

# Analytical Micromechanical Modelling of In-plane Shear Behaviour of Unidirectional Composite Plies

MSc Thesis

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Killian Caignie

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Cover: RGB colour representation of my student number.

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# Preface

This thesis manuscript marks the end of my time as a student at Delft University of Technology. Over a span of seven years, this journey has not only been shaped by academic development, but by a series of experiences that ultimately defined my interests, ambitions, and perspective as an engineer.

My bachelor years provided a broad foundation in aerospace engineering, though they were, to a large extent, marked by the disruptions of the COVID 19 pandemic. While this period resulted in a not so typical student experience, it also reinforced independence and adaptability, qualities that would prove valuable later on.

A defining moment came during my gap year as Chief of Structures at AeroDelft. Taking responsibility for structural design in a real engineering context shifted my perspective from studying theory to applying it under constraints. This experience played a key role in my decision to pursue a master's degree at the department of Aerospace Structures and Materials.

During this master's programme, I developed a strong interest in composite materials, largely inspired by the teachings and enthusiasm of my supervisor, Dr. Christos Kassapoglou. His ability to connect fundamental mechanics with practical applications made a lasting impression on me and significantly influenced the direction of both my internship and thesis.

Beyond Delft, my studies brought me to a range of places and environments. A minor in Electrical Engineering in Espoo, Finland, introduced me not only to a different academic culture and way of life but also to bouldering, a hobby that would later take an unexpected turn. An internship at Dronamics in Bulgaria provided hands on experience in an ambitious aerospace start up, as well as many memorable moments with the team.

That internship, however, ended abruptly due to a dislocated shoulder sustained while bouldering. This eventually led to surgery, during which thermoplastic PEEK and UHMWPE fibers were used in reconstruction. In a somewhat ironic twist, this has made me, in a literal sense, a composite structure myself, a fact I have found difficult to resist mentioning, be it in conversations with friends or in job applications.

I would like to express my sincere gratitude to my supervisor, Dr. Kassapoglou, for his guidance, patience, and continuous support throughout this work.

All of the above would not have been possible without the unwavering support of my parents, for which I am eternally grateful. I am equally grateful to my brother, my girlfriend, and friends for their encouragement, patience, and for providing balance outside of academic life. To all my friends around the world, I wish you all the very best.

*"One day, in retrospect, the years of struggle will strike you as the most beautiful."*

— Sigmund Freud

*Killian Caignie  
Delft, May 2026*



# Summary

This thesis presents a micromechanical modelling framework to predict the in-plane shear stress–strain response of unidirectional (UD) composite plies from constituent-level properties. The approach is based on a representative volume element (RVE) formulation in which the fiber–matrix system is idealised as a Timoshenko beam on an elastic foundation, allowing the reduction of the three-dimensional problem to a tractable one-dimensional formulation.

A deterministic analytical model is first developed to describe the elastic response of the RVE. Closed-form expressions for the effective shear stiffness and shear modulus are obtained and analysed as a function of fiber volume fraction. The formulation provides insight into the roles of fiber and matrix contributions, as well as the influence of modelling assumptions such as isotropy and shear correction factors. Supporting finite element analyses are used to examine the implications of these assumptions.

The formulation is subsequently extended to incorporate constituent failure mechanisms, including fiber fracture and matrix cracking. Local stress redistributions following fiber failure are captured through a superposition approach, enabling the construction of piecewise load–displacement responses at the RVE level. Different stiffness degradation mechanisms based on interfacial and radial cracking are considered and their influence on the mechanical response is evaluated.

To bridge the gap between micromechanical predictions and ply-level behaviour, a statistical homogenisation procedure is introduced. Variability in fiber volume fraction is represented using a probabilistic description, and the resulting distribution of local stiffnesses is homogenised to obtain an effective ply shear modulus. The final ply response is constructed by combining the contributions of intact and damaged regions through a homogenisation principle based on strain compatibility and a failure fraction, linking local degradation to the global stress–strain response.

The methodology is further extended to account for nonlinear matrix behaviour through an implicit constitutive relation, allowing the prediction of nonlinear shear response. The combined framework enables the construction of full stress–strain curves, incorporating both material nonlinearity and progressive damage.

Model predictions are compared with available experimental data and existing analytical approaches, and are used to assess the influence of modelling assumptions and constituent behaviour on the predicted response.

Overall, the thesis establishes a framework that links constituent properties, microstructural variability, and damage mechanisms to the macroscopic shear response of UD composite plies.

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# Nomenclature

## Abbreviations

| Abbreviation | Definition                       |
|--------------|----------------------------------|
| CDM          | Continuum Damage Mechanics       |
| CDF          | Cumulative Distribution Function |
| FEA          | Finite Element Analysis          |
| MADM         | Macro-Damage Mechanics           |
| PDF          | Probability Density Function     |
| RMSE         | Root Mean Square Error           |
| RoM          | Rule of Mixtures                 |
| RVE          | Representative Volume Element    |
| UD           | Unidirectional                   |

## Symbols

Unless stated otherwise, lengths are expressed in  $\mu\text{m}$ , stresses and moduli in GPa, and forces in mN.

| Symbol                             | Definition                                               | Unit                         |
|------------------------------------|----------------------------------------------------------|------------------------------|
| $A, A_f, A_m$                      | Cross-sectional areas                                    | $[\mu\text{m}^2]$            |
| $A_{\text{hex}}$                   | Hexagonal cross-sectional area                           | $[\mu\text{m}^2]$            |
| $A_{\text{RVE}}$                   | Cross-sectional area of the RVE                          | $[\mu\text{m}^2]$            |
| $A_{\mathcal{N}}, A_{\mathcal{W}}$ | Fitting parameters for normal and Weibull CDFs           | [-]                          |
| $b$                                | Effective width of RVE / foundation                      | $[\mu\text{m}]$              |
| $b_0$                              | Intercept of linear relation                             | [-]                          |
| $C_0$                              | Integration constant                                     | $[\mu\text{m GPa}]$          |
| $C_s$                              | Shear stiffness of fiber beam                            | $[\text{mN}]$                |
| $c, c_f$                           | Correction factors for shear modulus                     | [-]                          |
| $c_0, c_1, c_2$                    | Polynomial coefficients                                  | [-]                          |
| $d_f$                              | Fiber diameter                                           | $[\mu\text{m}]$              |
| $d_{\text{eff}}$                   | Effective recovery length                                | $[\mu\text{m}]$              |
| $D_b$                              | Fiber bending stiffness                                  | $[\text{mN } \mu\text{m}^2]$ |
| $E, E_f, E_m$                      | Young's moduli                                           | $[\text{GPa}]$               |
| $E_{\text{excess}}$                | Excess strain energy                                     | $[\text{mN } \mu\text{m}]$   |
| $f$                                | Failure fraction                                         | [-]                          |
| $f_f, f_m$                         | Fiber and matrix load fractions                          | [-]                          |
| $G$                                | Coefficient multiplying $d^2w/dx^2$ in moment field      | $[\text{mN } \mu\text{m}^2]$ |
| $G_f, G_m$                         | Fiber and matrix shear moduli                            | $[\text{GPa}]$               |
| $G_{12}$                           | Ply in-plane shear modulus                               | $[\text{GPa}]$               |
| $G_{m,0}$                          | Initial matrix shear modulus                             | $[\text{GPa}]$               |
| $G_{\text{RVE}}$                   | Effective RVE shear modulus                              | $[\text{GPa}]$               |
| $G_{II}$                           | Mode II energy release rate                              | $[\text{mN}/\mu\text{m}]$    |
| $H$                                | Coefficient multiplying $w$ in moment field              | $[\text{mN}]$                |
| $I$                                | Coefficient multiplying $dw/dx$ in shear-force field     | $[\text{mN}]$                |
| $I_f$                              | Fiber second moment of area                              | $[\mu\text{m}^4]$            |
| $J$                                | Coefficient multiplying $d^3w/dx^3$ in shear-force field | $[\text{mN } \mu\text{m}^2]$ |
| $k_0$                              | Fitting parameter                                        | [-]                          |

| Symbol                           | Definition                                            | Unit                                            |
|----------------------------------|-------------------------------------------------------|-------------------------------------------------|
| $k_{99}$                         | One-sided tolerance factor                            | [-]                                             |
| $k_n, k_\tau$                    | Foundation moduli                                     | [mN/ $\mu\text{m}^3$ ],<br>[mN/ $\mu\text{m}$ ] |
| $K$                              | Ramberg–Osgood parameter                              | [GPa]                                           |
| $K_w$                            | Winkler stiffness                                     | [mN/ $\mu\text{m}^2$ ]                          |
| $K_{\text{RVE}}$                 | Effective RVE shear stiffness                         | [mN]                                            |
| $L$                              | RVE length                                            | [ $\mu\text{m}$ ]                               |
| $L_{\text{int}}, L_{\text{rad}}$ | Interfacial and radial crack lengths                  | [ $\mu\text{m}$ ]                               |
| $m$                              | Slope of linear relation                              | [-]                                             |
| $M(x)$                           | Internal bending moment                               | [mN $\mu\text{m}$ ]                             |
| $\mathbf{M}^{(\pm)}$             | Material system matrix                                | [-]                                             |
| $n$                              | Ramberg–Osgood exponent                               | [-]                                             |
| $n_f, n_m$                       | Fiber and matrix shear correction factors             | [-]                                             |
| $n_{\text{RVE}}$                 | RVE shear correction factor                           | [-]                                             |
| $p$                              | Power-mean exponent                                   | [-]                                             |
| $q_0$                            | Distributed load                                      | [mN/ $\mu\text{m}$ ]                            |
| $Q(x)$                           | Internal shear force                                  | [mN]                                            |
| $Q_f, Q_m$                       | Fiber and matrix shear forces                         | [mN]                                            |
| $Q_{\text{RVE}}$                 | Shear force in RVE                                    | [mN]                                            |
| $r, r_f$                         | Radial coordinate and fiber radius                    | [ $\mu\text{m}$ ]                               |
| $R^{(\pm)}$                      | Ratio of ODE coefficients                             | [-]                                             |
| $R_{\text{eq}}$                  | Equivalent RVE radius                                 | [ $\mu\text{m}$ ]                               |
| $R_{\text{in}}, R_{\text{out}}$  | Inscribed and circumscribed radii                     | [ $\mu\text{m}$ ]                               |
| $R_i$                            | Rotational spring stiffness                           | [mN]                                            |
| $s$                              | Fiber spacing                                         | [ $\mu\text{m}$ ]                               |
| $S, S_f, S_m$                    | Shear strengths                                       | [GPa]                                           |
| $S_{12}$                         | Ply shear strength                                    | [GPa]                                           |
| $T_s$                            | Shear layer stiffness                                 | [mN]                                            |
| $t_{\text{m,eq}}$                | Equivalent matrix thickness                           | [ $\mu\text{m}$ ]                               |
| $u_i$                            | Displacement components                               | [ $\mu\text{m}$ ]                               |
| $u_{\text{excess}}$              | Excess strain energy density                          | [GPa]                                           |
| $U_{\text{tot}}$                 | Total strain energy                                   | [mN $\mu\text{m}$ ]                             |
| $U$                              | Coefficient multiplying $dw/dx$ in rotation field     | [-]                                             |
| $V$                              | Coefficient multiplying $d^3w/dx^3$ in rotation field | [ $\mu\text{m}^2$ ]                             |
| $v_f, v_m$                       | Fiber and matrix volume fractions                     | [-]                                             |
| $V_f, V_m, V_{\text{ply}}$       | Volumes                                               | [ $\mu\text{m}^3$ ]                             |
| $w(x)$                           | Transverse displacement                               | [ $\mu\text{m}$ ]                               |
| $w_0(x)$                         | Affine displacement                                   | [ $\mu\text{m}$ ]                               |
| $w_{\text{corr}}(x)$             | Correction displacement field                         | [ $\mu\text{m}$ ]                               |
| $w_f$                            | Crack displacement amplitude                          | [ $\mu\text{m}$ ]                               |
| $x, y, z$                        | Cartesian coordinates                                 | [ $\mu\text{m}$ ]                               |
| $X_t, X_c$                       | Tensile and compressive strengths                     | [GPa]                                           |
| $X_1, X_{99}$                    | Distribution percentiles                              | [-]                                             |
| $A, B, C, D$                     | Integration constants of governing ODE                | [ $\mu\text{m}$ ]                               |
| $\alpha, \beta$                  | Spatial wave numbers                                  | [ $\mu\text{m}^{-1}$ ]                          |
| $\alpha_\beta, \beta_\beta$      | Beta distribution shape parameters                    | [-]                                             |
| $\beta(x)$                       | Shear angle                                           | [rad]                                           |
| $\theta$                         | Circumferential coordinate (cylindrical system)       | [rad]                                           |
| $\delta_I, \delta_{II}$          | Crack opening displacements                           | [ $\mu\text{m}$ ]                               |
| $\Delta$                         | Discriminant of governing ODE                         | [ $\mu\text{m}^{-4}$ ]                          |
| $\epsilon_{11}, \epsilon_{13}$   | Strain components                                     | [-]                                             |
| $\epsilon_G$                     | Relative $G_{\text{RVE}}$ error                       | [%]                                             |
| $\gamma_{12}, \gamma_{12,0}$     | Shear strains                                         | [-]                                             |
| $\eta$                           | Mesh density parameter                                | [-]                                             |

| Symbol                | Definition                              | Unit                      |
|-----------------------|-----------------------------------------|---------------------------|
| $\eta_s$              | Stress partitioning parameter           | [-]                       |
| $\mu$                 | Mean value                              | [-]                       |
| $\nu, \nu_f, \nu_m$   | Poisson's ratios                        | [-]                       |
| $\Pi$                 | Total potential energy                  | [mN $\mu$ m]              |
| $\rho$                | Density                                 | [kg/m <sup>3</sup> ]      |
| $\sigma$              | Standard deviation                      | [-]                       |
| $\boldsymbol{\sigma}$ | Stress tensor                           | [GPa]                     |
| $\tau$                | Shear stress                            | [GPa]                     |
| $\phi, \psi$          | Spatial wave-number combinations        | [ $\mu$ m <sup>-3</sup> ] |
| $\Phi$                | Cumulative distribution function        | [-]                       |
| $\psi(x)$             | Beam rotation                           | [rad]                     |
| $c_i^{(\pm)}$         | Coefficient sets of ODE solution        | variable                  |
| $c_{i,w}$             | Displacement coefficient                | [-]                       |
| $c_{i,wx}$            | First derivative coefficient            | [ $\mu$ m <sup>-1</sup> ] |
| $c_{i,wx x}$          | Second derivative coefficient           | [ $\mu$ m <sup>-2</sup> ] |
| $c_{i,wx x x}$        | Third derivative coefficient            | [ $\mu$ m <sup>-3</sup> ] |
| $c_{i,wx x x x}$      | Fourth derivative coefficient           | [ $\mu$ m <sup>-4</sup> ] |
| $c_{i,\psi}$          | Rotation coefficient                    | [-]                       |
| $c_{i,Q}$             | Shear-force coefficient                 | [mN/ $\mu$ m]             |
| $c_{i,M}$             | Moment coefficient                      | [mN]                      |
| $c_{i,\epsilon_{11}}$ | Axial-strain coefficient                | [ $\mu$ m <sup>-1</sup> ] |
| $c_{i,\gamma_{13}}$   | Shear-strain coefficient                | [ $\mu$ m <sup>-1</sup> ] |
| $i$                   | Coefficient index ( $i = 1, \dots, 8$ ) | [-]                       |

# 1

## Introduction

This chapter introduces the context and motivation for this thesis work and defines the problem addressed. The research objective and corresponding questions are formulated, followed by a clarification of the scope and underlying assumptions. Finally, an overview of the thesis outline is provided to guide the reader through the subsequent chapters.

### 1.1. Engineering context

Fiber-reinforced composite materials have become indispensable in modern engineering applications due to their high specific stiffness and strength, as well as their tailorability to specific loading conditions. In particular, unidirectional (UD) composite plies form the fundamental building blocks of laminated structures used in aerospace, automotive, and energy sectors [1], [2].

Despite their widespread use, predicting the mechanical response of composite materials remains challenging due to their inherently heterogeneous microstructure [3]. The macroscopic behaviour of a ply emerges from the interaction between its fiber and matrix constituents across multiple length scales.

The designing composite plies and their application in structural design, require the selection of the length scale which is relevant for the problem at hand and determines the analysis type. Whereas fiber diameters are in the order of a few micrometers, complete composite structures, such as wind turbine blades, quickly span several tens of meters.

If one is to design a composite laminate, i.e. a stacked and bonded collection of plies, it is common to seek the effective stiffness properties of the heterogeneous ply that emerge from the fiber and matrix properties and attribute those to an equivalent homogeneous ply. In this manner, the laminate designer no longer has to consider individual fiber and matrix properties in their initial analyses. Using the same principle, the laminate itself, being a collection of plies with possibly different effective stiffness properties, can be attributed effective stiffness properties of its own. In such a way, it can be treated as a homogeneous structural part in further design without the consideration of the smaller length scales at first. The process of finding such equivalent homogeneous properties of a heterogeneous ensemble is commonly referred to as homogenization [3], [4].

However, for detailed analyses that require the consideration of failure and damage of laminated structures, the different length scales at which a laminate can be designed also come with different types of damages at each length scale. Typical damages in laminates ranging from smallest to largest are: fiber surface flaws, failure of fiber-matrix interface, fiber and matrix cracks, through-thickness failures such as holes and cracks and inter-ply failures such as delaminations [5]. In the ideal case, a damage model can track damage at the lowest scale, e.g. a matrix crack, as it evolves, such that its effect on a larger length scale, e.g. a single ply, can be accounted for. Consequently, a model that bridges many length scales can quickly become computationally expensive. To prevent this, equivalent aggregate flaws such as notches are commonly used to model the effect on macroscopic structures such as laminates, at the expense of capturing damage evolution at lower scales [5].



In light of an increased emphasis on sustainability, accurate fatigue-life predictions for laminate structures are required. Based on earlier work by Kassapoglou [6] on fatigue life prediction for metals, accurate stress-strain curves for the individual ply loadings within a laminate are required to extend this method to composites. While such stress-strain curves can be experimentally obtained, the large amount of ply types associated with different fiber and matrix systems, combined with variations in manufacturing and loading conditions, results in a purely experimental approach being impractical. This motivates the development of a physics based modelling framework capable of predicting the ply-level response from constituent properties, thereby reducing the experimental effort while retaining predictive capability.

In this work, in-plane shear loading of a UD ply is considered, which is known to produce a nonlinear response dominated by matrix behaviour and progressive damage mechanisms. Failure is modelled at the level of the fiber and matrix, and through a homogenization procedure, the corresponding macroscopic ply response is obtained. In this manner, the evolution of local damage is systematically translated into an effective constitutive response at the ply level. A similar underlying concept has been demonstrated for tensile loading of unidirectional plies by Socci and Kassapoglou [7], although the present work extends this idea to shear-dominated behaviour and associated failure mechanisms.

## 1.2. Problem statement

The central challenge addressed in this work is the prediction of the mechanical response of a composite unidirectional ply from its microscopic properties. More specifically, the goal is to construct an accurate stress-strain curve for a unidirectional ply subjected to in-plane shear loading, based on a bottom-up modelling approach, starting with the interaction between an individual fiber and its surrounding matrix.

Such a stress-strain relation is a key input for fatigue life prediction methods, in which the energy dissipated per loading cycle is related to the area enclosed by the stress-strain curve [6]. Existing work has demonstrated this approach for longitudinal tensile loading, while extensions to other loading cases remain incomplete or unpublished [8]. In particular, while various micromechanical approaches exist, accurately capturing the shear response of a ply within analytical, low-computational frameworks remains challenging.

## 1.3. Research objective and questions

The primary objective of this thesis is to determine whether the macroscopic in-plane shear behaviour of a composite ply, as shown in Figure 1.1, can be predicted from a micromechanical perspective, taking into account constituent properties, local failure, and damage evolution.

This objective leads to the following main research question:

Can the stress-strain response of a unidirectional composite ply under in-plane shear loading be predicted from constituent-level properties using a micromechanical modelling framework?

To support this, several sub-questions are addressed:

- How can representative volume elements (RVEs) be formulated to capture the shear response of fiber-matrix systems?
- How do local failure mechanisms, such as fiber failure and matrix cracking, influence the global stress-strain behaviour?
- How can the contributions of intact and damaged material regions be combined into a consistent homogenised response?

## 1.4. Scope and assumptions

This thesis focuses exclusively on the prediction of the stress-strain response of unidirectional composite plies under in-plane shear loading. Although the broader objective includes fatigue life prediction, practical limitations have restricted the scope of this thesis to the construction of the stress-strain re-

sponse itself. The focus is therefore placed on capturing the underlying mechanisms governing shear behaviour, rather than on subsequent fatigue analysis.

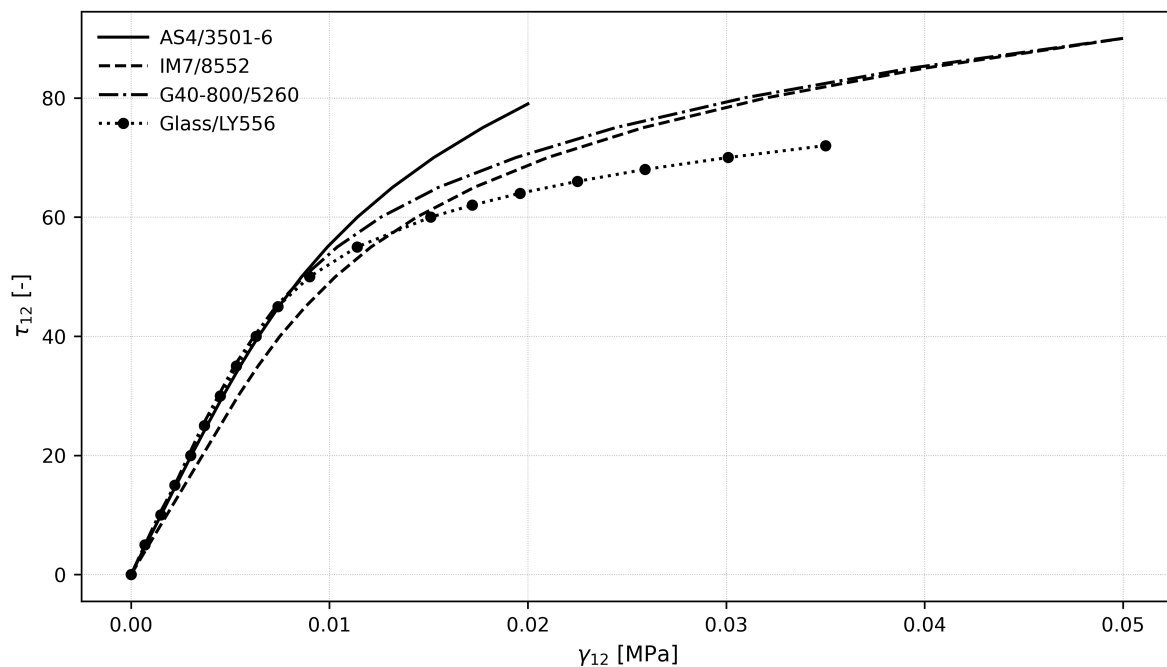
Rather than collecting all modelling assumptions in a single location, assumptions are introduced and justified throughout the thesis where they are most relevant. These include assumptions related to material behaviour, failure criteria, and homogenisation procedures.

## 1.5. Thesis outline

The structure of this thesis is as follows: Chapter 2 introduces relevant literature and the micromechanical background required to relate constituent behaviour to macroscopic properties. Chapter 3 presents the material systems and input data used for model construction and validation.

Chapter 4 develops the formulation of a fiber-matrix RVE and provides initial predictions of ply stiffness. Chapter 5 examines the RVE behaviour in more detail and presents comparative finite-element analyses. Chapter 6 introduces variability in fiber volume fraction to statistically refine the initial predictions of chapter 4. Chapter 7 considers the failure of fiber and matrix constituents and its effect on RVE stiffness.

These developments culminate in chapter 8, where macroscopic ply stress-strain curves are constructed. Finally, chapter 9 presents the conclusions and recommendations for future work.



**Figure 1.1:** In-plane shear stress-strain curves for considered materials

# 2

## Micromechanical Background and Literature Review

This chapter introduces the micromechanical background and modelling concepts relevant for describing the in-plane shear behaviour of unidirectional composite plies. Starting from the definition of composite microstructures and the concept of homogenisation, the role of representative volume elements in linking constituent behaviour to effective ply properties is described. Reduced-order micromechanical modelling approaches, including beam-based representations of fibers and elastic foundation models for the matrix, are then introduced to capture fiber–matrix interaction. Finally, the nonlinear in-plane shear response is discussed, highlighting the roles of matrix-dominated deformation, inelasticity, and damage, as well as existing modelling approaches and their limitations. This forms the basis for the development of the analytical framework presented in the subsequent chapters.

### 2.1. Motivation and Existing Frameworks

In recent work by Kassapoglou [6], a fatigue life prediction framework was developed based on the concept of dissipation of energy per unit volume during cyclic loading. The central idea of this energy dissipation is that fatigue failure occurs when the total energy dissipated over repeated loading cycles reaches the available energy to failure of the material, which corresponds to the area under the full stress-strain curve. This concept has been used in several earlier works and all of them conclude that the dissipated energy during cycling only occurs in the presence of plastic strains [6]. Whereas these works still relied on experimental results and simulations, Kassapoglou [6] provides a closed-form expression for the dissipated energy based on phenomenological properties of the loading and unloading of metallic specimens, showing very good agreement with test results.

An important implication of this formulation is that accurate fatigue predictions rely heavily on the quality of the underlying stress-strain curves used. In particular, small deviations from linear elastic behaviour have a significant influence on the predicted fatigue life. This highlights the necessity of physically accurate and high-resolution stress-strain characterisation when applying such energy-based fatigue models, while also showing that inherent scatter in experimentally obtained stress-strain data can reproduce the inherent scatter observed in fatigue lives [6].

In a first attempt at extending this framework to composites, Wojtuszewski et al. [9] accurately predicted the fatigue life of a thermoplastic bathtub fitting by first determining the critical stress to be in-plane compression by a detailed finite-element analysis, and then using the framework of Kassapoglou [6], which was derived for uni-axial loading.

The onset for the present work was the idea to extend the framework of Kassapoglou [6] to predict the fatigue life of composite laminates, by applying the principle of energy dissipation in the presence of plastic strains to each individual ply for their individual loading situation based on their stacking orientation, be it axial tension/compression, transverse tension/compression or, in case of the present

work, in-plane shear.

To obtain this in-plane shear stress-strain curve, a similar consistent approach to Kassapoglou [6] is sought, where only a few material constants from published stress-strain data, such as initial Young's modulus, yield strain and stress, are needed for accurate predictions, eliminating the need for fitting experimental fatigue data. Extending such a parsimonious approach to composites should result in a model that ideally only requires the basic material properties of its constituents, related to their stiffnesses and strengths.

While the approach is well established for metals, extending it to composite plies requires the consideration of the underlying micromechanical damage mechanisms that govern stiffness degradation. In this context, the work by Socci and Kassapoglou [7] provides a physically grounded analytical description of stiffness degradation in a unidirectional ply under uni-axial tension due to damage. Starting from the elastic response of the fiber and matrix, the model captures the stress redistribution induced by fiber fracture through shear transfer mechanisms, described by a shear-lag-type formulation. A key outcome is the introduction of an effective recovery length  $d_{\text{eff}}$ , representing the distance over which the fiber stress recovers to its far-field value after breakage, a common aspect of shear-lag-type formulations [1], [10], [11], [12].

The elevated shear stresses in the surrounding matrix then drive the initiation and growth of matrix damage, which are either fiber-matrix interfacial cracks or inter-fiber radial cracks. Conceptually, this approach has been successfully used by Esrail and Kassapoglou [13] to predict delamination sizes of laminates after low speed impact and by Huo et al. [14] to predict delamination sizes of laminates under static indentation. Subsequently, the energetically favourable crack configuration is determined based on an additional energy criterion.

The progressive accumulation of fiber cracks that cause the matrix cracking is statistically incorporated by treating the tensile fiber failure strain as a random variable, allowing the fraction of failed fibers to be expressed through a cumulative distribution function. This approach provides a continuous measure of damage at the fiber-matrix level and directly links the microscale variability to the macroscopic stress-strain response.

## 2.2. Micromechanics of Unidirectional Plies

To introduce the discipline of micromechanics a brief recap of what a composite material is, is in place: A composite material consists of two or more distinct constituents that remain macroscopically separable, typically a reinforcing phase embedded in a continuous matrix and is therefore heterogeneous. In contrast, metal alloys are for all practical purposes macroscopically homogeneous, as their constituents can no longer be distinguished by the naked eye [15]. The most important aspect of composite materials, and the reason for their existence, is that the resulting engineering material exhibits the best of its constituent properties, or even properties that neither constituent possesses, such that the characteristics required for a given design task can be achieved [4], [15].

In UD composite plies, aligned fibers act as the reinforcing constituent and are dispersed within a polymer matrix [2]. A key characteristic of the composite ply is the volume fractions of its constituents, defined as

$$v_f = \frac{V_f}{V_{\text{ply}}}, \quad (2.1)$$

$$v_m = \frac{V_m}{V_{\text{ply}}}, \quad (2.2)$$

where  $V_f$ ,  $V_m$ , and  $V_{\text{ply}}$  denote the fiber, matrix, and total ply volumes, respectively, and  $v_f$  and  $v_m$  are the corresponding volume fractions. By definition, assuming no voids,

$$v_f + v_m = 1. \quad (2.3)$$

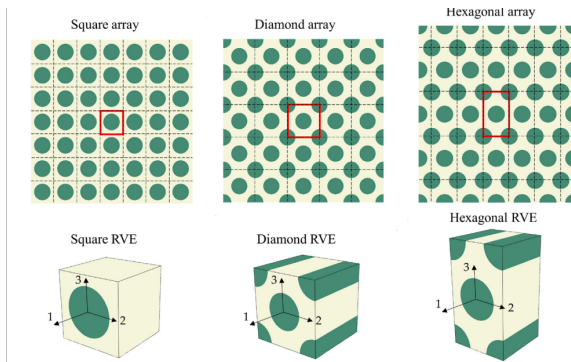
These fractions, together with the mechanical properties of the constituents, govern the resulting effective mechanical properties of the ply [16]. The study of how these effective properties arise from interactions at the microscopic scale is referred to as *micromechanics* [1], [15].

The determination of effective properties for a heterogeneous microstructure such as a UD ply is commonly referred to as homogenisation, and enables the representation of the material as an equivalent homogeneous medium with averaged properties. The study of composite material behaviour under this assumption is referred to as *macromechanics* [1], [15]. As briefly explained in chapter 1, extending the principle of homogenisation to laminates, the determination of effective properties for a stack of plies with arbitrary orientations is referred to as Classical Laminated Plate Theory (CLPT). This theory will not be covered or used in the remainder of this work, and the reader is referred to [17].

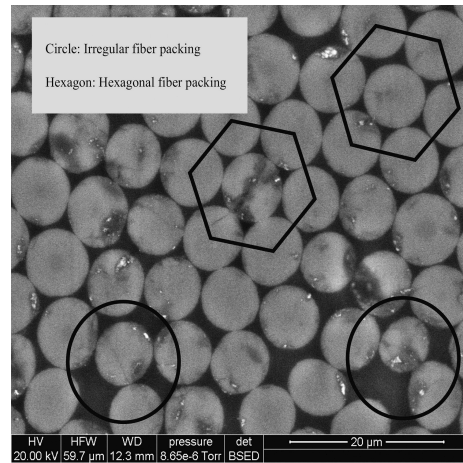
Experimentally, the effective properties of a UD ply can be measured from simple tests on specimens of adequate size. When using theoretical models, homogenization requires the existence of a representative volume element (RVE), defined as a material volume that is statistically representative of the whole UD ply and contains a sufficient number of fibers. Because of the heterogeneous nature of the UD ply, different physical phenomena occur at different length scales. At the smallest length scale relevant in the present work, the microstructure is governed by the size of the constituents, e.g. fiber diameter, while at the structural level, the response of the UD ply varies over length scales much larger than those of inhomogeneities such as the fibers [3].

The validity of the RVE concept relies on the separation of these length scales, commonly denoted by  $\ell_1$  and  $\ell_3$  and the introduction of a mesoscale  $\ell_2$ , such that  $\ell_1 \ll \ell_2 \ll \ell_3$  [18]. By averaging strain and stress fields over a domain of characteristic size  $\ell_2$ , a smooth macroscopic field is obtained that satisfies the governing elasticity equations in an average sense, thereby justifying the use of an RVE with dimensions  $\propto \ell_2$  for relating the constituents behaviour to effective properties of the UD ply [4], [18], [19].

An important consideration in defining an RVE and its size, is the spatial arrangement of fibers within the matrix. In idealised settings, fibers are often assumed to be periodically arranged, allowing a single unit cell with appropriate boundary conditions to serve as an RVE [1], [15], [20]. Common periodic arrangements and their RVEs are shown in Figure 2.1.



**Figure 2.1:** Periodic fiber arrangements and their RVEs [20]



**Figure 2.2:** SEM photograph showing irregular fiber arrangement [21]

In practice, real composite microstructures exhibit a certain degree of randomness in fiber positions due to manufacturing variability, such as fiber misalignment, resin flow, and defects [22], [23], [24], [25], as shown in Figure 2.2, which may require larger domains to achieve statistical representativeness [4], [21], [26], [27]. As a result, the choice of RVE is not unique and depends on the assumed level of microstructural idealisation, ranging from periodic unit cells to statistically representative random realizations. Several works already considering randomizing fiber arrays to reflect physical reality by generating RVEs for finite-element analyses using statistics derived from image processing [21], [28], [29], [30], [31].

The appropriate size and definition for an RVE is thus subject to ongoing discussions in the literature, but in the absence of a specific definition, the RVE is essentially a convenient theoretical concept that en-

ables the transition from the heterogeneous microstructure to an equivalent homogeneous description [3], [32], [33].

The transition from the heterogeneous microstructure to an equivalent homogeneous description requires a consistent definition of macroscopic stress and strain measures. Within the framework of homogenisation, these macroscopic quantities are obtained as volume averages of the corresponding local fields over the RVE [3], [4], [19].

Let the RVE occupy a volume  $V$ , which has a local stress field  $\sigma(\mathbf{x})$  that varies due to the presence of different constituents. The homogenised, or macroscopic, stress is then defined as the volume average [3]

$$\bar{\sigma} = \frac{1}{V} \int_V \sigma(\mathbf{x}) dV. \quad (2.4)$$

By subdividing the RVE into subdomains corresponding to the individual constituent phases  $r = 1, 2, \dots, n$ , each occupying a volume  $V_r$  with volume fraction  $v_r = V_r/V$ , the overall stress can be expressed as a weighted sum of the phase stresses. This yields

$$\bar{\sigma} = \sum_r v_r \bar{\sigma}_r \quad (2.5)$$

and shows that the macroscopic stress follows directly from the contributions of the individual constituents weighted by their volume fractions [3]. In case of the ideal UD ply,  $r = 2$ , corresponding to the fiber and matrix constituents with their respective volume fractions defined by Equation 2.1. Considerations of impurities or voids would result in  $r > 2$ .

An analogous relation holds for the strain field. These averaging relations provide the fundamental link between the microscale fields within the RVE and the effective properties used in a homogenised description of the material. The averaging relation of Equation 2.5 is formulated in terms of subdomains within the RVE, typically corresponding to the individual material constituents. More generally, however, it applies to any partition of a material domain into regions with distinct local responses [3].

This averaging procedure forms the basis for classical homogenisation approaches, including the Voigt [34] and Reuss [35] mixture rules, proved to be upper and lower bounds for estimating effective stiffnesses by Hill [36]. More refined bounds are provided by Hashin and Shtrikman [3], [37], [38].

Popular approximate methods for estimations of effective properties within these bounds are the Mori-Tanaka [39] scheme, self-consistent methods and the differential scheme [3]. In engineering practice, semi-empirical relations such as Halpin-Tsai [40], Tsai-Hahn [17] and Chamis [41] are often used for their simplicity and ease of application. However, many of these approaches, in assuming uniform or idealised microstructures, struggle with predicting matrix-dominated ply moduli consistently, such as  $G_{12}$  and  $E_2$  [1], [16], [31], [42]. In addition, these approaches primarily predict the initial effective properties of the UD ply, and do not directly capture mechanisms responsible for stiffness degradation and nonlinearity.

### 2.2.1. Beam-Based Micromechanical Modelling of Fiber-Matrix Interaction

To resolve local stress distribution mechanisms due to damage, reduced-order micromechanical models are introduced. Whereas the full theory of elasticity can describe the complete stress field of solid bodies, only few closed-form analytical solutions exist. Practical problems are therefore commonly addressed by introducing *a priori* assumptions regarding either the stress, strain, or displacement field. Beam theory represents one such reduction, in which the three-dimensional problem is reduced to determining the centreline deflection  $w(x)$ . Other quantities of interest, such as internal forces and bending moments, follow from this solution and its derivatives  $\frac{d^n w}{dx^n}$  [43], [44].

For bending problems induced by lateral loads, the Euler-Bernoulli beam theory is typically applied when the beam is slender, i.e. when its length-to-depth ratio is sufficiently large. In that case, bending strain energy dominates and shear deformation may be neglected [43]. It is therefore common in analytical micromechanical modelling of continuous fiber-reinforced composites to model the fibers as structural elements such as beams [7], [45], [46], [47].

Classical Euler-Bernoulli beam theory assumes that transverse shear deformation is negligible, which is appropriate for slender beams dominated by bending. However, for micromechanical problems involving in-plane shear of UD plies, the primary quantity of interest is the effective shear response of the fiber-matrix system. In this case, neglecting transverse shear deformation would be inconsistent with the underlying physics.

**Timoshenko beam theory** To account for these effects, shear-deformable beam theories such as Timoshenko beam theory are employed. In contrast to Euler-Bernoulli theory, it accounts for transverse shear deformation through an additional kinematic variable, introducing an independent rotation of the cross-section and allowing for non-zero transverse shear strains. While Timoshenko theory is often motivated for short or deep beams, it is also well suited for problems where shear compliance plays a significant role, irrespective of geometric slenderness.

As an illustrative example, the deflection of a simply supported beam of length  $L$  and depth  $h$  under uniform load  $q_0$  according to both theories reads [43]:

$$w^{\text{Euler-Bernoulli}}(x) = \frac{q_0 L^4}{24EI} \left\{ \left( \frac{x}{L} \right)^4 - \frac{3}{2} \left( \frac{x}{L} \right)^2 + \frac{5}{16} \right\}, \quad (2.6)$$

$$w^{\text{Timoshenko}}(x) = \frac{q_0 L^4}{24EI} \left\{ \left( \frac{x}{L} \right)^4 - \frac{3}{2} \left( \frac{x}{L} \right)^2 + \frac{5}{16} - \left( \frac{h}{L} \right)^2 \frac{E}{nG} \left( \left( \frac{x}{L} \right)^2 - \frac{1}{4} \right) \right\}. \quad (2.7)$$

The additional term in the Timoshenko solution reflects the contribution of transverse shear deformation, governed by the shear modulus  $G$  and shear correction factor  $n$ .

**Shear correction factor** The shear correction factor was introduced by Timoshenko [48] in the early development of what is now known as Timoshenko beam theory. In contrast to the classical Euler-Bernoulli beam formulation, Timoshenko theory accounts for transverse shear deformation, allowing it to describe the behaviour of moderately thick beams more accurately. However, the theory assumes that the transverse shear strain is constant through the cross-section, whereas the true shear stress distribution obtained from elasticity is generally non-uniform. To reconcile this discrepancy, Timoshenko introduced a shear correction factor that ensures equivalence between the shear strain energy predicted by the beam theory and the exact strain energy obtained from the actual stress distribution Dym and Shames [43] and Timoshenko [48].

Determining an appropriate value for the shear correction factor has since been the subject of extensive analytical and numerical investigation. Early derivations relied on energy-equivalence arguments based on assumed shear stress distributions. For example, the classical value of  $k = 5/6$  for rectangular cross-sections follows from matching the strain energy of a parabolic shear stress distribution to the uniform shear strain assumed in the beam theory. Later refinements tried to account more rigorously for the effects of the Poisson's ratio and the geometry of the cross-section. In particular, Cowper [49] derived expressions for the shear correction factor using three-dimensional elasticity solutions, resulting in values that depend explicitly on Poisson's ratio and the shape of the cross-section. Subsequent work by Hutchinson [50] revisited the derivation and clarified the relationship between different definitions of the factor used in literature.

Despite these developments, the shear correction factor cannot be regarded as a uniquely defined material or geometric constant. Its value depends on the theoretical framework used to derive it, the assumed shear stress distribution, and the specific definition adopted for the shear stiffness in the beam model. Different derivations based on energy equivalence, stress matching, or variational principles may therefore lead to different numerical values even for the same cross-section [43], [51]. Furthermore, when the concept is extended to layered, anisotropic, or heterogeneous structures, such as thick laminated sandwich beams [52], [53], the shear stress distribution becomes strongly influenced by the material architecture. In such cases the effective shear correction factor may deviate substantially from classical values and can even exceed unity, reflecting the fact that the simplified beam kinematics do not fully capture the underlying three-dimensional stress state.

**Elastic Foundations** Applications of beam-on-foundation models to micromechanical problems involving fiber-matrix interaction have primarily been found in the context of fiber buckling [45], [47], [54], [55]. In such formulations, the surrounding matrix is represented as a continuous support, commonly known as an elastic foundation, that restrains the deformation of the fiber, thereby influencing its stability behaviour.

Winkler [56] first developed the elastic-foundation beam model in 1867 as a way to represent the support provided by the soil beneath railway tracks and similar civil engineering structures. The soil support was modeled as a series of independent springs that react proportionally to the local beam deflections. As these springs act independently, the effect of the soil continuity is lost, and thus several so-called two-parameter models have been proposed to alleviate this, including the Pasternak, Vlasov, Filonenko-Borodich and Reissner foundations. These formulations introduce an additional parameter next to the independent Winkler springs, based on shear layer interactions, continuum soil derivations, membrane interactions or foundation bending stiffness, to better reflect the underlying physics [25], [57], [58], [59].

With these enhanced formulations, elastic foundation models have found a wide range of applications beyond civil engineering, including the design of composite sandwich panels in aerospace engineering [60], the evaluation of contact pressures in knee replacements in biomechanical engineering [61], and the design and analysis of micro-electronics [54].

## 2.3. Nonlinear In-Plane Shear Response and Modelling Approaches

In contrast to the predominantly linear elastic behaviour observed in axial loading of UD plies, the in-plane shear response is typically nonlinear from relatively small strain levels onwards [17], [62], [63]. The initial response of the ply, characterised by its shear modulus  $G_{12}$ , is governed by the matrix-dominated deformation. Since the shear stiffness of the matrix is typically much lower than that of the fibers, the overall response is controlled by the more compliant constituent [1], [16]. From a homogenisation perspective, so-called mixture rules which reflect the dominant role of the matrix in shear deformation, such as the Reuss estimate, provide first approximations for  $G_{12}$  [35].

Beyond the initial ply response, experimental observations consistently show a gradual reduction in shear modulus  $G_{12}$  prior to ultimate failure, rather than an abrupt loss of load-carrying capacity, resulting in the nonlinear shear stress-strain curves for the ply [64], [65].

### 2.3.1. Origin of nonlinearity: damage vs inelasticity

A commonly adopted interpretation attributes the nonlinear shear response of UD plies to the progressive accumulation of microdamage within the matrix and at the fiber-matrix interface. In this view, mechanisms such as plasticity, fiber-matrix debonding, matrix cracking, lead to a gradual degradation of stiffness as loading progresses [12], [66].

The initiation and growth of such damage are driven by local stress concentrations due to the heterogeneous microstructure, and are influenced by imperfections such as fiber breaks [1], [12], porosity [67], fiber waviness [1], [24], and resin-rich or resin-starved regions introduced during manufacturing [23], [68], [69], [70].

As fiber breaks provide a discrete type of failure due to their brittle nature [2], they provide a convenient starting point to analyse subsequent failure of the matrix in its vicinity due to load re-distributions. Such analyses are commonly referred to as shear lag problems [1], [10], [11], [12], [17]. Fiber strength, which drives fiber breaks, exhibits larger scatter than fiber moduli, because the former depends more on fiber defects [17]. This variability thus governs the onset of fiber fracture and subsequent stress redistributions in the matrix.

However, an alternative interpretation suggests that at least part of the observed nonlinearity originates from an inherent inelastic response of the matrix constituent itself, rather than from progressive damage. In this context, the matrix may show nonlinear elastic, viscoelastic, or viscoplastic behaviour. Consequently, the homogenised composite material can be considered to be inelastic of itself, and its modelling does not necessarily need explicit damage mechanisms [62], [71], [72]. An important distinction is that, in contrast with nonlinearities due to irreversible damage, nonlinearities due e.g. viscoelasticity may be largely reversible.



In practice, both mechanisms are likely to contribute to the overall response. At low strains, reduction in ply stiffness may be largely due to matrix inelasticity, while microdamages increasingly dominate at higher strains. However, experimentally distinguishing between these contributions remains challenging [73].

