) = z$
I_3' is parameterized by the negatively orientated circular arc σ_R' at angle π consisting of all $z = Re^{it}, t \in [0, \pi]$
I_4 is parameterized by the positively orientated circular arc σ_r at angle π consisting of all $z = re^{it}, t \in [0, \pi]$

Once again we can use Cauchy's Theorem (1.8) to find that:

$$\oint_{C_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz + \int_{\sigma_R'} f(z)\, dz + \int_{\sigma_r} f(z)\, dz = 0.$$

So just as before we will calculate $\lim_{R\to\infty} \int_{\sigma_R'} f(z)\, dz$ and $\lim_{r\to 0^+} \int_{\sigma_r} f(z)\, dz$. First, we calculate the integral over I_3':

$$\int_{\sigma_R'} \frac{1}{2iz} e^{iz} e^{-2\pi i\xi z}\, dz.$$

Call $h(z) = \frac{1}{2iz}$, then

$$\max_{z\in\sigma_R} \left| \frac{1}{2iz} \right| = \frac{1}{2R}.$$

Since $\lim_{R\to\infty} \frac{1}{2R} = 0$, we find by substituting $\theta \to -\theta$ in Jordan's General Lemma (1.10) that

$$\lim_{R\to\infty} \int_{\sigma'_R} \frac{1}{2iz} e^{iz(1-2\pi\xi)}\, dz = 0.$$

Next, we calculate the integral over I_4:

$$\int_{\sigma_r} \frac{1}{2iz} e^{iz(1-2\pi\xi)}\, dz = \frac{1}{i\pi} \int_{\sigma_r} \frac{\frac{\pi}{2} e^{iz(1-2\pi\xi)}}{z}\, dz,$$

which by the Shrinking Path Lemma (1.9) when taking the limit of $r \to 0^+$ is equal to $\frac{\pi}{2}$. Therefore

$$\lim_{r\to 0^+, R\to\infty} \left(-\int_{\sigma'_R} f(z)\, dz - \int_{\sigma_r} f(z)\, dz \right) = -\frac{\pi}{2}$$

Therefore

$$(i) = \begin{cases} \frac{\pi}{2} & \text{if } 1-2\pi\xi \geq 0 \\ -\frac{\pi}{2} & \text{if } 1-2\pi\xi < 0 \end{cases}$$

Similarly, we find the value of (ii):,

$$(ii) = \int_{C_i} \frac{1}{2iz} e^{iz(-1-2\pi\xi)}\, dz = \begin{cases} \frac{\pi}{2} & \text{if } -1-2\pi\xi > 0 \\ -\frac{\pi}{2} & \text{if } -1-2\pi\xi \leq 0 \end{cases}$$

Therefore,

$$\lim_{R\to\infty} \int_{\mathbb{R}} \frac{\sin(x)}{x} e^{-2\pi i\xi x}\, dx = (i) - (ii) = \begin{cases} \pi & \text{if } |2\pi\xi| \leq 1 \\ 0 & \text{if } |2\pi\xi| > 1 \end{cases}$$

1.4 Calculating the Fourier Transform of the Borwein Integrands Using Convolution

In the previous Section 1.3, we determined that the Fourier transform of $\frac{\sin(x)}{x}$ evaluated at zero is equal to B_0. To compute the Fourier transform of the other Borwein integrands, we require two components. First, we need the Fourier transform of $\frac{\sin(\frac{x}{m})}{\frac{x}{m}}$, where $m \in \mathbb{R}$. Second, we need to employ a technique known as "Convolution".

First, let's discuss the Fourier transform of $\frac{\sin(\frac{x}{m})}{\frac{x}{m}}$ for $m \in \mathbb{R}$. A concise explanation of the Fourier transform of this function is that it is similar to the Fourier transform of $\frac{\sin(x)}{x}$, but dilated by a factor of $\frac{1}{m}$. Therefore, the Fourier transform is given by the following expression:

$$\widehat{\delta^{\frac{1}{m}} f}(\xi) = \begin{cases} m\pi & \text{if } |2\pi\xi| \leq \frac{1}{m}, \\ 0 & \text{if } |2\pi\xi| > \frac{1}{m}. \end{cases} \tag{2}$$

Using Proposition 1.3, we can confirm (2) finding that the Fourier transform of $\widehat{\delta^{\frac{1}{m}} f}$ is as follows:

$$\widehat{\delta^{\frac{1}{m}} f} = (\frac{1}{m})^{-1} \delta^m \widehat{f} = m\delta^m \widehat{f}$$

Since $\widehat{f} = \begin{cases} \pi & \text{if } |2\pi\xi| \leq 1 \\ 0 & \text{if } |2\pi\xi| > 1 \end{cases}$, it follows that $m\delta^m \widehat{f} = \begin{cases} m\pi & \text{if } |2\pi\xi| \leq \frac{1}{m}, \\ 0 & \text{if } |2\pi\xi| > \frac{1}{m}. \end{cases}$ as expected.

Next, we introduce the concept of convolution:

Definition 1.11. *Given $f, g \in L^1(\mathbb{R})$, the convolution $f * g$ is defined as:*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy, \;\; x \in \mathbb{R} \tag{3}$$

One might question whether (1.11) is well-defined. In fact, it is well-defined almost everywhere, and this can be established using the following theorem from [5]:

Theorem 1.12 (Fubini's theorem)**.** *If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is integrable, then*

$$\int_{\mathbb{R}\times\mathbb{R}} f\,d(x \times y) = \int_{\mathbb{R}}\int_{\mathbb{R}} f\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} f\,dy\,dx.$$

The reasoning is then as follows: If $f, g \in L^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty}(f*g)(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\,dx.$$

By Theorem 1.12, this can be rewritten as:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\,dx\,g(y)\,dy.$$

Utilizing the translation invariance of the $L^1(\mathbb{R})$ norm, we can further rewrite this as:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\,dx\,g(y)\,dy \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x-y)|\,dx\,|g(y)|\,dy = \int_{-\infty}^{\infty} \|f\|_1 |g(y)|\,dy.$$

Since $\|f\|_1$ is a constant, this simplifies to:

$$\|f\|_1 \int_{-\infty}^{\infty} |g(y)|\,dy = \|f\|_1\|g\|_1 < \infty.$$

Therefore, since we considered the integral of $f * g$, we know that Definition 1.11 is well-defined almost everywhere.

Definition 1.13. *Given $f, g \in L^2(\mathbb{R})$, the convolution $f * g$ is defined as:*

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy. \tag{4}$$

Remark. Definition 1.13 is well-defined for the following reason: Given $f, g \in L^2(\mathbb{R})$, we know by using translation invariance and Hölder's Inequality 1.4 that $\|fg\|_1 \leq \|f\|_2\|g\|_2 < \infty$. Therefore, the product fg is in $L^1(\mathbb{R})$, and thus the convolution is well-defined.

To calculate the Fourier transform of the Borwein integrands, we will need a number of theorems, definitions, and propositions, all of which are derived from [3]. We will prove one of these, namely Proposition 1.15 These are as follows:

Definition 1.14. *Given a function f, the inverse Fourier transform $\check{f}$ is defined as:*

$$\check{f}(x) = \hat{f}(-x), \;\; x \in \mathbb{R}$$

Proposition 1.15. *Given f, g in $L^1(\mathbb{R})$ we have*

$$\widehat{f*g} = \hat{f}\hat{g}$$

Proof. Let $f, g \in L^1(\mathbb{R})$. By the definition of the convolution, we have

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

We seek the Fourier transform of $(f * g)(x)$:

$$\widehat{f * g}(\xi) = \int_{-\infty}^{\infty} (f * g)(x)e^{-2\pi i x\xi}\,dx.$$

Substituting the definition of the convolution into this integral, we get:

$$\widehat{f * g}(\xi) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy \right) e^{-2\pi i x\xi}\,dx.$$

Interchanging the order of integration, which is justified by Fubini's Theorem 1.12, we obtain:

$$\widehat{f * g}(\xi) = \int_{-\infty}^{\infty} g(y) \left(\int_{-\infty}^{\infty} f(x-y)e^{-2\pi i x\xi}\,dx \right) dy.$$

Now, we multiply the inner integral by $1 = e^{-2\pi i y\xi + 2\pi i y\xi}$ to obtain the following:

$$\int_{-\infty}^{\infty} f(x-y)e^{-2\pi i x\xi}\,du = e^{-2\pi i y\xi} \int_{-\infty}^{\infty} f(x-y)e^{-2\pi i (x-y)\xi}\,dx.$$

So

$$\widehat{f * g}(\xi) = \int_{-\infty}^{\infty} g(y)e^{-2\pi i y\xi} \int_{-\infty}^{\infty} f(x-y)e^{-2\pi i (x-y)\xi}\,dxdy.$$

Using translation invariance, we recognize that the inner integral is the Fourier transform of f, and thus the expression simplifies to:

$$\widehat{f * g}(\xi) = \int_{-\infty}^{\infty} g(y)e^{-2\pi i y\xi} \widehat{f}(\xi)\,dy.$$

Since $\widehat{f}(\xi)$ is independent of y, we can factor it out of the integral:

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \int_{-\infty}^{\infty} g(y)e^{-2\pi i y\xi}\,dy.$$

The remaining integral is the Fourier transform of g:

$$\int_{-\infty}^{\infty} g(y)e^{-2\pi i y\xi}\,dy = \widehat{g}(\xi).$$

Combining these results, we obtain:

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi).$$

Thus, we have proven that the Fourier transform of the convolution of f and g is equal to the product of their Fourier transforms.

□

Theorem 1.16. *Given $f, \hat{f} \in L^1(\mathbb{R})$*

$$\check{\hat{f}} = f = \hat{\check{f}}$$

Proposition 1.17. *Let f and g be functions in $L^2(\mathbb{R})$. Then the Fourier transform of the convolution $(f * g)$*

is equal to the product of the Fourier transforms of f and g. In other words,

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi),$$

where $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy$.

Theorem 1.18. *Given $f \in L^2(\mathbb{R})$*

$$\check{\widehat{f}} = f = \widehat{\check{f}}$$

Now that we understand convolution and have a number of theorems and definitions at our disposal, we can link convolution to the Fourier transform.

