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On the conjugacy class of the Fibonacci dynamical system

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ABSTRACT

We characterize the symbolical dynamical systems which are topologically isomorphic to the Fibonacci dynamical system. We prove that there are infinitely many injective primitive substitutions generating a dynamical system in the Fibonacci conjugacy class. In this class there are infinitely many dynamical systems not generated by a substitution. An example is the system generated by doubling the 0's in the infinite Fibonacci word.

1. Introduction

We study the Fibonacci substitution φ given by

$$\varphi: \quad 0 \to 01,\ 1 \to 0.$$

The infinite Fibonacci word w_{F} is the unique one-sided sequence (to the right) which is a fixed point of φ:

$$w_{\mathrm{F}} = 0100101001\dots.$$

We also consider one of the two two-sided fixed points x_{F} of φ^2:

$$x_{\mathrm{F}} = \dots 01001001 \cdot 0100101001\dots.$$

The dynamical system generated by taking the orbit closure of x_{F} under the shift map σ is denoted by (X_φ, σ).

The question we will be concerned with is: what are the substitutions η which generate a symbolic dynamical system topologically isomorphic to the Fibonacci dynamical system? Here *topologically isomorphic* means that there exists a homeomorphism $\psi: X_\varphi \to X_\eta$, such that $\psi\sigma = \sigma\psi$, where we denote the shift on X_η also by σ. In this case (X_η, σ) is said to be *conjugate* to (X_φ, σ).

This question has been completely answered for the case of constant length substitutions in the paper [2]. It is remarkable that there are only finitely many injective primitive substitutions of length L which generate a system conjugate to a given substitution of length L. Here a substitution α is called *injective* if $\alpha(a) \neq \alpha(b)$ for all letters a and b from the alphabet with $a \neq b$. When we extend to the class of all substitutions, replacing L by the Perron–Frobenius eigenvalue of the

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incidence matrix of the substitution, then the conjugacy class can be infinite in general. See [5] for the case of the Thue–Morse substitution. In the present paper we will prove that there are infinitely many injective primitive substitutions with Perron–Frobenius eigenvalue $\Phi = (1+\sqrt{5})/2$ which generate a system conjugate to the Fibonacci system—see Theorem 5.1.

In the non-constant length case some new phenomena appear. If one has an injective substitution α of constant length L, then all its powers α^n will also be injective. This is no longer true in the general case. For example, consider the injective substitution ζ on the alphabet $\{1,2,3,4,5\}$ given by

$$\zeta: \qquad 1 \to 12,\ 2 \to 3,\ 3 \to 45,\ 4 \to 1,\ 5 \to 23.$$

An application of Theorem 2.1 followed by a partition reshaping (see Section 4) shows that the system (X_ζ, σ) is conjugate to the Fibonacci system. However, the square of ζ is given by

$$\zeta^2: \qquad 1 \to 123,\ 2 \to 45,\ 3 \to 123,\ 4 \to 12,\ 5 \to 345,$$

which is *not* injective. To deal with this undesirable phenomenon we introduce the following notion. A substitution α is called a *full rank* substitution if its incidence matrix has full rank (non-zero determinant). This is a strengthening of injectivity, because obviously a substitution which is not injective can not have full rank. Moreover, if the substitution α has full rank, then all its powers α^n will also have full rank, and thus will be injective.

Another phenomenon, which does not exist in the constant length case, is that non-primitive substitutions ζ may generate uniquely defined minimal systems conjugate to a given system. For example, consider the injective substitution ζ on the alphabet $\{1,2,3,4\}$ given by

$$\zeta: \qquad 1 \to 12, \quad 2 \to 31, \quad 3 \to 4, \quad 4 \to 3.$$

With the partition reshaping technique from Section 4 one can show that the system (X_ζ, σ) is conjugate to the Fibonacci system (ignoring the system on two points generated by ζ). In the remainder of this paper we concentrate on primitive substitutions.

The structure of the paper is as follows. In Section 2 we show that all systems in the conjugacy class of the Fibonacci substitution can be obtained by letter-to-letter projections of the systems generated by so-called N-block substitutions. In Section 3 we give a very general characterization of symbolical dynamical systems in the Fibonacci conjugacy class, in the spirit of a similar result on the Toeplitz dynamical system in [4]. In Section 4 we introduce a tool which admits to turn non-injective substitutions into injective substitutions. This is used in Section 5 to show that the Fibonacci class has infinitely many primitive injective substitutions as members. In Section 6 we quickly analyze the case of a 2-symbol alphabet. Sections 7 and 8 give properties of equicontinuous factors and incidence matrices, which are used to analyze the 3-symbol case in Section 9. In the final Section 10 we show that the system obtained by doubling the 0's in the infinite Fibonacci word is conjugate to the Fibonacci dynamical system, but can not be generated by a substitution.

2. *N*-block systems and *N*-block substitutions

For any N the N-block substitution $\hat{\theta}_N$ of a substitution θ is defined on an alphabet of $p_\theta(N)$ symbols, where $p_\theta(\cdot)$ is the complexity function of the language $\mathcal{L}_\theta$ of θ (cf. [11, p. 95]). What is *not* in [11], is that this N-block substitution generates the N-block presentation of the system (X_θ, σ).

We denote the letters of the alphabet of the N-block presentation by $[a_1a_2\dots a_N]$, where $a_1a_2\dots a_N$ is an element from $\mathcal{L}_\theta^N$, the set of words of length N in the language of θ. The N-block presentation $(X_\theta^{[N]}, \sigma)$ emerges by applying an sliding block code Ψ to the sequences of X_θ, so Ψ is the map

$$\Psi(a_1a_2\dots a_N) = [a_1a_2\dots a_N].$$

We denote by ψ the induced map from X_θ to $X_\theta^{[N]}$:

$$\psi(x) = \dots \Psi(x_{-N},\dots,x_{-1})\Psi(x_{-N+1},\dots,x_0)\dots.$$

It is easy to see that ψ is a conjugacy, where the inverse is π_0 induced by the 1-block map (also denoted π_0) given by $\pi_0([a_1a_2\dots a_N]) = a_1$.