### 2.3.2. Modelling Approaches

Existing methodologies to predict the damage mechanics and subsequent response of composite materials, not restricted to the nonlinear shear response of UD plies, can broadly be categorised into three approaches: phenomenological, micromechanical and statistical [74]. Phenomenological models introduce damage variables or internal state parameters that are calibrated to experimental stress-strain data and are broadly referred to as macro-damage mechanics (MADM) or continuum damage mechanics (CDM) and may or not may explicitly model matrix nonlinearities [12], [65], [75], [76]. While these approaches can provide accurate fits, their predictive capability is inherently limited by the availability and quality of experimental data, and they often lack a direct link to the underlying physical mechanisms.

Micromechanical, physics-based approaches aim to explicitly resolve the micromechanical behaviour of the fiber-matrix system. These often include detailed finite element models of representative volume elements, which capture local stress concentrations and damage evolution mechanisms, and easily allow for modelling matrix nonlinearities with appropriate constitutive laws [29], [30], [77]. Although such models provide valuable insight into the governing mechanisms, they are computationally expensive and therefore not well suited for integration into efficient engineering design frameworks.

While micromechanical models do consider underlying physical mechanisms, they often employ simplifying assumptions regarding the geometry of defects to keep problems tractable and those may not always be supported by observations [74]. In addition, the effect of defect interactions is typically neglected or it undermines the definition of the RVE used. At initial loading, damage are dispersed and of a size in the order of the constituents, justifying the use of an RVE as if no such damage was present. But with increasing loading, damages grow and cluster, e.g. microcracks coalesce into a macro-crack, and may exceed the RVE size. Percolation theory can be used to study the evolution of the disordered material state to a state dominated by a single cluster of damages that eventually leads to final failure [74], [78]. Modelling the disordered material state is commonly done by assuming distributions of initial defects or strengths, which drives the shape of the stress-strain curve [74].

For fiber strength, the Weibull distribution is commonly used due to their brittle fracture [17], [78], [79]. Based on this, fiber bundle models describe the progressive failure of a collection of fibers under load, accounting for load redistributions following individual fiber breaks [79], [80]. Extension of these concepts to continuous fiber composites have been developed using shear-lag formulations to reasonably predict tensile ply failure by a RoM of longitudinal strength [17].

Although the Weibull distribution often fits experimental strength data well, its applicability to composites can suffer from physical inconsistencies. First, it assumes abrupt, catastrophic failure from a critical defect, whereas composites may fail gradually due to progressive fibre failure, load redistribution, and matrix cracking [81]. Macroscopically, such abrupt behaviour may be a reasonable approximation for fibre-dominated tensile loading, but is less appropriate for matrix-dominated loading, such as transverse tension and in-plane shear. Second, the associated size effect [17] is not always consistently observed [81]. Both of these observations suggest that failure is not purely governed by weakest-link statistics.

## 2.4. Summary and outlook

- Extending parsimonious modelling approaches from metals to composite plies requires consideration of micromechanical damage mechanisms governing stiffness degradation, with the work of Succi and Kassapoglou [8] providing a primary reference for uniaxial loading;
- Micromechanics provides the framework to relate constituent properties (fiber and matrix) and their volume fractions to effective ply properties through their interaction at the microscale;
- Homogenisation enables the replacement of a heterogeneous microstructure by an equivalent homogeneous medium, forming the basis for further analysis at scales larger than the microscale;

- The RVE is a central concept for homogenisation, justified by separation of length scales and statistical representativeness, but its definition and size are not unique and depend on the assumed microstructural idealisation (e.g. periodic vs. random fiber arrangements) and the presence and size of damages;
- Reduced-order models, such as beam-on-foundation models, can provide a physically interpretable way to capture fiber-matrix interaction and stress redistribution mechanisms at the microscale;
- The in-plane shear response of UD plies is inherently nonlinear and matrix-dominated, even at small strain levels;
- The in-plane shear response of UD plies is inherently nonlinear and dominated by the matrix, with stiffness degradation occurring progressively rather than through abrupt failure;
- Origins of progressive stiffness degradation under shear are ambiguous and motivate combining the mechanisms of matrix nonlinearity and damage such as fiber-matrix debonding and matrix cracking;
- Existing modelling approaches face a trade-off between phenomenological models, which are efficient but rely on experimental calibration, and physics-based models, which are more predictive but computationally expensive;
- This motivates the development of an analytical, micromechanics-based framework that captures the essential physics of fiber-matrix interaction and progressive degradation, while remaining sufficiently efficient for engineering applications.

# Material Systems and Input Data

This chapter introduces the materials systems and corresponding input data used throughout modelling and analysis. The main objective is to define a consistent set of data containing both constituent and ply properties. Relevant stiffness properties are required for the construction of the RVE of chapter 4, the finite-element analyses of chapter 5 and the statistically informed homogenisation of chapter 6. Relevant strength properties are required for the constituent failure analyses of chapter 7 and the final homogenisation towards macroscopic ply behaviour of chapter 8.

Material systems are selected based on the availability of experimentally obtained in-plane shear stress-strain data, which serve as the reference for model validation. Additionally, properties for both constituents, i.e. fibers and matrix, are required to formulate the micromechanical approach. Since no single source provides a complete dataset for all relevant properties, data from multiple literature sources is combined.

This combination of datasets introduces inherent inconsistencies, as material properties may be derived from different test methods, specimen conditions, or even slightly different material variants. The implications of these assumptions are addressed explicitly in the following sections, and should be considered when interpreting the predictive capability of the model.

## 3.1. Considered material systems

To evaluate and validate the proposed modelling framework, UD material systems for which both the majority of constituent properties as the in-plane shear stress strain curve are available were sought after. Kaddour et al. [64] presents four such materials in the context of the third World-Wide Failure Exercise (WWFE-III):

- IM7/8552 (carbon/epoxy)
- G40-800 (carbon/epoxy)
- AS4/3501-6 (carbon/epoxy)
- E-glass/LY556 (glass/epoxy)

However, the data for these materials does not include any fiber shear strengths. In their work on determining moisture and thermal expansion coefficients of UD materials, Cairns and Adams [82] included material properties for the following constituents:

- Hercules AS graphite fiber
- Owens-Corning S2 glass fiber
- Hercules 3501-6 epoxy

Based on constituent similarities, the data sets from Kaddour et al. [64] and Cairns and Adams [82] are combined, as to have a full set of material data for each UD material, covering both stiffness and strength. While no stress-strain curves for the Hercules AS/3501-6 and Owens-Corning/3501-6 are

available, their properties are used to assess parts of the methodology where useful. With regards to fiber failure statistics, the standard deviation  $\sigma_\epsilon$  of tensile failure strain for the fibers is taken from Socci and Kassapoglou [8], which allows for a probabilistic description of fiber failure, required for the final homogenisation procedure. All relevant material data is summarized in Table 3.1-Table 3.3. Footnotes highlight assumptions made which are elaborated upon in section 3.2.

**Table 3.1:** Relevant UD properties.

|                                                         | IM7/8552<br>[64] | G40-800/5260<br>[64] | AS4/3501-6<br>[64] | Glass/LY556<br>[64] |
|---------------------------------------------------------|------------------|----------------------|--------------------|---------------------|
| In-plane shear modulus $G_{12}$ [GPa]                   | 5.6              | 6.94                 | 6.6                | 5.83                |
| In-plane shear strength $S_{12}$ [MPa]                  | 90               | 91                   | 79                 | 73                  |
| Strain energy release rate Mode II $G_{II}$ [ $J/m^2$ ] | 800              | 900                  | 650                | 1500                |

**Table 3.2:** Relevant fiber properties.

|                           | IM7<br>[64]        | G40-800<br>[64]    | AS4<br>[64]        | Hercules AS4<br>[82] | Glass<br>[64]     | O-C Glass<br>[82] |
|---------------------------|--------------------|--------------------|--------------------|----------------------|-------------------|-------------------|
| $E_f$ [GPa]               | 276                | 290                | 231                | 220                  | 74                | 86.2              |
| $G_f$ [GPa]               | 27                 | 27                 | 15 <sup>5</sup>    | 34.5 <sup>5</sup>    | 30.8              | 35.3              |
| $\nu_f$ [-]               | 0.2                | 0.2                | 0.2                | 0.2                  | 0.2               | 0.22              |
| $d_f$ [ $\mu m$ ]         | 4.5 <sup>6</sup>   | 8                  | 7                  | 7 <sup>2</sup>       | 11                | 11 <sup>4</sup>   |
| $X_t$ [GPa]               | 5.18               | 5.18               | 3.5                | 3.1                  | 2.15              | 4.825             |
| $X_c$ [GPa]               | 3.2                | 3.2                | 3                  | 2.66                 | 1.45              | 3.25              |
| $S$ [GPa]                 | 0.172 <sup>1</sup> | 0.172 <sup>1</sup> | 0.172 <sup>1</sup> | 0.172                | 2.41 <sup>3</sup> | 2.41              |
| $\sigma_\epsilon$ [-] [7] | 0.00293            | 0.00316            | 0.00237            | -                    | 0.00617           | -                 |

<sup>1</sup> Taken from Hercules AS4/3501-6 [82]; both are carbon fiber.

<sup>2</sup> Assumed equal to AS4 fiber diameter.

<sup>3</sup> Taken from O-C Glass/3501-6 [82]; both are glass fiber.

<sup>4</sup> Assumed equal to Glass fiber diameter.

<sup>5</sup> Despite AS4 and Hercules AS technically referring to the same material, a vastly different  $G_f$  is reported.

<sup>6</sup> Datasheet manufacturer reports 5.2 $\mu m$  [83].

**Table 3.3:** Relevant matrix properties.

|             | 8552<br>[64] | 5260<br>[64] | 3501-6<br>[64] | LY556<br>[64] |
|-------------|--------------|--------------|----------------|---------------|
| $E_m$ [GPa] | 4.08         | 3.45         | 4.2            | 3.35          |
| $G_m$ [GPa] | 1.478        | 1.28         | 1.567          | 1.24          |
| $\nu_m$ [-] | 0.38         | 0.34         | 0.35           | 0.35          |
| $X_t$ [GPa] | 99e-3        | 70e-3        | 69e-3          | 80e-3         |
| $X_c$ [GPa] | 130e-3       | 130e-3       | 250e-3         | 120e-3        |
| $S$ [GPa]   | 57e-3        | 50e-3        | 57e-3          | 54e-3         |

<sup>3</sup> From tensile/shear test data.

<sup>4</sup> Estimated proportionally to matrix  $X_t$  based on AS4/3501-6.

## 3.2. Data Consistency and Assumptions

As the material used in this work is assembled from multiple literature sources, several underlying assumptions are made to obtain a complete and usable dataset. These assumptions are summarised below.

- **Combination of material datasets:** Ply-level shear properties and stress–strain curves are primarily taken from Kaddour et al. [64], while missing constituent-level parameters, such as fiber shear strength, are supplemented using data from Cairns and Adams [82]. Where direct constituent data is unavailable, properties from closely related materials are adopted, e.g. the fiber shear strengths of Cairns and Adams [82] are used to complete the data of Kaddour et al. [84], while fiber diameter data of Kaddour et al. [84] is used to complete the data of Cairns and Adams [82], as shown in Table 3.2. This approach assumes that the selected materials are sufficiently similar in composition and processing for their properties to be combined.
- **Matrix property discrepancies:** Additional notes on the 3501-6 matrix properties are made in Table 3.4, as Cairns and Adams [82] reported matrix properties from both tensile and shear tests, but showing differences with the data reported by Kaddour et al. [64]. As no stress-strain curves are available for Hercules AS/3501-6 and Owens-Corning/3501-6, the data from Kaddour et al. [64] for 3501-6 is used, as reflected in Table 3.3.
- **Fiber failure statistics:** In chapter 8, fiber failure is modelled as a stochastic process by introducing a distribution in failure strain. The standard deviation is taken from [7], while the mean value follows directly from the ratio of fiber strength to shear modulus. The fiber shear modulus itself is treated as deterministic.
- **Deterministic vs stochastic inputs:** All stiffness parameters are treated as deterministic, while variability is introduced only through the fiber failure criterion. This neglects any potential variability in matrix properties.

The above assumptions are made to construct a complete material description, but they also introduce uncertainty in the predicted response. Their influence on the model results will be revisited in later chapters.

**Table 3.4:** Additional note on 3501-6 epoxy matrix.

|             | 3501-6<br>Kaddour et al. [64] | 3501-6<br>Cairns and Adams [82] | Comments                                                                                                                          |
|-------------|-------------------------------|---------------------------------|-----------------------------------------------------------------------------------------------------------------------------------|
| $E_m$ [GPa] | 4.2                           | 4.27/5.79 <sup>3</sup>          | Compared to Cairns and Adams [82], Kaddour et al. [64] seems to report from tensile data.                                         |
| $G_m$ [GPa] | 1.567                         | 1.59/2.14 <sup>3</sup>          | Compared to Cairns and Adams [82], Kaddour et al. [64] seems to report from tensile data.                                         |
| $X_t$ [GPa] | 69e-3                         | 82.7e-3/170e-3 <sup>3</sup>     | Cairns and Adams [82] tensile test provides higher estimate.                                                                      |
| $X_c$ [GPa] | 250e-3                        | 299e-3/616e-3 <sup>4</sup>      | Cairns and Adams [82] tensile test provides higher estimate.; Proportional estimate for shear testing seems unlikely to be valid. |
| $S$ [GPa]   | 57e-3                         | 41.4e-3/84.8e-3 <sup>3</sup>    | Higher estimate used by Kaddour et al. [64]; no stress–strain curves available.                                                   |

<sup>3</sup> From tensile/shear test data.

<sup>4</sup> Estimated proportionally to matrix  $X_t$  based on AS4/3501-6.

# 4

## Deterministic Micromechanical Formulation

This chapter develops a deterministic micromechanical description of a fiber-matrix RVE subject to in-plane shear. An analytical solution of the governing boundary-value problem (BVP) is derived based on idealising the fiber-matrix RVE as a Timoshenko beam on an elastic foundation. From the obtained displacement and strain fields, an effective RVE shear stiffness is defined and subsequently used to predict the initial shear modulus of a unidirectional ply composed of identical RVEs. The resulting stiffness prediction provides the analytical baseline for further evaluation.

### 4.1. Representative Volume Element

As introduced in chapter 2 within the framework of multi-scale homogenisation, to descend from the homogenised description of a UD ply towards the micro-scale at which the governing fiber-matrix interaction mechanisms are resolved, it is appropriate to select a single fiber of circular diameter  $d_f$  surrounded by a portion of matrix as RVE under the assumption that the fibers are arranged in a periodic hexagonal pattern, as shown in Figure 4.1. As a consequence, the fiber volume fraction  $v_f$  is assumed constant throughout the ply. The portion of matrix surrounding the fiber is a function of  $v_f$  and is shaped hexagonally due to the assumed arrangement of the fibers. The center-to-center fiber spacing  $s$  in this hexagonal array is [7]:

$$s = d_f \sqrt{\frac{\sqrt{3}\pi}{6v_f}}. \quad (4.1)$$

#### 4.1.1. Geometric idealisation

To facilitate further derivations, the hexagonally shaped matrix portion is approximated by a hollow circular annulus with inner radius equal to the fiber radius, i.e.  $d_f/2$ , and an equivalent outer radius  $R_{eq}$  chosen such that the cross-sectional area of the hexagon is preserved, i.e.  $A_{RVE} = A_{hex}$ .

A regular hexagon may be described by its inner and outer radii, corresponding to the inscribed and circumscribed circles, respectively. These are related by

$$R_{in} = \frac{\sqrt{3}}{2} R_{out}. \quad (4.2)$$

The area of the hexagon can then be written as

$$A_{hex} = \frac{3\sqrt{3}}{2} R_{out}^2. \quad (4.3)$$

Setting  $s = 2R_{in}$  as shown in Figure 4.2, and using Equation 4.1-4.3, the equivalent radius becomes

$$R_{eq} = \frac{d_f \sqrt{v_f}}{2v_f}. \quad (4.4)$$

The corresponding fiber and matrix cross-sectional areas are

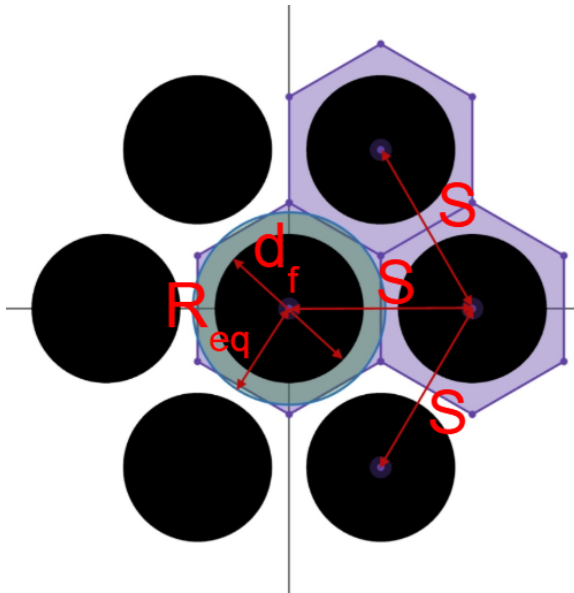
$$A_f = \frac{\pi d_f^2}{4}, \quad (4.5)$$

$$A_m = \pi R_{eq}^2 - A_f = \frac{\pi d_f^2}{4} \left( \frac{1}{v_f} - 1 \right). \quad (4.6)$$

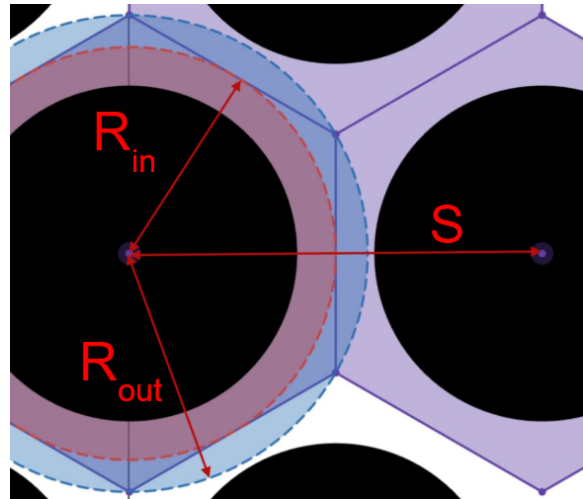
Using Equation 4.5 and 4.6, the total RVE area can be expressed in terms of the fiber diameter  $d_f$  and fiber volume fraction  $v_f$  as

$$A_{RVE} = A_f + A_m = \frac{\pi d_f^2}{4v_f}, \quad (4.7)$$

which is identical to the result obtained directly from the definition of the fiber volume fraction  $v_f = A_f/A_{RVE}$ .



**Figure 4.1:** Hexagonal fiber array containing 7 fibers of diameter  $d_f$ , separated by a distance of  $s$ . The hexagonal matrix portion around a single fiber is approximated by circular portion of radius  $R_{eq}$ .



**Figure 4.2:** Close up of neighbouring fibers separated by a distance of  $s$ . The radii of the inscribed and circumscribed circles of the hexagonal matrix portion are  $R_{in}$  and  $R_{out}$ .

#### 4.1.2. Matrix continuum idealisation

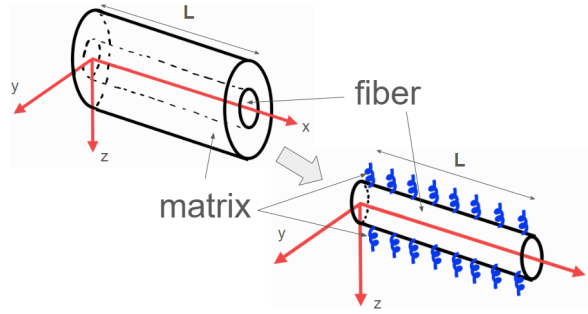
The matrix surrounding the fiber is initially treated as a homogeneous, linear elastic continuum undergoing small deformations. Under in-plane shear loading of the composite ply, the primary mechanical role of the matrix at the micromechanical scale is to mediate shear transfer between neighbouring fibers [1], [2], [10]. Consequently, the full three-dimensional stress distribution within the matrix is of secondary importance compared with the resultant shear interaction between fiber and matrix. This observation allows replacing the full continuum description of the matrix by an equivalent distributed elastic foundation introduced in subsequent sections.

### 4.1.3. Fiber continuum idealisation

The fiber is likewise modelled as a linear elastic but not necessarily isotropic continuum subject to small strains. Because transverse shear effects are non-negligible at the micromechanical length scale considered, a shear-deformable beam representation is used to capture the essential kinematics of the fiber response. When the RVE is then loaded in in-plane shear, the fiber experiences axial deformation directly linked to the cross-sectional rotation, which itself consists of a bending and a shear deformation. This motivates the adoption of a Timoshenko beam description in the reduced-order formulation developed in the following section.

## 4.2. Mechanical Abstraction as Timoshenko Beam on Elastic Foundation

To obtain an effective shear stiffness of the RVE, the geometrically idealised RVE from section 4.1 is further reduced by modelling the continuum fiber-matrix assembly as a beam on an elastic foundation, schematically shown in Figure 4.3.



**Figure 4.3:** Fiber-matrix RVE of length  $L$  to Timoshenko beam on elastic foundation with corresponding coordinate system

In light of the discussion on Timoshenko vs Euler-Bernoulli beam theory of subsection 2.2.1, Timoshenko beam theory is adapted here as to retain the shear deformations that it considers.

Therefore, the following displacement field is assumed [43]:

$$\begin{cases} u_1(x, y, z) &= -z \psi(x), \\ u_2(x, y, z) &= 0, \\ u_3(x, y, z) &= w(x), \end{cases} \quad (4.8)$$

where  $u_i$  denote the displacement components in the  $x$ ,  $y$ , and  $z$  directions, respectively. The coordinate  $x$  is aligned with the beam axis,  $z$  is taken positive downward, and  $y$  completes a right-handed coordinate system as shown in Figure 4.3. The transverse deflection of the beam centreline is given by  $w(x)$ .

In contrast to Euler-Bernoulli theory, the rotation of the cross-section  $\psi(x)$  is treated as an independent kinematic variable. The axial displacement  $u_1$  therefore depends on this cross-sectional rotation rather than directly on the slope of the deflection curve.

The rotation  $\psi(x)$  can be decomposed into the geometric slope of the centreline and an additional shear contribution as shown in Figure 4.4,

$$\psi(x) = \frac{dw}{dx} - \beta(x), \quad (4.9)$$

where  $\frac{dw}{dx}$  represents the inclination of the deflected centreline and  $\beta(x)$  denotes the shear angle. The difference between these two quantities gives rise to a non-zero transverse shear strain, which is retained in Timoshenko beam theory and is central to the present formulation.



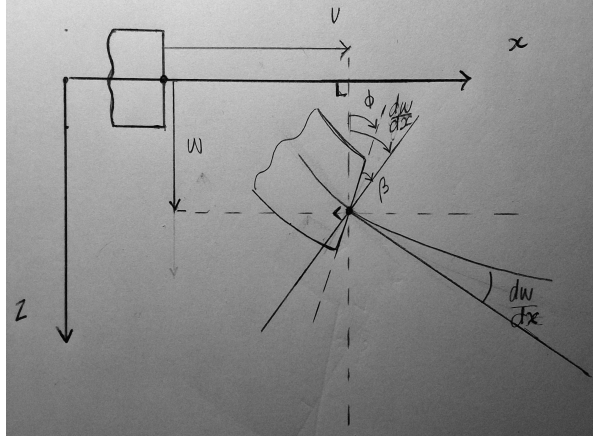


Figure 4.4: Coordinates deflected Timoshenko beam.

#### 4.2.1. Beam representation of the fiber

To model the continuum fiber as a beam instead, the fiber geometry and material properties are translated into the beam bending stiffness

$$D_b = E_f I_f, \quad (4.10)$$

where  $E_f$  is the fiber Young's modulus and  $I_f$  the second moment of area, corresponding to that of a beam with circular cross-section. The beam shear stiffness

$$C_s = n_f G_f A_f, \quad (4.11)$$

contains the fiber shear modulus  $G_f$ , its cross-sectional area  $A_f$  and  $n_f$  the appropriate shear correction factor for a circular cross-section taken from Cowper [49] such that

$$n_f = \frac{6(1 + \nu_f)}{7 + 6\nu_f} \quad (4.12)$$

where  $\nu_f$  is the fiber Poisson's ratio. Note how Equation 4.11 does assume isotropy, due to a lack of a closed-form expression for anisotropic beams.

#### 4.2.2. Elastic foundation representation of the matrix

With the fiber modeled as a beam, the surrounding matrix will be modeled as a so-called elastic foundation as introduced in subsection 2.2.1. Xia [44] formulated a generalized Timoshenko beam on an elastic foundation by modeling the foundation with three parameters:  $K_w$  as the Winkler spring stiffness,  $T_s$  as the shear stiffness of the shear layer, and  $R_i$  as the rotation stiffness of the rotation springs, as shown in Figure 4.5. The general form of these foundation parameters are as follows:

$$K_w = b k_n, \quad (4.13)$$

$$T_s = b k_\tau, \quad (4.14)$$

$$R_i = b \tau y_x^2, \quad (4.15)$$

where  $b$  is conventionally the width of the beam,  $k_n [N/m^3]$  and  $k_\tau [N/m]$  are the foundation moduli and  $y_x$  is the distance between the beam's neutral axis and its top, thus equaling  $d_f/2$ . The foundation moduli commonly involve the moduli of the foundation material, in this case  $E_m$  and  $G_m$ , and a parameter related to its thickness [59], [85]. The parameter  $\tau$  is supposed to represent the interfacial shear strength, which will be assumed to be equal to that of the matrix i.e.  $S_m$ .

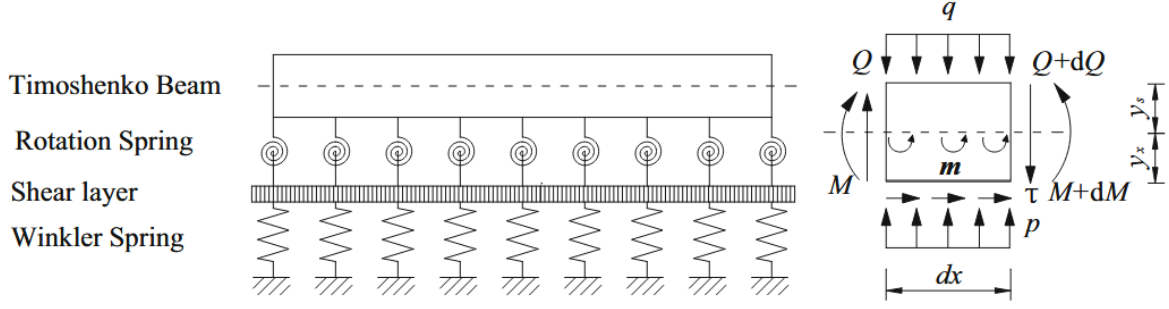


Figure 4.5: Timoshenko beam supported by generalized elastic foundation [44]

Translating the matrix into a three-parameter elastic foundation requires further idealisation of its geometry and material properties into those of the spring types considered. First, to preserve the correct supported length in this reduced model,  $b$  is taken to be equal to the fiber circumference  $\pi d_f$ , as illustrated in Figure 4.6, which is equivalent to doubling the amount of springs as to support the fiber from both above and below. The equivalent thickness of the elastic foundation is obtained from the equivalent matrix outer radius  $R_{eq}$  introduced in subsection 4.1.1. Accordingly, the foundation thickness is taken equal to that of the matrix circular annulus, i.e.

$$t_{m,eq} = R_{eq} - \frac{d_f}{2} = \frac{d_f}{2} \left( \frac{\sqrt{v_f}}{v_f} - 1 \right). \quad (4.16)$$

Comparing to literature, Equation 4.16 is near identical to the one used by Pimenta et al. [46] in their work on micromechanical kink-band formation:

$$t_{m,Pimenta} = \frac{d_f}{2} \left( \frac{\pi}{2\sqrt{3}v_f} - 1 \right) \approx \frac{d_f}{2} \left( 0.952 \frac{\sqrt{v_f}}{v_f} - 1 \right) \quad (4.17)$$

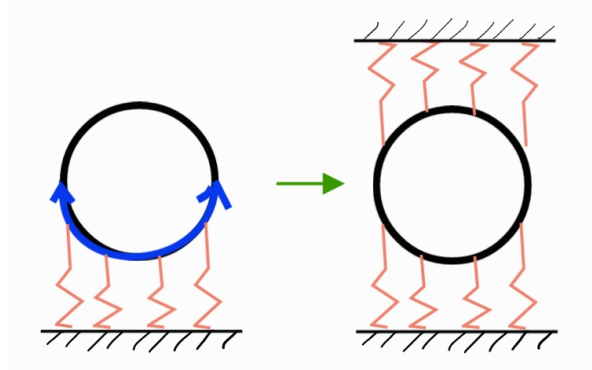


Figure 4.6: The fiber beam is supported by the matrix foundation over a length of  $b = \pi d_f$ . Simplified springs are shown.

To consistently incorporate  $t_{m,eq}$  into the expressions for  $K_w$ ,  $T_s$ , and  $R_i$ , a dimensional analysis of the total energy in Equation 4.21 is used. Since the unit of energy is  $N \cdot m$  and the integration is carried out over the beam length (unit  $m$ ), the integrand in Equation 4.21 must have units of  $N$ . Evaluating the terms involving  $K_w$ ,  $T_s$ , and  $R_i$  in terms of their dimensions gives

$$[K_w] = \frac{N}{[w^2/2]} = \frac{N}{m^2}, \quad (4.18a)$$

$$[T_s] = \frac{N}{\left[\frac{1}{2} \left(\frac{dw}{dx}\right)^2\right]} = N, \quad (4.18b)$$

$$[R_i] = \frac{N}{[\psi^2/2]} = N. \quad (4.18c)$$

To satisfy Equation 4.18, the expressions in Equation 4.13-4.15 are combined with  $t_{m,eq}$  such that

$$K_w = \frac{bE_m}{t_{m,eq}}, \quad (4.19a)$$

$$T_s = bG_m t_{m,eq}, \quad (4.19b)$$

$$R_i = \frac{b\psi^2 S_m}{t_{m,eq}}. \quad (4.19c)$$

Introducing  $b = \pi d_f$  into Equation 4.19a-4.19c, the elastic foundation parameters can be expressed in terms of  $t_{m,eq}$  or, using Equation 4.16, directly in terms of  $v_f$ :

$$\begin{aligned} K_w &= \frac{\pi d_f E_m}{t_{m,eq}} = 2\pi E_m \left( \frac{\sqrt{v_f} + v_f}{1 - v_f} \right), \\ T_s &= \pi d_f G_m t_{m,eq} = \frac{A_f}{2} G_m \left( \frac{\sqrt{v_f}}{v_f} - 1 \right), \\ R_i &= \frac{\pi d_f^3}{4t_{m,eq}} S_m = \frac{A_f}{2} S_m \left( \frac{\sqrt{v_f} + v_f}{1 - v_f} \right). \end{aligned} \quad (4.20)$$

Note that  $K_w$  is independent of the fiber geometry, whereas  $T_s$  and  $R_i$  scale with the fiber cross-sectional area. Using these equivalent stiffness parameters for both fiber and matrix, the governing differential equation of the coupled fiber-matrix response can be introduced.

### 4.3. Mathematical formulation of the fiber-matrix RVE

This section derives the analytical solution of the governing BVP describing the coupled fiber-matrix response. The solution procedure begins with the introduction of an appropriate ansatz, followed by determination of characteristic spatial parameters and integration constants. The resulting displacement, rotation, internal force fields are then obtained in closed form, providing the basis for subsequent stiffness evaluation.

#### 4.3.1. Governing differential equation

Having introduced the fiber as a beam and the matrix as an elastic foundation, each with their respective stiffnesses in the set up as shown in Figure 4.5, the total energy of the beam-foundation assembly equals [44]:

$$\Pi = \int \left[ \frac{D_b}{2} \left( \frac{d\psi}{dx} \right)^2 + \frac{C_s}{2} \left( \frac{dw}{dx} - \psi \right)^2 + \frac{K_w w^2}{2} + \frac{T_s}{2} \left( \frac{dw}{dx} \right)^2 + \frac{R_i \psi^2}{2} - qw - m\psi \right] dx \quad (4.21)$$

where  $m$  and  $q$  are the distributed moment  $m$  and load  $q$  as shown in Figure 4.5. Applying the variational principle to Equation 4.21 and setting the distributed moment and load to zero, yields the homogeneous ODE to be [44]:

$$\frac{d^4 w}{dx^4} - \frac{K_w D_b + R_i C_s + T_s C_s + R_i T_s}{D_b(C_s + T_s)} \frac{d^2 w}{dx^2} + \frac{K_w(R_i + C_s)}{D_b(C_s + T_s)} w = 0 \quad (4.22)$$

### 4.3.2. General solution of the governing equation

The solution of Equation 4.22 depends on the sign of the so-called characteristic parameter

$$\Delta = \Gamma^2 - 4\Phi = \left( \frac{K_w D_b + R_i C_s + T_s C_s + R_i T_s}{D_b(C_s + T_s)} \right)^2 - \frac{4K_w(C_s + R_i)}{D_b(C_s + T_s)}, \quad (4.23)$$

which in this case is a function of the fiber-matrix material system and the RVE idealisation. Two different ansätze for Equation 4.22 were proposed consisting of exponential and trigonometric or hyperbolic terms [44]:

For  $\Delta < 0$ , the lateral displacement takes the form of

$$w(x) = (Ae^{\alpha x} + Ce^{-\alpha x}) \cos \beta x + (Be^{\alpha x} + De^{-\alpha x}) \sin \beta x \quad (4.24)$$

with the unknown coefficients  $A, B, C$  and  $D$ . The spatial wave numbers  $\alpha$  and  $\beta$ , which dictate the exponential growth/decay and oscillatory behavior of the solution, respectively:

$$\alpha = \sqrt{\frac{\sqrt{\Phi}}{2} + \frac{\Gamma}{4}} \quad (4.25)$$

$$\beta = \sqrt{\frac{\sqrt{\Phi}}{2} - \frac{\Gamma}{4}} \quad (4.26)$$

And for  $\Delta > 0$  it takes the form of

$$w(x) = (Ae^{\alpha x} + Ce^{-\alpha x}) \cosh \beta x + (Be^{\alpha x} + De^{-\alpha x}) \sinh \beta x \quad (4.27)$$

with

$$\alpha = \sqrt{\frac{\sqrt{\Phi}}{2} + \frac{\Gamma}{4}} \quad (4.28)$$

$$\beta = \sqrt{-\frac{\sqrt{\Phi}}{2} + \frac{\Gamma}{4}} \quad (4.29)$$

For practical reasons, the case for  $\Delta = 0$  is omitted, as numerically speaking, Equation 4.23 will never be exactly 0 and thus either (4.24) or (4.27) will always apply.

Introducing the following shorthand combinations of stiffness terms:

$$U = \frac{C_s^2 - K_w D_b}{C_s(R_i + C_s)}, \quad (4.30a)$$

$$V = \frac{D_b(C_s + T_s)}{C_s(R_i + C_s)}, \quad (4.30b)$$

$$I = \frac{K_w D_b + R_i C_s + T_s C_s + R_i T_s}{R_i + C_s}, \quad (4.30c)$$

$$J = \frac{D_b(C_s + T_s)}{R_i + C_s}, \quad (4.30d)$$

$$H = \frac{K_w D_b}{C_s}, \quad (4.30e)$$

$$G = \frac{D_b(C_s + T_s)}{C_s}. \quad (4.30f)$$

The rotation  $\psi(x)$ , internal shear load  $Q(x)$  and internal moment  $M(x)$  are then as follows [44]:

$$\psi(x) = U \frac{dw}{dx} + V \frac{d^3 w}{dx^3}, \quad (4.31)$$

$$Q(x) = I \frac{dw}{dx} - J \frac{d^3 w}{dx^3}, \quad (4.32)$$

$$M(x) = Hw - G \frac{d^2 w}{dx^2}. \quad (4.33)$$

Due to the exponential-trigonometric/hyperbolic nature of Equation 4.24 and 4.27, and differentiation being a linear operator, physical quantities derived from the ODE solutions, which include  $\psi(x)$ ,  $Q(x)$ ,  $M(x)$ , tensorial axial strain  $\varepsilon_{11}(x)$  and tensorial shear strain  $\varepsilon_{13}(x)$ , have the same functional form:

$$\left[ (c_{1,f}^{(-)} A + c_{2,f}^{(-)} B) e^{\alpha x} + (c_{3,f}^{(-)} C + c_{4,f}^{(-)} D) e^{-\alpha x} \right] \cos \beta x + \left[ (c_{5,f}^{(-)} A + c_{6,f}^{(-)} B) e^{\alpha x} + (c_{7,f}^{(-)} C + c_{8,f}^{(-)} D) e^{-\alpha x} \right] \sin \beta x \quad (4.34)$$

or

$$\left[ (c_{1,f}^{(+)} A + c_{2,f}^{(+)} B) e^{\alpha x} + (c_{3,f}^{(+)} C + c_{4,f}^{(+)} D) e^{-\alpha x} \right] \cosh \beta x + \left[ (c_{5,f}^{(+)} A + c_{6,f}^{(+)} B) e^{\alpha x} + (c_{7,f}^{(+)} C + c_{8,f}^{(+)} D) e^{-\alpha x} \right] \sinh \beta x \quad (4.35)$$

where the subscript  $f(x)$  denotes the field quantity of interest,

$$f(x) \in \{\phi(x), Q(x), M(x), \varepsilon_{11}(x), \varepsilon_{13}(x)\},$$

and the constants  $c_{i,f}^{(\pm)}$  are determined by the sign of Equation 4.23 and by what quantity they express. Details of the derivations and tabulated results can be found in Appendix A.

### 4.3.3. Boundary conditions of the RVE

Since the fiber will be represented by a beam and the matrix by an elastic foundation supporting the beam, the fiber will be loaded directly, while the matrix will react indirectly by developing distributed shear and normal stresses to oppose the fiber deformation. The total applied shear force  $Q_{RVE}$  on the RVE is thus distributed over the fiber and the matrix depending on their respective load fractions, dictated by their shear stiffness  $G$  and cross-sectional area  $A$ . Under the assumption of uniform shear strain, the respective load fractions are:

$$f_i = \frac{n_i G_i A_i}{n_f G_f A_f + n_m G_m A_m} \quad i = f, m \quad (4.36)$$

such that the fiber and matrix load are:

$$Q_f = f_f Q_{RVE} \quad (4.37)$$

$$Q_m = f_m Q_{RVE} = (1 - f_f) Q_{RVE} \quad (4.38)$$

Plotting Equation 4.36 for the material data reported in chapter 3, as shown in Figure 4.7, it can be seen that the majority of the load is being carried by the fibers. At  $v_f = 0.6$ ,  $Q_f$  is at least 97.7% of  $Q_{RVE}$  for all materials, except for the AS4 fibers, with  $Q_f$  carrying about 95.8% of  $Q_{RVE}$  due to its comparably low  $G_f$ , as mentioned in chapter 3.

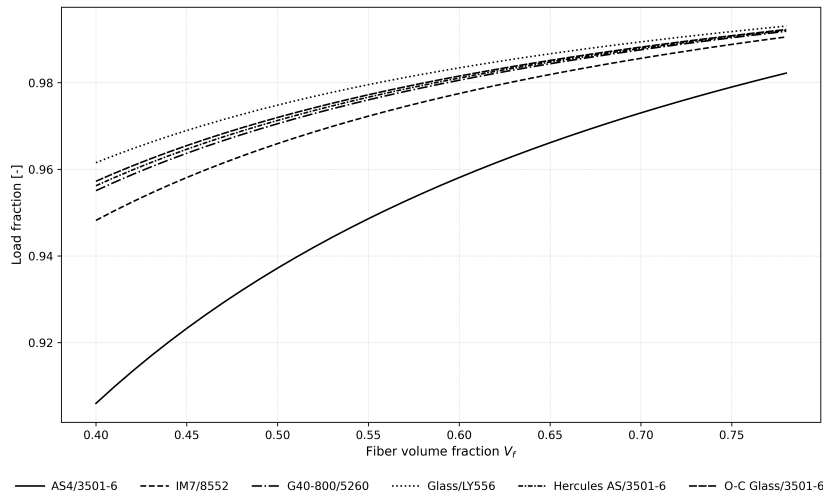


Figure 4.7: Fiber load fractions

Xia [44] proposed a calculation method based on a transfer-matrix formulation, in which the deflections and internal forces at successive discretized points along the beam are obtained from the inertia, damping, and stiffness properties between adjacent points. However, the practical implementation of this method is insufficiently documented and the accompanying appendix is suspected to contain errors. Attempts to reproduce the results resulted in solutions exhibiting exponential growth. Therefore, the governing ODE was re-solved for the particular case of applied shear loads and moments at the beam ends.

Assuming a state of simple shear in the fiber-matrix RVE at its ends ( $x = 0, x = L$ ), the following boundary conditions, visualized in Figure 4.8, are applicable for a load-dependent formulation:

$$\begin{cases} Q_0 = Q(x=0) = Q_f; \\ Q_L = Q(x=L) = Q_f; \\ M_0 = M(x=0) = 0; \\ M_L = M(x=L) = 0 \end{cases} \quad (4.39)$$

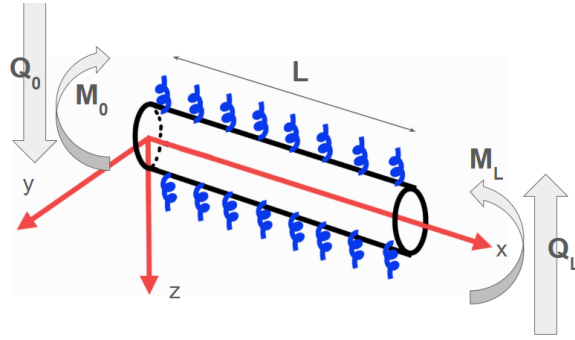


Figure 4.8: Fiber-matrix RVE with applied loadings at its ends.

#### 4.3.4. Statement of the boundary-value problem

Combining subsection 4.3.1- 4.3.3 the boundary value problem can be formally stated. The deformation of the fiber within the representative volume element is governed by the fourth-order differential equation given in Equation 4.22. Together with the boundary conditions defined in Equation 4.39, this forms the BVP for the displacement field  $w(x)$  over the interval  $x \in [0, L]$ .

The objective is therefore to determine the constants  $A$ ,  $B$ ,  $C$ , and  $D$  in the general solutions given by Equation 4.24 and Equation 4.27 such that the boundary conditions are satisfied.

### 4.4. Solution of the boundary-value problem

Following the formal statement of the BVP in subsection 4.3.4, this section presents the systems of equations needed to determine  $A$ ,  $B$ ,  $C$  and  $D$  and investigates the resulting displacement and internal force fields.

