Analysis of the response of a simply supported microbeam subject to an electric actuation

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Abstract

In this research a study of the response of a simply supported microbeam subject to an electric actuation is presented. A perturbation method called the method of multiple scales is explained and used to solve our problem. A model concerning the mid-plane stretching and an electric force with a direct and alternating current component is formulated. The method of multiple scales is used to construct a solution that is valid for a long time after the initial conditions. The effect of the frequency of the alternating current was studied by performing a stability analysis. The results show that for frequencies close to the eigenfrequency of the homogeneous problem, there is no stable equilibrium and resonance occurs. Furthermore, a start was made to study the effect of the damping coefficient. The results show that a smaller damping will always lead to resonance on a very small time scale. The results also indicate larger oscillations and an small increase of the importance of the non-linear terms for smaller damping. All results are validated by comparing them with a numeric solution and show excellent agreement.

Contents

1 Introduction

In the current society, technology becomes smaller and smaller every day. This leads to the need for devices that can serve as sensors or switches at a very small scale: the order of micrometers or nanometers. An example of such a device is a microbeam. One specific type of microbeams is a microbeam which is actuated by electricity. These microbeams are called micro- or nanoelectromechanical systems (MEMS/NEMS). MEMS can convert electrical current in mechanical motion and the other way around due to the fact that MEMS are very sensitive to the external forces we apply. Small changes in the surroundings can be detected and converted to a digital signal, making MEMS very good sensors. MEMS can for instance be used as accellerometers and are used in car airbags, the autopilot of an airplane and the Nintendo Wii [1], [2]. Among the many other applications are pressure sensors, injekt printers and microphones in portable devices [3]. Due to a damping, the oscillations of a microbeam fade out. If this happens, the microbeam will lose its applicability. However, its sensitivity to external forces can for certain frequencies of the current result in resonance, which prevents the fading out of the oscillations. When the electric actuation leads to oscillations larger than the restoring mechanical forces can handle, the microbeam might collapse. This is obviously undesirable. In order to be able to use MEMS, it is therefore important to have an understanding of how they move.

The behaviour of MEMS is often described by fourth-order non-linear partial differential equations. It is therefore usually not possible to solve the equation governing the microbeam exactly. Many research has been done on electrically actuated microbeams. Younis and Nayfeh studied the effect of certain parameters on the nonlinearity of the problem with the use of the method of multiple scales and for instance found that decreasing the damping coefficient increases the effect of th non-linear terms [4]. More recently Younis studied the behaviour of a microbeam subject to a direct voltage and two alternating voltages instead of only one [5]. This research may in the future lead to the application of MEMS as communication devices. Sapmaz e.a. have researched the application of carbon nantobues as nanoelctromechanical systems compared to silicon, which is mostly used as a material for microbeams [6]. As microbeams are just very small beams, a lot of the research by Boertjens and Van Horssen [7], [8], [9] on the resonance of weakly nonlinear beam equations with the method of multiple scales can be applied. Moreover, we can use their work on the validity of a solution constructed with perturbation methods.

In most of the previously mentioned works, a microbeam which was clamped at both sides was considered. In this research we will investigate the motion of a microbeam that is simply supported at both sides. Using supported boundary equations will simplify the solutions and we expect that it will have little effect on the accuracy compared to realistic microbeams. Moreover, in realistic MEMS, there is always a little flexibility so a clamped boundary condition is not perfect either [4]. Additionally, many of these previous works used a mode analysis to describe the frequency response. As this often neglects the internal resonance, we will not use this. Similar to Younis and Nayfeh [4], we will use the method of multiple scales to solve our equations. We will investigate for which frequencies of the electric actuation resonance will occur and how the microbeam behaves close to these frequencies. We will also discuss the role of the damping coefficient as this parameter is often unknown and can greatly influence the behaviour of the microbeam. The results will be verified with the help of numerical solutions.

This research will begin with explaining the method of multiple scales on the basis of the textbook of Holmes [10]. Some information about the method is given and an example is calculated to illustrate the method. Next in section 2.2 the Duffing equation will be treated. This is a single-degree-of-freedom system with a cubic nonlinearity. Investigating this equation will greatly help us to better understand the equation of our microbeam as they are quite similar. After that we will derive the equation of motion of our microbeam in section 3. First we make some assumptions about the beam to make the equation more pleasant in section 3.1. Then we will build up our equation of motion considering mid-plane stretching and a electric actuation by both a direct current and an alternating current in sections 3.2-3.6. Then we will simplify the equation before we solve it, for instance by making it dimensionless in section 3.7. Next, we will solve our equation of motion using the method of multiple scales. We will hereby consider three cases, namely: a forcing frequency away from all resonance frequency in section 4.1, a forcing frequency equal to the resonance frequency in section 4.2 and a forcing frequency close to the resonance frequency in section 4.3. We will compare the behaviour of the microbeam for each case. Lastly, in section 4.4 we will make a start with the case that the damping coefficient is one order smaller and see if in this case there are different frequencies which may lead to resonance.

2 Method of multiple scales

Most differential equations concerning real-life problems cannot be solved exactly. To construct a solution, it is possible to use a computer to (very accurately) approximate the problem numerically. However, the disadvantage of this is that this does not give much insight in the physical meaning of the solution. This insight can be achieved by approximating the solution analytically. Moreover, having both an analytical and a numerical solution helps checking the correctness of the solution. One way to do this is using so called perturbation methods. The crucial step for perturbation methods is to find a parameter in your problem which is very small compared to the other parameters. Perturbation methods can be applied to many physical problems since they are capable of handling non-linear, inhomogeneous and multidimensional problems. Besides giving more physical insight into the solution, perturbation methods can be used to find more efficient numerical algorithms as well. There are different perturbation methods with different applications, which can be found in the textbook of Holmes [10]. For the analysis of our problem, we will make use of a perturbation method called the method of multiple scales. The essence of this method is to introduce different time scales, which are assumed independent of each other. This may lead to the fact that what started as an ordinary differential equation, is transformed into a partial differential equation. However, this will sometimes help solving the equation. In this chapter we will first solve a textbook example with the method of multiple scales. Afterwards, we will investigate a more complicated equation, namely the Duffing-equation, a much studied single-degree-of-freedom system with a cubic nonlinearity, with the help of the method of multiple scales.

2.1 Example

Suppose we have the following problem for the function $\phi(t)$, which appears in the study of Josephson junctions [10]:

$$\begin{cases} \phi'' + \epsilon\left(1+\gamma\cos(\phi)\right)\phi' + \sin(\phi) = \alpha\epsilon, \\ \phi(0) = 0, \\ \phi'(0) = 0, \end{cases} \tag{1}$$

where γ is a positive constant and ϵ is considered to be a very small dimensionless parameter. We will make an approximation of the solution of this problem valid up to a timescale of $\frac{1}{\epsilon}$ using the method of multiple scales. Firstly, we expand $\phi(t) = \epsilon\phi_1(t) + \epsilon^2\phi_2(t) + ...$ Then $\cos(\phi) = 1 - \frac{1}{2}\epsilon^2\phi_1^2 + ...$ and $\sin(\phi) = \epsilon\phi_1 + \epsilon^2\phi_2 + ...$ and equation (1) becomes

$$\begin{cases} \epsilon\phi_1'' + \epsilon^2\phi_2 + ... + \epsilon\left(1+\gamma\left(1-\frac{1}{2}\epsilon^2\phi_2 + ...\right)\right)\left(\epsilon\phi_1' + \epsilon^2\phi_2' + ...\right) + \epsilon\phi_1 + \epsilon^2\phi_2 = \epsilon\alpha, \\ \phi_0(0) + \epsilon\phi_1(0) + ... = 0, \\ \phi_0'(0) + \epsilon\phi_1'(0) + ... = 0. \end{cases} \tag{2}$$

This type of expansion was already widely used before the method of multiple scales appeared. However, in the nineteenth century, Poincaré found that solving the equations of motion of planets with a regular expansion lead to large errors after a few rotations already [10]. To improve this method, we introduce two time-scales $t_1 = t$ and $t_2 = \epsilon t$. Assuming that $\phi(t) = \epsilon\phi_1(t_1,t_2) + \epsilon^2\phi_2(t_1,t_2) + ...$, equation (2) becomes:

$$\begin{cases} \left(\partial_{t_1}^2 + 2\epsilon\partial_{t_1}\partial_{t_2} + \epsilon^2\partial_{t_2}^2\right)\left(\epsilon\phi_1'' + \epsilon^2\phi_2 + ...\right) + \epsilon\left(1 + \gamma\left(1 - \frac{1}{2}\epsilon^2\phi_2 + ...\right)\right)(\partial_{t_1} + \\ \qquad \epsilon\partial_{t_2})\left(\epsilon\phi_1 + \epsilon^2\phi_2 + ...\right) = \epsilon\alpha, \\ \epsilon\phi_1(0,0) + \epsilon^2\phi_2(0,0) + ... = 0, \\ \left(\partial_{t_1} + \epsilon\partial_{t_2}\right)\left(\epsilon\phi_1 + \epsilon^2\phi_2 + ...\right)|_{(0,0)} = 0, \end{cases} \tag{3}$$

where $\partial_{t_i} = \frac{\partial}{\partial t_i}$. Since ϵ is very small, the different timescales do not influence each other for a long time. It is therefore a good approximation to assume that the two timescales are independent. Equation (3) therefore can be split into multiple equations. We would like to construct a solution which holds up to time-scale of $O(\frac{1}{\epsilon})$ by solving ϕ_1. We will equate terms with the same power of epsilon. The equation of order ϵ is:.

$$\begin{cases} \partial_{t_1}^2\phi_1 + \phi_1 = \alpha, \\ \phi_1(0,0) = 0, \\ \partial_{t_1}\phi_1(0,0) = 0. \end{cases} \tag{4}$$

This has as a solution:

$$\phi_1 = a_1(t_2)\sin(t_1) + b_1(t_2)\cos(t_1) + \alpha, \tag{5}$$

with

$$\begin{cases} a_1(0) = 0 \\ b_1(0) = -\alpha. \end{cases} \tag{6}$$

The equation of $O(\epsilon^2)$ is:

$$\begin{cases} \partial_{t_1}^2\phi_2 + 2\partial_{t_1}\partial_{t_2}\phi_1 + (1+\gamma)\,\partial_{t_1}\phi_1 + \phi_2 = 0, \\ \phi_2(0,0) = 0, \\ \partial_{t_2}\phi_1(0,0) + \partial_{t_1}\phi_2(0,0) = 0, \end{cases} \tag{7}$$

Substituting equation (5) in eqatuion (7) gives:

$$\begin{cases} \partial_{t_1}^2\phi_2 + \phi_2 = -2a_1'\cos(t_1) + 2b_1'\sin(t_1) - (1+\gamma)\left(a_1\cos(t_1) - b_1\sin(t_1)\right), \\ \phi_2(0,0) = 0, \\ \partial_{t_1}\phi_2(0,0) = -b_1'(0). \end{cases} \tag{8}$$

The solution to equation (8) is

$$\begin{aligned} \phi_2 = a_2(t_2)\sin(t_1) + b_2(t_2)\cos(t_1) - \frac{1}{2}\left(\left((\gamma+1)\,b_1 + 2b_1'\right)t_1 - (\gamma+1)\,a_1 - 2a_1'\right)\cos(t_1) - \\ \frac{1}{2}\left((\gamma+1)\,a_1 + 2a_1'\right)t_1\sin(t_1). \end{aligned} \tag{9}$$

We can see that after a long time the terms $((\gamma+1)b_1 + 2b_1')t_1\cos(t_1)$ and $\frac{1}{2}((\gamma+1)a_1 + 2a_1')t_1\sin(t_1)$ become very large. These terms are called secular terms [10]. When $\epsilon t \approx 1$, $\epsilon\phi_1$ and $\epsilon^2\phi_2$ are about as large. This means that our assumption that the timescales are independent is not accurate anymore. We can prevent these secular terms because we can still choose our $a_1(t_1)$ and $b_1(t_1)$. This is the big advantage of this method over doing a regular expansion $\phi = \epsilon\phi_1 + \epsilon^2\phi_1 + ...$ without defining t_1 and t_2, where we would not have this freedom. So we have

$$\begin{cases}(1+\gamma)b_1 + 2b_1' = 0,\\ (1+\gamma)a_1 + 2a_1' = 0,\end{cases} \tag{10}$$

where our initial conditions are the initial conditions of the $O(\epsilon)$-equation (6). The solution to this is

$$\begin{cases}b_1(t_2) = -\alpha e^{-\frac{(1+\gamma)t_2}{2}},\\ a_1(t_2) = 0.\end{cases} \tag{11}$$

So our first-term approximation is

$$\phi \approx \alpha\epsilon\left(1 - e^{-\frac{1+\gamma}{2}\epsilon t}\cos(t)\right). \tag{12}$$

In figure 1 we plot equation (12) and a numerical solution of (1) by using Euler forward [11]. We can see that the solutions are completely overlapping until one hundred seconds for $\epsilon = 0.01$. So our solution constructed with the method of multiple scales is indeed a good one for $t < \frac{1}{\epsilon}$. This fact can be proven [9].

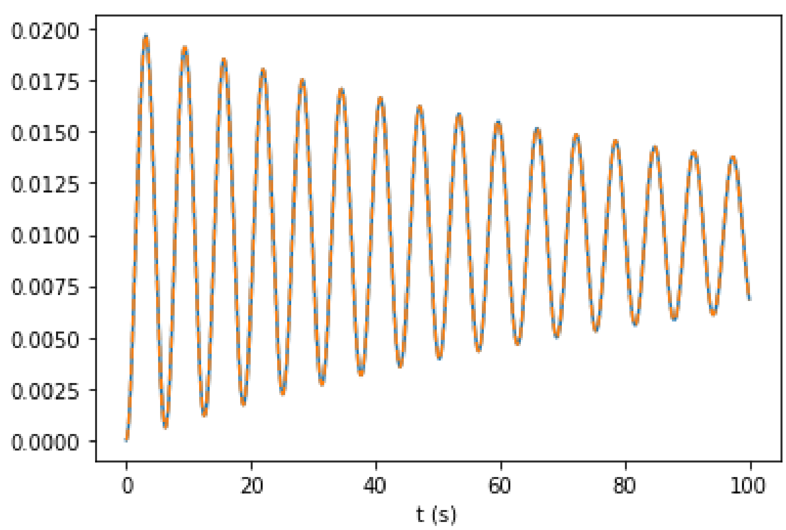


Figure 1: Solution of ϕ for $\epsilon = 0.01$ and $\alpha = \gamma = 1$. The orange dashed line is the solution obtained with the method of multiple scales. The blue solid line is the solution obtained with Euler forward with $\Delta t = 0.01$s.

In our example we used two timescales $t_1 = t$ and $t_2 = \epsilon t$. It is also possible to use more than two timescales such as $t_1 = \frac{t}{\epsilon}$ $t_2 = t$, $t_3 = \epsilon t$ and $t_4 = \epsilon^2 t$ to improve the solution. Another possibility to improve the solution is by making a second-term approximation instead of a first-term and solving ϕ_2 and adding it to the solution. This makes the solution more accurate for $t < \frac{1}{\epsilon}$ [10]. However, these methods do not make sure the solution is more accurate on a longer timescale. In some problems it is useful to define $t_1 = t$ and $t_2 = \epsilon^2 t$ [4]. With the use of matrix and vector notation, the method of multiple scales can be extended to more dimensions [10].

2.2 Duffing oscillator

Another well-known and much studied example in which the method of multiple scales can be applied is the Duffing oscillator. In this subsection we will first derive a general $O(1)$-solution with the method of multiple scales. Then we will consider how this solutions looks for different forcing frequencies. First, a forcing frequency away from the resonance frequency. Then, a forcing frequency equal and close to the resonance frequency. The Duffing oscillator is the most trivial non-linear generalization of the harmonic oscillator used to model damped and driven oscillators. Its general equation is [12]:

$$x'' + \delta x' + \beta x + \alpha x^3 = \gamma \cos(\omega t), \tag{13}$$

with $x(t)$ the displacement, $\delta > 0$ the damping coefficient, γ a forcing parameter and restoring force $-\beta x - \alpha x^3$. It considers the the next term in the expansion of the potential compared to a simple harmonic oscillator ($\alpha = 0$). For $\beta > 0$, the Duffing oscillator is a forced oscillator on a spring. When $\alpha > 0$ this is called a hardening spring and when $\alpha < 0$ this is called a softening spring. When $\beta < 0$ the Duffing oscillator is a point mass in a double well potential. Without the forcing term, the oscillations would slowly fade out due to the damping. Including the forcing term will maintain the oscillations and the system will converge to an equilibrium oscillation that depends strongly on the forcing frequency as we will show in this subsection. Let us consider a weakly forced oscillator. We apply a perturbation method and we take all terms that are not in the undamped and undriven simple harmonic oscillator small. We set $\alpha \to \epsilon\alpha$, $\gamma \to \epsilon\gamma$ and $\delta \to \epsilon\delta$, with ϵ a small parameter and $\beta = \omega_0^2$, with ω_0 the natural frequency of the undamped and undriven simple harmonic oscillator. Then equation (13) becomes:

$$x'' + \omega_0^2 x = \epsilon \left(-\delta x' - \alpha x^3 + \gamma \cos(\omega t)\right). \tag{14}$$

We let $x = x_0 + \epsilon x_1 + ...$ Equation (14) then becomes

$$x_0'' + \epsilon x_1'' + .. + \omega_0^2 \left(x_0 + \epsilon x_1 + ...\right) = \epsilon \left(-\delta \left(x_0' + \epsilon x_1' + ...\right) - \alpha (x_0 + \epsilon x_1 + ..)^3 + \gamma \cos(\omega t)\right). \tag{15}$$

We introduce the timescales $t_1 = t$ and $t_2 = \epsilon t$. Equation (15) then becomes:

$$\begin{aligned}\left(\partial_{t_1}^2 + 2\epsilon\partial_{t_1}\partial_{t_2} + \epsilon^2\partial_{t_2}^2\right)(x_0 + \epsilon x_1 + ...) + \omega_0^2(x_0 + \epsilon x_1 + ...) = \epsilon(-\delta(\partial_{t_1} + \epsilon\partial_{t_2})(x_0 + \\ \epsilon x_1 + ...) - \alpha((x_0 + \epsilon x_1 + ...)^3 + \gamma\cos(\omega t_1)).\end{aligned} \tag{16}$$

We then split the equation in a part of order 1 and a part of order ϵ. The $O(1)$-problem is:

$$\partial_{t_1}^2 x_0 + \omega_0^2 x_0 = 0. \tag{17}$$

This is the differential equation for a undamped and undriven simple harmonic oscillator. The solution of equation (17) is:

$$x_0 = a_0(t_2)\cos(\omega_0 t_1) + b_0(t_2)\sin(\omega_0 t_1). \tag{18}$$

The $O(\epsilon)$-problem is:

$$2\partial_{t_1}\partial_{t_2}x_0 + \partial_{t_1}^2 x_1 + \omega_0^2 x_1 = -\delta\partial_{t_1}x_0 - \alpha x_0^3 - \gamma\cos(\omega t_1). \tag{19}$$

Substituting equation (18) in equation (19) gives:

$$\begin{aligned} \partial_{t_1}^2 x_1 + \omega_0^2 x_1 = -2\left(-a_0'\omega_0\sin(\omega_0 t_1) + b_0'\omega_0\cos(\omega_0 t_1)\right) - \delta(-a_0\omega_0\sin(\omega_0 t_1) + \\ b_0\omega_0\cos(\omega_0 t_1)) - \alpha\left(a_0^3\cos^3(\omega_0 t_1) + 3a_0^2\cos^2(\omega_0 t_1)b_0\sin(\omega_0 t_1) + 3a_0\cos(\omega_0 t_1)b_0^2\sin^2(\omega_0 t_1) + \\ b_0^3\sin^3(\omega_0 t_1)\right) - \gamma\cos(\omega t_1). \end{aligned} \tag{20}$$

Using the trigonometric identities $\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$, $\cos^2(x)\sin(x) = \frac{1}{4}\sin(x) + \frac{1}{4}\sin(3x)$, $\cos(x)\sin^2(x) = \frac{1}{4}\cos(x) - \frac{1}{4}\cos(3x)$ and $\sin^3(x) = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$, equation (20) can be rewritten as

$$\begin{aligned} \partial_{t_1}^2 x_1 + \omega_0^2 x_1 = \left(2a_0'\omega_0 + \delta a_0\omega_0 - \frac{3}{4}\alpha a_0^2 b_0 - \alpha\frac{3}{4}b_0^3\right)\sin(\omega_0 t_1) + (-2b_0'\omega_0 - \delta b_0\omega_0 - \\ \frac{3}{4}\alpha b_0^2 a_0 - \alpha\frac{3}{4}a_0^3\Big)\cos(\omega_0 t_1) + \left(-\frac{1}{4}\alpha a_0^3 + \frac{3}{4}\alpha a_0 b_0^2\right)\cos(3\omega_0 t_1) + \Big(\frac{1}{4}\alpha b_0^3 - \\ \frac{3}{4}\alpha b_0 a_0^2\Big)\sin(3\omega_0 t_1) - \gamma\cos(\omega t_1). \end{aligned} \tag{21}$$

If in the non-homogeneous term on the right, the homogeneous solution $\sin(\omega_0 t)$ or $\cos(\omega t_1)$ occurs, we will obtain secular terms. In order to prevent this, we need that the coefficients before $\cos(\omega_0 t_1)$ and $\sin(\omega_0 t_1)$ are 0. We will research the role of the forcing term by considering different frequencies ω and investigate the behaviour of the solution. First, we will consider ω not close to the eigenfrequency of the homogeneous solution ω_0. Then we will consider ω close or equal to ω_0. In the last case we expect the oscillator to resonate due to the forcing.