The *N-block substitution* $\hat{\theta}_N$ is defined by requiring that for each word $a_1a_2\dots a_N$ the length of $\hat{\theta}_N([a_1a_2\dots a_N])$ is equal to the length L_1 of $\theta(a_1)$, and the letters of $\hat{\theta}_N([a_1a_2\dots a_N])$ are the Ψ-codings of the first L_1 consecutive N-blocks in $\theta(a_1a_2\dots a_N)$.

Theorem 2.1. *Let $\hat{\theta}_N$ be the N-block substitution of a primitive substitution θ. Let $(X_\theta^{[N]}, \sigma)$ be the N-block presentation of the system (X_θ, σ). Then $X_\theta^{[N]} = X_{\hat{\theta}_N}$.*

Proof. Let x be a fixed point of θ, and let $y = \psi(x)$, where ψ is the N-block conjugacy, with inverse π_0. The key equation is $\pi_0\,\hat{\theta}_N = \theta\,\pi_0$. This implies

$$\pi_0\,\hat{\theta}_N(y)=\theta\,\pi_0(y)=\theta\,\pi_0(\psi(x))=\theta(x)=x.$$

Applying ψ on both sides gives $\hat{\theta}_N(y)=\psi(x)=y$, i.e., y is a fixed point of $\hat{\theta}_N$. But then $X_\theta^{[N]}=X_{\hat{\theta}_N}$, by minimality of $X_\theta^{[N]}$. □

It is well known (see, e.g., [11, p. 105]) that $p_\varphi(N)=N+1$, so for the Fibonacci substitution φ the N-block substitution $\hat{\varphi}_N$ is a substitution on an alphabet of $N+1$ symbols.

We describe how one obtains $\hat{\varphi}_2$. We have $\mathcal{L}_\varphi^2=\{00,01,10\}$. Since 00 and 01 start with 0, and 10 with 1, we obtain

$$\hat{\varphi}_2:\quad [00]\mapsto[01][10],\ [01]\mapsto[01][10],\ [10]\mapsto[00],$$

reading off the consecutive 2-blocks from $\varphi(00)=0101$, $\varphi(01)=010$ and $\varphi(10)=001$. It is useful to recode the alphabet $\{[00],[01],[10]\}$ to the standard alphabet $\{1,2,3\}$. We do this in the order in which they appear for the first time in the infinite Fibonacci word $w_{\rm F}$—we call this the *canonical coding*, and will use the same principle for all N. For $N=2$ this gives $[01]\to 1$, $[10]\to 2$, $[00]\to 3$. Still using the notation $\hat{\varphi}_2$ for the substitution on this new alphabet, we obtain

$$\hat{\varphi}_2(1)=12\quad \hat{\varphi}_2(2)=3,\quad \hat{\varphi}_2(3)=12.$$

In this way the substitution is in standard form (cf. [2] and [6]).

3. The Fibonacci conjugacy class

Let F_n for $n=1,2,\dots$ be the Fibonacci numbers

$$F_1=1,\ F_2=1,\ F_3=2,\ F_4=3,\ F_5=5,\dots.$$

Theorem 3.1. *Let (Y,σ) be any subshift with infinite cardinality. Then (Y,σ) is topologically conjugate to the Fibonacci system (X_φ,σ) if and only if there exist $n\ge 3$ and two words B_0 and B_1 of length F_n and F_{n-1}, such that any y from Y is a concatenation of B_0 and B_1, and moreover, if $\cdots B_{x_{-1}}B_{x_0}B_{x_1}\cdots B_{x_k}\cdots$ is such a concatenation, then $x=(x_k)$ is a sequence from the Fibonacci system.*

Proof. First let us suppose that (Y,σ) is topologically isomorphic to the Fibonacci system. By the Curtis–Hedlund–Lyndon theorem, there exists an integer N such that Y is obtained by a letter-to-letter projection π from the N-block presentation $(X_\varphi^{[N]},\sigma)$ of the Fibonacci system. Now if B_0 and B_1 are two decomposition blocks of sequences from $X_\varphi^{[N]}$ of length F_n and F_{n-1}, then $\pi(B_0)$ and $\pi(B_1)$ are decomposition blocks of sequences from Y with lengths F_n and F_{n-1}, again satisfying the concatenation property. Here we use that since Y is infinite, the decomposition of a sequence y from Y in the two blocks $\pi(B_0)$ and $\pi(B_1)$ is unique, cf. [10]. So it suffices to prove the result for $X_\varphi^{[N]}$. Note that we may suppose that the integers N pass through an infinite subsequence; we will use $N=F_n$, where $n=3,4,\dots$. Useful to us are the *singular words* w_n introduced in [13]. The w_n are the unique words of length F_{n+1} having a different Parikh vector from all the other words of length F_{n+1} from the language of φ. Here $w_1=1$, $w_2=00$, $w_3=101$, and for $n\ge 4$

$$w_n=w_{n-2}w_{n-3}w_{n-2}.$$

The set of return words of w_n has only two elements which are $u_n=w_nw_{n+1}$ and $v_n=w_nw_{n-1}$ (see page 108 in [7]). The lengths of these words are $|u_n|=F_{n+3}$ and $|v_n|=F_{n+2}$. Let w_n^- be w_n with the last letter deleted. Define for $n\ge 5$

$$B_0=\Psi(u_{n-3}w_{n-3}^-),\quad B_1=\Psi(v_{n-3}w_{n-3}^-),$$

where Ψ is the N-block code from $\mathcal{L}_\varphi^N$ to $\mathcal{L}_{\varphi^{[N]}}$, with $N=F_{n-2}$. Then these blocks have the right lengths, and by Theorem 2.11 in [7], the two return words partition the infinite Fibonacci word $w_{\rm F}$ according to the infinite Fibonacci word—except for a prefix $r_{n,0}$:

$$w_{\rm F}=r_{n,0}u_nv_nu_nu_nv_nu_n\dots.$$

By minimality this property carries over to all two-sided sequences in the Fibonacci dynamical system.