#### 4.4.1. Determination unknown coefficients

Using boundary conditions of Equation 4.39, the following systems of equations can be formulated for both general solutions:

For  $\Delta < 0$ :

$$c_{1,Q}^{(-)}A + c_{2,Q}^{(-)}B + c_{3,Q}^{(-)}C + c_{4,Q}^{(-)}D = Q_f \quad (4.40)$$

$$\begin{aligned} & \left( c_{1,Q}^{(-)} \cos \beta L + c_{5,Q}^{(-)} \sin \beta L \right) e^{\alpha L} A + \left( c_{2,Q}^{(-)} \cos \beta L + c_{6,Q}^{(-)} \sin \beta L \right) e^{\alpha L} B \\ & + \left( c_{3,Q}^{(-)} \cos \beta L + c_{7,Q}^{(-)} \sin \beta L \right) e^{-\alpha L} C + \left( c_{4,Q}^{(-)} \cos \beta L + c_{8,Q}^{(-)} \sin \beta L \right) e^{-\alpha L} D = Q_f \end{aligned} \quad (4.41)$$

$$c_{1,M}^{(-)}A + c_{2,M}^{(-)}B + c_{3,M}^{(-)}C + c_{4,M}^{(-)}D = 0 \quad (4.42)$$

$$\begin{aligned} & \left( c_{1,M}^{(-)} \cos \beta L + c_{5,M}^{(-)} \sin \beta L \right) e^{\alpha L} A + \left( c_{2,M}^{(-)} \cos \beta L + c_{6,M}^{(-)} \sin \beta L \right) e^{\alpha L} B \\ & + \left( c_{3,M}^{(-)} \cos \beta L + c_{7,M}^{(-)} \sin \beta L \right) e^{-\alpha L} C + \left( c_{4,M}^{(-)} \cos \beta L + c_{8,M}^{(-)} \sin \beta L \right) e^{-\alpha L} D = 0 \end{aligned} \quad (4.43)$$

For  $\Delta > 0$ :

$$c_{1,Q}^{(+)}A + c_{2,Q}^{(+)}B + c_{3,Q}^{(+)}C + c_{4,Q}^{(+)}D = Q_f \quad (4.44)$$

$$\begin{aligned} & \left( c_{1,Q}^{(+)} \cosh \beta L + c_{5,Q}^{(+)} \sinh \beta L \right) e^{\alpha L} A + \left( c_{2,Q}^{(+)} \cosh \beta L + c_{6,Q}^{(+)} \sinh \beta L \right) e^{\alpha L} B \\ & + \left( c_{3,Q}^{(+)} \cosh \beta L + c_{7,Q}^{(+)} \sinh \beta L \right) e^{-\alpha L} C + \left( c_{4,Q}^{(+)} \cosh \beta L + c_{8,Q}^{(+)} \sinh \beta L \right) e^{-\alpha L} D = Q_f \end{aligned} \quad (4.45)$$

$$c_{1,M}^{(+)}A + c_{2,M}^{(+)}B + c_{3,M}^{(+)}C + c_{4,M}^{(+)}D = 0 \quad (4.46)$$

$$\begin{aligned} & \left( c_{1,M}^{(+)} \cosh \beta L + c_{5,M}^{(+)} \sinh \beta L \right) e^{\alpha L} A + \left( c_{2,M}^{(+)} \cosh \beta L + c_{6,M}^{(+)} \sinh \beta L \right) e^{\alpha L} B \\ & + \left( c_{3,M}^{(+)} \cosh \beta L + c_{7,M}^{(+)} \sinh \beta L \right) e^{-\alpha L} C + \left( c_{4,M}^{(+)} \cosh \beta L + c_{8,M}^{(+)} \sinh \beta L \right) e^{-\alpha L} D = 0 \end{aligned} \quad (4.47)$$

The linear systems of equations formed by Equation 4.40-4.43 and Equation 4.44-4.47 can be written in matrix form, such that by matrix inversion the solution for the unknown coefficients equals:

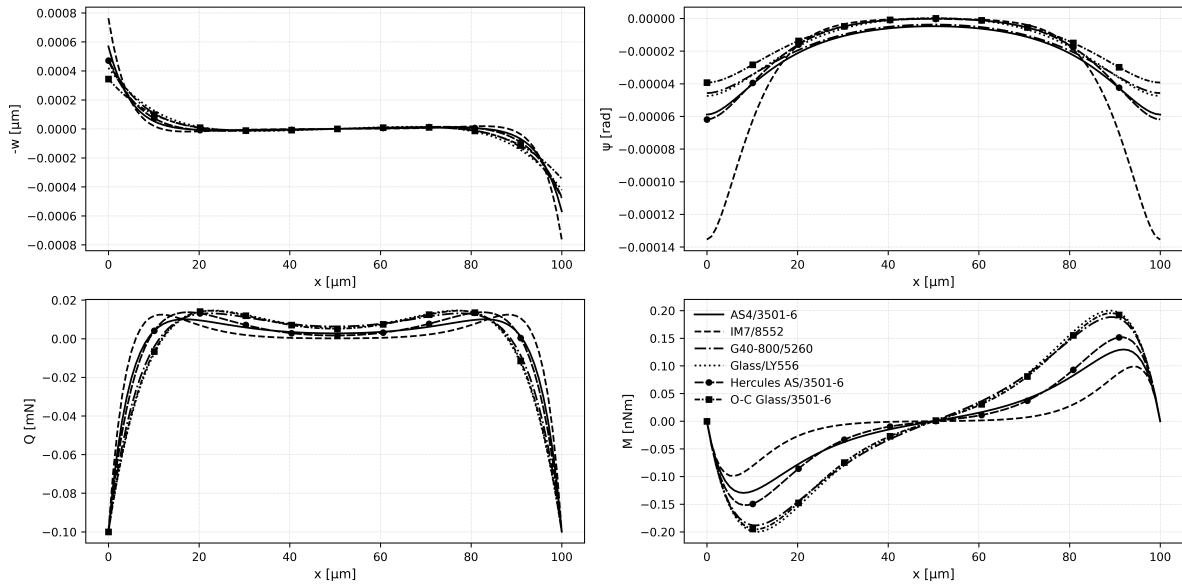
$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \mathbf{M}^{(\pm)-1} \begin{bmatrix} Q_f \\ Q_f \\ 0 \\ 0 \end{bmatrix}, \quad (4.48)$$

with  $\mathbf{M}^{(\pm)}$  being the material system matrix containing the constants  $c_{i,Q}^{(\pm)}$  and  $c_{i,M}^{(\pm)}$ , the fiber length  $L$  and the spatial wave numbers  $\alpha$  and  $\beta$ .

#### 4.4.2. Resulting displacement and internal force fields

Combining the displacement ansatz of Equation 4.24 or Equation 4.27, depending on the sign of  $\Delta$ , with the numerical solutions for the coefficients  $A$ ,  $B$ ,  $C$ , and  $D$  obtained in subsection 4.4.1, yields the response of the RVE for the material systems introduced in chapter 3. For an RVE length of  $L = 100\mu m$ , the resulting fields are shown in Figure 4.9. Since the fiber diameters are at most  $11\mu m$ , an RVE of  $100\mu m$  can still be treated as a beam.

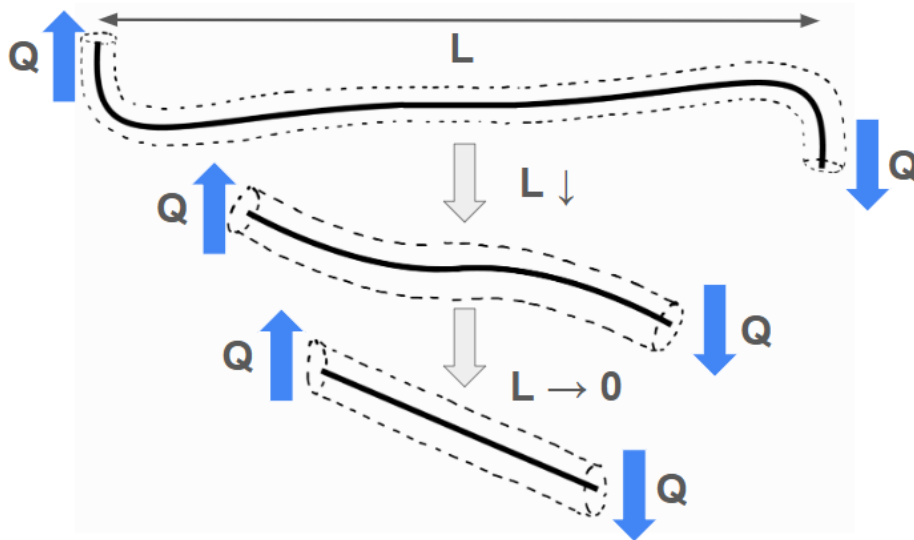
For this RVE length, the displacement  $w$  and rotation  $\psi$  are primarily concentrated at the RVE ends, in response to the imposed loading, and gradually return to their pre-loading values on the interior of the RVE. The imposed boundary conditions of Equation 4.39 are satisfied, with  $Q_{0/L} = Q_f = -0.1 mN$  and  $M_{0/L} = 0 nNm$ . Differences between material systems manifest mainly through variations in the magnitude and extent of internal loadings, reflecting the relative stiffnesses between the fiber as a beam and the matrix as elastic foundation.



**Figure 4.9:** Resulting displacement field  $w$ , cross-sectional rotation  $\psi$ , internal force  $Q$  and internal moment  $M$  for an applied force of  $Q = 0.1$  and an RVE length of  $L = 100\mu m$

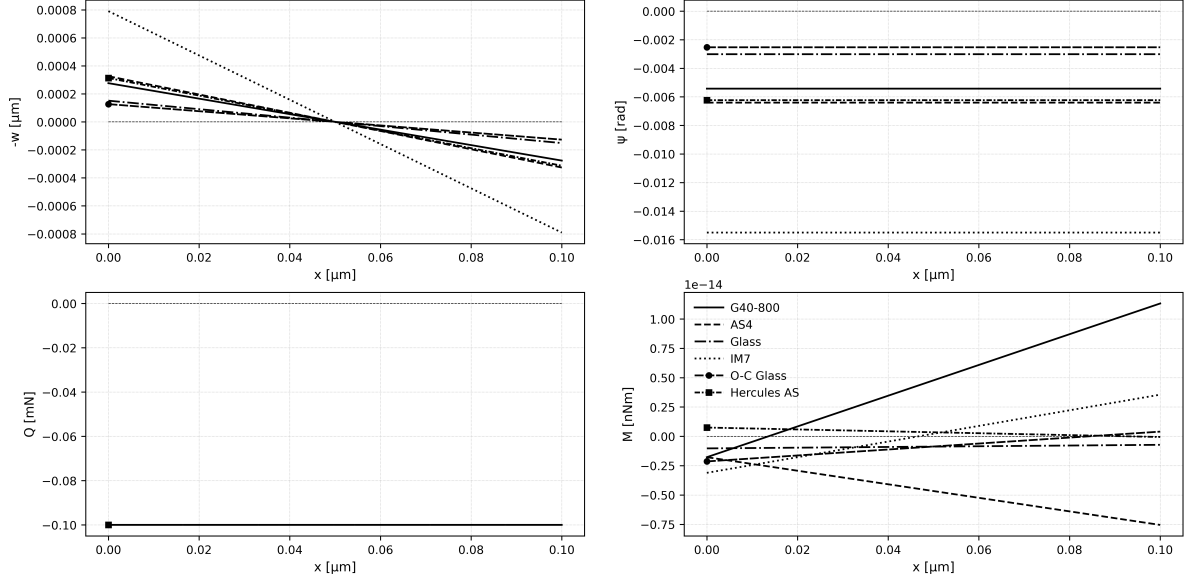
Two key trends may be identified. First, the displacement induced by the applied shear load decays away from the boundaries, indicating that the deformation is localised near the loading points. Second, the development of a non-zero internal bending moment, which increases from the boundary, reaches a maximum, and subsequently diminishes, implies that axial strains arise through bending effects, despite the nominally shear-dominated boundary conditions.

For shorter RVE lengths, the displacement and internal shear force no longer decay to zero within the domain. This observation motivates a reduction of the RVE length such that the displacement field approaches a linear variation and the internal shear force becomes nearly constant and equal to the applied shear load, thereby approximating a state of pure shear and reducing any bending. In the limiting case, taking  $L \rightarrow 0$  corresponds to evaluating the governing fields locally at  $x = 0$ . This procedure is shown schematically in Figure 4.10, and numerically in Figure 4.11 for an RVE length of  $L = 0.1\mu m$ . The resulting displacement field is linear, the internal shear force constant and the internal moment linear, since  $\frac{dM}{dx} \propto Q$ , but of negligible magnitude ( $10^{-14}$ ).



**Figure 4.10:** In the limit of decreasing the RVE length to zero, the RVE will be purely in shear.





**Figure 4.11:** Resulting displacement field  $w$ , cross-sectional rotation  $\psi$ , internal force  $Q$  and internal moment  $M$  for an applied force of  $Q = 0.1$  and an RVE length of  $L = 0.1\mu\text{m}$ .

## 4.5. Evaluation of the strain field

The analytical solution derived in subsection 4.3.4 provides the displacement and rotation fields of the fiber within the RVE. From these kinematic fields, the corresponding strain measures can be obtained through the small-strain relations of continuum mechanics.

### 4.5.1. Tensorial strain components along the RVE

Applying the small-strain kinematic relations [86]

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (4.49)$$

to the displacement field introduced in Equation 4.8 yields the tensorial strain components governing the deformation of the fiber. In particular, the axial strain component

$$\varepsilon_{11} = -z \frac{d\psi}{dx}, \quad (4.50)$$

and the tensorial shear strain component

$$\varepsilon_{13} = \frac{1}{2} \left( \frac{dw}{dx} - \psi \right), \quad (4.51)$$

follow directly from the spatial gradients of the displacement field.

The engineering shear strain is subsequently defined as

$$\gamma_{13} = 2\varepsilon_{13}. \quad (4.52)$$

An example distribution of these strain measures along the RVE length is shown in Figure 4.12, illustrating the localisation of shear-induced deformation near the boundaries and the corresponding evolution of axial strain associated with bending. Note how  $\varepsilon_{13}$  reaches a maximum and  $\varepsilon_{11}$  a minimum at  $x = 0$ , consistent with the applied boundary conditions.

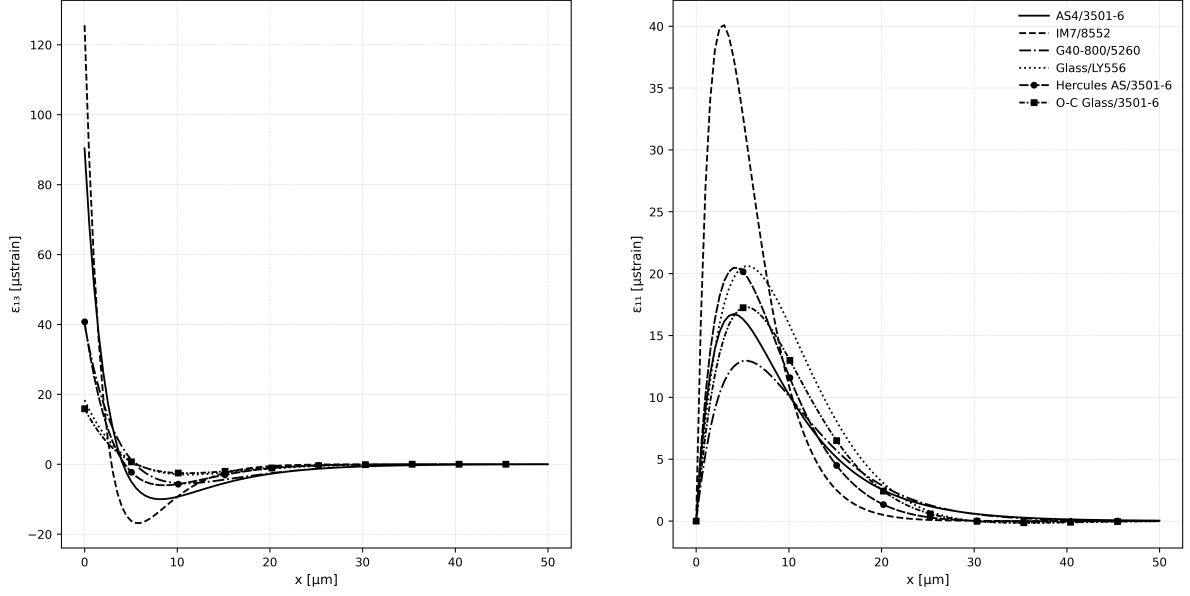


Figure 4.12: Distribution of tensorial shear strain and axial strain along the RVE.

#### 4.5.2. Shear strain at the RVE boundary

Having established the general form of the strain components, attention is now restricted to the shear strain at the RVE boundary, which directly links the applied shear load to the resulting deformation. The relevant deformation measure for the present problem is the engineering shear strain in the fiber–matrix system,  $\gamma_{13} = 2\varepsilon_{13}$ .

Evaluating this quantity using the displacement field of Equation 4.8 yields the shear strain at  $x = 0$  as

$$\gamma_{13} = 2\varepsilon_{13} = \left. \frac{dw}{dx} \right|_{x=0} - \psi(0). \quad (4.53)$$

Substituting the constitutive relation for the rotation field  $\psi(x)$  (Equation 4.31) and collecting common terms yields

$$\gamma_{13} = (1 - U) \left. \frac{dw}{dx} \right|_{x=0} - V \left. \frac{d^3w}{dx^3} \right|_{x=0}. \quad (4.54)$$

Substituting in the derivatives of the displacement field  $w(x)$  from Appendix A into Equation 4.54 and collecting common terms yields

$$\gamma_{13} = (A - C)e + (B + D)d. \quad (4.55)$$

where the parameters  $e$  and  $d$  depend on the characteristic parameter  $\Delta$  and thus differentiate between the two anzätze:

$$e = \begin{cases} \alpha(1 - U) - (\alpha^3 - 3\alpha\beta^2)V, & \text{if } \Delta < 0, \\ \alpha(1 - U) - (\alpha^3 + 3\alpha\beta^2)V, & \text{if } \Delta > 0, \end{cases} \quad d = \begin{cases} \beta(1 - U) - (-\beta^3 + 3\alpha^2\beta)V, & \text{if } \Delta < 0, \\ \beta(1 - U) - (\beta^3 + 3\alpha^2\beta)V, & \text{if } \Delta > 0. \end{cases} \quad (4.56)$$

It is noted that the kinematic description of the present beam formulation naturally results in the shear strain component  $\gamma_{13}$  rather than  $\gamma_{12}$ , since this follows directly from the standard beam coordinate system. As a result, the corresponding constitutive relation would formally involve the shear modulus  $G_{13}$ . However, for a UD ply, the material is commonly assumed to be transversely isotropic about the fiber direction [3]. This implies that the two directions in the transverse cross-sectional plane are materially equivalent, such that

$$G_{12} = G_{13}. \quad (4.57)$$

Since the beam axis of the present RVE is aligned with the fiber direction, this transverse isotropy applies directly to the current formulation. Therefore, although the beam kinematics naturally produce

$\gamma_{13}$ , this quantity may be associated consistently with the shear response described by  $G_{12}$ . Hence, for notational consistency in the remainder of this work,  $\gamma_{13}$  will be written as  $\gamma_{12}$ , such that:

$$\gamma_{12} = (A - C)e + (B + D)d \quad (4.58)$$

This relation now expresses the shear strain at the RVE boundary directly in terms of the constants of the analytical solution.

#### 4.5.3. Dependence on applied shear load

A slight rewrite of Equation 4.48 will allow for a reformulation of Equation 4.55:

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \mathbf{M}^{(\pm)-1} \begin{bmatrix} Q_f \\ Q_f \\ 0 \\ 0 \end{bmatrix} = Q_f \mathbf{M}^{(\pm)-1} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = Q_f \begin{bmatrix} A/Q_f \\ B/Q_f \\ C/Q_f \\ D/Q_f \end{bmatrix} = Q_f \begin{bmatrix} A' \\ B' \\ C' \\ D' \end{bmatrix} \quad (4.59)$$

And thus with  $A, B, C$  and  $D$  being linearly dependent on  $Q_f$  through Equation 4.59, Equation 4.58 can be written as:

$$\gamma_{12} = Q_f [(A' - C')e + (B' + D')d] \quad (4.60)$$

#### 4.5.4. Effective RVE shear stiffness

Equation 4.60 shows that the shear strain scales linearly with the applied shear load  $Q_f$ . Comparing to the standard mechanics-of-materials relation for Timoshenko shear deformation [43]:

$$\gamma = \frac{Q}{nGA} = \frac{Q}{K} \quad (4.61)$$

shows that Equation 4.60 plays an analogous role, with its bracketed term as an effective compliance such that the RVE shear stiffness  $K_{RVE}$  is equal to:

$$K_{RVE} = \frac{1}{(A' - C')e + (B' + D')d} \quad (4.62)$$

### 4.6. Reduced analytical formulation

Although Equation 4.62 provides an expression for the shear strain in terms of the constants  $A, B, C$ , and  $D$ , it relies on the full solution of the  $4 \times 4$  boundary-value system. Numerical evaluation of this system reveals a structural simplification that allows a more compact analytical formulation to be obtained. This observation is examined below and subsequently used to derive a reduced analytical solution for the RVE response.

#### 4.6.1. Numerical observation and physical interpretation

Inspection of the numerical solution of Equation 4.61 shows that the constants  $A$  and  $B$  are negligibly small for all material systems considered. This behaviour can be understood by examining the general solution expressions (Equations 4.26 and 4.29), where both constants are associated with terms proportional to  $e^{\alpha x}$ . Such terms would lead to an unbounded growth of the displacement field when moving away from the points of load application, which is not physically admissible for the present fiber-matrix configuration.

Consequently, the physically relevant solution is dominated by the exponentially decaying components, and it is therefore reasonable to set

$$A = B = 0,$$

*a priori*. This observation enables a reduction of the original BVP and motivates the search for a simplified analytical formulation.

### 4.6.2. Reduced boundary-value problem and closed-form solution

By imposing  $A = B = 0$ , the original  $4 \times 4$  system of Equation 4.50 reduces to the following  $2 \times 2$  system governing the remaining constants  $C$  and  $D$ :

$$\begin{bmatrix} c_{3,Q}^{(\pm)} & c_{4,Q}^{(\pm)} \\ c_{3,M}^{(\pm)} & c_{4,M}^{(\pm)} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} Q_f \\ 0 \end{bmatrix}. \quad (4.63)$$

The reduced system allows for a closed-form solution, yielding

$$C = Q_f \frac{c_{4,M}^{(\pm)}}{c_{3,Q}^{(\pm)} c_{4,M}^{(\pm)} - c_{4,Q}^{(\pm)} c_{3,M}^{(\pm)}}, \quad (4.64)$$

$$D = Q_f \frac{-c_{3,M}^{(\pm)}}{c_{3,Q}^{(\pm)} c_{4,M}^{(\pm)} - c_{4,Q}^{(\pm)} c_{3,M}^{(\pm)}}. \quad (4.65)$$

These expressions provide analytical values for the remaining solution constants and form the basis for deriving an explicit expression for the effective shear stiffness of the RVE.

## 4.7. Analytical expression for RVE shear stiffness

As Equation 4.55 is still valid, combining it with the condition that  $A = B = 0$  and Equation 4.64 and Equation 4.65 gives:

$$\gamma_{12} = Q_f \frac{-c_{4,M}^{(\pm)} e - c_{3,M}^{(\pm)} d}{c_{3,Q}^{(\pm)} c_{4,M}^{(\pm)} - c_{4,Q}^{(\pm)} c_{3,M}^{(\pm)}}, \quad (4.66)$$

such that:

$$K_{RVE} = \frac{c_{3,Q}^{(\pm)} - c_{4,Q}^{(\pm)} \left( \frac{c_{3,M}^{(\pm)}}{c_{4,M}^{(\pm)}} \right)}{-e - d \left( \frac{c_{3,M}^{(\pm)}}{c_{4,M}^{(\pm)}} \right)}. \quad (4.67)$$

Substituting in Equation 4.56, and defining the following shorthand notations,

$$R = \left( \frac{c_{3,M}^{(\pm)}}{c_{4,M}^{(\pm)}} \right) = \begin{cases} \frac{H/G - (\alpha^2 - \beta^2)}{2\alpha\beta}, & \text{if } \Delta < 0, \\ \frac{H/G - (\alpha^2 + \beta^2)}{2\alpha\beta}, & \text{if } \Delta > 0, \end{cases} \quad (4.68)$$

$$\phi = \begin{cases} -\alpha^3 + 3\alpha\beta^2, & \text{if } \Delta < 0, \\ -\alpha^3 - 3\alpha\beta^2, & \text{if } \Delta > 0, \end{cases} \quad \psi = \begin{cases} -\beta^3 + 3\alpha^2\beta, & \text{if } \Delta < 0, \\ \beta^3 + 3\alpha^2\beta, & \text{if } \Delta > 0. \end{cases} \quad (4.69)$$

Equation 4.67 can be generalized into:

$$K_{RVE} = \frac{I \left( \frac{\alpha + \beta R}{\phi - \psi R} \right) + J}{(1 - U) \left( \frac{\alpha + \beta R}{\phi - \psi R} \right) + V}, \quad (4.70)$$

with the appropriate selection of  $\alpha, \beta, \phi, \psi$  and  $R$  based on the sign of  $\Delta$ .

## 4.8. Initial ply stiffness prediction

Following the definition of the shear stiffness [43]:

$$K = nGA \quad (4.71)$$

with shear correction factor  $n$ , cross-sectional area  $A$  and shear modulus  $G$ , the RVE shear modulus would follow as:

$$G_{RVE} = \frac{K_{RVE}}{n_{RVE}A_{RVE}} \quad (4.72)$$

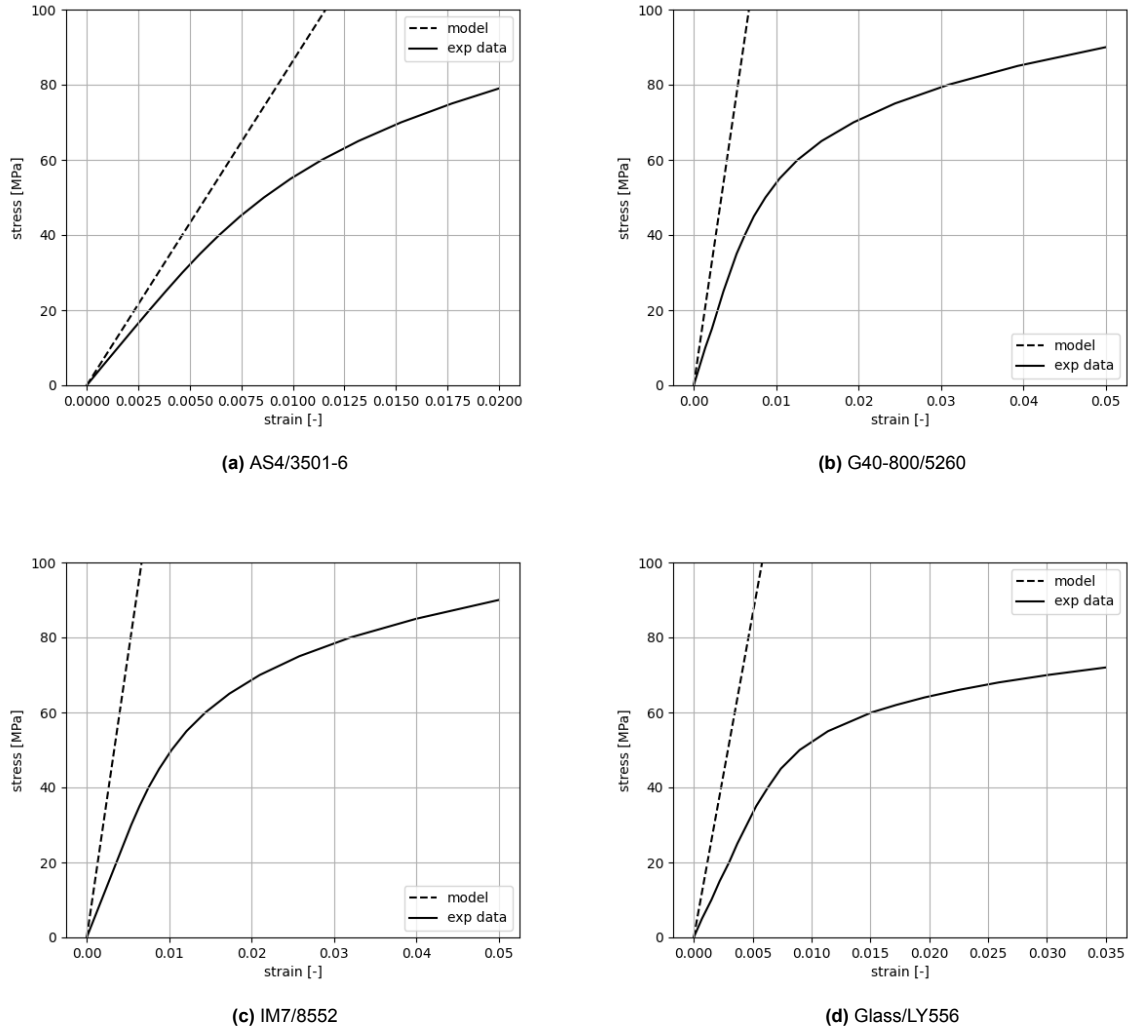
However, the shear correction factor of the RVE,  $n_{RVE}$  proved to be problematic to define.

In light of the general considerations outlined in subsection 2.2.1, the determination of an effective shear correction factor for the present RVE configuration would require a detailed resolution of the three-dimensional stress field and its relation to the assumed beam kinematics. Such an analysis is beyond the scope of the current formulation and would introduce an unnecessary level of complexity at this stage. For simplicity, the shear correction factor is therefore taken as

$$n_{RVE} = 1. \quad (4.73)$$

The implications of this assumption will be assessed in chapter 5.

Using Equation 4.72, an initial ply stiffness can be evaluated. Considering an undamaged ply consisting solely of fibers and matrix that correspond to the current RVE arrangement its linear behaviour is shown in Figure 4.13a-Figure 4.13d.



**Figure 4.13:** Idealized ply behaviour versus experimental data obtained from Kaddour et al. [84]

For all material systems considered, Equation 4.72 consistently overpredicts  $G_{12}$  of the ply. A relative comparison is shown in Table 4.1. Overprediction ranges from +30% to nearly +200% for AS4/3501-6 and Glass/LY556 respectively. Considering the shear moduli of the respective constituents, i.e.  $G_m$  and  $G_f$ , in comparison to  $G_{12}$  it appears reasonable to suggest that the fiber is contributing too much to the RVE shear stiffness and thus too much to the initial ply modulus.

**Table 4.1:** Comparison of experimental data with modeled results for selected UD systems. Data from Kaddour et al. [64]

| Material System | Experimental   | Timoshenko Fiber |           |
|-----------------|----------------|------------------|-----------|
|                 | $G_{12}$ [GPa] | $G_{12}$ [GPa]   | Error [%] |
| AS4/3501-6      | 6.60           | 8.62             | +30.7     |
| G40-800/5260    | 6.94           | 14.9             | +115      |
| IM7/8552        | 5.60           | 15.0             | +168      |
| Glass/LY556     | 5.83           | 17.2             | +195      |
| Average         | -              | -                | +127.2    |
| Std. dev.       | -              | -                | 64.3      |

## 4.9. Summary and outlook

The main findings of this chapter can be summarised as follows:

- An analytical solution describing the coupled shear response of the fiber-matrix RVE was obtained.
- The derived formulation enables direct extraction of an effective RVE shear stiffness and corresponding ply-level shear modulus.
- The deterministic prediction significantly overestimates the expected initial ply shear modulus, indicating an excessive stiffness contribution of the fiber.

While the deterministic formulation establishes a rigorous analytical baseline, the discrepancy in predicted stiffness remains unsolved. To investigate the origin of this inconsistency, the following chapter performs systematic analytical comparisons and finite-element studies across fiber volume fractions and RVE geometries.

# 5

## Analytical Comparison and Numerical Evidence

This chapter evaluates the deterministic stiffness prediction through comparison with analytical bounds and FEA RVE simulations. The behaviour of the analytical model is examined for the full fiber volume fraction range and contrasted with Voigt- and Reuss-type responses. FEA analyses are then used to investigate the influence of RVE geometry at fixed fiber volume fraction, providing numerical evidence for variability in effective stiffness. At the same time, the finite element analysis serves as an alternative model which provides insight into the interaction between matrix and fibers.

### 5.1. Analytical behaviour of deterministic RVE stiffness

Before interpreting the overprediction of  $G_{12}$  by  $G_{RVE}$ , as identified in section 4.8, the behaviour of the underlying  $K_{RVE}$  is examined. This allows assessment of whether the reduction of the three-dimensional fiber-matrix interaction problem to a one-dimensional beam-on-elastic-foundation model preserves the essential stiffness characteristics.

#### 5.1.1. Limit behaviour of the deterministic stiffness

In the limit of  $v_f \rightarrow 0$ , the properties of the matrix should be obtained, such that by definition of the shear stiffness

$$\lim_{v_f \rightarrow 0} K_{RVE}(v_f) = (nGA)_m. \quad (5.1)$$

However to obtain  $v_f \rightarrow 0$ , it is not that the fiber disappears, but that the matrix area  $A_m$  becomes unbounded as illustrated in Figure 5.1. Consequently Equation 5.1 becomes unbounded and does not provide any meaningful insight. Instead, the limit of

$$\lim_{v_f \rightarrow 0} \frac{K_{RVE}(v_f)}{A_{RVE}(v_f)} = n_m G_m, \quad (5.2)$$

should remain bounded, as  $G_m$  is a material property and the shear correction factor  $n_m$  depends on shape, not size of the matrix annulus.

The limit of vanishing fiber volume fraction provides insight into the physical consistency of the deterministic stiffness formulation. In this regime, the RVE is dominated by matrix deformation, and the effective shear modulus would be expected to recover the matrix shear modulus  $G_m$ , taking into account the shear correction factor  $n_m$ . However, due to the abstraction of the surrounding matrix as an elastic foundation of infinite radial extent, the deterministic formulation does not reproduce this limit exactly, such that



$$\lim_{v_f \rightarrow 0} \frac{K_{RVE}(v_f)}{A_{RVE}(v_f)} \approx G_m. \quad (5.3)$$

This deviation highlights a fundamental modelling consequence of reducing the three-dimensional matrix domain to a one-dimensional foundation representation. The consequence of Equation 5.3 can be observed in Figure 5.9-5.14.

Conversely, the limit of  $v_f \rightarrow 1$  corresponds to an RVE consisting solely of the fiber. In this case, as the fiber geometry is included exactly, the deterministic stiffness asymptotically approaches the shear stiffness of the fiber beam:

$$\lim_{v_f \rightarrow 1} K_{RVE}(v_f) = C_s. \quad (5.4)$$

Attempts were made to provide analytical solutions for Equation 5.3 and Equation 5.4, which proved to be rather complicated due to nested dependencies on  $v_f$  and branching behaviour due to the sign of  $\Delta$ .

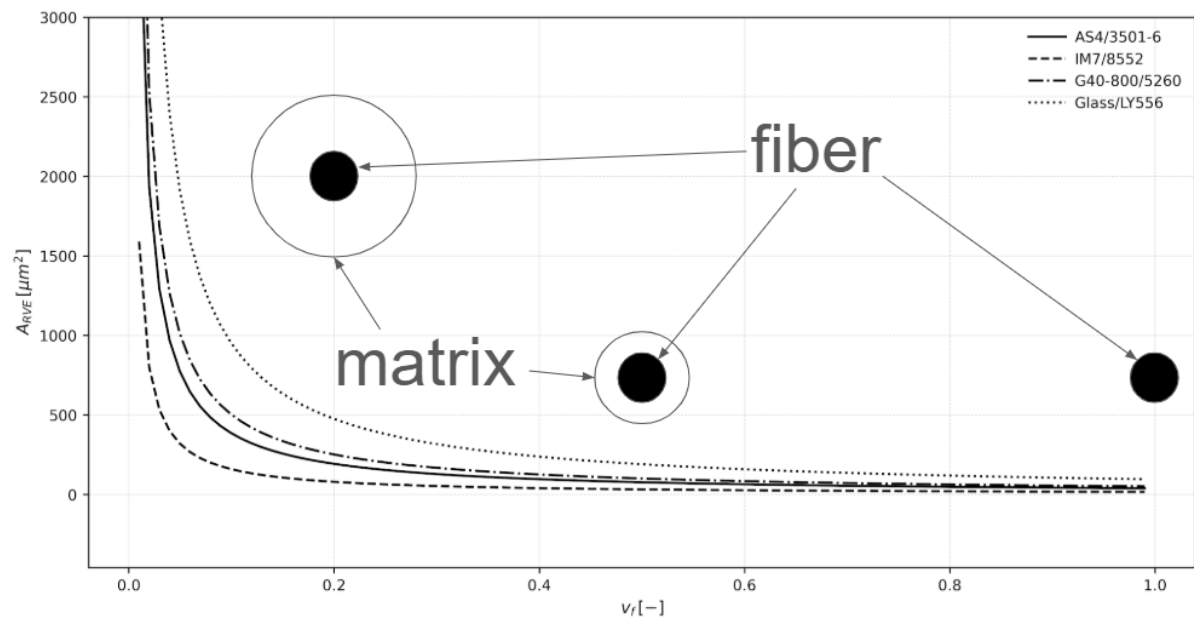


Figure 5.1:  $A_{RVE}$  as a function of  $v_f$

### 5.1.2. Comparison with concentric-cylinder analytical solutions

Steinboeck et al. [87] derived a general analytical expression for the shear stiffness of cylindrical beams composed of radially inhomogeneous materials. A particular case of this formulation corresponds to a circular cross-section consisting of two concentric layers. Applying this result to the present RVE yields:

$$K_{Steinboeck,num} = 3A_{RVE}E_m(1+\nu)\left(\frac{E_f}{E_m}(1+v_f)+1-v_f\right)\left(\left(\frac{E_f}{E_m}-1\right)v_f^2+1\right)^2 \quad (5.5)$$

$$K_{Steinboeck,denom} = \left(\left(\frac{E_f}{E_m}\right)^2 v_f^3(1+v_f) + \frac{E_f}{E_m}(1+2v_f^2-v_f^3-2v_f^4) + (1-v_f^2)^2\right)(7+14\nu+8\nu^2) \quad (5.6)$$

$$+ 4\left(1 - \frac{E_f}{E_m}\right)v_f(1-v_f)^2(5+10\nu+4\nu^2)$$

$$K_{Steinboeck} = \frac{K_{Steinboeck,num}}{K_{Steinboeck,denom}} \quad (5.7)$$

Here,  $\nu$  denotes the Poisson's ratio. In the original formulation,  $\nu$  was assumed radially constant, as the expression was derived for metallic beams. Therefore,  $\nu$  will be interpreted as either  $\nu_f$  or  $\nu_m$ , as shown in Figure 5.2-5.7. A comparison between  $K_{RVE}$  (Equation 4.70) and  $K_{Steinboeck}$  (Equation 5.7) can now be made.

Qualitatively, both expressions exhibit similar behaviour. For low  $v_f$ , a vertical asymptote is observed, as both depend on  $A_{RVE}$ , which tends to infinity as  $v_f \rightarrow 0$ . Conversely, for high  $v_f$ , both expressions approach a plateau, reflecting the increasing dominance of the fiber contribution.  $K_{RVE}$  also does not have a minimum at approximately  $v_f = 0.1$  but monotonically decreases towards  $v_f = 1$ .

Quantitatively,  $K_{RVE}$  predicts a far lower shear stiffness for the carbon materials, as shown in Figure 5.2-5.5, while the glass materials provide a close approximation from  $v_f = 0.4$  onwards. To investigate this difference, the limiting behaviour of the shear stiffness  $K_{Steinboeck}$  is examined, as was done for  $K_{RVE}$  in subsection 5.1.1. Although Equation 5.7 considers a constant Poisson's ratio  $\nu$  while the RVE defined in chapter 4 does not, in the limit of the RVE only containing a fiber. i.e.  $v_f \rightarrow 1$ ,  $\nu \rightarrow \nu_f$ .

The limit of Equation 5.7 for an RVE containing only the fiber, i.e.  $v_f \rightarrow 1$ , equals

$$\lim_{v_f \rightarrow 1} K_{Steinboeck} = \frac{3A_f E_f (1 + \nu)}{7 + 14\nu + 8\nu^2}. \quad (5.8)$$

Recognizing that

$$(7 + 14\nu + 8\nu^2) \approx (7 + 15\nu + 8\nu^2) = (1 + \nu)(7 + 8\nu), \quad (5.9)$$

Equation 5.8 can be simplified to

$$\lim_{v_f \rightarrow 1} K_{Steinboeck} \approx \frac{3A_f E_f}{7 + 8\nu}. \quad (5.10)$$

Employing the assumption of material isotropy, this can be further rewritten as

$$\lim_{v_f \rightarrow 1} K_{Steinboeck} \approx \frac{6G_f A_f (1 + \nu)}{7 + 8\nu}, \quad (5.11)$$

which is very close to the fiber shear stiffness

$$C_s = \frac{6G_f A_f (1 + \nu_f)}{7 + 6\nu_f}, \quad (5.12)$$

defined by Equation 4.11 in chapter 4 and corresponds to the limit of  $v_f \rightarrow 1$  for  $K_{RVE}$ , as shown by Equation 5.4. Referring to Equation 5.10 as  $K_{S,v_f \rightarrow 1}$  and to Equation 5.11 as  $K_{S,iso}$ , a comparison is made in Table 5.1, with  $\nu = \nu_f$ . Errors with respect to  $C_s$  from Equation 5.12 are computed for both  $K_{S,v_f \rightarrow 1}$  and  $K_{S,iso}$ , showing that  $K_{S,iso}$  closely recovers  $C_s$  for all materials (average  $-4.69\%$ , std. dev.  $0.17\%$ ), whereas  $K_{S,v_f \rightarrow 1}$  only recovers  $C_s$  closely for the glass materials.

This highlights a similarity between  $K_{Steinboeck}$  and  $K_{RVE}$ , namely that both formulations are based on isotropic constitutive assumptions. While  $K_{RVE}$  adopts isotropy *a priori* for both fiber and matrix and therefore recovers  $C_s$  in the limit,  $K_{Steinboeck}$  considers a radially inhomogeneous beam composed of isotropic layers. In the limiting case of a single-layer cylinder, the radial inhomogeneity disappears and the formulation likewise reduces to that of a homogeneous isotropic beam.

As both models assume isotropy of the fiber material, the Steinboeck model recovers  $C_s$  closely for the glass fibers, whether isotropy is enforced (Equation 5.11) or not (Equation 5.10), since the relation between  $E_f$ ,  $G_f$ , and  $\nu_f$  is approximately isotropic for glass fibers. For carbon fibers, however, the strong material anisotropy breaks this relation, and the Steinboeck limit only recovers  $C_s$  when isotropy is artificially enforced, as was done for  $K_{RVE}$ . This indicates that the agreement observed for glass is not coincidental, but shows compatibility between the model assumptions and material symmetry, whereas the discrepancy for carbon highlights that both formulations inherently rely on material isotropy and therefore cannot fully capture anisotropic shear response.

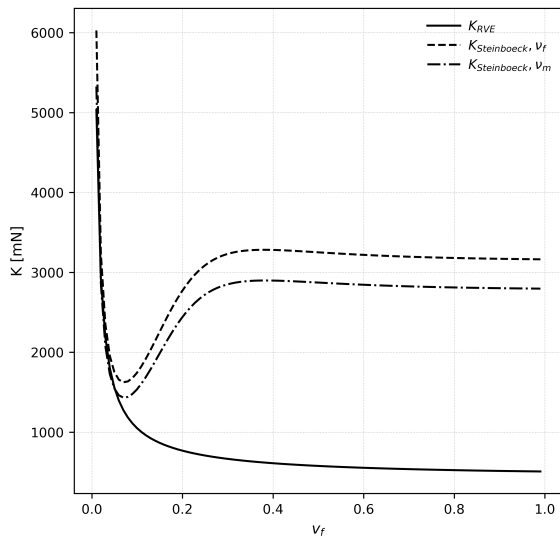
Overall, the  $K_{RVE}$  model shows close agreement with  $K_{Steinboeck}$  for  $v_f \geq 0.4$  for glass fibers, primarily because the assumption of isotropy is reasonably valid. The remaining discrepancies are likely

attributable to differences in Poisson's ratio between fiber and matrix, which are assumed to be equal in the Steinboeck formulation. For carbon fibers, however, the limit  $K_{RVE} \rightarrow C_s$ , which inherently assumes fiber isotropy, only matches the Steinboeck result when isotropy is explicitly enforced. This demonstrates that both models are unable to capture the anisotropic shear behaviour of carbon fibers due to their built-in assumption of isotropy. Therefore, the applicability of both formulations is limited to material systems for which an isotropic representation of the shear response is sufficiently accurate.

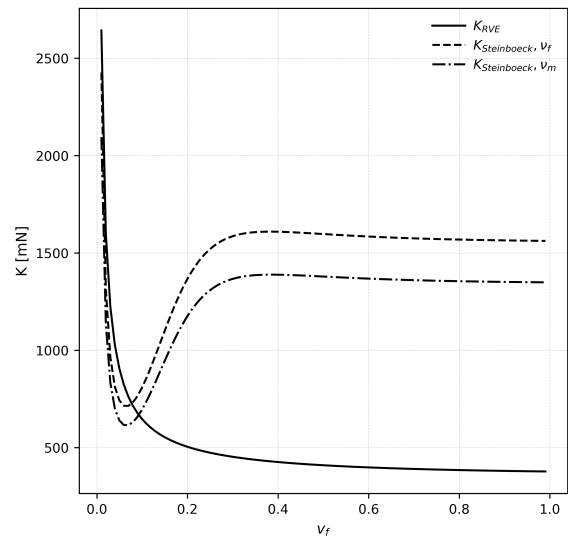
Consequently,  $K_{Steinboeck}$  does not provide an alternative model that would improve upon the initial stiffness predictions for carbon materials for  $v_f = 0.6$  of section 4.8. In fact, the observed overestimation would be even larger, as  $K_{Steinboeck} \gg K_{RVE}$ . For the glass system, no substantial difference at  $v_f = 0.6$  is expected as  $K_{Steinboeck} \approx K_{RVE}$ .