To begin with, let $\omega \neq \omega_0$, this means that the forcing frequency is not close the eigenfrequency of the system. We therefore do not expect resonance. The prevention of secular terms implies:

$$\begin{cases} 2a_0'\omega_0 + \delta a_0\omega_0 - \frac{3}{4}\alpha a_0^2 b_0 - \alpha\frac{3}{4}b_0^3 = 0, \\ -2b_0'\omega_0 - \delta b_0\omega_0 - \frac{3}{4}\alpha b_0^2 a_0 - \alpha\frac{3}{4}a_0^3 = 0. \end{cases} \tag{22}$$

To solve this system of equations, we switch to polar coordinates:

$$\begin{cases} a_0(t_2) = r(t_2)\cos(\phi(t_2)), \\ b_0(t_2) = r(t_2)\sin(\phi(t_2)), \end{cases} \tag{23}$$

with $r(t_2)$ a real positive function representing the amplitude and $\phi(t_2)$ a real function representing the phase. Substituting these formulas in equation (22) results in:

$$\begin{cases} 2\left(r'\cos(\phi) - r\sin(\phi)\phi'\right)\omega_0 + \delta r\cos(\phi)\omega_0 - \frac{3}{4}\alpha r^3\cos^2(\phi)\sin(\phi) - \alpha\frac{3}{4}r^3\sin^3(\phi) = 0, \\ -2\left(r'\sin(\phi) + r\cos(\phi)\phi'\right)\omega_0 - \delta r\sin(\phi)\omega_0 - \frac{3}{4}\alpha r^3\sin^2(\phi)\cos(\phi) - \alpha\frac{3}{4}r^3\cos^3(\phi) = 0. \end{cases} \tag{24}$$

Multiplying the first equation with $\cos(\phi)$ and subtracting the second equation multiplied with $\sin(\phi)$ and using trigonometric identities, results in:

$$2\omega_0 r' + \delta r\omega_0 = 0. \tag{25}$$

The general solution to equation (25) is:

$$r(t_2) = c_1 e^{-\frac{\delta}{2}t_2}. \tag{26}$$

Multiplying the first equation of (24) with $\sin(\phi)$ and adding the second equation multiplied with $\cos(\phi)$, results in:

$$-2\omega_0 r\phi' - \alpha\frac{3}{4}r^3 = 0. \tag{27}$$

Which has as a general solution:

$$\phi(t_2) = \frac{3\alpha}{8\omega_0\delta}c_1^2 e^{-\delta t_2} + c_2. \tag{28}$$

Thus the solution to the Duffing-equation for $\omega \neq \omega_0$ is:

$$x_0 = c_1 e^{-\frac{\delta}{2}\epsilon t}\cos\left(\frac{3\alpha}{8\omega_0\delta}c_1^2 e^{-\delta\epsilon t} + c_2\right)\cos(\omega_0 t) + c_1 e^{-\frac{\delta}{2}\epsilon t}\sin\left(\frac{3\alpha}{8\omega_0\delta}c_1^2 e^{-\delta\epsilon t} + c_2\right)\sin(\omega_0 t), \tag{29}$$

where c_1 and c_2 depend on the initial conditions of the Duffing-oscillator. We can see that there are no terms that are multiplied with t so we can indeed conclude that the solution does not become large and that no resonance will occur. The damping is even making the solution converging to zero as can be seen by negative exponent with δt, where $\delta > 0$. This agrees with what we expected.

Next we consider a frequency ω close to ω_0. We would like to investigate for which frequencies the solution becomes very large. We use the following expansion: $\omega = \omega_0 + \epsilon\Omega$, where Ω is a detuning parameter which tells us how close to the resonance frequency we are. Then $\cos(\omega t_1) = \cos(\omega_0 t_1)\cos(\Omega t_2) - \sin(\omega_0 t_1)\sin(\Omega t_2)$ Then in order to prevent secular terms in (21), we need:

$$\begin{cases} 2a_0'\omega_0 + \delta a_0\omega_0 - \frac{3}{4}\alpha a_0^2 b_0 - \alpha\frac{3}{4}b_0^3 + \gamma\sin(\Omega t_2) = 0, \\ -2b_0'\omega_0 - \delta b_0\omega_0 - \frac{3}{4}\alpha b_0^2 a_0 - \alpha\frac{3}{4}a_0^3 - \gamma\cos(\Omega t_2) = 0. \end{cases} \tag{30}$$

We will consider two cases. First we will look at pure resonance, so $\Omega = 0$. Then we will consider $\Omega \neq 0$ and see if we can find values of Ω for which the behaviour of the oscillator changes. For now we set $\Omega = 0$, since this means pure resonance we expect our solution to become very large. For $\Omega = 0$, equation (30) becomes:

$$\begin{cases} 2a_0'\omega_0 + \delta a_0\omega_0 - \frac{3}{4}\alpha a_0^2 b_0 - \alpha\frac{3}{4}b_0^3 = 0, \\ -2b_0'\omega_0 - \delta b_0\omega_0 - \frac{3}{4}\alpha b_0^2 a_0 - \alpha\frac{3}{4}a_0^3 - \gamma = 0. \end{cases} \tag{31}$$

since this equation cannot be solved exactly with the same method as above, we will look when the system is at equilibrium and investigate what the solution looks like in a neighbourhood of the equilibrium. We can then also check if the system has any stable equilibria. To find the equilibrium points, we say $a_0' = 0$ and $b_0' = 0$. Then we have:

$$\begin{cases} \delta a_{0,eq}\omega_0 - \frac{3}{4}\alpha a_{0,eq}^2 b_{0,eq} - \alpha\frac{3}{4}b_{0,eq}^3 = 0, \\ -\delta b_{0,eq}\omega_0 - \frac{3}{4}\alpha b_{0,eq}^2 a_{0,eq} - \alpha\frac{3}{4}a_{0,eq}^3 - \gamma = 0. \end{cases} \tag{32}$$

The only real solution of this is:

$$\begin{cases} a_{0,eq} = -\frac{3\alpha\gamma R}{4(\delta^2\omega^2 + \frac{9}{16}\alpha^2 R^2)}, \\ b_{0,eq} = \frac{-\gamma\delta\omega_0}{\delta^2\omega^2 + \frac{9}{16}\alpha^2 R^2}, \end{cases} \tag{33}$$

with $R = \sqrt[3]{\frac{8\gamma}{9\alpha^2} + \sqrt{\frac{64\gamma^2}{81\alpha^4} + \frac{4096\delta^6\omega_0^6}{19683\alpha^6}}} + \sqrt[3]{\frac{8\gamma}{9\alpha^2} - \sqrt{\frac{64\gamma^2}{81\alpha^4} + \frac{4096\delta^6\omega_0^6}{19683\alpha^6}}}$. To study the stability of $a_{0,eq}$ and $b_{0,eq}$, we use a local linearization around the equilibrium [13]. We set $u = a_0 - a_{0,eq}$ and $v = b_0 - b_{0,eq}$. Then,

$$\begin{cases} u' = a_0' = \frac{1}{2\omega_0}\left[-\delta\omega_0(u + a_{0,eq}) + \frac{3}{4}\alpha(u + a_{0,eq})^2(v + b_{0,eq}) + \frac{3}{4}\alpha(v + b_{0,eq})^3\right], \\ v' = b_0' = \frac{-1}{2\omega_0}\left[\delta\omega_0(v + b_{0,eq}) + \frac{3}{4}\alpha(v + b_{0,eq})^2(u + a_{0,eq}) + \frac{3}{4}\alpha(u + a_{0,eq})^3 + \gamma\right]. \end{cases} \tag{34}$$

Using equation (32) and neglecting terms of order 2 and higher, equation (34) in matrix notation becomes:

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \frac{1}{2\omega_0}\begin{bmatrix} -\delta\omega_0 + \frac{3}{2}\alpha a_{0,eq}b_{0,eq} & \frac{3}{4}\alpha a_{0,eq}^2 + \frac{9}{4}\alpha b_{0,eq}^2 \\ -\frac{3}{4}\alpha b_{0,eq}^2 - \frac{9}{4}\alpha a_{0,eq}^2 & -\delta\omega_0 - \frac{3}{2}\alpha a_{0,eq}b_{0,eq} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{35}$$

The eigenvalues of this matrix are

$$\lambda = -\frac{\delta}{2} \pm i\frac{3|\alpha|}{8\omega_0}\sqrt{3}R. \tag{36}$$

since $\delta > 0$, the real part of the eigenvalues is negative, the equilibrium is therefore stable for all values of the parameters [13]. We apply Euler forward [11] to equation (31) to obtain a numerical solution. This is plotted in figure 2 for $\alpha = \gamma = \delta = \omega_0 = 1$. Equation (33) gives for these values of the parameters $a_0 = -0.429$ and $b_0 = -0.756$, which agree with the values to which the numerical solution converges. From the phase plot it also becomes clear that the equilibrium is indeed stable. We can see that the amplitude, the absolute value of a_0 and b_0, initially grows, but that after a while it decreases again. We can interpret this as first having a rapidly growing solution due to resonance but after a while the damping coefficient will make sure that the deviation does not become too large. The oscillations do not fade out completely as was the case for a forcing frequency not close to the resonance frequency. This all nicely agrees with what we would expect.

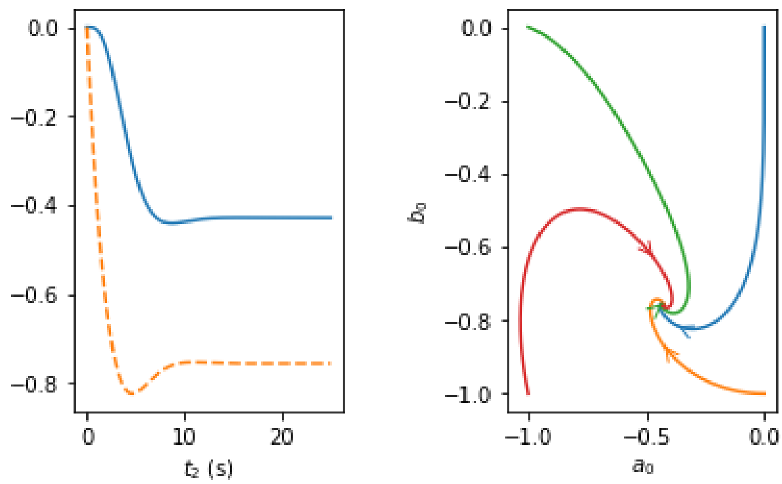


Figure 2: a_0 and b_0 for pure resonance with $\alpha = \gamma = \delta = \omega_0 = 1$ obtained by using Euler forward with $\Delta t_2 = 0.0025$s. On the left a_0 is plotted as a solid blue line and b_0 as a dashed orange line against time with initial conditions $a_0(0) = 0$ and $b_0(0) = 0$. On the right is a phase plot of a_0 and b_0 for different initial conditions.

When $\Omega \neq 0$, we will again perform a stability analysis. We will investigate if we can find any equilibria and for which Ω these equilibria are stable. We use polar coordinates (23) in (30), multiply the first equation with $\cos(\phi)$ and subtracting the second equation multiplied with $\sin(\phi)$ and multiply the first equation with $\sin(\phi)$ and adding the second equation multiplied with $\cos(\phi)$ to obtain:

$$\begin{cases} 2\omega_0 r' + \delta r\omega_0 + \gamma \sin(\Omega t_2)\cos(\phi) + \gamma\cos(\Omega t_2)\sin(\phi) = 0, \\ -2\omega_0 r\phi' - \frac{3}{4}\alpha r^3 + \gamma\sin(\Omega t_2)\sin(\phi) - \gamma\cos(\Omega t_2)\cos(\phi) = 0. \end{cases} \tag{37}$$

This is the same as:

$$\begin{cases} 2\omega_0 r' + \delta r\omega_0 + \gamma\sin(\Omega t_2 + \phi) = 0, \\ -2\omega_0 r\phi' - \frac{3}{4}\alpha r^3 - \gamma\cos(\Omega t_2 + \phi) = 0. \end{cases} \tag{38}$$

We set $\psi = \phi + \Omega t_2$, then equation (38) becomes:

$$\begin{cases} 2\omega_0 r' + \delta r\omega_0 + \gamma\sin(\psi) = 0, \\ 2\omega_0 r(\psi' - \Omega) + \frac{3}{4}\alpha r^3 + \gamma\cos(\psi) = 0. \end{cases} \tag{39}$$

To find the equilibria, set $r' = \psi' = 0$. Then:

$$\begin{cases} \delta r_{eq}\omega_0 + \gamma\sin(\psi_{eq}) = 0, \\ -2\omega_0 r_{eq}\Omega + \frac{3}{4}\alpha r_{eq}^3 + \gamma\cos(\psi_{eq}) = 0. \end{cases} \tag{40}$$

We bring the sine and cosine to the other side, square the equation and add them to obtain a equation for r_{eq}:

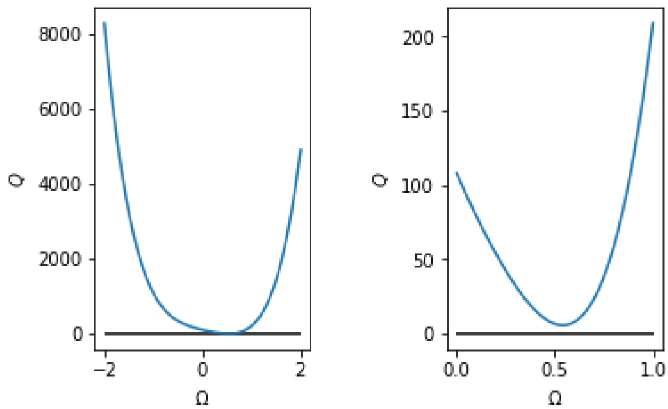


Figure 3: Q against Ω for $\alpha = \gamma = \delta = \omega_0 = 1$ and the line Q = 0. On the left for Ω from -2 to 2 and on the right the part closest to zero enlarged.

$$\frac{9}{16}\alpha^2 r_{eq}^6 - 3\alpha\omega_0\Omega r_{eq}^4 + \omega_0^2\left(\delta^2 + 4\Omega^2\right) r_{eq}^2 - \gamma^2 = 0. \tag{41}$$

If $Q = 4(\frac{16\omega_0^2}{9\alpha^2}(\delta^2 - \frac{3}{4}\Omega^2))^3 + 27(\frac{256\omega_0^3}{81\alpha^3}\Omega(\delta^2 + \frac{4}{9}\Omega^2) - \frac{16\gamma^2}{9\alpha^2})^2 > 0$, there is one real positive solution for r_{eq}, namely:

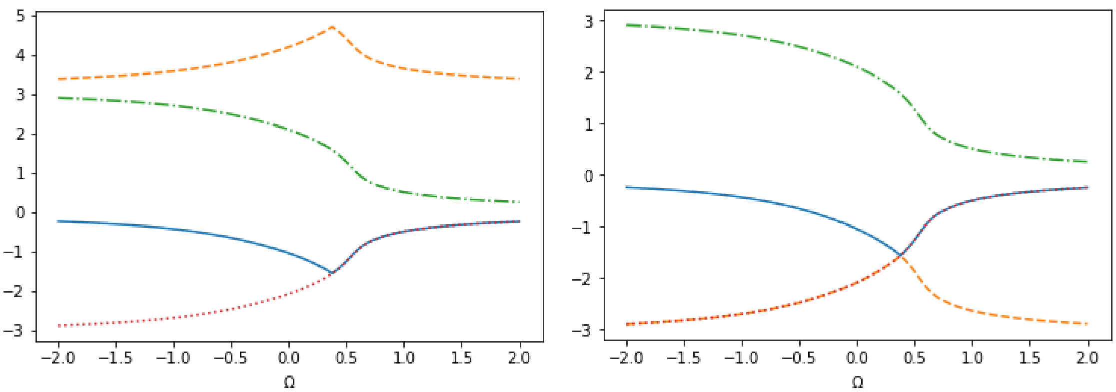
$$r_{eq} = \sqrt{S + \frac{16\omega_0\Omega}{9\alpha}}, \tag{42}$$

with $S = \sqrt[3]{\frac{8\gamma^2}{9\alpha^2} - \frac{128\omega_0^3}{81\alpha^3}\Omega(\delta^2 + \frac{4}{9}\Omega^2) + \sqrt{T}} + \sqrt[3]{\frac{8\gamma^2}{9\alpha^2} - \frac{128\omega_0^3}{81\alpha^3}\Omega(\delta^2 + \frac{4}{9}\Omega^2) - \sqrt{T}}$ and $T = \frac{64\gamma^4}{81\alpha^4} - \frac{2048\gamma^2\omega_0^3}{729\alpha^5}\Omega(\delta^2 + \frac{4}{9}\Omega^2) + \frac{16384\omega_0^6}{6561\alpha^6}\Omega^2(\delta^2 + \frac{4}{9}\Omega^2)^2 + \frac{4096\omega_0^6}{19683\alpha^6}(\delta^2 - \frac{4}{3}\Omega^2)^3$. Q is plotted for $\alpha = \gamma = \delta = \omega_0 = 1$ against Ω in figure 3. We can see that for our values of the parameters we always have one real solution. For a different choice of parameters, Q could be smaller than zero. Then our equation has three real solutions for r_{eq} which can be positive.

The possible values of ψ_{eq} are

$$\begin{cases} \psi_{eq,1} = \arcsin\left(-\frac{\delta\omega_0}{\gamma} r_{eq}\right) + 2k\pi, k \in \mathbb{Z}, \\ \psi_{eq,2} = \pi - \psi_{eq,2} + 2k\pi, k \in \mathbb{Z}, \\ \psi_{eq,3} = \arccos\left(\frac{2\omega_0 r_{eq}\Omega}{\gamma} - \frac{3\alpha r_{eq}^3}{4\gamma}\right) + 2k\pi, k \in \mathbb{Z}, \\ \psi_{eq,4} = -\psi_{eq,3} + 2k\pi, k \in \mathbb{Z}. \end{cases} \tag{43}$$

For ψ_{eq} to be an equilibrium of (40), it needs to satisfy one of $\psi_{eq,1}$ and $\psi_{eq,2}$ and one of $\psi_{eq,3}$ and $\psi_{eq,4}$. The four values are plotted against Ω in figure 4 for the example values or our parameters. Here we can see that $\psi_{eq,4}$ is always and equilibrium. $\psi_{eq,1}$ is an equilibrium for $\Omega \geq 0.375$ and $\psi_{eq,2}$ is an equilibrium for $\Omega < 0.375$. The fact that $\psi_{eq,4}$ is always an equilibrium can be proven and is similar to the proof in the appendix for the case in section 4.3.



(a) (b)

Figure 4: The ψ_{eq} against Ω for $\alpha = 1$, $\gamma = 1$, $\delta = 1$ and $\omega_0 = 1$. The solid blue line is $\psi_{eq,1}$, the dashed orange line is $\psi_{eq,2}$ on the left and $2\pi - \psi_{eq,2}$ on the right, the dash-dot green line is $\psi_{eq,3}$ and the dotted red line is $\psi_{eq,4}$.

