

Maximal inequalities for stochastic convolutions in 2-smooth Banach spaces and applications to stochastic evolution equations

van Neerven, Jan; Veraar, Mark

DOI

[10.1098/rsta.2019.0622](https://doi.org/10.1098/rsta.2019.0622)

Publication date

2020

Document Version

Final published version

Published in

Philosophical transactions. Series A, Mathematical, physical, and engineering sciences

Citation (APA)

van Neerven, J., & Veraar, M. (2020). Maximal inequalities for stochastic convolutions in 2-smooth Banach spaces and applications to stochastic evolution equations. *Philosophical transactions. Series A, Mathematical, physical, and engineering sciences*, 378(2185), 1-21. <https://doi.org/10.1098/rsta.2019.0622>

Important note

To cite this publication, please use the final published version (if applicable).
Please check the document version above.

Copyright

Other than for strictly personal use, it is not permitted to download, forward or distribute the text or part of it, without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license such as Creative Commons.

Takedown policy

Please contact us and provide details if you believe this document breaches copyrights.
We will remove access to the work immediately and investigate your claim.

Green Open Access added to TU Delft Institutional Repository

'You share, we take care!' - Taverne project

<https://www.openaccess.nl/en/you-share-we-take-care>

Otherwise as indicated in the copyright section: the publisher is the copyright holder of this work and the author uses the Dutch legislation to make this work public.

Review



Cite this article: van Neerven J, Veraar M. 2020 Maximal inequalities for stochastic convolutions in 2-smooth Banach spaces and applications to stochastic evolution equations. *Phil. Trans. R. Soc. A* **378**: 20190622. <http://dx.doi.org/10.1098/rsta.2019.0622>

Accepted: 4 July 2020

One contribution of 13 to a theme issue 'Semigroup applications everywhere'.

Subject Areas:

analysis

Keywords:

maximal inequalities, stochastic convolutions, 2-smooth Banach spaces, stochastic evolution equations, semigroups of operators, evolution families

Author for correspondence:

Jan van Neerven

e-mail: J.M.A.M.vanNeerven@tudelft.nl

Maximal inequalities for stochastic convolutions in 2-smooth Banach spaces and applications to stochastic evolution equations

Jan van Neerven and Mark Veraar

Delft University of Technology, Faculty EEMCS/DIAM, PO Box 5031, 2600 GA Delft, The Netherlands

JvN, 0000-0003-4577-7908; MV, 0000-0003-3167-7471

This paper presents a survey of maximal inequalities for stochastic convolutions in 2-smooth Banach spaces and their applications to stochastic evolution equations.

This article is part of the theme issue 'Semigroup applications everywhere'.

1. Introduction

This paper presents an overview of maximal inequalities for Banach space-valued stochastic processes $(u_t)_{t \in [0, T]}$ of the form

$$u_t = \int_0^t S(t, s) g_s \, dW_s, \quad t \in [0, T], \quad (1.1)$$

where $(S(t, s))_{0 \leq s \leq t \leq T}$ is a strongly continuous evolution family acting on a Banach space X , $(W_t)_{t \in [0, T]}$ is a (cylindrical) Brownian motion defined on a probability space Ω , and $(g_t)_{t \in [0, T]}$ is a stochastic process taking values in X (in the case of a Brownian motion $(W_t)_{t \in [0, T]}$) or in a space of operators acting from H to X (in the case of a cylindrical Brownian motion whose covariance is given by the inner product of a Hilbert space H), defined on the same probability space Ω . The stochastic integral in (1.1) is the Banach space-valued extension of the classical Itô stochastic integral. In the important special case $S(t, s) = S(t - s)$ arising from a one-parameter semigroup of operators $(S(t))_{t \geq 0}$, the stochastic integral (1.1) takes the form of a stochastic convolution. This justifies our slight abuse of terminology to also

refer to (1.1) as a stochastic convolution. In addition to reviewing the literature on this topic, some new contributions are included as well.

Under a *maximal inequality* for $(u_t)_{t \in [0, T]}$, we understand a bound on the random variable

$$u^*(\omega) = \sup_{t \in [0, T]} \|u_t(\omega)\|, \quad \omega \in \Omega.$$

Maximal inequalities are important in the theory of stochastic evolution equations, where the mild solution of the time-dependent inhomogeneous stochastic evolution equation

$$\begin{cases} du_t = A(t)u_t dt + g_t dW_t, & t \in [0, T], \\ u_0 = 0, \end{cases}$$

is of the form (1.1) provided one assumes that the operator family $(A(t))_{t \in [0, T]}$ generates the evolution family $(S(t, s))_{0 \leq s \leq t \leq T}$ in a suitable sense. The availability of a maximal inequality in this setting typically implies that the solution process $(u_t)_{t \in [0, T]}$ has a continuous version.

In the present paper, we limit ourselves to maximal estimates of the Burkholder type, where u^* is estimated in terms of a square function norm analogous to the one occurring in the classical Burkholder maximal inequality for continuous-time martingales. Different techniques to obtain pathwise continuous solutions, such as developed in [1, 2], will not be discussed here.

At present, two theories of Itô stochastic calculus in Banach spaces are available: for 2-smooth Banach spaces [3] and for UMD Banach spaces [4, 5]. Both approaches are surveyed in [6]. Each of the two approaches has its advantages and disadvantages. The construction of the stochastic integral in 2-smooth Banach spaces is fairly elementary and its use in the theory of stochastic evolution equations is straightforward, but its applicability covers only half of the L^p -scale (namely the exponents $2 \leq p < \infty$) [7, 8]. It replaces the basic Itô isometry with a one-sided estimate which necessarily entails some loss in precision. This manifests itself in questions relating to maximal regularity, which cannot be fully treated with this theory. The stochastic integral in UMD Banach spaces covers the full reflexive L^p -scale (exponents $1 < p < \infty$) and leads to two-sided estimates for the stochastic integrals, but due to the more subtle form of the expressions involved it requires some constraints on the properties of the stochastic processes under consideration. In practice, this entails that the theory can be applied effectively to evolution equations in the parabolic setting only, but in that setting a full-fledged maximal regularity theory is available [9–12]. The 2-smooth theory is applicable beyond the parabolic setting and covers the case of arbitrary C_0 -evolution families.

In order to keep this paper at a reasonable length, we will exclusively deal with maximal estimates in the 2-smooth setting. Maximal estimates for stochastic convolutions in 2-smooth Banach spaces are useful in applications to stochastic partial differential equations. Typically, one takes X to be L^p , the Bessel potential space $H^{s,p}$, or the Besov space $B_{p,q}^s$; as will be explained in example 2.1 these spaces are 2-smooth if $2 \leq p, q < \infty$. Maximal estimates in the setting of UMD spaces are covered in [13] and the follow-up works [10, 11]. Maximal inequalities for stochastic convolutions in 2-smooth Banach spaces with respect to other noise processes than (cylindrical) Brownian motions, such as Poissonian noise, are discussed in [14–16] and the references therein; see also survey [17] for the Hilbertian case.

An important motivation to study stochastic partial differential equations in the setting of 2-smooth Banach spaces comes from the fact that estimates in Sobolev and Besov spaces with high integrability exponent p (which are 2-smooth) can be combined with Sobolev embedding results to obtain further integrability and regularity properties of solutions. This plays a key role in many papers (see [8, 9] and references therein).

2. Preliminaries

We assume familiarity with the basic notions of probability theory and stochastic analysis. This preliminary section fixes notation following the references [6, 18, 19] where unexplained

terminology can be found. All random variables and stochastic processes are assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which we fix once and for all. We work over the real scalar field.

(a) Stochastic preliminaries

When X is a Banach space, an X -valued *random variable* is a strongly measurable function $\phi : \Omega \rightarrow X$, i.e. a function that can be approximated $\mathbb{P}$ -almost surely by a sequence of $\mathcal{F}$ -measurable simple functions with values in X . The adjective ‘ X -valued’ will usually be omitted; depending on the context, random variables can be real- or vector-valued. The expected value of an integrable random variable ϕ is denoted by $\mathbb{E}\phi = \int_{\Omega} \phi \, d\mathbb{P}$. For $0 < p \leq \infty$, we denote by $L^p(\Omega; X)$ the (quasi-) Banach space of strongly measurable functions $\phi : \Omega \rightarrow X$ such that $\mathbb{E}\|\phi\|^p < \infty$, with the usual adjustment for $p = \infty$, and by $L^0(\Omega; X)$ the space of all strongly measurable functions $\phi : \Omega \rightarrow X$ endowed with the metric topology induced by convergence in measure. In dealing with elements of these spaces, it is always understood that we identify random variables that equal almost surely. When $0 \leq p \leq \infty$ and $\mathcal{G}$ is a sub- σ -algebra of $\mathcal{F}$, we denote by $L^p(\Omega, \mathcal{G}; X)$ the closed subspace of $L^p(\Omega; X)$ of all elements that are strongly measurable as random variables defined on $(\Omega, \mathcal{G}, \mathbb{P}|_{\mathcal{G}})$. The conditional expectation of a random variable ϕ given $\mathcal{G}$ is denoted by $\mathbb{E}_{\mathcal{G}}(\phi)$ or $\mathbb{E}(\phi|\mathcal{G})$.

A *filtration* is a family $(\mathcal{F}_t)_{t \in [0, T]}$ of sub- σ -algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \leq t$. A *process* is a family of X -valued random variables $(\phi_t)_{t \in [0, T]}$. It is called *adapted* if for every $t \in [0, T]$ the random variable ϕ_t is strongly measurable as a random variable on $(\Omega, \mathcal{F}_t, \mathbb{P}|_{\mathcal{F}_t})$. A process ϕ is called a *martingale* if $\mathbb{E}(\phi_t|\mathcal{F}_s) = \phi_s$ almost surely whenever $s \leq t$. Discrete filtrations and martingales are defined similarly, replacing the index set $[0, T]$ by a finite set $\{0, 1, \dots, N\}$.

The *progressive σ -algebra* on $[0, T] \times \Omega$ is the σ -algebra $\mathcal{P}$ generated by sets of the form $B \times F$ with $B \in \mathcal{B}([0, t])$ and $F \in \mathcal{F}_t$, where t ranges over $[0, T]$. A process ϕ is said to be *progressively measurable* if it is strongly measurable with respect to $\mathcal{P}$. Two processes ϕ, ψ are called *versions* of each other if for every $t \in [0, T]$ we have $\phi_t = \psi_t$ almost surely; the exceptional set is allowed to depend on t . A process ϕ is said to have a *continuous version* if it has a pathwise continuous version ψ , i.e. a version such that for all $\omega \in \Omega$ the path $t \mapsto \psi_t(\omega)$ is continuous.