**Table 5.1:** Comparison of fiber shear stiffness predicted by  $K_{Steinboeck}$  with prediction assuming material isotropy.

| Material system            | $K_{S,v_f \rightarrow 1}$ [mN] | $K_{S,iso}$ [mN] | $C_s$ [mN] | Error $K_{S,v_f \rightarrow 1}$ [%] | Error $K_{S,iso}$ [%] |
|----------------------------|--------------------------------|------------------|------------|-------------------------------------|-----------------------|
| AS4/3501-6                 | 3101                           | 483.3            | 506.9      | 511.8                               | -4.66                 |
| IM7/8552                   | 1531                           | 360.0            | 377.0      | 306.1                               | -4.51                 |
| G40-800/5260               | 5085                           | 1136             | 1191.7     | 326.7                               | -4.67                 |
| Hercules AS/3501-6         | 2953                           | 1112             | 1165.8     | 153.3                               | -4.62                 |
| Glass/LY556                | 2453                           | 2450             | 2570       | -4.55                               | -4.67                 |
| Owens-Corning Glass/3501-6 | 2805                           | 2803             | 2951       | -4.95                               | -5.01                 |
| Average                    | —                              | —                | —          | 214.7                               | -4.69                 |
| Std. dev.                  | —                              | —                | —          | 186.6                               | 0.17                  |



**Figure 5.2:** Shear stiffness comparison for AS4/3501-6



**Figure 5.3:** Shear stiffness comparison for IM7/8552

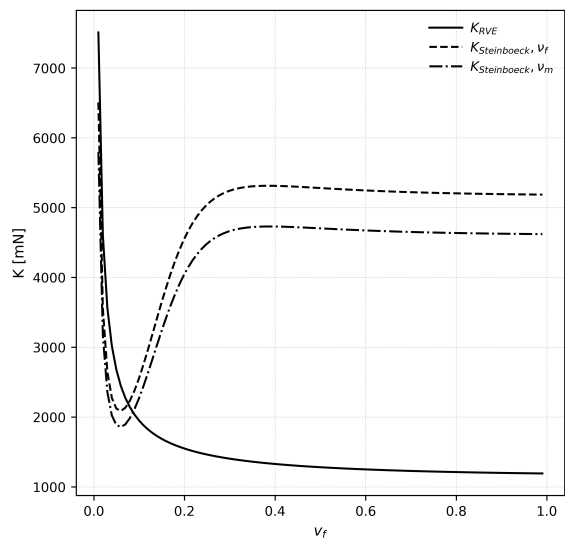


Figure 5.4: Shear stiffness comparison for G40-800/5260

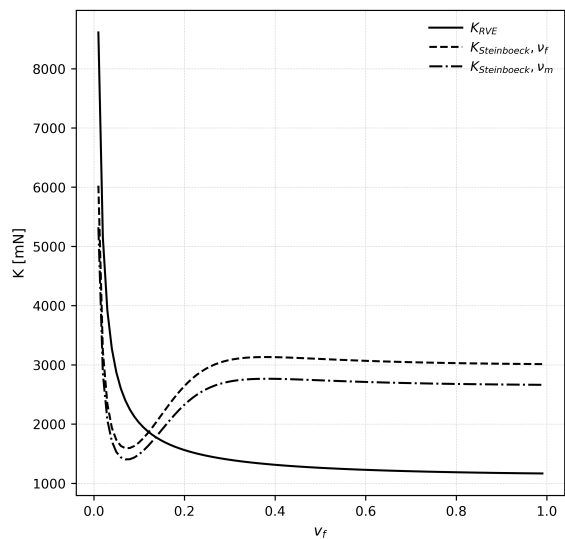


Figure 5.5: Shear stiffness comparison for Hercules AS/3501-6

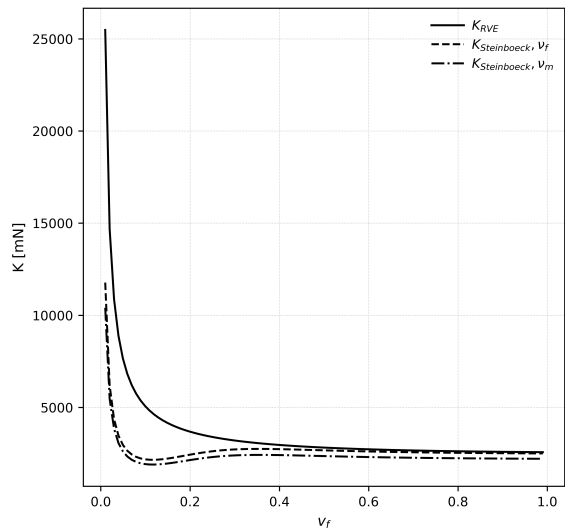


Figure 5.6: Shear stiffness comparison for Glass/LY556

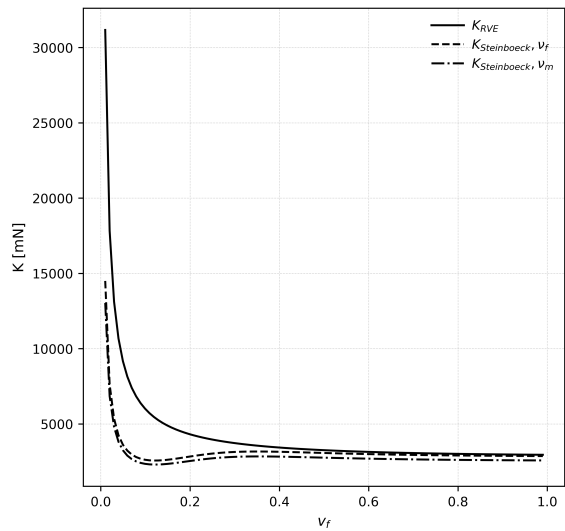


Figure 5.7: Shear stiffness comparison for Owens-Corning Glass/3501-6

## 5.2. Deterministic RVE shear modulus and relation to classical bounds

To shed light on the consistent over-estimation of  $G_{RVE}$  observed in section 4.8 for selected material systems with  $v_f = 0.6$ , the resulting  $G_{RVE}$ , as defined by Equation 4.72, is evaluated over the full range  $v_f \in [0, 1]$ , as shown in Figure 5.9-5.14. It can be clearly seen that  $G_{RVE}$  behaves similarly to a rule-of-mixtures (RoM) model, which is commonly used to predict composite moduli based on the elastic properties of the constituent phases and their relative volume fractions within the composite.

### 5.2.1. Relation to Voigt-Reuss bounds

Hill [36] showed, using variational energy principles for heterogeneous linear elastic media, that assuming uniform strain and uniform stress fields yields rigorous upper and lower bounds on the effective elastic moduli, known respectively as the Voigt and Reuss bounds. Consequently, any physically admissible micromechanical model must predict effective properties that lie between these two limits [1], [36].

The Voigt and Reuss bounds originate from early averaging schemes for polycrystalline materials, proposed by Voigt [34] under a uniform strain assumption and by Reuss [35] under a uniform stress assumption. Applied to a two-phase UD composite ply to estimate  $G_{12}$ , one obtains

$$G_{12,Voigt} = v_f G_f + v_m G_m, \quad (5.13)$$

$$G_{12,Reuss} = \left( \frac{v_f}{G_f} + \frac{v_m}{G_m} \right)^{-1}, \quad (5.14)$$

where  $v_m = 1 - v_f$  denotes the matrix volume fraction. Results for Equation 5.13 and 5.14 are shown in Figure 5.9-5.14. First approximations of  $G_{12}$  for UD plies are commonly made by using Equation 5.14, due to the experimentally observed nonlinear transition between matrix and fiber shear modulus [16].

Any RoM to estimate  $G_{12}$ , which includes  $G_{RVE}$ , is therefore expected to behave non-linearly in a Reuss-like manner, rather than linearly in a Voigt-like manner. However, inspection of Figure 5.9-5.14 shows that  $G_{RVE}$  behaves quite linear and closer to the Voigt bound than the Reuss bound.

At low fiber volume fractions, it produces either slight under- or overpredictions compared to Voigt while consistently underpredicting the response at high  $v_f$ . In fact, since  $A_{RVE} \rightarrow A_f$  and  $K_{RVE} \rightarrow C_s$  as  $v_f \rightarrow 1$ , as shown by Equation 4.7 and 5.4,  $G_{RVE}$  underpredicts the fiber shear modulus according to

$$\frac{G_{RVE}(v_f = 1)}{G_f} = n_f, \quad (5.15)$$

which is a direct consequence of assuming the shear correction factor  $n_{RVE} = 1$  for all  $v_f \in [0, 1]$  in section 4.8.

Therefore, under the assumption that  $n_{RVE} = 1$ ,  $G_{RVE}$  derived from a 1-D Timoshenko fiber beam on a matrix elastic foundation suggests that the fiber and matrix interaction is closer to a state of uniform strain (Voigt), rather than uniform stress (Reuss).

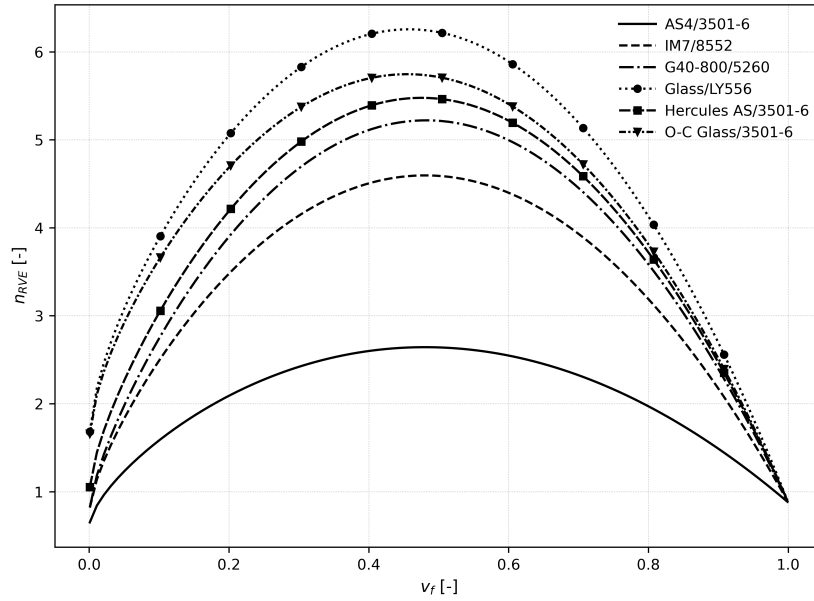
Correcting the Voigt-like behaviour of  $G_{RVE}$  closer to the desired Reuss-like behaviour could be considered as simple as weakening the assumption on the shear correction factor  $n_{RVE}$ . Intermediately re-defining the shear correction factor to be

$$n_{RVE} = \frac{(G_{RVE})_{n_{RVE}=1}}{G_{12,Reuss}}, \quad (5.16)$$

such that  $\frac{K_{RVE}}{n_{RVE} A_{RVE}} = G_{RVE} = G_{12,Reuss}$ , the resulting shear correction factor is shown in Figure 5.8. The immediate observation is that for practically any  $v_f$  a shear correction factor exceeding unity would be required, with the exception of  $v_f \rightarrow 1$ , as explained by Equation 5.15.

Recalling the discussion on the origin and definitions of the shear correction factor as covered in subsection 2.2.1, it is decided that using Equation 5.16 to formulate a shear correction factor of the present RVE is not allowed for the reasons that

1. The rather simplified approach to obtain  $n_{RVE}$  in this manner is inconsistent with other, established methods;
2. No other examples of shear correction factors greatly exceeding unity, even  $> 6$  in the case of Glass/LY556, are found in literature;
3. For a given  $v_f$ , which locks the RVE geometry for any material considered, this hypothetical shear correction factor shows rather large discrepancies across the materials under consideration.



**Figure 5.8:** Hypothetical  $n_{RVE}$  to correct the behaviour of  $G_{RVE}$

The existence of a shear correction factor  $\leq 1$  for the present RVE is not denied, but it does not form the solution to correcting  $G_{RVE}$  to behave closer to a non-linear, Reuss-like behaviour. However, it is admitted that the formulation of  $G_{RVE}$  from the 1-D Timoshenko fiber beam on a matrix elastic foundation would benefit from a shear correction factor  $n_{RVE} = f(v_f, \text{material})$ , as to stay consistent with the definition of shear stiffness  $K = nGA$ .

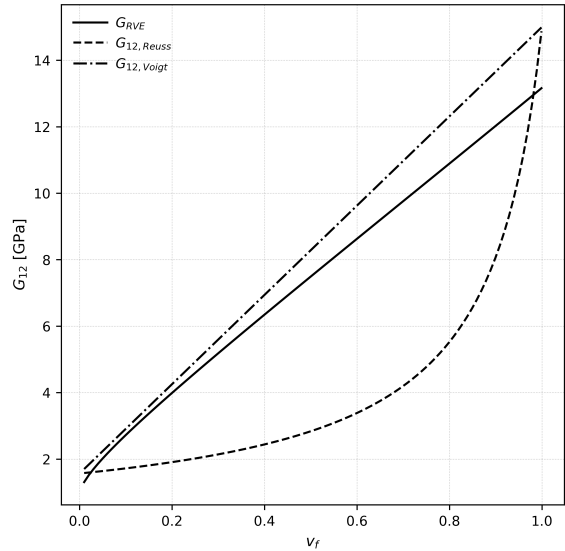


Figure 5.9: Shear modulus comparison for AS4/3501-6

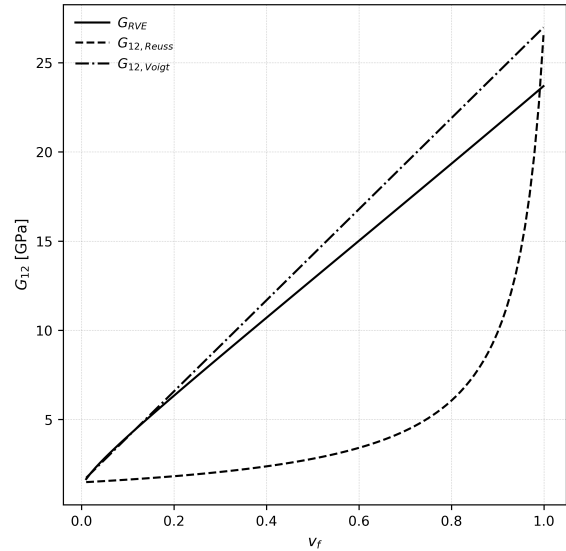


Figure 5.10: Shear modulus comparison for IM7/8552

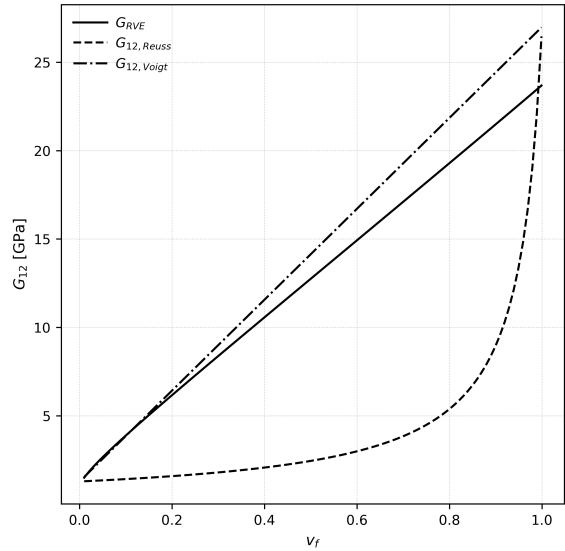


Figure 5.11: Shear modulus comparison for G40-800/5260

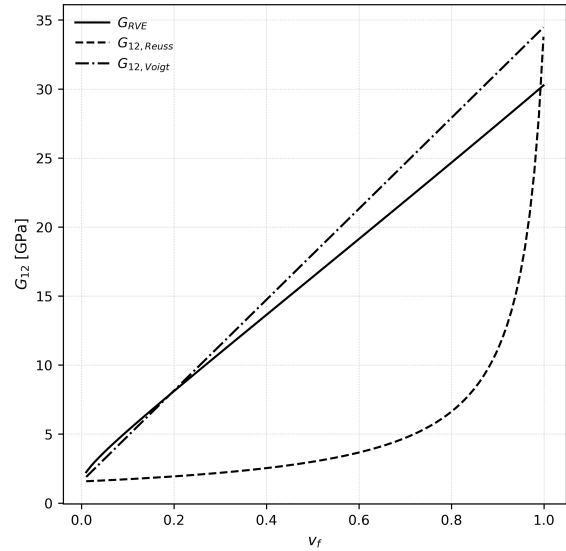


Figure 5.12: Shear modulus comparison for Hercules AS/3501-6

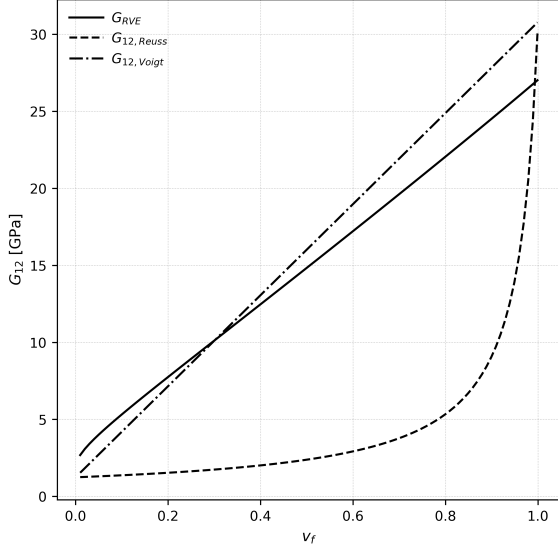


Figure 5.13: Shear modulus comparison for Glass/LY556

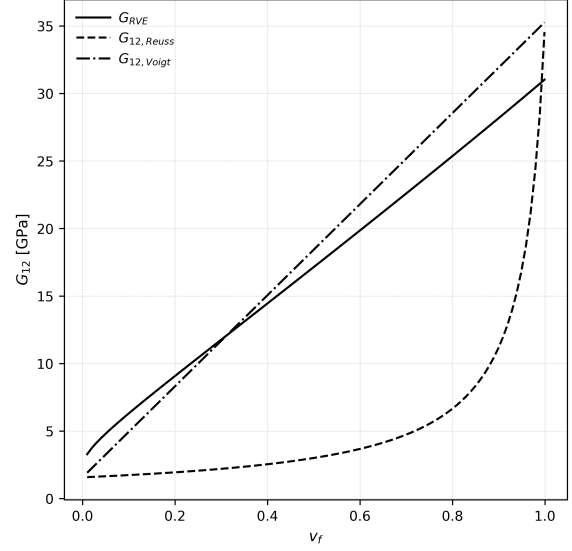


Figure 5.14: Shear modulus comparison for Owens-Corning Glass/3501-6

### 5.3. FEA evaluation of deterministic stiffness predictions

To complement the analytical assessment of the deterministic formulation, FEA simulations are performed on representative fiber–matrix RVEs. These simulations serve both to verify the stiffness extraction methodology and to investigate the influence of geometric configuration on the predicted shear response.

#### 5.3.1. Validation of shear stiffness extraction methodology

A FEA RVE model was developed to verify the numerical procedure used to extract shear stiffness and effective shear modulus. The RVE geometry corresponds to a cylindrical fiber–matrix configuration representative of the AS4/3501-6 material system. For convenience, all geometric dimensions were modelled in millimetres while preserving the correct relative length scales.

The fiber diameter was taken as

$$d_f = 7 \text{ mm},$$

with an equivalent matrix thickness of

$$t_{m,eq} = 1.018 \text{ mm},$$

resulting in an overall RVE diameter of

$$d_{RVE} = 9.037 \text{ mm}.$$

This corresponds to a fiber volume fraction of approximately

$$\frac{A_f}{A_{RVE}} = v_f = 0.6.$$

The RVE length was chosen equal to its diameter, producing a short-beam configuration consistent with the assumptions underlying Timoshenko beam theory. The final geometry can be seen in Figure 5.15 and 5.16.

The domain was discretised using three-dimensional SOLID186 elements [88] with a uniform element size of 0.25 mm, resulting in approximately cubic elements. A conformal mesh across the fiber–matrix interface was enforced through a shared topology approach, which is equivalent to assuming a perfectly bonded interface.



Prior to introducing distinct fiber and matrix material properties, the full geometry was modelled as a homogeneous isotropic solid using structural steel properties, summarized in Table 5.2. This verification step allows direct comparison between the finite- element response and classical beam theory predictions, thereby isolating the numerical stiffness extraction procedure from microstructural modelling effects.

**Table 5.2:** Material properties of structural steel [89]

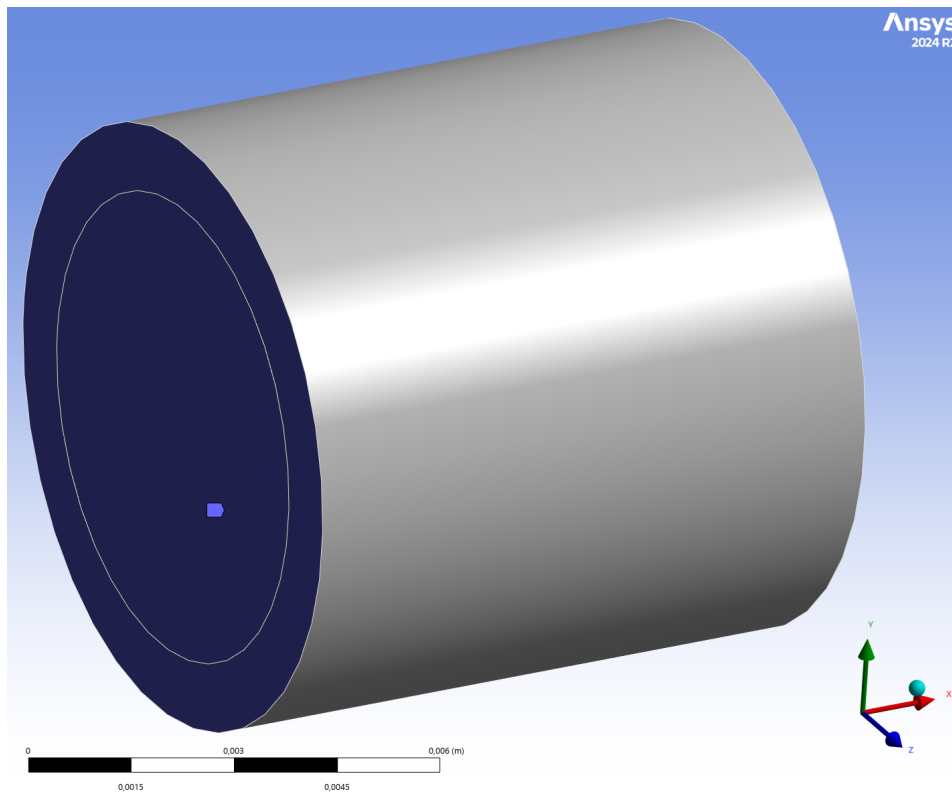
| Property              | Value                  |
|-----------------------|------------------------|
| Density $\rho$        | 7850 kg/m <sup>3</sup> |
| Young's modulus $E$   | 200 GPa                |
| Poisson's ratio $\nu$ | 0.30                   |
| Bulk modulus $K$      | 166.67 GPa             |
| Shear modulus $G$     | 76.92 GPa              |

#### 5.3.1.1. Boundary conditions and loading

The RVE was fully clamped at one end by constraining all degrees of freedom. At the opposite end, a prescribed transverse displacement of

$$\delta = 0.1 \text{ mm}$$

was applied in the positive  $y$ -direction. This loading configuration induces a nominal shear deformation of the short cylindrical beam.



**Figure 5.15:** RVE clamped end

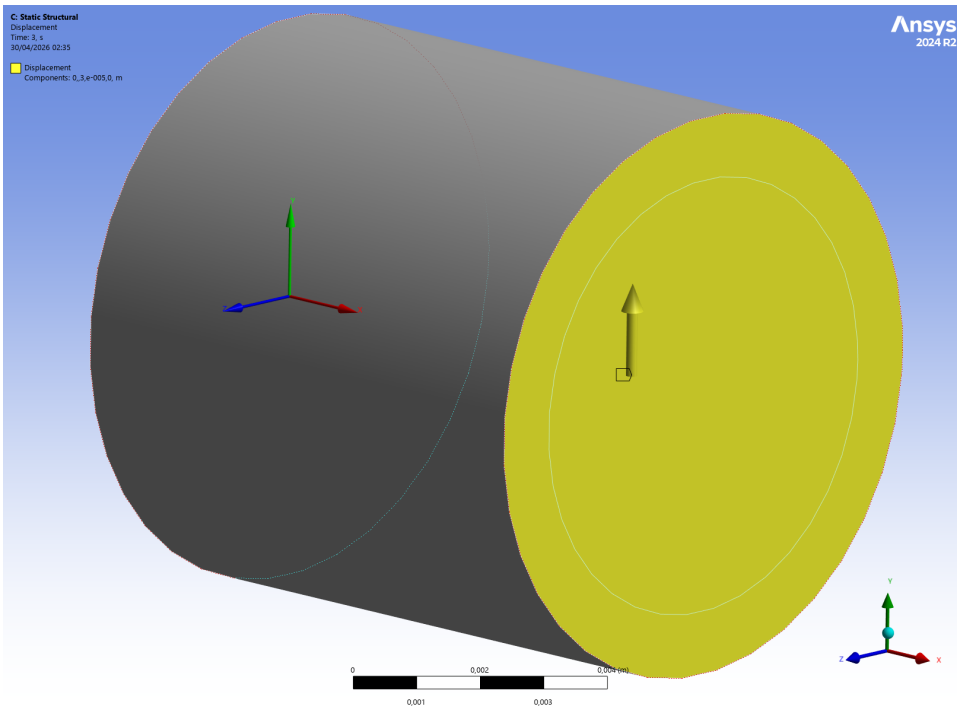


Figure 5.16: RVE displaced end

5.3.1.2. Strain field characteristics and Saint-Venant considerations

The resulting shear strain field exhibited pronounced extrema near the loaded and clamped ends, as labelled in Figure 5.17. These localised effects arise from the application of displacement boundary conditions and do not represent the uniform deformation state of interest. Consistent with Saint-Venant’s principle, strain evaluation was therefore performed at the mid-length cross-section of the RVE, sufficiently far from both boundaries, as illustrated in Figure 5.18.

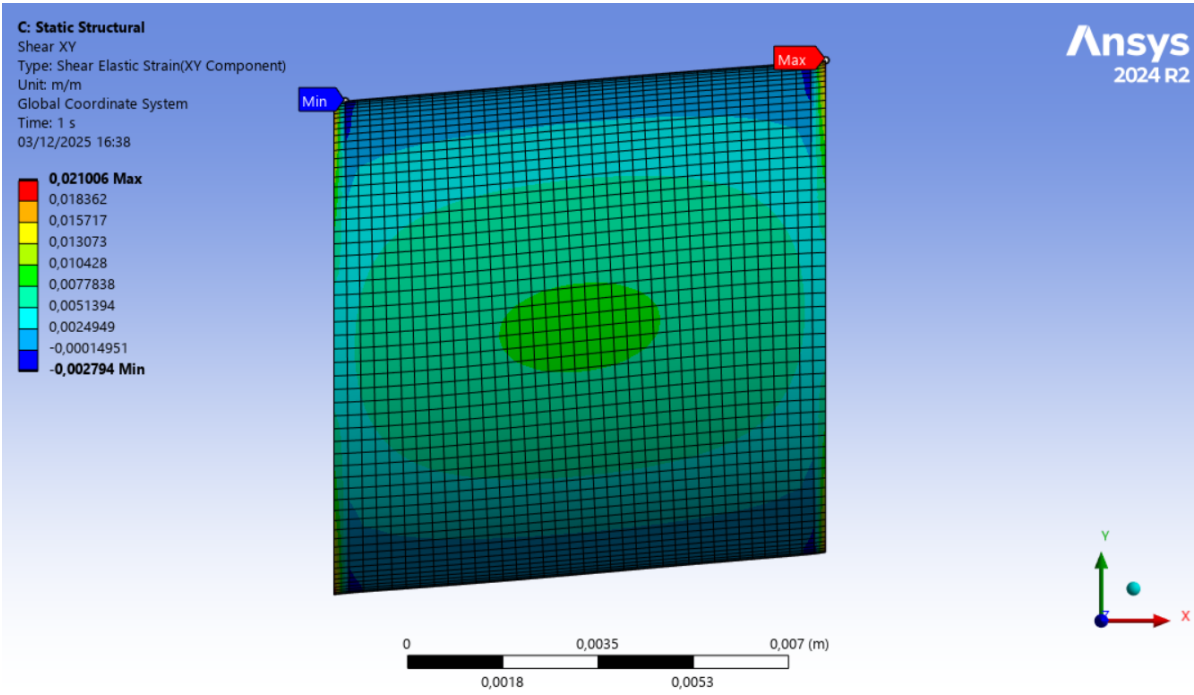


Figure 5.17: Strain field of deformed RVE

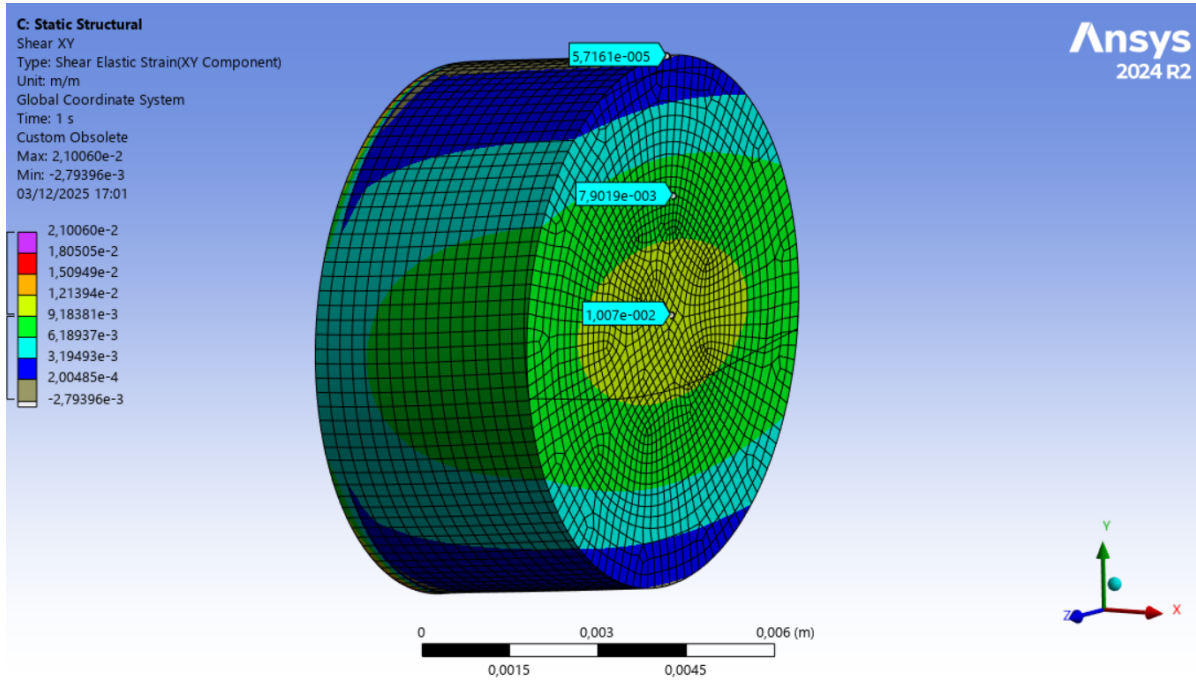


Figure 5.18: Strain field at RVE midplane

At the midplane, the shear strain distribution varied from near-zero values at the top and bottom edge of the cross-section to maximum values at its center. This spatial variation is consistent with classical beam theory, where the actual shear stress distribution is non-uniform and typically exhibits parabolic behaviour [43]. As Timoshenko beam theory assumes an uniform shear strain distribution over the cross-section, there should be at least two locations at which strain predictions under this assumption match with the parabolic prediction. This is schematically shown in Figure 5.19. The strain prediction by the FEA that should approximately match the constant Timoshenko strain was determined to be  $7902 \mu\text{strain}$ , as shown labeled in Figure 5.18.

#### 5.3.1.3. Comparison with Timoshenko beam theory

To relate the FEA strain field to a global stiffness measure, Timoshenko beam theory was adopted. In this framework, an average shear strain is defined assuming a constant shear distribution across the cross-section, such that

$$\gamma = \frac{Q}{(nGA)_{RVE}}, \quad (5.17)$$

where  $Q$  is the transverse shear force,  $n$  is the shear correction factor,  $G$  the material shear modulus, and  $A$  the cross-sectional area.

From the steel verification model, the reaction force extracted at the clamped end was

$$Q = 3.45 \times 10^4 \text{ N},$$

with a Cowper shear correction factor of [49]

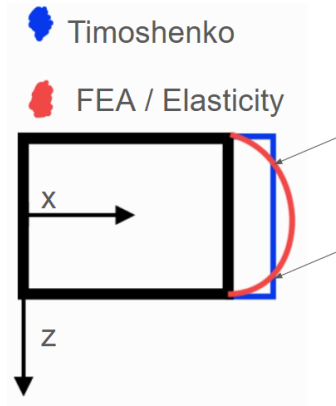
$$n_{RVE} = 0.886.$$

Using the material shear modulus of structural steel (Table 5.2) and the RVE cross-sectional area

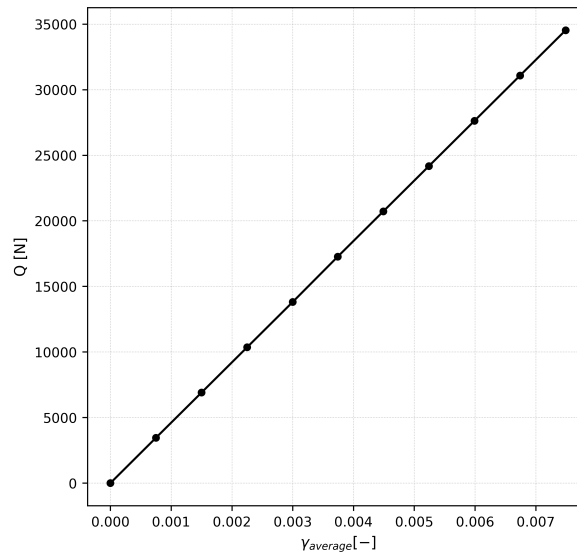
$$A_{RVE} = 64.14 \text{ mm}^2,$$

the corresponding average shear strain predicted by Timoshenko theory is

$$\gamma \approx 7.894 \times 10^{-3},$$



**Figure 5.19:** Constant shear strain by Timoshenko theory vs parabolic shear strain elasticity



**Figure 5.20:** Reaction force–average shear strain relation for the steel cylindrical RVE

which agrees excellently with the mid-section strain obtained from the finite- element simulation ( $-0.1\%$  rel. diff.).

This agreement confirms that the average-strain assumption of Timoshenko beam theory provides an adequate representation of the global shear response of the FEA model.

#### 5.3.1.4. Extraction of shear stiffness and modulus

To further verify the numerical stiffness extraction methodology, the loading of  $\delta = 0.1\text{mm}$  was applied in multiple steps and the corresponding reaction forces and mid-section average shear strains were recorded. The shear stiffness  $K$  was obtained from the slope of the resulting  $Q$ – $\gamma$  relation shown in Figure 5.20, and the effective shear modulus was subsequently computed as

$$G = \frac{K}{nA}. \quad (5.18)$$

The shear modulus retrieved from this back-calculation equaled  $G = 81.1\text{ GPa}$ , which deviates by  $5.4\%$  from the reference material value in Table 5.2. Despite the absence of a mesh convergence study, this level of agreement was deemed sufficient to validate the numerical methodology. The FEA model is therefore considered capable of extracting shear and effective shear modulus with acceptable accuracy and the same methodology is subsequently applied to multi-material fiber–matrix RVEs.

### 5.3.2. Effective stiffness of concentric cylindrical fiber-matrix RVEs

Following verification of the method to extract effective shear stiffness and modulus, the concentric cylinder RVE was re-analysed as a two-phase fiber–matrix system. The simulations aim to evaluate the predicted shear stiffness over a range of fiber volume fractions and to assess the physical admissibility of the resulting homogenised shear modulus.

The matrix was modelled as isotropic, with Young’s modulus and Poisson’s ratio taken from manufacturer data, as reported in chapter 3.

The fiber was assumed to exhibit isotropic elastic behaviour, which by definition has two independent elastic constants [3], such that

$$G = \frac{E}{2(1 + \nu)}, \quad (5.19)$$

However, while the fiber is assumed to be isotropic in the present analytical work, in reality the Young’s modulus, shear modulus, and Poisson’s ratio are all independent and known from material data. Since selecting either two elastic constants and calculate the other, differs from data, the material was instead implemented as orthotropic with identical properties in all principal directions:

$$E_x = E_y = E_z, \quad \nu_{xy} = \nu_{yz} = \nu_{xz}, \quad G_{xy} = G_{yz} = G_{xz}. \quad (5.20)$$

This modelling assumption is noted as requiring further evaluation. In analogy with the analytical model, the shear correction factor was assumed to equal unity,

$$n_{RVE} = 1, \quad (5.21)$$

such that the extracted effective shear modulus follows from the relation

$$G_{RVE} = \frac{K_{RVE}}{A_{RVE}} \quad (5.22)$$

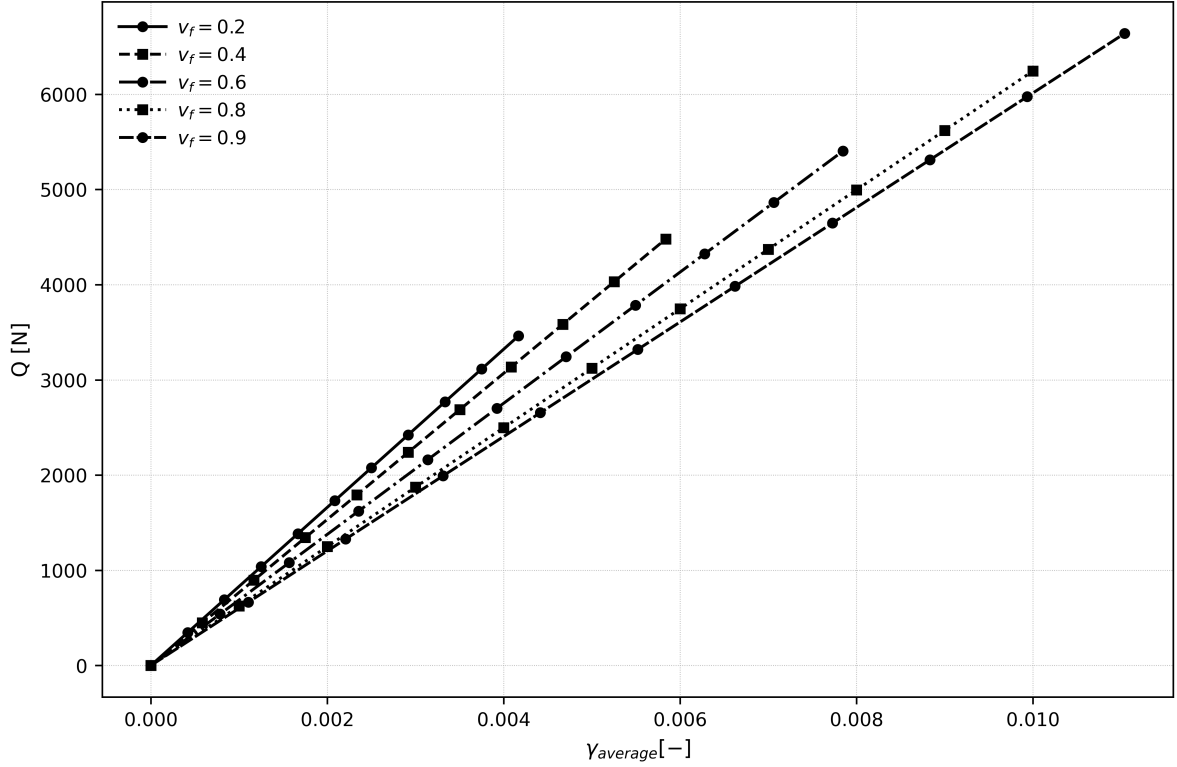
As with the analytical model, the fiber geometry is kept constant, so that the matrix annulus is increased in size to obtain the required fiber volume fraction  $v_f$ . As  $A_m$  becomes unbounded for  $v_f \rightarrow 0$ , as pointed out in section 5.1, a lower limit of  $v_f = 0.2$  is adopted to keep the simulation at a reasonable size. To further limit computational time, the analyses were restricted to,

$$v_f = [0.2, 0.4, 0.6, 0.8, 0.9].$$

The shear stiffness of each cylindrical RVE was obtained from the slope of the reaction force–average shear strain relation, shown in Figure 5.21, using the methodology established in subsection 5.3.1 with meshing following the mesh convergence study documented in Appendix B. The following observations can be made: First, an increase in shear stiffness is observed with decreasing fiber volume fraction. Although this appears counterintuitive, it follows from geometric scaling. Since

$$K_{RVE} \propto A_{RVE}, \quad A_{RVE} \propto \frac{1}{v_f},$$

a low-fiber-volume RVE possesses a larger cross-sectional area and therefore exhibits a higher global shear stiffness despite having a lower effective shear modulus  $G_{RVE}$ . This trend was also observed in section 5.1, with the analytical  $K_{RVE}$  becoming unbounded for  $v_f \rightarrow 0$ .



**Figure 5.21:**  $Q - \gamma$  for various  $v_f$  for cylindrical fiber-matrix RVE.

Furthermore, low-fiber-volume RVEs experience a lower maximum average shear strain for an equal imposed end displacement. This behaviour can be interpreted from an energy perspective. For a displacement-controlled cantilever beam, the FEA solution minimizes the total strain energy

$$U = \frac{1}{2} Q \delta, \quad (5.23)$$

by adjusting the reaction force  $Q$ . Since [43]

$$\delta = Q \left( \frac{L^3}{3EI} + \frac{L}{K} \right)_{RVE}, \quad (5.24)$$

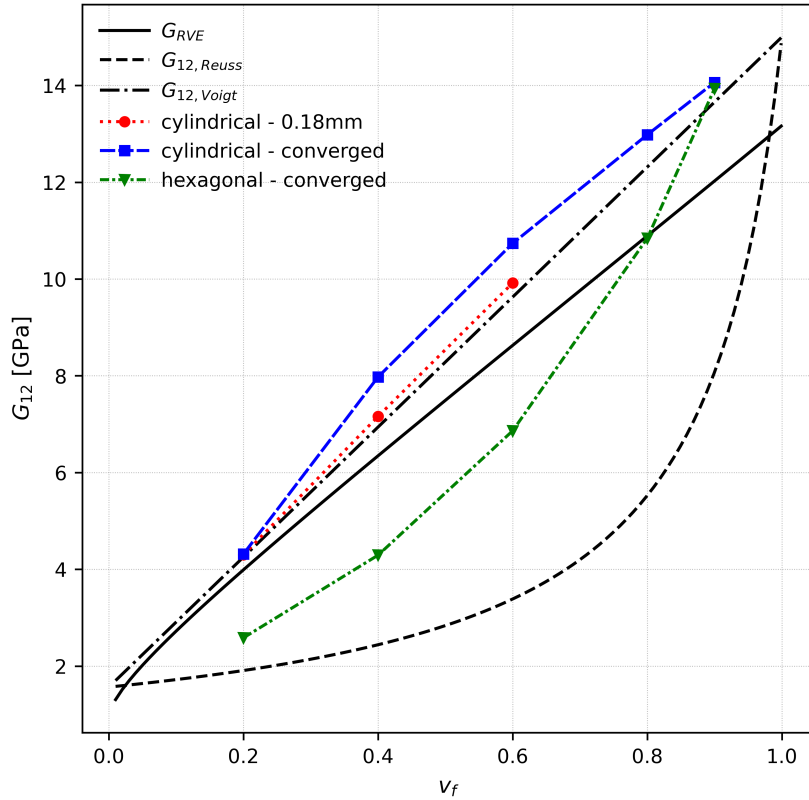
the total energy may be written as

$$U = \frac{1}{2} Q \left( \frac{L^3}{3EI} + \frac{L}{K} \right)_{RVE}. \quad (5.25)$$

Thus, although the bending contribution remains small for short beam lengths, an increase in shear stiffness shifts a fraction of the deformation energy into bending. Consequently, a smaller portion of the total deformation is accommodated through shear, resulting in reduced average shear strains for the same imposed displacement.

The effective shear modulus extracted from the cylindrical RVE is now compared with the classical Voigt-Reuss bounds and the analytical solution as was done in subsection 5.2.1.

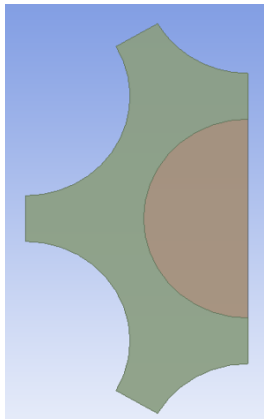
While an initial coarse mesh (0.18mm, based on rough estimates) produces a Voigt-like response, the converged solutions, for which mesh convergency studies are included in Appendix B, are exceeding the upper Voigt bound. Following Hill's reasoning [1], [36], the cylindrical RVE, under the current modelling assumptions, does not produce a physically admissible model for extraction of an effective shear modulus  $G_{RVE}$ .



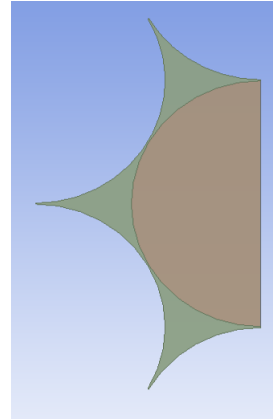
**Figure 5.22:** Comparison of FEA cylindrical RVE, FEA hexagonal RVE, analytical RVE and classical bounds.