As above we linearize around the equilibria to determine the stability. Set $u = r - r_{eq}$ and $v = \psi - \psi_{eq}$, write (39) in matrix notation, use (40) and neglect terms of higher order to obtain:

$$\begin{bmatrix} u' \\ rv' \end{bmatrix} = \frac{-1}{2\omega_0} \begin{bmatrix} \delta\omega_0 & \gamma\cos(\psi_{eq}) \\ \frac{9}{4}\alpha r_{eq}^2 - 2\omega_0\Omega & -\gamma\sin(\psi_{eq}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{44}$$

The eigenvalues of this matrix are:

$$\lambda = -\frac{\delta}{2}(1 + r_{eq}) \pm \frac{1}{2\omega_0}\sqrt{\delta^2\omega_0^2(1 - r_{eq})^2 + (9\gamma\alpha r_{eq}^2 - 2\omega_0\Omega)\cos(\psi_{eq})}. \tag{45}$$

If the term under the square root is negative, the solution is stable since $\delta > 0$ and $r_{eq} > 0$. The only case when we have a $\lambda > 0$ and the solution is unstable, is when:

$$\left(9\gamma\alpha^2 r_{eq}^2 - 2\omega_0\Omega\right)\cos(\psi_{eq}) > 4\omega_0^2\delta^2 r_{eq}. \tag{46}$$

As an example, we set $\gamma = \omega_0 = \alpha = \delta = 1$, and solve the eigenvalues numerically. In figure 5 the largest eigenvalue of the matrix is plotted for different Ω. We see that we always have a stable equilibrium for our choice of parameters.

In figure 6 r and ψ are plotted against time for two different Ω by using the fourth order Runge-Kutta method (RK4 method) [11] on equation (39). For $\Omega = 2$, the solution should converges to $r_{eq} = 0.245$ and $\psi_{eq} = -0.248 + k2\pi$, for $k \in \mathbb{Z}$. This agrees with figure 6 for $k = 1$. For $\Omega = -1$, the solution should converge to $r_{eq} = 0.424$ and $\psi_{eq} = 3.58 + k2\pi$ This agrees with with figure 6 for $k = -1$.

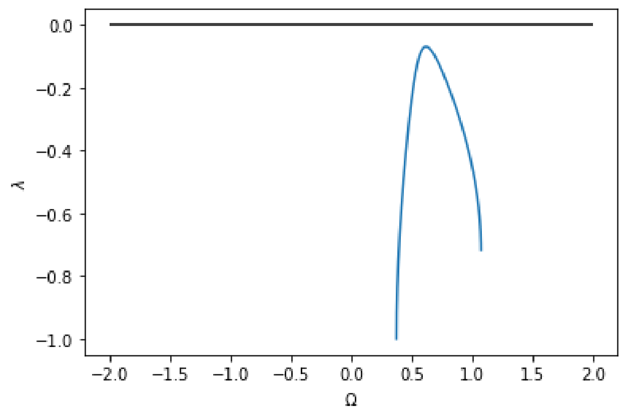


Figure 5: The largest eigenvalue plotted against Ω for $\alpha = \gamma = \delta = \omega_0 = 1$ with a horizontal line at $\lambda = 0$. If Ω becomes small or large, the solution becomes complex.

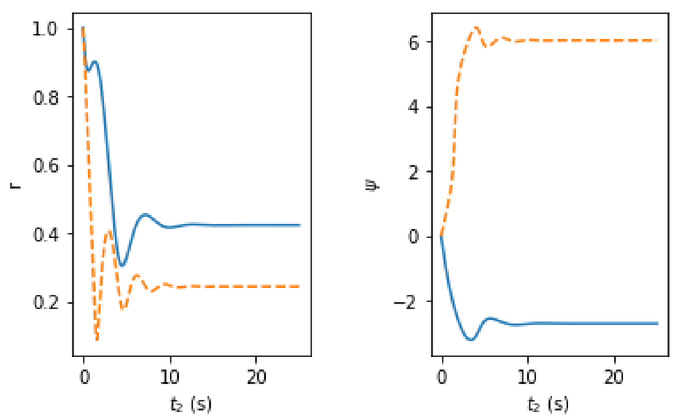


Figure 6: r and ψ against t_2 for $\alpha = \gamma = \delta = \omega_0 = 1$, $r(0) = 1$ and $\psi(0) = 0$ obtained by using RK4 method with $\Delta t_2 = 0.0025$s. The solid blue line is $\Omega = -1$ and the dashed orange line is $\Omega = 2$.

In figure 7 r is plotted against ψ. We can see that depending on the initial condition, the solution converges to the stable equilibria. Based on the figure we would expect extra unstable equilibria, for instance between the blue and purple line in the right figure.

If we go back to our original coordinates r and ϕ, we obtain figure 8. Here we can see that we have no equilibrium point. a_0 and b_0 do not become large since r does converge, but they do keep oscillating since ϕ does not converge. As with the case $\Omega = 0$, we can interpret this as first having a rapidly growing oscillations which is then stopped by the damping but does not fade out completely. We can see that r_{eq} for $\Omega = 0$ is larger than r_{eq} for $\Omega = -1$, which again larger that r_{eq} for $\Omega = 2$. So we can conclude that the closer we are to the resonance frequency the larger the oscillations remain, which is what we would expect.

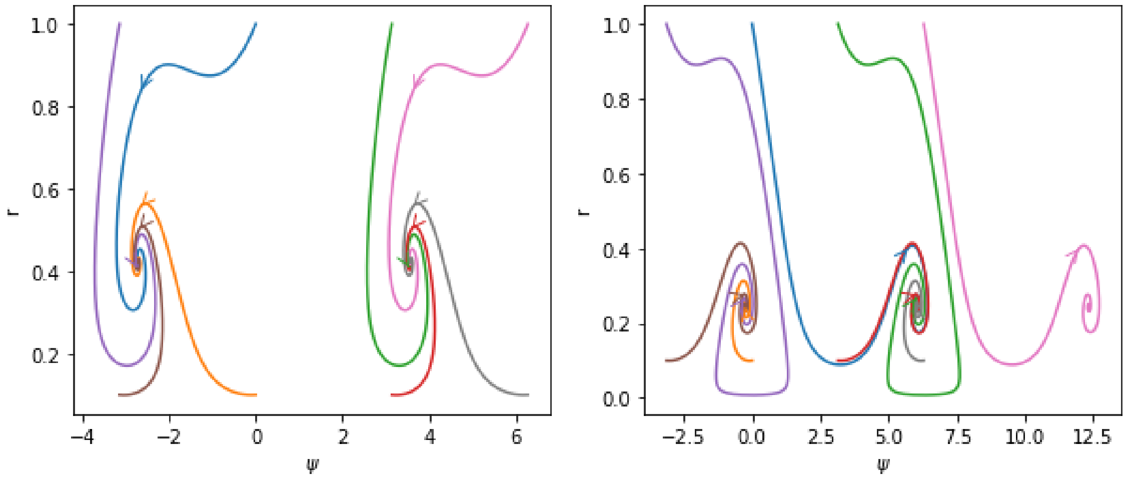


Figure 7: r against ψ for different initial conditions. $\alpha = \gamma = \delta = \omega_0 = 1$. On the left $\Omega = -1$, on the right $\Omega = 2$.

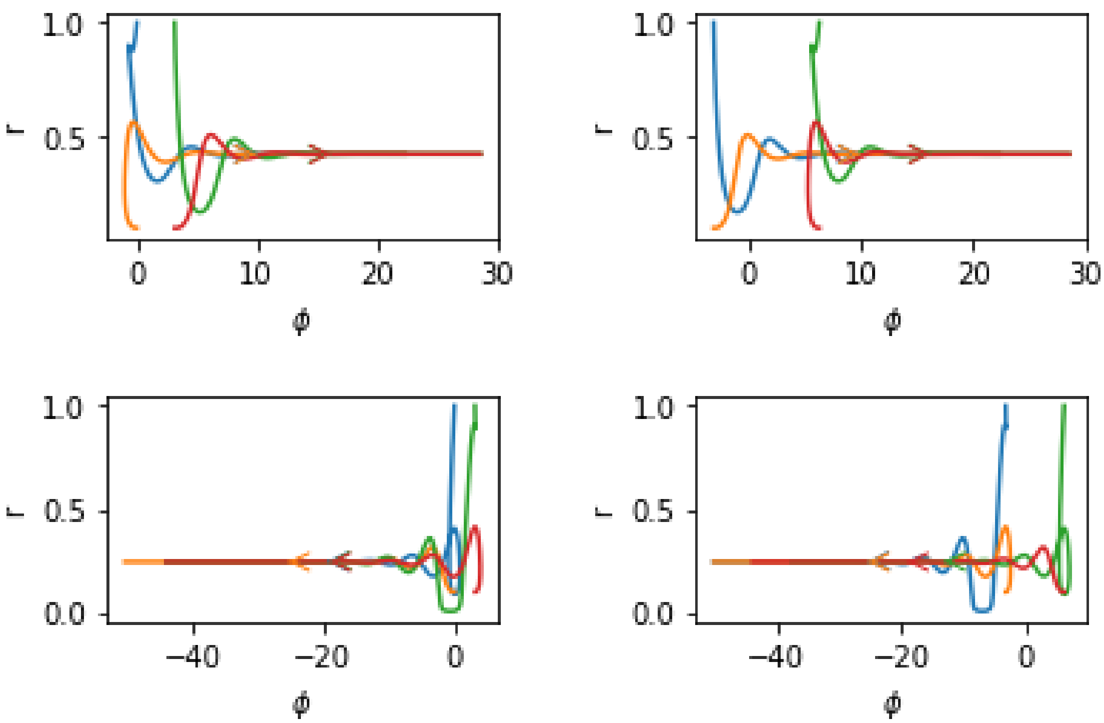


Figure 8: r against ϕ for different initial conditions, with $\alpha = \gamma = \delta = \omega_0 = 1$. For the top row $\Omega = -1$ and for the bottom row $\Omega = 2$.

3 Problem formulation

In this section we will derive the equation of motion of the deflection of a simply supported microbeam as sketched in figure 9. First we will make some assumptions about the beam in section 3.1. Then in sections 3.2-3.6 we will formulate the equation of motion by starting with the equation of motion of a string and then step by step adding dimensions, internal forces and external forces. In section 3.7 we will simplify the obtained equation so that we can solve it in the next chapter.

3.1 Assumptions

Consider a microbeam with a length l in the x-direction, width b in the y-direction and thickness h in the z-direction. It is simply supported at $x = 0$ and $x = l$. A stationary electrode, completely overlapping the area of the microbeam is placed at a distance d in the z-direction. This electrode actuates the beam with a direct current (DC)-component V_p and an alternating current (AC)-component $v(t)$. The beam is subject to a viscous damping per unit length $\hat{c}$. Due to this actuation the beam will deflect. We let $w(x,t)$ denote the transverse deflection of the plate in the negative z-direction. We introduce the following symbols: ρ the density of the beam, E the beams Young's modulus, I its moment of inertia, ν the Poisson ratio, T the tension in the beam and ϵ_r the relative dielectric constant of the medium between the beam and the electrode. For certain frequencies of the AC-component, the beam may resonate. In order to use the microbeam we need to know how it moves.

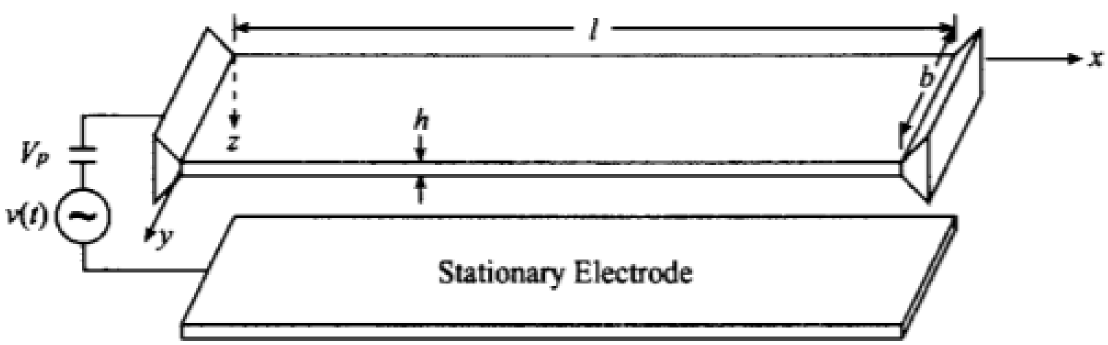


Figure 9: A schematic drawing of a simply supported microbeam [4].

We make a few assumptions about our microbeam. First, we assume that it is uniform along the width. Second, we assume that its length is much bigger than its width. These two assumptions results in the assumption that the deflection of the beam is uniform in the y-direction. Additionally, we assume the width is much bigger than the height, so that we can assume our beam is thin. Furthermore, we assume the height is much bigger than the deflection. This makes sure that the field lines of the electric field are perpendicular to the microbeam. This is a valid assumption since the beam collapses if the deflection becomes as large as the height. Since the microbeam is small, we assume its weight is small, this makes sure that we can neglect gravity. We also neglect the influence of transverse shear deformation and rotary inertia, this is called Euler-Bernoulli theory

[14]. Because the beam is thin, we can neglect the stress in the z-direction. Moreover, we can assume that the middle plane of the plate does not undergo in-plane deformation. We therefore say that we can approximate the movement of the three dimensional plate as a two dimensional plate by looking at the middle of the plate. This extension of Euler-Bernoulli theory, is called the Kirchoff approximation [15]. With these approximations we can assume that the deflection in the x-direction and y-direction is small.

3.2 One-dimensional string

In the following sections we will derive the equation of motion by building it up step by step. We would like to obtain the same equation as Younis and Nayfeh started with for their microbeam [4]. First, we will consider the beam as a one-dimensional string stretched in the x-direction in this section. Second, we will add the y dimension and consider a plate in section 3.3. Third, in section 3.4 we will consider a thin beam and apply Euler-Bernoulli theory. After that we will add the external forces in section 3.5 and lastly in section 3.6 we add the horizontal displacement with the help of the Kirchoff approximation. We begin with considering an infinitesimally thin segment of the string between x and $x + \Delta x$ as in figure 10.

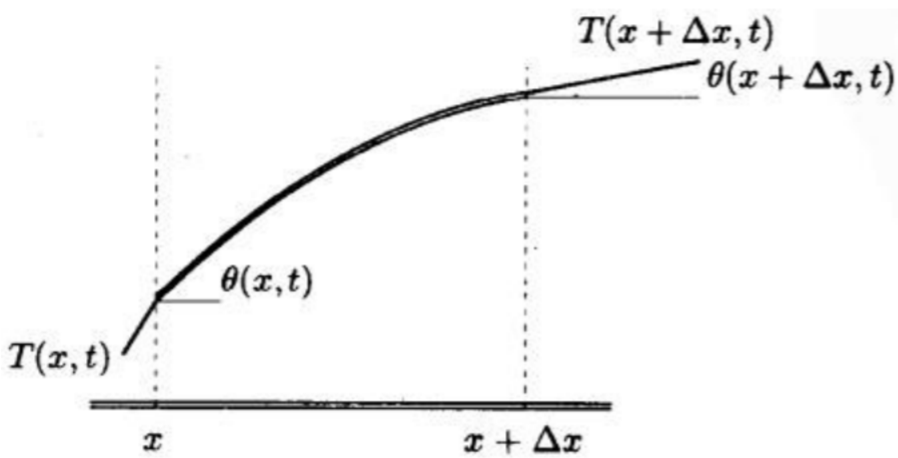


Figure 10: A schematic drawing of the stretching of a small segment of a string [16].

We assume the string has density ρ and denote the cross-section by $A = bh$. We use Newton's second law:

$$\vec{F} = m\frac{\partial^2 \vec{x}}{\partial t^2}. \tag{47}$$

We assume that the string is flexible and offers no resistance to bending. This means that we can say that the only force is the tension at the endpoints in the direction tangent to the string denoted by T. To obtain the different components of the tension, we have to know the angle θ between the x-axis and the string. Letting w denote the vertical displacement, the vertical component of Newton's law becomes:

$$\rho A\Delta x\frac{\partial^2 w}{\partial t^2} = T\sin(\theta(x+\Delta x,t)) - T\sin(\theta(x,t)) + \Delta x f(x,t), \tag{48}$$

where we let $f(x,t)$ denote external force per unit length. since we assumed the deflection is small, the angle is small as well. So we can neglect the horizontal component of Newton's law. Dividing both sides of equation (48) by Δx and letting $\Delta x \to 0$, we get:

$$\rho A\frac{\partial^2 w}{\partial t^2} = T\frac{\partial}{\partial x}\sin(\theta(x,t)) + f(x,t). \tag{49}$$

For small angles

$$\sin(\theta) \approx tan(\theta) = \frac{\partial w}{\partial x}, \tag{50}$$

so equation (49) becomes:

$$\rho A\frac{\partial^2 w}{\partial t^2} = T\frac{\partial^2 w}{\partial x^2} + f(x,t). \tag{51}$$

3.3 Plate

Now we have our equation of motion for a one-dimensional string. In this section we will extend the problem to two dimensions. We add the y-direction and we obtain a plate. Newton's second law still holds, but now becomes:

$$\begin{aligned}\rho h\Delta x\Delta y\frac{\partial^2 w}{\partial t^2} = T\left[\Delta y\frac{\partial w}{\partial x}(x+\Delta x,y) - \Delta y\frac{\partial w}{\partial x}(x,y) + \Delta x\frac{\partial w}{\partial y}(x,y+\Delta y) - \Delta x\frac{\partial w}{\partial y}(x,y)\right] + \\ \Delta x\Delta y f(x,y,t),\end{aligned} \tag{52}$$

where f now denotes the force per unit area. Dividing both sides by $\Delta x\Delta y$, and letting $\Delta x \to 0$ and $\Delta y \to 0$, we get:

$$\rho h\frac{\partial^2 w}{\partial t^2} = T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = T\nabla^2 w + f(x,y,t). \tag{53}$$

This is our equation of motion for a plate. We see that this is very similar to our one dimensional equation.

3.4 Thin beam

In this section we add a third dimension: the thickness. We will now have to include the internal forces we earlier neglected. We consider a plate as in figure 11 with stress components σ_{xx}, $\sigma_{xy} = \sigma_{yx}$, σ_{xz}, σ_{yz} and σ_{yy}. The stress in the direct z-direction σ_{zz} is assumed 0 because the plate was assumed thin.

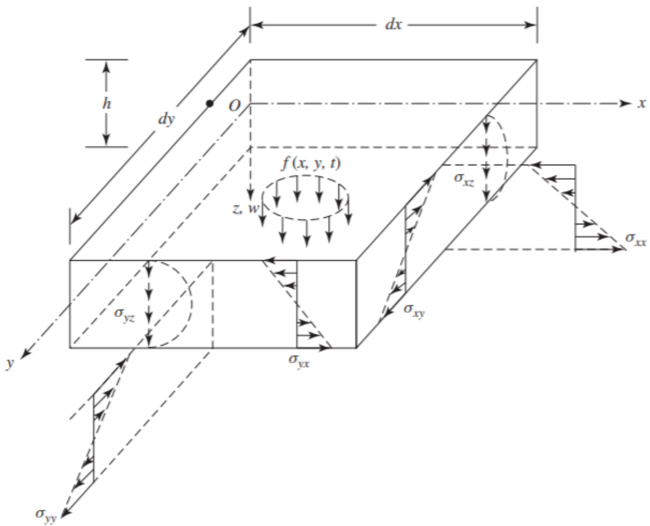


Figure 11: A schematic drawing of a plate with its internal forces [14].

We can define force and moment components per unit length:

$$\begin{cases} M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z dz, \\ M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} z dz, \\ M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xy} z dz = M_{yx}, \\ Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} dz, \\ Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yz} dz. \end{cases} \tag{54}$$

We note that if M_x acts on one side of the plate, $M_x + \Delta M_x = M_x + \frac{\partial M_x}{\partial x}\Delta x$ acts on the other side. Newton's law in the z-direction then becomes:

$$\rho h \Delta x \Delta y \frac{\partial^2 w}{\partial t^2} = \left(Q_x + \frac{\partial Q_x}{\partial x}\Delta x\right)\Delta y + \left(Q_y + \frac{\partial Q_y}{\partial y}\Delta y\right)\Delta x - Q_x \Delta y - Q_y \Delta x + f \Delta x \Delta y, \tag{55}$$

where f is the intensity of the external distributed load. Again dividing by $\Delta x \Delta y$ and letting $\Delta x \to 0$ and $\Delta y \to 0$, we get:

$$\rho h \frac{\partial^2 w}{\partial t^2} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + f(x, y, t). \tag{56}$$

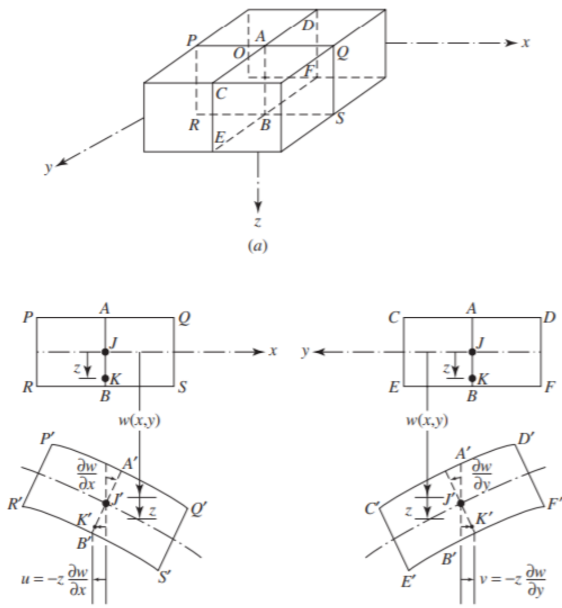


Figure 12: A schematic drawing of the displacement of a small segment of the plate [14].