First, let's fix $f, g \in L^2(\mathbb{R})$. According to Proposition 1.17, we have:

$$\widehat{f * g} = \widehat{f}\widehat{g}.$$

Taking the inverse Fourier transform on both sides, we find:

$$f * g = \widecheck{\widehat{f}\widehat{g}}$$

Since Proposition 1.17 holds for all f and g, it also holds for $\check{f}$ and $\check{g}$. Putting all these pieces together, we find the following:

$$\begin{aligned} \check{f} * \check{g}(x) = \widecheck{\widehat{\check{f}}\widehat{\check{g}}}(x) &= \widecheck{fg}(x) \\ \iff \int_{-\infty}^{\infty} \check{f}(x-y)\check{g}(y)\,dy &= \widecheck{fg}(x) \\ \iff \int_{-\infty}^{\infty} \widehat{f}(-x+y)\widehat{g}(-y)\,dy &= (\widehat{\delta^{-1}f)(\delta^{-1}}g)(x). \end{aligned}$$

Therefore, if we wish to know $(\widehat{\delta^{-1}f)(\delta^{-1}}g)$, all we need is $(\delta^{-1}f) * (\delta^{-1}g)$. So, if we know the Fourier transforms of separate functions, we also know the Fourier transform of the product of the functions. Lastly, we remark that this holds for the product of any number of functions. Given $f_0, f_1, \ldots, f_k$, we can say that

$$(\delta^{-1}f_0)(\widehat{\delta^{-1}f_1)\cdots}(\delta^{-1}f_k) = (\widehat{\delta^{-1}f_0}) * (\widehat{\delta^{-1}f_1}) * \cdots * (\widehat{\delta^{-1}f_k})$$

by using $f = f_0$ and $g = f_1 \cdots f_k$ and repeating the process k times.

Now we will specifically consider the Borwein integrands and their Fourier transforms. To do this, we first define our functions as follows for $k = 0, 1, 2, \ldots$:

$$f_k(x) = \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}} \tag{5}$$

$$b_k = \prod_{i=0}^{k} f_i \tag{6}$$

$$B_k = \int_{-\infty}^{\infty} b_k\,dx \tag{7}$$

Remark. f_k is even for all $k \in \mathbb{N}$, and the product of even functions is again even. Therefore, b_k is even too. As a result, we know that

$$(\delta^{-1}f_0)(\widehat{\delta^{-1}f_1)\cdots}(\delta^{-1}f_k) = (\widehat{\delta^{-1}f_0}) * (\widehat{\delta^{-1}f_1}) * \cdots * (\widehat{\delta^{-1}f_k})$$

is the same as

$$\widehat{f_0 f_1 \cdots f_k} = \widehat{f_0} * \widehat{f_1} * \cdots * \widehat{f_k}.$$

Using what we have learned so far, the value of B_k can be found by using the following:

$$B_k = \widehat{b_k}(0) = (\widehat{f_0} * \widehat{f_1} * \cdots * \widehat{f_k})(0).$$

Call $G_k(x) = \widehat{b_k}(x) = (\widehat{f_0} * \widehat{f_1} * \cdots * \widehat{f_k})(x)$. Then $G_k(0) = B_k$, so we can analyze the behaviour of B_k by examining $G_k(0)$. Using Definition 2 and WolframAlpha to compute G_1, we find the closed form of the first two functions:

$$G_0(\xi) = \widehat{b_0}(\xi) = \widehat{f_0}(\xi) = \begin{cases} \pi & \text{if } |2\pi\xi| \le 1 \\ 0 & \text{if } |2\pi\xi| > 1 \end{cases}$$

$$\begin{aligned} G_1(\xi) = \widehat{f_0} * \widehat{f_1}(\xi) &= \int_{-\infty}^{\infty} \widehat{f}(\xi - y)\widehat{g}(y)\, dy \\ &= -\frac{\pi}{2}\left((1 - 3\pi\xi)\operatorname{sgn}\left(\frac{2}{3} - 2\pi\xi\right) + (3\pi\xi - 2)\operatorname{sgn}\left(\frac{4}{3} - 2\pi\xi\right) + 3\pi\xi\operatorname{sgn}\left(2\pi\xi + \frac{2}{3}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{2}{3}\right)\right. \\ &\quad \left. - 3\pi\xi\operatorname{sgn}\left(2\pi\xi + \frac{4}{3}\right) - 2\operatorname{sgn}\left(2\pi\xi + \frac{4}{3}\right)\right) \end{aligned}$$

After G_1 the closed form becomes too convoluted to be of use. In the next section we will analyze the graphical nature of the functions G_0 through G_7.

1.5 A Graphical Analysis of the Fourier Transformations of the Borwein Integrands

In the previous section, we derived analytical expressions for the Fourier transforms of the first two Borwein integrands. However, in their analytical form, it is not immediately clear what the functions look like. Therefore, in this section, we will perform a graphical analysis of them. The code for generating these figures can be found in Appendix 4.1.

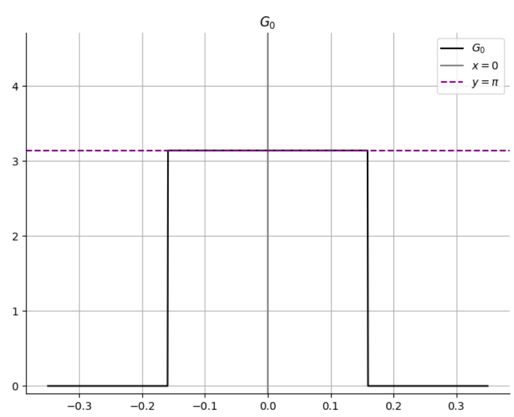


Figure 5: $G_0; G_0(0) = \pi$

The graph of G_0 is plotted in Figure 5. Notably, it takes values of either 0 or π. An important feature of this graph is the "plateau" of length $\frac{1}{\pi}$ centered around zero, where the function has a value exactly equal to π. This feature will be important later and will reoccur as a significant characteristic in the upcoming graphs.

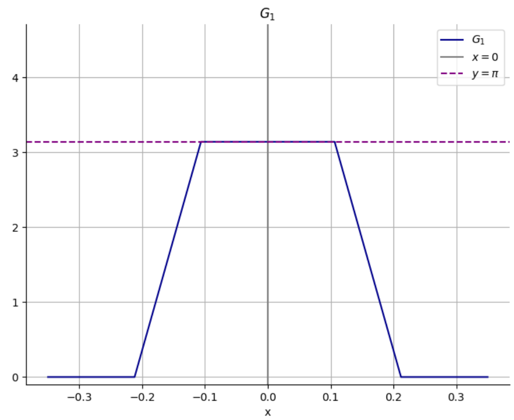
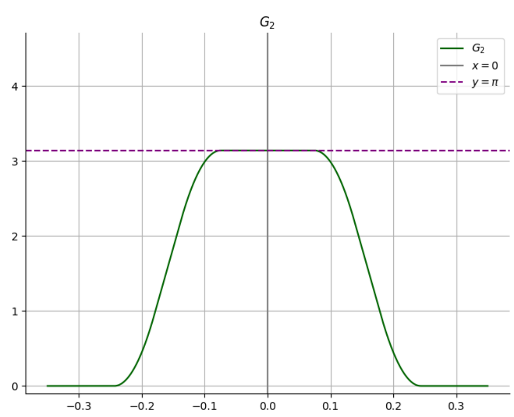


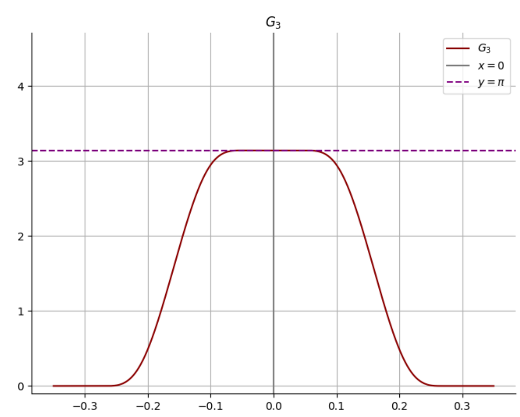
Figure 6: $G_1; G_1(0) = \pi$

Figure 6 shows the plot of G_1. Recall that G_1 is formed using convolution, which in a sense takes the moving average of G_0 using a window of width $\frac{1}{3\pi}$. The important effect of this is that G_1 still has a plateau centered around $x = 0$ where the function is equal to π, but this plateau has shrunk in length by $\frac{1}{3\pi}$. Now the plateau has a length of $\frac{1}{\pi}(1 - \frac{1}{3})$.

Hereafter, each G_i is formed by taking the moving average of G_{i-1} using a window of width $\frac{1}{2i+1}$. The result of this process can be seen in Figure 7a through Figure 9b, where the plateau of G_i has a length of $\frac{1}{\pi}(1 - \frac{1}{3} - \cdots - \frac{1}{2i+1})$. While this plateau exists and is always centered around zero, we know that the value of $B_i = \pi$. However, once $1 - \frac{1}{3} - \cdots - \frac{1}{2i+1}$ becomes less than zero, the plateau vanishes and the value of B_i will be less than π. Notably, this happens for the first time when $i = 7$, which breaks the "pattern". Namely, $1 - \frac{1}{3} - \cdots - \frac{1}{15} \approx -0.02180$ and $B_7 = G_7(0) \approx \pi - 0.0000000000462$ instead of π.

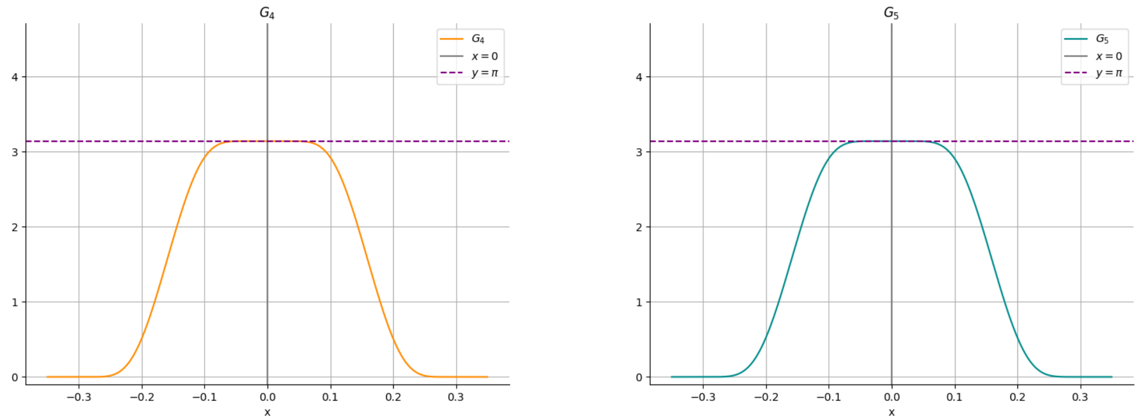


(a) $G_2(0) = \pi$



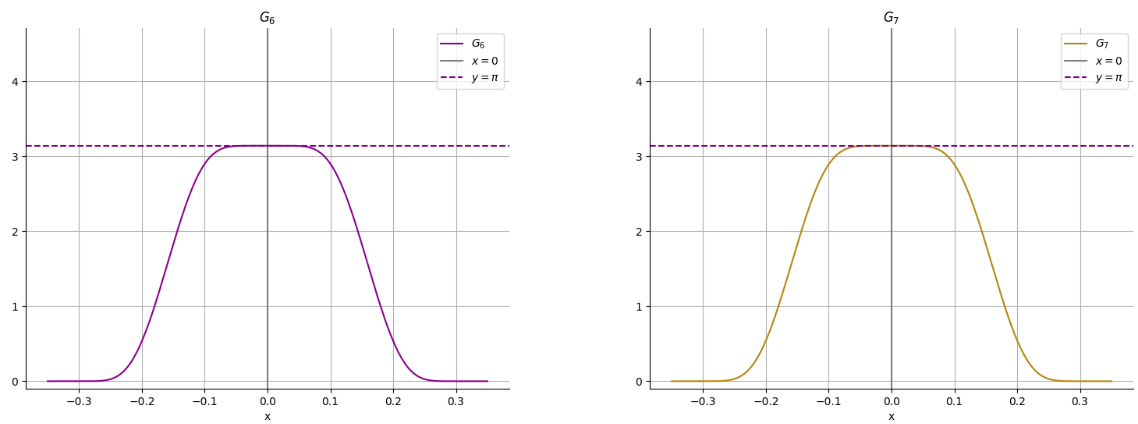
(b) $G_3(0) = \pi$

Figure 7: G_2 and G_3



(a) $G_4(0) = \pi$ (b) $G_5(0) = \pi$

Figure 8: G_4 and G_5



(a) $G_6(0) = \pi$ (b) $G_7(0) = \pi - 0.0000000000462$

Figure 9: G_6 and G_7

1.6 The Closed Form of The Borwein Integrals

In the previous section, we explored why the apparent pattern of the Borwein integrals "broke". What then, is the pattern? Or rather, is it possible to find a closed form for B_k for any $k \in \mathbb{N}$? This section discusses this question and investigates the behaviour of B_k as $k \to \infty$. D. and J. Borwein, in Theorem 2 of their paper [2], found the closed form of a generalized version of the Borwein integrals. The result is as follows:

Let $a_0, a_1, \ldots, a_n$ be complex numbers with $n \geq 1$. For each of the 2^n ordered n-tuples $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \{-1, 1\}^n$, define:

$$b_\gamma := a_0 + \sum_{k=1}^{n} \gamma_k a_k, \quad \epsilon_\gamma := \prod_{k=1}^{n} \gamma_k.$$

(i) If $a_0, a_1, \ldots, a_n$ are real, then:

$$\int_0^\infty \prod_{k=0}^{n} \frac{\sin(a_k x)}{x} \, dx = \frac{\pi}{2} \frac{1}{2^n n!} \sum_{\gamma \in \{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).$$

where the sign function is defined as:

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Note that the Borwein integrals are almost in the form of (i). If we use $a_k = \frac{1}{2k+1}$ then we find the following:

$$\int_0^\infty \prod_{k=0}^n \frac{\sin(a_k x)}{x}\, dx = \int_0^\infty \prod_{k=0}^n \frac{\sin\left(\frac{x}{2k+1}\right)}{x}\, dx$$

Then to find the Borwein integrals we multiply by $1 = \frac{2k+1}{2k+1}$:

$$\frac{\pi}{2}\frac{1}{2^n n!}\sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma). = (\prod_{k=0}^n \frac{2k+1}{2k+1}) \int_0^\infty \prod_{k=0}^n \frac{\sin\left(\frac{x}{2k+1}\right)}{x}\, dx = (\frac{1}{\prod_{k=0}^n 2k+1}) \int_0^\infty \prod_{k=0}^n \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}\, dx$$

Therefore:

$$\int_0^\infty \prod_{k=0}^n \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}\, dx = (\prod_{k=0}^n 2k+1)\frac{\pi}{2}\frac{1}{2^n n!}\sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).$$

Lastly, since $\frac{\sin(a_k x)}{x}$ is an even function, the integral over $-\infty$ to ∞ is equal to twice the integral over 0 to ∞. Therefore, we have the following:

$$B_n = \int_{-\infty}^\infty \prod_{k=0}^n \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}\, dx = (\prod_{k=0}^n 2k+1)\frac{\pi}{2^n n!}\sum_{\gamma\in\{-1,1\}^n} \epsilon_\gamma b_\gamma^n \operatorname{sign}(b_\gamma).$$

Now that we have a closed-form expression for the value of B_n, a natural next step is to ask if there is a limit as $n \to \infty$. The evaluation of this limit does exist and can be found with the help of computers. J. Cook [7] found that the limit was equal to approximately $\pi - 0.0000704$.

2 The Nahin Integrals

We know from the previous chapter that the Borwein integrals are not entirely unique; however, even the general version restricted itself to functions composed of sine functions. In this section, we will consider similar functions that depend on cosine, perform a similar analysis on them, and compare the results to the Borwein integrals. These functions were found in the preface of the book "Inside Interesting Integrals" [8] by P. Nahin. In the preface of that book, Nahin used the following functions as an example of an apparent pattern which breaks after a certain point:

$$h_n : \mathbb{R} \to \mathbb{R}, \quad h_n(x) = \frac{\sin(4x)}{x} \prod_{k=0}^{n} \cos\left(\frac{x}{k+1}\right)$$

We will refer to these functions as the Nahin functions. While they have similarities to the Borwein integrands, there are also several noticeable differences. We will explore these differences while analyzing the Nahin functions. Furthermore, we will define the integral over these functions as the "Nahin integrals":

Definition 2.1. *The Nahin integrals are a sequence of integrals of the following form:*

$$H_n = \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \prod_{k=0}^{n} \cos\left(\frac{x}{k+1}\right) dx, \quad n \in \mathbb{N}$$

The values of the first 31 integrals are as follows:

$$H_0 = \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \cos\left(\frac{x}{1}\right) dx = \pi$$
$$H_1 = \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \cos\left(\frac{x}{1}\right) \cos\left(\frac{x}{2}\right) dx = \pi$$
$$\cdots = \pi$$
$$H_{29} = \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \cos\left(\frac{x}{1}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{3}\right) \ldots \cos\left(\frac{x}{30}\right) dx = \pi$$
$$H_{30} = \int_{-\infty}^{\infty} \frac{\sin(4x)}{x} \cos\left(\frac{x}{1}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{3}\right) \ldots \cos\left(\frac{x}{31}\right) dx \approx \pi - 0.000000003589792... \approx 3.14177$$

This is similar to the behaviour we observed with the Borwein integrals, where the integral equals π for the first few terms and then starts to decrease slowly. However, it is not immediately clear why this occurs by simply examining the functions. Therefore, just as before, we will plot the functions and analyze their graphs.

2.1 A Graphical Analysis of the Nahin Functions

In this section, we graph the first two Nahin functions, h_0 and h_1, as well as h_{29} and h_{30}, since it is between the last two that the 'pattern' breaks. By graphing these functions, we hope to identify a pattern in the graphs that will explain the observed phenomenon. The code for these figures can be found in Appendix 4.1.