### 5.3.3. Effective stiffness of hexagonal fiber-matrix RVEs

To investigate the effect of assumed microstructural geometry, the FEA procedure for the cylindrical RVE was repeated with an RVE representing a hexagonal fiber packing. This provides a more realistic approximation of the local fiber arrangement and allows for comparing the effect of the geometry for a given fiber volume fraction. Example geometries are shown in Figure 5.23-5.24.



**Figure 5.23:** Half hexagonal model  $v_f = 0.6$



**Figure 5.24:** Half hexagonal model  $v_f = 0.9$

To reduce the element count, only half of the RVE is modelled. Since this yields half the reaction force compared to the full configuration, the shear response of the complete hexagonal model is obtained by appropriate scaling. Verification of this approach, performed on the cylindrical model, is included in Appendix B.

Compared to the cylindrical model, the hexagonal model shows a clearly different trend in the extracted effective shear modulus  $G_{RVE}$ , as illustrated in Figure 5.22. Compared to the full cylindrical model, the hexagonal model remains largely in between the Voigt and Reuss bounds, with the exception of  $v_f = 0.9$ , where it approaches the cylindrical result, due to the diminishing contribution of the matrix.

#### 5.3.4. Influence of modelling assumptions

Two modelling assumptions used in the FEA analyses require further consideration: the representation of the fiber material as orthotropic with identical principal properties, and the assumption of a unit shear correction factor for the RVE.

For an isotropic elastic solid, only two independent elastic constants exist, typically taken as Young's modulus  $E$  and Poisson's ratio  $\nu$ , with the shear modulus following from

$$G = \frac{E}{2(1 + \nu)}. \quad (5.26)$$

However, for high-modulus fibers, material data are often reported in terms of  $E$ ,  $\nu$ , and  $G$ , which do not exactly satisfy the isotropic constraint. In the present FEA analyses, these three constants are therefore retained by adopting an orthotropic material definition with identical properties in each principal direction. This preserves the measured values while relaxing the strict isotropy constraint.

As a consequence, the local stress–strain response may differ slightly from that of an ideal isotropic material, which can influence the predicted load transfer behaviour within the fiber–matrix system, particularly in shear-dominated deformation states where the relative magnitudes of  $E$  and  $G$  govern the partitioning of bending and shear energy. The resulting effective stiffness should therefore be interpreted as conditional on this constitutive representation rather than as a uniquely defined material property.

Similarly, the assumption  $n_{RVE} = 1$  implies that the average shear strain extracted from the FEA model corresponds directly to the effective shear modulus definition adopted in the analytical formulation. In reality, the shear correction factor depends on the cross-sectional stress distribution and therefore on both geometry and material heterogeneity [51], [90]. Neglecting this effect introduces an additional source of modelling uncertainty. A further remark on the shear correction factor relates to the difference between the cylindrical and hexagonal model. For a fixed fiber volume fraction and fixed fiber geometry, both RVEs correspond to the same homogenised cross-sectional area. Differences in the extracted global shear stiffness may therefore be interpreted as reflecting differences in the effective distribution of shear deformation associated with each geometric idealisation.

If the true effective shear modulus is also assumed to depend only on fiber volume fraction and not on the specific RVE geometry, as is commonly the case for RoM, then the observed difference in stiffness may be interpreted through the corresponding shear correction factors. Under this assumption, the ratio of the shear correction factors equals that of the ratio of extracted stiffnesses,

$$\frac{n_{\text{hex}}}{n_{\text{cyl}}} = \frac{K_{\text{hex}}}{K_{\text{cyl}}}. \quad (5.27)$$

This expression is not used as a correction to either predictions, but serves as a possible interpretation for the differences between the cylindrical and hexagonal model which is likely due to geometry-induced differences in the non-uniform shear stress field and the associated shear correction needed. Recalling the intermediate re-definition of the shear correction factor from subsection 5.2.1,

$$n_{RVE} = \frac{(G_{RVE})_{n_{RVE}=1}}{G_{12,Reuss}},$$

this shear correction factor interpreted as the correction from  $G_{RVE}$  towards  $G_{12,Reuss}$  can also be computed for the FEA models, shown in Figure 5.25. The analytical hypothetical  $n_{RVE}$  for AS4/3501-6 from subsection 5.2.1 is included. Again, all hypothetical  $n_{RVE}$  greatly exceed unity and are therefore rejected on the same grounds as listed in subsection 5.2.1. The FEA results being nested around the analytical result for AS4/3501-6, provides confidence that FEA results for the other material results would overestimate the hypothetical  $n_{RVE}$  to the same extent as the analytical results of Figure 5.8.



Again, the existence of a consistent  $n_{RVE}$  is not denied, but it is not expected to correct the FEA  $G_{RVE}$  adequately to Reuss-like behaviour.

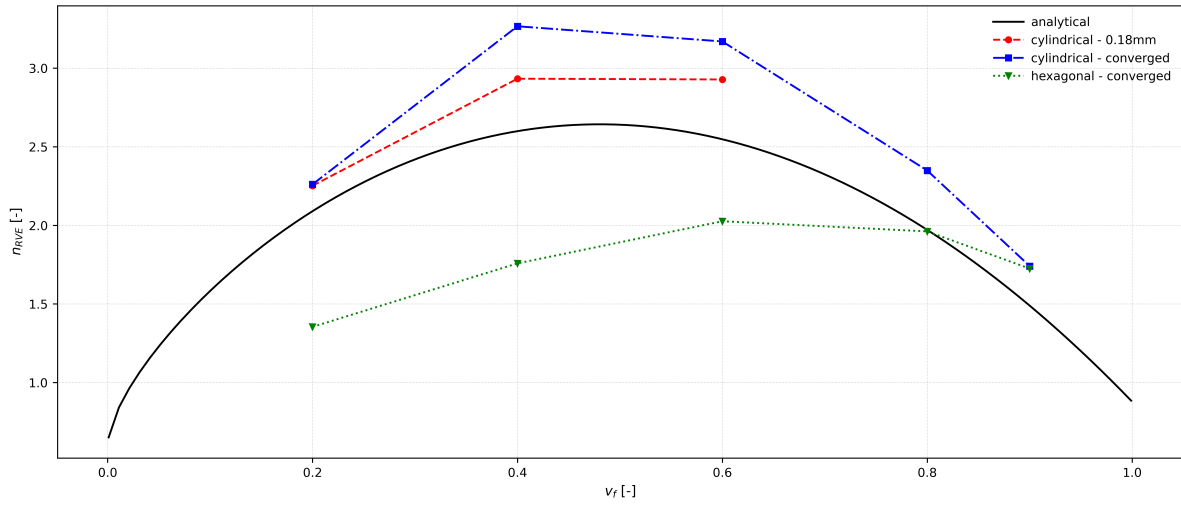


Figure 5.25: Hypothetical  $n_{RVE}$  to correct the FEA  $G_{RVE}$

Taken together, these assumptions indicate that the FEA estimates should not be interpreted as quantitatively exact predictions of the homogenised shear modulus. Instead, their value is more in revealing trends and sensitivities associated with geometric idealisations.

## 5.4. Summary and outlook

The main findings of this chapter can be summarised as follows:

- The deterministic analytical model exhibits a Voigt-like stiffness behaviour across the fiber volume fraction range, while a Reuss-like stiffness behaviour is sought for;
- FEA analyses show similar qualitatively trends and demonstrate that the extraction of effective shear stiffness does not only depend on the constituent properties and RVE area but also on the local geometry;
- In particular, the FEA of the concentric cylinder showed similar behaviour as the analytical model but does exceed classical bounds under the present modelling assumptions, while the hexagonal model does not;
- The majority of the FEA results are to be used qualitatively rather than quantitatively, due to constraint of not knowing the exact shear correction factor for heterogeneous beams.

Because the deterministic formulation fails to reproduce the experimentally determined initial shear stiffness and does not behave like common RoMs, a modification of the governing constituent moduli is required. Rather than introducing empirical adjustments to correct predictions to observed ply behaviour, the following chapter develops a statistically informed homogenisation framework to derive effective modulus correction factors inspired by the observed effects of micromechanical geometry of the RVEs.

# Statistical Homogenisation of Initial Shear Stiffness

This chapter develops a statistical informed homogenisation framework to reconcile the deterministic micromechanical stiffness prediction with experimentally observed ply behaviour. Variability in local fiber volume fraction and RVE geometry is represented through a probabilistic description, while local shear stiffness is evaluated using concentric-cylinder-type relations. The resulting stochastic homogenisation is employed to identify effective correction factors for the constituent moduli governing the deterministic formulation, in analogy with established Tsai-Hahn modulus corrections [17]. These corrected moduli provide a consistent description of the initial ply shear stiffness and establish the basis for subsequent modelling of stiffness degradation.

## 6.1. Background and motivation

The deterministic micromechanical formulation developed in chapter 4 provides an analytical estimate of the RVE shear stiffness and modulus. However, comparison with experimental ply-level shear data indicates that this formulation systematically overestimates the effective stiffness. In particular, the contribution of the fiber shear modulus is found to be excessively dominant when compared to classical mixture relations.

A straightforward remedy would be to introduce correction factors in  $K_{RVE}$  that transform  $G_{RVE}$  into the Reuss RoM. However, such an approach would undermine the work that went into expressing the fiber-matrix RVE as a Timoshenko beam on elastic foundation. Moreover, even though the Reuss RoM is a common first estimate for  $G_{12}$  [16], it is observed to underpredict experimentally measured  $G_{12}$  in many material systems as shown in Table 6.1. This underestimation of matrix-dominated ply moduli, such as  $G_{12}$  and  $E_2$ , has been reported by others [1], [16], [31].

**Table 6.1:** Comparison of  $G_{12}$  data vs  $G_{12}$  estimated by Voigt RoM

| Material     | $G_{12}^{data}$ [GPa] | $G_{12}^{Reuss}$ [GPa] | rel. diff. [%] |
|--------------|-----------------------|------------------------|----------------|
| AS4/3501-6   | 6.6                   | 3.93                   | -40.4          |
| IM7/8552     | 5.6                   | 3.41                   | -39.1          |
| G40-800/5260 | 6.94                  | 2.99                   | -57.0          |
| Glass/LY556  | 5.83                  | 2.92                   | -49.9          |

This indicates that the discrepancy cannot be resolved by simply selecting between classical RoMs. Instead, a physically motivated homogenisation framework is required that captures the underlying micromechanical mechanisms governing shear stiffness.

In this context, the Tsai-Hahn RoM offers an improved deterministic description by incorporating geometric features of the local fiber arrangement. Nevertheless, such models, including the current  $G_{RVE}$  model so far (see assumptions chapter 4), still assume a uniform microstructure with fiber being arranged in either square or hexagonal grids and therefore neglect variability in local fiber volume fraction and RVE geometry, as introduced in chapter 2. Since real composite plies exhibit spatial fluctuations in fiber packing, these effects must be accounted for to obtain consistent description of initial shear stiffness.

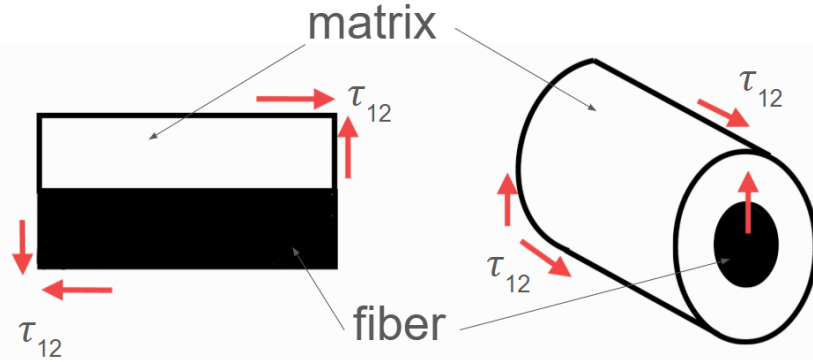
The objective of this chapter is therefore to extend the deterministic micromechanical formulation into statistically informed homogenisation framework capable of reproducing experimentally observed stiffnesses more closely.

## 6.2. Deterministic geometric corrections

The discrepancies identified in section 6.1 indicate that classical mixture bounds are insufficient to describe the effective in-plane shear stiffness of unidirectional plies. While such bounds provide useful limiting estimates, they do not account for the geometric constraints imposed by the local fiber–matrix arrangement. Deterministic micromechanical models that incorporate idealised representations of this geometry therefore offer a more physically informed description of stress transfer mechanisms. Among these, the formulation of Tsai and Hahn introduces in-situ corrections to constituent moduli based on simplified microstructural considerations, providing an intermediate level of modelling between classical mixture rules and fully resolved micromechanical analyses.

### 6.2.1. Tsai-Hahn in-situ corrected moduli

Improved deterministic estimates can be obtained by considering the local fiber–matrix geometry. While the Reuss model assumes a plate-like arrangement of the constituent phases, Tsai and Hahn [17] adopted a concentric-cylinder representation, in which each fiber is fully surrounded by the matrix. A schematic comparison of these idealisations is shown in Figure 6.1.



**Figure 6.1:** Plate arrangement vs concentric cylinder arrangement of fiber and matrix

On this basis, Tsai and Hahn introduced *in-situ* corrected moduli for the fiber and matrix phases through a semi-empirical formulation. These corrections account for the constraint imposed by the matrix on each individual fiber, in contrast to Reuss-type RVE, which idealises the composite as two plates of equal thickness representing the fiber and matrix phases. The resulting corrected shear moduli are expressed as

$$G_f^* = (v_f + \eta_s v_m) G_f, \quad (6.1)$$

$$G_m^* = \left( \frac{v_f}{\eta_s} + v_m \right) G_m. \quad (6.2)$$

with the stress partitioning parameter  $\eta_s$  defined as

$$\eta_s = \frac{\bar{\tau}_{matrix}}{\bar{\tau}_{fiber}} \quad 0 < \eta_s \leq 1. \quad (6.3)$$

A semi-empirical approach showed  $\eta_s$  to be nearly independent of  $v_f$  and closely approximated by [17]

$$\eta_s = \frac{1}{2} \left( 1 + \frac{G_m}{G_f} \right). \quad (6.4)$$

The corrected moduli  $G_f^*$  and  $G_m^*$  are then to be used in the Reuss RoM, Equation 5.14, such that:

$$G_{12}^{Tsai-Hahn} = \left( \frac{v_f}{G_f^*} + \frac{v_m}{G_m^*} \right)^{-1} \quad (6.5)$$

Table 6.2 shows that both Reuss and Tsai-Hahn mixture relations systematically underestimate the in-plane shear modulus, while the Tsai formulation consistently reduces the modelling error relative to the Reuss rule of mixtures, yielding improvements in predictive accuracy ranging from approximately 23% to 69% depending on the material system.

**Table 6.2:** Comparison of experimental  $G_{12}$  with Voigt and Tsai-Hahn estimates

| Material     | $G_{12}^{\text{data}}$ [GPa] | $G_{12}^{\text{Reuss}}$ [GPa] | rel. diff. [%] | $G_{12}^{\text{T-H}}$ [GPa] | rel. diff. [%] | improvement [%] |
|--------------|------------------------------|-------------------------------|----------------|-----------------------------|----------------|-----------------|
| AS4/3501-6   | 6.60                         | 3.93                          | -40.4          | 4.54                        | -31.2          | 22.8            |
| IM7/8552     | 5.60                         | 3.41                          | -39.1          | 4.92                        | -12.1          | 69.1            |
| G40-800/5260 | 6.94                         | 2.99                          | -57.0          | 4.35                        | -37.3          | 34.6            |
| Glass/LY556  | 5.83                         | 2.92                          | -49.9          | 4.32                        | -26.0          | 47.9            |

## 6.3. Microstructural variability in unidirectional plies

As introduced in chapter 2, real composite microstructures exhibit spatial variability in fiber arrangement which influence local stiffnesses. The following section introduces the origins of such variability and methods to describe it.

### 6.3.1. Origin of variability in fiber volume fraction

In practical composite manufacturing processes, fiber packing is inherently non-uniform. Local variations in fiber volume fraction arise from several sources, including:

- resin flow [23], [68], [69];
- fiber bundle geometry and clustering [22], [91];
- fiber waviness [1], [24];
- manufacturing defects such as voids and resin-rich zones [67].

These variations manifest visually as deviations from the idealised hexagonal fiber packing, as illustrated in Figure 6.2. Consequently, spatial fluctuations occur in the local fiber volume fraction, which may lead to local over- or underestimation of the nominal ply volume fraction typically reported by material suppliers. Such local variations in fiber content directly translate into corresponding local variations in stiffness.

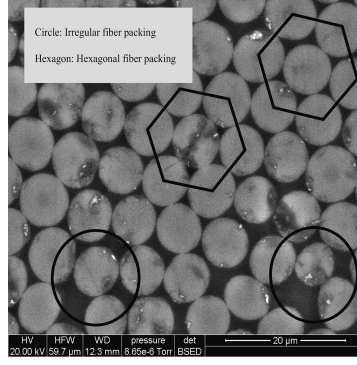


Figure 6.2: SEM image of a CFRP [21]

### 6.3.2. Probabilistic descriptions of local fiber packing

Analytical homogenisation models typically assume uniform microstructure and therefore neglect statistical variability. Several studies have proposed probabilistic approaches to represent local fiber volume fraction distributions [4], [19], [26], [29]

Sanei and Fertig [27] evaluated several probability distributions against experimentally measured data on fiber volume fraction. The four best-performing distributions, in decreasing order of goodness of fit, were the Weibull, Generalized Extreme Value, Extreme Value, and Beta distributions. However, with the exception of the Beta distribution, these distributions possess unbounded support extending to infinity, which is not physically consistent with the finite packing limits of fiber-reinforced composites.

For this reason, the Beta distribution is adopted in the present work. It provides a bounded representation of local fiber–volume–fraction variability that remains consistent with the geometric constraints imposed by fiber packing.

To model the variability in fiber volume fraction, a scaled Beta distribution is adopted over the physically admissible interval

$$0 \leq v_f \leq v_{f,\max}, \quad v_{f,\max} = \frac{\pi}{2\sqrt{3}}. \quad (6.6)$$

where  $v_{f,\max}$  corresponds to the maximum packing density of circles in a hexagonal arrangement. Let the random variable  $V_f$  denote the fiber volume fraction. It is assumed that

$$V_f \sim \text{Beta}(\alpha_\beta, \beta_\beta; 0, v_{f,\max}), \quad (6.7)$$

that is, a Beta distribution scaled from the unit interval to  $[0, v_{f,\max}]$  controlled by shape parameters  $\alpha_\beta$  and  $\beta_\beta$ . Its mean is prescribed as

$$\mu = \mathbb{E}[V_f] = v_{f,ply}. \quad (6.8)$$

To estimate the standard deviation of this scaled beta distribution on  $[0, v_{f,\max}]$ , it is postulated that there exists an auxiliary normal distribution with unknown mean  $\mu_N$  but with an equal standard deviation  $\sigma$ , as shown in Figure 6.3. This normal distribution is characterised by the following percentiles:

$$X_1 = \mu_N - k_{99}\sigma, \quad (6.9)$$

$$X_{99} = \mu_N + k_{99}\sigma. \quad (6.10)$$

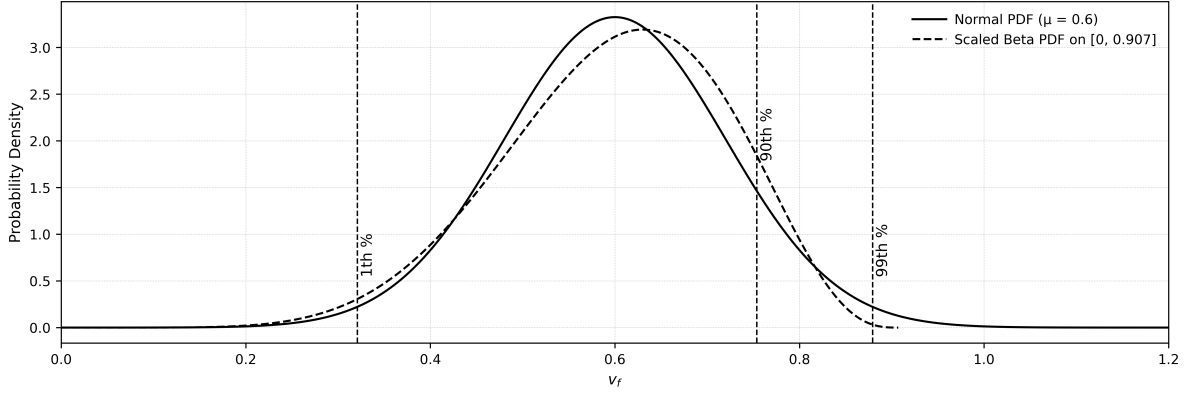
where  $k_{99}$  is the one-sided tolerance limit factor for normal distributions [92]. Eliminating  $\mu_N$  by subtracting Equation 6.9 from Equation 6.10 yields an estimate for the standard deviation:

$$\sigma = \frac{X_{99} - X_1}{2k_{99}}, \quad (6.11)$$

To close the problem, it is assumed that the minimum and maximum of the scaled beta distribution can be identified with  $X_1$  and  $X_{99}$ , respectively. Under this assumption, the selected percentile pair uniquely determines  $\sigma$  with:

$$\sigma = \frac{v_{f,\max}}{2k_{99}}, \quad (6.12)$$

with  $k_{99} = 2.326$  [92].



**Figure 6.3:** Example of a normal distribution postulated to have the same standard deviation as the required beta distribution.

Using the mean and variance relations of the scaled Beta distribution, the corresponding shape parameters are written as

$$\alpha_\beta = \frac{v_{f,\text{ply}}}{v_{f,\text{max}}} \left[ \frac{2k_{99} v_{f,\text{ply}}}{v_{f,\text{max}}^2} \left( 1 - \frac{v_{f,\text{ply}}}{v_{f,\text{max}}} \right) - 1 \right], \quad (6.13)$$

$$\beta_\beta = \alpha_\beta \left( \frac{v_{f,\text{max}}}{v_{f,\text{ply}}} - 1 \right). \quad (6.14)$$

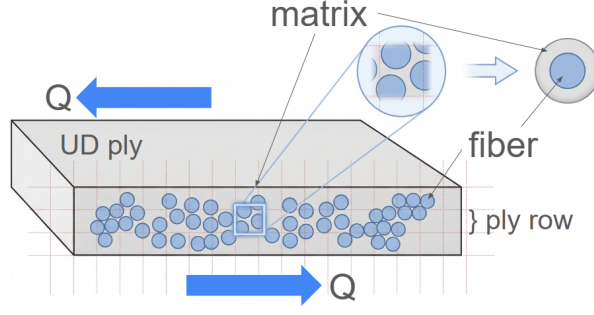
The shape of the Beta distribution is thus governed by Equation 6.13 and Equation 6.14, which control both the skewness and concentration of probability mass within the bounded interval. When  $\alpha_\beta = \beta_\beta > 1$ , the distribution is symmetric and unimodal, with increasing values leading to stronger concentration around the mean. If  $\alpha_\beta > \beta_\beta$ , the distribution is left-skewed, indicating a higher probability of larger fiber volume fractions, whereas  $\alpha_\beta < \beta_\beta$  produces right-skewness and emphasises lower values. For  $\alpha_\beta, \beta_\beta < 1$ , the distribution becomes U-shaped, concentrating probability near the bounds, while values close to unity yield an approximately uniform distribution [93]. Accordingly, the choice of  $\alpha_\beta$  and  $\beta_\beta$  enables flexible representation of physically plausible variability patterns in local fiber packing.

## 6.4. Statistical homogenisation of ply shear modulus

Including variability in the fiber volume fraction as per section 6.3 requires a local evaluation of stiffness. The following sections describe the means to do so by discretizing the ply and evaluating the stiffness, taking into account local geometry, and then obtain a ply shear modulus through strain compatibility.

### 6.4.1. Element-based ply representation

The ply is idealised as an assembly of small elements, each characterised by a local fiber volume fraction. Within each element, the microstructure is represented by an equivalent concentric-cylinder geometry, as shown in Figure 6.4, preserving its fiber volume fraction. The local shear modulus of each element is then evaluated by using the Tsai-Hahn RoM (Equation 6.24).



**Figure 6.4:** Subdivision of the ply cross-section in elements. Each element is then matched to a concentric-cylinder model with equal fiber volume fraction.

#### 6.4.2. Homogenisation via strain compatibility

Selecting an arbitrary row randomly positioned in the ply thickness as shown in Figure 6.4, the shear modulus  $G_i$  of each row element  $i$  is defined by Equation 6.24. All elements are assumed to experience the same shear strain:

$$\gamma = \frac{\tau_i}{G_i} = \frac{Q_i}{A_i G_i}, \quad (6.15)$$

where  $Q_i$  is the force carried by the  $i$ th cell and  $A_i$  is the corresponding element area, assumed to be identical for all cells.

From shear-strain compatibility, the force in each element is therefore

$$Q_i = \frac{A_i G_i}{\sum_j A_j G_j} Q, \quad (6.16)$$

where  $Q$  denotes the total force applied to the row of  $N$  elements. Substituting this expression into the definition of  $\gamma$  yields

$$\gamma = \frac{Q}{\sum_j A_j G_j}. \quad (6.17)$$

This strain must also equal the strain of the entire row, i.e.

$$\gamma = \frac{Q}{G_{\text{row}} \sum_j A_j}. \quad (6.18)$$

Equating both expressions gives

$$\frac{Q}{\sum_j A_j G_j} = \frac{Q}{G_{\text{row}} \sum_j A_j}. \quad (6.19)$$

Cancelling  $Q$  and using the assumption  $A_i = A = \text{const.}$ , so that  $\sum_j A_j = NA$ , leads to

$$G_{\text{row}} = \frac{A \sum_j G_j}{\sum_j A_j} = \frac{A \sum_j G_j}{NA} = \frac{1}{N} \sum_j G_j. \quad (6.20)$$

Hence, the shear modulus of a row is obtained as the arithmetic mean of the shear moduli of its constituent elements. If the row characterised by Equation 6.20 is assumed to be statistically independent of its position through the ply thickness, the ply may be regarded as a stacking of rows with identical stiffness. Consequently,

$$G_{12,\text{ply}} = G_{\text{row}} \quad (6.21)$$

### 6.5. Generalised mixture laws

Classical mixture relations such as the Voigt and Reuss models may be interpreted as limiting cases of a generalised power-mean formulation. This viewpoint enables interpolation between different mixture behaviours through a continuous parameter that governs the relative dominance of the constituent moduli. This approach has only been reported in a limited number of studies [94], [95].

### 6.5.1. Power-mean formulation

Considering Equation 5.14 and Equation 5.13, the in-plane shear modulus  $G_{12}$  may be expressed as a weighted power mean of the constituent shear moduli,

$$G_{12} = M_p(G_f, G_m; v_f) = \begin{cases} G_f^{v_f} G_m^{v_m}, & p = 0, \\ (v_f G_f^p + v_m G_m^p)^{1/p}, & p \neq 0, |p| < 1. \end{cases} \quad (6.22)$$

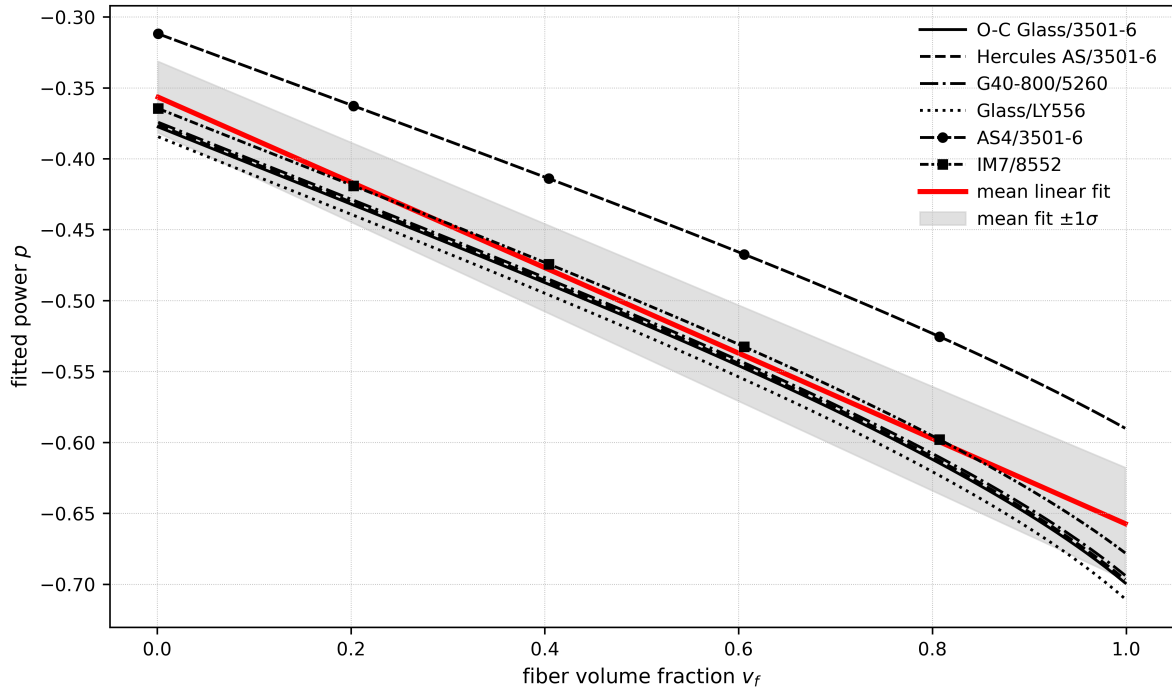
where  $v_m = 1 - v_f$  and  $p$  is the continuous parameter that allows for interpolation between the classical bounds, as this formulation recovers the classical bounds as special cases,

$$G_{12} = \begin{cases} G_{12, \text{Voigt}}, & p = 1, \\ G_{12, \text{Reuss}}, & p = -1. \end{cases} \quad (6.23)$$

Accordingly, the Voigt and Reuss bounds correspond to the weighted harmonic and weighted arithmetic means of the constituent moduli, respectively.

### 6.5.2. Tsai-Hahn as power-mean

It can be shown numerically that other RoM, such as the one proposed by Tsai and Hahn with corrected constituent moduli as explained in subsection 6.2.1, can be expressed in terms of an effective power-mean representation with uncorrected moduli and  $p$  as a function of fiber volume fraction  $v_f$ . In case of the Tsai-Hahn RoM,  $p(v_f)$  was found to be a near linear function, as shown in Figure 6.5. In fact, with the exception of AS4/3501-6, all the other materials tend to collapse to a single  $p(v_f)$ .



**Figure 6.5:** The parameter  $p$  as a function of  $v_f$  to represent the Tsai-Hahn RoM as a power-mean

Linear regression results presented in Table 6.3 indicate that the power-mean exponent  $p$  required to reproduce Tsai's shear modulus relation exhibits a remarkably weak dependence on the specific material system considered. Across the six investigated fiber-matrix combinations, the fitted slopes and intercepts cluster tightly around the mean values

$$p(v_f) \approx \bar{m} v_f + \bar{b} = (-0.303 \pm 0.014) v_f - (0.356 \pm 0.023),$$



where the quoted uncertainties represent the standard deviation across materials. This limited scatter suggests that, to first order, the variation of the effective in-plane shear modulus predicted by Tsai-Hahn’s formulation may be interpreted as arising from a nearly universal interpolation between constituent moduli, governed primarily by the fiber volume fraction rather than the absolute magnitude of the fiber–matrix stiffness contrast. As4/3501-6 as an outlier is suspected to be so due to its low  $G_f$ , as mentioned in chapter 3. And thus the Tsai-Hahn RoM with corrected moduli can be approximated by the following power mean with uncorrected moduli:

$$G_{12}^{Tsai-Hahn} = \left( \frac{v_f}{G_f^*} + \frac{v_m}{G_m^*} \right)^{-1} \approx (v_f G_f^{-0.303v_f-0.356} + v_m G_m^{-0.303v_f-0.356})^{\frac{1}{-0.303v_f-0.356}} \quad (6.24)$$

Physically, this observation implies that the Tsai-Hahn relation behaves analogously to a generalised power-mean mixture law with an exponent that varies approximately linearly with  $v_f$ , while remaining largely insensitive to the specific composite system. The residual deviations observed between material systems are therefore attributable to higher-order nonlinear effects embedded in the original semi-empirical formulation. Consequently, the power-mean representation provides a compact surrogate model that captures the dominant structure of Tsai-Hahn’s shear modulus prediction while offering a more transparent parametric interpretation of mixture behaviour.

**Table 6.3:** Linear fit parameters of  $p(v_f) \approx m v_f + b$  obtained by matching the power-mean shear modulus to the Tsai-Hahn RoM.

| Material system      | Slope $m$ | Intercept $b$ | $R^2$ [%] |
|----------------------|-----------|---------------|-----------|
| O-C Glass / 3501-6   | −0.310003 | −0.368051     | 99.3      |
| Hercules AS / 3501-6 | −0.309302 | −0.366695     | 99.4      |
| G40-800 / 5260       | −0.308419 | −0.364999     | 99.4      |
| Glass / LY556        | −0.313247 | −0.374468     | 99.3      |
| AS4 / 3501-6         | −0.273224 | −0.306405     | 99.8      |
| IM7 / 8552           | −0.303580 | −0.355979     | 99.5      |
| Mean                 | −0.302963 | −0.356100     | -         |
| Std. deviation       | 0.014016  | 0.023403      | -         |

## 6.6. Calibration of a statistically informed power–mean law

Sections 6.3–6.5 established the three key ingredients required to construct a statistically informed mixture description of the ply shear modulus:

- the bounded probabilistic representation of local fiber volume fraction variability (section 6.3),
- the element-based homogenisation result linking ply stiffness to the mean stiffness of its constituents while considering geometric corrections (section 6.4),
- and the interpretation of mixture relations as a generalised power–mean interpolation between constituent moduli (section 6.5).

The present section combines these results into a compact deterministic representation that reproduces the statistically homogenised ply response. The resulting exponent function  $p(v_f)$ , obtained from the 3-step process described below, provides a quantitative measure of the deviation from classical mixture bounds induced by spatial variability in fiber packing. Its determination does not really on any experimental data beyond the shear moduli of the ply constituents.

### 6.6.1. Statistically averaged reference stiffness

For a prescribed nominal fiber volume fraction  $v_f$ , the local microstructure is modelled through the random variable  $V_f$  introduced in section 6.3. The corresponding statistically averaged shear modulus based on a beta-distributed fiber volume fraction is defined as

$$G_{12}^{\text{stat}}(v_f) = \mathbb{E} [G_{12}^{\text{Tsai-Hahn}}(V_f)], \quad (6.25)$$

where  $\mathbb{E}[\cdot]$  denotes the expectation operator with respect to the probability distribution of the random variable  $V_f$  and  $G_{12}^{\text{Tsai-Hahn}}$  is the deterministic concentric-cylinder estimate used at the element level.

This expectation corresponds directly to the ply-level homogenisation result derived in Section 6.4.

### 6.6.2. Identification of an effective mixture exponent

The statistically averaged modulus  $G_{12}^{\text{stat}}(v_f)$  is now represented using the power–mean mixture formulation introduced in section 6.5. For each nominal fiber volume fraction, an effective exponent  $p$  is determined by enforcing

$$p(v_f) = \arg \min_p \left[ G_{12}^{\text{stat}}(v_f) - G_{12}^{(p)}(v_f) \right]^2, \quad (6.26)$$

where  $G_{12}^{(p)}$  denotes the power–mean prediction.

This procedure yields a discrete mapping between fiber volume fraction and the mixture exponent implied by statistically varying local fiber packing.

### 6.6.3. Continuous representation of the exponent function

To obtain a smooth constitutive description, the calibrated exponent values are approximated by a polynomial relation,

$$p(v_f) \approx c_2 v_f^2 + c_1 v_f + c_0, \quad (6.27)$$

whose coefficients are determined through least–squares regression.

In constructing this fit, only exponent values associated with statistically admissible fiber–volume–fraction distributions are used. Specifically, the corresponding Beta–distribution shape parameters introduced in section 6.3 are required to satisfy

$$\alpha_\beta > 1, \quad \beta_\beta > 1, \quad (6.28)$$

ensuring that the local fiber volume fraction distribution remains unimodal within the physically admissible interval. This restriction excludes cases in which the probability density becomes U-shaped or strongly boundary–dominated, which would be inconsistent with the assumed microstructural variability of typical unidirectional plies.

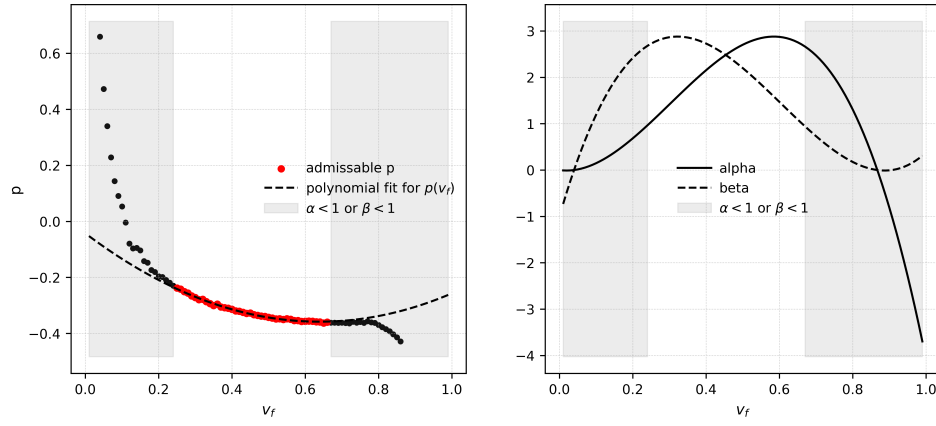
Substitution of Equation 6.27 into the power–mean formulation of Section 6.5 yields a deterministic approximation of the statistically homogenised ply shear modulus.

## 6.7. Results for considered material systems

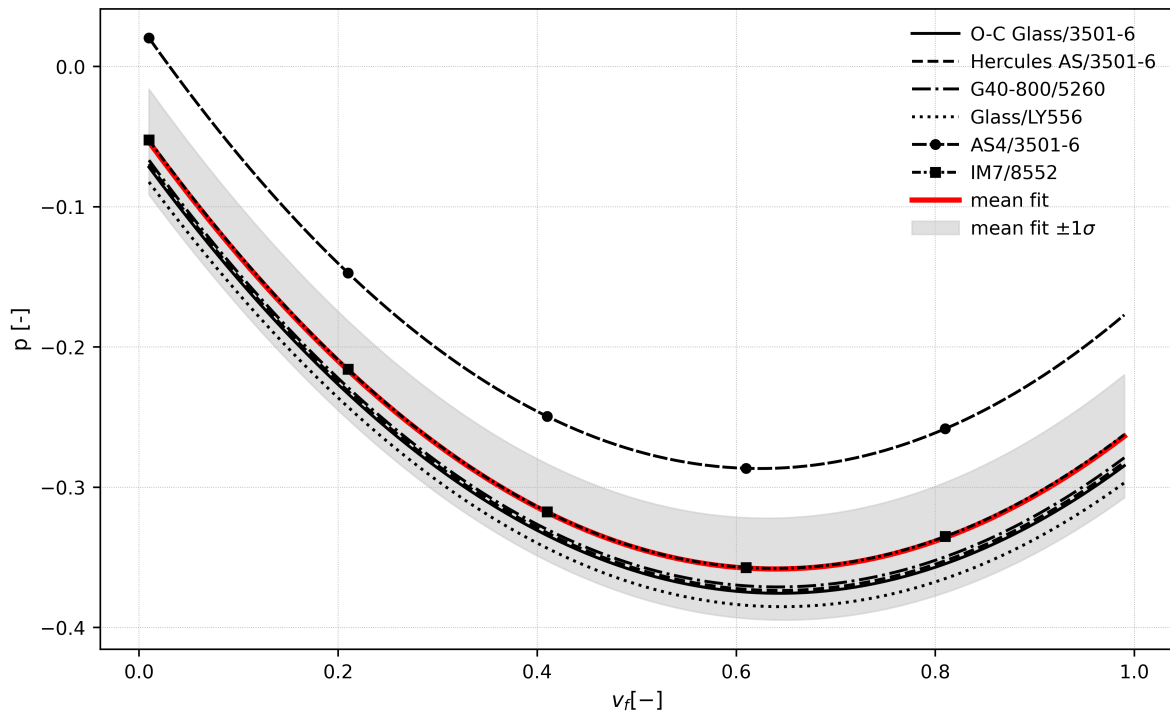
The calibration procedure described in section 6.6 was applied to the material systems introduced in chapter 3. For each system, the statistically homogenised modulus  $G_{12}^{\text{stat}}(v_f)$  was evaluated and used to determine the corresponding exponent function  $p(v_f)$ .

### 6.7.1. Calibrated exponent functions

An example of the discrete exponent values obtained from Equation 6.26 are shown in Figure 6.6 for IM7/8552. The admissible values of  $p$  correspond to the range of fiber volume fraction that resulted in  $\alpha_\beta, \beta_\beta > 1$  to ensure a unimodal Beta distribution. The fitted polynomial coefficients according to Equation 6.27 for the considered material systems, as shown in Figure 6.6 for IM7/8552, are summarised in Figure 6.7 and Table 6.4.



**Figure 6.6:** Calibrated power-mean exponents as a function of fiber volume fraction for IM7/8552.



**Figure 6.7:** The parameter  $p$  as a function of  $v_f$  to deterministically represent the statistically informed shear modulus  $G_{12}$

**Table 6.4:** Polynomial fit coefficients for  $p(v_f)$  across materials

| Material           | $c_2$    | $c_1$     | $c_0$     | $R^2$ [%] |
|--------------------|----------|-----------|-----------|-----------|
| O-C Glass/3501-6   | 0.757431 | -0.974705 | -0.062156 | 99.5      |
| Hercules AS/3501-6 | 0.759330 | -0.976267 | -0.059027 | 99.5      |
| G40-800/5260       | 0.761651 | -0.978168 | -0.057159 | 99.5      |
| Glass/LY556        | 0.748022 | -0.966900 | -0.072826 | 99.6      |
| AS4/3501-6         | 0.815734 | -1.017293 | -0.030415 | 99.2      |
| IM7/8552           | 0.773143 | -0.987430 | -0.042678 | 99.5      |
| Mean               | 0.769218 | -0.983461 | -0.044055 | -         |
| Std                | 0.022070 | 0.016284  | 0.034464  | -         |

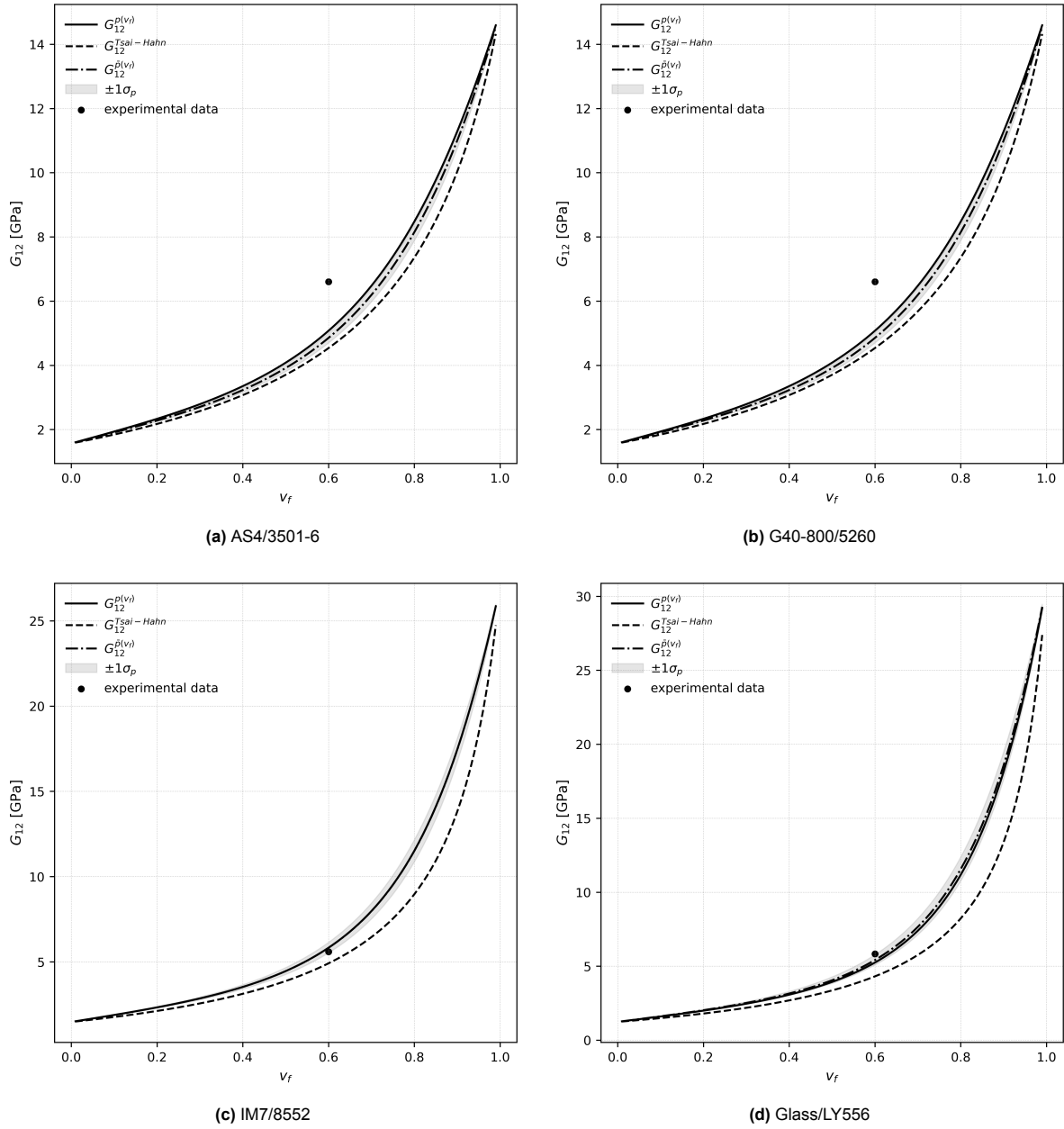
The limited scatter observed across material systems indicates that the influence of statistical microstructural variability on mixture behaviour is governed primarily by the fiber volume fraction, rather than by the absolute stiffness contrast between constituents. A similarly weak material dependence was already identified for the representation of the Tsai-Hahn relation as a generalised power-mean (see section 6.5). Consequently, the statistically averaged modulus  $G_{12}^{\text{stat}}$  of any material considered in Table 6.4 may be approximated with sufficient accuracy by adopting the following polynomial representation for the power-mean exponent,

$$\bar{p}(v_f) \approx (0.7692 \pm 0.022) v_f^2 + (-0.9835 \pm 0.016) v_f + (-0.044055 \pm 0.034464), \quad (6.29)$$

which provides a smooth deterministic surrogate for the statistically calibrated exponent.