For the equilibrium around the x-axis, we have:

$$\left(Q_y + \frac{\partial Q_y}{\partial y}\Delta y\right)\Delta x \Delta y = \left(M_y + \frac{\partial M_y}{\partial y}\Delta y\right)\Delta x + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\Delta x\right)\Delta y - M_y \Delta x - M_{xy}\Delta y - f\Delta x \Delta y \frac{\Delta x}{2}. \tag{57}$$

We divide by $\Delta x \Delta y$ and let $\Delta x \to 0$ and $\Delta y \to 0$. We can therefore neglect the terms involving a product of Δx and Δy and obtain:

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}. \tag{58}$$

Similarly:

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y}. \tag{59}$$

Let us consider a small element of our plate such as in figure 12 and look at the point K. Because transverse shear deformation is neglected, we can assume that the lines P'R', A'B' and Q'S' and the lines C'E', A'K' and D'F' remain straight.

If the displacement of K parallel to the x-axis is given by u and parallel to the y-axis by v, then:

$$\begin{cases} u = -z\frac{\partial w}{\partial x}, \\ v = -z\frac{\partial w}{\partial y}. \end{cases} \tag{60}$$

The linear strain-displacements are:

$$\begin{cases} \epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \\ \epsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}, \\ \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}. \end{cases} \tag{61}$$

Assuming the plate is in a state of plane stress, its relation with strain is:

$$\begin{cases} \sigma_{xx} = \frac{E}{1-\nu^2}\epsilon_{xx} + \frac{\nu E}{1-\nu^2}\epsilon_{yy}, \\ \sigma_{yy} = \frac{E}{1-\nu^2}\epsilon_{yy} + \frac{\nu E}{1-\nu^2}\epsilon_{xx}, \\ \sigma_{xy} = G\epsilon_{xy}, \end{cases} \tag{62}$$

With E Young's Modulus, G the shear Modulus and ν the Poisson ratio. If we substitute (61) in (62) and then (62) in (54), we get:

$$\begin{cases} M_x = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \\ M_y = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \\ M_{xy} = M_{yx} = -(1-\nu)\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^2 w}{\partial x \partial y}. \end{cases} \tag{63}$$

Substituting this in (58) and (59) results in:

$$\begin{cases} Q_x = -\frac{Eh^3}{12(1-\nu^2)}\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^w}{\partial y^2}\right), \\ Q_y = -\frac{Eh^3}{12(1-\nu^2)}\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^w}{\partial y^2}\right). \end{cases} \tag{64}$$

Substituting these equation in (56), gives:

$$\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h\frac{\partial^2 w}{\partial t^2} = f(x,y,t). \tag{65}$$

Multiplying both sides with b gives:

$$\frac{EI}{1-\nu^2}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = bf(x,y,t), \tag{66}$$

where $I = \frac{1}{12}bh^3$ is the moment of inertia. Because we assumed that the deflection in the y-direction is small, we can ignore the derivatives to y to obtain:

$$\frac{EI}{1-\nu^2}\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = bf(x,t). \tag{67}$$

3.5 External forces

Now that we have our equation of motion for a thin beam with Euler-Bernoulli theory, we will add the external forces. Because we assumed the mass of the microbeam is small, we can neglect gravity. If we would like to take into account gravity, this can be done by incorporating it in our oscillation $w(x,t)$ by using a substitution [9]. For now the only relevant external force is the electric actuation. To calculate the contribution of the force due to the electrode, we use the energy of a capacitor [17]:

$$W = \frac{1}{2}CV^2, \tag{68}$$

where C is the capacitance given by [17]:

$$C = \frac{A\epsilon_0}{d}, \tag{69}$$

where ϵ_0 is the dielectric constant in vacuum and d is the distance between the plates. In our case the voltage is composed of both a static component V_p and a dynamic component $v(t)$. So the energy of the capacitor is in our case:

$$W = \frac{A\epsilon_0}{2(d-w)}(V_p + v(t))^2. \tag{70}$$

So the force due to the electrode is:

$$\vec{F} = -\nabla W = \frac{A\epsilon_0}{2(d-w)^2}(V_p + v(t))^2 \hat{z}, \tag{71}$$

where an extra minus sign occurs because the direction of w and of z are opposite. If we are not in vacuum but in a medium with relative dielectric constant ϵ_r, this force becomes:

$$\vec{F} = \frac{\epsilon_0 \epsilon_r}{2} A \frac{(V_p + v(t))^2}{(d-w)^2} \hat{z}. \tag{72}$$

Adding this to (67) gives:

$$\frac{EI}{1-\nu^2}\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = b\frac{\epsilon_0 \epsilon_r}{2}\frac{(V_p + v(t))^2}{(d-w)^2}. \tag{73}$$

Adding a damping with damping coefficient per length $\hat{c}$ gives:

$$\frac{EI}{1-\nu^2}\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + \hat{c}\frac{\partial w}{\partial t} = b\frac{\epsilon_0 \epsilon_r}{2}\frac{(V_p + v(t))^2}{(d-w)^2}. \tag{74}$$

3.6 Horizontal displacement

Equation (74) is our equation of motion for our thin beam with its relevant external forces. Finally we add the horizontal displacement of the beam. We will do this by largely following the method described in the book of Kauderer [15]. We look at the (x,z)-plane of our beam. Let P be a point in space with coordinates (x, z) and Q be the point at $(x + \Delta x, z + \Delta z)$. Suppose P moves to P' with $u(x, t)$ the x-component and $w(x, t)$ the z-component of the displacement vector. Then using a Taylor expansion of the displacement vector from Q to Q', we get:

$$\begin{cases} u_Q = u_P + (\frac{\partial u}{\partial x})_P \Delta x + \left(\frac{\partial u}{\partial z}\right)_P \Delta z + h.o.t., \\ w_Q = w_P + \left(\frac{\partial w}{\partial x}\right)_P \Delta x + (\frac{\partial w}{\partial z})_P \Delta z + h.o.t. \end{cases} \tag{75}$$

Then the distance between P' and Q' is:

$$\Delta l'^2 = [(x+\Delta x + u_Q) - (x+u_P)]^2 + [(z+\Delta z + w_Q) - (z+w_P)]^2 = \\ (1+\lambda_{xx})\Delta x^2 + (1+\lambda_{zz})\Delta z^2 + 2\lambda_{zx}\Delta z \Delta x + h.o.t., \tag{76}$$

where:

$$\begin{cases} \lambda_{xx} = 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2, \\ \lambda_{zz} = 2\frac{\partial w}{\partial z} + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2, \\ \lambda_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}. \end{cases} \tag{77}$$

If Q has coordinates $(x+\Delta x, z)$ then

$$\frac{\Delta l'^2}{\Delta l^2} = 1 + \lambda_{xx} + f(\Delta x). \tag{78}$$

We denote the strain in the x-direction as

$$\epsilon_{x0} = \lim_{\Delta x \to 0} \frac{\Delta l'}{\Delta l} - 1 = \sqrt{1+\lambda_{xx}} - 1 \approx \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{79}$$

where we used a first order Taylor-expansion. The total strain now is:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2}. \tag{80}$$

Using Hooke's law:

$$U_{pot} = \frac{1}{2}E\iint_A \epsilon_x^2 dydz = \frac{1}{2}EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + \frac{1}{2}EI\left(\frac{\partial^2 u}{\partial x^2}\right)^2, \tag{81}$$

where $\iint_A z^2 dydz = \frac{bh^3}{12} = I$ and the cross term is zero because we integrate over a symmetrical object, so $\iint_A zdydz = 0$. The total strain work in the beam is then:

$$U = \frac{1}{2}EA\int_0^l \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 dx + \frac{1}{2}EI\int_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx. \tag{82}$$

The kinetic energy of the beam without inertia is:

$$E_k = \frac{1}{2}\rho A\int_0^l \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dx. \tag{83}$$

Using equation (82) and (83) the Hamiltonian becomes:

$$H = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \rho A\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] dxdt. \tag{84}$$

Using the Hamiltonian principle that the variation of H is 0, we obtain:

$$\begin{cases} \rho A \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] = 0, \\ \rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} - EA \frac{\partial}{\partial x} \left[\frac{\partial w}{\partial x} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right] = 0. \end{cases} \tag{85}$$

Using the Kirchhoff approximation, we can say that the x-component of the velocity $\frac{\partial u}{\partial t}$ is small and therefore the acceleration as well, from the first equation of (85) we obtain:

$$\frac{\partial}{\partial x} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] = 0. \tag{86}$$

Looking at equation (79) it follows that ϵ_{x0} cannot be a function of x anymore, but only of t. Thus integrating $\epsilon_{x,0}$ from 0 to l with respect to x gives:

$$\epsilon_{x,0}(t) = \frac{1}{l} \int_0^l \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] dx = \frac{1}{l} \left[u(l,t) - u(0,t) + \frac{1}{2} \int_0^l \left(\frac{\partial w}{\partial x} \right)^2 dx \right]. \tag{87}$$

Substituting this in the second equation of (85) gives:

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = \frac{EA}{l} \left[u(l,t) - u(0,t) + \frac{1}{2} \int_0^l \left(\frac{\partial w}{\partial x} \right)^2 dx \right] \frac{\partial^2 w}{\partial x^2}. \tag{88}$$

The term $\frac{EA}{l}[u(l,t) - u(0,t)]$ is equal to the tension T we had in equation (51) for a string. The total equation of motion now is:

$$\frac{EI}{1-\nu^2} \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + \hat{c} \frac{\partial w}{\partial t} = \left[\frac{EA}{2l(1-\nu^2)} \int_0^l \left(\frac{\partial w}{\partial x} \right)^2 dx + T \right] \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \epsilon_0 \epsilon_r b \frac{(V_p + v(t))^2}{(d-w)^2}. \tag{89}$$

This is is the same equation as in the paper of Younis and Nayfeh [4]. What is different, however, are the boundary conditions. Younis and Nayfeh considerd a clamped beam, but we will consider a simply supported beam. These boundary conditions will make the calculations in the next chapter easier while it will not have much effect on the outcome. Furthermore, having completely clamped boundary conditions is not perfectly realistic either, since the boundary can often move a little bit [4]. We therefore have the following boundary conditions:

$$\begin{cases} w(0,t) = 0, \\ w(l.t) = 0, \\ \frac{\partial^2 w}{\partial x^2}(0,t) = 0, \\ \frac{\partial^2 w}{\partial x^2}(l,t) = 0. \end{cases} \tag{90}$$

3.7 Simplification

In this subsection we will simplify the equation derived in the previous sections by reducing the amount of parameters, making them dimensionless and see which of them are small. This will make it easier to solve our problem in the next section. To start, we make the parameters dimensionless. We let $w^* = \frac{w}{d}$, $x^* = \frac{x}{l}$ and $t^* = \frac{t}{\tau}$, with $\tau = \sqrt{\frac{\rho b h l^4 (1-\nu^2)}{EI}}$. Equation (89) then becomes:

$$\frac{\partial^4 w^*}{\partial x^{*4}} + \frac{\partial^2 w^*}{\partial t^{*2}} + c\frac{\partial w^*}{\partial t^*} = (\alpha_1 \Gamma(w^*, w^*) + N)\frac{\partial^2 w^*}{\partial x^{*2}} + \alpha_2 \frac{(V_p + v(t^*))^2}{(1-w^*)^2}, \tag{91}$$

with $c = \frac{\hat{c}l^4(1-\nu^2)}{EI\tau}$, $\alpha_1 = 6(\frac{d}{h})^2$, $\mathrm{N} = \frac{Tl^2(1-\nu^2)}{EI}$, $\alpha_2 = \frac{6\epsilon_0\epsilon_r l^4(1-\nu^2)}{Eh^3d^3}$ and $\Gamma(f,g) = \int_0^1 \frac{\partial f}{\partial x^*}\frac{\partial g}{\partial x^*}dx^*$. These parameters are dimensionless except for α_1. This is because α_1 is multiplied with the voltage, which will be a important control parameter and is therefore mentioned explicitly. Equation (91) is a nondimensional partial differential equation with both linear and non-linear terms and external excitation terms. The boundary conditions in the dimensionless parameters are:

$$\begin{cases} w^*(0,t) = 0, \\ w^*(1.t) = 0, \\ \frac{\partial^2 w^*}{\partial x^{*2}}(0,t^*) = 0, \\ \frac{\partial^2 w^*}{\partial x^{*2}}(1,t^*) = 0. \end{cases} \tag{92}$$

Typical sizes of our parameters are [4],[5],[6],[17]: $\rho = 10^3\mathrm{kg/m^3}$, $b = 10^{-5}\mathrm{m}$, $h = 10^{-6}\mathrm{m}$, $l = 10^{-4}\mathrm{m}$, $\nu^2 \approx 0$, $E = 10^{12}\mathrm{Pa}$, $d = 10^{-6}\mathrm{m}$, $T = 10^{-2}\mathrm{N}$, $\epsilon_0 = 10^{-11}\mathrm{C^2/Nm^2}$, $\epsilon_r \approx 1$ and $V_p = 10^0\mathrm{V}$. $\hat{c}$ is often unknown, we hope that it is small, so that our beam does not stop oscillating too fast. In section 4.4 we will investigate the role of the size of $\hat{c}$, for now we take $\hat{c} = 10^{-4}\mathrm{kg/ms}$. Then $I = 10^{-24}\mathrm{m}^4$ and $\tau = 10^{-6}\mathrm{s}$. So $c = 10^{-2}$, $\alpha_1 = 10^1$, $N = 10^2$ and $\alpha_2 = 10^{-2}\mathrm{V}^{-2}$. since w is small compared to d, we can say that $w^* \approx 0.01$. This is the same order as $\alpha_2 V_p^2$ and c. We let $w^* = \epsilon w_1 + \epsilon^2 w_2 + ...$ and set $\alpha_2 V_p^2 = \epsilon V_0^2$ and $c = \epsilon c^*$, with V_0 and c^* in the order of 10^0. $v(t)$ is small compared to V_p. We can therefore say $v(t) = \epsilon A \sin(\Omega t^*)$, where A and Ω are the amplitude and frequency of the applied voltage. We make a Taylor-expansion of $\frac{1}{(1-w)^2}$ around $w = 0$:

$$\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + 4w^3 + ... \tag{93}$$

Then we can reformulate (91) as:

$$\begin{gathered} \epsilon\frac{\partial^4 w_1}{\partial x^{*4}} + \epsilon^2\frac{\partial^4 w_2}{\partial x^{*4}} + ... + \epsilon\frac{\partial^2 w_1}{\partial t^{*2}} + \epsilon^2\frac{\partial^2 w_2}{\partial t^{*2}} + ... + \epsilon^2 c^*\frac{\partial w_1}{\partial t^*} + \epsilon^3 c^*\frac{\partial w_2}{\partial t^*} + ... = \\ \alpha_1\left(\epsilon^2\Gamma(w_1,w_1) + 2\epsilon^3\Gamma(w_1,w_2) + ...\right)\left(\epsilon\frac{\partial^2 w_1}{\partial x^{*2}} + \epsilon^2\frac{\partial^2 w_2}{\partial x^{*2}} + ...\right) + \epsilon N\frac{\partial^2 w_1}{\partial x^{*2}} + \\ \epsilon^2 N\frac{\partial^2 w_2}{\partial x^{*2}} + ... + \epsilon(V_0 + \epsilon A\sin(\Omega t^*))^2\left(1 + 2\epsilon w_1 + 2\epsilon^2 w_2 + 3\epsilon^2 w_1^2 + ...\right) \end{gathered} \tag{94}$$

To have a well-formulated problem we need initial conditions. As investigating the influence of the initial conditions is not part of this research, we consider a beam that is initially at rest. So:

$$\begin{cases} \epsilon w_1(x^*,0) + \epsilon^2 w_2(x^*,0) + ... = 0, \\ \epsilon\frac{\partial w_1}{\partial t^*}(x^*,0) + \epsilon^2\frac{\partial w_2}{\partial t^*}(x^*,0) + ... = 0. \end{cases} \tag{95}$$

4 Solution to the problem

In this section we will solve the problem formulated in the previous section with the method of multiple scales. We would like to investigate which values of the frequency of the voltage Ω will lead to resonance. We will therefore consider different cases for different values of Ω. First we will derive a general expression for the $O(\epsilon)$- and $O(\epsilon^2)$-problem and the energy of the beam. We will formulate a solvability condition which we will then solve for different values of Ω in the next sections. In section 4.1 we consider Ω not close to the resonance frequency, in section 4.2 we consider Ω equal to the resonance frequency and in section 4.3 we consider Ω close to the resonance frequency. After this we will make a beginning with the consideration of c one order smaller in 4.4 and see how the solutions changes and if we can find a different resonance frequency compared to the larger c. We start with equations (92), (94) and (95). For convenience, we drop the stars and obtain:

$$\begin{cases} \epsilon\dfrac{\partial^4 w_1}{\partial x^4} + \epsilon^2\dfrac{\partial^4 w_2}{\partial x^4} + \epsilon\dfrac{\partial^2 w_1}{\partial t^2} + \epsilon^2\dfrac{\partial^2 w_2}{\partial t^2} + \epsilon^2 c\dfrac{\partial w_1}{\partial t} + \epsilon^3 c\dfrac{\partial w_2}{\partial t} + ... = \alpha_1\left(\epsilon^2\Gamma(w_1,w_1) + \right. \\ \qquad \left. 2\epsilon^3\Gamma(w_1,w_2) + ...\right)\left(\epsilon\dfrac{\partial^2 w_1}{\partial x^2} + \epsilon^2\dfrac{\partial^2 w_2}{\partial x^2} + ...\right) + \epsilon N\dfrac{\partial^2 w_1}{\partial x^2} + \epsilon^2 N\dfrac{\partial^2 w_2}{\partial x^2} + ... + \\ \qquad \epsilon(V_0 + \epsilon A\sin(\Omega t))^2\left(1 + 2\epsilon w_1 + 2\epsilon^2 w_2 + 3\epsilon^2 w_1^2 + ...\right), \\ \epsilon w_1(0,t) + \epsilon^2 w_2(0,t) + ... = 0, \\ \epsilon w_1(1,t) + \epsilon^2 w_2(0,t) + ... = 0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(0,t) + \epsilon^2\frac{\partial^2 w_2}{\partial x^2}(0,t) + ... = 0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(1,t) + \epsilon^2\frac{\partial^2 w_2}{\partial x^2}(1,t) + ... = 0, \\ \epsilon w_1(x,0) + \epsilon^2 w_2(x,0) + ... = 0, \\ \epsilon\frac{\partial w_1}{\partial t}(x,0) + \epsilon^2\frac{\partial w_2}{\partial t}(x,0) + ... = 0. \end{cases} \tag{96}$$