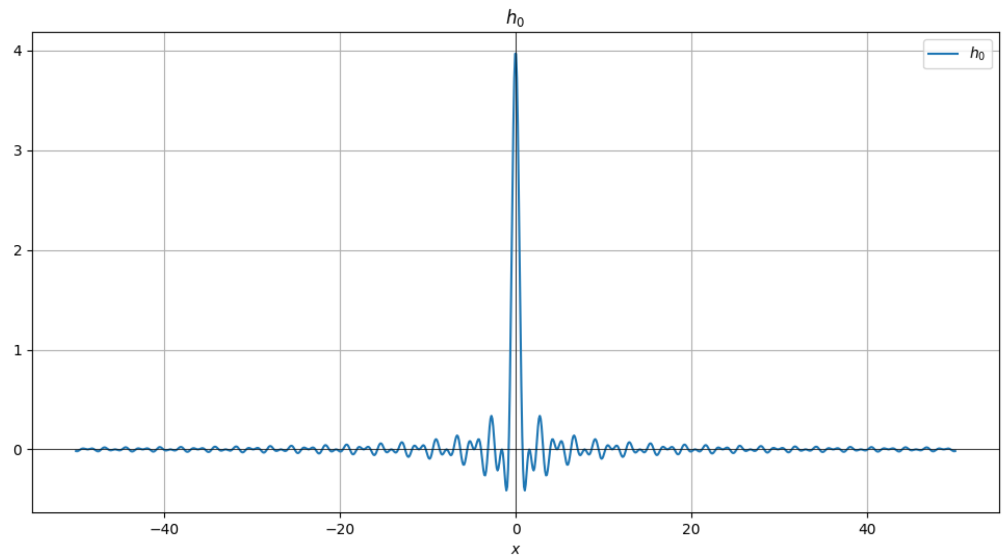


Figure 10: h_0

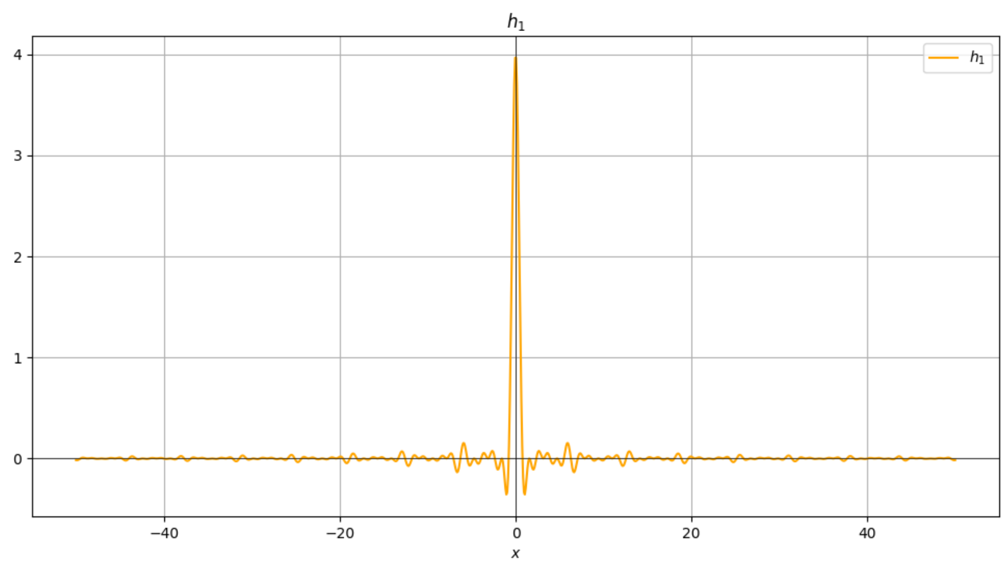


Figure 11: h_1

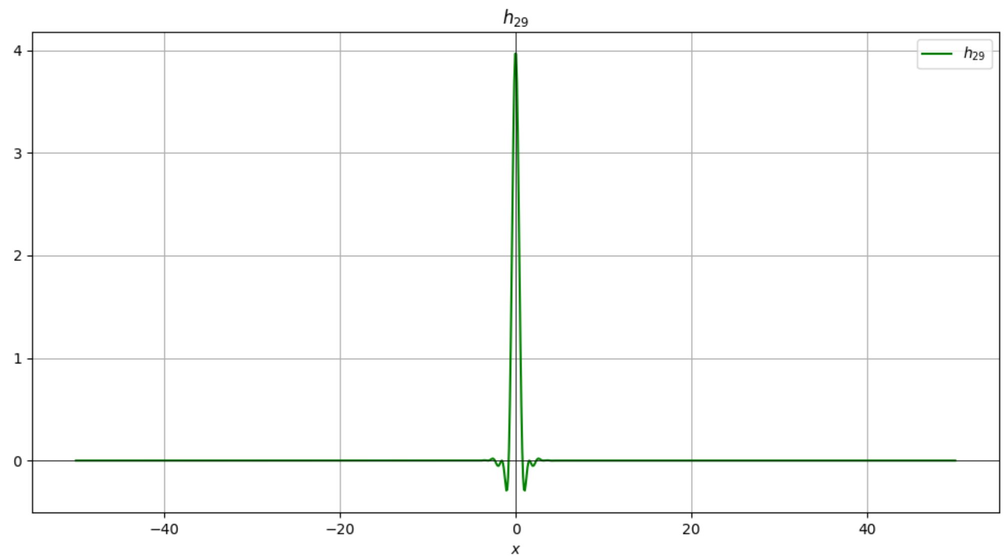


Figure 12: h_{29}

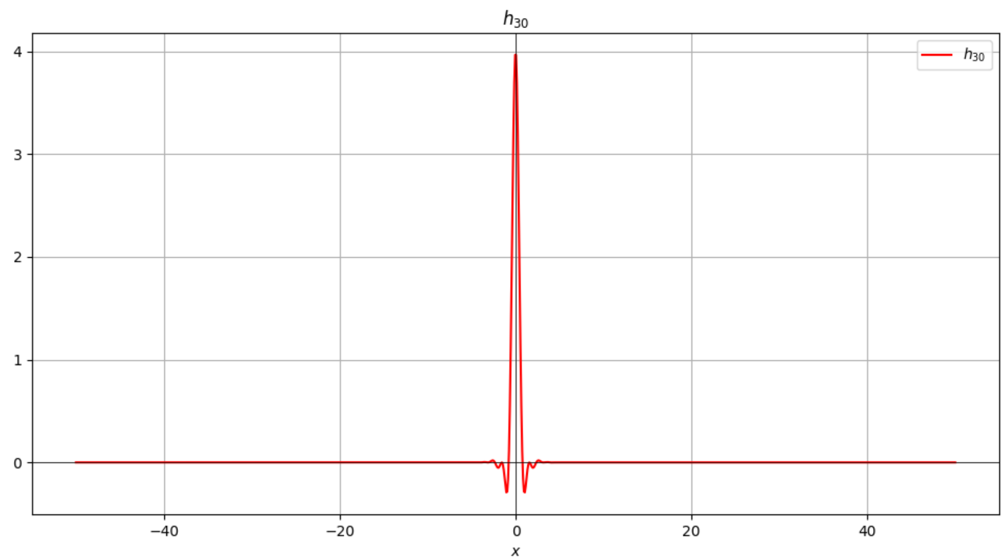


Figure 13: h_{30}

Noticeably, in Figure 10, similar to the Borwein integrands, there is a spike in the center of the graph where the function approaches four. Furthermore, there are smaller oscillations around the x-axis that die out as the function approaches $\pm\infty$. The spike in the graph is retained in each figure, but the oscillations become more damped with each successive function. Just as with the Borwein integrals, there is no apparent explanation for why the first thirty integrals are equal to π, but the thirty-first is not. Therefore, we will use the same approach as before by calculating the Fourier transform of the Nahin integrands in the following section.

2.2 The Fourier Transform of the Nahin Integrands

Just as with the Borwein integrals, we will look at the Fourier transform of the Nahin functions in order to make sense of the apparent phenomena. The first three Fourier transforms are as follows:

$$\begin{aligned}\widehat{h_0}(\xi) &= \int_{-\infty}^{\infty} \left(\frac{\sin(4x)}{x} \prod_{k=1}^{1} \cos\left(\frac{x}{k}\right) \right) e^{-2\pi i \xi x}\, dx \\ &= \frac{\pi}{4} \left(\operatorname{sgn}(3 - 2\pi\xi) + \operatorname{sgn}(5 - 2\pi\xi) + \operatorname{sgn}(2\pi\xi + 3) + \operatorname{sgn}(2\pi\xi + 5) \right)\end{aligned}$$

$$\begin{aligned}\widehat{h_1}(\xi) &= \int_{-\infty}^{\infty} \left(\frac{\sin(4x)}{x} \prod_{k=1}^{2} \cos\left(\frac{x}{k}\right) \right) e^{-2\pi i \xi x}\, dx \\ &= \frac{\pi}{8}\Bigg(\operatorname{sgn}\left(\frac{5}{2} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{7}{2} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{9}{2} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{11}{2} - 2\pi\xi\right) \\ &\quad + \operatorname{sgn}\left(2\pi\xi + \frac{5}{2}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{7}{2}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{9}{2}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{11}{2}\right) \Bigg)\end{aligned}$$

$$\begin{aligned}\widehat{h_2}(\xi) &= \int_{-\infty}^{\infty} \left(\frac{\sin(4x)}{x} \prod_{k=1}^{3} \cos\left(\frac{x}{k}\right) \right) e^{-2\pi i \xi x}\, dx \\ &= \frac{\pi}{16}\Bigg(\operatorname{sgn}\left(\frac{13}{6} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{17}{6} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{19}{6} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{23}{6} - 2\pi\xi\right) \\ &\quad + \operatorname{sgn}\left(\frac{25}{6} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{29}{6} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{31}{6} - 2\pi\xi\right) + \operatorname{sgn}\left(\frac{35}{6} - 2\pi\xi\right) \\ &\quad + \operatorname{sgn}\left(2\pi\xi + \frac{13}{6}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{17}{6}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{19}{6}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{23}{6}\right) \\ &\quad + \operatorname{sgn}\left(2\pi\xi + \frac{25}{6}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{29}{6}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{31}{6}\right) + \operatorname{sgn}\left(2\pi\xi + \frac{35}{6}\right) \Bigg)\end{aligned}$$

As we can see, the formulae for $\widehat{h_n}$ become quite convoluted quite quickly, where each terms seems to have twice as many sign functions than the term before it. Once again we will turn to the graphs of these functions to try and explain the behaviour:

2.3 A Graphical Explanation of the Fourier Transform of the Nahin Functions

This section contains the graphical explanation of the Fourier transform of the Nahin functions. The Fourier transforms of the first six functions were computed using WolframAlpha and then plotted using Python, the code for which can be found in Appendix 4.1. The Fourier transform of the i-th function requires approximately 2^{i+2} computations, making the calculation of the Fourier transform of the thirtieth function (where the apparent pattern breaks) impossible using a normal laptop. However, further study could calculate this using a supercomputer or by approximating it numerically. We did not calculate the Fourier transform numerically because of the Gibbs phenomenon, which causes oscillations near discontinuities in the function. Since the Fourier transforms of the Nahin functions have many discontinuities, this would result in a large number of oscillations, making it impossible to discover the true behaviour of the function. Nevertheless, we can draw some conclusions, knowing that the pattern breaks at $i = 30$ by studying the graphical behaviour of the Fourier transforms of the first six functions.

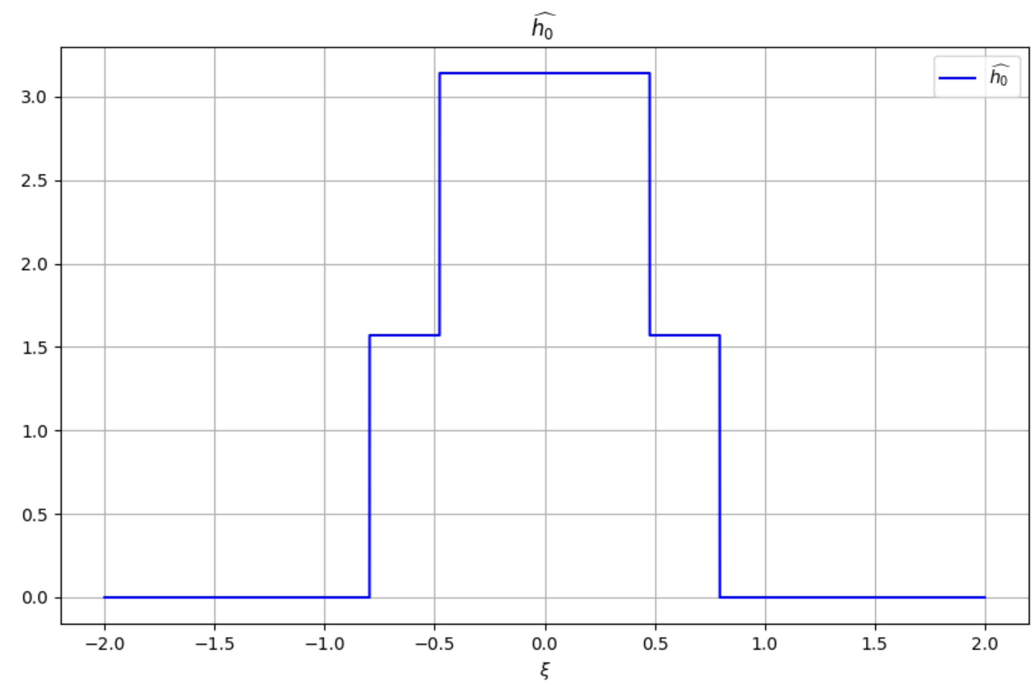


Figure 14: Fourier transform of $\widehat{h_0}$

Just as with the Borwein integrals, we know that the integral over the Nahin functions is equal to the Fourier transform at the point 0. Furthermore, just as before, we find a plateau where the function is equal to π centered around $\xi = 0$. Therefore, by analyzing the behaviour of the plateau in the first six functions, we can deduce the behaviour of the subsequent functions. For $\widehat{h_0}$, which can be seen in Figure 14, the plateau has a width $p_0 \approx 0.9549$.

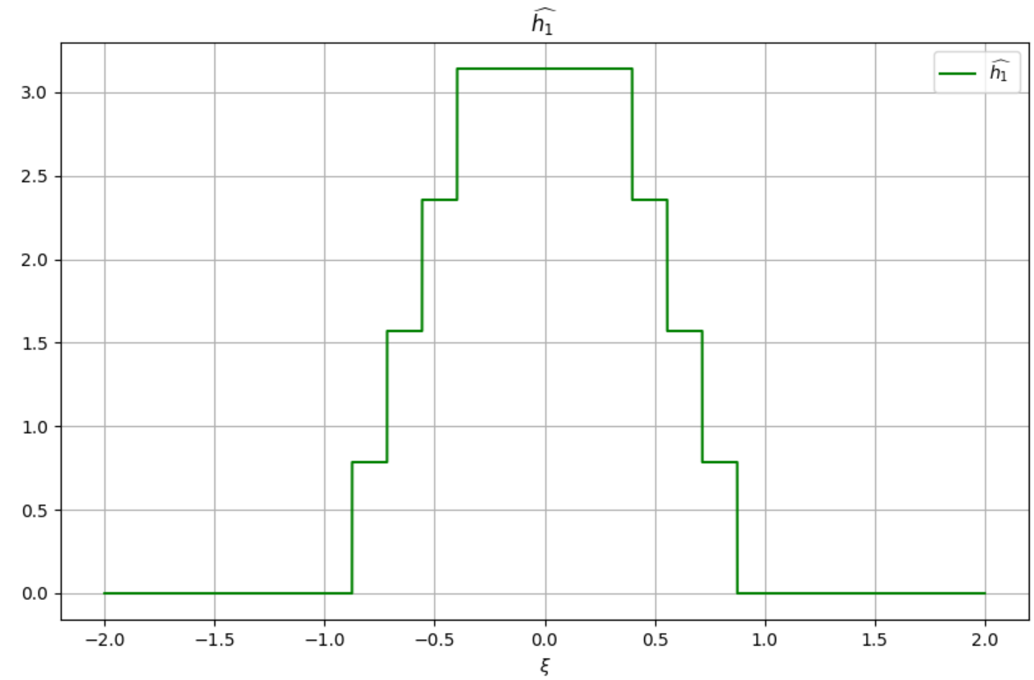


Figure 15: Fourier transform of $\widehat{h_1}$

Figure 15 depicts the graph of $\widehat{h_1}$, where the function is equal to π for $\xi \in [\approx -0.3979, \approx 0.3979]$. The plateau then has a width $p_1 \approx 0.7958$. Therefore, the plateau has shrunk by approximately 0.1592.