Next, we extend the notion of a Hilbert–Schmidt operator to the Banach space setting. The reader is referred to [19, ch. 9] and [20] for systematic treatments. Let $\mathcal{H}$ be a Hilbert space and X be a Banach space. The space of finite rank operators from $\mathcal{H}$ into X is denoted by $\mathcal{H} \otimes X$. For a finite rank operator $R \in \mathcal{H} \otimes X$, say $R = \sum_{n=1}^N h_n \otimes x_n$ with $(h_n)_{n=1}^N$ orthonormal in $\mathcal{H}$ and $(x_n)_{n=1}^N$ a sequence in X (we can always represent R in this way by a Gram–Schmidt orthogonalization argument), we define

$$\|R\|_{\gamma(\mathcal{H}, X)}^2 = \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2,$$

where $(\gamma_n)_{n=1}^N$ is a sequence of independent standard Gaussian random variables. The norm $\|\cdot\|_{\gamma(\mathcal{H}, X)}$ is well defined, and the completion of $\mathcal{H} \otimes X$ with respect to this norm is denoted by $\gamma(\mathcal{H}, X)$. The natural inclusion mapping $\mathcal{H} \otimes X \subseteq \mathcal{L}(\mathcal{H}, X)$ extends to an injective and contractive inclusion mapping $\gamma(\mathcal{H}, X) \subseteq \mathcal{L}(\mathcal{H}, X)$. A linear operator in $\mathcal{L}(\mathcal{H}, X)$ is said to be *γ -radonifying* if it belongs to $\gamma(\mathcal{H}, X)$. For Hilbert spaces X , the identity mapping on $\mathcal{H} \otimes X$ extends to an isometrical isomorphism

$$\gamma(\mathcal{H}, X) \simeq \mathcal{L}_2(\mathcal{H}, X),$$

where $\mathcal{L}_2(\mathcal{H}, X)$ is the space of Hilbert–Schmidt operators from $\mathcal{H}$ to X . Another important instance where an explicit identification is available is the case $X = L^p(S; Y)$ with $(S, \mathcal{A}, \mu)$ a measure space, $1 \leq p < \infty$, and Y a Banach space; the mapping $h \otimes (f \otimes y) \mapsto f \otimes (h \otimes y)$ sets up an isomorphism of Banach spaces

$$\gamma(\mathcal{H}, L^p(S; Y)) \simeq L^p(S; \gamma(\mathcal{H}, Y)).$$

(b) 2-Smooth Banach spaces

A Banach space X is called (p, D) -smooth, where $p \in [1, 2]$ and $D \geq 0$ is a constant, if for all $x, y \in X$, we have

$$\|x + y\|^p + \|x - y\|^p \leq 2\|x\|^p + 2D^p\|y\|^p. \quad (2.1)$$

A Banach space is called p -smooth if it is (p, D) -smooth for some $D \geq 0$. The case $x = 0$ demonstrates that the constant in (2.1) necessarily satisfies $D \geq 1$. For $p = 2$, the defining condition is a generalized parallelogram identity.

Example 2.1. Here are some examples of 2-smooth Banach spaces:

- Every Hilbert space is 2-smooth (with $D = 1$, by the parallelogram identity).
- The space $L^p(\mu)$ is 2-smooth if and only if $2 \leq p < \infty$ (and in that case we may take $D = \sqrt{p-1}$, see [21, Proposition 2.1]). More generally, the space $L^p(\mu; X)$ is 2-smooth if and only if X is 2-smooth and $2 \leq p < \infty$ (in that case, if X is $(2, D)$ -smooth, then $L^p(\mu; X)$ is $(2, D\sqrt{p-1})$ -smooth [22]; for an earlier result in this direction, see [23]).
- For any $s \in \mathbb{R}$, the Bessel potential space $H^{s,p}(\mathbb{R}^d)$ is 2-smooth if and only if $2 \leq p < \infty$ (and in that case we may take $D = \sqrt{p-1}$). Indeed, this space is isometrically isomorphic to $L^p(\mathbb{R}^d)$, the isometry being given by the Fourier multiplier $(1 + |\xi|^2)^{s/2}$.
- For any $s \in \mathbb{R}$, the Besov space $B_{p,q}^s(\mathbb{R}^d)$ (equipped with its Littlewood–Paley norm) is 2-smooth if and only if $2 \leq p, q < \infty$ (and in that case we may take $D = \sqrt{(p-1)(q-1)}$). Indeed, with this norm, Littlewood–Paley theory identifies $B_{p,q}^s(\mathbb{R}^d)$ isometrically with a closed subspace of $\ell^p(L^q(\mathbb{R}^d))$. Analogous considerations apply to the Triebel–Lizorkin spaces $F_{p,q}^s(\mathbb{R}^d)$, and for both scales the results extend to more general open domains $\mathcal{O} \subseteq \mathbb{R}^d$ (with the same constant D if one uses the quotient norm of [24, Section 4.2.1]).
- For any $k \in \mathbb{N}$, the Sobolev space $W^{k,p}(\mathcal{O})$ is 2-smooth if and only if $2 \leq p < \infty$ (and in that case we may take $D = \sqrt{p-1}$ if we use the norm $\|f\|_{W^{k,p}(\mathcal{O})}^p = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_p^p$).
- The Schatten trace ideal C^p is 2-smooth if and only if $2 \leq p < \infty$ (and in that case we may take $D = \sqrt{p-1}$, see [25]).

A Banach space X is said to have *martingale type* $p \in [1, 2]$ if there exists a constant $C \geq 0$ such that

$$\mathbb{E}\|f_N\|^p \leq C^p \left(\mathbb{E}\|f_0\|^p + \sum_{n=1}^N \|f_n - f_{n-1}\|^p \right), \quad (2.2)$$

for all X -valued L^p -martingales $(f_n)_{n=0}^N$. The case $N = 0$ demonstrates that the constant in (2.2) necessarily satisfies $C \geq 1$. It is a fundamental result due to Pisier [26] (see also [27–29]) that, for any $p \in [1, 2]$, up to equivalence of norms, a Banach space is p -smooth if and only if it has martingale type p . The advantage of p -smoothness over martingale type p is that the former is an isometric condition, whereas the latter is isomorphic. We will encounter various maximal inequalities for semigroups or evolution families of contractions acting on 2-smooth Banach spaces. Such results cannot be expected to have a counterpart in martingale type 2-spaces, unless they hold more generally for uniformly bounded C_0 -semigroups, the point being that contractivity is typically not preserved under passing to equivalent norms.

3. Maximal inequalities for indefinite stochastic integrals

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot | \cdot)$. An $\mathcal{H}$ -isnormal process is a mapping $W : \mathcal{H} \rightarrow L^2(\Omega)$ with the following two properties:

- (i) for all $h \in \mathcal{H}$ the random variable Wh is Gaussian;
- (ii) for all $h_1, h_2 \in \mathcal{H}$ we have $\mathbb{E}(Wh_1 \cdot Wh_2) = (h_1 | h_2)$.

For $h = 0$, we interpret $W0$ as the Dirac measure concentrated at 0. From (ii), it easily follows that $\mathcal{H}$ -isonormal processes are linear, and this in turn implies that for all $h_1, \dots, h_N \in \mathcal{H}$ the $\mathbb{R}^N$ -valued random variable $(Wh_1, \dots, Wh_N)$ is jointly Gaussian, i.e., $(Wh)_{h \in \mathcal{H}}$ is a Gaussian process; see [30] for the details.

If W is an $L^2(0, T)$ -isonormal process, the process $(W\mathbf{1}_{(0,t]})_{t \in [0, T]}$ is a standard Brownian motion. This prompts us to define, for a Hilbert space H , a *cylindrical H -Brownian motion* as an $L^2(0, T; H)$ -isonormal process. In what follows the Hilbert space H will be considered to be fixed and we will consider a fixed cylindrical H -Brownian motion W . Following standard usage in the literature we will write

$$W_t h := W(\mathbf{1}_{(0,t]} \otimes h), \quad t \in [0, T], \quad h \in H.$$

For each $h \in H$, $(W_t h)_{t \in [0, T]}$ is a Brownian motion, which is standard if and only if h has norm one; two such Brownian motions corresponding to $h_1, h_2 \in H$ are independent if and only if h_1 and h_2 are orthogonal. A cylindrical H -Brownian motion W is said to be *adapted* to a given filtration $(\mathcal{F}_t)_{t \in [0, T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ if $W_t h \in L^2(\Omega, \mathcal{F}_t)$ for all $t \in [0, T]$ and $h \in H$. In what follows, we will always assume that a filtration has been fixed and that W is adapted to it.

A stochastic process $\Phi : [0, T] \times \Omega \rightarrow \gamma(H, X)$ is called an *adapted finite rank step process* if there exist $0 = s_0 < s_1 < \dots < s_n = T$, random variables $\xi_{ij} \in L^\infty(\Omega, \mathcal{F}_{s_{j-1}}) \otimes X$ (the subspace of $L^\infty(\Omega; X)$ of strongly $\mathcal{F}_{s_{j-1}}$ -measurable random variables taking values in a finite-dimensional subspace of X) for $i = 1, \dots, m$ and $j = 1, \dots, n$, and an orthonormal system $h_1, \dots, h_m$ in H such that

$$\Phi = \sum_{j=1}^n \mathbf{1}_{(s_{j-1}, s_j]} \otimes \sum_{i=1}^m h_i \otimes \xi_{ij}.$$

The *stochastic integral process* associated with Φ is then defined by

$$\int_0^t \Phi_s \, dW_s := \sum_{j=1}^n \sum_{i=1}^m (W_{s_j \wedge t} - W_{s_{j-1} \wedge t}) h_i \otimes \xi_{ij}, \quad t \in [0, T].$$

Since $s \mapsto W_s h$, being a Brownian motion, has a continuous version, it follows that the process $t \mapsto \int_0^t \Phi_s \, dW_s$ has a continuous version. Such versions will always be used in the sequel.

The following elementary upper bound for the stochastic integral of X -valued elementary adapted processes with respect to the cylindrical Brownian motion W , due to Neidhardt [3], extends the Itô isometry to 2-smooth Banach spaces. It is important to note that the proposition only provides an upper bound. It can be shown that this upper bound is an equivalence of norms if and only if X is isomorphic to a Hilbert space [6]. Indeed, it is this one-sidedness of the bound which constitutes the main limitation of the Itô stochastic integral in 2-smooth Banach spaces compared to its competitor for UMD Banach spaces.

Proposition 3.1 (Neidhardt). *Let X be a $(2, D)$ -smooth Banach space. Then, for all adapted finite rank step processes $\Phi : [0, T] \times \Omega \rightarrow \gamma(H, X)$,*

$$\mathbb{E} \left\| \int_0^T \Phi_t \, dW_t \right\|^2 \leq D^2 \|\Phi\|_{L^2(\Omega; L^2(0, T; \gamma(H, X)))}^2.$$

Since the adapted finite rank step processes are dense in the closed subspace $L^2_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$ of all progressively measurable processes in $L^2(\Omega; L^2(0, T; \gamma(H, X)))$, the estimate of Proposition 3.1 permits the extension of the stochastic integral to processes $\Phi \in L^2_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$. By Doob's maximal inequality the resulting stochastic integral process $t \mapsto \int_0^t \Phi_s \, dW_s$ has a continuous version which satisfies the maximal estimate

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t \Phi_s \, dW_s \right\|^2 \leq 4D^2 \|\Phi\|_{L^2(\Omega; L^2(0, T; \gamma(H, X)))}^2. \quad (3.1)$$

By a standard localization argument the mapping $\Phi \mapsto \int_0^t \Phi_s \, dW_s$ can be further extended to a continuous mapping from $L^0_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$ into $L^0(\Omega; C[0, T; X])$. Here, and in other

instances below, the subscript $\mathcal{P}$ designates the closed subspace of all progressively measurable process in a given space of processes.

In the scalar-valued setting, it is a classical result of Burkholder, with later refinements by Davis and Gundy, that the maximal inequality (3.1) admits an extension with L^2 -norms over Ω replaced by L^p -norms with constants of order $O(\sqrt{p})$ as $p \rightarrow \infty$. The problem of extending the Burkholder–Davis–Gundy inequality to 2-smooth Banach spaces has been considered by many authors [7,8,31–34]. The optimal asymptotic dependence of the constant in these inequalities for $p \rightarrow \infty$ was first obtained by Seidler [35], who proved the following result.

Theorem 3.2 (Seidler). *Let X be a $(2, D)$ -smooth Banach space and let $0 < p < \infty$. For all $\Phi \in L^p_{\mathcal{P}}(\Omega; L^2(0, T; \gamma(H, X)))$, the process $(\int_0^t \Phi_s \, dW_s)_{t \in [0, T]}$ has a continuous version which satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t \Phi_s \, dW_s \right\|^p \leq C_{p,D}^p \|\Phi\|_{L^p(\Omega; L^2(0, T; \gamma(H, X)))}^p$$

where $C_{p,D}$ is a constant only depending on p and D . For $2 \leq p < \infty$, one may take $C_{p,D} = C_D \sqrt{p}$, where C_D is a constant only depending on D .

The proof is based on an extension to 2-smooth Banach spaces of the classical Burkholder–Rosenthal inequality due to Pinelis [21]. Tracking and optimizing constants in this reference one finds that the choice $C_D = 10D$ will do (see [22] for the details).