For the converse, let Y be a Fibonacci concatenation system as above. Let $C_0=\varphi^{n-2}(0)$ and $C_1=\varphi^{n-2}(1)$. We define a map g from (Y,σ) to a subshift of $\{0,1\}^{\mathbb{Z}}$ by

$$g:\quad \cdots B_{x_{-1}}B_{x_0}B_{x_1}\cdots B_{x_k}\cdots\ \mapsto\ \cdots C_{x_{-1}}C_{x_0}C_{x_1}\cdots C_{x_k}\cdots,$$

respecting the position of the 0th coordinate. By the uniqueness of the decomposition (cf. [10]), g is well defined. Since $|C_0|=|B_0|$ and $|C_1|=|B_1|$, g commutes with the shift. Also, g is obviously continuous. Moreover, since for any sequence x in the Fibonacci system $\varphi^{n-2}(x)$ is again a sequence in the Fibonacci system, $g(Y)\subseteq X_\varphi$. So, by minimality, (X_φ,σ) is a factor of (Y,σ). Since g is invertible, with continuous inverse, (Y,σ) is in the conjugacy class of the Fibonacci system. □

Example. The case $(F_n, F_{n-1}) = (13, 8)$. Then $n = 7$, so we have to consider the singular word $w_4 = 00100$ of length 5.

The set of 5-blocks is {01001, 10010, 00101, 01010, 10100, 00100}.

These will be coded by the canonical coding Ψ to the standard alphabet $\{1, 2, 3, 4, 5, 6\}$. Note that $\Psi(w_4) = 6$. Further, $w_3 = 101$ and $w_5 = 10100101$. So $u_n = 0010010100101$ and $v_n = 00100101$. Applying Ψ gives the two decomposition blocks $B_0 = 6123451234512$ and $B_1 = 61234512$.

4. Reshaping substitutions

We call a language preserving transformation of a substitution a reshaping. An example is the prefix-suffix change used in [5]. Here we consider a variation which we call a *partition reshaping*.

We give an example of this technique. Take the N-block representation of the Fibonacci system for $N = 4$. All five 4-blocks occur consecutively at the beginning of the Fibonacci word w_{F} as {0100, 1001, 0010, 0101, 1010}. The canonical coding to $\{1, 2, 3, 4, 5\}$ gives the 4-block substitution $\hat{\varphi}_4$:

$$\hat{\varphi}_4: \qquad 1 \to 12,\ 2 \to 3,\ 3 \to 45,\ 4 \to 12,\ 5 \to 3.$$

Its square is equal to

$$\hat{\varphi}_4^2: \qquad 1 \to 123,\ 2 \to 45,\ 3 \to 123,\ 4 \to 123,\ 5 \to 45.$$

Since the letters 1, 2, and 3, respectively 4 and 5, in the two blocks $B_0 = 123$ and $B_1 = 45$ only occur in the language in these two blocks, this permits to do a partition reshaping. Symbolically this can be represented by

$$\begin{array}{ccccccccccccc}
1 & & & 2 & 3 & & & \qquad & 4 & & & 5 & \\
\downarrow & & & \downarrow & \downarrow & & & & \downarrow & & & \downarrow & \\
1 & 2 & & 3 & 4 & & 5 & & 1 & & 2 & 3 & \\
1 & 2 & \| & 3 & 4 & \| & 5 & & 1 & \| & 2 & 3 & \|
\end{array}$$

Here the third line gives the images $\hat{\varphi}_4(B_0) = \hat{\varphi}_4(123) = 12345$ and $\hat{\varphi}_4(B_1) = \hat{\varphi}_4(45) = 123$; the fourth line gives a *another* partition of these two words in three, respectively two subwords from which the new substitution η can be read of:

$$\eta: \qquad 1 \to 12,\ 2 \to 34,\ 3 \to 5,\ 4 \to 1,\ 5 \to 23.$$

What we gain is that the partition reshaped substitution η generates the same language as $\hat{\varphi}_4$, but that η is injective—in this example it is even of full rank.

In general, one looks for one or more *isolated words* in the language of a substitution η, i.e., words $w = w_1 \dots w_m$ such that the letter $a = w_i$ can only occur in the language as $w_1 \dots w_{i-1} a \dots w_m$ for $i = 1, \dots, m$. This implies in particular that all letters in an isolated word are different, as in the two isolated words 123 and 45 in the example above. Partition reshapings ζ of η are obtained by partitioning the image word $\eta(w)$ of an isolated word $w = w_1 \dots w_m$ into m words, and assigning these words to the letters w_1 to w_m.

Lemma 4.1. *Let η be a primitive substitution that permits a partition reshaping ζ, and let ζ be primitive. Then $\lambda_{\mathrm{PF}}(\eta) = \lambda_{\mathrm{PF}}(\zeta)$, where $\lambda_{\mathrm{PF}}(\eta)$ and $\lambda_{\mathrm{PF}}(\zeta)$ are the Perron–Frobenius eigenvalues of η and ζ.*

Proof. By the Perron–Frobenius theorem there exist words v in $\mathcal{L}_\eta$ such that $|\eta(v)|$ is arbitrarily close to $\lambda_{\mathrm{PF}}(\eta)|v|$. As we may assume that in v only complete isolated words w occur, we have $\zeta(v) = \eta(v)$, and in particular $|\zeta(v)| = |\eta(v)|$. So another application of the PF-theorem gives that $\lambda_{\mathrm{PF}}(\zeta) = \lambda_{\mathrm{PF}}(\eta)$. □

5. The Fibonacci class has infinite cardinality

Theorem 5.1. *There are infinitely many primitive injective substitutions with Perron–Frobenius eigenvalue the golden mean that generate dynamical systems topologically isomorphic to the Fibonacci system.*

We will explicitly construct infinitely many primitive injective substitutions whose systems are topologically conjugate to the Fibonacci system. The topological conjugacy will follow from the fact that the systems are N-block codings of the Fibonacci system, where N will run through the numbers $F_n - 1$. As an introduction we look at $n = 5$, i.e., we consider the blocks of length $N = F_5 - 1 = 4$. With the canonical coding of the N-blocks we obtain the 4-block substitution $\hat{\varphi}_4$—see Section 4:

$$\hat{\varphi}_4: \qquad 1 \to 12,\ 2 \to 3,\ 3 \to 45,\ 4 \to 12,\ 5 \to 3.$$

An *interval* I starting with $a \in A$ is a word of length L of the form

$$I = a, a+1, ..., a+L-1.$$

Note that $\hat{\varphi}_4(123) = 12345$, and $\hat{\varphi}_4(45) = 123$, and these four words are intervals.