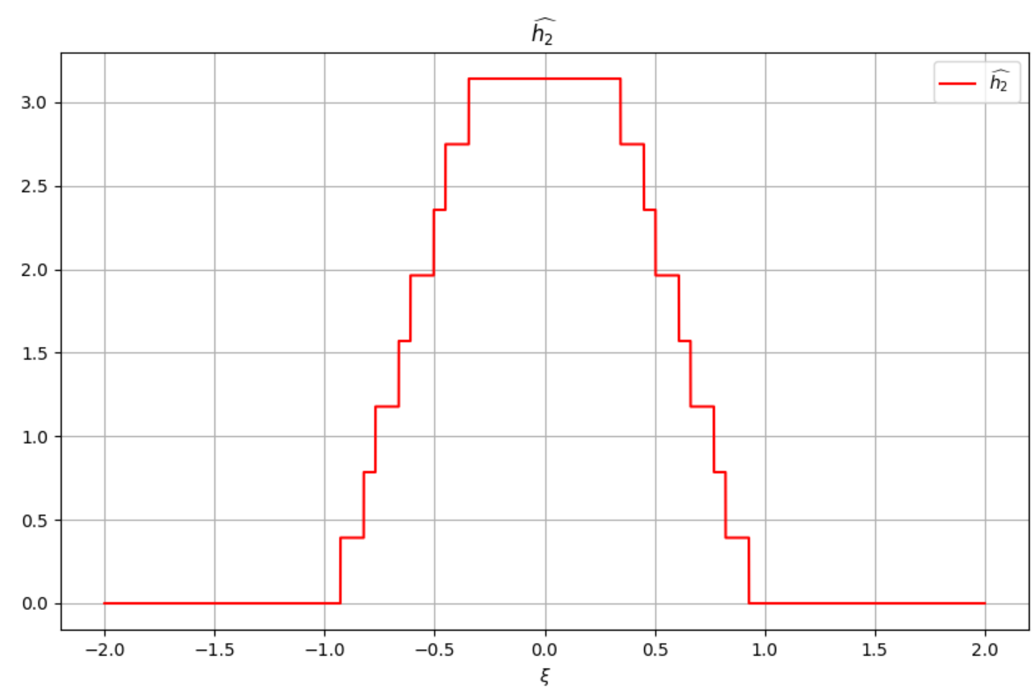


Figure 16: Fourier transform of $\widehat{h_2}$

Next, for $\widehat{h_2}$ seen in Figure 16, the function is equal to π for $\xi \in [\approx -0.3448, \approx 0.3448]$. The plateau then has a width $p_2 \approx 0.6897$, which is approximately 0.1061 less than p_1.

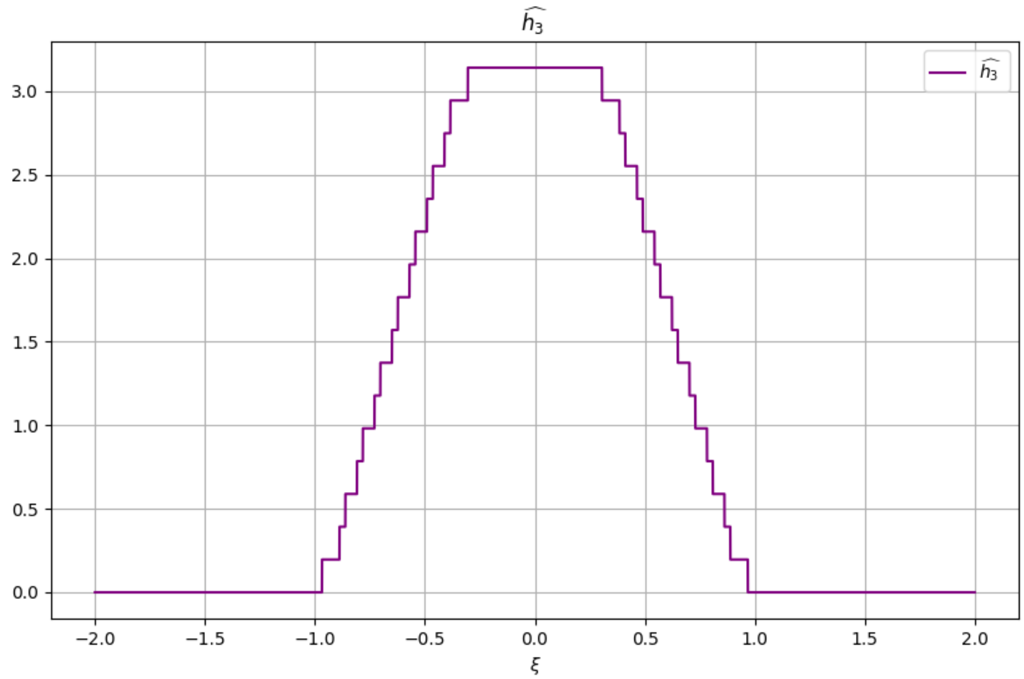


Figure 17: Fourier transform of $\widehat{h_3}$

For $\widehat{h_3}$, the function is equal to π for $\xi \in [\approx -0.3050, \approx 0.3050]$ and is depicted in Figure 17. The plateau then has a width $p_3 \approx 0.6101$, meaning the plateau has shrunk by approximately 0.0796.

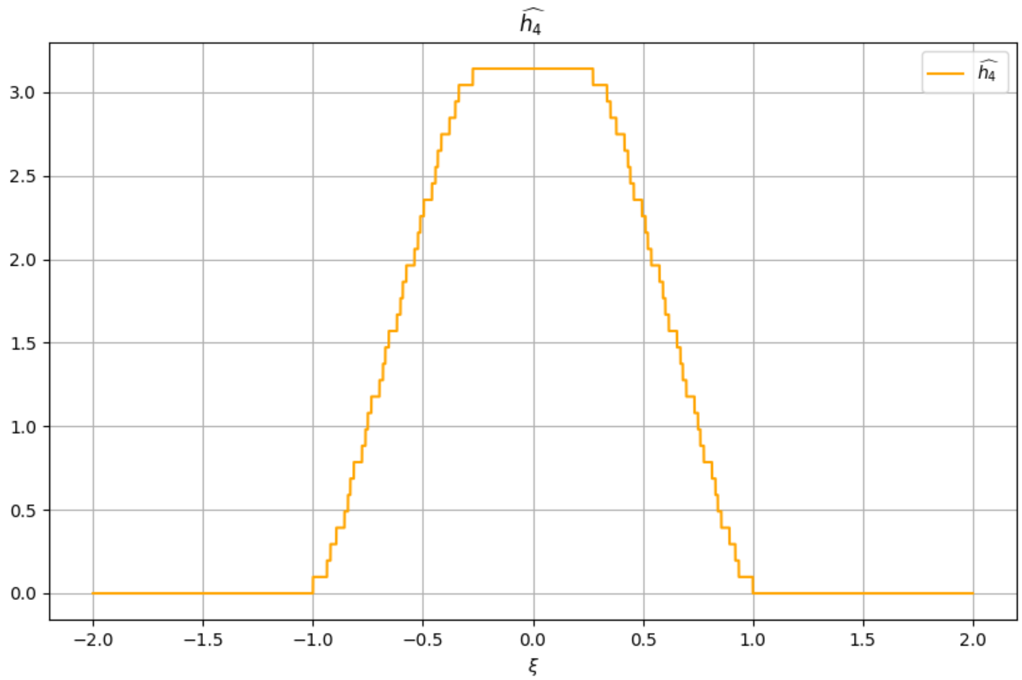


Figure 18: Fourier transform of $\widehat{h_4}$

Next, for $\widehat{h_4}$, the function is depicted in Figure 18 and is equal to π for $\xi \in [\approx -0.2732, \approx 0.2732]$. The plateau then has a width $p_4 \approx 0.5464$, which is approximately 0.0636 less than p_3.

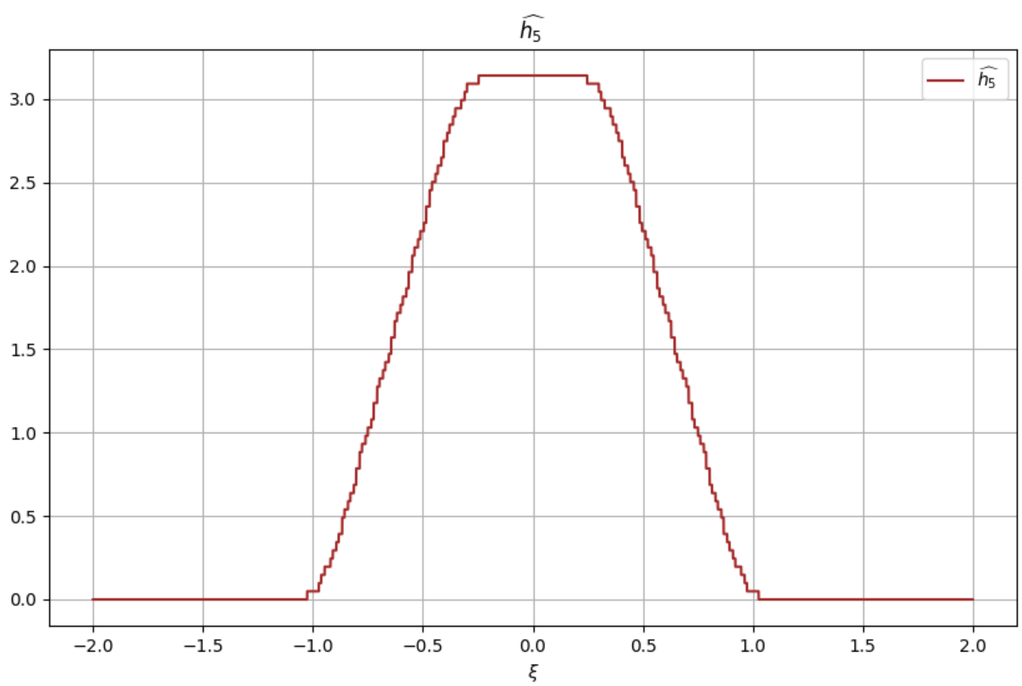


Figure 19: Fourier transform of $\widehat{h_5}$

Finally, for $\widehat{h_5}$ seen in Figure 19, the function is equal to π for $\xi \in [\approx -0.2467, \approx 0.2467]$. The plateau then has a width $p_5 \approx 0.4934$, which is approximately 0.0530 less than p_4.