### 6.7.2. Comparison with classical mixture relations

The effective shear modulus predicted using the calibrated power-mean is compared against the deterministic Tsai-Hahn RoM in Figure 6.8a-6.8d. Included is the  $G_{12}^{\bar{p}(v_f)}$  RoM based on Equation 6.29 with bands of  $\pm\sigma_p$ . The improvement achieved by incorporating statistical variability in fiber volume fraction is quantified in Table 6.5.



**Figure 6.8:** Idealized ply behaviour versus experimental data obtained from Kaddour et al. [84]

**Table 6.5:** Comparison of experimental  $G_{12}$  with Tsai-Hahn and power-mean estimates

| Material     | $G_{12}^{data}$ [GPa] | $G_{12}^{Tsai-Hahn}$ [GPa] | rel. diff. [%] | $G_{12}^{p(v_f)}$ [GPa] | rel. diff. [%] | improvement[%] |
|--------------|-----------------------|----------------------------|----------------|-------------------------|----------------|----------------|
| AS4/3501-6   | 6.60                  | 4.54                       | -31.2          | 5.08                    | -23.1          | 26.0           |
| IM7/8552     | 5.60                  | 4.92                       | -12.1          | 5.83                    | +4.13          | 64.5           |
| G40-800/5260 | 6.94                  | 4.35                       | -37.3          | 5.23                    | -24.7          | 33.8           |
| Glass/LY556  | 5.83                  | 4.32                       | -26.0          | 5.25                    | -10.0          | 61.5           |

The results demonstrate that accounting for spatial variability in fiber packing leads to a systematic reduction in modelling error, thereby providing a physically motivated extension of deterministic mixture relations. A similar trend was found by Yuanchen Huang et al. [31].

From a modelling perspective, the proposed approach therefore provides a compact extension classical

mixture relations, retaining their analytical simplicity while incorporating the essential physics associated with microstructural randomness. This makes the resulting formulation particularly suitable for integration into multi-scale constitutive frameworks and engineering design tools.

### 6.7.3. Validation using materials with multiple $v_f$

The new RoM is tested for a glass filament-epoxy resin system, as reported by Öchsner [16]. Obtaining the new RoM for any different material system is done by following the 3-step procedure of section 6.6. Its constituent properties can be found in Table 6.7. The comparison presented in Table 6.6 and Figure 6.9 demonstrates that the proposed power-mean formulation provides a consistently improved prediction of the measured shear modulus relative to the Tsai-Hahn relation. Whereas the Tsai-Hahn curve systematically underestimates the experimental values across the investigated range of fiber volume fractions, the statistically calibrated power-mean relation yields predictions that more closely follow both the curvature and magnitude of the measured trend.

A key observation is that the experimental data points are effectively nested around the deterministic prediction  $G_{12}^{p(v_f)}$ , with some of the measurements falling within the uncertainty band associated with the statistical variation of the exponent. This supports the interpretation that deviations from the classic RoMs are not random modelling artefacts, but rather arise from spatial variability in fiber packing and the resulting non-uniform stress. Local clustering and variations in matrix constraint alter the load-sharing between the fibers and matrix, such that the apparent shear stiffness is higher compared to predictions based on uniform stress assumptions, such as Reuss.

The improvement with respect to the Tsai-Hahn formulation is most pronounced at intermediate fiber volume fractions, where the curvature of the effective mixture response is governed by the competition between matrix-dominated shear transfer and fiber-dominated stiffness contributions. In this regime, deterministic mixture laws tend to underpredict the stiffness due to their implicit assumption of homogeneous phase interaction. By incorporating statistical variability in the local fiber volume fraction, the present model effectively modifies the mixture curvature, thereby providing a more realistic representation of the underlying micromechanical response.

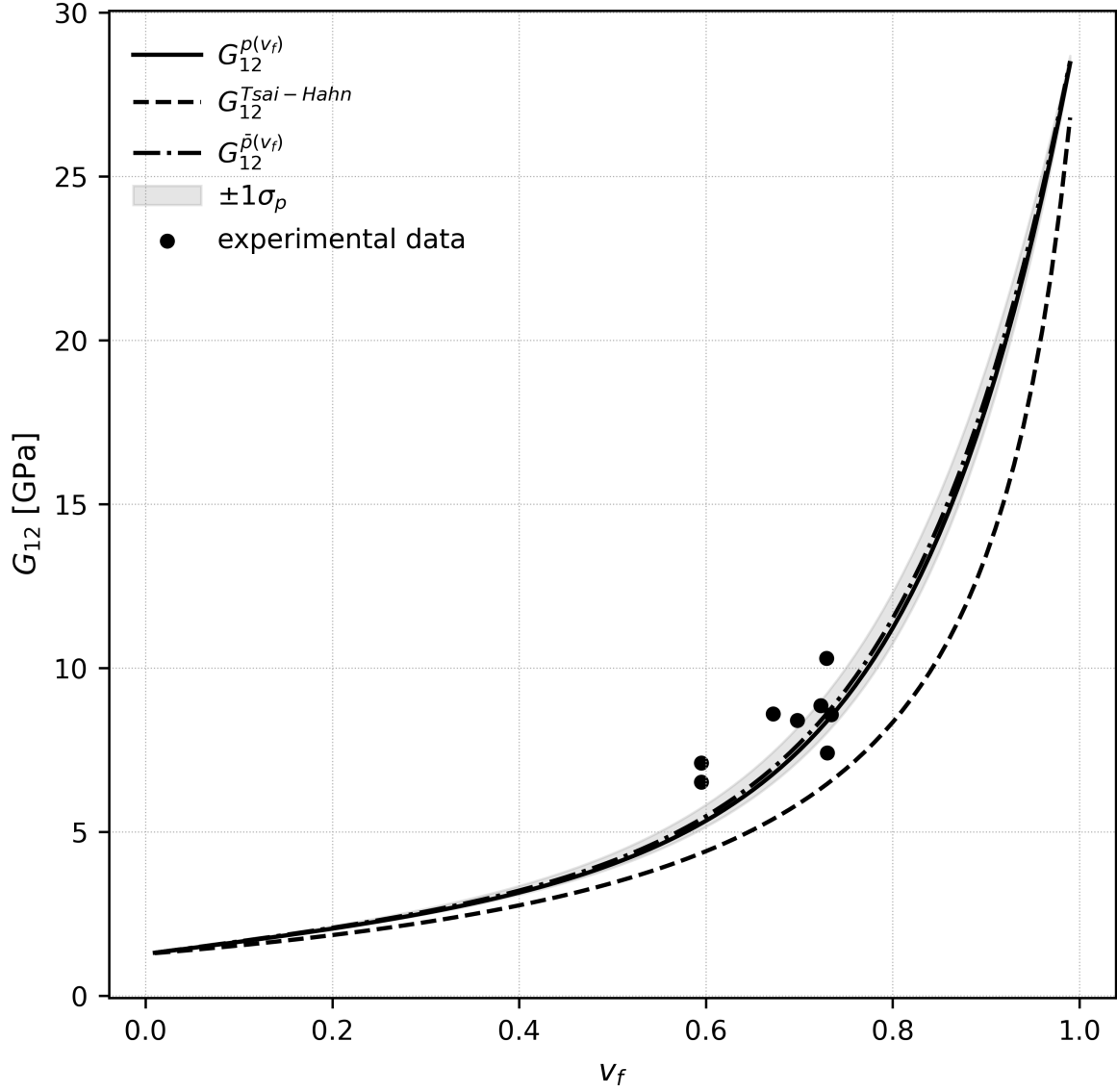
Overall, the results confirm that the proposed statistical homogenisation approach gives a compact yet effective extension of classical mixture relations, retaining their analytical simplicity while incorporating the essential physics associated with microstructural randomness.

**Table 6.6:** Measured shear modulus and power-mean prediction

| $v_f$ | $G_{12}^{\text{data}}$ [GPa] [16] | $G_{12}^p$ [GPa] | rel. diff. [%] |
|-------|-----------------------------------|------------------|----------------|
| 0.595 | 6.52                              | 5.26             | -19.2          |
| 0.595 | 7.11                              | 5.26             | -26.0          |
| 0.672 | 8.60                              | 6.78             | -21.1          |
| 0.698 | 8.41                              | 7.43             | -11.7          |
| 0.723 | 8.86                              | 8.17             | -7.79          |
| 0.729 | 10.3                              | 8.36             | -18.8          |
| 0.730 | 7.41                              | 8.39             | +13.2          |
| 0.734 | 8.58                              | 8.53             | -0.58          |

**Table 6.7:** Fiber and matrix properties of the glass filament-epoxy resin system [16], expressed in GPa. The shear modulus is obtained from the isotropic relation  $G = E/[2(1 + \nu)]$ .

| Property  | Fiber  | Matrix |
|-----------|--------|--------|
| $E$ [GPa] | 73.084 | 3.447  |
| $\nu$ [-] | 0.22   | 0.35   |
| $G$ [GPa] | 29.953 | 1.277  |



**Figure 6.9:** Statistically calibrated power-mean and Tsai-Hahn vs data for a glass filament-epoxy system. Data from [16]

## 6.8. RVE stiffness correction

The statistically calibrated power-mean relation derived in section 6.6 provides an effective ply-level shear modulus that accounts for spatial variability in fiber packing. However, the deterministic RVE formulation developed in chapter 4 predicts an effective shear stiffness that remains systematically higher than this statistically inferred behaviour, as was shown in chapter 5.

To ensure consistency between the micromechanical RVE model and the statistically homogenised ply response, a correction of the constituent shear moduli entering the RVE formulation is introduced.

### 6.8.1. Consistency condition

Let the deterministic RVE model provide an effective shear stiffness  $K_{RVE}$  as covered by Equation 4.70 under the assumption that  $n_{RVE} = 1$  (Equation 4.73) and corresponding modulus

$$G_{RVE}(G_f, G_m, v_f) = \frac{K_{RVE}(G_f, G_m, v_f)}{A_{RVE}}. \quad (6.30)$$

Consistency with the statistically homogenised ply behaviour is enforced by requiring

$$G_{\text{RVE}}(G_f^*, G_m^*, v_f) = G_{12}^{\text{stat}}(v_f), \quad (6.31)$$

where  $G_{12}^{\text{stat}}(v_f)$  is the calibrated power–mean mixture relation derived in section 6.7.

### 6.8.2. Correction of constituent moduli

Following the idea of Tsai–Hahn in-situ modulus corrections, the effective stiffness response of the deterministic RVE model is adjusted through multiplicative modification of the constituent shear moduli.

In the most general form, a scalar correction factor may be applied to both phases,

$$G_f^* = c G_f, \quad G_m^* = c G_m, \quad (6.32)$$

thereby preserving the relative stiffness contrast while modifying the overall stiffness predicted by the RVE formulation.

However, since the deterministic RVE idealisation primarily overpredicts stiffness, and thus an overly strong contribution of the fiber is present, an alternative correction strategy is to modify only the fiber shear modulus,

$$G_f^* = c_f G_f, \quad G_m^* = G_m. \quad (6.33)$$

This approach attributes the discrepancy between deterministic RVE predictions and statistically homogenised ply behaviour predominantly to the idealised representation of the fiber response, while retaining the matrix properties.

Substitution into Equation 6.31 yields a scalar nonlinear equation for the correction factor  $c$  at each fiber volume fraction. The correction factor are then determined numerically by solving

$$G_{\text{RVE}}(c G_f, c G_m, v_f) - G_{12}^{\text{stat}}(v_f) = 0, \quad (6.34)$$

or

$$G_{\text{RVE}}(c_f G_f, G_m, v_f) - G_{12}^{\text{stat}}(v_f) = 0, \quad (6.35)$$

which defines a one–dimensional root-finding problem. The resulting corrected constituent moduli ensure that the deterministic RVE formulation reproduces the statistically inferred ply stiffness while retaining the analytical structure of the micromechanical model. Results for both correction strategies are shown in Figure 6.10 and 6.11 over the full fiber volume fraction range and Table 6.8 contains the corrections to match at  $v_f = 0.6$  specifically. Note how correction factors for both constituents are consistently very close to the fiber correction and slightly larger.

**Table 6.8:** Comparison of correction strategies required to reconcile deterministic RVE predictions with statistically calibrated ply shear modulus.

| Material system | $c_f$ (fiber only) | $c$ (fiber & matrix) |
|-----------------|--------------------|----------------------|
| AS4/3501-6      | 0.561              | 0.594                |
| IM7/8552        | 0.367              | 0.393                |
| G40-800/5260    | 0.330              | 0.354                |
| Glass/LY556     | 0.287              | 0.312                |
| Average         | 0.386              | 0.413                |
| Std. dev.       | 0.121              | 0.122                |



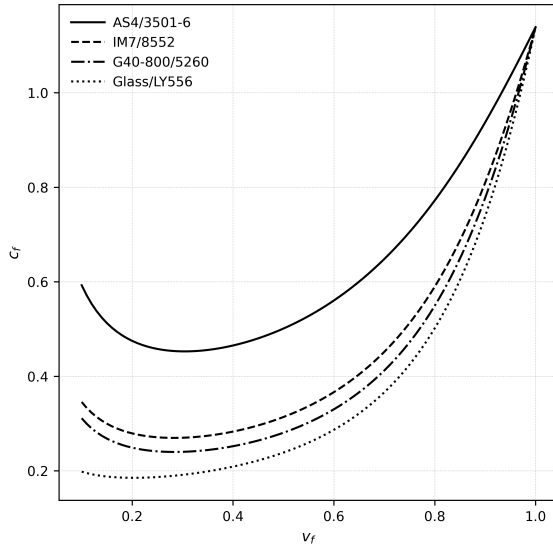


Figure 6.10: Corrections  $c_f$  for  $G_f$

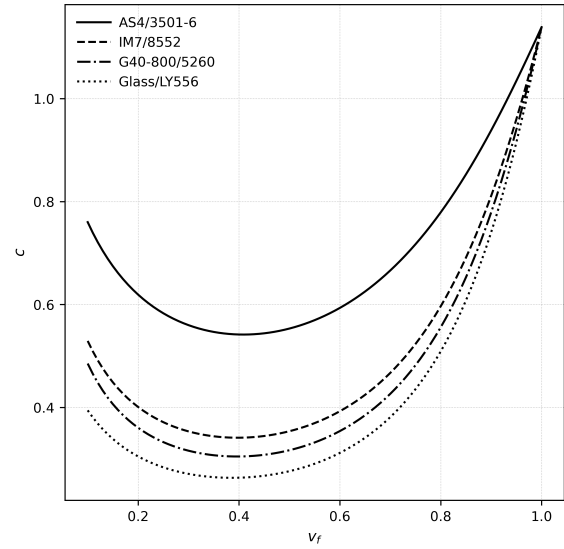


Figure 6.11: Corrections  $c$  for  $G_f$  and  $G_m$

## 6.9. Summary and outlook

The main findings of this chapter can be summarised as follows:

- Classical mixture relations were shown to underpredict the initial in-plane shear stiffness of unidirectional plies, indicating the need for an enhanced homogenisation approach.
- Spatial variability in local fiber volume fraction was introduced using a bounded probabilistic description based on a scaled Beta distribution, representing physically admissible variations in fiber packing.
- Statistical homogenisation of locally evaluated shear stiffnesses led to an effective ply response that can be accurately represented by a power-mean mixture law with a volume-fraction-dependent exponent.
- The calibrated exponent function was found to exhibit only weak material dependence, suggesting that the dominant driver of the effective mixture behaviour is the fiber volume fraction rather than constituent stiffness contrast.
- The statistically informed ply response was translated into consistency conditions for the deterministic RVE formulation through multiplicative corrections of the constituent shear moduli.
- Both uniform constituent scaling and fiber-only correction strategies were shown to restore agreement between the deterministic RVE stiffness and the statistically homogenised ply response.

The present framework provides a consistent description of the initial, undamaged shear stiffness of the ply. In the following chapter, this corrected initial stiffness serves as the reference state for investigating micromechanical failure mechanisms and the associated progressive degradation of RVE stiffness.

## Constituent Failure and Local Stiffness Degradation

With the statistically corrected initial shear stiffness established in preceding chapter, the remaining challenge is to describe the onset of stiffness degradation during shear loading. This chapter therefore investigates the micromechanical failure mechanisms of the fiber-matrix RVE subjected to increasing shear deformation. Failure criteria for the fiber and matrix phases are evaluated using the strain fields obtained from the analytical formulation with corrected moduli as per chapter 6, and stress redistribution following fiber fracture is approximated to identify radial and interfacial cracking processes in the matrix. Based on these mechanisms, the distinct degraded stiffness states are defined at the RVE level, forming the mechanical foundation for modelling progressive stiffness reduction at the ply scale.

### 7.1. Pre-failure kinematic state of the RVE

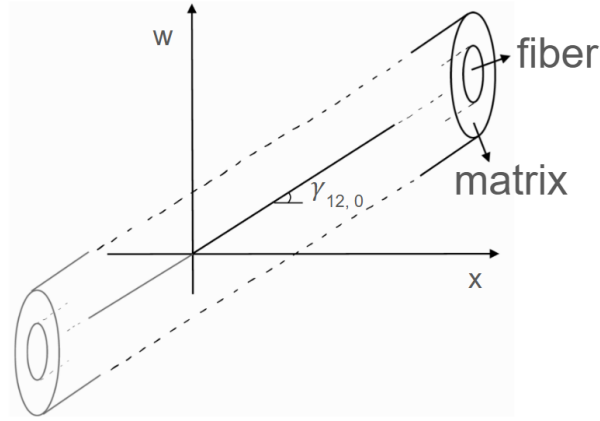
Prior to the onset of local failure, the RVE is subjected to a uniform engineering shear strain  $\gamma_{12,0}$  as established in chapter 4,

$$\gamma_{12,0} = \frac{Q_f}{K_{RVE}} = Q_f \left( \frac{(1-U) \left( \frac{\alpha + \beta R}{\phi - \psi R} \right) + V}{I \left( \frac{\alpha + \beta R}{\phi - \psi R} \right) + J} \right), \quad (7.1)$$

with the appropriate selection of  $\alpha, \beta, \phi, \psi$  and  $R$  based on the sign of the characteristic parameter  $\Delta$  (Equation 4.23). Under the assumption of small deformations, this engineering shear strain corresponds to the centerline displacement field,

$$w_0(x) = \gamma_{12,0}x, \quad (7.2)$$

illustrated in Figure 7.1. For convenience, a different coordinate system is employed, with the lateral displacement  $w$  positive upward. A conceptual view of the cylindrical RVE as idealized in chapter 4 is sketched around the fiber displacement. This is the reference state for all post-failure solutions. Development of the RVE displacement and strain field evolution upon failure will focus on the first quadrant of the coordinate system, with the third quadrant being symmetric with respect to the origin.



**Figure 7.1:** Pre-failure fiber axis displacement due to uniform shear strain  $\gamma_{12,0}$ .

## 7.2. Constituent failure criteria

This section defines the onset of failure for the constituents and relates it to the far-field loading of the RVE. A strain-based criterion is used, based on assumed strain compatibility at the fiber-matrix interface, and expressed in terms of the strength-to-stiffness ratio of each constituent.

Using the load fractions of fiber and matrix, this criterion is reformulated in terms of failure loads and stresses at the RVE level. Resulting expressions are then evaluated for the considered material systems, highlighting whether failure is governed by the fiber or the matrix.

### 7.2.1. Formulation failure criterion

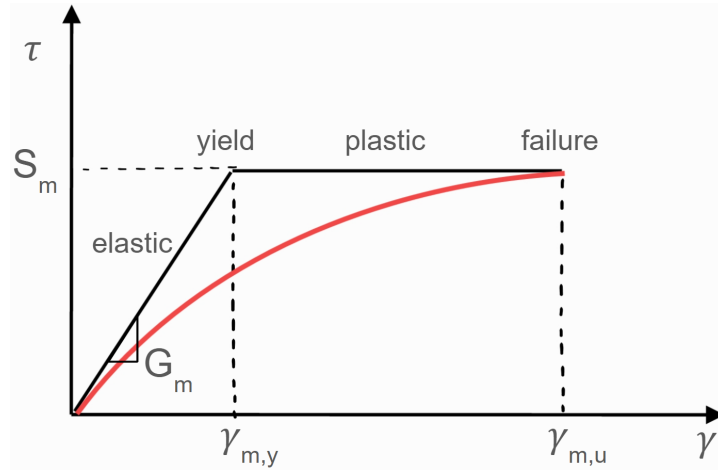
As the fiber is modelled as a Timoshenko beam, the uniform shear strain  $\gamma_{12,0}$  of its centreline is constant over the fiber cross-section. Consequently, the outer surface of the fiber experiences the same shear strain. At this surface, the fiber-matrix interface is located. Under the assumption of perfect bonding, the shear strain in the fiber is therefore equal to the shear strain in the matrix at the interface.

The onset of shear failure in either constituent is described by the condition

$$\frac{G_i \gamma_{12}}{S_i} = 1, \quad i = f, m, \quad (7.3)$$

where  $G_i$  and  $S_i$  denote the shear modulus and shear strength of the respective phase. Equation (7.3) follows from the linear elastic relation  $\tau_i = G_i \gamma_{12}$  and expresses failure as the point at which the shear stress in the constituent reaches its strength.

For the fiber, which is treated as linear elastic and brittle [2], this criterion provides a suitable representation of shear failure. For the matrix, which may behave inelastic, Equation (7.3) is interpreted as defining the onset of irreversible damage, consistent with an elastic-ideally plastic material model, illustrated in Figure 7.2.



**Figure 7.2:** Conceptual ideal elastic-plastic matrix model corresponding to failure criterion of Equation 7.3.

Recalling that the applied shear load carried by the fiber is given by

$$Q_f = K_{RVE} \gamma_{12,0}, \quad (7.4)$$

and using the definition of the fiber load fraction from Chapter 4,  $Q_f = f_f Q_{RVE}$ , Equation (7.3) can be rearranged to obtain the far-field RVE load corresponding to the local failure of constituent  $i$ ,

$$Q_{RVE, failure, i} = \frac{1}{f_f} \left( \frac{S}{G} \right)_i K_{RVE} \quad i = f, m. \quad (7.5)$$

The corresponding nominal shear stress at the RVE level is

$$\tau_{RVE, failure, i} = \frac{Q_{RVE, failure, i}}{n_{RVE} A_{RVE}} = \frac{1}{A_{RVE} f_f} \left( \frac{S}{G} \right)_i K_{RVE} = \frac{G_{RVE}}{f_f} \left( \frac{S}{G} \right)_i \quad i = f, m, \quad (7.6)$$

where the assumption of  $n_{RVE} = 1$  is used, as in previous chapters.

At the instant of failure, the local shear stress in the failing constituent satisfies either

$$\tau_f = S_f, \quad (7.7)$$

or

$$\tau_m = S_m, \quad (7.8)$$

such that Equation 7.5 or 7.6 relates the far-field RVE loading to the constituent-level strength condition.

### 7.2.2. Results for the considered material systems

Results from Equation 7.5 and Equation 7.6 depend strongly on the strength-to-stiffness ratio  $(S/G)_i$  of the constituents. The shear strength, shear modulus, and corresponding ratios are summarised in Table 7.1 for the fibers and in Table 7.2 for the matrix. As the fiber shear modulus  $G_f$  is corrected in subsection 6.8.2 to  $G_f^*$  to match experimental ply stiffness, this correction is consistently carried through the remainder of the framework and is therefore included here.

From Equation 7.6, noting that  $(S/G)_i$  is dimensionless, it effectively represents a shear failure strain, consistent with Equation 7.3, and is reported as such. For the fiber, which is assumed linear elastic, this provides a reasonable estimate of the ultimate shear strain, not directly reported in Kaddour et al. [64]. For the matrix,  $(S/G)_m$  corresponds to the yield strain  $\gamma_{m,y}$  shown in Figure 7.2 and therefore  $(S/G)_m < \gamma_{m,u}$  as seen in Table 7.2.

Comparing  $(S/G)_f$  and  $(S/G)_m$  across the materials shows that failure loads and stresses from Equation 7.5 and Equation 7.6 are governed by fiber failure for the carbon systems, i.e.  $(S/G)_f < (S/G)_m$ ,

and by matrix failure for the glass systems, i.e.  $(S/G)_m < (S/G)_f$ . Applying the corrected fiber shear modulus  $G_f^*$  increases the fiber ultimate shear strain, but does not alter these trends.

**Table 7.1:** Fiber shear strength, shear modulus, corrected modulus, and strength-to-stiffness ratios, adapted from [64], [82] as per chapter 3.

| Material system | $S_f$ [MPa] | $G_f$ [GPa] | $G_f^*$ [GPa] | $S_f/G_f$ [%] | $S_f/G_f^*$ [%] |
|-----------------|-------------|-------------|---------------|---------------|-----------------|
| Hercules AS     | 172.0       | 34.5        | -             | 0.499         | -               |
| AS4             | 172.0       | 15.0        | 8.42          | 1.15          | 2.05            |
| IM7             | 172.0       | 27.0        | 9.91          | 0.637         | 1.74            |
| G40-800         | 172.0       | 27.0        | 8.91          | 0.637         | 1.93            |
| OC Glass        | 2410        | 35.3        | -             | 6.83          | -               |
| Glass/LY556     | 2410        | 30.8        | 8.84          | 7.82          | 27.3            |

**Table 7.2:** Matrix shear strength, shear modulus, strength-to-stiffness ratios and ultimate shear strain, [64], as per chapter 3.

| Material system | $S_m$ [MPa] | $G_m$ [GPa] | $S_m/G_m$ [%] | $\gamma_{m,u}$ [%] |
|-----------------|-------------|-------------|---------------|--------------------|
| Hercules AS     | 57          | 1.567       | 3.64          | 4.0                |
| AS4             | 57          | 1.567       | 3.64          | 4.0                |
| IM7             | 57          | 1.478       | 3.86          | 5.1                |
| G40-800         | 50          | 1.280       | 3.91          | 5.0                |
| OC Glass        | 57          | 1.567       | 3.64          | 4.0                |
| Glass/LY556     | 54          | 1.240       | 4.36          | 6.0                |

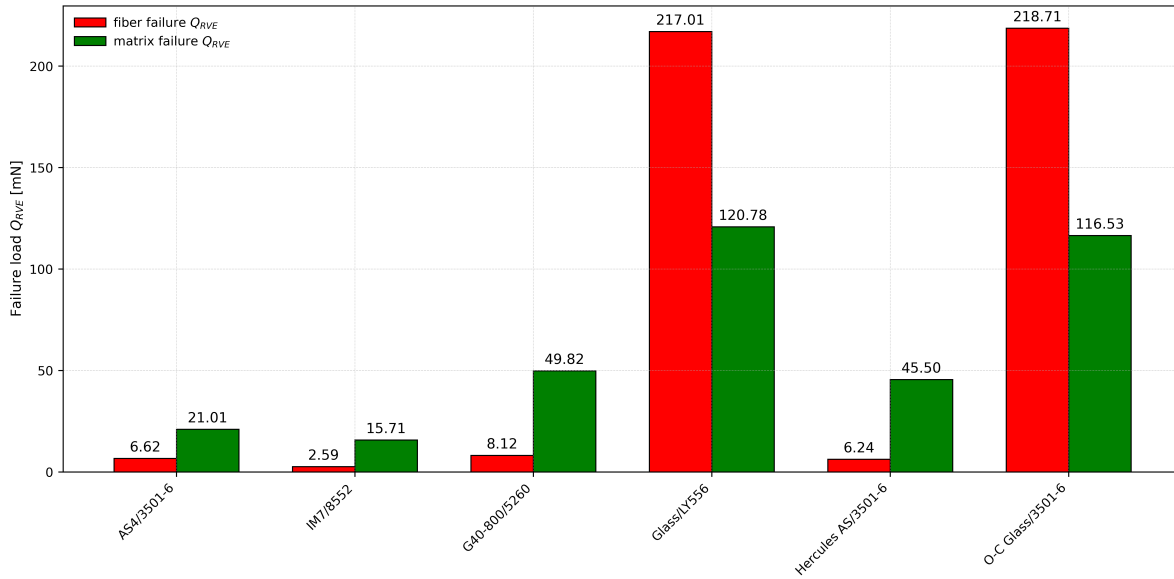
The resulting failure loads and stresses for the material systems with uncorrected fiber shear moduli are shown in the bar charts of Figure 7.3-7.4. Several observations can be made:

For the failure loads,  $Q_{RVE, failure, i}$ :

- Despite a large difference in  $G_f$ , both AS4 and Hercules AS exhibit comparable failure loads, reflecting their overall similarity;
- Carbon systems show failure loads approximately one order of magnitude lower than glass systems, consistent with the failure strains defined in Table 7.1 and 7.2;
- The relatively low fiber failure load of IM7/8552 is associated with its small fiber diameter.

For the failure stresses,  $\tau_{RVE, failure, i}$ :

- Carbon fiber failure stresses cluster around 98.84 MPa, with a standard deviation of 2.58 MPa;
- Glass failure stresses are remarkably high, particularly when considered at the ply level.



**Figure 7.3:**  $Q_{RVE, failure, i}$  for considered material systems, uncorrected.

The reason for including the uncorrected material systems in this failure analysis is that it was initially performed prior to establishing the correction strategy. It was observed that the carbon failure stresses  $\tau_{RVE, failure, f}$  are closely clustered and lie in the vicinity of the ply in-plane shear strength  $S_{12, ply}$ .

A comparison is made with methods by Huang [96][97], based on the Bridging model (Br) and its extension accounting for stress concentrations (Br- $K_t$ ). Results and associated errors are summarised in Table 7.3.

The main observation is that the Br model performs best, while the Br- $K_t$  model does not provide an improvement and is outperformed by  $\tau_{RVE, failure, f}$ . It should be noted that  $S_f$  is taken equal for all carbon materials due to limited data availability. Introducing a statistical spread in  $S_f$  is expected to improve the relative error of  $\tau_{RVE, failure, f}$ , yielding both under- and overestimates.

Recalling the discussion in chapter 6, where different RoMs reflect different fiber-matrix interaction mechanisms ranging from uniform strain to uniform stress, the uncorrected  $G_{RVE}$  corresponds to a response closer to uniform strain. In contrast, more accurate representations of the ply initial shear modulus  $G_{12}$  tend towards a uniform stress assumption, which motivated the corrected  $G_{RVE}$ .

Since Equation 7.6, when evaluated with the uncorrected  $G_{RVE}$ , appears to approximate  $S_{12, ply}$ , it may be indicative that the effective fiber-matrix interaction evolves during loading. In this view, different forms of  $G_{RVE}$ , whether corrected or uncorrected, could be associated with different stages of the loading process, such as initial deformation versus conditions approaching failure.

**Table 7.3:** Comparison of experimental and modeled results for selected UD systems.

| Material System    | $S_{12, ply}$ [MPa] [64] | $\tau_{RVE, failure, f}$ | error [%] | Br    | error [%] | Br- $K_t$ | error [%] |
|--------------------|--------------------------|--------------------------|-----------|-------|-----------|-----------|-----------|
| AS4/3501-6         | 79                       | 103.26                   | +30.7     | 90.19 | +14.17    | 127.18    | +61.1     |
| G40-800/5260       | 91                       | 96.97                    | +6.56     | 83.02 | -8.78     | 123.21    | +35.4     |
| IM7/8552           | 90                       | 97.88                    | +8.76     | 94.03 | +4.48     | 138.58    | +54.0     |
| Hercules AS/3501-6 | 79 <sup>1</sup>          | 97.23                    | +23.1     | 94.80 | +20.0     | 140.97    | +78.5     |
| Average            | -                        | 98.84                    | 17.3      | -     | 11.9      | -         | 57.3      |
| Std. dev.          | -                        | 2.58                     | 9.7       | -     | 5.9       | -         | 15.5      |

<sup>1</sup> Assumed equal to that of AS4/3501-6.

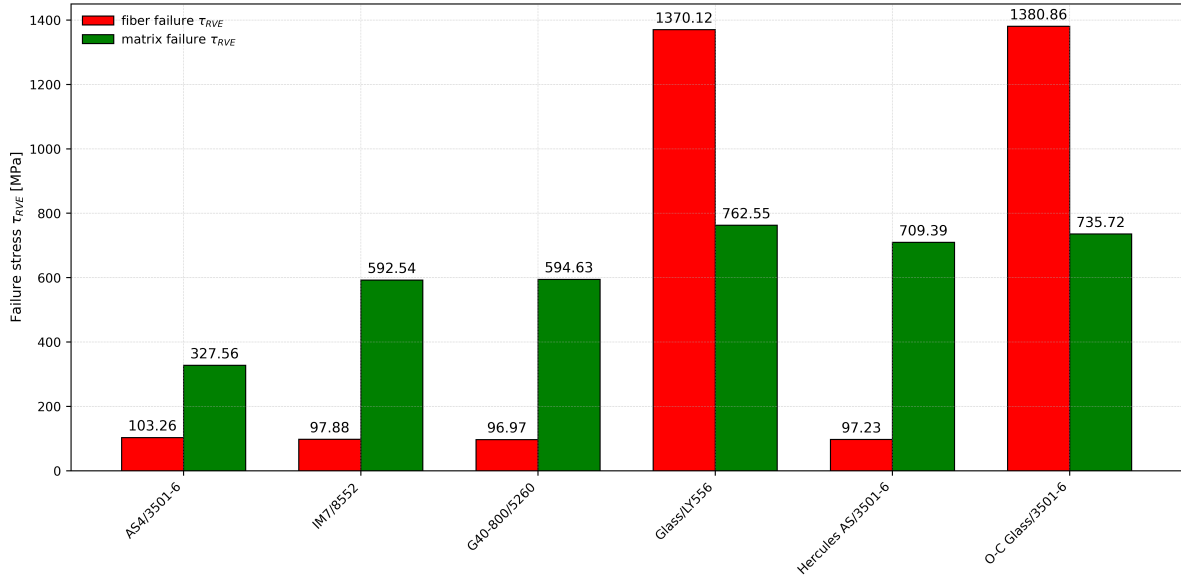


Figure 7.4:  $\tau_{RVE, failure, i}$  for considered material systems, uncorrected.

It should also be emphasised that the strength parameters  $S_i$  are inherently more uncertain than the elastic moduli. Shear strength measurements are sensitive to the specific test configuration and the extent to which the experimental loading state corresponds to the idealised conditions assumed in the present micromechanical model.

The resulting failure loads and stresses for the material systems with corrected fiber shear moduli  $G_f$ , as per chapter 6, are shown in the bar charts of Figure 7.5-7.6. Several observations can be made:

For the failure loads,  $Q_{RVE, failure, i}$ :

- Fiber failure loads are only slightly increased w.r.t. the uncorrected case. Apparently, from Equation 7.5,  $K_{RVE}(G_f^*)/G_f^* \approx K_{RVE}(G_f)/G_f$ ;
- Matrix failure loads show substantial decrease as they are affected only through the decrease of  $K_{RVE}(G_f^*)$  w.r.t.  $K_{RVE}(G_f)$ .

For the failure stresses,  $\tau_{RVE, failure, i}$ , the same trends apply as for the failure loads.

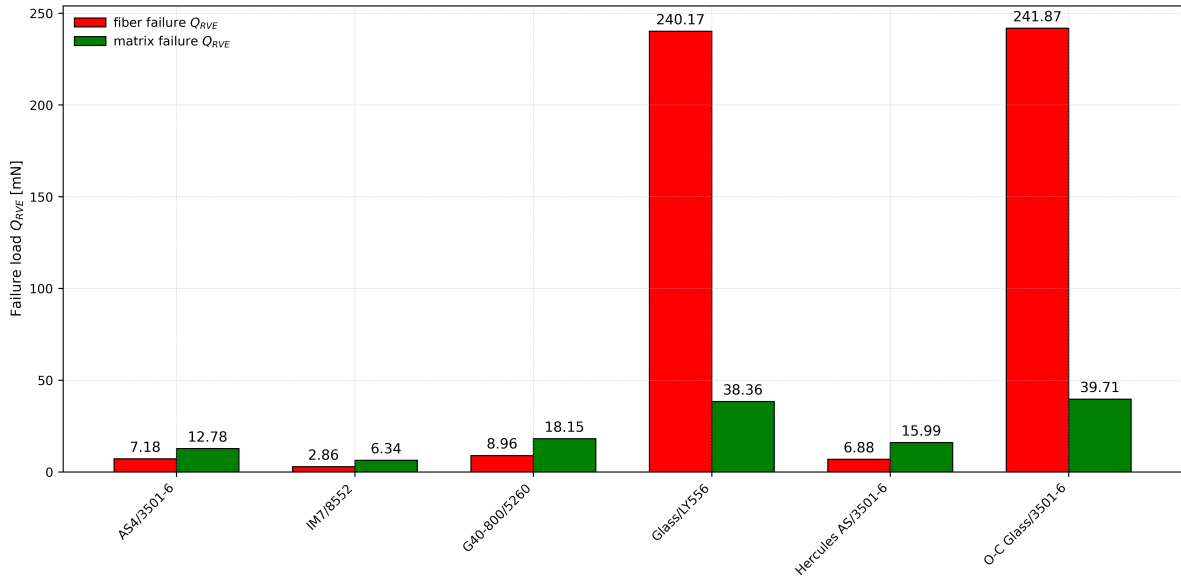


Figure 7.5:  $Q_{RVE, failure, i}$  for considered material systems, corrected.

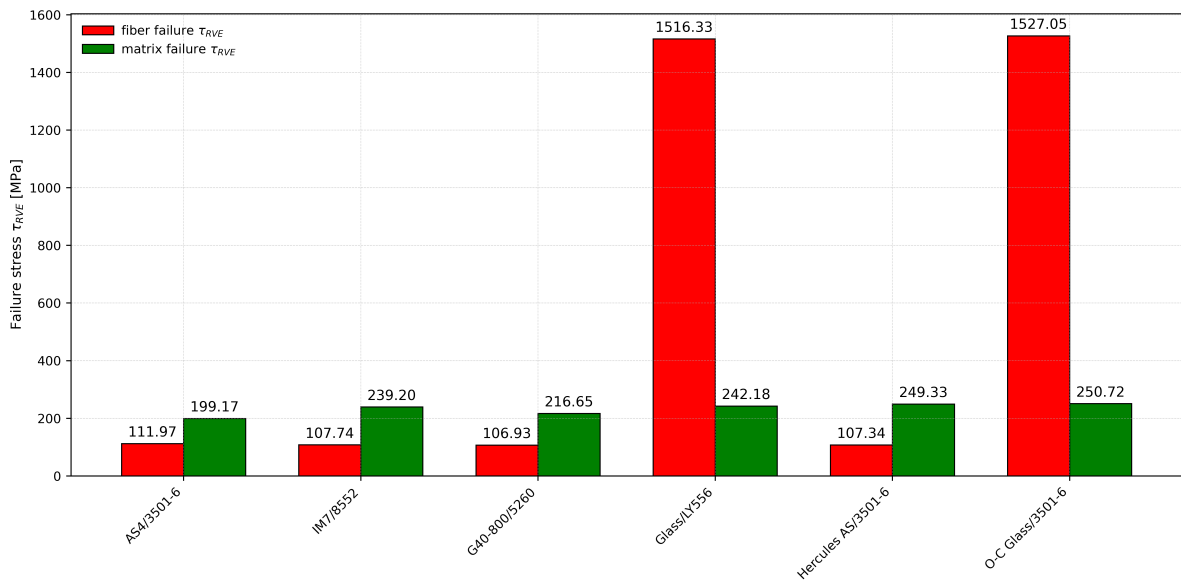


Figure 7.6:  $\tau_{RVE, failure, i}$  for considered material systems, corrected.

In the subsequent general analysis, it is assumed that the fiber has failed and is fully cracked, even though the present results indicate that for the glass systems the matrix would constitute the critical phase and at comparatively high stress levels. This simplifying assumption is introduced to enable a consistent treatment of the post-failure behaviour across all material systems.

The rationale for this choice is as follows. First, the fiber shear strength  $S_f$  is interpreted as a mean value of a normally distributed strength. Consequently, under the iso-strain assumption, where the fibers carry the dominant share of the applied load, individual fibers are expected to fracture progressively during ply loading. In this statistical framework, a subset of fibers will fail at relatively low global strain levels, leading to the early formation of discrete fiber cracks. This idea will be expanded upon in chapter 8.

The present analysis therefore focuses on the mechanics associated with fiber cracking as the governing local damage mechanism. The implications of this assumption, and its suitability for carbon versus

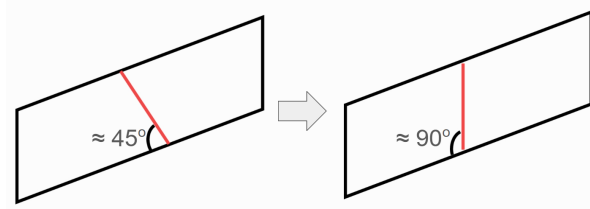


glass systems, will be revisited in the discussion of the subsequent material-specific results.

### 7.3. Load re-distribution due to fiber crack

As introduced in subsection 7.2.2, the mechanism of initial damage considered in the present analysis will be that of a discrete fiber crack.

In analogy with tensile failure, fiber fracture is associated with the presence of local flaws distributed along the fiber length [98]. Under shear loading, failure would in general occur along an inclined fracture plane. However, for the purposes of the present one-dimensional micromechanical model, the crack is idealised as a straight transverse discontinuity normal to the fiber axis, as illustrated in Figure 7.7.

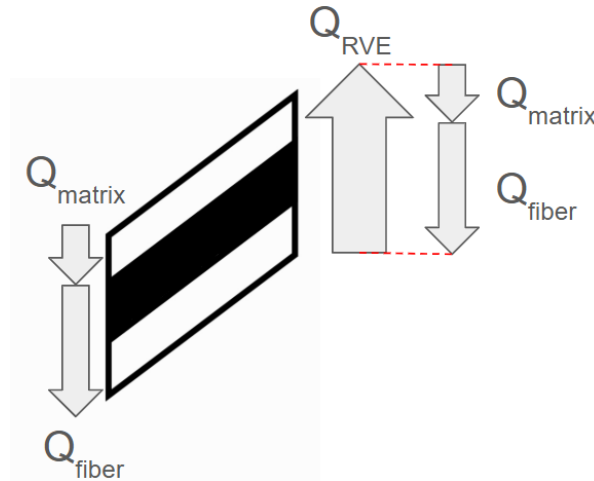


**Figure 7.7:** Considering the small angle approximation, the  $45^\circ$  angled crack will be approximated by a  $90^\circ$  angled crack.

Prior to fiber fracture, the RVE is subjected to a constant engineering shear strain  $\gamma_{12,0}$ , defined by Equation 7.1. The corresponding displacement field is given by Equation 7.2. Under the assumption of perfect bonding and equal strain in both constituents, the applied shear load is partitioned between the fiber and the matrix according to the load fractions introduced in chapter 4. Along the RVE  $x$ -axis this partitioning can be expressed as

$$Q_{\text{RVE}} = Q_f + Q_m. \quad (7.9)$$

Equation 7.9 is schematically shown in Figure 7.8. Note that this relation reflects the constitutive load sharing between the fiber and matrix rather than a local force equilibrium involving a moment arm.



**Figure 7.8:** Total applied shear force  $Q_{\text{RVE}}$  is carried by the fiber and matrix in proportion to their relative stiffnesses.