4. Maximal inequalities for stochastic convolutions

A family $(S(t, s))_{0 \leq s \leq t \leq T}$ of bounded linear operators on a Banach space X is called a C_0 -evolution family indexed by $[0, T]$ if the following conditions are satisfied:

- (1) $S(t, t) = I$ for all $t \in [0, T]$;
- (2) $S(t, r) = S(t, s)S(s, r)$ for all $0 \leq r \leq s \leq t \leq T$;
- (3) the mapping $(t, s) \rightarrow S(t, s)$ is strongly continuous on the set $\{0 \leq s \leq t \leq T\}$.

Under the assumption that X is $(2, D)$ -smooth, for processes $g \in L^0_{\mathcal{P}}(\Omega; L^2(0, T; \gamma(H, X)))$, we consider the stochastic convolution process $(u_t)_{t \in [0, T]}$ defined by

$$u_t := \int_0^t S(t, s) g_s \, dW_s, \quad t \in [0, T].$$

As explained in the Introduction, the nomenclature ‘stochastic convolution’ is justified by the important special case where the evolution family arises from a semigroup of operators.

The remainder of this paper is dedicated to surveying the following two problems:

- to find conditions guaranteeing that u has a continuous version which satisfies the Burkholder type L^p -maximal inequality

$$\mathbb{E} \sup_{t \in [0, T]} \|u_t\|^p \leq C_{p,X}^p \|g\|_{L^p(\Omega; L^2(0, T; \gamma(H, X)))}^p \quad (4.1)$$

- if this is the case, to determine whether the constant $C_{p,X}$ is of order $O(\sqrt{p})$ as $p \rightarrow \infty$.

In (4.1) and in the rest of the paper, we do not distinguish notationally between u and its continuous version. The right-hand side of (4.1) is motivated by Theorem 3.2, which gives (4.1) in the special case of the trivial family $S(t, s) \equiv I$ with $O(\sqrt{p})$ dependence of the constant as $p \rightarrow \infty$.

A number of general remarks can be made at this point.

Remark 4.1. In many applications, the evolution family is generated by a family $(A(t))_{t \in [0, T]}$ of closed operators on X , in the sense made precise in §5b. In this case, the process u can be

interpreted as the mild solution to the stochastic differential equation

$$du_t = A(t)u_t dt + g_t dW_t, \quad u(0) = 0. \quad (4.2)$$

If u is a *strong solution* of (4.2), i.e. if for all $t \in [0, T]$ one has that $t \mapsto A(t)u_t$ belongs to $L^1(0, t; X)$ almost surely and

$$u_t = \int_0^t A(s)u_s ds + \int_0^t g_s dW_s \quad \text{almost surely,} \quad (4.3)$$

(by the stochastic Fubini theorem this happens, e.g. when u_t is $\mathcal{D}(A(t))$ -valued and both u and Au belong to $L^0_{\mathcal{D}}(\Omega; L^1(0, T; X))$), then it is easy to see that u has a continuous version, namely the process defined by the right-hand side of (4.3) once a continuous version of the stochastic integral $\int_0^t g_s dW_s$ has been selected.

Remark 4.2. If u has a version satisfying (4.1) for all $g \in L^p_{\mathcal{D}}(\Omega; L^q(0, T; \gamma(H, X)))$, for certain fixed $0 < p < \infty$ and $1 \leq q \leq \infty$, a standard localization argument shows that for all $g \in L^0_{\mathcal{D}}(\Omega; L^q(0, T; \gamma(H, X)))$ the process u has a continuous version. Moreover, an application of Lenglart's inequality [36, Proposition IV.4.7] implies that (4.1) (with p replaced by r) extends to all exponents $0 < r \leq p$.

For general C_0 -evolution families, and even for C_0 -semigroups, the problem of proving the existence of a continuous version is open even when X is a Hilbert space. In §4a and 4b, we will discuss two techniques to approach this problem: the factorization method of Da Prato, Kwapien and Zabczyk, and the dilation method of Hausenblas and Seidler. Both methods also lead to maximal inequalities. In the case of the factorization method, this inequality is weaker than (4.1); the dilation method gives (4.1) with optimal asymptotic dependence of the constant. In §5, we will see that for C_0 -evolution families of *contractions*, a continuous version always exists and (4.1) holds with optimal asymptotic dependence of the constant.

One of the reasons for insisting on asymptotic $O(\sqrt{p})$ -dependence of the constant is that it implies Gaussian tail estimates. This is an immediate consequence of the special case $\alpha = 2$ of the following elementary lemma.

Lemma 4.3. Let ξ be a non-negative random variable and suppose there exist $\alpha > 0$ and $C \geq 1$ such that $\mathbb{E}\xi^p \leq C^p p^{p/\alpha}$ for all $p \geq \alpha$. Then setting $\sigma^2 = eC^\alpha$ one has

$$\mathbb{P}(\xi \geq r) \leq 3 \exp(-r^\alpha / (\alpha \sigma^2)), \quad r > 0.$$

Proof. By Markov's inequality, $\mathbb{P}(\xi \geq r) \leq r^{-p} \mathbb{E}\xi^p \leq (C/r)^p p^{p/\alpha}$. If $e^{-1}(r/C)^\alpha \geq \alpha$, we can set $p = e^{-1}(r/C)^\alpha$ to obtain $\mathbb{P}(\xi \geq r) \leq e^{-p/\alpha} = \exp(-r^\alpha / (\alpha \sigma^2))$. If $e^{-1}(r/C)^\alpha < \alpha$, then $\mathbb{P}(\xi \geq r) \leq 1 \leq 3e^{-1} \leq 3 \exp(-r^\alpha / (\alpha \sigma^2))$. ■

Indeed, applying the lemma to $\xi = \sup_{t \in [0, T]} \|\int_0^t S(t, s)g_s dW_s\|$ and $\alpha = 2$, we obtain the following general result:

Corollary 4.4. Let $(S(t, s))_{0 \leq s \leq t \leq T}$ be a C_0 -evolution family of contractions on a $(2, D)$ -smooth Banach space X and let $g \in L^\infty_{\mathcal{D}}(\Omega; L^q(0, T; \gamma(H, X)))$ with $1 \leq q \leq \infty$. If the maximal inequality

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t S(t, s)g_s dW_s \right\|^p \leq C^p (\sqrt{p})^p \|g\|_{L^\infty(\Omega; L^q(0, T; \gamma(H, X)))}^p$$

holds for all $2 \leq p < \infty$, where C is a constant independent of p , then the process $(\int_0^t S(t, s)g_s dW_s)_{t \in [0, T]}$ has a continuous version which satisfies the Gaussian tail estimate

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left\| \int_0^t S(t, s)g_s dW_s \right\| \geq r \right) \leq 2 \exp \left(-\frac{r^2}{2\sigma^2} \right), \quad r > 0,$$

where $\sigma^2 = eC^2 \|g\|_{L^\infty(\Omega; L^q(0, T; \gamma(H, X)))}^2$.

This method of getting Gaussian tail estimates gives rather poor bounds on the variance. In §5, we will discuss another method which, when applied to Theorem 5.2, gives a bound that is close to being optimal.

(a) The factorization method

The so-called factorization method was introduced by Da Prato, Kwapień and Zabczyk [37] to prove the existence of a continuous version for stochastic convolutions with C_0 -semigroups defined on a Hilbert space and was extended to C_0 -evolution families by Seidler [38]. It is based on the formula

$$\int_r^t (t-s)^{\alpha-1} (s-r)^{-\alpha} ds = \frac{\pi}{\sin(\pi\alpha)},$$

from which one deduces the following identity, valid for $0 < \alpha < 1/2$:

$$\frac{\pi}{\sin \pi\alpha} \int_0^t S(t,s) g_s dW_s = \int_0^t (t-s)^{\alpha-1} S(t,s) \left(\int_0^s (s-r)^{-\alpha} S(s,r) g_r dW_r \right) dr.$$

For $2 < p < \infty$ and $\frac{1}{p} < \alpha < \frac{1}{2}$, the process $R_\alpha(s) := \int_0^s (s-r)^{-\alpha} S(s,r) g_r dW_r$ belongs to $L^p(0, T; L^p(\Omega; X))$, which we identify with $L^p(\Omega; L^p(0, T; X))$, and then use the fact that the mapping $R_\alpha \mapsto \int_0^t (t-s)^{\alpha-1} S(t,s) R_\alpha(s) ds$ maps the latter space into $L^p(\Omega; C([0, T]; X))$. *Mutatis mutandis* this method extends to the more general setting of 2-smooth Banach spaces. By bookkeeping the norm estimates and tracking constants, and performing a standard localization argument, the following result is obtained.

Theorem 4.5 (Factorization method). *Let $(S(t,s))_{0 \leq s \leq t \leq T}$ be a C_0 -evolution family on a $(2, D)$ -smooth Banach space X and let $2 < q < \infty$. For all $g \in L^0_{\mathcal{D}}(\Omega; L^q(0, T; \gamma(H, X)))$, the process $(\int_0^t S(t,s) g_s dW_s)_{t \in [0, T]}$ has a continuous version. For $g \in L^p_{\mathcal{D}}(\Omega; L^q(0, T; \gamma(H, X)))$ with $0 < p \leq q$, this version satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t S(t,s) g_s dW_s \right\|^p \leq C_{p,q,D,T}^p M^p \|g\|_{L^p(\Omega; L^q(0, T; \gamma(H, X)))}^p,$$

where $M = \sup_{0 \leq s \leq t \leq T} \|S(t,s)\|$. For $p = q$, one may take $C_{p,p,D,T} = DK_p \sqrt{p} T^{\frac{1}{2} - \frac{1}{p}}$ with $\limsup_{p \rightarrow \infty} K_p < \infty$.

It is important to observe that the estimate is phrased in terms of the norm of $L^p(\Omega; L^q(0, T; \gamma(H, X)))$, rather than $L^p(\Omega; L^2(0, T; \gamma(H, X)))$ as in the Burkholder type estimate (4.1). On the other hand, in contrast to the results of §5 where contractivity is required, Theorem 4.5 is applicable to arbitrary C_0 -evolution families.

(b) The dilation method

In this subsection, we discuss an abstract version of a dilation technique due to the Hausenblas and Seidler [14, 39]. In their original formulation for C_0 -contraction semigroups on Hilbert spaces, the key idea is to use the Sz.-Nagy dilation theorem [40] to dilate the semigroup to a unitary C_0 -group $(U(t))_{t \in \mathbb{R}}$ on a larger Hilbert space. Extending g to this larger Hilbert space as well and using the group property to write

$$\int_0^t U(t-s) g_s dW_s = U(t) \int_0^t U(-s) g_s dW_s,$$

the stochastic integral on right-hand side can be estimated by means of Theorem 3.2, or rather, its special case for Hilbert spaces X . This then gives the result. Still in the setting of Hilbert spaces X , the method can be extended *mutatis mutandis* to the situation where $g dW$ is replaced by dM , where M is an arbitrary X -valued continuous local martingale.

There is no obvious way to extend the Hausenblas–Seidler argument to general C_0 -semigroups or to C_0 -evolution families. Moreover, the Sz.-Nagy dilation theorem is a Hilbert space theorem. To overcome both problems, the next definition introduces an abstract dilation framework.