This is a property that holds in general. First we need the fact that the first F_n words of length $F_n - 1$ in the fixed point of φ are all different. This result is given by Theorem 2.8 in [3]. We code these $N+1$ words by the canonical coding to the letters $1, 2, \dots, F_n$. We then have

$$\hat{\varphi}_N(12...F_{n-1}) = 12\dots F_n, \qquad \hat{\varphi}_N(F_{n-1}+1, \dots F_n) = 12\dots F_{n-1}. \tag{1}$$

This can be seen by noting that $\pi_0\hat{\varphi}_N^n = \varphi^n\pi_0$, for all n, and that the fixed point of φ starts with $\varphi^{n-2}(0)\varphi^{n-3}(0)$.

We continue for $n \geq 5$ with the construction of a substitution $\eta = \eta_n$ which is a partition reshaping of $\hat{\varphi}_N$. The F_n letters in the alphabet $A^{[N]}$ are divided in three species, S, M and L (for Small, Medium and Large).

$$\mathrm{S} := 1, ..., F_{n-3}, \quad \mathrm{M} := F_{n-3}+1, ..., F_{n-1}, \quad \mathrm{L} := F_{n-1}+1, ..., F_n.$$

Note that $\mathrm{Card}\,\mathrm{M} = F_{n-1} - F_{n-3} = F_{n-2} = F_n - F_{n-1} = \mathrm{Card}\,\mathrm{L}$.

An important role is played by $a_\mathrm{M} := F_{n-3}+1$, the smallest letter in M, and $a_\mathrm{L} := F_{n-1}+1$, the smallest letter in L. For the letters in M (except for a_M) the rules are very simple:

$$\eta(a) = a + F_{n-2}$$

(i.e., a single letter obtained by addition of the two integers). The first letter in M has the rule

$$\eta(a_\mathrm{M}) = \eta(F_{n-3}+1) = F_{n-1}, F_{n-1}+1 = F_{n-1}, a_\mathrm{L}.$$

The images of the letters in L are intervals of length 1 or 2, obtained from a partition of the word $12\dots F_{n-1}$. Their lengths are coming from $\varphi^{n-4}(0)$, rotated once (put the 0 in front at the back). This word is denoted $\rho(\varphi^{n-4}(0))$. The choice of this word is somewhat arbitrary, other choices would work. The properties of $v := \rho(\varphi^{n-4}(0))$ which matter to us are

(V1) $\ell := |v| = F_{n-2}$.

(V2) $v_1 = 1$, $v_\ell = 0$.

(V3) v does not contain any 11.

Now the images of the letters in L are determined by v according to the following rule: $|\eta(a_\mathrm{L} + k - 1)| = 2 - v_k$, for all $k = 1, \dots, F_{n-2}$. Note that this implies in particular that for all $n \geq 5$ one has by property (V2)

$$\eta(a_\mathrm{L}) = \eta(F_{n-1}+1) = 1, \qquad \eta(F_n) = F_{n-1} - 1, F_{n-1}.$$

The images of the letters in S are then obtained by choosing the lengths of the $\eta(a)$ in such a way that the coarsest common refinement of the induced partitions of the images of S and L is the singleton partition (this is always possible in a unique way).

Example. The case $n = 7$, so $F_n = 13$, $F_{n-1} = 8$, and $F_{n-2} = 5$.

Then $\mathrm{S} = \{1, 2, 3\}$, $\mathrm{M} = \{4, 5, 6, 7, 8\}$, $\mathrm{L} = \{9, 10, 11, 12, 13\}$.

Rules for M: $4 \to 89$, $5 \to 10$, $6 \to 11$, $7 \to 12$, $8 \to 13$. Now

$$\varphi^3(0) = 01001 \Rightarrow v = 10010 \Rightarrow \text{the partition is } 1|23|45|6|78.$$

This partition gives the following rules for L:

$$9 \to 1, \ 10 \to 23, \ 11 \to 45, \ 12 \to 6, \ 13 \to 78.$$

The induced partition for the images of the letters in S is $|12|34|567|8$, yielding rules

$$1 \to 12, \ 2 \to 34, \ 3 \to 567.$$

In summary we obtain the substitution $\eta = \eta_7$ given by:

$$\mathrm{S}: \begin{cases} 1 & \to 1,2 \\ 2 & \to 3,4 \\ 3 & \to 5,6,7 \end{cases} \qquad \mathrm{M}: \begin{cases} 4 & \to 8,9 \\ 5 & \to 10 \\ 6 & \to 11 \\ 7 & \to 12 \\ 8 & \to 13 \end{cases} \qquad \mathrm{L}: \begin{cases} 9 & \to 1 \\ 10 & \to 2,3 \\ 11 & \to 4,5 \\ 12 & \to 6 \\ 13 & \to 7,8. \end{cases}$$

The substitution η is primitive because you 'can go' from the letter 1 to any letter and from any letter to the letter 1. This gives irreducibility; there is primitivity because periodicity is impossible by the first rule $1 \to 1, 2$.

The substitution η has full rank because any unit vector

$$e_a = (0, \dots, 0, 1, 0, \dots, 0)$$

is a linear combination of rows of the incidence matrix M_η of η. For $a \in \mathrm{L} \setminus \{9\}$ this combination is trivial, and for the other letters this is exactly forced by the choice of lengths in such a way that the coarsest common refinement of the induced partitions of the images of S and L is the singleton partition. In more detail: denote the a^{th} row of M_η by R_a. Then $e_1 = R_9$, and thus $e_2 = R_1 - R_9$, $e_3 = R_{10} - e_2 = R_{10} - R_1 + R_9$, etc.

The argument yielding the property of full rank will hold in general for all $n \geq 5$. To prove primitivity for all n we need some more details.