#### 7.3.1. Redistribution of load at the crack location

Now, it is assumed that the principle of load partitioning between fiber and matrix remains upon cracking of the fiber. Once a fiber crack forms, the fiber's cross-section turns into a stress-free surface at that location. Cracking can occur at any random location along the fiber, which is tied to statistics of fiber strength. Here it is assumed that a crack appears at  $x = 0$ , in the coordinate system of Figure 7.1. Consequently, the fiber can no longer sustain shear load at this location, such that

$$Q_f(x = 0) = 0. \quad (7.10)$$

Let  $Q_{f,0}$  and  $Q_{m,0}$  denote the pre-crack load contributions of the fiber and matrix, respectively. The redistribution of load along the fiber axis may then be expressed as

$$Q_f(x) = Q_{f,0} - \Delta Q_f(x), \quad (7.11)$$

$$Q_m(x) = Q_{m,0} + \Delta Q_f(x), \quad (7.12)$$

such that the total applied load remains unchanged,

$$Q_{\text{RVE}} = Q_f(x) + Q_m(x). \quad (7.13)$$

At the crack location this implies

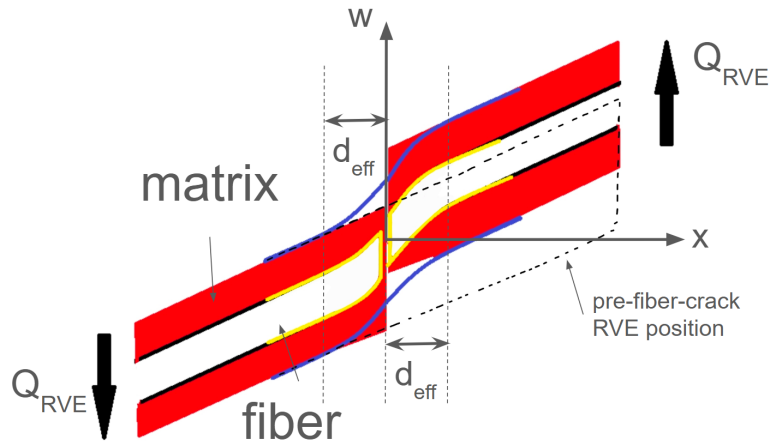
$$Q_{f,0} - \Delta Q_f(0) = 0, \quad Q_{m,0} + \Delta Q_f(0) = Q_{\text{RVE}}. \quad (7.14)$$

The released fiber load  $\Delta Q_f(x)$  is therefore transferred to the surrounding matrix and subsequently redistributed along the fiber axis.

### 7.3.2. Qualitative deformation mechanism

To provide physical insight into the redistribution process, consider the internal side view shown in Figure 7.9 of the RVE subjected to the far-field shear strain  $\gamma_{12,0}$ . Prior to fiber fracture, the applied shear load is shared between the fiber and the matrix.

Following fiber fracture, the load previously carried by the fiber at the crack location must be transferred to the surrounding matrix. Since the matrix is more compliant than the combined fiber–matrix system, this local increase in load results in an additional deformation in the vicinity of the crack. Through interfacial coupling, this increased matrix deformation imposes a transverse displacement on the fiber.



**Figure 7.9:** Qualitative description of change in deformation due to appearance fiber crack.

A qualitative interpretation of this behaviour may be obtained by considering the RVE as a shear-loaded segment held at its ends. Upon formation of the fiber crack, the sudden loss of fiber load-carrying capacity at the crack plane leads to a local increase in matrix deformation, causing a small but finite shift in the end displacement required to maintain equilibrium with the externally applied load. This shift reflects the increased compliance of the locally degraded fiber–matrix system. A schematic view of the RVE upon the fiber cracking is shown in Figure 7.9.

The resulting deformation field may therefore be interpreted as the superposition of the pre-crack uniform far-field shear deformation and a local perturbation induced by the imposed displacement at the crack location.

### 7.3.3. Post-fracture fiber displacement

Prior to fracture, the deformation is described by the uniform far-field shear displacement  $w_0$  given by Equation 7.2. Following fracture, the local deformation field is represented by

$$w(x) = w_0(x) - w_f + w_{\text{corr}}(x), \quad (7.15)$$

where  $w_f$  is the small, finite shift of the end displacement, described in subsection 7.3.2, and  $w_{\text{corr}}(x)$  denotes the correction displacement induced by the loss of fiber load-carrying capacity at the crack, connecting the crack displacement with the far-field displacement.

The correction displacement field is obtained by re-solving the Timoshenko beam formulation introduced in chapter 4, with boundary conditions reflecting the presence of the crack.

The boundary conditions at the crack plane are taken as

- $w_{\text{corr}}(0) = w_f$ ,
- $M(0) = 0$ ,

where  $w_f$  denotes the fiber-tip displacement associated with the increased deformation of the surrounding matrix, as discussed in subsection 7.3.2. Following the reduced boundary-value procedure described in subsection 4.6.2, the remaining constants  $C$  and  $D$  are obtained from

$$\begin{bmatrix} 1 & 0 \\ c_{3,M}^{(\pm)} & c_{4,M}^{(\pm)} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} w_f \\ 0 \end{bmatrix}. \quad (7.16)$$

From the first row, it follows directly that

$$C = w_f, \quad (7.17)$$

and accordingly

$$D = - \left( \frac{c_{3,M}^{(\pm)}}{c_{4,M}^{(\pm)}} \right) w_f = -R^{(\pm)} w_f, \quad (7.18)$$

where  $c_{3,M}^{(\pm)}$  and  $c_{4,M}^{(\pm)}$  can be found in Appendix A or the shorthand  $R$  from Equation 4.68 is used.

The correction displacement field therefore takes the form

$$w_{\text{corr}}(x) = w_f \left( \cos(\beta x) - \left( \frac{c_{3,M}^{(-)}}{c_{4,M}^{(-)}} \right) \sin(\beta x) \right) e^{-\alpha x} = w_f \left( \cos(\beta x) - R^{(-)} \sin(\beta x) \right) e^{-\alpha x}, \quad (7.19)$$

or

$$w_{\text{corr}}(x) = w_f \left( \cosh(\beta x) - \left( \frac{c_{3,M}^{(+)}}{c_{4,M}^{(+)}} \right) \sinh(\beta x) \right) e^{-\alpha x} = w_f \left( \cosh(\beta x) - R^{(+)} \sinh(\beta x) \right) e^{-\alpha x}, \quad (7.20)$$

where the presence of the term  $e^{-\alpha x}$  will ensure that for increasing distance from the crack

$$w(x) \rightarrow w_0(x) + w_f \quad \text{as } x \rightarrow \infty. \quad (7.21)$$

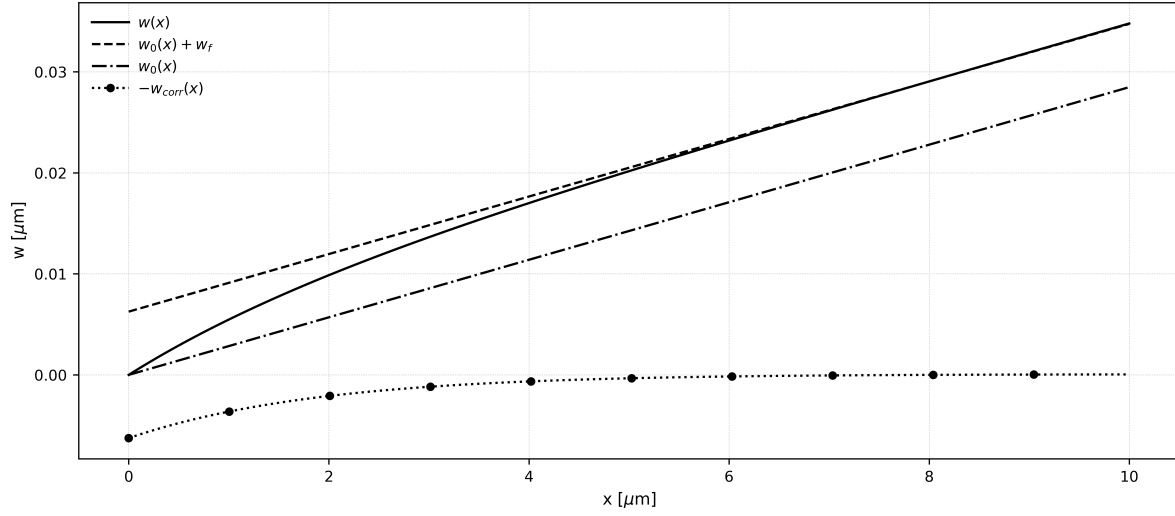
and at the crack, i.e.  $x = 0$

$$w(x) = 0. \quad (7.22)$$

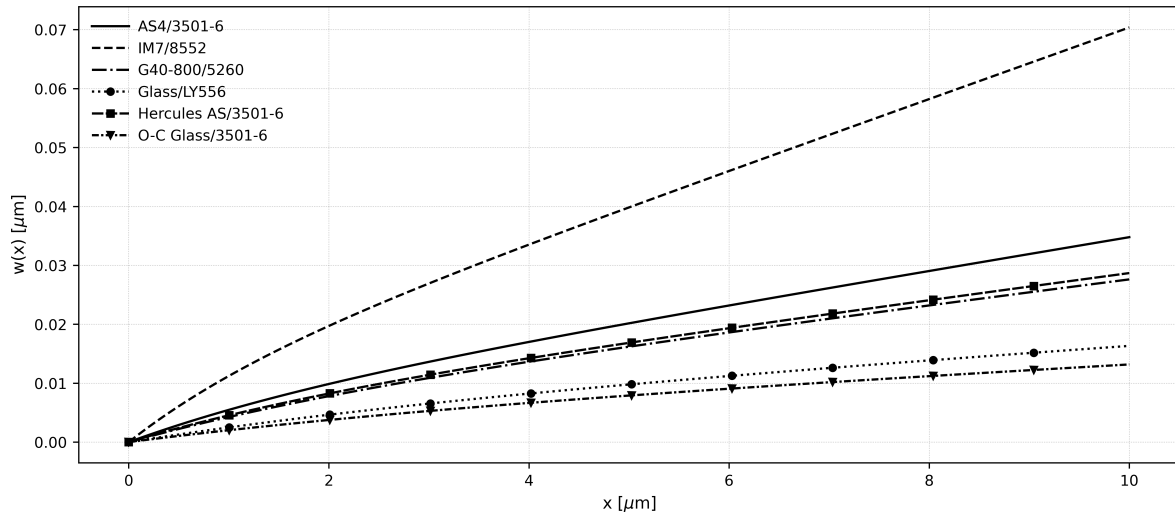
Since  $w_{\text{corr}}(0)$  corresponds to the pure-shear solution derived in chapter 4, combination of Equation 7.17 with the analytical expression for  $C$  obtained previously (Equation 4.64) yields

$$w_f = \frac{Q_f}{(-\alpha - \beta R)I + (-\phi + \psi R)J}, \quad (7.23)$$

and thus the displacement field of Equation 7.15 can be expressed as a function of the far field applied load  $Q_f$  or  $Q_{RVE}$ . An example of  $w(x)$  and its components are shown in Figure 7.10 for  $Q_{RVE} = 1mN$ . For the material systems considered,  $w(x)$  fields are compared in Figure 7.11.



**Figure 7.10:** Example post-fracture displacement field  $w(x)$  and its components for  $Q_{RVE} = 1mN$ .



**Figure 7.11:** Post-fracture displacement field  $w(x)$  for  $Q_{RVE} = 1mN$  for the considered material systems.

#### 7.3.4. Recovery of far-field load

Based on the solutions for  $C$  and  $D$  from Equation 7.17 and 7.18, and using the shorthand  $R^\pm$ , the released fiber load  $\Delta Q_f$  then follows from Equation 4.34 or Equation 4.35,

$$\Delta Q_f = w_f \left( \left( c_{3,Q}^{(-)} - c_{4,Q}^{(-)} R^{(-)} \right) \cos \beta x + \left( c_{7,Q}^{(-)} - c_{8,Q}^{(-)} R^{(-)} \right) \sin \beta x \right) e^{-\alpha x} \quad (7.24)$$

or

$$\Delta Q_f = w_f \left( \left( c_{3,Q}^{(+)} - c_{4,Q}^{(+)} R^{(+)} \right) \cosh \beta x + \left( c_{7,Q}^{(+)} - c_{8,Q}^{(+)} R^{(+)} \right) \sinh \beta x \right) e^{-\alpha x}, \quad (7.25)$$

where  $c_{3,Q}^{(\pm)}$ ,  $c_{4,Q}^{(\pm)}$ ,  $c_{7,Q}^{(\pm)}$  and  $c_{8,Q}^{(\pm)}$  can be found in Appendix A. The form of Equation 7.24 and 7.25 ensures that at increasing distance from the crack the load, given by Equation 7.11 gradually recovers towards

the far-field value,

$$Q_f(x) \rightarrow Q_{f,0}, \quad (7.26)$$

while at the crack, the fiber is unable to sustain any load:

$$Q_f(x=0) = 0, \quad (7.27)$$

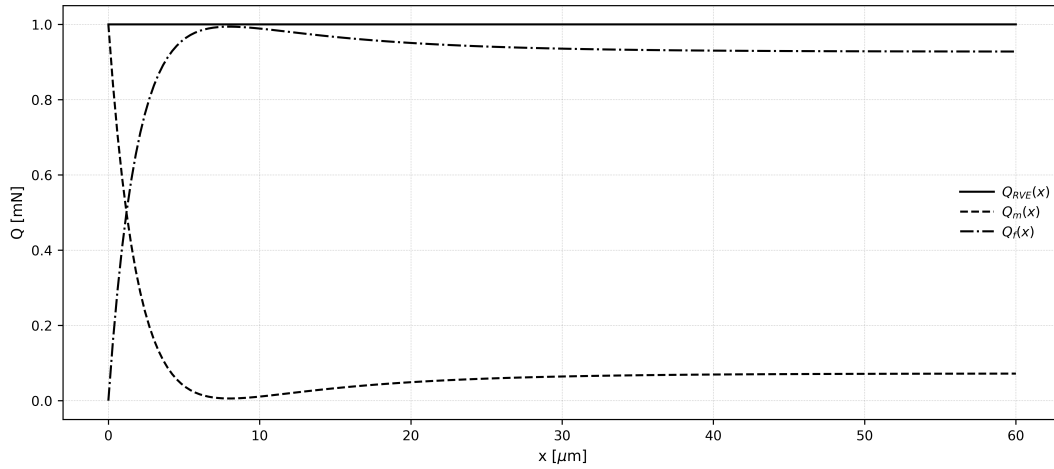
At the same time, the matrix is overloaded according to Equation 7.12, such that at the crack

$$Q_m(x=0) = Q_{RVE}, \quad (7.28)$$

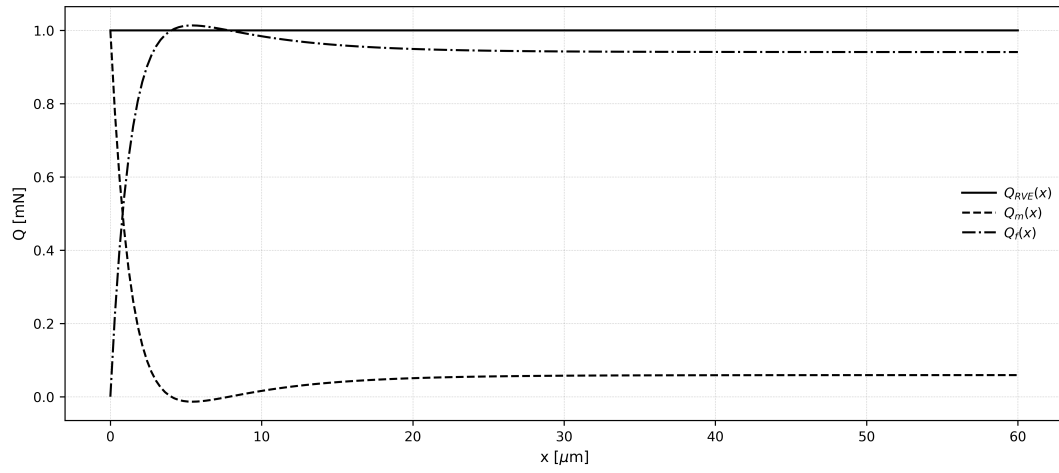
and gradually recovers towards its far field value,

$$Q_m(x) \rightarrow Q_{m,0}, \quad (7.29)$$

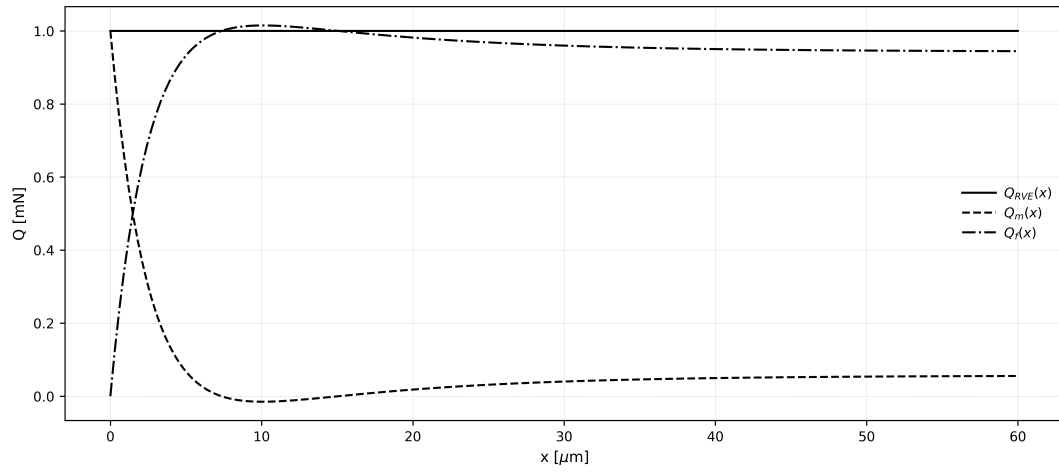
The load re-distribution results for the corrected material systems are shown in Figure 7.12-7.17. Although the results are similar accross the materials, one important feature is noticed: with the exception of AS4/3501-6, all material systems have a recovering  $Q_f(x)$  that locally overshoots the far-field applied  $Q_{RVE}$  load, with the glass-systems being most pronounced. Technically that means where  $Q_f(x) > Q_{RVE}$  another fiber crack would occur, causing a cascading failure. This would be the case for constant fiber shear strength  $S_f$  along the fiber length. However, as mentioned before, interpreting  $S_f$  as statistically distributed around a mean value among different fibers, it is also reasonable to assume that the local shear strength along a fiber varies as a function of distributed flaws. For now, the assumption will be made that no significant flaw is within close range of the flaw that caused the initial fiber crack. Another reason for doing so, is that such an overshoot of the recovering load is essentially a function of the effectiveness of the elastic foundation in representing the matrix for the loading considered. So far, no measure of that effectiveness can be quantified.



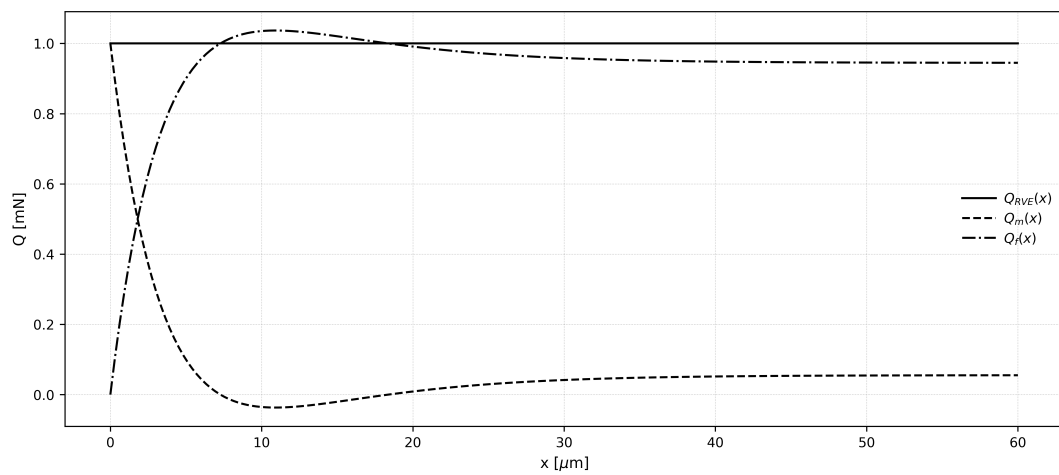
**Figure 7.12:** Load re-distribution between fiber and matrix upon fiber crack for AS4/3501-6,  $Q_{RVE} = 1mN$ .



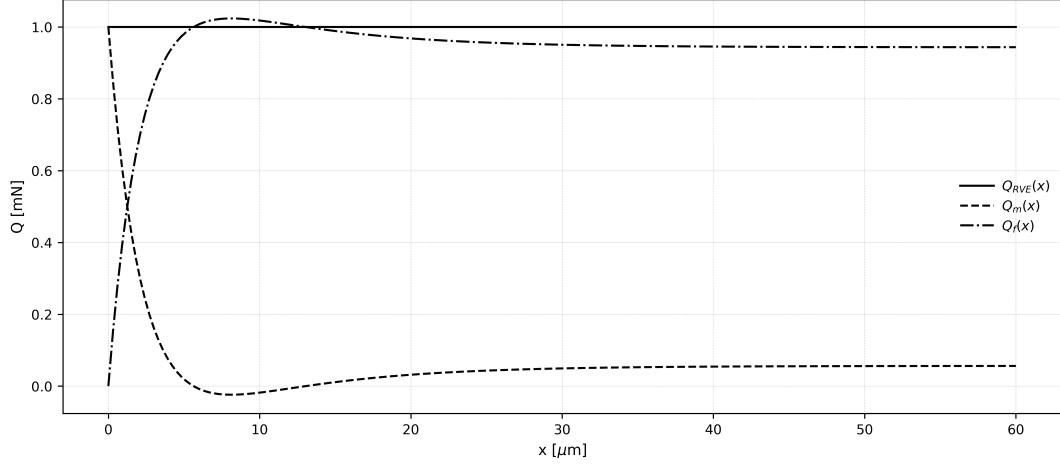
**Figure 7.13:** Load re-distribution between fiber and matrix upon fiber crack for IM7/8852,  $Q_{RVE} = 1\text{mN}$ .



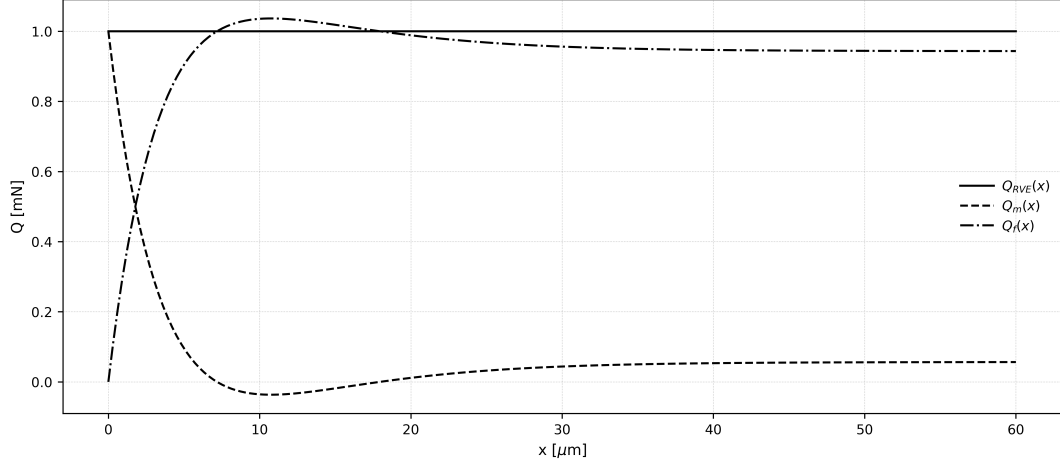
**Figure 7.14:** Load re-distribution between fiber and matrix upon fiber crack for G40-800/5260,  $Q_{RVE} = 1\text{mN}$ .



**Figure 7.15:** Load re-distribution between fiber and matrix upon fiber crack for Glass/LY556,  $Q_{RVE} = 1\text{mN}$ .



**Figure 7.16:** Load re-distribution between fiber and matrix upon fiber crack for Hercules AS/3501-6,  $Q_{RVE} = 1mN$ .



**Figure 7.17:** Load re-distribution between fiber and matrix upon fiber crack for O-C Glass/3501-6,  $Q_{RVE} = 1mN$ .

### 7.3.5. Interface shear stress distribution

Having established the overloading of the matrix by Equation 7.12, the corresponding shear stress over the cross-section will be used to approximate the shear stress at the fiber-matrix interface as

$$\tau_0(x) = \frac{Q_m(x)}{n_m A_m}, \quad (7.30)$$

where  $n_m$  denotes the matrix shear correction factor and  $A_m$  the effective matrix shear area, such that  $\tau_0(x)$  is a scaled  $Q_m(x)$  field as shown in Figure 7.12-7.17, retaining the same shape.

Substituting either expressions for  $Q_m(x)$  based on the sign of  $\Delta$  into Equation 7.30, the interface shear stress field is either

$$\tau_0^{(-)}(x) = \frac{Q_{RVE}}{n_m A_m} \left[ f_m + \frac{f_f e^{-\alpha x}}{(-\alpha - \beta R)I + (-\phi + \psi R)J} \left( (c_{3,Q}^{(-)} - c_{4,Q}^{(-)} R^{(-)}) \cos \beta x + (c_{7,Q}^{(-)} - c_{8,Q}^{(-)} R^{(-)}) \sin \beta x \right) \right], \quad (7.31)$$

or

$$\tau_0^{(+)}(x) = \frac{Q_{RVE}}{n_m A_m} \left[ f_m + \frac{f_f e^{-\alpha x}}{(-\alpha - \beta R)I + (-\phi + \psi R)J} \left( (c_{3,Q}^{(+)} - c_{4,Q}^{(+)} R^{(+)}) \cosh \beta x + (c_{7,Q}^{(+)} - c_{8,Q}^{(+)} R^{(+)}) \sinh \beta x \right) \right]. \quad (7.32)$$

The spatial variation of  $\tau_0(x)$  governs the initiation of matrix cracking mechanisms and forms the starting point for the radial stress field analysis presented in the following section. As  $\tau_0(x)$  also evolves towards a far field value, the redistribution of shear load following fiber fracture may be characterised by an effective length scale  $d_{\text{eff}}$ , defined as the axial distance over which the excess matrix stress becomes negligible according to a prescribed criterion,

$$\tau_0(x = d_{\text{eff}}) = \tau_{\text{threshold}}, \quad (7.33)$$

where  $\tau_{\text{threshold}}$  is defined as a near far-field matrix stress. In the present work,

$$\tau_{\text{threshold}} = 0.99 \tau_{\text{farfield}}. \quad (7.34)$$

Due to the nature of Equation 7.31 and 7.32, no closed form solution exists for  $d_{\text{eff}}$  and is therefore numerically solved for.

The effective length  $d_{\text{eff}}$  thus represents the spatial extent over which the local perturbation induced by fiber fracture influences the deformation of the RVE, as is conceptually demonstrated in Figure 7.9.

## 7.4. Matrix cracking due to fiber crack

Following the formation of a fiber crack, the local loss of load-carrying capacity in the fiber must be compensated by an increased load transfer to the surrounding matrix. As established in Section 7.3, this redistribution results in an elevated shear stress at the fiber–matrix interface. The resulting stress field in the matrix governs the initiation and evolution of matrix cracking mechanisms.

### 7.4.1. Matrix stress field induced by fiber crack

Having determined the redistribution of shear load following fiber fracture, the matrix shear stress field is now evaluated. At the fiber–matrix interface, the matrix experiences an increased shear stress  $\tau_0(x)$ , which varies along the fiber axis as the load carried by the fiber gradually recovers to its farfield value.

To determine the radial distribution of this stress, an intermediate cylindrical coordinate system  $(r, \theta, x)$  is introduced, aligned with the fiber axis. Under the assumptions that

- the shear stress field is axisymmetric, i.e.  $\partial(\cdot)/\partial\theta = 0$ ,
- axial normal stresses in the matrix are negligible for  $r > r_f$  due to  $E_f \gg E_m$ , where  $r_f$  denotes the fiber radius, i.e.  $r_f = d_f/2$ .

the axial equilibrium equation in cylindrical coordinates [86]

$$\frac{\partial \tau_{rx}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta x}}{\partial \theta} + \frac{\partial \sigma_x}{\partial x} + \frac{\tau_{rx}}{r} = 0, \quad (7.35)$$

reduces to

$$\frac{\partial \tau_{rx}}{\partial r} + \frac{\tau_{rx}}{r} = 0. \quad (7.36)$$

This first-order ordinary differential equation can be solved by separation of variables, yielding

$$\tau_{rx}(r) = \frac{C_0}{r}, \quad (7.37)$$

where  $C_0$  is determined from the boundary condition at the fiber–matrix interface.

Let  $\tau_0(x)$  denote the shear stress at the interface,

$$\tau_0(x) = \tau(x, r_f). \quad (7.38)$$

Substituting  $r = r_f$  into Equation 7.37 gives

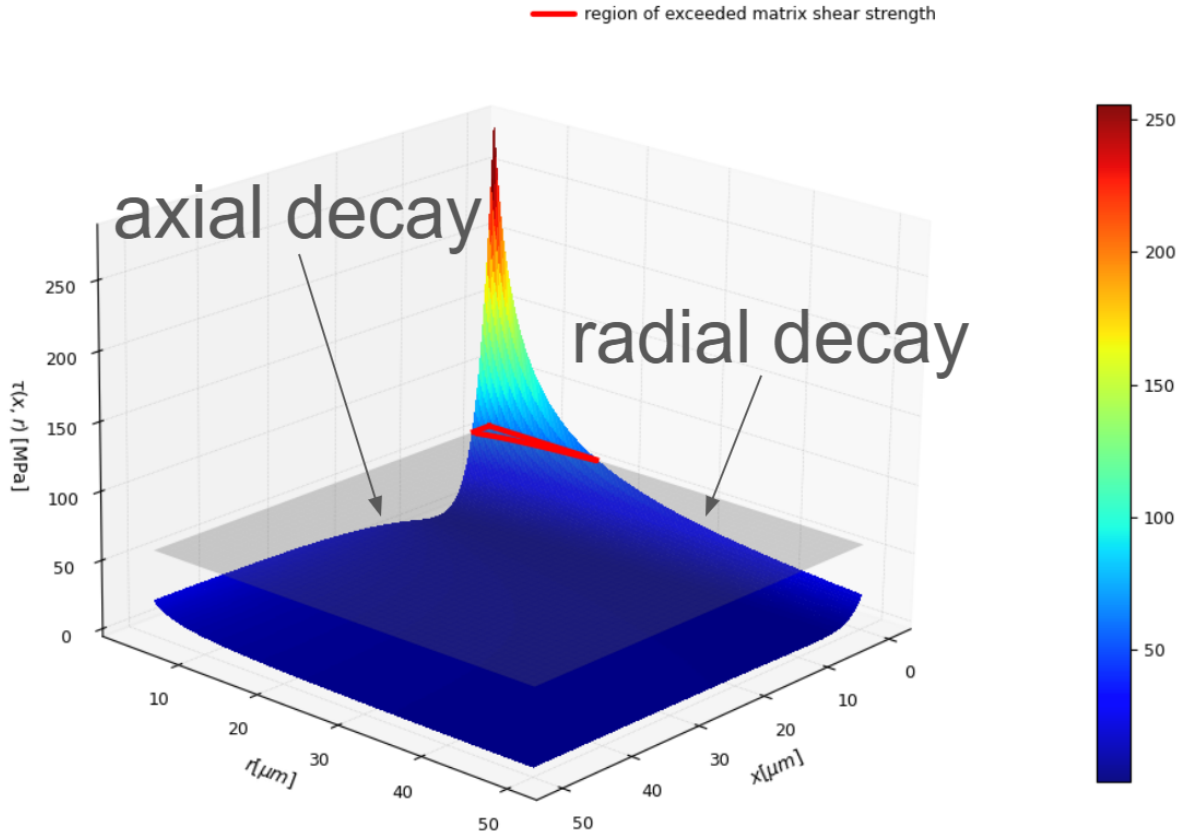
$$\tau_0(x) = \frac{C_0}{r_f}, \quad (7.39)$$



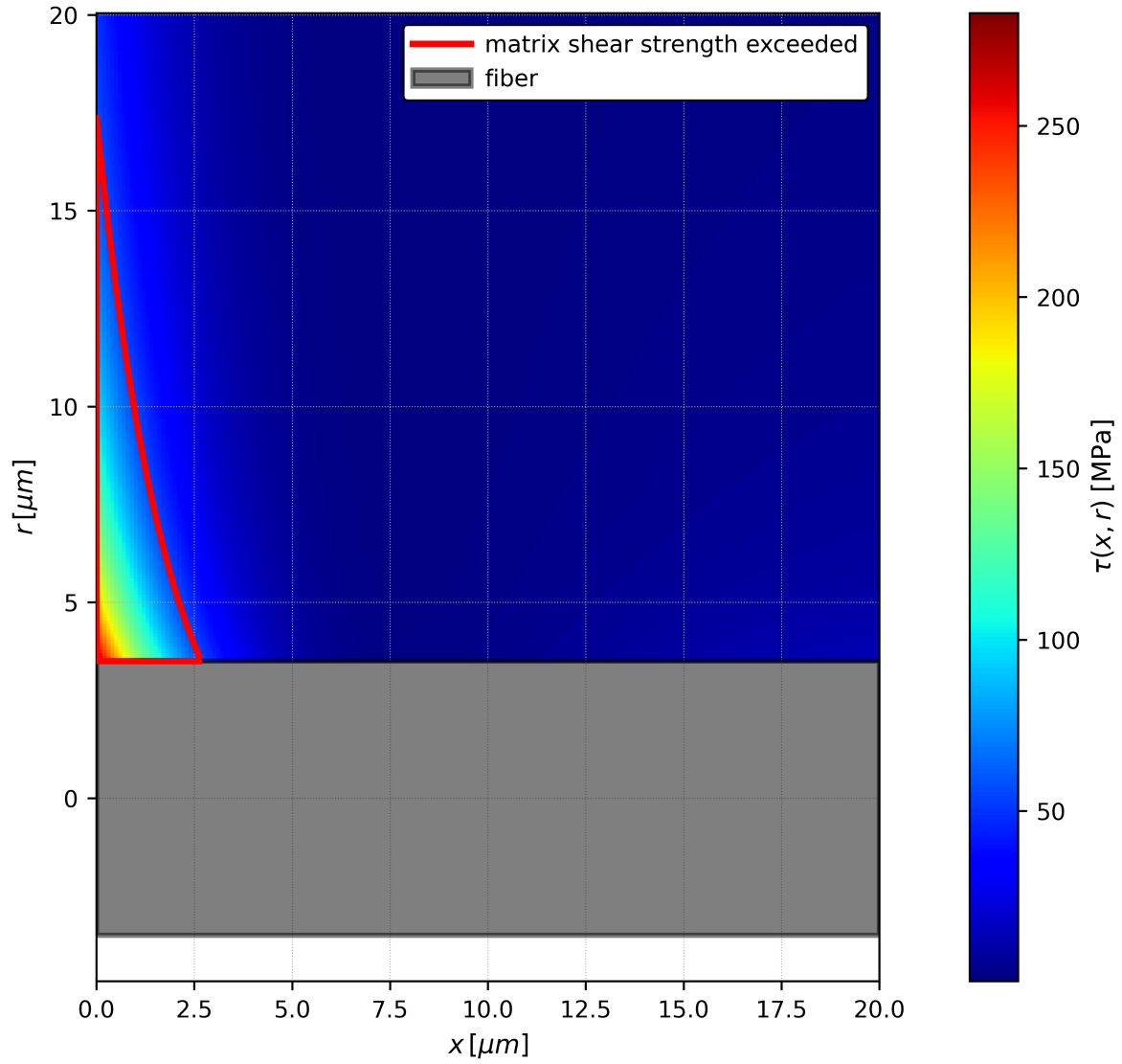
so that  $C_0 = \tau_0(x) r_f$ . The shear stress in the surrounding matrix then becomes

$$\tau(x, r) = \tau_0(x) \frac{r_f}{r}. \quad (7.40)$$

An example shear stress field in the matrix is shown in Figure 7.18, highlighting both the axial decay along the fiber and the radial decay outwards.



**Figure 7.18:** Shear stress field  $\tau(x, r)$



**Figure 7.19:** Detailed view of shear stress field  $\tau(x, r)$  exceeding the matrix strength  $S_m$

#### 7.4.2. Radial extent of the overloaded matrix region

Matrix failure is assumed to occur when the magnitude of the shear stress exceeds the matrix shear strength  $S_m$ ,

$$|\tau(x, r)| > S_m. \quad (7.41)$$

For a given axial position  $x$ , the radial extent of the region in which the matrix strength is exceeded is therefore obtained from

$$|\tau_0(x)| \frac{r_f}{r} = S_m, \quad (7.42)$$

leading to

$$r_{\text{over}}(x) = r_f \frac{|\tau_0(x)|}{S_m}. \quad (7.43)$$

The overloaded matrix region thus occupies the annular domain

$$r \in [r_f, r_{\text{over}}(x)]. \quad (7.44)$$

An example of the shear stress field exceeding the matrix strength is shown in Figure 7.19 by highlighting  $r_{\text{over}}(x)$ , where the fiber geometry is added for additional clarity.

### 7.4.3. Excess strain energy in the matrix

The shear strain energy density in the matrix is given by

$$u(x, r) = \frac{\tau(x, r)^2}{2G_m}. \quad (7.45)$$

To quantify only the energy stored beyond the matrix shear strength, an excess energy density is introduced as

$$u_{\text{excess}}(x, r) = \begin{cases} \frac{\tau(x, r)^2 - S_m^2}{2G_m}, & |\tau(x, r)| > S_m, \\ 0, & |\tau(x, r)| \leq S_m. \end{cases} \quad (7.46)$$

Using

$$\tau(x, r) = \tau_0(x) \frac{r_f}{r}, \quad (7.47)$$

the radial boundary of the overloaded region follows from

$$|\tau(x, r_{\text{over}})| = S_m, \quad (7.48)$$

such that

$$r_{\text{over}}(x) = \frac{|\tau_0(x)| r_f}{S_m}. \quad (7.49)$$

The excess strain energy stored in the overloaded matrix region on one side of the broken fiber is then

$$E_{\text{excess}} = \int_0^{x_S} \int_{r_f}^{r_{\text{over}}(x)} u_{\text{excess}}(x, r) 2\pi r dr dx. \quad (7.50)$$

where  $x_S$  is such that  $|\tau_0(x_S)| = S_m$ . As with  $d_{\text{eff}}$ , no closed form solution exist for  $x_S$  and is therefore numerically solved for.

Substituting the stress field gives

$$E_{\text{excess}} = \frac{\pi}{G_m} \int_0^{x_S} \int_{r_f}^{r_{\text{over}}(x)} \left( \frac{\tau_0(x)^2 r_f^2}{r} - S_m^2 r \right) dr dx. \quad (7.51)$$

Carrying out the radial integration yields

$$E_{\text{excess}} = \frac{\pi}{G_m} \int_0^{x_S} \left[ \tau_0(x)^2 r_f^2 \ln \left( \frac{r_{\text{over}}(x)}{r_f} \right) - \frac{S_m^2}{2} (r_{\text{over}}(x)^2 - r_f^2) \right] dx. \quad (7.52)$$

Using  $r_{\text{over}}(x) = |\tau_0(x)| r_f / S_m$ , this becomes

$$E_{\text{excess}} = \frac{\pi r_f^2}{G_m} \int_0^{x_S} \left[ \tau_0(x)^2 \ln \left( \frac{|\tau_0(x)|}{S_m} \right) - \frac{1}{2} (\tau_0(x)^2 - S_m^2) \right] dx, \quad |\tau_0(x)| > S_m. \quad (7.53)$$

Introducing the threshold condition explicitly, the final expression may be written as

$$E_{\text{excess}} = \frac{\pi r_f^2}{G_m} \int_0^{x_S} \begin{cases} \tau_0(x)^2 \ln \left( \frac{|\tau_0(x)|}{S_m} \right) - \frac{1}{2} (\tau_0(x)^2 - S_m^2), & |\tau_0(x)| > S_m, \\ 0, & |\tau_0(x)| \leq S_m. \end{cases} dx. \quad (7.54)$$

Once more due to the nature of Equation 7.31 and 7.32, no closed form solution exists for  $E_{\text{excess}}$  either, due to presence of the logarithmic term.

#### 7.4.4. Matrix cracking mechanisms

The excess strain energy generated in the overloaded matrix region provides a driving force for the initiation and growth of matrix cracks and is used to determine crack sizes. This method has been successfully applied by Socci and Kassapoglou [7] to the case of UD plies under tensile loading. Two primary cracking modes are considered [7]:

- interfacial cracking, characterised by debonding along the fiber–matrix interface over a length  $L_{\text{int}}$ , schematically shown in Figure 7.20
- radial cracking, characterised by matrix fracture propagating outward from the fiber surface over a length  $L_{\text{rad}}$ , schematically shown in Figure 7.21.

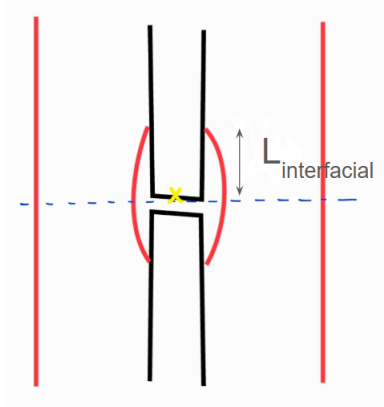


Figure 7.20: Interfacial matrix crack geometry

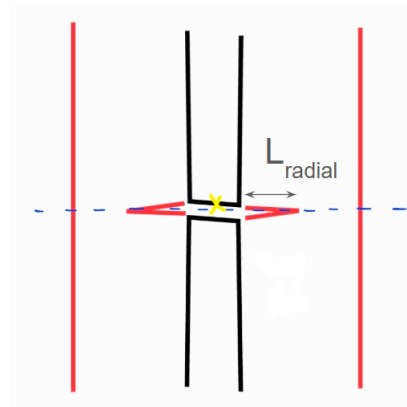


Figure 7.21: Radial matrix crack geometry

Denoting by  $\Gamma_{\text{int}}$  and  $\Gamma_{\text{rad}}$  the fracture energy per unit area of interfacial and radial crack respectively, the energy to create one of these crack types is set equal to the excess energy in the matrix due to the fiber crack:

$$E_{\text{excess}} = \begin{cases} \pi d_f L_{\text{int}} \Gamma_{\text{int}}, \\ \frac{\pi}{4} \left[ (d_f + 2L_{\text{rad}})^2 - d_f^2 \right] \Gamma_{\text{rad}}. \end{cases} \quad (7.55)$$

where the geometry involved is illustrated in detail in Figure 7.22.

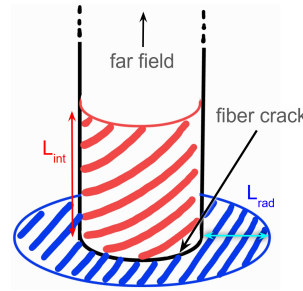


Figure 7.22: Crack lengths  $L_{\text{int}}$  and  $L_{\text{rad}}$  corresponding to an interfacial and radial crack respectively.

Solving Equation 7.55 for  $L_{\text{int}}$  and  $L_{\text{rad}}$  and setting  $\Gamma_{\text{int}} = \Gamma_{\text{rad}} = G_{II}$ , which is the Mode II strain energy release rate for the considered UD systems, the crack lengths are

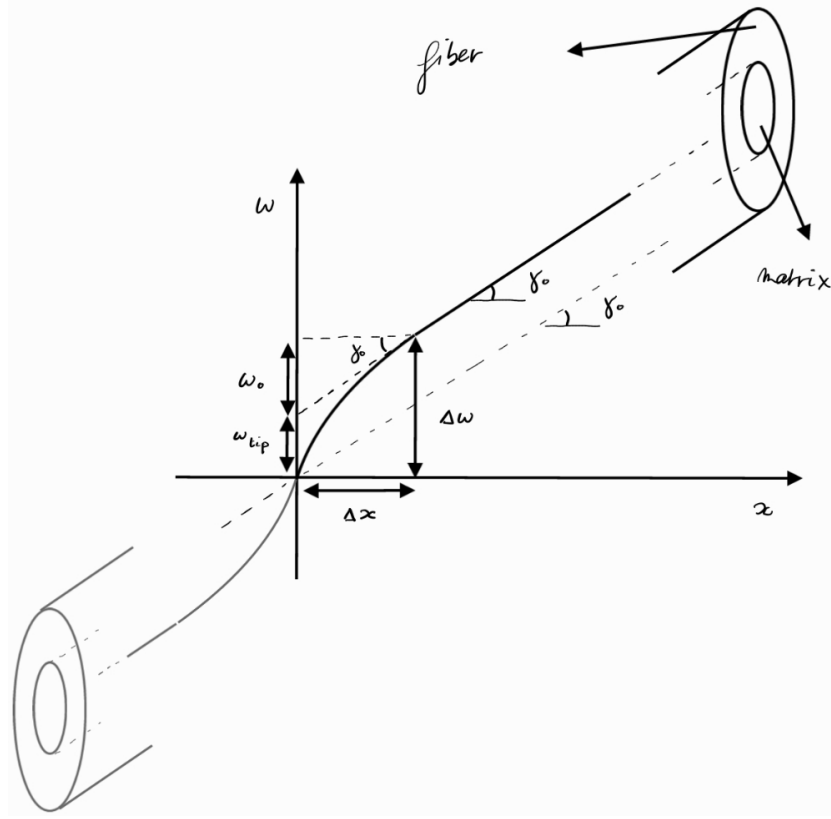
$$L_{\text{int}} = \frac{E_{\text{excess}}}{\pi d_f G_{II,m}}, \quad (7.56)$$

and

$$L_{\text{rad}} = \frac{1}{2} \left( \sqrt{\frac{4E_{\text{excess}}}{\pi G_{II,m}} + d_f^2} - d_f \right). \quad (7.57)$$

## 7.5. Effective RVE concept

Previous sections have covered how a fiber crack introduces a perturbation to the otherwise uniform shear deformation field. While the far field loading remains unchanged, the redistribution of load from the fractured fiber to the surrounding matrix results in an increased local compliance in the vicinity of the crack. This local compliance modifies the displacement field over a finite axial extent, quantified by  $d_{\text{eff}}$ , beyond which the deformation returns the uniform far field state. A conceptual view of displacement fields as shown in Figure 7.11, is presented in Figure 7.23.