Definition 4.6. A C_0 -evolution family $(S(t, s))_{0 \leq s \leq t \leq T}$ on a Banach space X is said to:

- (1) *admit an invertible dilation on the Banach space Y* , if there exist strongly continuous functions $J : [0, T] \rightarrow \mathcal{L}(X, Y)$ and $Q : [0, T] \rightarrow \mathcal{L}(Y, X)$ such that

$$S(t, s) = Q(t)J(s) \quad \text{for all } 0 \leq s \leq t \leq T.$$

- (2) *admit an approximate invertible dilation on the sequence of Banach spaces $(Y_n)_{n \geq 1}$* , if there exist strongly continuous functions $J_n : [0, T] \rightarrow \mathcal{L}(X, Y_n)$ and $Q_n : [0, T] \rightarrow \mathcal{L}(Y_n, X)$ such that $\sup_{n \geq 1} \sup_{t \in [0, T]} \|J_n(t)\| < \infty$, $\sup_{n \geq 1} \sup_{t \in [0, T]} \|Q_n(t)\| < \infty$, and

$$S(t, s)x = \lim_{n \rightarrow \infty} Q_n(t)J_n(s)x \quad \text{for all } 0 \leq s \leq t \leq T \text{ and } x \in X.$$

Example 4.7. A sufficient condition for the existence of an invertible dilation is that every operator $S(t, s)$ be invertible, in which case we can take $Y = X$, $Q(t) = S(t, 0)$, and $J(s) = S(s, 0)^{-1}$.

Example 4.8. A C_0 -semigroup $(S(t))_{t \geq 0}$ is said to *dilate to a C_0 -group* if there exist a C_0 -group $(U(t))_{t \in \mathbb{R}}$ on a Banach space Y and bounded operators $J \in \mathcal{L}(X, Y)$ and $Q \in \mathcal{L}(Y, X)$ such that $S(t) = QU(t)J$ for all $t \geq 0$. In this case, the operators $Q(t) := QU(t)$ and $J(s) := U(-s)J$ define an invertible dilation in the sense of definition 4.6. In cases of interest, it is often possible to construct group dilations which preserve certain features of interest:

- If $(S(t))_{t \geq 0}$ is a C_0 -semigroup of contractions on a Hilbert space X , then a unitary group dilation exists on a Hilbert space Y . This is the content of the Sz.-Nagy dilation theorem.
- If $(S(t))_{t \geq 0}$ is a C_0 -semigroup of positive contractions on an L^p -space with $1 < p < \infty$, then a group dilation of positive contractions exists on another L^p -space. This is the content of Fendler's theorem [41].
- If the negative generator $-A$ has a bounded H^∞ -calculus on of angle $< \frac{1}{2}\pi$ on any Banach space X , then a group dilation exists on the Banach space $\gamma(L^2(\mathbb{R}), X)$. This result is essentially due to [42] and stated in its present form in [27]. If X is 2-smooth, then so is $\gamma(L^2(\mathbb{R}), X)$.

Further dilation results can be found in [43–45]. As far as we know, no extensions of these results are known for evolution families. We also do not know whether every C_0 -semigroup has an (approximate) invertible dilation in the sense of definition 4.6, or whether in the cases that such a dilation exists there also exists a group dilation. Here, it is important that the space Y should enjoy similar geometric properties to X , such as Hilbertianity, 2-smoothness, or UMD.

Example 4.9. We now give an example where an approximate dilation can be constructed. Let X and X_1 be Hilbert spaces, with X_1 continuously and densely embedded in X , and let $A \in C([0, T]; \mathcal{L}(X_1, X))$ be such that there exist constants $c > 0$ and $C \geq 0$ such that

$$c\|x\|_{X_1} \leq \|x\|_X + \|A(t)x\|_X \leq C\|x\|_{X_1}, \quad t \in [0, T], \quad x \in X_1.$$

Suppose further that for all $t \in [0, T]$ the operator $A(t)$ generates a C_0 -contraction semigroup $(S^t(s))_{s \geq 0}$ and that for all $s_0, s_1, t_0, t_1 \in [0, T]$ the operators $S^{t_0}(s_0)$ and $S^{t_1}(s_1)$ and their adjoints commute. Then A generates a C_0 -evolution family $(S(t, s))_{0 \leq s \leq t \leq T}$ of contractions on X in the sense of [46, Theorem 5.3.1] or [47, Theorem 4.4.1].

Setting $t_k^n = kT/n$ and $I_k^n = [t_k^n, t_{k+1}^n)$ (with endpoint included if $k = n - 1$), from the proof of the theorems just cited one infers $S(t, s)x = \lim_{n \rightarrow \infty} S_n(t, s)x$ for all $0 \leq s \leq t \leq T$ and $x \in X$, where

$$S_n(t, s) = \begin{cases} S_k^n(t - s), & \text{if } s, t \in I_k^n; \\ S_\ell^n(t - t_\ell^n) \left(\prod_{j=k+1}^{\ell-1} S_j^n(T/n) \right) S_k^n(s - t_k^n), & \text{if } s \in I_k^n \text{ and } t \in I_\ell^n \text{ with } k \leq \ell. \end{cases}$$

It is easy to check that $(S_n(t, s))_{0 \leq s \leq t \leq T}$ is a C_0 -evolution family of contractions. By the assumption that the contraction semigroups $(S^t(s))_{s \geq 0}$ commute among themselves and with their adjoints, it follows from [40, Proposition 9.2] that there exist a Hilbert space Y and contractions $J \in \mathcal{L}(X, Y)$

and $Q \in \mathcal{L}(Y, X)$, as well as commuting isometric C_0 -groups $(U^t(s))_{s \in \mathbb{R}}$ on Y such that for all $s_1, \dots, s_n$ and $t_1, \dots, t_n \in [0, T]$ we have

$$S^{t_1}(s_1) \dots S^{t_n}(s_n) = QU^{t_1}(s_1) \dots U^{t_n}(s_n)J. \quad (4.4)$$

For $0 \leq s \leq t \leq T$, we define the operators $U_n(t, s)$ by

$$U_n(t, s) := \begin{cases} U_k^n(t-s), & \text{if } s, t \in I_k^n; \\ U_\ell^n(t-t_\ell^n) \left(\prod_{j=k+1}^{\ell-1} U_j^n(T/n) \right) U_k^n(s-t_k^n), & \text{if } s \in I_k^n \text{ and } t \in I_\ell^n \text{ with } k \leq \ell. \end{cases}$$

Then $(U_n(t, s))_{0 \leq s \leq t \leq T}$ is C_0 -evolution family of invertible operators, and by (4.4) we have

$$S_n(t, s) = QU_n(t, s)J, \quad 0 \leq s \leq t \leq T.$$

It follows that there exists an approximate invertible dilation given by $Q_n(t) = QU_n(t, 0)$ and $J_n(s) = U_n(s, 0)^{-1}J$.

The next theorem extends the Hausenblas–Seidler dilation theorem to evolution families on 2-smooth Banach spaces. By example 4.8, it is applicable to C_0 -semigroups on 2-smooth Banach spaces whose negative generator has a bounded H^∞ -calculus of angle $< \frac{1}{2}\pi$. The resulting maximal inequality, with $O(\sqrt{p})$ dependence of the constant as $p \rightarrow \infty$, was obtained independently in [13, 35]. Some of the maximal estimates in the latter paper are valid for a class of processes strictly larger than $L^0_{\mathcal{F}}(\Omega; L^2(0, T; \gamma(H, X)))$, but with best-known constant of order $O(p)$ instead of $O(\sqrt{p})$.

Theorem 4.10 (Dilation). *Let $(S(t, s))_{0 \leq s \leq t \leq T}$ be a C_0 -evolution family on a $(2, D)$ -smooth Banach space X which admits an approximate invertible dilation on a sequence of $(2, D)$ -smooth Banach spaces $(Y_n)_{n \geq 1}$. For all $0 < p < \infty$ and $g \in L^p_{\mathcal{F}}(\Omega; L^2(0, T; \gamma(H, X)))$, the process $(\int_0^t S(t, s)g_s dW_s)_{t \geq 0}$ has a continuous version which satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t S(t, s)g_s dW_s \right\|^p \leq C_{p,D}^p C_J^p C_Q^p \|g\|_{L^p(\Omega; L^2(0, T; \gamma(H, X)))}^p,$$

where $C_J = \sup_{n \geq 1} \sup_{t \in [0, T]} \|J_n(t)\|$ and $C_Q = \sup_{n \geq 1} \sup_{t \in [0, T]} \|Q_n(t)\|$. For $2 \leq p < \infty$, one may take $C_{p,D} = 10D\sqrt{p}$.

Proof. By Remark 4.2, it suffices to consider the case $2 \leq p < \infty$, and by a limiting argument it even suffices to consider the case $2 < p < \infty$.

Let us first assume that $g \in L^p_{\mathcal{F}}(\Omega; L^p(0, T; \gamma(H, X)))$. For such processes, Theorem 4.5 implies the existence of a continuous version. By monotone convergence, it suffices to prove the maximal estimate with suprema taken over finite sets $\pi \subseteq [0, T]$. For $t \in \pi$, write

$$u_t = \int_0^t S(t, s)g_s dW_s = \lim_{n \rightarrow \infty} \int_0^t Q_n(t)J_n(s)g_s dW_s = \lim_{n \rightarrow \infty} Q_n(t) \int_0^t J_n(s)g_s dW_s,$$

where the limit is taken in $L^p(\Omega; X)$ by dominated convergence. Using that π is finite, by taking suitable subsequences we may assume the above limit holds pointwise on $\pi \times \Omega_0$, where $\Omega_0 \subseteq \Omega$ is a measurable set with $\mathbb{P}(\Omega_0) = 1$. Therefore, for all $t \in \pi$, pointwise on Ω_0 , we have

$$\|u_t\|_X \leq C_Q \liminf_{n \rightarrow \infty} \left\| \int_0^t J_n(s)g_s dW_s \right\|_{Y_n}.$$

Taking the supremum over $t \in \pi$, upon taking $L^p(\Omega)$ -norms, we obtain

$$\begin{aligned}
 \mathbb{E} \sup_{t \in \pi} \|u_t\|^p &\leq C_Q^p \mathbb{E} \sup_{t \in \pi} \liminf_{n \rightarrow \infty} \left\| \int_0^t J_n(s) g_s \, dW_s \right\|_{Y_n}^p \\
 &\leq C_Q^p \mathbb{E} \liminf_{n \rightarrow \infty} \sup_{t \in \pi} \left\| \int_0^t J_n(s) g_s \, dW_s \right\|_{Y_n}^p \\
 &\leq C_Q^p \liminf_{n \rightarrow \infty} \mathbb{E} \sup_{t \in \pi} \left\| \int_0^t J_n(s) g_s \, dW_s \right\|_{Y_n}^p \quad \text{by Fatou's Lemma} \\
 &\leq (10D\sqrt{p}C_Q)^p \liminf_{n \rightarrow \infty} \|J_n g\|_{L^p(\Omega; L^2(0, T; \gamma(H, Y_n)))}^p \quad \text{by Theorem 3.2} \\
 &\leq (10D\sqrt{p}C_Q)^p \|g\|_{L^p(\Omega; L^2(0, T; \gamma(H, X)))}^p.
 \end{aligned}$$

This gives the result for processes $g \in L^p_{\mathcal{D}}(\Omega; L^p(0, T; \gamma(H, X)))$. The general case of processes $g \in L^p_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$ follows from it by approximation. ■

Remarkably, the method of dilations has been used [48,49] to derive maximal inequalities also for the case of stochastic Volterra equations on Hilbert spaces.

5. The contractive case

Up to this point, we have considered general C_0 -evolution families. In the present section, we take a closer look at the special case of C_0 -evolution families of contractions. By a standard rescaling argument, the results of this section extend to the situation where, for some $\lambda \geq 0$, one has

$$\|S(t, s)\| \leq e^{\lambda(t-s)}, \quad 0 \leq s \leq t \leq T.$$

An additional term $e^{\lambda T}$ has then to be added on the right-hand side of the estimates.

(a) The main result

We begin with a general result on the existence of continuous versions. It extends a result stated in [50] for Hilbert spaces and continuous square integrable martingales, to $(2, D)$ -smooth Banach spaces and Brownian motion. Replacing Hilbertian L^2 -estimates by [22, Lemma 2.2] and Proposition 3.1, the original argument can be generalized and leads to the following result with D^4 instead of D^2 ; an additional approximation argument permits the passage to D^2 .