We know that the plateau will eventually vanish at $\widehat{h_{30}}$ since $\widehat{h_{30}}(0) < \pi = \widehat{h_{29}}$. We know this because, just like with the Borwein integrals, the plateau where each $\widehat{h_n}$ is equal to π is centered around zero. Furthermore, since each $\widehat{h_n}$ is formed by taking an average of the previous one by convolution, and the function value on the plateau is also the maximum value that the function takes, the plateau will either stay the same length or become smaller. Since $\widehat{h_{30}}(0) < \pi$, the plateau must have vanished since the function value at zero is no longer π. We therefore know that the plateau will vanish. However, we do not yet have a mathematical expression for when the plateau will vanish or the length of the plateau at $\widehat{h_n}$. With the Borwein integrals, we knew that the coefficients were the determining factor; however, in this case, we do not see a direct link between the coefficients of the functions and the length of the plateau. In the next section, we will discover the mathematical explanation for this phenomenon.

2.4 A Further Analysis of the Nahin Phenomena Leading to a Mathematical Explanation

In this section, we will determine the mathematical explanation for the Nahin phenomenon. To do this, we will first rewrite the Nahin functions in the following manner:

$$h_n(x) = \frac{\sin(4x)}{x}\prod_{k=0}^{n}\cos\left(\frac{x}{k+1}\right) = 4\frac{\sin(4x)}{4x}\prod_{k=0}^{n}\frac{e^{i\frac{1}{k+1}x} + e^{i(-\frac{1}{k+1})x}}{2} = \frac{4}{2^{n+1}}\frac{\sin(4x)}{4x}\prod_{k=0}^{n}(e^{i\frac{1}{k+1}x} + e^{i(-\frac{1}{k+1})x})$$

Next, we introduce the notation $\gamma := (\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n) \in \{-1, 1\}^{n+1}$ and define $c := (c_0, c_1, c_2, \dots, c_n)$, where $c_k = \frac{1}{k+1}$ for $k = 0, 1, 2, \dots, n$. Then we can continue to rewrite the Nahin functions as follows:

$$\frac{4}{2^{n+1}}\frac{\sin(4x)}{4x}\prod_{k=0}^{n}(e^{i\frac{1}{k+1}x} + e^{i(-\frac{1}{k+1})x}) = \frac{4}{2^{n+1}}\frac{\sin(4x)}{4x}\sum_{\gamma\in\{-1,1\}^{n+1}} e^{i(\gamma\cdot c)x}$$

Using the above formulation of the Nahin functions, we can then rewrite the Fourier transform of them in the following manner:

$$\begin{aligned}
\widehat{h_n}(\xi) &= \int_{-\infty}^{\infty}\left(\frac{\sin(4x)}{x}\prod_{k=0}^{n}\cos\left(\frac{x}{k+1}\right)\right)e^{-2\pi i\xi x}\,dx \\
&= \int_{-\infty}^{\infty}\frac{4}{2^{n+1}}\frac{\sin(4x)}{4x}\left(\sum_{\gamma\in\{-1,1\}^{n+1}} e^{i(\gamma\cdot c)x}\right)e^{-2\pi i\xi x}\,dx \\
&= \frac{4}{2^{n+1}}\sum_{\gamma\in\{-1,1\}^{n+1}}\int_{-\infty}^{\infty}\frac{\sin(4x)}{4x}e^{i(\gamma\cdot c)x-2\pi i\xi x}\,dx \\
&= \frac{4}{2^{n+1}}\sum_{\gamma\in\{-1,1\}^{n+1}}\int_{-\infty}^{\infty}\frac{\sin(4x)}{4x}e^{i(\gamma\cdot c-2\pi\xi)x}\,dx \\
&= \frac{4}{2^{n+1}}\sum_{\gamma\in\{-1,1\}^{n+1}}\widehat{f_{\frac{1}{4}}}\left(\xi - \frac{\gamma\cdot c}{2\pi}\right)
\end{aligned}$$

Where we use a $f_m(x) = \frac{\sin\left(\frac{x}{m}\right)}{\frac{x}{m}}, m \in \mathbb{R}$. With this formulation, we have now found a direct link between the Nahin functions and the Borwein functions! Namely, the Fourier transform of the Nahin functions can be written as the sum of dilated Fourier transforms of the integrand of B_0.

We continue our analysis by recalling from Definition 2 that the Fourier transform of $\delta^{\frac{1}{4}}f:\ \delta^{\frac{1}{4}}f(x) = \frac{\sin(4x)}{4x}$ is given by:

$$\widehat{f_{\frac{1}{4}}}(\xi - \frac{\gamma\cdot c}{2\pi}) = \begin{cases} \frac{\pi}{4} & \text{if } |2\pi\xi - \gamma\cdot c| \leq 4, \\ 0 & \text{if } |2\pi\xi - \gamma\cdot c| > 4. \end{cases}$$

So we know the following:

$$\widehat{h_n}(\xi) = \frac{4}{2^{n+1}} \sum_{\gamma\in\{-1,1\}^{n+1}} \widehat{f_{\frac{1}{4}}}(\xi - \frac{\gamma\cdot c}{2\pi})$$

Notably, $\widehat{h_n}(\xi)$ is only equal to π if for every $\gamma \in \{-1,1\}^{n+1}$, we have $|2\pi\xi - \gamma\cdot c| \leq 4$. In this case,

$$\widehat{h_n}(\xi) = \frac{4}{2^{n+1}} \cdot 2^{n+1}\frac{\pi}{4} = \pi.$$

Furthermore, since the plateau is centered around zero, as soon as there is a $\gamma\cdot c$ for which $|2\pi\cdot 0 - \gamma\cdot c| > 4$ holds, we know that the plateau will be gone since then $\widehat{h_n}(0) < \pi$. Note that $|2\pi\cdot 0 - \gamma\cdot c| > 4$ is the same as $|\gamma\cdot c| > 4$, and that if this is not the case, then for all $|\gamma\cdot c|$:

$$\gamma\cdot c \in \left[-\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right), 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right].$$

So, the first time that $|\gamma\cdot c|$ can be larger than four is when $\gamma\cdot c$ is largest in magnitude, which is when the following holds:

$$\gamma\cdot c = \pm\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

Putting it all together, we find that the plateau will vanish when there is a $\gamma\cdot c$ such that $|\gamma\cdot c| > 4$. The first $\gamma\cdot c$ for which this will happen is:

$$\gamma\cdot c = \pm\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right).$$

Thus, when n is such that

$$\left|\pm\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)\right| = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} > 4,$$

the plateau will no longer exist, and the value of $H_n = \widehat{h_n}(0)$ will be less than π.

At the beginning of this section, we knew that H_{29} was equal to π, but H_{30} was not, and now we know why. Namely,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30} \approx 3.99498\ldots < 4,$$

and so $H_{29} = \pi$. However,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{31} \approx 4.02724\ldots > 4,$$

and thus $H_{30} < \pi$.

Now we know why the 'pattern' in the Nahin functions breaks, and we end this section by giving the closed form value of the Nahin functions for an arbitrary n:

$$H_n = \widehat{h_n}(0) = \frac{4}{2^{n+1}} \sum_{\gamma\in\{-1,1\}^{n+1}} \widehat{f_{\frac{1}{4}}}\left(-\frac{\gamma\cdot c}{2\pi}\right)$$

where γ, c, and $\widehat{f_m}$ are defined as before.

3 Conclusion and Discussion

In conclusion, this paper has provided a comprehensive analysis of the Borwein integrands through various methods, including analytical and graphical approaches, and Fourier analysis. While the apparent phenomena remained unexplained when looking at the Borwein integrands themselves, a Fourier analysis of the integrands offered an explanation for these phenomena and the integrals' overall behaviour.

Another interesting aspect of the Borwein integrands is their connection to the Random Harmonic Series and Probability Theory. For further study, see [9].

Extending the same methodology to the Nahin functions resulted in a similar explanation of their behaviour, as well as a closed-form expression for the value of each Nahin integral. Further research could still be conducted to discover the behaviour of the Nahin integral as n approaches infinity, namely, whether it approaches zero or whether it, like the Borwein integrals, approaches some number between zero and π.

Further studies could build on the results of this paper and on the work of D. and J. Borwein [2] by finding closed-form expressions of general integrals of the form:

$$\int_{-\infty}^{\infty} \prod_{k_1=0}^{n_1} \cos(b_{k_1} x) \prod_{k_2=0}^{n_2} \sin(b_{k_2} x) \prod_{k_3=0}^{n_3} \frac{b_{k_3} x}{x} \, dx, \quad n_1, n_2, n_3, b_{k_1}, b_{k_2}, b_{k_3} \in \mathbb{N}$$

Lastly, for more instances of interesting phenomena in mathematics, see Section 1.4: "High Precision Fraud" of [10].

4 Appendices

The following appendices contain codes from Python and TexWorks which were created with the help of ChatGPT. Some codes are too long for the margin of the page, in that case the code which extends beyond the page has been moved to the next line and ## has been inserted at the start of that line to indicate this change. To correctly run the code, simply remove the ## and reattach the two lines. As a last remark, ChatGPT was also used as a help in the making of part of the proof of Proposition 1.17 and used as a language tool to correct errors and improve the flow of the paper.