Proposition 5.1. *The substitution $\eta = \eta_n$ is primitive for all $n \geq 5$.*

Proof. The proposition will be proved if we show that for all $a \in A$ the letter a will occur in some iteration $\eta^k(1)$, and conversely, that for all $a \in A$ the letter 1 will occur in some iteration $\eta^k(a)$. The first part is easy to see from the fact that $\eta(1) = 1, 2$ and that $\eta^2(1, \dots, F_{n-2}) = 1, \dots, F_n - 1$, plus $\eta^2(a_\mathrm{M}) = F_n, 1$. For the second part, we show that A) for any $a \in \mathrm{M} \cup \mathrm{L}$ a letter from S will occur in $\eta^k(a)$ in $k \leq \mathrm{Card}\ \mathrm{M} \cup \mathrm{L}$ steps (see Lemma 5.1) and B), that for any $a \in \mathrm{S}$ the letter 1 will occur in $\eta^k(a)$ in $k \leq 2\mathrm{Card}\ A$ steps (see Lemma 5.2). □

Lemma 5.1. *Let $f : A \to A$ be the map that assigns the first letter of $\eta^2(a)$ to a. Then f is strictly decreasing on $\mathrm{L} \cup \mathrm{M} \backslash \{a_\mathrm{M}\}$.*

Proof. First we consider f on L. We have

$$\eta^2(a_\mathrm{L} \dots F_n) = \eta(1, \dots, F_{n-1} - 1, F_{n-1}) = 1 \dots F_n.$$

Since

$$\eta^2(F_n) = \eta(F_{n-1} - 1, F_{n-1}) = F_{n-1} - 1 + F_{n-2}, F_{n-1} + F_{n-2} = F_n - 1, F_n,$$

we obtain $f(F_n) = F_n - 1 < F_n$. This implies that also the previous letters in L are mapped by f to a smaller letter.

Next we consider f on $\mathrm{M} \backslash \{a_\mathrm{M}\}$. Here

$$\eta^2(a_\mathrm{M} + 1, \dots, F_{n-1}) = \eta(a_\mathrm{L} + 1, \dots, F_n) = 2, 3, \dots, F_{n-1}.$$

Now

$$\eta^2(F_{n-1}) = \eta(F_n) = F_{n-1} - 1, F_{n-1}.$$

So we obtain $f(F_{n-1}) = F_{n-1} - 1 < F_{n-1}$. This implies that also the previous letters in M are mapped by f to a smaller letter. □

Lemma 5.2. *For all $a \in \mathrm{S}$ there exists $k \leq 2\,\mathrm{Card}\,A$ such that the letter 1 occurs in $\eta^k(a)$.*

Proof. The substitution η^2 maps intervals I to intervals $\eta^2(I)$, provided I does not contain a_M or a_L. By construction, since the $\eta(b)$ for $b \in \mathrm{L}$ have length 1 or 2, the length of $\eta(a)$ for $a \in \mathrm{S}$ is 2 or 3, and so $\eta(a)$ contains a word $c, c+1$ for some $c \in A$. Since $\rho\varphi^{(n-4)}(0)$ does not contain two consecutive 1's (property (V3)), the image $\eta^2(c, c+1)$ has at least length 3. Since[1] any word of length at least 3 in the language of η contains an interval of length 2, the length increases by at least 1 if you apply η^2. It follows that for all $n \geq 5$ and all $a \in \mathrm{S}$ one has $|\eta^{2n+1}(a)| \geq n + 2$. But then after less than Card A steps a letter a_M or a letter a_L must occur in $\eta^{2n+1}(a)$. This implies that the letter 1 occurs in $\eta^{2n+3}(a)$, since both $\eta^2(a_\mathrm{M})$ and $\eta^2(a_\mathrm{L})$ contain a 1. □

Proof of Theorem 5.1. The theorem follows from the construction and the full rank argument given above combined with Proposition 5.1 and Lemma 4.1. □

6. The 2-symbol case

On an alphabet of two symbols, modulo a permutation of the symbols, the only possible incidence matrix with Perron–Frobenius eigenvalue the golden mean is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. There are two substitutions with this incidence matrix: Fibonacci φ, and reverse Fibonacci φ_R, defined by

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[1] This follows from the fact that any word in the language of η occurs in some concatenation of the two words $12 \dots F_n$ and $12 \dots F_{n-1}$.

$$\varphi_{\mathrm{R}}: \qquad 0 \to 10,\ 1 \to 0.$$

These two substitutions are essentially different, as they have different standard forms (see [6] for the definition of standard form).

However, it follows directly from Tan Bo's criterion in his paper [12] that φ_{R} and φ have the same language,[2] but then they also generate the same system. Conclusion: the conjugacy class of Fibonacci with Perron–Frobenius eigenvalue the golden mean restricted to two symbols consists of Fibonacci and reverse Fibonacci (up to a permutation of symbols).

7. The Fibonacci system and the golden mean rotation

A topological dynamical system (X, τ) is called *equicontinuous* if the family of maps $\{\tau^n : n \in \mathbb{Z}\}$ is uniformly equicontinuous.

Let T be the mapping from the unit circle $\mathbb{T}_1$ to itself defined by $Tt = t + \gamma \mod 1$, where $\gamma = (\sqrt{5} - 1)/2$ is the small golden mean. This, being an irrational rotation, is indeed an equicontinuous dynamical system – the usual distance metric is an invariant metric under the mapping. The factor map from the Fibonacci dynamical system (X_φ, σ) to $(\mathbb{T}_1, T)$ is given by requiring that the cylinder sets $\{x : x_0 = 0\}$ and $\{x : x_0 = 1\}$ are mapped to the intervals $[0, \gamma]$ and $[\gamma, 1]$ respectively, and requiring equivariance. If we take any point of $\mathbb{T}_1$ not of the form $n\gamma \mod 1$ (n any integer), then the corresponding sequence is unique. If, however, we use an element in the orbit of γ, then for this point there will be two codes, a “left” one and a “right” one.

We want to understand more generally why two or more points map to a single point. Suppose x and y are two points of a system (X, τ) that map to two points x' and y' in an equicontinuous factor. Then for any power of T (the map of the factor system) the distance between $T^n(x')$ and $T^n(y')$ is just equal to the distance between x' and y'. So x and y map to the same point x' if either all x_n and y_n are equal for sufficiently large n, or all x_n and y_n are equal for sufficiently large $-n$. We say that x and y are respectively *right asymptotic* or *left asymptotic*

A pair of letters (b, a) is called a *cyclic pair* of a substitution α if ba is an element of the language of α, and for some integer m

$$\alpha^m(b) = \dots b \quad \text{and} \quad \alpha^m(a) = a \dots .$$

Such a pair gives an infinite sequence of words $\alpha^{mk}(ba)$ in the language of α, which—if properly centered—converge to an infinite word which is a fixed point of α^m. With a slight abuse of notation we denote this word by $\alpha^\infty(b) \cdot \alpha^\infty(a)$.