Proposition 5.1. *Let $(S(t, s))_{0 \leq s \leq t \leq T}$ be a C_0 -evolution family of contractions on a $(2, D)$ -smooth Banach space X . For all $g \in L^0_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$, the process $(\int_0^t S(t, s) g_s \, dW_s)_{t \in [0, T]}$ has a continuous version. If $g \in L^2_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$, then it satisfies the following tail estimate for all $r > 0$:*

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left\| \int_0^t S(t, s) g_s \, dW_s \right\| \geq r \right) \leq \frac{D^2}{r^2} \|g\|_{L^2(\Omega; L^2(0, T; \gamma(H, X)))}^2.$$

By combining the discretization technique used in the proof of this proposition with a version of a theorem of Pinelis [21] used in the proof of Theorem 3.2, the following L^p -maximal inequality has been recently obtained in [22].

Theorem 5.2. *Let $(S(t, s))_{0 \leq s \leq t \leq T}$ be a C_0 -evolution family of contractions on a $(2, D)$ -smooth Banach space X and let $0 < p < \infty$. For all $g \in L^p_{\mathcal{D}}(\Omega; L^2(0, T; \gamma(H, X)))$, the process $(\int_0^t S(t, s) g_s \, dW_s)_{t \in [0, T]}$ has a continuous version which satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^t S(t, s) g_s \, dW_s \right\|^p \leq C_{p,D}^p \|g\|_{L^p(\Omega; L^2(0, T; \gamma(H, X)))}^p.$$

For $2 \leq p < \infty$, one may take $C_{p,D} = 10D\sqrt{p}$.

Theorem 5.2 is in some sense definitive, in that it applies to arbitrary C_0 -evolution families of contractions and gives the correct order $O(\sqrt{p})$ of the constant; as such it is new even for Hilbert spaces X . It is also new for C_0 -semigroups of contractions in 2-smooth Banach spaces.

Theorem 5.2 has a long history with contributions by many authors. Here, we will only review the semigroup approach; L^2 -maximal inequalities for monotone stochastic evolution equations with random coefficients in Hilbert spaces are older and go back to [51] and [52]. For an exposition and further references to the literature the reader is referred to [53]. The first author to use semigroup methods to derive L^2 -maximal inequalities is Kotelenetz [50] who obtained path continuity in the more general situation where term $g dW$ is replaced by an arbitrary continuous square integrable X -valued martingale. This paper also contains a weak type estimate similar to the one of Proposition 5.1. Still for Hilbert spaces and $p = 2$, (4.1) was first proved in [54,55] using Itô's formula applied to the C^2 -function $x \mapsto \|x\|^2$ under further assumptions on the evolution family. For C_0 -contraction semigroups, these results were extended to exponents $2 \leq p < \infty$ by Tubaro [56], who applied Itô's formula to the mapping $x \mapsto \|x\|^p$ which for Hilbert is twice continuously Fréchet differentiable. The extension to exponents $0 < p < 2$ was subsequently obtained by Ichikawa [57]. Tubaro's method of proof was revisited by Brzeźniak & Peszat [58], who extended it to Banach spaces X with the property that for some $2 \leq p < \infty$ the mapping $x \mapsto \|x\|^p$ is twice continuously Fréchet differentiable and the first and second Fréchet derivatives are bounded by constant multiples of $\|x\|^{p-1}$ and $\|x\|^{p-2}$, respectively. Spaces with this property are 2-smooth and include $L^q(\mu)$ for $2 \leq q \leq p < \infty$ and the Besov spaces $B_{q,r}^s(\mathbb{R}^d)$ for $2 \leq q \leq r \leq p < \infty$. In the converse direction, it is not true that all 2-smooth spaces satisfy the twice differentiability condition; an abstract counterexample follows from [59, Theorem 3.9] (see also [60, Example 1.1]). In the Besov scale, the twice differentiability condition is unclear if $2 \leq r < q < \infty$, even though the space $B_{q,r}^s(\mathbb{R}^d)$ is 2-smooth in that case, too. The approach based on Itô's formula extends to evolution families, but has the general drawback that it does not seem to give the optimal growth rate $O(\sqrt{p})$ of the constant as expected from the Burkholder–Davis–Gundy inequalities as $p \rightarrow \infty$. As discussed in §4b, for C_0 -contraction semigroups on Hilbert spaces a new proof of the maximal inequality for exponents $0 < p < \infty$ giving growth of order $O(\sqrt{p})$ was obtained by Hausenblas & Seidler [39].

The approach via Itô's formula was once more revisited in [60], where it was finally extended to arbitrary 2-smooth Banach spaces by exploiting the fact that, in such spaces, for $2 \leq p < \infty$ the mapping $x \mapsto \|x\|^p$ is once continuously Fréchet differentiable with a Lipschitz continuous derivative. As it turns out, this already suffices to prove a version of the Itô formula with the help of which the argument can be completed. This approach, however, does not seem to give the optimal p -dependence of the constant as $p \rightarrow \infty$.

(b) The Itô formula approach revisited

The aim of the present subsection is to present the Itô formula approach to maximal estimates for stochastic convolutions. In comparison with Theorem 5.2, it does not lead to new results (in fact we need stronger assumptions on the evolution family and obtain non-optimal asymptotic dependence of the constant), but this approach has the merit that it can be extended to *random* C_0 -evolution families of contractions. To the best of our knowledge, for this setting no maximal L^p -estimates of the form (4.1) in 2-smooth spaces are available in the literature. For stochastic evolution equations with random coefficients in Hilbert spaces subject to monotonicity conditions, L^2 -maximal inequalities go back to [51,52]; for an exposition and further reference see [53]. Some extensions to the case $p \neq 2$ have been obtained recently in [61].

In order to avoid technicalities that would obscure the line of argument we present our main results for non-random evolution families and indicate the changes that have to be made in the Ω -dependent case in Remark 5.8. Rather than discussing the maximal L^p -inequality in [60] in detail, we will provide a detailed proof of a Gaussian tail estimate. The rationale of this choice is that this estimate cannot be deduced (e.g. via Lemma 4.3) from the result of [60] due to the fact that it does not provide the correct order $O(\sqrt{p})$ of the constant. The result presented here is new, in that

it generalizes [58, Theorem 1.2] to arbitrary 2-smooth Banach spaces. A further novel feature of our result is that it gives an improved bound on the variance.

As in [16,58] the idea is to apply Itô's formula to $h_\lambda : X \rightarrow [0, \infty)$ given by

$$h_\lambda(x) := (1 + \lambda\|x\|^2)^{1/2}, \quad x \in X.$$

The function h_λ is Fréchet differentiable and

$$h'_\lambda(x) = \frac{\lambda q'(x)}{(1 + \lambda\|x\|^2)^{1/2}},$$

where $q(x) := \|x\|^2$ is known to be Fréchet differentiable (see [60]) with $q'(0) = 0$ and

$$q'(x) = 2\|x\|n_x, \quad x \neq 0, \quad (5.1)$$

where n_x is the Fréchet derivative of $\|\cdot\|$ at $x \neq 0$ and satisfies $\|n_x\| = 1$. Although h_λ is generally not C^2 , the following 'Itô inequality' holds:

Theorem 5.3. *Let X be a 2-smooth Banach space and let $(a_t)_{t \in [0, T]}$ and $(g_t)_{t \in [0, T]}$ be processes in $L^0_{\mathcal{P}}(\Omega; L^1(0, T; X))$ and $L^0_{\mathcal{P}}(\Omega; L^2(0, T; \gamma(H, X)))$, respectively. Fix $x \in X$ and let the process $(\xi_t)_{t \in [0, T]}$ be given by*

$$\xi_t := x + \int_0^t a_s \, ds + \int_0^t g_s \, dW_s.$$

Then, almost surely, for all $t \in [0, T]$, we have

$$h_\lambda(\xi_t) \leq h_\lambda(x) + \int_0^t \langle a_s, h'_\lambda(\xi_s) \rangle \, ds + \int_0^t h'_\lambda(\xi_s) \circ g_s \, dW_s + \frac{1}{2} D^2 \lambda \|g\|_{L^2(0, t; \gamma(H, X))}^2. \quad (5.2)$$

Proof. We proceed in three steps.

Step 1. First suppose that a and the operators in the range of g take values in a fixed finite-dimensional subspace Y of X . Then ξ also takes its values in Y . Now, we regularize the norm as in [21, Lemma 2.2]. Let μ be a centred Gaussian measure with support $\text{supp}(\mu) = Y$. Fix $\varepsilon > 0$ and let $q_\varepsilon : Y \rightarrow \mathbb{R}$ be given by

$$q_\varepsilon(x) := \int_Y \|x - \varepsilon y\|^2 \, d\mu(y).$$

Then by [21, Lemma 2.2], the function q_ε has Fréchet derivatives of all orders, and

$$|q_\varepsilon(x)^{1/2} - \|x\|| \leq \varepsilon, \quad \|(q_\varepsilon^{1/2})'(x)\| \leq 1, \quad q_\varepsilon''(x)(v, v) \leq 2D^2\|v\|^2. \quad (5.3)$$

Moreover, $q'_\varepsilon(x) = \int_Y q'(x - \varepsilon y) \, d\mu(y)$ and from (5.1) and the dominated convergence theorem we obtain $q'_\varepsilon(x) \rightarrow q'(x)$ as $\varepsilon \downarrow 0$. Writing $q_\varepsilon = q_\varepsilon^{1/2} q_\varepsilon^{1/2}$, differentiation by the product rule gives

$$\|q'_\varepsilon(x)\| = 2\|(q_\varepsilon^{1/2})'(x)\| q_\varepsilon^{1/2}(x) \leq 2q_\varepsilon^{1/2}(x). \quad (5.4)$$

It follows that the function $h_{\lambda, \varepsilon} : Y \rightarrow \mathbb{R}$ given by

$$h_{\lambda, \varepsilon}(x) := (1 + \lambda q_\varepsilon(x))^{1/2}, \quad x \in Y,$$

has Fréchet derivatives of all orders, and

$$h'_{\lambda, \varepsilon}(x) = \frac{\lambda q'_\varepsilon(x)}{2(1 + \lambda q_\varepsilon(x))^{1/2}}, \quad h''_{\lambda, \varepsilon}(x)(y, y) = \frac{\lambda q''_\varepsilon(x)(y, y)}{2(1 + \lambda q_\varepsilon(x))^{1/2}} - \frac{\lambda^2 \langle y, q'_\varepsilon(x) \rangle^2}{4(1 + \lambda q_\varepsilon(x))^{3/2}}.$$

Therefore, by (5.3) and (5.4), for all $x \in Y$, one has $h_{\lambda, \varepsilon}(x) \rightarrow h_\lambda(x)$, $h'_{\lambda, \varepsilon}(x) \rightarrow h'_\lambda(x)$ as $\varepsilon \downarrow 0$, and

$$\|h'_{\lambda, \varepsilon}(x)\| \leq \sqrt{\lambda}, \quad h''_{\lambda, \varepsilon}(x)(y, y) \leq D^2 \lambda \|y\|^2. \quad (5.5)$$

Step 2. By the Itô formula,

$$h_{\lambda, \varepsilon}(\xi_t) = h_{\lambda, \varepsilon}(x) + \int_0^t \langle a_s, h'_{\lambda, \varepsilon}(\xi_s) \rangle \, ds + \int_0^t h'_{\lambda, \varepsilon}(\xi_s) \circ g_s \, dW_s + \frac{1}{2} \int_0^t h''_{\lambda, \varepsilon}(\xi_s)(g_s, g_s) \, ds. \quad (5.6)$$

Since

$$\frac{1}{2} \int_0^t h''_{\lambda,\varepsilon}(\xi_s)(g_s, g_s) \, ds \leq \frac{1}{2} D^2 \lambda \|g\|_{L^2(0,t;\gamma(H,X))}^2 \quad \text{almost surely,} \quad (5.7)$$

this proves (5.2) with $h_{\lambda,\varepsilon}$ instead of h_λ .