We introduce the two timescales $t_0 = t$ and $t_1 = \epsilon t$, to obtain:

$$\begin{cases} \epsilon\dfrac{\partial^4 w_1}{\partial x^4} + \epsilon^2\dfrac{\partial^4 w_2}{\partial x^{*4}} + ... + \left(\dfrac{\partial^2}{\partial t_0^2} + 2\epsilon\dfrac{\partial^2}{\partial t_0\partial t_1} + \epsilon^2\dfrac{\partial^2}{\partial t_1^2}\right)(\epsilon w_1 + \epsilon^2 w_2 ...) + \epsilon c\left(\dfrac{\partial}{\partial t_0} + \right. \\ \qquad \left. \epsilon\dfrac{\partial}{\partial t_1}\right)(\epsilon w_1 + \epsilon^2 w_2 + ...) = \alpha_1\left(\epsilon^2\Gamma(w_1,w_1) + 2\epsilon^3\Gamma(w_1,w_2) + ...\right)\left(\epsilon\dfrac{\partial^2 w_1}{\partial x^2} + \right. \\ \left. \epsilon^2\dfrac{\partial^2 w_2}{\partial x^2} + ...\right) + \epsilon N\dfrac{\partial^2 w_1}{\partial x^2} + \epsilon^2 N\dfrac{\partial^2 w_2}{\partial x^2} + ... + \epsilon(V_0 + \epsilon A\sin(\Omega t_0))^2\left(1 + 2\epsilon w_1 + 2\epsilon^2 w_2 + \right. \\ \qquad \left. 3\epsilon^2 w_1^2 + ...\right), \\ \epsilon w_1(0,t_0,t_1) + \epsilon^2 w_2(0,t_0,t_1) + ... = 0, \\ \epsilon w_1(1,t_0,t_1) + \epsilon^2 w_2(1,t_1,t_0) + ... = 0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(0,t_0,t_1) + \epsilon^2\frac{\partial^2 w_2}{\partial x^2}(0,t_0,t_1) + ... = 0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(1,t_0,t_1) + \epsilon^2\frac{\partial^2 w_2}{\partial x^2}(1,t_0,t_1) + ... = 0, \\ \epsilon w_1(x,0,0) + \epsilon^2 w_2(x,0,0) + ... = 0, \\ \left(\frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{t_1}\right)(\epsilon w_1 + \epsilon^2 w_2 + ...)\,|_{(x,0,0)} = 0. \end{cases} \tag{97}$$

We note that we do not have an $O(1)$ problem and that the $O(\epsilon)$ problem does not depend on

t_1. The electric actuation appears from $O(\epsilon^2)$. We will assume that terms with different powers of ϵ are independent and first consider the $O(\epsilon)$-problem:

$$\begin{cases} \frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^2 w_1}{\partial t_0^2} = N\frac{\partial^2 w_1}{\partial x^2} + V_0^2, \\ w_1(0,t_0,t_1) = w_1(1,t_0,t_1) = \frac{\partial^2 w_1}{\partial x^2}(0,t_0,t_1) = \frac{\partial^2 w_1}{\partial x^2}(1,t_0,t_1) = 0, \\ w(x,0,0) = \frac{\partial w_1}{\partial t_0}(x,0,0) = 0. \end{cases} \tag{98}$$

We use separation of variables on the homogeneous problem to obtain:

$$\begin{cases} \frac{d^4\phi(x)}{dx^4} - N\frac{d^2\phi(x)}{dx^2} = \lambda\phi(x), \\ \phi(0) = \phi(1) = \frac{d^2\phi}{dx^2}(0) = \frac{d^2\phi}{dx^2}(0) = 0. \end{cases} \tag{99}$$

The solution to this is:

$$\begin{cases} \phi_n(x) = \sin(n\pi x), \\ \lambda_n = Nn^2\pi^2 + n^4\pi^4. \end{cases} \tag{100}$$

To solve the time-dependent part of w_1 in (98), we use the method of eigenfunction expansion [16] and assume that the solution is of the form:

$$w_1(x,t_0,t_1) = \sum_{n=1}^{\infty} a_n(t_0,t_1)\phi_n(x). \tag{101}$$

Filling in our initial conditions, multiplying both sides with ϕ_m and integrating over x from 0 to 1, we obtain:

$$\begin{cases} a_n(0,0) = 0, \\ \frac{\partial a_n}{\partial t_0}(0,0) = 0. \end{cases} \tag{102}$$

We substitute in (101) in (98) and use (99) to obtain:

$$\sum_{n=1}^{\infty}\left[\lambda_n a_n + \frac{\partial^2 a_n}{\partial t_0^2}\right]\phi_n(x) = V_0^2. \tag{103}$$

Again multiplying both sides with ϕ_m and integrating over x from 0 to 1, results in the following differential equation for $a_n(t_0,t_1)$:

$$\lambda_n a_n + \frac{\partial^2 a_n}{\partial t_0^2} = \frac{\int_0^1 V_0^2\phi_n dx}{\int_0^1 \phi_n^2 dx} = \begin{cases} \frac{4V_0^2}{n\pi}, \text{ n is odd}, \\ 0, \text{ n is even}. \end{cases} \tag{104}$$

The solution to this is:

$$a_n(t_0,t_1) = a_{n,0}(t_1)\sin(\sqrt{\lambda_n}t_0) + b_{n,0}(t_1)\cos(\sqrt{\lambda_n}t_0) + \frac{4V_0^2}{n\pi\lambda_n}\mathbb{1}_{\{\text{n is odd}\}}. \tag{105}$$

Initial conditions (102) give:

$$\begin{cases} a_{n,0}(0) = 0, \\ b_{n,0}(0) = -\frac{4V_0^2}{n\pi\lambda_n}\mathbb{1}_{\{\text{n is odd}\}}. \end{cases} \tag{106}$$

The $O(\epsilon^2)$-problem is:

$$\begin{cases} \frac{\partial^4 w_2}{\partial x^4} + 2\frac{\partial^2 w_1}{\partial t_0 \partial t_1} + \frac{\partial w_2}{\partial t_0^2} + c\frac{\partial w_1}{\partial t_0} = N\frac{\partial w_2}{\partial x^2} + 2V_0^2 w_1 + 2V_0 A\sin(\Omega t_0), \\ w_2(0,t_0,t_1) = w_2(1,t_0,t_1) = \frac{\partial^2 w_2}{\partial x^2}(0,t_1,t_2) = \frac{\partial^2 w_2}{\partial x^2}(1,t_0,t_1) = 0, \\ w_2(x,0,0) = 0, \\ \frac{\partial w_2}{\partial t_0}(0,0) + \frac{\partial w_1}{\partial t_1}(0,0) = 0. \end{cases} \tag{107}$$

We can see that the homogeneous problem for w_2 is the same as for w_1. Therefore we use a similar expansion as before and assume w_2 is of the form:

$$w_2(x,t_0,t_1) = \sum_{n=1}^{\infty} b_n(t_0,t_1)\sin(n\pi x). \tag{108}$$

Filling in equation (101) and (105) in (107), the inhomogeneous term is:

$$\begin{aligned} -2\frac{\partial^2 w_1}{\partial t_0 \partial t_1} - c\frac{\partial w_1}{\partial t_0} + 2V_0^2 w_1 + 2V_0 A\sin(\Omega t_0) = \sum_{n=1}^{\infty}\Big[-2\sqrt{\lambda_n}a'_{n,0}\cos(\sqrt{\lambda_n}t_0) + \\ 2\sqrt{\lambda_n}b'_{n,0}\sin(\sqrt{\lambda}t_0) - c\sqrt{\lambda_n}a_{n,0}\cos(\sqrt{\lambda_n}t_0) + c\sqrt{\lambda_n}b_{n,0}\sin(\sqrt{\lambda_n}t_0) + \\ 2V_0^2 a_{n,0}\sin(\sqrt{\lambda_n}t_0) + 2V_0^2 b_{n,0}\cos(\sqrt{\lambda_n}t_0) + \frac{8V_0^4}{n\pi\lambda_n}\mathbb{1}_{\{\text{n is odd}\}}\Big]\phi_n(x) + 2V_0 A\sin(\Omega t_0) = \\ \sum_{n=1}^{\infty} B_n(t_0,t_1)\phi_n(x) + 2V_0 A\sin(\Omega t_0). \end{aligned} \tag{109}$$

Similar to equation (104) we obtain a differential equation for $b_n(t_1,t_2)$:

$$\lambda_n b_n + \frac{\partial^2 b_n}{\partial t_0^2} = B_n(t_0,t_1) + \frac{8V_0 A\sin(\Omega t_0)}{n\pi}\mathbb{1}_{\{\text{n is odd}\}}. \tag{110}$$

The homogeneous solution to this differential equation is:

$$b_n(t_0,t_1) = a_{n,1}(t_1)\sin(\sqrt{\lambda_n}t_0) + b_{n,1}(t_1)\cos(\sqrt{\lambda_n}t_0). \tag{111}$$

For a good first term approximation, we do not need to solve $b_n(t_0,t_1)$. Rather, we need the condition that it does not lead to secular terms. In order to prevent these secular terms, we need that the inhomogeneous term does not contain the homogeneous solution. In the next subsections we will see what this condition implies for different Ω. To see if resonance occurs, it is interesting to look at the energy of the microbeam as well. If there is no resonance, we expect the energy to stay finite. For resonance we expect that the energy of the microbeam will continue growing. To calculate the energy, we multiply equation (91) with $\frac{\partial w}{\partial t}$ and rewrite it, to obtain:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial w}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{1}{2}N\left(\frac{\partial w}{\partial x}\right)^2\right)+c\left(\frac{\partial w}{\partial t}\right)^2+$$
$$\alpha_1\int_0^1\left(\frac{\partial w}{\partial x}\right)^2dx\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\frac{\partial^3 w}{\partial x^3}-\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^2 w}{\partial x^2}\right)- \tag{112}$$
$$\left(\alpha_1\int_0^1\left(\frac{\partial w}{\partial x}\right)^2dx+N\right)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\frac{\partial w}{\partial x}\right)-\alpha_2(V_p+v(t))^2\frac{\partial}{\partial t}\frac{1}{1-w}=0.$$

To obtain a ordinary derivative with respect to t, we integrate this equation from 0 to 1 over x and use our boundary conditions to obtain:

$$\frac{d}{dt}\left(\int_0^1\frac{1}{2}\left(\frac{\partial w}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{1}{2}N\left(\frac{\partial w}{\partial x}\right)^2-\frac{\alpha_2V_p^2}{1-w}dx+\frac{\alpha_1}{2}\left(\int_0^1\left(\frac{\partial w}{\partial x}\right)^2dx\right)^2\right)+$$
$$c\int_0^1\left(\frac{\partial w}{\partial t}\right)^2dx-\alpha_2\int_0^1\left(2V_pv(t)+v(t)^2\right)\frac{\partial}{\partial t}\frac{1}{1-w}dx=0. \tag{113}$$

Then our energy is:

$$E(t)=C+\int_0^1-\alpha_2\left(2V_pv(t)+v(t)^2\right)\frac{1}{1-w}+\int_0^t c\left(\frac{\partial w}{\partial t}\right)^2+2\alpha_2(V_p+v(t))\frac{\frac{\partial v}{\partial t}}{1-w}dtdx. \tag{114}$$

since our beam is initially at rest we let $C=0$. Moreover, because w is small, we can approximate $\frac{1}{1-w}$ by $1+w$. Using this and equation (101), equation (114) becomes:

$$E(t)=-\alpha_2\left(2V_pv(t)+v(t)^2\right)\left(1+\sum_{\text{n odd}}\frac{2a_n}{n\pi}\right)$$
$$+\int_0^t\frac{c}{2}\sum_{n=1}^{\infty}\left(\frac{da_n}{dt}\right)^2+2\alpha_2(V_p+v(t))\frac{\partial v}{\partial t}\left(1+\sum_{\text{n odd}}\frac{2a_n}{n\pi}\right)dt, \tag{115}$$

where

$$\frac{da_n}{dt}=\epsilon a'_{n,0}(t_1)\sin(\sqrt{\lambda_n}t_0)+\epsilon b'_{n,0}(t_1)\cos(\sqrt{\lambda_n}t_0)+\sqrt{\lambda_n}a_{n,0}(t_1)\cos(\sqrt{\lambda_n}t_0)-$$
$$\sqrt{\lambda_n}b_{n,0}(t_1)\sin(\sqrt{\lambda_n}t_0). \tag{116}$$

4.1 Not resonance frequency

As with the Duffing equation we separate three cases of our forcing frequency. In this section we consider the the case that Ω is not close to any of the eigenfrequencies of the first order solution.

In section 4.2 we consider Ω equal to the eigenfrequency and in section 4.3 we consider Ω close to the eigenfrequency. So we begin with Ω not close to any of the eigenfrequencies of the first order solution. For this case we do not expect any resonance. Due to the static component of the current, we do not expect that the solution fades out completely but converges to an equilibrium deflection. Furthermore we expect the energy of the microbeam to stay small. In this case we can only have secular terms due to $B_n(t_0, t_1)$ and in order to prevent these, we need:

$$\begin{cases} -2\sqrt{\lambda_n}a'_{n,0} - c\sqrt{\lambda_n}a_{n,0} + 2V_0^2 b_{n,0} = 0, \\ 2\sqrt{\lambda_n}b'_{n,0} + cb_{n,0}\sqrt{\lambda_n} + 2V_0^2 a_{n,0} = 0, \end{cases} \tag{117}$$

with initial conditions (106). In matrix notation this is:

$$\begin{bmatrix} a'_{n,0} \\ b'_{n,0} \end{bmatrix} = \begin{bmatrix} -\frac{c}{2} & \frac{V_0^2}{\sqrt{\lambda_n}} \\ -\frac{V_0^2}{\sqrt{\lambda_n}} & -\frac{c}{2} \end{bmatrix} \begin{bmatrix} a_{n,0} \\ b_{n,0} \end{bmatrix}. \tag{118}$$

The eigenvalues of this matrix are:

$$\mu_1 = -\frac{c}{2} + i\frac{V_0^2}{\sqrt{\lambda_n}}, \ \mu_2 = -\frac{c}{2} - i\frac{V_0^2}{\sqrt{\lambda_n}} \tag{119}$$

and the eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \ v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}. \tag{120}$$

So the solution to (117) is

$$\begin{bmatrix} a_{n,0} \\ b_{n,0} \end{bmatrix} = e^{-\frac{c}{2}t_1} \left(c_1 e^{i\frac{V_0^2}{\sqrt{\lambda_n}}t_1} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-i\frac{V_0^2}{\sqrt{\lambda_n}}t_1} \begin{bmatrix} i \\ 1 \end{bmatrix} \right). \tag{121}$$

Taking the real part of (121) and using initial conditions (106) we obtain:

$$\begin{cases} a_{n,0}(t_1) = -\frac{4V_0^2}{n\pi\lambda_n} e^{-\frac{c}{2}t_1} \sin\left(\frac{V_0^2}{\sqrt{\lambda_n}}t_1\right) \mathbb{1}_{\{\text{n is odd}\}}, \\ b_{n,0}(t_1) = -\frac{4V_0^2}{n\pi\lambda_n} e^{-\frac{c}{2}t_1} \cos\left(\frac{V_0^2}{\sqrt{\lambda_n}}t_1\right) \mathbb{1}_{\{\text{n is odd}\}}. \end{cases} \tag{122}$$

In figure 13 we plot the solution of w_1. We limit ourselves to the first 10 components of w_1. Since all terms are divided by λ_n and λ_n grows with n to the fourth power, this is a valid approximation. For large timescales, $a_{n,0}$ and $b_{n,0}$ become close to zero because the oscillations fade away due to the damping in the problem. The solution then converges to the static solution. This agrees with our expectation.

We fill in (122) in (115) to obtain the energy of the microbeam for this case. We use Euler forward [11] to obtain figure 14. We can see that the growth of the energy indeed stops. The energy does not become zero as the bending due to the static current remain. This is again in agreement with what we would expect.

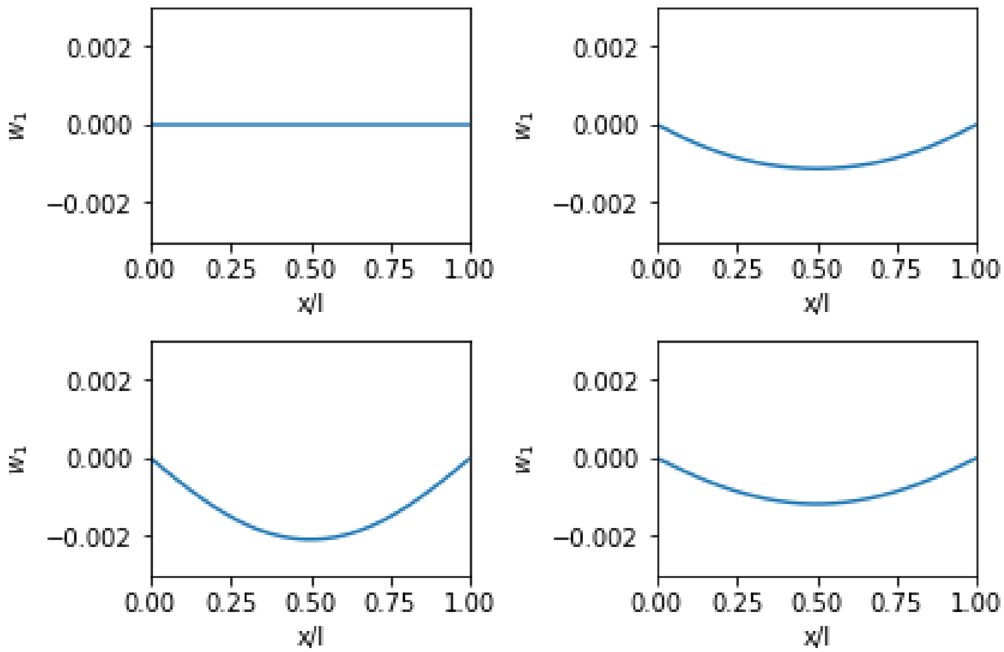


Figure 13: The sum of the first ten components of w_1 against $\frac{x}{l}$ at $t = 0$s, $\frac{250.5\pi}{\sqrt{\lambda_1}}$s, $\frac{501\pi}{\sqrt{\lambda_1}}$s and $\frac{751.5\pi}{\sqrt{\lambda_1}}$s for $c = 1$, $N = 100$, $V_0 = 1$V, $\epsilon = 0.01$ and $A = 1$. Ω is not close to $\sqrt{\lambda_n}$ for any n.

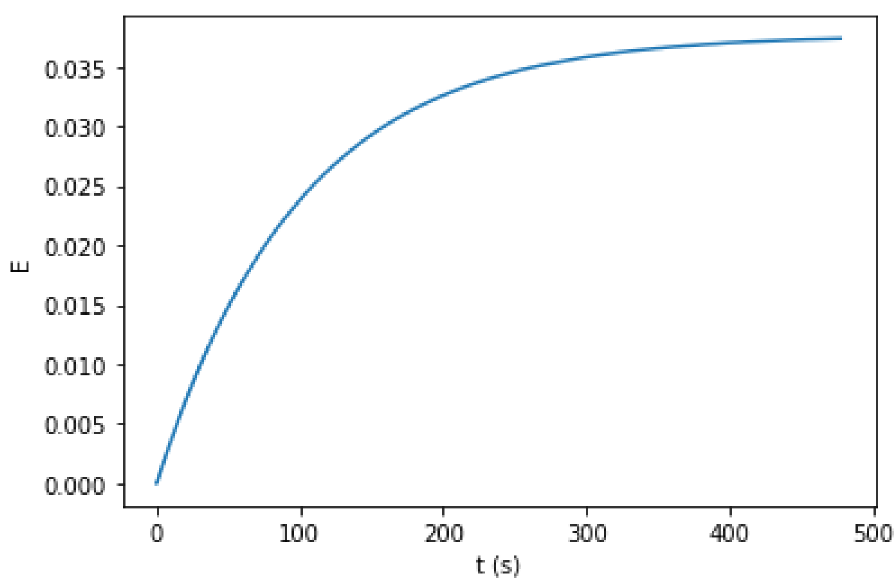


Figure 14: Energy against time if Ω is not close to any of the $\sqrt{\lambda_n}$ for $c = 1$, $N = 100$, $V_0 = 1$V, $\epsilon = 0.01$ and $A = 1$ obtained by using Euler forward with $\Delta t = 0.06$s

4.2 Exactly resonance frequency

In this subsection we will consider Ω equal to one of the eigenfrequencies of the homogeneous solution: we take $\Omega = \sqrt{\lambda_N}$, for N fixed, then the frequency of the alternating current is equal to a eigenfrequency of the solution. Since the even components of the solution are zero, if N is even, nothing changes compared to the solution in the previous subsection. If N is odd, we expect resonance to occur and predict that our oscillations and our energy become very large. We choose $N = 1$. Then in order to prevent secular terms we need to solve the following system for $n = 1$:

$$\begin{cases} -2\sqrt{\lambda_1}a'_{1,0} - c\sqrt{\lambda_1}a_{1,0} + 2V_0^2 b_{1,0} = 0, \\ 2\sqrt{\lambda_1}b'_{1,0} + cb_{1,0}\sqrt{\lambda_1} + 2V_0^2 a_{1,0} + \frac{8AV_0}{\pi} = 0. \end{cases} \tag{123}$$

For $n \neq 1$, we have the same solution as (122). The solution for $n = 1$ is $a_{1,0}(t_1) = a_{1,0,h}(t_1) + a_{1,0,p}(t_1)$ and $b_{1,0}(t_1) = b_{1,0,h}(t_1) + b_{1,0,p}(t_1)$. Where $a_{1,0,h}(t_1)$ and $b_{1,0,h}(t_1)$ are given by $a_{n,0}$ and $b_{n,0}$ in (122). Using the method of variation of parameters [13], $a_{1,0,p}(t_1)$ and $b_{1,0,p}(t_1)$ are found to be:

$$\begin{cases} a_{1,0,p}(t_1) = \frac{\frac{4AV_0}{\pi\sqrt{\lambda_1}}}{1+\frac{4V_0^4}{c^2\lambda_1}} \left[\frac{2}{c}e^{-\frac{c}{2}t_1}\sin\left(\frac{V_0^2}{\sqrt{\lambda_1}}t_1\right) - \frac{4V_0^2}{c^2\sqrt{\lambda_1}} + \frac{4V_0^2}{c^2\sqrt{\lambda_1}}e^{-\frac{c}{2}t_1}\cos\left(\frac{V_0^2}{\sqrt{\lambda_1}}t_1\right)\right], \\ b_{1,0,p}(t_1) = \frac{\frac{4AV_0^2}{\pi\sqrt{\lambda_1}}}{1+\frac{4V_0^4}{c^2\lambda_1}} \left[-\frac{2}{c} - \frac{4V_0^2}{c^2\sqrt{\lambda_1}}e^{-\frac{c}{2}t_1}\sin\left(\frac{V_0^2 t_1}{\sqrt{\lambda_1}}\right) + \frac{2}{c}e^{-\frac{c}{2}t_1}\cos\left(\frac{V_0^2 t_1}{\sqrt{\lambda_1}}\right)\right]. \end{cases} \tag{124}$$

In both $a_{1,0,p}$ and $b_{n,0,p}$, there is a term without a negative exponent with the damping coefficient. So both terms will not vanish at large times. In figure 15 we plot the solution of w_1 for this case. We can see that the solution indeed becomes much larger compared to figure 13, which is what we would expect from resonance.