4.1 Appendix A: Python Code for Plotting the Borwein and Nahin Integrands and the Fourier Transform of the Borwein and Nahin Integrands

The following is the code which plots the Borwein integrands:

```
import numpy as np
import matplotlib.pyplot as plt

# Define the individual B_i functions
def B_1(x):
    return np.sin(x / 1) / (x / 1)

def B_2(x):
    return (np.sin(x / 1) / (x / 1)) * (np.sin(x / 3) / (x / 3))

def B_6(x):
    product = 1
    for k in range(7):
        product *= np.sin(x / (2*k + 1)) / (x / (2*k + 1))
    return product

def B_7(x):
    product = 1
    for k in range(8):
        product *= np.sin(x / (2*k + 1)) / (x / (2*k + 1))
    return product

# Define the range of x values (in terms of pi)
x = np.linspace(-9*np.pi, 9*np.pi, 1000)

# Define the functions and their corresponding labels and colors
functions = [(B_1, 'B_1', 'blue'), (B_2, 'B_2', 'green'), (B_6, 'B_6', 'red'), (B_7, 'B_7', 'purple')]

# Plot each function separately
for func, label, color in functions:
    y = func(x)
    plt.figure()
    plt.plot(x, y, color=color, label=label)
    plt.xlim([-24, 24])
    plt.grid(True)
    plt.xticks(np.arange(-7*np.pi, 8*np.pi, np.pi),
```

```
                [f'{k}' for k in range(-7, 8)])
    plt.show()
```

Next is the code which plots the Fourier transforms of the first nine Borwein integrands:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import convolve

# Define rect function with width 1
def rect(x):
    return np.where(np.logical_and(x >= -0.5, x <= 0.5), np.pi, 0)

# Define f_0(x) as rect(x)
def f_0(x):
    return rect(x)

# Define a function to generate a rectangular window
def rectangular_window(x, width):
    return np.where(np.logical_and(x >= -width/2, x <= width/2), 1/width, 0)

# Define f_n(x) as the moving average of f_{n-1}(x) using the specified window size
def moving_average(f_prev, window_size):
    f_convolved = convolve(f_prev, rectangular_window(x, window_size), mode='same')
    return f_convolved * (np.pi / np.max(f_convolved))

# Define the range of x values
x = np.linspace(-1, 1, 1001)

# Initialize f_1(x) as f_0(x)
f_1_values = f_0(x)

# Define the window sizes for each function
window_sizes = [1/3, 1/5, 1/7, 1/9, 1/11, 1/13, 1/15, 1/17]

# Define more distinct colors for the plots
colors = ['darkblue', 'darkgreen', 'darkred', 'darkorange', 'darkcyan', 'darkmagenta', 'darkgoldenrod',

# Plot the rect function separately
plt.figure(figsize=(8, 6))
plt.plot(x, f_1_values, color='black', label='$G_0$')
plt.axvline(x=0, color='grey', linestyle='-', label='$x = 0$')  # Add grey vertical line at x=0
plt.axhline(y=np.pi, color='purple', linestyle='--', label='$y = \pi$')  # Add purple horizontal line a
plt.ylim(-0.1, 1.5 * np.pi)  # Set y-axis range
plt.title('$G_0$')
plt.xlabel('x')
plt.gca().spines['top'].set_visible(False)  # Hide top spine
plt.gca().spines['right'].set_visible(False)  # Hide right spine
plt.grid(True)  # Show grid lines
plt.legend(loc='upper right')  # Move legend to upper right
```

```
plt.show()

# Compute and plot each function on separate graphs
for i, (window_size, color) in enumerate(zip(window_sizes, colors), 1):
    f_n_values = moving_average(f_1_values, window_size)
    f_1_values = f_n_values

    # Plot the function with default white background and different color for each plot
    plt.figure(figsize=(8, 6))
    plt.plot(x, f_n_values, color=color, label=f'$G_{i}$')
    plt.axvline(x=0, color='grey', linestyle='-', label='$x = 0$')  # Add grey vertical line at x=0
    plt.ylim(-0.1, 1.5 * np.pi)  # Set y-axis range
    plt.axhline(y=np.pi, color='purple', linestyle='--', label='$y = \pi$')  # Add purple horizontal li
    plt.title(f'$G_{i}$ ')
    plt.xlabel('x')
    plt.gca().spines['top'].set_visible(False)  # Hide top spine
    plt.gca().spines['right'].set_visible(False)  # Hide right spine
    plt.grid(True)  # Show grid lines
    plt.legend(loc='upper right')
    plt.show()
```

Next is the code which plots the Nahin integrands:

```
import numpy as np
import matplotlib.pyplot as plt

def H_i(x, i):
    """Compute the integrand H_i(x) = (sin(4x)/x) * product(cos(x/(k+1)) for k=0 to i"""
    if x == 0:
        return 4  # Handle the singularity at x=0
    product_term = np.prod([np.cos(x/(k+1)) for k in range(i+1)])
    return (np.sin(4*x)/x) * product_term

# Define x range for plotting
x = np.linspace(-50, 50, 1000)

# Compute H_i(x) for i=0, 1, 29, 30
H_0 = np.array([H_i(xi, 0) for xi in x])
H_1 = np.array([H_i(xi, 1) for xi in x])
H_29 = np.array([H_i(xi, 29) for xi in x])
H_30 = np.array([H_i(xi, 30) for xi in x])

# Plot H_0(x)
plt.figure(figsize=(12, 6))
plt.plot(x, H_0, label='$h_0(x)$')
plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.title('$h_0(x)$')
plt.xlabel('$x$')
plt.grid(True)
```

```
plt.legend()
plt.show()

# Plot H_1(x)
plt.figure(figsize=(12, 6))
plt.plot(x, H_1, label='$h_1(x)$', color='orange')
plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.title('$h_1(x)$')
plt.xlabel('$x$')
plt.grid(True)
plt.legend()
plt.show()

# Plot H_29(x)
plt.figure(figsize=(12, 6))
plt.plot(x, H_29, label='$h_{29}(x)$', color='green')
plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.title('$h_{29}(x)$')
plt.xlabel('$x$')
plt.grid(True)
plt.legend()
plt.show()

# Plot H_30(x)
plt.figure(figsize=(12, 6))
plt.plot(x, H_30, label='$h_{30}(x)$', color='red')
plt.axhline(0, color='black', linewidth=0.5)
plt.axvline(0, color='black', linewidth=0.5)
plt.title('$h_{30}(x)$')
plt.xlabel('$x$')
plt.grid(True)
plt.legend()
plt.show()
```

Lastly, the code which plots the Fourier transform of the Nahin functions:

```
import numpy as np
import matplotlib.pyplot as plt
import sympy as sp

# Define the functions
def H_0(w):
    return np.pi/4 * (np.sign(3 - 2*np.pi*w) + np.sign(5 - 2*np.pi*w) + np.sign(2*np.pi*w + 3)
    ## + np.sign(2*np.pi*w + 5))

def H_1(w):
    return np.pi/8 * (np.sign(5/2 - 2*np.pi*w) + np.sign(7/2 - 2*np.pi*w) + np.sign(9/2 - 2*np.pi*w)
    ##+ np.sign(11/2 - 2*np.pi*w) + np.sign(2*np.pi*w + 5/2) + np.sign(2*np.pi*w + 7/2) +
```

```
    ## np.sign(2*np.pi*w + 9/2) + np.sign(2*np.pi*w + 11/2))

def H_2(w):
    return np.pi/16 * (np.sign(13/6 - 2*np.pi*w) + np.sign(17/6 - 2*np.pi*w) + np.sign(19/6 -
    ##2*np.pi*w) + np.sign(23/6 - 2*np.pi*w) + np.sign(25/6 - 2*np.pi*w) + np.sign(29/6 -
    ##2*np.pi*w) + np.sign(31/6 - 2*np.pi*w) + np.sign(35/6 - 2*np.pi*w) + np.sign(2*np.pi*w +
    ##13/6) + np.sign(2*np.pi*w + 17/6) + np.sign(2*np.pi*w + 19/6) + np.sign(2*np.pi*w + 23/6)
    ##+ np.sign(2*np.pi*w + 25/6) + np.sign(2*np.pi*w + 29/6) + np.sign(2*np.pi*w + 31/6) +
    ##np.sign(2*np.pi*w + 35/6))

def H_3(w):
    return np.pi/32 * (np.sign(23/12 - 2*np.pi*w) + np.sign(29/12 - 2*np.pi*w) + np.sign(31/12
    ##- 2*np.pi*w) + np.sign(35/12 - 2*np.pi*w) + np.sign(37/12 - 2*np.pi*w) + np.sign(41/12 -
    ##2*np.pi*w) + np.sign(43/12 - 2*np.pi*w) + np.sign(47/12 - 2*np.pi*w) + np.sign(49/12 -
    ##2*np.pi*w) + np.sign(53/12 - 2*np.pi*w) + np.sign(55/12 - 2*np.pi*w) + np.sign(59/12 -
    ##2*np.pi*w) + np.sign(61/12 - 2*np.pi*w) + np.sign(65/12 - 2*np.pi*w) + np.sign(67/12 -
    ##2*np.pi*w) + np.sign(73/12 - 2*np.pi*w) + np.sign(2*np.pi*w + 23/12) + np.sign(2*np.pi*w
    ##+ 29/12) + np.sign(2*np.pi*w + 31/12) + np.sign(2*np.pi*w + 35/12) + np.sign(2*np.pi*w +
    ##37/12) + np.sign(2*np.pi*w + 41/12) + np.sign(2*np.pi*w + 43/12) + np.sign(2*np.pi*w +
    ##47/12) + np.sign(2*np.pi*w + 49/12) + np.sign(2*np.pi*w + 53/12) + np.sign(2*np.pi*w +
    ##55/12) + np.sign(2*np.pi*w + 59/12) + np.sign(2*np.pi*w + 61/12) + np.sign(2*np.pi*w +
    ##65/12) + np.sign(2*np.pi*w + 67/12) + np.sign(2*np.pi*w + 73/12))

# Defining H_4
def H_4(w):
    coefficients = [
        103/60, 127/60, 133/60, 143/60, 157/60, 163/60, 167/60, 173/60,
        187/60, 193/60, 197/60, 203/60, 217/60, 223/60, 227/60, 233/60,
        247/60, 253/60, 257/60, 263/60, 277/60, 283/60, 287/60, 293/60,
        307/60, 313/60, 317/60, 323/60, 337/60, 347/60, 353/60, 377/60
    ]
    return (np.pi / 64) * sum(np.sign(c - 2 * np.pi * w) + np.sign(2 * np.pi * w + c)
    ## for c in coefficients)

# Define the simpler sign function
def sign_func(x):
    return np.sign(x)

# Define H_5 based on the given formula

def H_5(w):
    I = 1j
    sqrt_pi_over_2 = np.sqrt(np.pi / 2)

    def complex_sign_part(a, b):
        return (-I/2 * (-I * sqrt_pi_over_2 * sign_func(a - 2 * np.pi * w) +
        ##I * sqrt_pi_over_2 * sign_func(b - 2 * np.pi * w)))