It remains to let $\varepsilon \downarrow 0$ in each of the terms in (5.6), except the last one which is estimated using (5.7). By path-continuity of the integrals it suffices to prove convergence for every fixed $t \in [0, T]$.

Convergence of the first two terms in (5.6) is clear from the preliminaries in Step 1. For the third and fourth terms, we can apply the pointwise convergence and the dominated convergence theorem (using the bound (5.5)) to obtain $\langle a_s, h'_{\lambda,\varepsilon}(\xi) \rangle \rightarrow \langle a_s, h'_\lambda(\xi) \rangle$ in $L^1(0, t)$ almost surely and $h'_{\lambda,\varepsilon}(\xi) \circ g \rightarrow h'_\lambda(\xi) \circ g$ in $L^2(0, t; H)$ almost surely to obtain the required convergence. This completes the proof in the finite-dimensional case.

Step 3. In the general case, let $(a_n)_{n \geq 1}$ be a sequence of simple functions and $(g_n)_{n \geq 1}$ be a sequence of finite rank adapted step processes such that $a_n \rightarrow a$ in $L^1(0, t; X)$ and $b_n \rightarrow b$ in $L^2(0, t; \gamma(H, X))$ almost surely. Let $\xi_n(t) := x + \int_0^t a_{n,s} \, ds + \int_0^t g_n \, dW$. Then $\xi_n \rightarrow \xi$ in $L^0(\Omega; C([0, t]; X))$, and by passing to a subsequence we may suppose that $\xi_n \rightarrow \xi$ in $C([0, t]; X)$ almost surely. By Step 1, (5.2) holds with (a, g, ξ) replaced by (a_n, g_n, ξ_n) . Since h'_λ is uniformly bounded and Lipschitz with constant $D^2 \lambda$ (this follows from the second estimate in (5.5) and letting $\varepsilon \downarrow 0$), by dominated convergence we obtain $\langle a_n, h'_\lambda(\xi_n) \rangle \rightarrow \langle a, h'_\lambda(\xi) \rangle$ in $L^1(0, t; X)$ almost surely and $h'_\lambda(\xi_{n,s}) \circ g_{n,s} \rightarrow h'_\lambda(\xi_s) \circ g_s$. Letting $n \rightarrow \infty$, we obtain (5.2) for (a, g, ξ) . ■

For the remainder of the paper, we assume that the following hypothesis is satisfied.

Hypothesis 5.4. $(S(t, s))_{0 \leq s \leq t \leq T}$ is a C_0 -evolution family of contractions and $(A(t))_{t \in [0, T]}$ is a family of closed operators, acting on the same Banach space X . They enjoy the following properties:

- (1) For all $t \in [0, T]$, we have $(0, \infty) \subseteq \varrho(A(t))$ and there exist constant $M \geq 1$ such that

$$\|\lambda(\lambda - A(t))^{-1}\| \leq M, \quad t \in [0, T], \lambda > 0.$$

- (2) For all $t \in [0, T]$ and $\lambda \in (0, \infty)$, we have

$$\sup_{0 \leq s \leq t \leq T} \|A(t)S(t, s)R(\lambda, A(s))\| < \infty.$$

- (3) For all $s \in [0, T]$ and $x \in \mathcal{D}(A(s))$, we have $S(\cdot, s)x \in W^{1,1}(s, T; X)$ and, for almost all $t \in [s, T]$,

$$\frac{d}{dt} S(t, s)x = A(t)S(t, s)x.$$

Remark 5.5. It is folklore in the theory of evolution families that if hypothesis 5.4 holds and each operator $A(t)$ is the generator of a C_0 -semigroup, then (1) holds with $M = 1$.

Condition (1) means that the operators $-A(t)$ are sectorial, uniformly with respect to $t \in [0, T]$. Condition (2) expresses that $S(t, s)$ maps $\mathcal{D}(A(s))$ into $\mathcal{D}(A(t))$ with control on the norms uniformly with respect to $0 \leq s \leq t \leq T$. Condition (3) connects the operators $A(t)$ with $S(t, s)$ in the same way as a generator is connected to a semigroup of operators. These conditions are satisfied in many applications (e.g. [46, 47, 62–65]).

We are now ready to state the main result of this section. Under the additional assumption of hypothesis 5.4, it provides another proof of the Gaussian tail estimate that can be obtained by combining Theorem 5.2 with Corollary 4.4. The bound on the variance σ^2 obtained from that argument, namely $100eD^2\|g\|_\infty^2$, is improved here to $2D^2M^2\|g\|_\infty^2$, where M is the constant in hypothesis 5.4(1). By Remark 5.5 have $M = 1$ in the case of a C_0 -evolution family of contractions.

Theorem 5.6 (Gaussian tail estimate). Let X be a $(2, D)$ -smooth Banach space and suppose that $(S(t, s))_{0 \leq s \leq t \leq T}$ is a C_0 -evolution family of contractions satisfying hypothesis 5.4. For all $g \in$

$L^\infty(\Omega; L^2(0, T; \gamma(H, X)))$, the process $(\int_0^t S(t, s) g_s \, dW_s)_{t \in [0, T]}$ has a continuous version which satisfies

$$\mathbb{P} \left(\sup_{t \in [0, T]} \left\| \int_0^t S(t, s) g_s \, dW_s \right\| \geq r \right) \leq 3 \exp \left(-\frac{r^2}{2\sigma^2} \right),$$

for all $r > 0$, where $\sigma^2 = 2D^2 M^2 \|g\|_{L^\infty(\Omega; L^2(0, T; \gamma(H, X)))}^2$.

Proof. The main idea is that Theorem 5.3 provides the right estimate to generalize the proof of Brzeźniak & Peszat's [58, Theorem 1.2]. The proof will use some additional facts from stochastic analysis which are all standard and can be found in [36, 66].

Step 1. Let us first assume that $g \in L^\infty(\Omega; L^2(0, T; \gamma(H, X)))$ is such that for every $t \in [0, T]$, we have $g_t \in \mathcal{D}(A(t))$ and $t \mapsto A(t)g_t$ belongs to $L^2(0, T; \gamma(H, X))$ almost surely. Under this assumption, we claim that u is a strong solution, i.e. almost surely we have

$$u_t = \int_0^t A(s)u_s \, ds + \int_0^t g_s \, dW_s, \quad t \in [0, T]. \quad (5.8)$$

This means that the assumptions of Theorem 5.3 are satisfied with $a_t = A(t)u_t$. In order to prove (5.8), we set $u_t := \int_0^t S(t, s)g_s \, dW_s$. Then

$$A(t)u_t = \int_0^t A(t)S(t, s)g_s \, dW_s,$$

almost surely, since $A(t)S(t, s)R(1, A(s))$ is uniformly bounded by part (2) of the hypothesis. A standard argument involving the stochastic Fubini theorem and the formula

$$\int_r^t A(s)S(s, r)x \, ds = S(t, r)x - x, \quad x \in \mathcal{D}(A(r)),$$

(which follows from part (3) of the hypothesis) implies that for all $t \in [0, T]$ the identity (5.8) holds almost surely. By path continuity, almost surely the identity holds for all $t \in [0, T]$. This concludes the proof of the claim.

Step 2. Since $t \mapsto h_\lambda(S(t, s)x) = (1 + \lambda \|S(t, s)x\|^2)^{1/2}$ is non-increasing by the contractivity of $S(t, s)$, for all $x \in \mathcal{D}(A(t))$, we have

$$\langle A(t)S(t, s)x, h'_\lambda(S(t, s)x) \rangle = \frac{d}{dt} h_\lambda(S(t, s)x) \leq 0.$$

Therefore, setting $t = s$, we obtain $\langle A(s)x, h'_\lambda(x) \rangle \leq 0$ for almost all $s \in [0, T]$. Hence, by Theorem 5.3 applied with $a_s = A(s)u_s$ and $x = 0$, and noting that $h_\lambda(0) = 1$,

$$h_\lambda(u_t) \leq 1 + \int_0^t h'_\lambda(u_s) \circ g_s \, dW_s + \frac{1}{2} \lambda D^2 \|g\|_{L^2(0, t; \gamma(H, X))}^2. \quad (5.9)$$

Below we will several times use that $\|h'_\lambda(x)\| \leq \lambda^{1/2}$ (see (5.5)).

The quadratic variation of the process defined by $N_t := \int_0^t h'_\lambda(u_s) \circ g_s \, dW_s$ is given by $[N]_t = \int_0^t \|h'_\lambda(u_s) \circ g_s\|_H^2 \, ds$. Therefore, the process defined by $Z_t := e^{N_t - \frac{1}{2}[N]_t}$ is a local martingale by Itô's formula and

$$Z_t = 1 + \int_0^t Z_s h'_\lambda(u_s) \circ g_s \, dW_s.$$

Since Z is non-negative, it is a supermartingale and therefore $\mathbb{E}(Z_t) \leq 1$. Since $[N]_t \leq \lambda \|g\|_{L^\infty(\Omega; L^2(0, T; \gamma(H, X)))}^2$ almost surely, it is standard to check Z is actually a martingale with

$\mathbb{E}(Z_t) = \mathbb{E}(Z_0) = 1$ for all $t \in [0, T]$ (this follows for instance from Novikov's condition). We can rewrite (5.9) in the form (using that $D \geq 1$)

$$h_\lambda(u_t) \leq 1 + \log Z_t + \frac{1}{2} \int_0^t \|h'_\lambda(u_s) \circ g_s\|_H^2 ds + \frac{1}{2} \lambda D^2 \|g\|_{L^2(0,t;\gamma(H,X))}^2 \leq 1 + \log Z_t + \lambda C_g,$$

where $C_g := D^2 \|g\|_{L^\infty(\Omega; L^2(0,T;\gamma(H,X)))}^2$. Setting $\phi_\lambda(r) = (1 + \lambda r^2)^{1/2}$, Doob's inequality gives

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in [0,T]} \|u_t\| \geq r \right) &= \mathbb{P} \left(\sup_{t \in [0,T]} h_\lambda(u_t) \geq \phi_\lambda(r) \right) \\ &= \mathbb{P} \left(\sup_{t \in [0,T]} \log Z_t \geq \phi_\lambda(r) - 1 - \lambda C_g \right) \\ &= \mathbb{P} \left(\sup_{t \in [0,T]} Z_t \geq \exp(\phi_\lambda(r) - 1 - \lambda C_g) \right) \\ &\leq \exp(1 + \lambda C_g - \phi_\lambda(r)) \mathbb{E} Z_T = \exp(1 + \lambda C_g - \phi_\lambda(r)). \end{aligned}$$

If $r^2 \geq 2C_g$, choose $\lambda > 0$ so that $1 + \lambda r^2 = r^4/(4C_g^2)$. Then $\phi_\lambda(r) = r^2/(2C_g)$ and

$$\mathbb{P} \left(\sup_{t \in [0,T]} \|u_t\| \geq r \right) \leq \exp \left(1 + \frac{r^2}{4C_g^2} C_g - \frac{r^2}{2C_g} \right) = \exp \left(1 - \frac{r^2}{4C_g} \right) \leq 3 \exp \left(-\frac{r^2}{4C_g} \right).$$

If $0 < r^2 < 2C_g$, we have the trivial inequality

$$\mathbb{P} \left(\sup_{t \in [0,T]} \|u_t\| \geq r \right) \leq 1 \leq 3e^{-1/2} \leq 3 \exp \left(-\frac{r^2}{4C_g} \right).$$

This proves the result under the additional assumption on g made at the beginning of Step 1.