If we substitute these $a_{n,0}$ and $b_{n,0}$ in (115) and use Euler forward to compute the energy, we obtain figure 16. Here we can see that the energy indeed continues growing with time as we would expect from resonance. If the energy of the beam becomes more than it can handle, the beam will collapse.

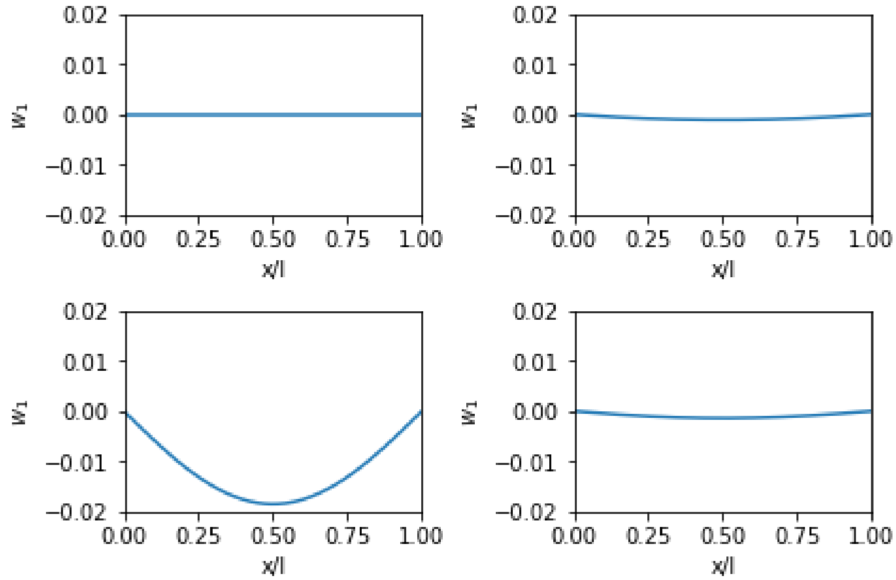


Figure 15: The sum of the first ten components of w_1 against $\frac{x}{l}$ at $t = 0$s, $\frac{250.5\pi}{\sqrt{\lambda_1}}$s, $\frac{501\pi}{\sqrt{\lambda_1}}$ and $\frac{751.5\pi}{\sqrt{\lambda_1}}$s for $c = 1$, $N = 100$, $V_0 = 1$V, $\epsilon = 0.01$, A=1 and $\Omega = \sqrt{\lambda_1}$.

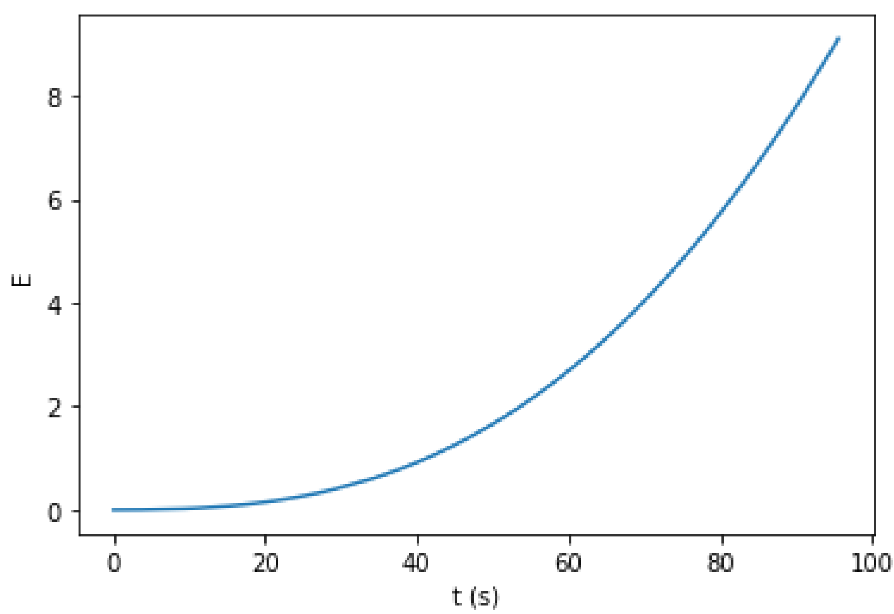


Figure 16: Energy against time if $\Omega = \sqrt{\lambda_1}$, $c = 1$, $N = 100$, $V_0 = 1$V, $\epsilon = 0.01$ and $A = 1$ obtained by using Euler forward with $\Delta t = 0.012$s.

4.3 Close to resonance frequency

Lastly we consider the case that the excitation frequency Ω is close to the resonance frequency of the microbeam. We let $\Omega = \sqrt{\lambda_N} + \epsilon\omega$, with N fixed, where ω is a tuning parameter. The smaller ω is, the closer we are to the resonance frequency. We will not just calculate the solution of the movement of the mircobeam, but rather we will study the behaviour of the beam by looking at equilibira and their stability. We will investigate what happens for different values of ω and if there are ω for which the stability of the equilibria changes. We expect that for smaller ω the oscillations of the beam become larger. For $N \neq n$ we have the same equations as section 4.1. If N is odd, nothing changes compared to section 4.1. Using trigonometric identities, $\sin(\Omega t_0) = \sin(\sqrt{\lambda_N} t_0)\cos(\omega t_1) + \cos(\sqrt{\lambda_N} t_0)\sin(\omega t_1)$ we need the following equation to prevent secular terms for $n = N$:

$$\begin{cases} -2\sqrt{\lambda_N} a'_{N,0} - c\sqrt{\lambda_N} a_{N,0} + 2V_0^2 b_{N,0} + \frac{8V_0 A}{N\pi} \mathbb{1}_{\{\text{N is odd}\}} \sin(\omega t_1) = 0, \\ 2\sqrt{\lambda_N} b'_{n,0} + c b_{n,0}\sqrt{\lambda_N} + 2V_0^2 a_{N,0} + \frac{8V_0 A}{N\pi} \mathbb{1}_{\{\text{N is odd}\}} \cos(\omega t_1) = 0. \end{cases} \tag{125}$$

We introduce polar coordinates $a_{N,0}(t_1) = r_N(t_1)\cos(\phi_N(t_1))$ and $b_{N,0}(t_1) = r_N(t_1)\sin(\phi_N(t_1))$, with $r_N(t_1)$ a real-valued positive function representing the amplitude and $\phi_n(t_1)$ a real-valued function representing the phase. Then (125) transforms into:

$$\begin{cases} -2\sqrt{\lambda_N}(r'_N\cos(\phi_N) - r_N\sin(\phi_N)\phi'_N) - c\sqrt{\lambda_N} r_N\cos(\phi_N) + 2V_0^2 r_N\sin(\phi_N) + \\ \qquad \frac{8V_0 A}{N\pi}\mathbb{1}_{\{\text{N is odd}\}}\sin(\omega t_1) = 0, \\ 2\sqrt{\lambda_N}(r'_N\sin(\phi_N) + r_N\cos(\phi_N)\phi'_N) + c\sqrt{\lambda_N} r_N\sin(\phi_N) + 2V_0^2 r_N\cos(\phi_N) + \\ \qquad \frac{8V_0 A}{N\pi}\mathbb{1}_{\{\text{N is odd}\}}\cos(\omega t_1) = 0. \end{cases} \tag{126}$$

Initial conditions (106) give

$$\begin{cases} r_N(0)\cos(\phi_N(0)) = 0, \\ r_N(0)\sin(\phi_N(0)) = -\frac{4V0^2}{N\pi\lambda_N}\mathbb{1}_{\{\text{N is odd}\}}. \end{cases} \tag{127}$$

If N is even, $r_N(t_1) = 0$ is a solution. So $a_{n,0}(t_1) = 0$ and $b_{n,0}(t_1) = 0$ and we have (122) as a solution, just as we expected. If N is odd, the first equation gives $r_N(0) = 0 \vee \cos(\phi_N(0)) = 0$, but the former would make the first equation 0 as well. So $\cos(\phi_N(0)) = 0$. This means $\phi_N(0) = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. This means $\sin(\phi_N(0)) = 1 \vee \sin(\phi_N(0)) = -1$, but the former would mean that $r_N(0)$ is negative which is impossible. From now on, we assume N is odd. The initial conditions are:

$$\begin{cases} r_N(0) = \frac{4V_0^2}{N\pi\lambda_N}, \\ \phi_N(0) = -\frac{\pi}{2}. \end{cases} \tag{128}$$

To obtain a differential equation for r_N and ϕ_N we use similar calculations as in chapter 2. Multiplying the first equation of (126) with $\sin(\phi_N)$ and adding the second equation multiplied with $\cos(\phi_N)$ and using trigonometric identities gives a differential equation for ϕ_N. Multiplying the first equation with $-\cos(\phi_N)$ and adding the second equation multiplied with $\sin(\phi_N)$ and using trigonometric identities gives a differential equation for r_N.

$$\begin{cases} 2\sqrt{\lambda_n} r_N \phi_N' + 2V_0^2 r_N + \frac{8V_0 A}{N\pi} \cos(\phi_N - \omega t_1) = 0, \\ 2\sqrt{\lambda_N} r_N' + c\sqrt{\lambda_N} r_N + \frac{8V_0 A}{N\pi} \sin(\phi_N - \omega t_1) = 0. \end{cases} \tag{129}$$

We introduce $\psi_N = \phi_N - \omega t_1$, then (129) transforms into:

$$\begin{cases} 2\sqrt{\lambda_N} r_N (\psi_N' + \omega) + 2V_0^2 r_N + \frac{8V_0 A}{N\pi} \cos(\psi_N) = 0, \\ 2\sqrt{\lambda_N} r_N' + c\sqrt{\lambda_N} r_N + \frac{8V_0 A}{N\pi} \sin(\psi_N) = 0. \end{cases} \tag{130}$$

In equilibrium the derivatives with respect to time are zero and eqaution (130) results in:

$$\begin{cases} 2\sqrt{\lambda_N} r_{N,eq} \omega + 2V_0^2 r_{N,eq} + \frac{8V_0 A}{N\pi} \cos(\psi_{N,eq}) = 0, \\ c\sqrt{\lambda_N} r_{N,eq} + \frac{8V_0 A}{N\pi} \sin(\psi_{N,eq}) = 0. \end{cases} \tag{131}$$

Bringing the sine and cosine to the other side, squaring the equations and adding them gives:

$$\left(4\lambda_N \omega^2 + 8V_0^2 \sqrt{\lambda_N} \omega + 4V_0^4 + c^2 \lambda_N\right) r_{N,eq}^2 = \frac{64 V_0^2 A^2}{N^2 \pi^2}. \tag{132}$$

This has a solution if:

$$4\lambda_N \omega^2 + 8V_0^2 \sqrt{\lambda_N} \omega + 4V_0^4 + c^2 \lambda_N > 0. \tag{133}$$

This holds for all $\omega \in \mathbb{R}$. The solution of (132) is:

$$r_{N,eq} = \frac{8V_0 A}{N\pi \sqrt{4\lambda_N \omega^2 + 8V_0^2 \sqrt{\lambda_N} \omega + 4V_0^4 + c^2 \lambda_N}}. \tag{134}$$

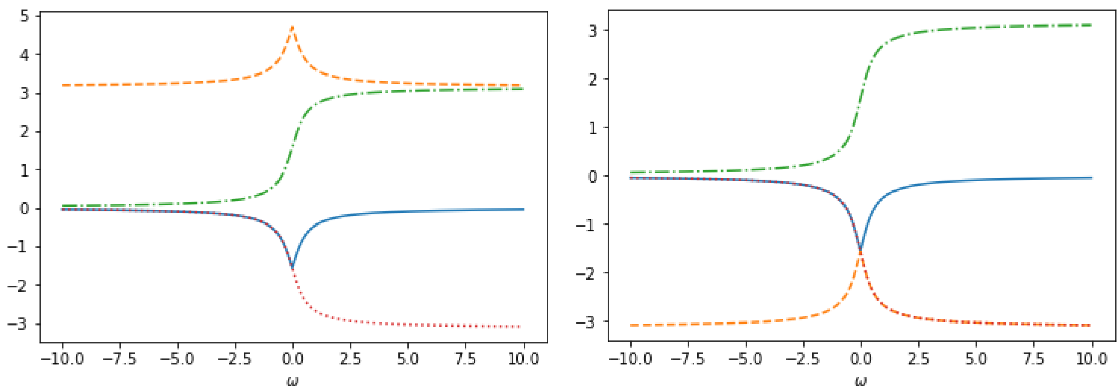
The possible values for $\psi_{N,eq}$ are

$$\begin{cases} \psi_{N,eq,1} = \arcsin\left(-\frac{cN\pi\sqrt{\lambda_N} r_{N,eq}}{8V_0 A}\right) + 2k\pi, k \in \mathbb{Z}, \\ \psi_{N,eq,2} = \pi - \psi_{N,eq,1} + 2k\pi, k \in \mathbb{Z}, \\ \psi_{N,eq,3} = \arccos\left(\frac{-V_0 N\pi r_{eq}}{4A} - \frac{\sqrt{\lambda_N}\omega N\pi r_{eq}}{4V_0 A}\right) + 2k\pi, k \in \mathbb{Z}, \\ \psi_{N,eq,4} = -\psi_{N,eq,3} + 2k\pi, k \in \mathbb{Z}. \end{cases} \tag{135}$$

For $\psi_{N,eq}$ to be an equilibrium of (131), it needs to satisfy one of $\psi_{N,eq,1}$ and $\psi_{N,eq,2}$ and one of $\psi_{N,eq,3}$ and $\psi_{N,eq,4}$. The four values are plotted against ω in figure 17 for our example values. Here we can see that $\psi_{1,eq,4}$ is always an equilibrium. $\psi_{N,eq,1}$ is an equilibrium for $\omega \leq -0.07$ and $\psi_{N,eq,2}$ is an equilibrium for $\omega > -0.07$. The fact that $\psi_{N,eq,4}$ is always an equilibrium is proven in the appendix.

To determine the stability we will linearize locally around the equilibrium [13]. We introduce $u_N = r_N - r_{N,eq}$ and $v_N = \psi_N - \psi_{N,eq}$, write (130) in matrix notation and use (131) similarly to section 2, to obtain:

$$\begin{bmatrix} u_N' \\ r_N v_N' \end{bmatrix} = \begin{bmatrix} -\frac{c}{2} & -\frac{4V_0 A}{N\pi\sqrt{\lambda_N}} \cos(\psi_{N,eq}) \\ -\omega - \frac{V_0^2}{\sqrt{\lambda_N}} & \frac{4V_0 A}{N\pi\sqrt{\lambda_N}} \sin(\psi_{N,eq}) \end{bmatrix} \begin{bmatrix} u_N \\ v_N \end{bmatrix} + h.o.t. \tag{136}$$



(a) (b)

Figure 17: The $\psi_{1,eq}$ against ω for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1$V. The solid blue line is $\psi_{1,eq,1}$, the dashed orange line is $\psi_{1,eq,2}$ on the left and $2\pi - \psi_{1,eq,2}$ on the right, the dash-dot green line is $\psi_{1,eq,3}$ and the dotted red line is $\psi_{1,eq,4}$.

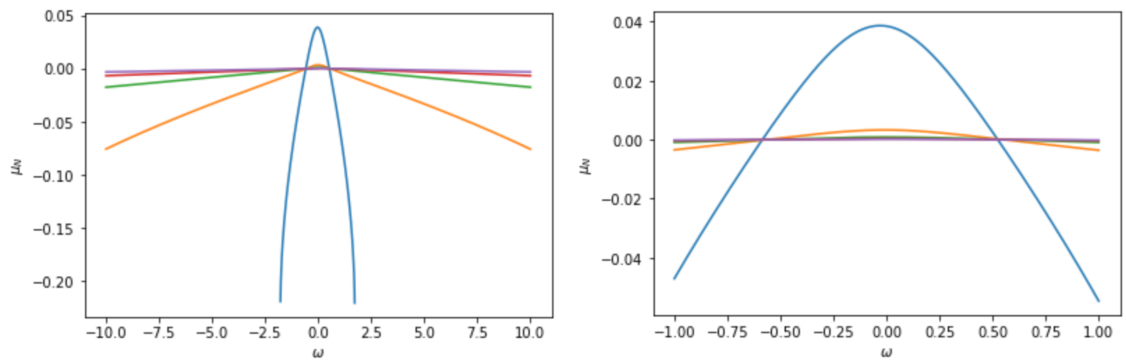
N	ω_1	ω_2
1	-0.585	0.524
3	-0.566	0.550
5	-0.562	0.555
≥ 7	-0.560	0.557

Table 1: The bifurcation points of the two eigenvalues for different N.

The eigenvalues of this matrix are:

$$\mu_N = \frac{2V_0A}{N\pi\sqrt{\lambda_N}} - \frac{c}{4} \pm \frac{1}{2}\sqrt{\left(\frac{c}{2} - \frac{4V_0A}{N\pi\sqrt{\lambda_N}}\sin(\psi_{N,eq})\right)^2 + \left(\frac{16\omega V_0 A}{N\pi\sqrt{\lambda_N}} + \frac{16V_0^3A}{N\pi\lambda_N}\right)\cos(\psi_{N,eq})}. \quad (137)$$

Considering the sizes of the parameters, the term $\frac{4V_0A}{N\pi\sqrt{\lambda_N}} - \frac{c}{2}$ is smaller than 0. So when the eigenvalue is complex, the equilibrium is stable. We solve this numerically to determine for which ω the solution changes. In figure 18 the eigenvalues are plotted against ω. The solution is unstable for $\omega_1 \leq \omega \leq \omega_2$, where ω_1 and ω_2 are given by table 1. ω_1 and ω_2 are called bifurcation points [13]. In figure 19 we plot how the position and stability of the equilibrium changes with ω.



(a) (b)

Figure 18: The first 5 odd largest eigenvalues of the matrix for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1$V. In subfigure (a) for ω from -10 to 10 and in subfigure (b) for ω from -1 to 1.