For the Fibonacci substitution φ, $(0, 0)$ and $(1, 0)$ are cyclic pairs, and the two synchronized points $\varphi^\infty(0) \cdot \varphi^\infty(0)$ and $\varphi^\infty(1) \cdot \varphi^\infty(0)$, are right asymptotic so they map to the same point in the equicontinuous factor $(\mathbb{T}_1, T)$.

Because of these considerations we now define Z-triples. Let η be a primitive substitution. Call three points x, y, and z in X_η a *Z-triple* if they are generated by three cyclic pairs of the form (b, a), (b, d) and (c, d), where $a, b, c, d \in A$. Then x, y, and z are mapped to the same point in an equicontinuous factor.

Theorem 7.1. *Let (X_η, σ) be any substitution dynamical system topologically isomorphic to the Fibonacci dynamical system. Then there do not exist Z-triples in X_η.*

Proof. Since (X_η, σ) is topologically isomorphic to (X_φ, σ), it has $(\mathbb{T}_1, T)$ as equicontinuous factor, and the factor map is at most 2-to-1. Suppose (b, a), (b, d) and (c, d) gives a Z-triple x, y, z in X_η. Noting that

$$x = \eta^\infty(b) \cdot \eta^\infty(a), \quad y = \eta^\infty(b) \cdot \eta^\infty(d)$$

are left asymptotic, and $y = \eta^\infty(b) \cdot \eta^\infty(d)$ and $z = \eta^\infty(c) \cdot \eta^\infty(d)$ are right asymptotic, this would give a contradiction. □

Example. Let η be the substitution given by

$$\eta: \qquad 1 \to 12,\ 2 \to 34,\ 3 \to 5,\ 4 \to 1,\ 5 \to 23.$$

Then η generates a system that is topologically isomorphic to the Fibonacci system (η is the substitution at the end of Section 4). Quite remarkably, η^6 admits 5 fixed points generated by the cyclic pairs $(1, 2)$, $(2, 3)$, $(3, 1)$, $(4, 5)$ and $(5, 1)$. Note however, that no three of these form a Z-triple.

8. Fibonacci matrices

Let $\mathcal{F}_r$ be the set of all non-negative primitive $r \times r$ integer matrices, with Perron–Frobenius eigenvalue the golden mean $\Phi = (1 + \sqrt{5})/2$.

We have seen already that $\mathcal{F}_2$ consists of the single matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

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[2] This follows also directly from the well-known formula $\varphi_{\mathrm{R}}^{2n}(0)\,10 = 01\,\varphi^{2n}(0)$ for all $n \geq 1$ (see [1, p.17]).

Theorem 8.1. *The class $\mathcal{F}_3$ essentially consists of the three matrices*

$$\mathrm{M}_1 = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,1 \\ 1\,1\,0 \end{pmatrix}, \ \mathrm{M}_2 = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,2\,0 \end{pmatrix}, \ \textit{and}\ \mathrm{M}_3 = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,1 \\ 1\,0\,1 \end{pmatrix}.$$

Here essentially means that in each class of 6 matrices corresponding to the permutations of the $r = 3$ symbols, one representing member has been chosen (actually corresponding to the smallest standard form of the substitutions having that matrix).

Proof. Let M be a non-negative primitive 3×3 integer matrix, with Perron–Frobenius eigenvalue the golden mean $\Phi = (1 + \sqrt{5})/2$. We write

$$M = \begin{pmatrix} a\,b\,c \\ d\,e\,f \\ g\,h\,i \end{pmatrix}.$$

The characteristic polynomial of M is $\chi_M(u) = u^3 - Tu^2 + Fu - D$, where $T = a + e + i$ is the trace of M, and

$$F = ae + ai + ei - bd - cg - fh, \quad D = aei + bfg + cdh - afh - bdi - ceg. \tag{2}$$

Of course D is the determinant of M. Since Φ is an eigenvalue of M, and we consider matrices over the integers, $u^2 - u - 1$ has to be a factor of χ_M. Performing the division we obtain

$$\chi_M(u) = \big(u - (T - 1)\big)\big(u^2 - u - 1\big) + (2 + F - T)u + 1 - T - D,$$

and requiring that the remainder vanishes, yields

$$F = T - 2, \quad D = 1 - T. \tag{3}$$

Note that the third eigenvalue equals $\lambda_3 = T - 1$. From the Perron–Frobenius theorem follows that this has to be smaller than Φ in absolute value, and since it is an integer, only $\lambda_3 = -1, 0, 1$ are possible. Thus there are only 3 possible values for the trace of M: $T = 0$, $T = 1$ and $T = 2$.

The smallest row sum of M has to be smaller than the PF-eigenvalue Φ (well known property of primitive non-negative matrices). Therefore M has to have one of the rows $(0, 0, 1)$, $(0, 1, 0)$ or $(0, 0, 1)$. Also, because of primitivity of M, the 1 in this row can not be on the diagonal. By performing permutation conjugacies of the matrix we may then assume that M has the form

$$M = \begin{pmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{pmatrix}.$$

The equation (2) combined with (3) then simplifies to

$$T - 2 = F = ei - d - fh, \quad 1 - T = D = fg - di. \tag{4}$$

Case $T = 0$
In this case $e = i = 0$, so (4) simplifies to

$$-2 = F = -d - fh, \quad 1 = D = fg. \tag{5}$$

Then $f = g = 1$, and so $d + h = 2$. This gives three possibilities leading to the matrices $\mathrm{M}_1, \mathrm{M}_2$ and a third matrix, which is permutation conjugate to M_2.