Step 3. In the general case set $g_t^{(n)} := nR(n, A(t))g_t$ for $n \geq 1$. For all $t \in [0, T]$ and $\omega \in \Omega$, we have $\|g_t^{(n)}(\omega)\| \leq M\|g_t(\omega)\|$ and consequently $C_{g^{(n)}} \leq M^2 C_g$ by sectoriality. It follows that $g^{(n)} \rightarrow g$ in $L^2(\Omega; L^2(0, T; \gamma(H, X)))$. Therefore, using any of the known maximal tail or L^p -estimates (e.g. Proposition 5.1), we infer that the corresponding stochastic convolutions satisfy $u^{(n)} \rightarrow u$ in $L^0(\Omega; C([0, T]; X))$. This implies the tail estimate in the general case. ■

Remark 5.7 (L^p -bounds). A variant of the Itô inequality of Theorem 5.3 can be proven for $\|x\|^p$ with $p \geq 2$. Then, in the same way as [60, Theorem 1.2] (due to the time dependence in A some modifications are required in the approximation argument which are similar to the ones in the proof of Theorem 5.6), it is possible to recover the conclusion of Theorem 5.2. Tracking the constant $C_{p,D}$, this proof does not seem to give the correct order $O(\sqrt{p})$ as $p \rightarrow \infty$, however.

Remark 5.8 (Random evolution families). We now indicate how Theorem 5.6 and the result pointed at in Remark 5.7 can be generalized to *random* evolution families. To make this notion precise, we assume that for all $\omega \in \Omega$ a family $(A(t, \omega))_{t \in [0,T]}$ of closed operators on X is given, as well as a C_0 -evolution family $(S(t, s, \omega))_{0 \leq s \leq t \leq T}$ satisfying hypothesis 5.4, with estimates uniform in $\omega \in \Omega$. We furthermore assume that for all $0 \leq s \leq t \leq T$ and $x \in X$, the random variable $S(t, s, \cdot)x$ is strongly $\mathcal{F}_t$ -measurable. In what follows, we will suppress the ω -dependence from our notation whenever it is convenient.

Under these assumptions, it is not even clear how the problem should be stated to begin with, because the stochastic convolution integral $\int_0^t S(t, s)g_s dW_s$ is not well defined in general. The reason is that the random variables $S(t, s)x$ are assumed to be $\mathcal{F}_t$ -measurable rather than $\mathcal{F}_s$ -measurable, and therefore the integrand will not be progressively measurable in general.

To overcome this problem one notes that in the Ω -independent case, for sufficiently regular g one has the almost sure identity

$$u_t := \int_0^t S(t,s)g_s \, dW_s = S(t,0) \int_0^t g_r \, dW_r - \int_0^t S(t,s)A(s) \left(\int_s^t g_r \, dW_r \right) ds, \quad t \in [0, T]. \quad (5.10)$$

Following [67], in the Ω -dependent case, we *define* the process $(u_t)_{t \in [0, T]}$ to be given by the expression on the right-hand side of (5.10) and refer to it as the *pathwise mild solution* of the problem $du_t = A(t)u_t \, dt + g_t \, dW_t$. This formula has the merit of avoiding adaptedness issues and $(u_t)_{t \in [0, T]}$ can be shown to be progressively measurable. Pathwise mild solutions were extensively studied in the parabolic setting in [67]; under the assumptions spelled out below the general case can be treated along similar lines. The problem of extending Theorem 5.2 and the result mentioned in Remark 5.7 can now be formulated as proving suitable Gaussian tail estimates and L^p -bounds for the random variable

$$u^* := \sup_{t \in [0, T]} \|u_t\|,$$

with u_t given by the right-hand side of (5.10).

The proof of Theorem 5.6 can be repeated as soon as a suitable analogue of (5.8) is available. This is the case under the following additional technical assumptions. We assume that there exist continuous and dense embeddings of Banach spaces $Y_2 \hookrightarrow Y_1 \hookrightarrow Y_0 = X$ such that pointwise on Ω the following conditions are satisfied almost surely:

- (1) for all $0 \leq s \leq t \leq T$, we have $S(t,s)Y_1 \subseteq Y_1$;
- (2) for all $t \in [0, T]$, we have $Y_1 \subseteq \mathcal{D}(A(t))$ and, for $i \in \{0, 1\}$, we have $A(t)Y_{i+1} \subseteq Y_i$ boundedly and the process $A|_{Y_{i+1}} : [0, T] \times \Omega \rightarrow \mathcal{L}(Y_{i+1}, Y_i)$ is uniformly bounded and progressively measurable;
- (3) for all $0 \leq s \leq t \leq T$ and $y \in Y_1$, we have $S(t, \cdot)y \in W^{1,1}(0, t; X)$ and

$$\frac{d}{ds} S(t,s)y = -S(t,s)A(s)y.$$

Examples where this holds are discussed in [46,47,62–65] cited earlier.

Suppose first that g is an adapted finite rank step process g with values in Y_2 . The process u defined by the right-hand side of (5.10) then satisfies $u_t \in \mathcal{D}(A(t))$ and the process $(t, \omega) \mapsto A(t, \omega)u_t(\omega)$ is uniformly bounded. Moreover, as in [67, Proposition 4.2 and Theorem 4.7] and [68, Appendix] one checks that the analogue of (5.8) holds, i.e. almost surely we have

$$u_t = \int_0^t A(s)u_s \, ds + \int_0^t g_s \, dW_s, \quad t \in [0, T].$$

By following the proof of Theorem 5.6, a Gaussian tail estimate for u^* is obtained under the assumption that g is an adapted finite rank step process with values in Y_2 .

One would like to derive the general case by means of an approximation argument, but this is not straightforward due to the L^∞ -norm in the expression for the variance. Rather, we will first extend the L^p -estimate of Remark 5.7 to random evolution families first for adapted finite rank step processes g . In the L^p -case, the density argument can be carried out which, for arbitrary $g \in L^p_{\mathcal{F}}(\Omega; L^2(0, T; \gamma(H, X)))$, gives a limiting process $(u_t)_{t \in [0, T]}$ satisfying the L^p -maximal estimate. To identify the limiting process as a solution in a weak sense to the evolution equation at hand, we make the additional assumption that there exists a dense linear subspace $F \subseteq X^*$ satisfying $F \subseteq \mathcal{D}((A(s))^*)$ for all $s \in [0, T]$ and $\omega \in \Omega$ and that for all $x^* \in F$ the mapping $(s, \omega) \mapsto (A(s, \omega))^* x^*$ is strongly measurable and uniformly bounded. Under this assumption, it is straightforward to check that the limiting process u is a weak solution in the following sense: for all $x^* \in F^*$, almost

surely we have

$$\langle u_t, x^* \rangle = \int_0^t \langle u_s, A(s)^* x^* \rangle ds + \int_0^t x^* \circ g_s^* dW_s, \quad t \in [0, T].$$

Finally, to prove the Gaussian tail estimate for general $g \in L^\infty_{\mathcal{F}}(\Omega; L^2(0, T; \gamma(H, X)))$ one checks that there exists adapted finite rank step processes $(g^{(n)})$ with values in Y_2 such that $\|g^{(n)}\|_{L^2(0, T; \gamma(H, X))} \leq \|g\|_{L^2(0, T; \gamma(H, X))}$ almost surely and $g^{(n)} \rightarrow g$ in $L^2(\Omega; L^2(0, T; \gamma(H, X)))$. Then the previously mentioned L^p -estimate for $p=2$ implies that the corresponding solutions satisfy $u^{(n)} \rightarrow u$ in $L^2(\Omega; C([0, T]; X))$. Therefore, the Gaussian tail estimate for the pair (g, u) follows from the one for $(g^{(n)}, u^{(n)})$ with constant $\sigma^2 = 2D^2 \|g\|_{L^\infty(\Omega; L^2(0, T; \gamma(H, X)))}^2$.

6. Open questions

Question 6.1. Let $(S(t))_{t \geq 0}$ be a C_0 -semigroup on a 2-smooth Banach space X and let W be a standard real-valued Brownian motion with values. Does the process $(\int_0^t S(t-s)g_s dW_s)_{t \geq 0}$ have a continuous version for every progressively measurable $g \in L^2(\Omega; L^2(0, T; X))$?

This question is open even in the case of Hilbert spaces X . More generally, one could pose the above question for H -cylindrical Brownian motions and processes $g \in L^0(\Omega; L^2(0, T; \gamma(H, X)))$. A possible approach could be to prove that whenever a Banach space X is 2-smooth and $(S(t))_{t \geq 0}$ is a given C_0 -semigroup on X , then there exists an equivalent 2-smooth norm with respect to which the semigroup is contractive.

Question 6.2. Which C_0 -semigroups and C_0 -evolution families admit an (approximate) invertible dilation?

As explained in Theorem 4.10, one can deduce maximal inequalities for stochastic convolutions in case C_0 -semigroups/evolution families admit an approximate invertible dilation.

Question 6.3. Is there a ‘supert theory’ containing both the theories of stochastic integration in 2-smooth spaces and UMD spaces as special cases?

Presumably, such a theory would be based on some decoupling principle, in which case one expects it to apply to the class of Banach spaces satisfying a suitable decoupling inequality. For more details on decoupling and its use in the theory of stochastic integration the reader is referred to [69] and references therein. A possible approach could be to answer the next open question:

Question 6.4. Does 2-smoothness of X imply the existence of a finite constant $C \geq 0$ such that for all $n \geq 1$ and all functions $f_j: \{-1, 1\}^{j-1} \rightarrow X$ for $1 \leq j \leq n$ one has the decoupling inequality

$$\mathbb{E} \left\| \sum_{j=1}^n r_j f_j(r_1, \dots, r_{j-1}) \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{j=1}^n \tilde{r}_j f_j(r_1, \dots, r_{j-1}) \right\|^2 ?$$

Here, $(r_j)_{j \geq 1}$ is a sequence of independent Rademacher random variables (a Rademacher random variable is a random variable taking the values $+1$ and -1 with equal probability $\frac{1}{2}$) and $(\tilde{r}_j)_{j \geq 1}$ is an independent copy of $(r_j)_{j \geq 1}$. This decoupling inequality is satisfied in UMD spaces.

For the class of Banach spaces with type 2, which contains the class of all 2-smooth Banach spaces as a proper subclass, question 6.4 has a negative answer. Indeed, James [70] constructed a non-reflexive space X with type 2. If the decoupling inequality would hold in this space, then [18, Propositions 4.3.5 and 4.3.13] imply that X has martingale type 2 and hence is reflexive, a contradiction.

Data accessibility. This article has no additional data.

Authors' contributions. Both authors contributed equally to this paper.

Competing interests. The authors declare that they have no competing interests.

Funding. M.C. is supported by the VIDI subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO).

Acknowledgements. We thank the anonymous referees for helpful comments.