    terms = [
```

```
        (complex_sign_part(-129/20, 31/20) + complex_sign_part(-89/20, 71/20)) / 2,
        (complex_sign_part(-109/20, 51/20) + complex_sign_part(-69/20, 91/20)) / 2,
        (complex_sign_part(-347/60, 133/60) + complex_sign_part(-227/60, 253/60)) / 2,
        (complex_sign_part(-287/60, 193/60) + complex_sign_part(-167/60, 313/60)) / 2,
        (complex_sign_part(-119/20, 41/20) + complex_sign_part(-79/20, 81/20)) / 2,
        (complex_sign_part(-99/20, 61/20) + complex_sign_part(-59/20, 101/20)) / 2,
        (complex_sign_part(-317/60, 163/60) + complex_sign_part(-197/60, 283/60)) / 2,
        (complex_sign_part(-257/60, 223/60) + complex_sign_part(-137/60, 343/60)) / 2,
        (complex_sign_part(-121/20, 39/20) + complex_sign_part(-81/20, 79/20)) / 2,
        (complex_sign_part(-101/20, 59/20) + complex_sign_part(-61/20, 99/20)) / 2,
        (complex_sign_part(-323/60, 157/60) + complex_sign_part(-203/60, 277/60)) / 2,
        (complex_sign_part(-263/60, 217/60) + complex_sign_part(-143/60, 337/60)) / 2,
        (complex_sign_part(-111/20, 49/20) + complex_sign_part(-71/20, 89/20)) / 2,
        (complex_sign_part(-91/20, 69/20) + complex_sign_part(-51/20, 109/20)) / 2,
        (complex_sign_part(-293/60, 187/60) + complex_sign_part(-173/60, 307/60)) / 2,
        (complex_sign_part(-233/60, 247/60) + complex_sign_part(-113/60, 367/60)) / 2,
        (complex_sign_part(-367/60, 113/60) + complex_sign_part(-247/60, 233/60)) / 2,
        (complex_sign_part(-307/60, 173/60) + complex_sign_part(-187/60, 293/60)) / 2,
        (complex_sign_part(-109/20, 51/20) + complex_sign_part(-69/20, 91/20)) / 2,
        (complex_sign_part(-89/20, 71/20) + complex_sign_part(-49/20, 111/20)) / 2,
        (complex_sign_part(-337/60, 143/60) + complex_sign_part(-217/60, 263/60)) / 2,
        (complex_sign_part(-277/60, 203/60) + complex_sign_part(-157/60, 323/60)) / 2,
        (complex_sign_part(-99/20, 61/20) + complex_sign_part(-59/20, 101/20)) / 2,
        (complex_sign_part(-79/20, 81/20) + complex_sign_part(-39/20, 121/20)) / 2,
        (complex_sign_part(-343/60, 137/60) + complex_sign_part(-223/60, 257/60)) / 2,
        (complex_sign_part(-283/60, 197/60) + complex_sign_part(-163/60, 317/60)) / 2,
        (complex_sign_part(-101/20, 59/20) + complex_sign_part(-61/20, 99/20)) / 2,
        (complex_sign_part(-81/20, 79/20) + complex_sign_part(-41/20, 119/20)) / 2,
        (complex_sign_part(-313/60, 167/60) + complex_sign_part(-193/60, 287/60)) / 2,
        (complex_sign_part(-253/60, 227/60) + complex_sign_part(-133/60, 347/60)) / 2,
        (complex_sign_part(-91/20, 69/20) + complex_sign_part(-51/20, 109/20)) / 2,
        (complex_sign_part(-71/20, 89/20) + complex_sign_part(-31/20, 129/20)) / 2,
    ]

    result = np.sqrt(np.pi / 2) * sum(terms) / 32
    return result*2

# Generate xi values
xi_values = np.linspace(-2, 2, 1000000)

# Calculate y values for each function
y_values_0 = H_0(xi_values)
y_values_1 = H_1(xi_values)
y_values_2 = H_2(xi_values)
y_values_3 = H_3(xi_values)
y_values_4 = H_4(xi_values)
y_values_5 = H_5(xi_values)

# Plot the functions
```

```
plt.figure(figsize=(10, 6))
plt.plot(xi_values, y_values_0, label='$\\widehat{h_0}$', color='blue')
plt.xlabel('$\\xi$')
plt.title('$\\widehat{h_0}$')
plt.grid(True)
plt.legend()
plt.show()

plt.figure(figsize=(10, 6))
plt.plot(xi_values, y_values_1, label='$\\widehat{h_1}$', color='green')
plt.xlabel('$\\xi$')
plt.title('$\\widehat{h_1}$')
plt.grid(True)
plt.legend()
plt.show()

plt.figure(figsize=(10, 6))
plt.plot(xi_values, y_values_2, label='$\\widehat{h_2}$', color='red')
plt.xlabel('$\\xi$')
plt.title('$\\widehat{h_2}$')
plt.grid(True)
plt.legend()
plt.show()

plt.figure(figsize=(10, 6))
plt.plot(xi_values, y_values_3, label='$\\widehat{h_3}$', color='purple')
plt.xlabel('$\\xi$')
plt.title('$\\widehat{h_3}$')
plt.grid(True)
plt.legend()
plt.show()

plt.figure(figsize=(10, 6))
plt.plot(xi_values, y_values_4, label='$\\widehat{h_4}$', color='orange')
plt.xlabel('$\\xi$')
plt.title('$\\widehat{h_4}$')
plt.grid(True)
plt.legend()
plt.show()

plt.figure(figsize=(10, 6))
plt.plot(xi_values, y_values_5, label='$\\widehat{h_5}$', color='brown')
plt.xlabel('$\\xi$')
plt.title('$\\widehat{h_5}$')
plt.grid(True)
plt.legend()
plt.show()
# Find the indices where y values are equal to pi
indices_0 = np.where(np.isclose(y_values_0, np.pi))[0]
```

```
indices_1 = np.where(np.isclose(y_values_1, np.pi))[0]
indices_2 = np.where(np.isclose(y_values_2, np.pi))[0]
indices_3 = np.where(np.isclose(y_values_3, np.pi))[0]
indices_4 = np.where(np.isclose(y_values_4, np.pi))[0]
indices_5 = np.where(np.isclose(y_values_5, np.pi))[0]

# Print the leftmost and rightmost xi values where the function is equal to pi
if indices_0.size > 0:
    print("For $\\widehat{h_0}$, xi values where the function is equal to pi are:",
    ##xi_values[indices_0[0]], "and",
    ##xi_values[indices_0[-1]], "And the plateau has width:",
    ##xi_values[indices_0[-1]] - xi_values[indices_0[0]])
else:
    print("For $\\widehat{h_0}$, the function is never equal to pi.")
if indices_1.size > 0:
    print("For $\\widehat{h_1}$, xi values where the function is equal to pi are:",
    ##xi_values[indices_1[0]], "and",
    ##xi_values[indices_1[-1]], "And the plateau has width:",
    ##xi_values[indices_1[-1]] - xi_values[indices_1[0]])
else:
    print("For $\\widehat{h_1}$, the function is never equal to pi.")
if indices_2.size > 0:
    print("For $\\widehat{h_2}$, xi values where the function is equal to pi are:",
    ##xi_values[indices_2[0]], "and",
    ##xi_values[indices_2[-1]], "And the plateau has width:",
    ##xi_values[indices_2[-1]] - xi_values[indices_2[0]])
else:
    print("For $\\widehat{h_2}$, the function is never equal to pi.")
if indices_3.size > 0:
    print("For $\\widehat{h_3}$, xi values where the function is equal to pi are:",
    ##xi_values[indices_3[0]], "and",
    ##xi_values[indices_3[-1]], "And the plateau has width:",
    ##xi_values[indices_3[-1]] - xi_values[indices_3[0]])
else:
    print("For $\\widehat{h_3}$, the function is never equal to pi.")
if indices_4.size > 0:
    print("For $\\widehat{h_4}$, xi values where the function is equal to pi are:",
    ##xi_values[indices_4[0]], "and",
    ##xi_values[indices_4[-1]], "And the plateau has width:",
    ##xi_values[indices_4[-1]] - xi_values[indices_4[0]])
else:
    print("For $\\widehat{h_4}$, the function is never equal to pi.")
if indices_5.size > 0:
    print("For $\\widehat{h_5}$, xi values where the function is equal to pi are:",
    ##xi_values[indices_5[0]], "and", xi_values[indices_5[-1]], "And the plateau has width:",
    ##xi_values[indices_5[-1]] - xi_values[indices_5[0]])
else:
    print("For $\\widehat{h_5}$, the function is never equal to pi.")
```

4.2 Appendix B: Code Used for Creating Contours

Firstly the contour above the x-axis:

```
\documentclass{article}
\usepackage{tikz}
\usetikzlibrary{arrows.meta, decorations.markings}

\begin{document}

\begin{center}
    \begin{tikzpicture}[scale=0.7]
        % Axes
        \draw (-3.5,0) -- (3.5,0);
        \draw (0,-1.5) -- (0,3.5); % Shortened negative y-axis

        % Large semi circle
        \begin{scope}[very thick,blue, decoration={markings, mark=at position 0.5 with
        ##{\arrowreversed{Stealth[length=2mm]}}}]
            \draw[postaction={decorate}] (180:3cm) arc (180:0:3cm);
        \end{scope}

        % Small semi circle
        \begin{scope}[very thick,blue, decoration={markings, mark=at position 0.5 with
        ##{\arrowreversed{Stealth[length=2mm]}}}]
            \draw[postaction={decorate}] (0:0.5cm) arc (0:180:0.5cm);
        \end{scope}

        % Lines between -r and -R, and between r and R
        \draw[very thick, blue, {Stealth[length=2mm]}-] (-0.6,0) -- (-3,0);
        \draw[very thick, blue, -{Stealth[length=2mm]}] (0.6,0) -- (3,0);

        % Markings for r, R, -r, -R
        \draw (.6,-0.15) node[below] {$r$};
        \draw (3,0) node[below right] {$R$};
        \draw (-.6,-0.11) node[below] {$-r$};
        \draw (-3,0) node[below left] {$-R$};

        % Labels for I_1, I_2, I_3, I_4
        \draw (-2,-0.5) node[below] {$I_1$};
        \draw (2,-0.5) node[below] {$I_2$};
        \draw (0,3.5) node[above] {$I_3$};
        \draw (0,0.6) node[above] {$I_4'$};
    \end{tikzpicture}
\end{center}

\end{document}
```

Next the contour below the x-axis:

```
\documentclass{article}
\usepackage{tikz}
\usetikzlibrary{arrows.meta, decorations.markings}

\begin{document}

\begin{center}
    \begin{tikzpicture}[scale=0.7]
        % Axes
        \draw (-3.5,0) -- (1.5,0);
        \draw (0,-3.5) -- (0,1.5); % Further shortened negative y-axis

        % Large semi circle
        \begin{scope}[very thick,blue, decoration={markings, mark=at position 0.5 with
        ##{\arrowreversed{Stealth[length=2mm]}}}]
            \draw[postaction={decorate}] (180:3cm) arc (180:360:3cm); % Changed angle to 360
        \end{scope}

        % Small semi circle (changed angle to 360)
        \begin{scope}[very thick,blue, decoration={markings, mark=at position 0.5 with
        ##{\arrow{Stealth[length=2mm]}}}]
            \draw[postaction={decorate}] (180:0.5cm) arc (180:360:0.5cm);
        \end{scope}

        % Lines between -r and -R, and between r and R
        \draw[very thick, blue, {Stealth[length=2mm]}-] (-0.6,0) -- (-3,0);
        \draw[very thick, blue, -{Stealth[length=2mm]}] (0.6,0) -- (3,0);

        % Markings for r, R, -r, -R
        \draw (0.6,0.65) node[below] {$r$};
        \draw (3,0) node[below right] {$R$};
        \draw (-0.6,0.71) node[below] {$-r$};
        \draw (-3,0) node[below left] {$-R$};

        % Labels for I_1, I_2, I_3, I_4
        \draw (-2,0.35) node[above] {$I_1$};
        \draw (2,0.35) node[above] {$I_2$};
        \draw (0,-3.5) node[below] {$I_3'$};
        \draw (0.6,-0.4) node[below] {$I_4$};
    \end{tikzpicture}
\end{center}

\end{document}
```

References

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