Case $T = 1$
In this case $e = 1, i = 0$, or $e = 0, i = 1$.
First case: $e = 1, i = 0$. Now (4) simplifies to

$$-1 = F = -d - fh, \quad 0 = D = fg. \tag{6}$$

Then $g = 0$, since $f = 0$ is not possible because of primitivity. But $g = 0$ also contradicts primitivity, as $d + fh = 1$, gives either $d = 0$ or $h = 0$.
Second case: $e = 0, i = 1$. Now (4) simplifies to

$$-1 = F = -d - fh, \quad 0 = D = fg - d. \tag{7}$$

Then $d = 0$ would imply that $f = h = 1$. But, as $g > 0$ because of primitivity, we get a contradiction with $fg = d = 0$.

On the other hand, if $d > 0$, then $d = 1$ and $f = 0$ or $h = 0$. But $fg = d = 1$ gives $f = g = 1$, so $h = 0$, and we obtain the matrix M_3.

Case $T=2$
In this case (4) becomes

$$0=F=ei-d-fh,\quad -1=D=fg-di. \tag{8}$$

Since $ei=0$ would lead to $f=0$, which is not allowed by primitivity, what remains is $e=1, i=1$. Then, substituting $d=fg+1$ in the first equation gives $0=f(g+h)$. But both $f=0$ and $g=h=0$ contradict primitivity.

Final conclusion: there are three matrices in $\mathcal{F}_3$, modulo permutation conjugacies. □

Remark. It is well-known that the PF-eigenvalue lies between the smallest and the largest row sum of the matrix. One might wonder whether this largest row sum is bounded for the class $\mathcal{F}=\cup_r\mathcal{F}_r$. Actually the class $\mathcal{F}_r$ contains matrices with some row sum equal to $r-1$ for all $r\ge 3$: take the matrix M with $M_{1,j}=1$ for $j=2,\dots,r$, $M_{2,2}=1$ and $M_{i,i+1}=1$, for $i=2,\dots,r-1$, $M_{r,1}=1$ and all other entries 0.

Now note that $(1,\Phi,\dots,\Phi)$ is a left eigenvector of M with eigenvalue Φ (since $\Phi^2=1+\Phi$). Since the eigenvector has all entries positive, it must be a PF-eigenvector (well known property of primitive, non-negative matrices), and hence M is in $\mathcal{F}_r$.

9. The 3-symbol case

Theorem 9.1. *On a three letter alphabet $\{a,b,c\}$ there are two primitive injective substitutions η and ζ with Perron–Frobenius eigenvalue the golden mean that generate dynamical systems topologically isomorphic to the Fibonacci system. These are given*[3] *by*

$$\eta(a)=b,\ \eta(b)=ca,\ \eta(c)=ba,\quad \zeta(a)=b,\ \zeta(b)=ac,\ \zeta(c)=ab.$$

Proof. The possible matrices for candidate substitutions with PF-eigenvalue the golden mean are given in Theorem 8.1. Let us first consider the matrix M_1. There are four substitutions with this matrix as incidence matrix:

$$\eta_1:\ a\to b,\ b\to ca,\ c\to ba,\qquad \eta_2:\ a\to b,\ b\to ca,\ c\to ab,$$
$$\eta_3:\ a\to b,\ b\to ac,\ c\to ba,\qquad \eta_4:\ a\to b,\ b\to ac,\ c\to ab.$$

Here $\eta_1=\eta$. To prove that the system of η is conjugate to the Fibonacci system consider the letter-to-letter map π given by

$$\pi(a)=1,\quad \pi(b)=\pi(c)=0.$$

Then π maps X_η onto X_φ, because $\pi\circ\eta=\varphi\circ\pi$. Moreover, π is a conjugacy, since if $x\neq y$ and $\pi(x)=\pi(y)$, then there is a k such that $x_k=b$ and $y_k=c$. But the words of length 2 in the language of η are ab, ba, bc and ca, implying that $x_{k-1}=a$ and $y_{k-1}=b$, contradicting $\pi(x)=\pi(y)$.

Since ζ is the time reversal of η, and we know already that the system of φ_{R} is conjugate to the Fibonacci system, the system generated by $\eta_4=\zeta=\eta_{\mathrm{R}}$ is conjugate to the Fibonacci system.

It remains to prove that η_2 and η_3 generate systems that are *not* conjugate to the Fibonacci system. Again, since η_3 is the time reversal of η_2, it suffices to do this for η_2. The language of η_2 contains the words ab, bb and bc. These words generate fixed points of η_2^6 in the usual way. But these three fixed points form a Z-triple, implying that the system of η_2 can not be topologically isomorphic to the Fibonacci system (see Theorem 7.1).

The next matrix we have to consider is M_2. There are three substitutions with this matrix as incidence matrix:

$$\eta_1:\ a\to b,\ b\to c,\ c\to abb,\qquad \eta_2:\ a\to b,\ b\to c,\ c\to bab,$$
$$\eta_3:\ a\to b,\ b\to c,\ c\to bba.$$

Again, the system of η_1 contains a Z-triple generated by ab, bb and bc. So this system is not conjugate to the Fibonacci system, and neither is the one generated by η_3 (time reversal of η_1). The system generated by η_2 behaves similarly to the Fibonacci system, *but* is has an eigenvalue -1 (it has a two-point factor via the projection $a,c\to 0, b\to 1$).

Finally, we have to consider the matrix M_3. There are four substitutions with this matrix as incidence matrix:

$$\eta_1:\ a\to b,\ b\to ac,\ c\to ac,\qquad \eta_2:\ a\to b,\ b\to ac,\ c\to ca,$$
$$\eta_3:\ a\to b,\ b\to ca,\ c\to ac,\qquad \eta_4:\ a\to b,\ b\to ca,\ c\to ca.$$

Here η_1 and η_4 generate systems conjugate to the Fibonacci system, but the substitutions are not injective. The substitution η_2 has all 9 words of length 2 in its language, and all of these generate fixed points of η_2^6. So the system of η_2 is certainly not topologically isomorphic to the Fibonacci system. The proof is finished, since η_3 is the time reversal of η_2. □

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[3] Standard forms: replace a, b, c by $1, 2, 3$.