1. Brzeźniak Z, Goldys B, Imkeller P, Peszat S, Priola E, Zabczyk J. 2010 Time irregularity of generalized Ornstein-Uhlenbeck processes. *C. R. Math. Acad. Sci. Paris* **348**, 273–276. (doi:10.1016/j.crma.2010.01.022)
2. Iscoe I, Marcus MB, McDonald D, Talagrand M, Zinn J. 1990 Continuity of l^2 -valued Ornstein-Uhlenbeck processes. *Ann. Probab.* **18**, 68–84. (doi:10.1214/aop/1176990938)
3. Neidhardt AL. 1978 *Stochastic integrals in 2-uniformly smooth Banach spaces*. PhD thesis, University of Wisconsin.
4. van Neerven JMAM, Veraar MC, Weis LW. 2007 Stochastic integration in UMD Banach spaces. *Ann. Probab.* **35**, 1438–1478. (doi:10.1214/009117906000001006)
5. van Neerven JMAM, Veraar MC, Weis LW. 2008 Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal.* **255**, 940–993. (doi:10.1016/j.jfa.2008.03.015)
6. van Neerven JMAM, Veraar MC, Weis LW. 2015 Stochastic integration in Banach spaces—a survey. In *Stochastic analysis: A series of lectures*, volume 68 of *Progress in Probability*. Birkhäuser Verlag.
7. Brzeźniak Z. 1995 Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.* **4**, 1–45. (doi:10.1007/BF01048965)
8. Brzeźniak Z. 1997 On stochastic convolution in Banach spaces and applications. *Stoch. Stoch. Rep.* **61**, 245–295. (doi:10.1080/17442509708834122)
9. Agresti A, Veraar MC. 2020 Nonlinear parabolic stochastic evolution equations in critical spaces. Part I: Stochastic maximal regularity and local existence. (<http://arxiv.org/abs/2001.00512>)
10. van Neerven JMAM, Veraar MC, Weis LW. 2012 Maximal L^p -regularity for stochastic evolution equations. *SIAM J. Math. Anal.* **44**, 1372–1414. (doi:10.1137/110832525)
11. van Neerven JMAM, Veraar MC, Weis LW. 2012 Stochastic maximal L^p -regularity. *Ann. Probab.* **40**, 788–812. (doi:10.1214/10-AOP626)
12. Portal P, Veraar MC. 2019 Stochastic maximal regularity for rough time-dependent problems. *Stoch. Partial Differ. Equ. Anal. Comput.* **7**, 541–597. (doi:10.1007/s40072-019-00134-w)
13. Veraar MC, Weis LW. 2011 A note on maximal estimates for stochastic convolutions. *Czechoslovak Math. J.* **61**, 743–758. (doi:10.1007/s10587-011-0023-0)
14. Hausenblas E, Seidler J. 2008 Stochastic convolutions driven by martingales: maximal inequalities and exponential integrability. *Stoch. Anal. Appl.* **26**, 98–119. (doi:10.1080/07362990701673047)
15. Zhu J, Brzeźniak Z, Hausenblas E. 2017 Maximal inequalities for stochastic convolutions driven by compensated Poisson random measures in Banach spaces. *Ann. Inst. Henri Poincaré Probab. Stat.* **53**, 937–956. (doi:10.1214/16-AIHP743)
16. Zhu J, Brzeźniak Z, Liu W. 2019 Maximal inequalities and exponential estimates for stochastic convolutions driven by Lévy-type processes in Banach spaces with application to stochastic quasi-geostrophic equations. *SIAM J. Math. Anal.* **51**, 2121–2167. (doi:10.1137/18M1169011)
17. Marinelli C, Röckner M. 2014 On maximal inequalities for purely discontinuous martingales in infinite dimensions. In *Séminaire de Probabilités XLVI*, pp. 293–315. Springer.
18. Hytönen TP, van Neerven JMAM, Veraar MC, Weis LW. 2016 *Analysis in Banach Spaces. Volume I: martingales and Littlewood–Paley Theory*. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 63. New York, NY: Springer.
19. Hytönen TP, van Neerven JMAM, Veraar MC, Weis LW. 2017 *Analysis in Banach spaces. Volume II: probabilistic methods and operator theory*. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. 67. New York, NY: Springer.
20. van Neerven JMAM. 2010 γ -Radonifying operators—a survey. In *The AMSI-ANU Workshop on spectral theory and harmonic analysis*, volume 44 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 1–61. Austral. Nat. Univ., Canberra.
21. Pinelis I. 1994 Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.* **22**, 1679–1706. (doi:10.1214/aop/1176988477)
22. van Neerven JMAM, Veraar MC. 2020 Maximal inequalities for stochastic convolutions and pathwise uniform convergence of time discretisation schemes. (<http://arxiv.org/abs/2006.06964>)
23. Figiel T. 1976 On the moduli of convexity and smoothness. *Studia Math.* **56**, 121–155. (doi:10.4064/sm-56-2-121-155)

24. Triebel H. 1978 *Interpolation theory, function spaces, differential operators*. North-Holland Mathematical Library, vol. 18. Amsterdam, the Netherlands: North-Holland Publishing Co.
25. Ball K, Carlen EA, Lieb EH. 1994 Sharp uniform convexity and smoothness inequalities for trace norms. *Invent. Math.* **115**, 463–482. (doi:10.1007/BF01231769)
26. Pisier G. 1975 Martingales with values in uniformly convex spaces. *Israel J. Math.* **20**, 326–350. (doi:10.1007/BF02760337)
27. Hytönen TP, van Neerven JMAM, Veraar MC, Weis LW. 2018 *Analysis in Banach spaces. Volume IV: stochastic analysis*. 20xx. In preparation.
28. Wenzel J. 2005 Strong martingale type and uniform smoothness. *J. Convex Anal.* **12**, 159–171.
29. Woyczyński WA. 2019 *Geometry and martingales in Banach spaces*. Boca Raton, FL: CRC Press.
30. Nualart D. 2006 *The Malliavin calculus and related topics*. Probability and its Applications (New York), 2nd edn. Berlin, Germany: Springer.
31. Brzeźniak Z. 2003 Some remarks on Itô and Stratonovich integration in 2-smooth Banach spaces. In *Probabilistic methods in fluids*, pp. 48–69. World Sci. Publishing, River Edge, NJ.
32. Dettweiler E. 1989 On the martingale problem for Banach space valued stochastic differential equations. *J. Theoret. Probab.* **2**, 159–191. (doi:10.1007/BF01053408)
33. Dettweiler E. 1991 Stochastic integration relative to Brownian motion on a general Banach space. *Doğa Mat.* **15**, 58–97.
34. Ondreját M. 2004 Uniqueness for stochastic evolution equations in Banach spaces. *Dissertationes Math. (Rozprawy Mat.)* **426**, 63. (doi:10.4064/dm426-0-1)
35. Seidler J. 2010 Exponential estimates for stochastic convolutions in 2-smooth Banach spaces. *Electron. J. Probab.* **15**, 1556–1573. (doi:10.1214/EJP.v15-808)
36. Revuz D, Yor M. 1999 *Continuous martingales and Brownian motion*. Grundlehren der Mathematischen Wissenschaften, 3rd edn, vol. 293. Berlin, Germany: Springer.
37. Da Prato G, Kwapien S, Zabczyk J. 1987 Regularity of solutions of linear stochastic equations in Hilbert spaces. *Stochastics* **23**, 1–23. (doi:10.1080/17442508708833480)
38. Seidler J. 1993 Da Prato-Zabczyk's maximal inequality revisited. I. *Math. Bohem.* **118**, 67–106.
39. Hausenblas E, Seidler J. 2001 A note on maximal inequality for stochastic convolutions. *Czechoslovak Math. J.* **51**, 785–790. (doi:10.1023/A:1013717013421)
40. Sz.-Nagy B, Foias C, Bercovici H, Kérchy L. 2010 *Harmonic analysis of operators on Hilbert space*, Universitext, second, enlarged edn. New York, NY: Springer.
41. Fendler G. 1997 Dilations of one parameter semigroups of positive contractions on L^p spaces. *Canad. J. Math.* **49**, 736–748. (doi:10.4153/CJM-1997-036-x)
42. Fröhlich AM, Weis LW. 2006 H^∞ calculus and dilations. *Bull. Soc. Math. France* **134**, 487–508. (doi:10.24033/bsmf.2520)
43. Arhancet F, Fackler S, Le Merdy C. 2017 Isometric dilations and H^∞ calculus for bounded analytic semigroups and Ritt operators. *Trans. Am. Math. Soc.* **369**, 6899–6933. (doi:10.1090/tran/6849)
44. Fackler S, Glück J. 2019 A toolkit for constructing dilations on Banach spaces. *Proc. Lond. Math. Soc.* (3) **118**, 416–440. (doi:10.1112/plms.12201)
45. Le Merdy C. 1996 On dilation theory for c_0 -semigroups on Hilbert space. *Indiana Univ. Math. J.* **45**, 945–959. (doi:10.1512/iumj.1996.45.1191)
46. Pazy A. 1983 *Semigroups of linear operators and applications to partial differential equations*. Applied Mathematical Sciences, vol. 44. New York, NY: Springer.
47. Tanabe H. 1979 *Equations of evolution*. Monographs and Studies in Mathematics, vol. 6. Boston, MA: Pitman (Advanced Publishing Program).
48. Peszat S, Zabczyk J. 2014 Time regularity for stochastic Volterra equations by the dilation theorem. *J. Math. Anal. Appl.* **409**, 676–683. (doi:10.1016/j.jmaa.2013.07.055)
49. Schnaubelt R, Veraar MC. 2017 Regularity of stochastic Volterra equations by functional calculus methods. *J. Evol. Equ.* **17**, 523–536. (doi:10.1007/s00028-016-0365-z)
50. Kotelenetz P. 1982 A submartingale type inequality with applications to stochastic evolution equations. *Stochastics* **8**, 139–151. (doi:10.1080/17442508208833233)
51. Pardoux E. 1975 Équations aux dérivées partielles stochastiques de type monotone. In *Séminaire sur les Équations aux Dérivées Partielles* (1974–1975), III, Exp. No. 2, p. 10.
52. Krylov NV, Rozovskii BL. 1981 Stochastic evolution equations. *J. Soviet Math.* **14**, 1233–1277. (doi:10.1007/BF01084893)
53. Liu W, Röckner M. 2015 *Stochastic partial differential equations: an introduction*. Cham: Universitext. Springer.

54. Ichikawa A. 1982 Stability of semilinear stochastic evolution equations. *J. Math. Anal. Appl.* **90**, 12–44. (doi:10.1016/0022-247X(82)90041-5)
55. Kotelenetz P. 1984 A stopped Doob inequality for stochastic convolution integrals and stochastic evolution equations. *Stoch. Anal. Appl.* **2**, 245–265. (doi:10.1080/07362998408809036)
56. Tubaro L. 1984 An estimate of Burkholder type for stochastic processes defined by the stochastic integral. *Stoch. Anal. Appl.* **2**, 187–192. (doi:10.1080/07362998408809032)
57. Ichikawa A. 1986 Some inequalities for martingales and stochastic convolutions. *Stoch. Anal. Appl.* **4**, 329–339. (doi:10.1080/07362998608809094)
58. Brzeźniak Z, Peszat S. 2000 Maximal inequalities and exponential estimates for stochastic convolutions in Banach spaces. In *Stochastic processes, physics and geometry: new interplays, I* (Leipzig, 1999), vol. 28 of *CMS Conf. Proc.*, pp. 55–64. Amer. Math. Soc., Providence, RI.
59. Leonard IE, Sundaresan K. 1974 Geometry of Lebesgue-Bochner function spaces—smoothness. *Trans. Am. Math. Soc.* **198**, 229–251. (doi:10.2307/1996757)
60. van Neerven JMAM, Zhu J. 2011 A maximal inequality for stochastic convolutions in 2-smooth Banach spaces. *Electron. Commun. Probab.* **16**, 689–705. (doi:10.1214/ECP.v16-1677)
61. Neelima, Šiška D. 2019 Coercivity condition for higher order moments for nonlinear SPDEs and existence of solution under local monotonicity. *Stochastics* **8**, 1–32. (doi:10.1080/17442508.2019.1650043)
62. Acquistapace P, Terreni B. 1987 A unified approach to abstract linear nonautonomous parabolic equations. *Rend. Sem. Mat. Univ. Padova* **78**, 47–107.
63. Amann H. 1995 *Linear and quasilinear parabolic problems. Vol. I*, vol. 89 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA. Abstract linear theory.
64. Lunardi A. 1995 *Analytic semigroups and optimal regularity in parabolic problems*. Progress in Nonlinear Differential Equations and their Applications, vol. 16. Basel: Birkhäuser Verlag.
65. Yagi A. 2010 *Abstract parabolic evolution equations and their applications*. Springer Monographs in Mathematics. Berlin, Germany: Springer.
66. Kallenberg O. 2002 *Foundations of modern probability*. Probability and its Applications (New York), 2nd edn. New York, NY: Springer.
67. Pronk M, Veraar MC. 2014 A new approach to stochastic evolution equations with adapted drift. *J. Differ. Equ.* **256**, 3634–3683. (doi:10.1016/j.jde.2014.02.014)
68. Kuehn C, Neamțu A. 2020 Pathwise mild solutions for quasilinear stochastic partial differential equations. *J. Differ. Equ.* **269**, 2185–2227.
69. Cox SG, Geiss S. 2018 On decoupling in Banach spaces. (<http://arxiv.org/abs/1805.12377>)
70. James RC. 1978 Nonreflexive spaces of type 2. *Israel J. Math.* **30**, 1–13. (doi:10.1007/BF02760825)