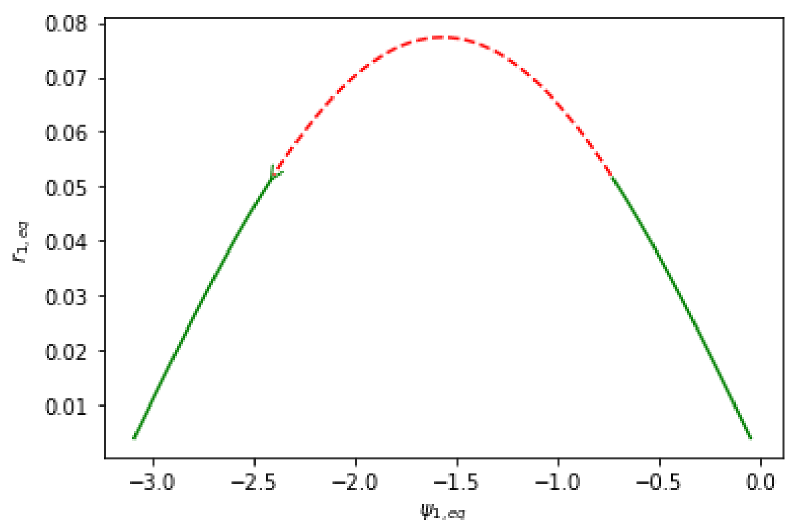


Figure 19: $r_{1,eq}$ against $\psi_{1,eq}$ for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1$V for ω from -10 to 10. The dashed red line means the equilibrium is unstable, the solid green line means the equilibrium is stable.

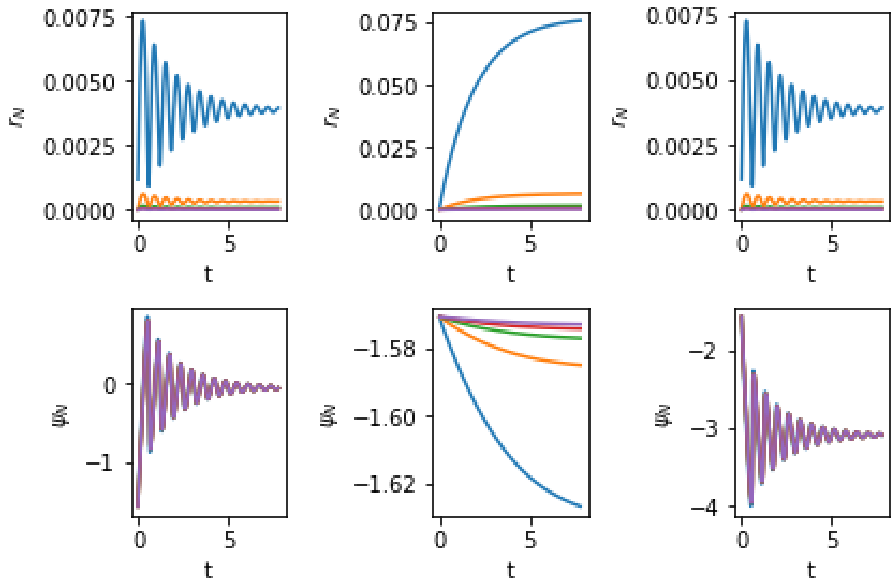


Figure 20: The first 5 odd r_n and ψ_n against time for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1$V. Left for $\omega = -10$, in the middle for $\omega = 0$ and right for $\omega = 10$. The values are determined using the RK4 method with $\Delta t = 0.012$s.

We will compare our obtained results to numeric results. We can obtain the solutions for r_N and ψ_N by applying the fourth order method of Runge-Kutta (RK4-method) [11] to equation (130). We then obtain figure 20. We compare them to the equilibria from (134) and (135). For $\omega = -10$ and $N = 1$, the equilibrium should be equal to $(0.0039, -0.050 + k2\pi, k \in \mathbb{Z})$. This is in excellent agreement with the left plot of figure 20 for $k = 0$. For $\omega = 10$ and $N = 1$, the equilibrium should be equal to $(0.0039, 3.19 + k2\pi, k \in \mathbb{Z})$). This is in excellent agreement with the right plot of figure 20 for $k = -1$. For $\omega = 0$ the equilibrium should be unstable. In figure 21 we plot the phase plots of r and ψ and the phase plots of our original coordinates r and ϕ. We can see that the oscillations converge to an oscillation with constant amplitude due to a balance between the electric actuation and the damping. In figure 22 we plot how our solution w_1 looks like for $N = 1$. We can see that if $\omega = 0$, we have the same solution as in subsection 4.2, which is what we would expect. We can also observe that if ω is smaller and we are closer to the resonance frequency, that then w_1 becomes larger. This is in agreement with what we expected.

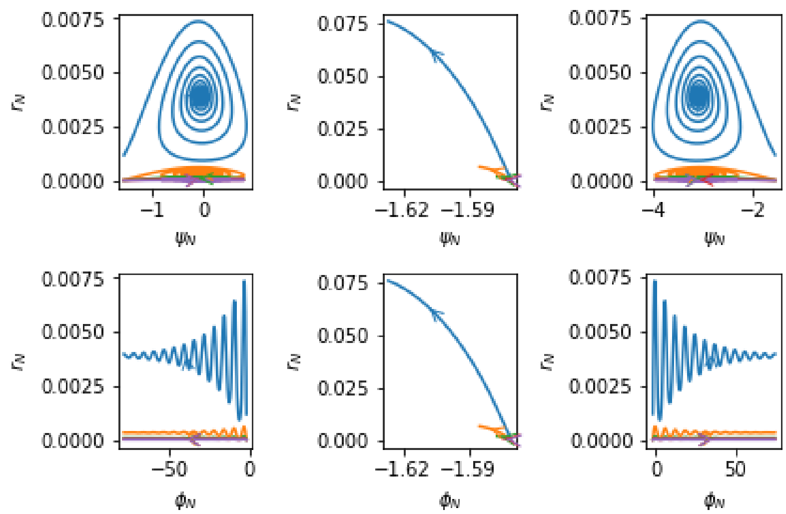


Figure 21: The first 5 odd r_n against ψ_n and r_n against ϕ_n for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1$V. Left for $\omega = -10$, in the middle for $\omega = 0$ and right for $\omega = 10$. The values are determined using the RK4-method with $\Delta t = 0.012$s.

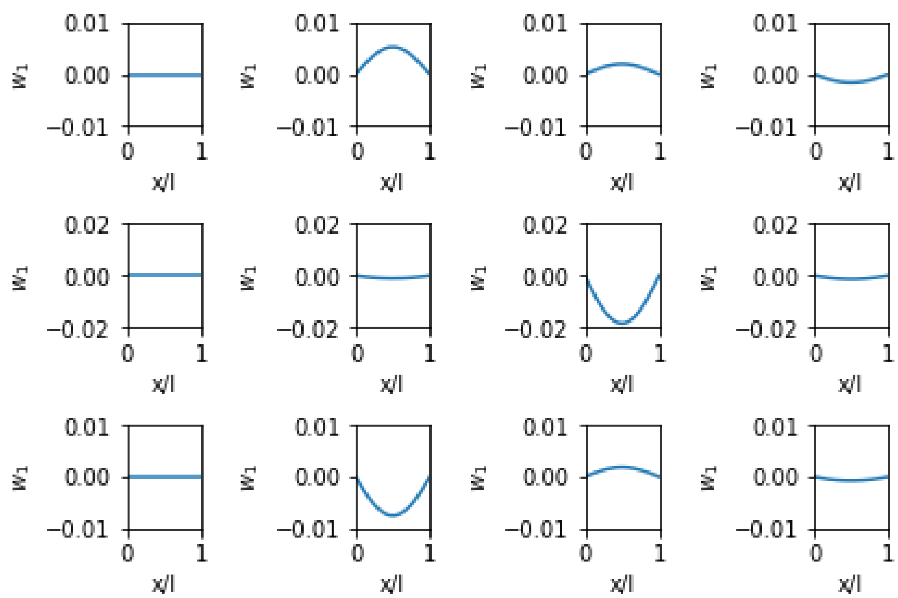


Figure 22: The first ten components of w_1 against $\frac{x}{l}$ for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1$V at $t = 0$s, , $\frac{250.5\pi}{\sqrt{\lambda_1}}$s, $\frac{501\pi}{\sqrt{\lambda_1}}$s and $\frac{751.5\pi}{\sqrt{\lambda_1}}$s. Above for $\omega = -10$, in the middle for $\omega = 0$ and below for $\omega = 10$. The values are determined using the RK4-method with $\Delta t = 0.012$s.

4.4 Smaller damping

In this section we will make a first step towards researching the effect of the damping coefficient by studying the same problem formulated in section 3 by equation (91), but now with the damping an order smaller. Since the damping is unknown, but we hope that it is small so that our oscillations do not fade away too fast, it is interesting to consider this case. To study the effect of the damping we will have to consider the $O(\epsilon^3)$-problem as well. We assume that Ω is not close to any $\sqrt{\lambda_n}$ and see if we can find a different resonance. In contrast to sections 4.1-4.3, we will not separate three types of frequency but only a frequency close to our new resonance frequency. Our new equation of motion is:

$$\begin{cases} \epsilon\frac{\partial^4 w_1}{\partial x^4}+\epsilon^2\frac{\partial^4 w_2}{\partial x^4}+\epsilon^3\frac{\partial^4 w_3}{\partial x^4}+\epsilon\frac{\partial^2 w_1}{\partial t^2}+\epsilon^2\frac{\partial^2 w_2}{\partial t^2}+\epsilon^3\frac{\partial^2 w_3}{\partial t^2}+\epsilon^3 c\frac{\partial w_1}{\partial t}+...= \\ \qquad \alpha_1\epsilon^3\Gamma(w_1,w_1)\frac{\partial^2 w_1}{\partial x^2}+\epsilon N\frac{\partial^2 w_1}{\partial x^2}+\epsilon^2 N\frac{\partial^2 w_2}{\partial x^2}+\epsilon^3 N\frac{\partial^2 w_3}{\partial x^2}+...+ \\ \qquad \epsilon(V_0+\epsilon A\sin(\Omega t))^2\left(1+2\epsilon w_1+2\epsilon^2 w_2+3\epsilon^2 w_1^2+...\right), \\ \epsilon w_1(0,t)+\epsilon^2 w_2(0,t)+\epsilon^3 w_3(0,t)+...=0, \\ \epsilon w_1(1,t)+\epsilon^2 w_2(0,t)+\epsilon^3 w_3(0,t)+...=0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(0,t)+\epsilon^2\frac{\partial^2 w_2}{\partial x^2}(0,t)+\epsilon^3\frac{\partial^2 w_3}{\partial x^2}(0,t)+...=0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(1,t)+\epsilon^2\frac{\partial^2 w_2}{\partial x^2}(1,t)+\epsilon^3\frac{\partial^2 w_3}{\partial x^2}(1,t)+...=0, \\ \epsilon w_1(x,0)+\epsilon^2 w_2(x,0)+\epsilon^3 w_3(x,0)+...=0, \\ \epsilon\frac{\partial w_1}{\partial t}(x,0)+\epsilon^2\frac{\partial w_2}{\partial t}(x,0)+\epsilon^3\frac{\partial w_3}{\partial t}(x,0)+...=0. \end{cases} \tag{138}$$

As before, we will use the method of mulitple scales. Introducing $t_0 = t$ and $t_1 = \epsilon t$, gives:

$$\begin{cases} \epsilon\frac{\partial^4 w_1}{\partial x^4}+\epsilon^2\frac{\partial^4 w_2}{\partial x^4}+\epsilon^3\frac{\partial^4 w_3}{\partial x^4}+\epsilon\frac{\partial^2 w_1}{\partial t_0^2}+2\epsilon^2\frac{\partial^2 w_1}{\partial t_0\partial t_1}+\epsilon^3\frac{\partial^2 w_1}{\partial t_1^2}+\epsilon^2\frac{\partial^2 w_2}{\partial t_0^2}+ \\ 2\epsilon^3\frac{\partial^2 w_2}{\partial t_0\partial t_1}+\epsilon^3\frac{\partial^2 w_3}{\partial t_0^2}+\epsilon^3 c\frac{\partial w_1}{\partial t_0}+...=\alpha_1\epsilon^3\Gamma(w_1,w_1)\frac{\partial^2 w_1}{\partial x^2}+\epsilon N\frac{\partial^2 w_1}{\partial x^2}+\epsilon^2 N\frac{\partial^2 w_2}{\partial x^2} \\ \qquad +\epsilon^3 N\frac{\partial^2 w_3}{\partial x^2}+...+\epsilon(V_0+\epsilon A\sin(\Omega t_0))^2\left(1+2\epsilon w_1+2\epsilon^2 w_2+3\epsilon^2 w_1^2+...\right), \\ \epsilon w_1(0,t_0,t_1)+\epsilon^2 w_2(0,t_0,t_1)+\epsilon^3 w_3(0,t_0,t_1)+...=0, \\ \epsilon w_1(1,t_0,t_1)+\epsilon^2 w_2(0,t_0,t_1)+\epsilon^3 w_3(0,t_0,t_1)+...=0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(0,t_0,t_1)+\epsilon^2\frac{\partial^2 w_2}{\partial x^2}(0,t_0,t_1)+\epsilon^3\frac{\partial^2 w_3}{\partial x^2}(0,t_0,t_1)+...=0, \\ \epsilon\frac{\partial^2 w_1}{\partial x^2}(1,t_0,t_1)+\epsilon^2\frac{\partial^2 w_2}{\partial x^2}(1,t_0,t_1)+\epsilon^3\frac{\partial^2 w_3}{\partial x^2}(1,t_0,t_1)+...=0, \\ \epsilon w_1(x,0,0)+\epsilon^2 w_2(x,0,0)+\epsilon^3 w_3(x,0,0)+...=0, \\ \epsilon\frac{\partial w_1}{\partial t_0}(x,0,0)+\epsilon^2\frac{\partial w_1}{\partial t_1}(x,0,0)+\epsilon^2\frac{\partial w_2}{\partial t_0}(x,0,0)+\epsilon^3\frac{\partial w_2}{\partial t_1}(x,0,0)+\epsilon^3\frac{\partial w_3}{\partial t_0}(x,0,0)+...=0. \end{cases} \tag{139}$$

Our $O(\epsilon)$-problem is the same as before in equation (98). For $O(\epsilon^2)$ the only difference compared to equation (107) is that $c = 0$. We substitute this in (122) and obtain:

$$\begin{cases} a_{n,0}(t_1) = -\frac{4V_0^2}{n\pi\lambda_n}\sin\left(\frac{V_0^2}{\sqrt{\lambda_n}}t_1\right)\mathbb{1}_{\{\text{n is odd}\}}, \\ b_{n,0}(t_1) = -\frac{4V_0^2}{n\pi\lambda_n}\cos\left(\frac{V_0^2}{\sqrt{\lambda_n}}t_1\right)\mathbb{1}_{\{\text{n is odd}\}}. \end{cases} \tag{140}$$

We would like to solve w_2 as well, so we have to compute the initial conditions:

$$\begin{cases} w_2(x,0,0) = 0, \\ \frac{\partial w_2}{\partial t_0}(x,0,0) = -\frac{\partial w_1}{\partial t_1}(x,0,0) = 0. \end{cases} \tag{141}$$

To construct our differential equation for $b_n(t_0,t_1)$, we use (109) and (117) to obtain:

$$\lambda_n b_n + \frac{\partial^2 b_n}{\partial t_0^2} = \left(\frac{8V_0^4}{n\pi\lambda_n} + \frac{8V_0 A\sin(\Omega t_0)}{n\pi}\right)\mathbb{1}_{\{\text{n is odd}\}}. \tag{142}$$

Which has as a solution:

$$\begin{aligned} b_n(t_0,t_1) = a_{n,1}(t_1)\sin(\sqrt{\lambda_n}t_0) + b_{n,1}(t_1)\cos(\sqrt{\lambda_n}t_0) - \\ \frac{8V_0(A\lambda_n^2\sin(\Omega t_0) - V_0^3(\Omega^2-\lambda_n))}{\pi n\lambda_n^2(\Omega^2-\lambda_n)}\mathbb{1}_{\{\text{n is odd}\}}. \end{aligned} \tag{143}$$

The initial conditions give:

$$\begin{cases} a_{n,1}(0) = 0, \\ b_{n,1}(0) = -\frac{8V_0^4}{\pi n\lambda_n^2}\mathbb{1}_{\text{n is odd}}. \end{cases} \tag{144}$$

$O(\epsilon^3)$-problem is:

$$\begin{aligned} \frac{\partial^4 w_3}{\partial x^4} + \frac{\partial^2 w_1}{\partial t_1^2} + 2\frac{\partial^2 w_2}{\partial t_0\partial t_1} + \frac{\partial^2 w_3}{\partial t_0^2} + c\frac{\partial w_1}{\partial t_0} = \alpha_1\Gamma(w_1,w_1)\frac{\partial^2 w_1}{\partial x^2} + N\frac{\partial^2 w_3}{\partial x^2} + 2V_0^2 w_2 + \\ 3V_0^2 w_1^2 + 4V_0 A w_1\sin(\Omega t_0) + A^2\sin^2(\Omega t_0). \end{aligned} \tag{145}$$

The homogeneous equation is the same as before. Using equations (108) and (143), the inhomogeneous part of equation (145) is:

$$\begin{aligned} -\frac{\partial^2 w_1}{\partial t_1^2} - 2\frac{\partial^2 w_2}{\partial t_0\partial t_1} - c\frac{\partial w_1}{\partial t_0} + \alpha_1\Gamma(w_1,w_1)\frac{\partial^2 w_1}{\partial x^2} + 2V_0^2 w_2 + 3V_0^2 w_1^2 + 4V_0 A w_1\sin(\Omega t_0) + \\ A^2\sin^2(\Omega t_0) = \sum_{n=1}^{\infty}\Big[-a_{n,0}''\sin(\sqrt{\lambda_n}t_0) - b_{n,0}''\cos(\sqrt{\lambda_n}t_0) - 2a_{n,1}'\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}t_0) \\ +2b_{n,1}'\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}t_0) - c\sqrt{\lambda_n}a_{n,0}\cos(\sqrt{\lambda_n}t_0) + c\sqrt{\lambda_n}b_{n,0}\sin(\sqrt{\lambda_n}t_0) - \\ \alpha_1 n^2\pi^2\Gamma(w_1,w_1)a_n + 2V_0^2 a_{n,1}\sin(\sqrt{\lambda_n}t_0) + 2V_0^2 b_{n.1}\cos(\sqrt{\lambda_n}t_0) - \\ \frac{16V_0^3(A\lambda_n^2\sin(\Omega t_0) - V_0^3(\Omega^2-\lambda_n))}{\pi n\lambda_n^2(\Omega^2-\lambda_n)}\mathbb{1}_{\{nisodd\}} + 4V_0 A\sin(\Omega t_0)a_{n,0}\sin(\sqrt{\lambda_n}t_0) + \\ 4V_0 A\sin(\Omega t_0)b_{n,0}\cos(\sqrt{\lambda_n}t_0) + \frac{16AV_0^3}{n\pi\lambda_n}\sin(\Omega t_0)\mathbb{1}_{\{nisodd\}}\Big]\sin(n\pi x) + 3V_0^2 w_1^2 + \\ A^2\sin^2(\Omega t_0) = \sum_{n=1}^{\infty} C_n(t_0,t_1)\sin(n\pi x) + 3V_0^2 w_1^2 + A^2\sin^2(\Omega t_0). \end{aligned} \tag{146}$$

Because we have the same homogeneous equation, we let $w_3 = \sum_{n=1}^{\infty} c_n(t_0, t_1)\sin(n\pi x)$. Then we have the following differential equation for $c_n(t_1, t_2)$:

$$\begin{gathered}\lambda_n c_n + \frac{\partial^2 c_n}{\partial t_0^2} = C_n(t_0, t_1) + \frac{4A^2\sin^2(\Omega t_0)}{n\pi}\mathbb{1}_{\{\text{n odd}\}} - \\ \frac{24V_0^2}{\pi}\sum_{\text{n+k+m odd}} \frac{kmn}{n^4 - 2n^2(m^2+k^2) + (m^2-k^2)^2} a_k(t_0, t_1)a_m(t_0, t_1).\end{gathered} \tag{147}$$

If $N = 100$ it is proven in the appendix that $\forall k, m, n \in \mathbb{N}$:

$$\sqrt{\lambda_k} + \sqrt{\lambda_m} \neq \sqrt{\lambda_n}. \tag{148}$$

.