Remark. An interesting question is: what is the set $\mathcal{C}_{\varphi,3}$ of *all* primitive injective substitutions on 3 symbols that generate systems conjugate to the Fibonacci system? Of course all the powers of η and ζ are in $\mathcal{C}_{\varphi,3}$, where η and ζ are the two substitutions in Theorem 9.1. Another element of $\mathcal{C}_{\varphi,3}$ is $a \to c$, $b \to abab$, $c \to cab$, which is a reshaping of η^2. An application of Theorem 3.1 with the words $B_0 = abc$, $B_1 = ab$ gives that the substitution $a \to ab$, $b \to cab$, $c \to abc$ is also in $\mathcal{C}_{\varphi,3}$.

An obvious candidate for $\mathcal{C}_{\varphi,3}$ is the substitution $\theta := \zeta \circ \eta$ given by

$$\theta(a) = ac,\ \theta(b) = abb,\ \theta(c) = acb.$$

The projection π given by $\pi(a) = 0$, $\pi(b) = \pi(c) = 1$ yields the Fibonacci system as a factor (it gives an *amalgamation* as considered in [2]: $\pi \circ \theta = \beta \circ \pi$, where β is the substitution given by $\beta(0) = 01$, $\beta(1) = 011$, which after mirroring is conjugate to the square of the Fibonacci substitution φ). *However*, the projection π is not injective, as can be seen by considering the two sequences of words $\theta^n(b)$ and $\theta^n(c)$: for all n these are equal except at one position where they are b respectively c. So $\pi(\theta^n(b)) = \pi(\theta^n(c))$, and taking an appropriate limit leads to two different sequences with the same projection. Thus the system generated by this θ is *not* in the conjugacy class of the Fibonacci system. Interestingly, if we make the same choice in the 2-symbol case, i.e., consider the substitution $\theta := \varphi \circ \varphi_{\mathrm{R}}$ given by

$$\varphi \circ \varphi_{\mathrm{R}}(0) = 001,\ \varphi \circ \varphi_{\mathrm{R}}(1) = 01,$$

then θ is conjugate (as a morphism!) to φ^2, and so (X_θ, σ) is conjugate to (X_φ, σ).

10. Letter-to-letter maps

By the Curtis–Hedlund–Lyndon theorem all members in the conjugacy class of the Fibonacci system can be obtained by applying letter-to-letter maps π to N-block presentations $(X^{[N]}, \sigma)$. Here we analyze the case $N = 2$. The 2-block presentation of the Fibonacci system is generated by (see Section 2) the 2-block substitution

$$\hat{\varphi}_2(1) = 12 \quad \hat{\varphi}_2(2) = 3, \quad \hat{\varphi}_2(3) = 12.$$

There are (modulo permutations of the symbols) three letter-to-letter maps from $\{1, 2, 3\}$ to $\{0, 1\}$. Two of these project onto the Fibonacci system, as they are projections on the first respectively the second letter of the 2-blocks. The third is π given by

$$\pi(1) = 0, \quad \pi(2) = 0, \quad \pi(3) = 1.$$

What is the system (Y, σ) with $Y = \pi\big(X^{[2]}\big)$?

First note that (Y, σ) is conjugate to the Fibonacci system since π is clearly invertible. Secondly, we note that the points in Y can be obtained by doubling the 0's in the points of the Fibonacci system. This holds because $\pi(12) = 00$, $\pi(3) = 1$, but also

$$\pi(\hat{\varphi}_2(12)) = \pi(123) = 001,\ \pi(\hat{\varphi}_2(3)) = \pi(12) = 00.$$

Thirdly, we claim that the system (Y, σ) cannot be generated by a substitution. This follows from the fact that the sequences in Y contain the word 0000, but no other fourth powers. This is implied by the 4th power-freeness of the Fibonacci word, proved in [9].

A fourth property is that the sequence y^+ obtained by doubling the 0's in w_{F} (where w_{F} is the infinite Fibonacci word) is given by

$$y_n^+ = [(n+2)\Phi] - [n\Phi] - [2\Phi], \qquad \text{for } n \geq 1,$$

according to [8] (here $[\cdot]$ denotes the floor function). For a proof, write

$$y_n^+ = [(n+2)\Phi] - [(n+1)\Phi] + [(n+1)\Phi] - [n\Phi] - 3,$$

and use that $([(n+1)\Phi] - [n\Phi])$ is the well known way to obtain the Fibonacci word on the letters 2 and 1.

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References

[1] Jean Berstel, Fibonacci words – a survey, in: G. Rozenberg, A. Salomaa (Eds.), The Book of L, Springer Verlag, 1986, pp. 13–26.
[2] Ethan M. Coven, F. Michel Dekking, Michael S. Keane, Topological conjugacy of constant length substitution dynamical systems, Indag. Math. 28 (2017) 91–107.
[3] Wai-Fong Chuan, Hui-Ling Ho, Locating factors of the infinite Fibonacci word, Theoret. Comput. Sci. 349 (2005) 429–442.
[4] E. Coven, M. Keane, M. LeMasurier, A characterization of the Morse minimal set up to topological conjugacy, Ergodic Theory Dynam. Systems 28 (2008) 1443–1451.
[5] Michel Dekking, On the structure of Thue–Morse subwords, with an application to dynamical systems, Theoret. Comput. Sci. 550 (2014) 107–112.
[6] F. Michel Dekking, Morphisms, symbolic sequences, and their standard forms, J. Integer Seq. 19 (2016) 16.1.1.
[7] Yuke Huang, ZhiYing Wen, The sequence of return words of the Fibonacci sequence, Theoret. Comput. Sci. 593 (2015) 106–116.
[8] The On-Line Encyclopedia of Integer Sequences, founded by N.J.A. Sloane, sequence A123740.
[9] Juhani Karhumäki, On cube-free ω-words generated by binary morphisms, Discrete Appl. Math. 5 (1983) 279–297.
[10] J. Karhumäki, J. Manuch, W. Plandowski, A defect theorem for bi-infinite words, Theoret. Comput. Sci. 292 (2003) 237–243.
[11] Martine Queffélec, Substitution Dynamical Systems – Spectral Analysis, 2nd ed., Lecture Notes in Mathematics, vol. 1294, Springer, Berlin, 2010.
[12] Bo Tan, Mirror substitutions and palindromic sequences, Theoret. Comput. Sci. 389 (2007) 118–124.
[13] Z.-X. Wen, Z.-Y. Wen, Some properties of the singular words of the Fibonacci word, European J. Combin. 15 (1994) 587–598.