This means that a sine or cosine with a frequency equal to the sum of eigenfrequencies will not lead to secular terms. If for instance $N = \frac{272}{28}\pi^2 \approx 96$, then $\sqrt{\lambda_4} + \sqrt{\lambda_4} = \sqrt{\lambda_6}$. In this case there will be a lot more secular terms we will not consider now. At the moment, we assume $N = 100$ and therefore:

$$\begin{gathered}\sum_{\text{n+k+m odd}} \frac{kmn}{n^4 - 2n^2(m^2+k^2) + (m^2-k^2)^2} a_k(t_0, t_1)a_m(t_0, t_1) = \\ 2\sum_{\text{m odd}} \frac{n^2}{m^3 - 4mn^2}\frac{4V_0^2}{m\pi\lambda_m}(a_{n,0}\sin(\sqrt{\lambda_n}t_0) + b_{n,0}\cos(\sqrt{\lambda_n}t_0)) + N.S.T.,\end{gathered} \tag{149}$$

where N.S.T. stands for terms that do not produce secular terms. The function Γ may give rise to secular terms as well. Filling in w_1, we obtain:

$$\Gamma(w_1, w_1) = \frac{1}{2}\sum_{k=1}^{\infty} a_k^2 k^2 \pi^2, \tag{150}$$

so:

$$\begin{gathered}\Gamma(w_1, w_1)a_n = \left(\frac{1}{2}n^2\pi^2\left[\frac{1}{4}b_{n,0}^2 + \frac{1}{4}a_{n,0}^2 + \frac{32V_0^4}{n^2\pi^2\lambda_n^2}\mathbb{1}_{\{\text{n is odd}\}}\right] + \frac{1}{2}\sum_{m=1}^{\infty} m^2\pi^2\left[\frac{1}{2}a_{m,0}^2 + \right.\right. \\ \left.\left.\frac{1}{2}b_{m,0}^2 + \frac{16V_0^4}{m^2\pi^2\lambda_m^2}\mathbb{1}_{\{\text{m is odd}\}}\right]\right)(b_{n,0}\cos(\sqrt{\lambda_n}t_0) + a_{n,0}\sin(\sqrt{\lambda_n}t_0)) + N.S.T.\end{gathered} \tag{151}$$

Now to see if there are any Ω besides the $\sqrt{\lambda_n}$ which may lead to resonance, we rewrite the terms in (146) which contain Ω.

$$\begin{cases} 4V_0 A\sin(\Omega t_0)a_{n,0}\sin(\sqrt{\lambda_n}t_0) = 2V_0 A a_{n,0}(\cos((\Omega - \sqrt{\lambda_n})t_0) - \cos((\Omega + \sqrt{\lambda_n})t_0)), \\ 4V_0 A\sin(\Omega t_0)b_{n,0}\cos(\sqrt{\lambda_n}t_0) = 2V_0 A b_{n,0}(\sin((\Omega - \sqrt{\lambda_n})t_0) + \sin((\Omega + \sqrt{\lambda_n})t_0)). \end{cases} \tag{152}$$

$$\frac{4A^2\sin^2(\Omega t_0)}{n\pi} = \frac{2A^2}{n\pi}(1 - \cos(2\Omega t_0)). \tag{153}$$

Equation (152) gives extra secular terms when $\Omega = 0$, a direct current instead of an alternating current, or if $\Omega = 2\sqrt{\lambda_n}$, a superharmonic resonance frequency. Equation (153) gives extra secular terms when $\Omega = \frac{1}{2}\sqrt{\lambda_n}$, a subharmonic resonance frequency. We choose $\Omega = 2\sqrt{\lambda_N} + \epsilon\omega$ for N fixed, with ω a detuning parameter to describe the behaviour near $2\sqrt{\lambda_N}$. Then $\cos((\Omega-\sqrt{\lambda_N})t_0) = \cos(\sqrt{\lambda_N}t_0)\cos(\omega t_1) - \sin(\sqrt{\lambda_n}t_0)\sin(\omega t_1)$ and $\sin((\Omega-\sqrt{\lambda_N})t_0) = \sin(\sqrt{\lambda_N}t_0)\cos(\omega t_1) + \cos(\sqrt{\lambda_N}t_0)\sin(\omega t_1)$. To prevent secular terms from occurring in the inhomogeneous term of (147) we need:

$$\begin{cases} 2b'_{n,1}\sqrt{\lambda_n} + 2V_0^2 a_{n,1} + S_1 - \frac{8V_0^3 A}{n\pi\lambda_n}\cos\left(\frac{V_0^2 t_1}{\sqrt{\lambda_N}} + \omega t_1\right)\mathbb{1}_{\{n=N\}} = 0, \\ -2a'_{n,1}\sqrt{\lambda_n} + 2V_0^2 b_{n,1} + S_2 - \frac{8V_0^3 A}{n\pi\lambda_N}\sin\left(\frac{V_0^2 t_1}{\sqrt{\lambda_N}} + \omega t_1\right)\mathbb{1}_{\{n=N\}} = 0, \end{cases} \tag{154}$$

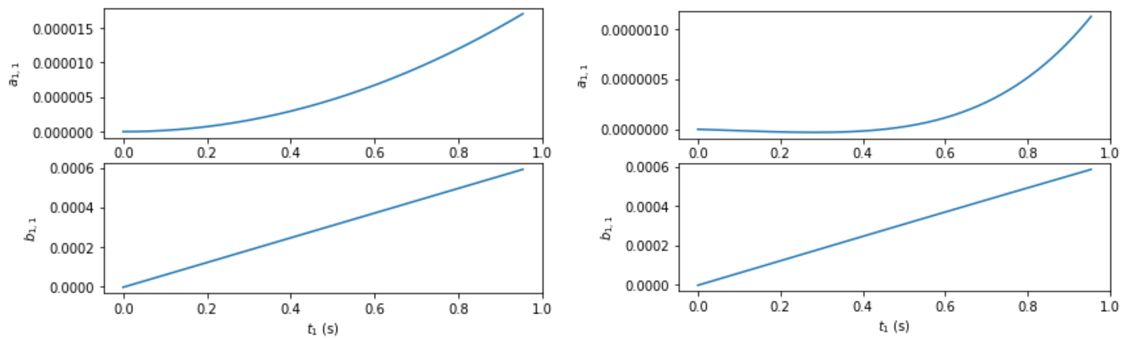
with

$$\begin{cases} S_1 = -\dfrac{4V_0^6}{n\pi\lambda_n^2}\sin\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right) - \dfrac{4cV_0^2}{n\pi\sqrt{\lambda_n}}\cos\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right) + 72\alpha_1\dfrac{V_0^6 n\pi}{\lambda_n^3}\sin\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right) + \\ \qquad \dfrac{48nV_0^6}{\pi\lambda_n}\displaystyle\sum_{\text{m is odd}}\left(\dfrac{\alpha_1\pi^2}{\lambda_m^2} + \dfrac{16}{m\pi^2\lambda_m(m^3-4mn^2)}\right)\sin\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right), \text{ for n odd,} \\ S_2 = -\dfrac{4V_0^6}{n\pi\lambda_n^2}\cos\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right) + \dfrac{4cV_0^2}{n\pi\sqrt{\lambda_n}}\sin\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right) + 72\alpha_1\dfrac{V_0^6 n\pi}{\lambda_n^3}\cos\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right) + \\ \qquad \dfrac{48nV_0^6}{\pi\lambda_n}\displaystyle\sum_{\text{m is odd}}\left(\dfrac{\alpha_1\pi^2}{\lambda_m^2} + \dfrac{16}{m\pi^2\lambda_m(m^3-4mn^2)}\right)\cos\left(\dfrac{V_0^2 t_1}{\sqrt{\lambda_n}}\right), \text{ for n odd} \end{cases} \tag{155}$$

and initial conditions (144). This equation can be solved. To see for which ω the solution is unstable, we consider $n = N$ and investigate the numerators in the solution that depend on ω. When these become 0, the solution will become large. The numerators depending on ω are

$$\begin{cases} V_0^2\lambda_N^{\frac{13}{2}} n\pi\left(\sqrt{\lambda_N}\omega + 2V_0^2\right)\left(2\lambda_N^{\frac{13}{2}}V_0^2 + \lambda_N^7\omega\right), \\ 2V_0^2 + \sqrt{\lambda_N}\omega, \\ V_0^2\pi n\left(4\lambda_N^{\frac{27}{2}}V_0^2\omega + 4V_0^4\lambda_N^{13} + \lambda_N^{14}\omega^2\right), \end{cases} \tag{156}$$

or constant multiples of this. These are zero when $\omega = \frac{-2V_0^2}{\sqrt{\lambda_n}}$. The homogeneous solutions of $a_{n,1}$ and $b_{n,1}$ are $-\frac{8V_0^4}{\pi n\lambda_n^2}\sin(\frac{V_0^2 t_1}{\sqrt{\lambda_n}})$ and $-\frac{8V_0^4}{\pi n\lambda_n^2}\cos(\frac{V_0^2 t_1}{\sqrt{\lambda_n}})$. We can see that S_1 and S_2 always contain these secular terms and the solution is therefore always unstable. This does not have to be a problem since this solution holds on a $\frac{1}{\epsilon}$ timescale for which $a_{1,1}$ and $b_{1,1}$ are still small. If $\omega = \frac{-2V_0^2}{\sqrt{\lambda_N}}$ then $\cos(\frac{V_0^2 t_1}{\sqrt{\lambda_N}} + \omega t_1)$ and $\sin(\frac{V_0^2 t_1}{\sqrt{\lambda_N}} + \omega t_1)$ are a multiple of the homogeneous solution as well. The solution of $a_{,1}$ and $b_{1,1}$ is plotted in figure 23. We can see that for $\omega = -\frac{2V_0^2}{\sqrt{\lambda_N}}$ $a_{1,1}$ grows faster than for $\omega = 1$, which is what we would expect.



(a) (b)

Figure 23: $a_{1,1}$ and $b_{1,1}$ against t_1 for $N = 100$, $A = 1$, $c = 1$ and $V_0 = 1\text{V}$ obtained using the RK4 method for $\Delta t_1 = 0.00012\text{s}$. Left for $\omega = -\frac{V_0^2}{\sqrt{\lambda_N}}$ and right for $\omega = 1$.

In figure 24 the amplitude of the oscillation r is plotted against Ω for c of $O(\epsilon)$ as in sections 4.1-4.3 and for c of $O(\epsilon^2)$ as in this section. We can see that for smaller c the amplitude of the oscillation becomes larger than for larger c as we would expect because the damping limits the oscillations. We can furthermore observe that the peak in the amplitude is much narrower for smaller c. This is because the effect of the damping grows with the velocity of the oscillation and therefore especially limits the fast oscillations near resonance. According to Younis and Nayfeh [4], the nonlinear terms should become more important when the damping decreases. We can indeed see that there is an small increase in the effect of the nonlinear terms for the smaller c.

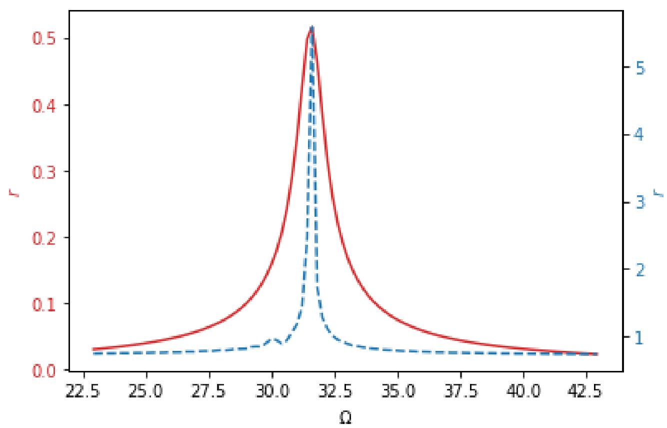


Figure 24: r against Ω for $\alpha = 200$, $N = 100$, $V_0^2 = 45$ and $A = \epsilon = 1$. The red solid line and the left axis correspond to $c = O(\epsilon)$. The blue dashed line and the right axis correspond to $c = O(\epsilon^2)$. The solution for $O(\epsilon^2)$ is obtained with the RK4-method with $\Delta t = 0.012\text{s}$.

5 Conclusions

In this research we considered the response of a simply supported microbeam subject to an applied axial load, accounting for mid-plane stretching, actuated by an electric actuation. We neglected shear deformation and rotary inertia. In order to solve our equation of motion we used the method of multiple scales. We have studied this method extensively. With this method we have constructed a solution to motion of the microbeam which is valid up to times $O(\frac{1}{\epsilon})$, where ϵ is a small dimensionless parameter.

During the research we have looked at the behavior of the microbeam for different frequencies of the alternating current of the electric actuation: not close to the eigenfrequencies of the system, exactly the same as one of the eigenfrequencies of the system and close to one of the eigenfrequencies of the system. We determined the eigenfrequencies of the system up to order ϵ as $\sqrt{\lambda_n} = \sqrt{Nn^2\pi^2 + n^4\pi^4}$, where N is a parameter depending on the bending stiffness, length, Poisson ratio and tension of the beam.

First we looked at a damping of $O(\epsilon)$. For the frequency of the alternating current Ω far away from the $\sqrt{\lambda_n}$ we found that due to the damping our solution remained small. We considered the energy of the system as well and have seen that the energy converged to a maximum value which was due to the static current. For Ω equal to one of the $\sqrt{\lambda_n}$ we have seen that resonance phenomena occur. We have found that the solution becomes very large and that the energy of the system continues growing with time. For Ω close to the eigenfrequencies of the system, we performed a stability analysis on the equilibrium points. We found that for our example values of the parameters for $\Omega \in [\sqrt{\lambda_n} - 0.00585, \sqrt{\lambda_n} + 0.00560]$ there is no stable equilibrium of our problem. This interval can be calculated for all values of the parameters. Physically this means that the damping will not extinguish the oscillations completely and the microbeam will not lose its applicability. We have solved our problem numerically and have seen excellent agreement with our conclusions using the method of multiple scales.

Next we made a start to the study of a damping of $O(\epsilon^2)$. Here we found that on a $\frac{1}{\epsilon}$ timescale resonance phenomena always occur and that resonance due to the electric actuation can occur for subharmonic, $\Omega = \frac{1}{2}\sqrt{\lambda_n}$, and superharmonic, $\Omega = 2\sqrt{\lambda_n}$, frequencies. Furthermore, we found that for a smaller damping the amplitude of the oscillations is larger and that the effect of the nonlinear terms is slightly more important. This agrees with what Younis and Nayfeh found for clamped boundary conditions [4].

For the derivation of our equation of motion we made some assumptions which lead to very good approximations of the solution in the cases we considered. In contrast to most of the previous research [4], [5] we did not apply mode analysis, as this often neglects internal resonance. Furthermore, we considered simply supported boundary conditions instead of clamped boundary conditions. As there is some flexibility at the boundaries, neither of them is a perfect model. Using simply supported boundary conditions leads to nicer calculations to handle and the results differ very little. This research can be used as a first step towards cases in which our approximations might not be valid. For some MEMS, for instance, the length is about as large as the width. This would mean that we cannot assume that the deflection is uniform in the y-direction. This would give an extra dimension to the problem and result in an equation for a membrane. As can be seen from equation

(53), the solution of a membrane would be very similar to the solution of a string such as the one presented in this research. Additionally, even more physical quantities such as shear deformation and rotary inertia could be taken into account. However, this will have very little effect on the qualitative behaviour of the frequency response of the microbeam and is therefore not the aim of this research.

For further research it might be interesting to use 4.4 as a starting point to study more extensively the case if the damping is $O(\epsilon^2)$. In this research we only considered resonance phenomena for the superharmonic case that Ω is close to $2\sqrt{\lambda_n}$. The subharmonic case that Ω is close to $\frac{1}{2}\sqrt{\lambda_n}$ has not yet been studied. Furthermore, we have proven that for $N = 100$ $\sqrt{\lambda_k} + \sqrt{\lambda_m} \neq \sqrt{\lambda_n}$ $\forall k, m, n \in \mathbb{N}$. We have also shown that for different N this is not the case and even more resonance frequencies might be found. Additionally, the effect of the nonlinear terms for smaller damping can be investigated more elaborately.

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Appendix

Theorem 1. $-arc\cos\left(\frac{-V_0N\pi r_{eq}}{4A} - \frac{\sqrt{\lambda_N\omega N\pi r_{eq}}}{4V_0A}\right)$ *is always a solution of* (131).

Proof. The equation trivially satisfies the first equation of (131). For the second equation, consider

$$\sin\left(-arc\cos\left(\frac{-V_0N\pi r_{eq}}{4A} - \frac{\sqrt{\lambda_N\omega N\pi r_{eq}}}{4V_0A}\right)\right) =$$
$$-\sin\left(arc\cos\left(\frac{-V_0N\pi r_{eq}}{4A} - \frac{\sqrt{\lambda_N\omega N\pi r_{eq}}}{4V_0A}\right)\right) =$$
$$-\sqrt{1-\left(\frac{-V_0N\pi r_{eq}}{4A} - \frac{\sqrt{\lambda_N\omega N\pi r_{eq}}}{4V_0A}\right)^2} =$$
$$-\sqrt{\frac{16V_0^2A^2 - V_0^4N^2\pi^2r_{eq}^2 - 2V_0^2N^2\pi^2r_{eq}^2\omega\sqrt{\lambda_N} - \lambda_N\omega^2N^2\pi^2r_{eq}^2}{16V_0^2A^2}} =$$
$$\frac{-1}{4V_0A}\sqrt{16V_0^2A^2 - \left(V_0^4 + 2V_0^2\omega\sqrt{\lambda_N} + \lambda_n\omega^2\right)N^2\pi^2r_{eq}^2}.$$

Using (134), we obtain

$$-\frac{N\pi r_{eq}}{4V_0A}\sqrt{\lambda_N\omega^2 + 2V_0\sqrt{\lambda_N}\omega + V_0^4 + \frac{1}{4}c^2\lambda_N - V_0^4 - 2V_0^2\omega\sqrt{\lambda_N} - \lambda_N\omega^2} = -\frac{N\pi r_{eq}c\sqrt{\lambda_N}}{8V_0A},$$

which satisfies the second equation. □

Theorem 2. *Assume* λ *is as in eqaution* (100) *and* $N = 100$. $\sqrt{\lambda_n} \neq \sqrt{\lambda_m} + \sqrt{\lambda_k}\ \forall k, m, n \in \mathbb{N}$

Proof. Suppose $\sqrt{\lambda_n} = \sqrt{\lambda_m} + \sqrt{\lambda_k}$. If we square both sides, we obtain:

$$\lambda_n = \lambda_m + 2\sqrt{\lambda_m}\sqrt{\lambda_k} + \lambda_m.$$

Moving the term with the roots to one side and the other terms to the other and squaring them again gives

$$4\lambda_m\lambda_n = \lambda_n^2 - 2\lambda_n\lambda_m - 2\lambda_n\lambda_k + \lambda_m^2 + 2\lambda_m\lambda_k + \lambda_k^2.$$

Filling in (100) gives

$$10000n^4\pi^4 + 200n^6\pi^6 - 20000n^2k^2\pi^4 - 200n^2k^4\pi^6 - 20000n^2m^2\pi^4 - 200n^2m^4\pi^6 + n^8\pi^8$$
$$-200n^4k^2\pi^6 - 2n^4k^4\pi^8 - 200m^2n^4\pi^6 - 2n^4m^4\pi^8 + 100000k^4\pi^4 + 200k^6\pi^6 + 20000k^2m^2\pi^4$$
$$+200k^2m^4\pi^6 + k^8\pi^8 + 200k^4m^2\pi^6 + 2k^4m^4\pi^8 + 10000m^4\pi^4 + 200m^6\pi^6 + m^8\pi^8 =$$
$$40000k^2m^2\pi^4 + 400k^2m^4\pi^6 + 400k^4m^2\pi^6 + 4k^4m^4\pi^8.$$

Since k, m, n are all natural numbers, for the terms before the same power of π the equality has to hold. For π^8 this means:

$$4k^4m^4 = n^8 - 2n^4k^4 - 2n^4m^4 + k^8 + 2k^4m^4 + m^8 = \left(n^4 - k^4 - m^4\right)^2$$

$$\implies n^4 - k^4 - m^4 = \pm 2k^2m^2$$

$$\implies n^4 = k^4 \pm 2k^2m^2 + m^4 = \left(k^2 \pm m^2\right)^2$$

$$\implies n^2 = \pm\left(k^2 \pm m^2\right).$$

Since $n \geq m > 0$ and $n \geq k > 0$ this has to mean

$$n^2 = k^2 + m^2.$$

For π^4 we have

$$4k^2m^2 = n^4 - 2n^2k^2 - 2n^2m^2 + k^4 + 2k^2m^2 + m^4 = \left(n^2 - k^2 - m^2\right)^2$$

$$\implies \pm 2km = n^2 - k^2 - m^2 = 0$$

$$\implies k = 0 \vee m = 0,$$

which is impossible. □