**Figure 7.23:** General displacement of fiber axis due to fiber break.

To account for this behaviour within a homogenised framework, the concept of an *effective RVE* is introduced. This effective RVE represents the portion of the microstructure over which the post-fracture perturbation significantly influences the deformation field. In the most general case, this effective RVE spans the distance  $\Delta x$  starting from the fiber crack, as shown in Figure 7.23.

As before, let the far-field shear strain prior to fracture be denoted by  $\gamma_{12,0}$  and thus  $\Delta x = d_{\text{eff}}$ . Over the effective length  $d_{\text{eff}}$ , the corresponding uniform far-field displacement is

$$w_0 = \gamma_{12,0} d_{\text{eff}}, \quad (7.58)$$

as constructed in Figure 7.23. Denoting the additional displacement at the crack location by  $w_{\text{tip}}$ , as shown in Figure 7.23, the average shear strain over the effective RVE may be approximated as

$$\gamma_{\text{eff}} \approx \frac{\Delta w}{\Delta x} = \frac{w_0 + w_{\text{tip}}}{d_{\text{eff}}} = \gamma_{12,0} + \frac{w_{\text{tip}}}{d_{\text{eff}}}. \quad (7.59)$$

where  $w_{\text{tip}} = w_f$  from Equation 7.23. This relation shows that fiber fracture leads to an increase in the local average strain relative to the far-field value, causing the local increase in compliance.

This local increase in compliance is now referred to as a local decrease in stiffness instead. Subsequently, the degradation of the effective stiffness of the RVE is represented by a modified displacement-load relations by averaging the post-damage deformation over the effective length  $d_{\text{eff}}$ .

First, for an RVE which does not contain a fiber crack and is henceforth referred to as an unbroken RVE, let the externally applied shear load be denoted by  $Q_{RVE}$  as before, such that the displacement-load relation is

$$Q_{RVE, \text{unbroken}} = \frac{K_{RVE}}{f_f} \frac{w_0}{d_{\text{eff}}}, \quad (7.60)$$

where  $K_{RVE}$  is the stiffness of the intact RVE derived previously. Following damage initiation, additional local displacements modify this relation, resulting in degraded stiffness states which are characterised below.

As discussed in subsection 7.3.2 and implemented in 7.3.3, the far field displacement of the RVE undergoes a small but finite shift. Thus, as illustrated in Figure 7.23,

$$w_{0, \text{broken}} = w_0 + w_{\text{tip}} = w_0 + w_f \quad (7.61)$$

Substituting in Equation 7.60 and 7.23 into Equation 7.61 and denoting by  $Q_{RVE, \text{broken}}$  the far field load that locally caused a fiber crack:

$$w_{0, \text{broken}} = f_f \left( \frac{Q_{RVE, \text{broken}}}{K_{RVE}} d_{\text{eff}} + \frac{Q_{RVE, \text{broken}}}{(-\alpha - \beta R)I + (-\phi + \psi R)J} \right), \quad (7.62)$$

so that

$$Q_{RVE, \text{broken}} = \frac{1}{f_f} \frac{w_{0, \text{broken}}}{\frac{d_{\text{eff}}}{K_{RVE}} + \frac{1}{(-\alpha - \beta R)I + (-\phi + \psi R)J}}. \quad (7.63)$$

Since

$$\frac{1}{\frac{d_{\text{eff}}}{K_{RVE}}} > \frac{1}{\frac{d_{\text{eff}}}{K_{RVE}} + \frac{1}{(-\alpha - \beta R)I + (-\phi + \psi R)J}}, \quad (7.64)$$

it follows that, for a given  $w_0 = w_{0, \text{broken}}$ ,

$$Q_{RVE, \text{unbroken}} > Q_{RVE, \text{broken}}. \quad (7.65)$$

Thus, for the same applied displacement, the unbroken fiber carries more load, consistent with the expected stiffness reduction after fiber fracture.

## 7.6. Effective RVE for the interfacial crack

From a kinematic perspective, the presence of an interfacial crack may be interpreted as an extension of the post-fracture bending-type deformation of the fiber derived in section 7.3, as shown in Figure 7.24 so that the average shear strain over the effective RVE may be approximated as

$$\gamma_{\text{eff}, \text{int}} \approx \frac{\Delta w}{\Delta x} = \frac{w_0 + w_{\text{tip}}}{\Delta x} = \gamma_{12,0} \left( \frac{1}{1 + L_{\text{int}}/d_{\text{eff}}} \right) + \frac{w_{\text{tip}}}{d_{\text{eff}} + L_{\text{int}}}. \quad (7.66)$$

where,

$$w_{\text{tip}} = w_f + w_{\text{int}}. \quad (7.67)$$

Equation 7.61 can now be adjusted to account for the interfacial crack such that

$$w_{0, \text{broken}, \text{int}} = w_0 + w_{\text{tip}} = w_0 + w_f + w_{\text{int}} \quad (7.68)$$

Here it is assumed that the additional displacement due to presence of the interfacial crack can be approximated by a one-dimensional beam solution for the matrix portion of length  $L_{\text{int}}$  that carries the

total load  $Q_{RVE}$  since the fiber is traction free for the distance of  $L_{int}$  starting from the fiber crack. Therefore [43]

$$w_{int} = Q_{RVE} \left( \frac{L_{int}^3}{3E_m I_m} + \frac{L_{int}}{n_m G_m A_m} \right). \quad (7.69)$$

Substituting in Equation 7.60, 7.23 and 7.69 into Equation 7.68 and denoting by  $Q_{RVE, broken, int}$  the far field load that locally caused a fiber crack and subsequent interfacial matrix crack:

$$w_{0, broken, int} = Q_{RVE, broken, int} \left( \frac{f_f}{K_{RVE}} d_{eff} + \frac{f_f}{(-\alpha - \beta R)I + (-\phi + \psi R)J} + \frac{L_{int}^3}{3E_m I_m} + \frac{L_{int}}{n_m G_m A_m} \right), \quad (7.70)$$

Hence,

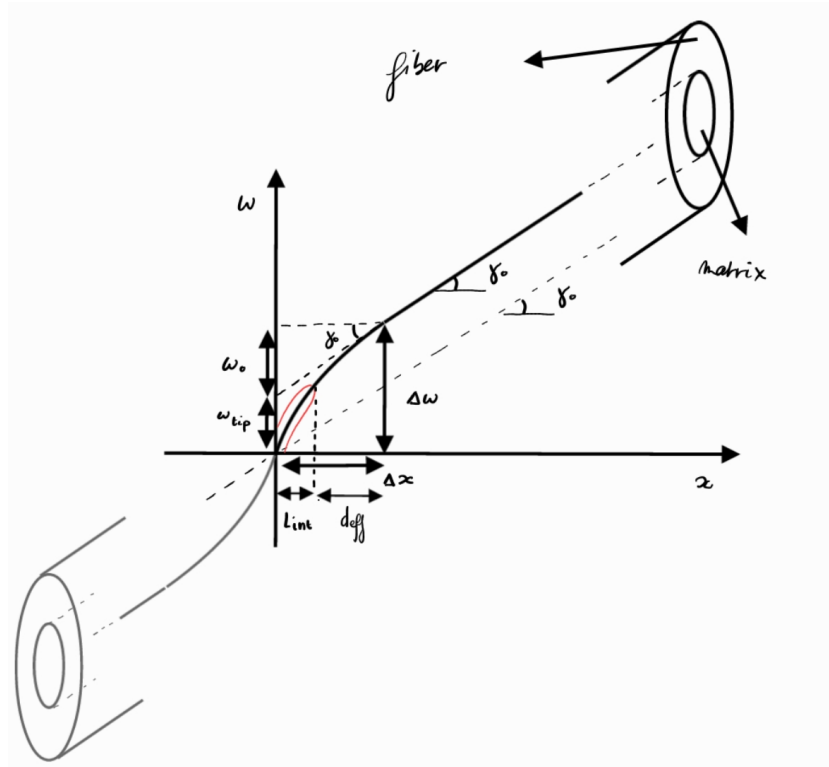
$$Q_{RVE, broken, int} = \frac{w_{0, broken, int}}{\left( \frac{f_f}{K_{RVE}} d_{eff} + \frac{f_f}{(-\alpha - \beta R)I + (-\phi + \psi R)J} + \frac{L_{int}^3}{3E_m I_m} + \frac{L_{int}}{n_m G_m A_m} \right)} \quad (7.71)$$

The denominator therefore increases further, implying that less load is carried by the broken fiber when interfacial cracking is included.

Note that

$$L_{int} = L_{int}(Q_{RVE}), \quad (7.72)$$

so the response is no longer strictly linear.



**Figure 7.24:** Displacement due to fiber break and interfacial crack. Debonded length is shown in red.

## 7.7. Effective RVE for the radial crack

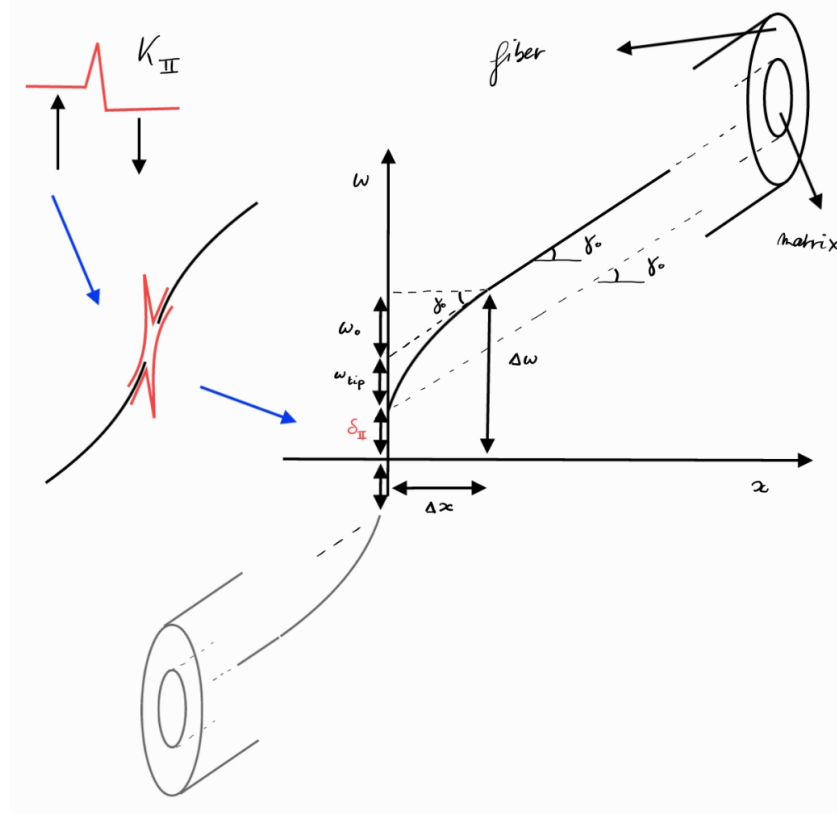
Unlike interfacial cracks, radial cracks introduce an opening displacement within the matrix, resulting in an additional local compliance that is not directly associated with fiber bending. This mechanism may therefore be represented by a crack-opening displacement term, denoted here by  $\delta_{II}$ , as shown in Figure 7.25.

The corresponding effective shear strain over the effective RVE becomes

$$\gamma_{\text{eff,rad}} \approx \frac{\Delta w}{\Delta x} = \frac{w_0 + w_{\text{tip}}}{\Delta x} = \gamma_{12,0} + \frac{w_{\text{tip}}}{d_{\text{eff}}}. \quad (7.73)$$

where

$$w_{\text{tip}} = w_f + \delta_{II} \quad (7.74)$$



**Figure 7.25:** Displacement due to fiber break and radial crack

In analogy with Socci and Kassapoglou [7],  $\delta_{II}$  is interpreted as the mode II crack opening displacement associated with transversely shearing a hollow, thick-walled cylinder as shown in Figure 7.26. Socci and Kassapoglou [7] used an analytical solution from Nied and Erdogan [99] of the form

$$\delta_{I,Nied} = k_0 \left( \frac{1 - \nu_m}{G_m} \right) \left( s - \frac{d_f}{2} \right) \sigma_0 \quad (7.75)$$

where  $\sigma_0$  is the far-field applied normal stress,  $s$  the center-to-center fiber distance of Equation 4.1 and  $k_0$  is a tabulated constant depending on the geometry and the numerical solution process involved. A graphical comparison is made in Figure 7.26.



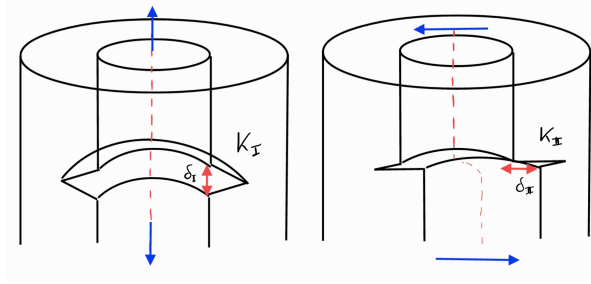


Figure 7.26: Mode I vs Mode II on thick-walled cylinders.

The complexity of the solution derived by Nied and Erdogan [99] motivated the use of an approximation instead, based on 2D solutions from Paris [100]:

$$\delta_{I,Paris} = \frac{4L_r}{E} V_1(L_r/s) \sigma_0 \quad (7.76)$$

$$\delta_{II,Paris} = \frac{4L_r}{E} V_1(L_r/s) \tau_0 \quad (7.77)$$

where

$$V_1\left(\frac{L_r}{s}\right) = \frac{1}{\left(\frac{\pi L_r}{2s}\right)} \left\{ 0.454 \sin\left(\frac{\pi L_r}{2s}\right) - 0.065 \left[\sin\left(\frac{\pi L_r}{2s}\right)\right]^3 - 0.007 \left[\sin\left(\frac{\pi L_r}{2s}\right)\right]^5 + \cosh^{-1} \left[ \sec\left(\frac{\pi L_r}{2s}\right) \right] \right\} \quad (7.78)$$

The associated geometries of these 2D solutions are shown in Figure 7.27 and 7.28.

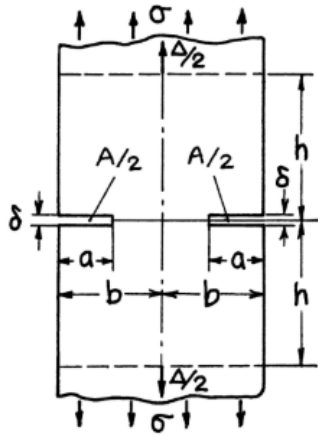


Figure 7.27: Geometry corresponding to 2D mode I crack opening

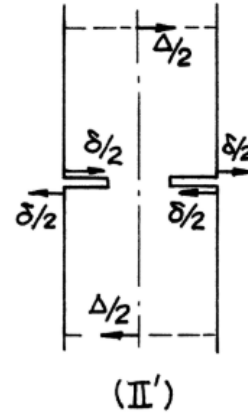


Figure 7.28: Geometry corresponding to 2D mode II crack opening

Now, to approximate the crack opening displacement  $\delta_{II}$  as shown in Figure 7.26, it is assumed that

$$\frac{\delta_{I,Nied}}{\delta_{II,Nied}} \approx \frac{\delta_{I,Paris}}{\delta_{II,Paris}}, \quad (7.79)$$

such that the following approximation can be constructed:

$$\delta_{II,Nied} \approx k_0 \left( \frac{1 - \nu_m}{G_m} \right) \left( s - \frac{d_f}{2} \right) \tau_0 \quad (7.80)$$

where

$$\tau_0 = \frac{Q_{RVE}}{n_m A_m} \quad (7.81)$$

The value  $k_0$  is a function of fiber geometry, fiber spacing and radial crack size:

$$k_0 = f(L_{rad}(Q_{RVE}), s, d_f), \quad (7.82)$$

and the values reported by Nied and Erdogan [99] are shown in Table 7.4.

For values of  $L_{rad}/s$  and  $d_f/2s$  that do not coincide with the tabulated entries in Table 7.4, the corresponding value of  $k_0$  is obtained by linear interpolation between the nearest available data points. Interpolation is performed independently in both directions, effectively resulting in a bilinear interpolation over the tabulated grid.

When either  $L_{rad}/s$  or  $d_f/2s$  falls outside the range covered by the table, the value is clipped to the nearest available bound. That is, values below the minimum tabulated range are set to the minimum entry, and values above the maximum range are set to the maximum entry. This ensures that no extrapolation beyond the provided data is introduced.

**Table 7.4:** Tabulated values of  $k_0$  as a function of  $L_{rad}/s$  and  $d_f/2s$

| $\frac{L_{rad}}{s} \setminus \frac{d_f}{2s}$ | 0.0   | 0.1   | 0.2   | 0.3   | 0.4   | 0.5   | 0.6   | 0.7   | 0.8   | 0.9   | 1.0   |
|----------------------------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1                                          | 0.128 | 0.212 | 0.244 | 0.261 | 0.272 | 0.281 | 0.287 | 0.292 | 0.297 | 0.302 | 0.310 |
| 0.2                                          | 0.258 | 0.373 | 0.441 | 0.488 | 0.524 | 0.555 | 0.582 | 0.608 | 0.635 | 0.666 | 0.732 |
| 0.3                                          | 0.390 | 0.526 | 0.626 | 0.707 | 0.776 | 0.841 | 0.906 | 0.973 | 1.050 | 1.150 | 1.410 |
| 0.4                                          | 0.531 | 0.686 | 0.817 | 0.935 | 1.046 | 1.158 | 1.277 | 1.412 | 1.580 | 1.821 | 2.614 |
| 0.5                                          | 0.689 | 0.864 | 1.029 | 1.189 | 1.351 | 1.523 | 1.718 | 1.954 | 2.267 | 2.765 | 4.950 |
| 0.6                                          | 0.877 | 1.077 | 1.283 | 1.493 | 1.716 | 1.966 | 2.261 | 2.636 | 3.166 | 4.091 | 9.965 |

Equation 7.61 can now be adjusted to account for the radial crack such that

$$w_{0,broken,rad} = w_0 + w_{tip} = w_0 + w_f + \delta_{II} \quad (7.83)$$

Substituting in Equation 7.60, 7.23 and 7.80 into Equation 7.83 and denoting by  $Q_{RVE,broken,rad}$  the far field load that locally caused a fiber crack and subsequent radial matrix crack:

$$w_{0,broken,rad} = Q_{RVE,broken,rad} \left( \frac{f_f}{K_{RVE}} d_{eff} + \frac{f_f}{(-\alpha - \beta R)I + (-\phi + \psi R)J} + \frac{k_0}{n_m A_m} \left( \frac{1 - \nu_m}{G_m} \right) \left( s - \frac{d_f}{2} \right) \right), \quad (7.84)$$

Hence,

$$Q_{RVE,broken,rad} = \frac{w_{0,broken,rad}}{\left( \frac{f_f}{K_{RVE}} d_{eff} + \frac{f_f}{(-\alpha - \beta R)I + (-\phi + \psi R)J} + \frac{k_0}{n_m A_m} \left( \frac{1 - \nu_m}{G_m} \right) \left( s - \frac{d_f}{2} \right) \right)} \quad (7.85)$$

Again, the denominator increases further w.r.t to RVE containing just the fiber crack, implying that less load is carried by the broken fiber when radial cracking is included.

Similar to the interfacial cracking, the response is also no longer strictly linear as

$$k_0 = f(Q_{RVE}). \quad (7.86)$$

## 7.8. Results for considered material systems

In the previous sections, constitutive  $w$ - $Q_{RVE}$  relations were derived for the different states of the RVE, namely:

- the pristine (unbroken) RVE,
- the RVE containing a broken fiber,
- the RVE containing a broken fiber with either an interfacial crack or a radial crack.

Each of these states is associated with a distinct stiffness, reflecting the evolving load transfer mechanisms between fiber and matrix. In order to obtain a physically meaningful response for a given material system, these individual relations must be combined into a single load path that captures the sequence of damage events. Displacement control is assumed.

### 7.8.1. Construction of the RVE load path

The RVE response is constructed in a piecewise manner, based on two characteristic events:

1. Fiber failure: as mentioned in section 7.3, the mechanism of initial damage considered is a discrete fiber crack, transitioning the RVE from the pristine to a degraded state,
2. Matrix failure: an upper bound beyond which the RVE can no longer sustain load is defined, based on the far field matrix stress.

**Step 1: Pristine response** For small applied displacements, the RVE follows the linear relation derived for the unbroken configuration, given by Equation 7.60, which, as long as the fiber did not break, is valid whether constant  $Q$  or constant  $w$  is assumed.

**Step 2: Fiber failure and stiffness transition** Fiber failure occurs at the critical load  $Q_{\text{RVE,failure,f}}$ , i.e. the far-field RVE load causing the local failure of the fiber given by Equation 7.5, determined from the fiber strength  $S_f$ . At this point, the load path transitions from the pristine relation to the response of a broken-fiber RVE, given by Equation 7.63. This transition comes with a reduction in stiffness, reflecting the loss of load-carrying capacity of the RVE.

Depending on the damage mechanism considered, the post-failure response follows either:

- the broken-fiber relation given by Equation 7.63 (in case of no matrix cracking, i.e.  $L_{\text{int}}(Q_{\text{RVE,failure,f}}) = 0$  or  $L_{\text{rad}}(Q_{\text{RVE,failure,f}}) = 0$ ), or
- the broken-fiber relation with additional deformation due to interfacial or radial cracking, given by either Equation 7.71 or Equation 7.85.

**Step 3: Matrix failure cutoff** As the applied displacement increases further, the increased matrix shear stress  $\tau_0(x)$  given by Equation 7.31 or 7.32 obtains a far field stress that equals the matrix shear strength  $S_m$ , such that the length  $x_S$  over which the matrix has failed, starting from the fiber crack, runs over the full length of the fiber, resulting in catastrophic failure.

In the present formulation, this is implemented as a cutoff at this ultimate load:

$$Q_{\text{RVE}} \leq Q_{\text{RVE,matrix,ult}} = \frac{1}{f_m}(n_m A_m S_m) \quad (7.87)$$

Once  $Q_{\text{RVE,matrix,ult}}$  is reached, a vertical drop in the load is introduced in the  $w$ - $Q$  response, representing complete loss of load-carrying capacity of the RVE.

It is recognized here that a different failure cutoff for the radially cracked RVE could also be defined, such that  $L_{\text{rad}} \geq t_{\text{m,eq}}$ , and consider its effect on neighbouring RVEs, but is currently not implemented.

### 7.8.2. Resulting piecewise response

Combining the above steps, the RVE load path is defined by three regimes, regardless of matrix crack type:

For interfacial matrix cracking:

$$Q_{\text{RVE,int}}(w) = \begin{cases} \text{Equation 7.60} & Q_{\text{RVE}} < Q_{\text{RVE,failure,f}}, \\ \begin{cases} \text{Equation 7.63,} & \text{if } L_{\text{int}} = 0, \\ \text{Equation 7.71,} & \text{if } L_{\text{int}} > 0, \end{cases} & Q_{\text{RVE,failure,f}} \leq Q_{\text{RVE}} \leq Q_{\text{RVE,matrix,ult}}, \\ 0, & Q_{\text{RVE}} > Q_{\text{RVE,matrix,ult}}, \end{cases} \quad (7.88)$$

and for radial matrix cracking:

$$Q_{\text{RVE,rad}}(w) = \begin{cases} \text{Equation 7.60} & Q_{\text{RVE}} < Q_{\text{RVE,failure,f}}, \\ \begin{cases} \text{Equation 7.63,} & \text{if } L_{\text{rad}} = 0, \\ \text{Equation 7.85,} & \text{if } L_{\text{rad}} > 0, \end{cases} & Q_{\text{RVE,failure,f}} \leq Q_{\text{RVE}} \leq Q_{\text{RVE,matrix,ult}}, \\ 0, & Q_{\text{RVE}} > Q_{\text{RVE,matrix,ult}}. \end{cases} \quad (7.89)$$

This construction results in a characteristic response consisting of:

- an initial linear regime,
- a stiffness reduction at fiber failure,
- a continued, lower-stiffness loading phase governed by crack evolution,
- a final abrupt drop at matrix failure.

An example is shown in Figure 7.29 including both crack types.

### 7.8.3. Crack preference

Once the fiber cracks, the RVE has two loadpath options to follow, either the one associated with interfacial cracks or the one associated with radial cracks. To determine its preference, the minimization of the energy [7]

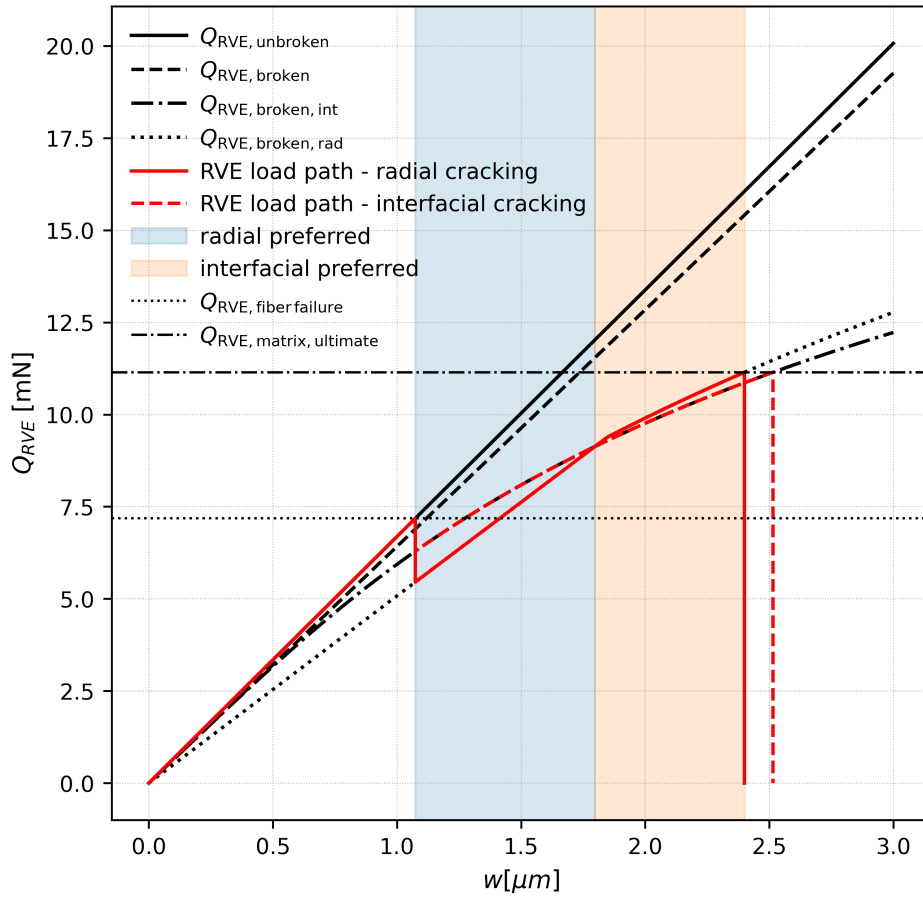
$$U = \int_0^w Q_{\text{RVE}}(w') dw', \quad (7.90)$$

is considered. Once  $w > w_{Q_{\text{RVE,failure,f}}}$ , an interfacial crack is preferred whenever

$$Q(w')_{\text{RVE,broken,int}} < Q(w')_{\text{RVE,broken,rad}}, \quad (7.91)$$

and a radial crack is preferred when

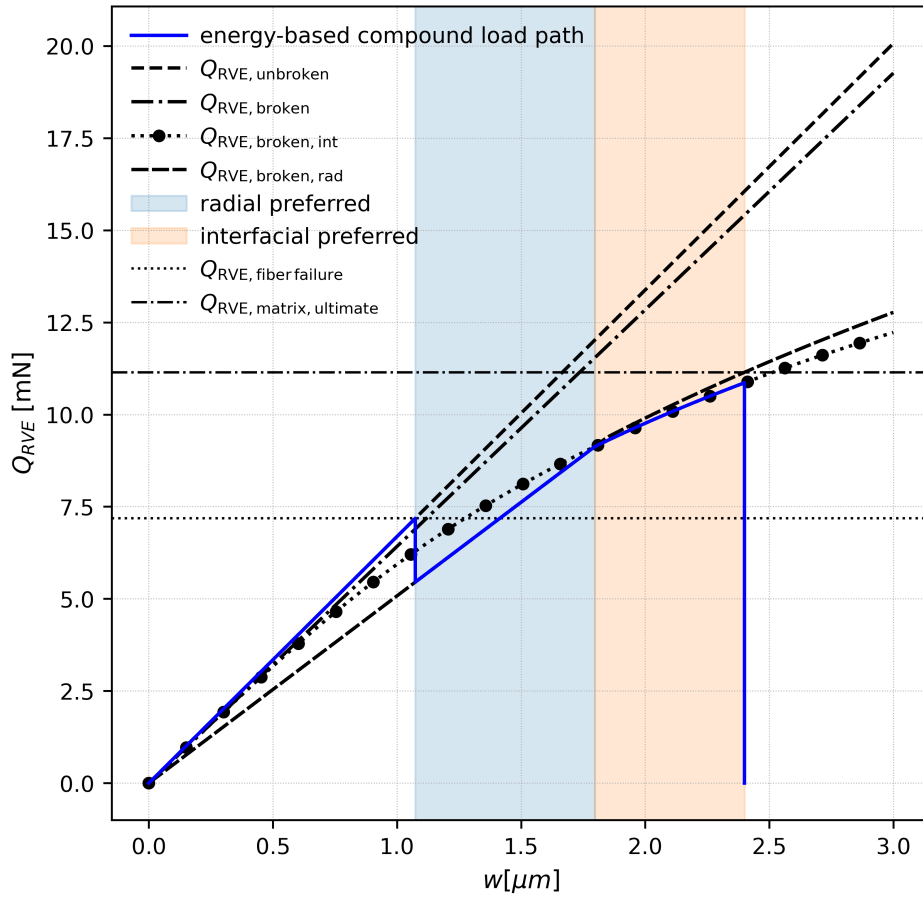
$$Q(w')_{\text{RVE,broken,rad}} < Q(w')_{\text{RVE,broken,int}}. \quad (7.92)$$



**Figure 7.29:** Piecewise RVE loadpath due to discrete fiber crack and matrix cracking for AS4/3501-6.

Regions of preferred cracking mechanisms are shown in Figure 7.29. In this case, in the current framework, the radial crack is initially preferred upon the breaking of the fiber, with a switch to interfacial cracks halfway between  $Q_{\text{RVE, failure, f}}$  and  $Q_{\text{RVE, matrix, ult}}$ , and a final switch when the RVE load path for radial cracking is cut off by  $Q_{\text{RVE, matrix, ult}}$ , as  $Q(w')_{\text{RVE, broken, rad}} = 0 < Q(w')_{\text{RVE, broken, int}}$ .

As this energy-based crack-switching mechanism is a function of the accuracy of the cracking models, with the radial cracking containing more approximations, results compared to just following either the interfacial crack or radial crack path is to be evaluated. A compound RVE load path based on crack preference is shown separately in Figure 7.30.

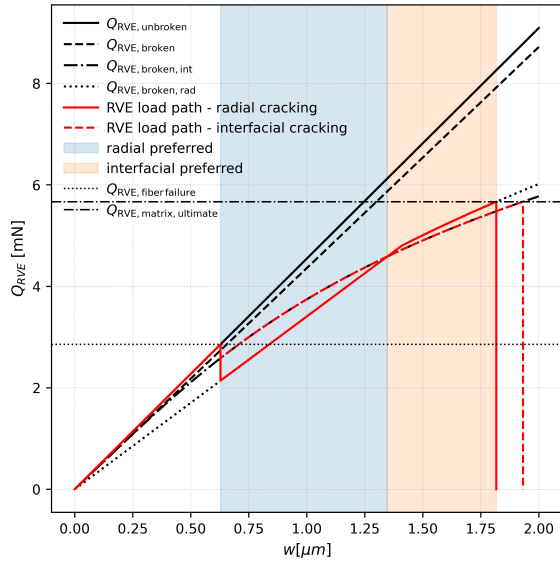


**Figure 7.30:** Piecewise compound RVE loadpath based on crack preference for AS4/3501-6.

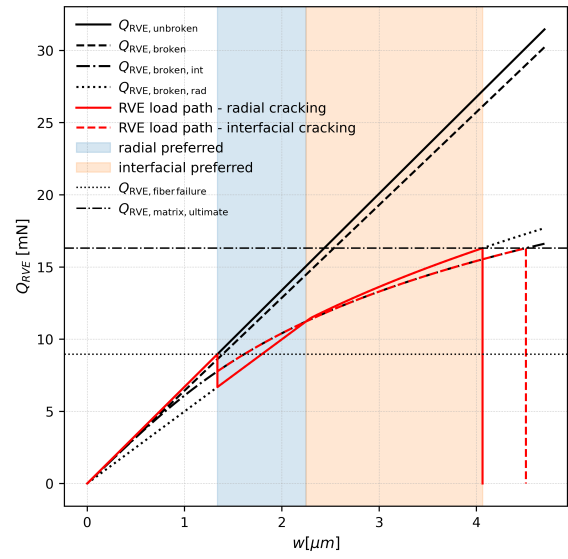
#### 7.8.4. Application to material systems

The above procedure of subsection 7.8.2 is applied to each considered material system and shown in Figure 7.31-Figure 7.35. AS4/3501-6 has already been considered in Figure 7.29. The same distinction between carbon and glass material systems as discussed in subsection 7.2.2 is apparent, and thus the current framework does not work for the glass-based material systems, since  $Q_{\text{RVE,failure,f}} > Q_{\text{RVE,failure,m}}$  and thus technically  $Q_{\text{RVE,matrix,ult}}$  is undefined. Another observation is that upon fiber failure, matrix cracking occurs immediately, as seen by the vertical drop in load settling on the nonlinear  $Q_{\text{RVE,broken,int}}(w)$  or  $Q_{\text{RVE,broken,rad}}(w)$  relation.

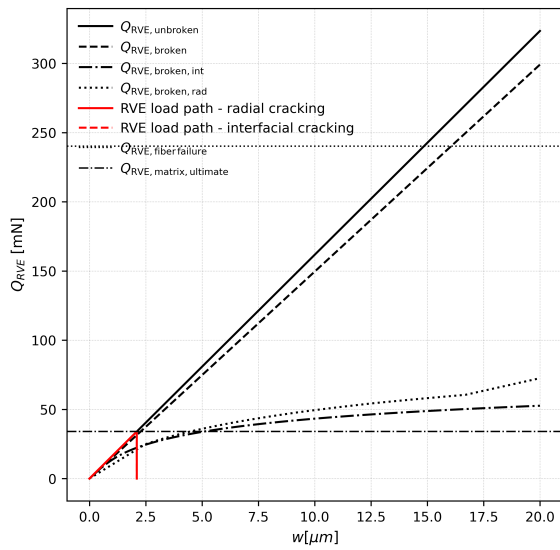
An important note is that all RVE loadpaths shown still assume the mean fiber shear strength  $S_f$ , as discussed in subsection 7.2.2 and reported in chapter 3. Variations of  $S_f$  will be covered in chapter 8.



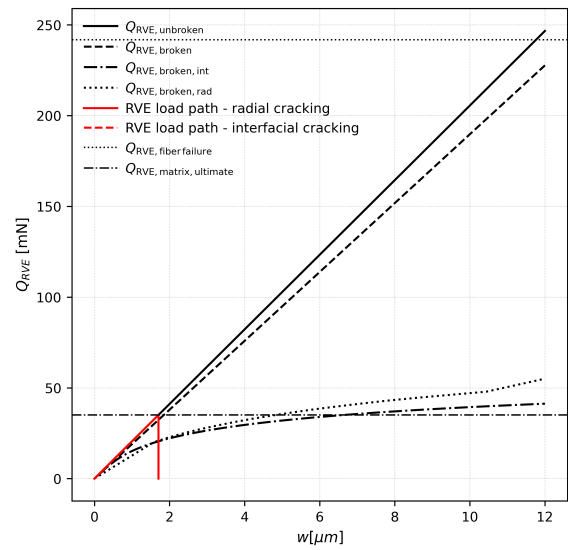
**Figure 7.31:** Piecewise RVE loadpath due to discrete fiber crack and matrix cracking for IM7/8552.



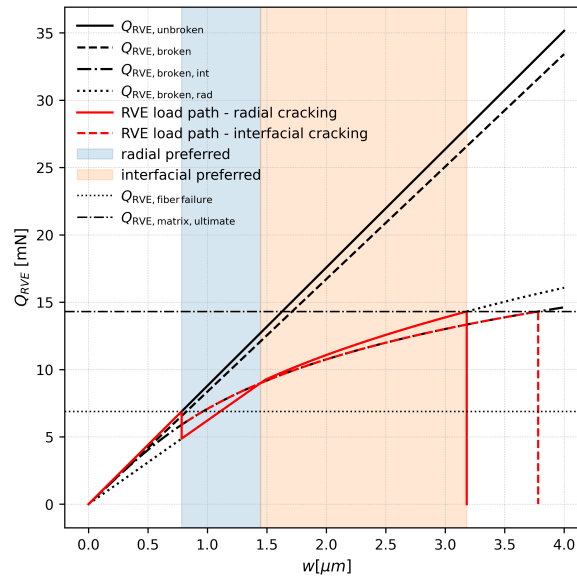
**Figure 7.32:** Piecewise RVE loadpath due to discrete fiber crack and matrix cracking for G40-800/5260.



**Figure 7.33:** Piecewise RVE loadpath due to discrete fiber crack and matrix cracking for Glass/LY556.



**Figure 7.34:** Piecewise RVE loadpath due to discrete fiber crack and matrix cracking for O-C Glass/3501-6



**Figure 7.35:** Piecewise RVE loadpath due to discrete fiber crack and matrix cracking for Hercules/3501-6

## 7.9. Summary and outlook

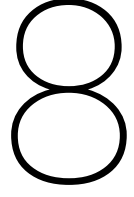
The principal findings of this chapter are:

- Constituent failure is governed by the relative strength-to-stiffness ratios of fiber and matrix, leading to fiber-dominated failure in carbon systems and matrix-dominated failure in glass systems;
- Fiber fracture causes a local redistribution of load to the surrounding matrix, resulting in elevated interfacial shear stresses that can trigger secondary matrix cracking mechanisms;
- Two distinct matrix cracking modes are identified, i.e. interfacial debonding and radial cracking, each introducing additional compliance through different physical mechanisms;
- These damage mechanisms can be consistently incorporated into the analytical framework through the concept of an effective RVE, enabling the definition of discrete degraded stiffness states based on local displacement contributions;
- By combining the individual constitutive responses of pristine, fiber-broken, and matrix-cracked RVEs, a piecewise load-displacement relation is constructed, capturing the sequence of damage events from initial loading to final failure;
- The resulting RVE load path exhibits three characteristic regimes: an initial linear response, a stiffness reduction upon fiber failure, and a final abrupt loss of load-carrying capacity at matrix failure, with intermediate behaviour governed by crack evolution and crack-type preference.

Together, these results establish the micromechanical degradation mechanism required to extend the corrected initial stiffness formulation toward prediction of stiffness evolution during loading.

While the present chapter defines the local failure mechanisms and associated degraded stiffness states, the macroscopic ply response depends on how these damaged RVEs accumulate statistically during loading but also how the matrix behaves. The following chapter therefore introduces a probabilistic description of progressive failure and includes the nonlinear effects of the matrix to develop a homogenised formulation for the evolving shear stiffness and nonlinear stress-strain behaviour of the ply.





# Statistical Evolution of Damage and Ply Stiffness

This chapter formulates a statistical model describing the progressive degradation of unidirectional ply stiffness during shear loading. Fiber strength variability is represented probabilistically, enabling prediction of the evolving fractions of the intact and degraded representative volume elements. The global ply shear modulus is obtained through a generalised weighted homogenisation of the corresponding stiffness states, leading to a prediction of the nonlinear shear stress-strain response. Predictions are immediately validated against experimental data and their performance is assessed. Methods for model calibration are discussed.

## 8.1. Objective and Scope

The purpose of this chapter is to extend the deterministic effective RVE response covered in chapter 7 to ply-level behaviour such that a macroscopic stress-strain response of a UD ply for in-plane shear can be constructed. The key idea is that the ply consists of a statistical ensemble of RVEs. Within this ensemble, RVEs exist in different states which include pristine RVEs with fiber and matrix undamaged, and RVEs that are degraded due to a fiber crack and subsequent matrix interfacial or radial cracking.

The ply response during loading is obtained as a weighted combination of RVE responses, where the weights are governed by statistical fiber failure. The central expression for this homogenisation concept is

$$\tau_{\text{ply}}(w) = f(w) \tau_{\text{RVE,broken}}(w) + (1 - f(w)) \tau_{\text{RVE,pristine}}(w) \quad (8.1)$$

where  $w$  is the RVE displacement,  $f$  denotes the fraction of degraded RVEs as a function of  $w$ , and  $\tau_{\text{RVE}}$  is associated with either the pristine or degraded, as a function of  $w$ . In this form, the ply response is constructed as a weighted average of RVE responses, consistent with the general volume averaging framework introduced in section 2.2, here interpreted over an ensemble of RVEs rather than within a single RVE with separate constituents.

As a consequence,  $\tau_{\text{RVE,pristine}}(w)$  and  $\tau_{\text{RVE,broken}}(w)$  must provide upper and lower bounds to the ply response  $\tau_{\text{ply}}(w)$  or else it cannot be the average response of its constituents.

Obtaining  $\tau_{\text{RVE}}$  follows from:

$$\tau_{\text{RVE}}(w) = \frac{Q_{\text{RVE}}(w)}{A_{\text{RVE}}} \quad (8.2)$$

where  $A_{\text{RVE}}$  is obtained from Equation 4.7 and the different  $Q_{\text{RVE}}(w)$  relations available to obtain RVE shear stresses, such as Equation 8.2, were derived in chapter 7, corresponding to the following RVE states:

- Pristine/unbroken linear response given by Equation 7.60;

- Containing a broken fiber and interfacial matrix cracking, resulting in a nonlinear response given by Equation 7.88;
- Containing a broken fiber and radial matrix cracking, resulting in a nonlinear response given by Equation 7.89.
- Compound expression from energy-based crack-switching mechanism as per subsection 7.8.3.

Two contributions to nonlinearity of the ply response are considered. First, the damage-induced nonlinearity covered by chapter 7 and second, the matrix material nonlinearity, introduced in section 8.2 and detailed out in section 8.3.

## 8.2. Model Assumptions

The construction of the ply-level response from RVE behaviour relies on the following modelling assumptions:

- **Uniform displacement** All RVEs are assumed to experience the same applied displacement

$$w_{\text{RVE}} = w_{\text{applied}}, \quad (8.3)$$

which allows for the direct comparison of RVE loadpaths at equal  $w$ . This can be interpreted as a uniform strain assumption (Voigt), as the macroscopic far-field strain is defined as:

$$\gamma_{12} = \frac{w_{\text{RVE}}}{d_{\text{eff}}}, \quad (8.4)$$

where  $d_{\text{eff}}$  is a constant value for a given abstraction of the fiber-matrix RVE as a Timoshenko beam on an elastic foundation and therefore independent of the damage present. The effect of damage comes from the different load-carrying capability of the RVE due to presence of fiber and matrix cracking.

- **No load redistribution** Upon breaking of a fiber within an RVE, it is assumed that no significant load redistribution occurs. The reasoning is as follows: Consider a bundle of 7 intact fibers in a hexagonal packing. Let this bundle be represented by a collection of 7 pristine RVEs to which a total far-field load of

$$Q_{\text{applied}} = 7Q_{\text{RVE}} \quad (8.5)$$

is applied. Upon breaking of the central fiber, a local re-distribution occurs such that

$$Q_{\text{RVE,broken}} = 0; \quad (8.6)$$

$$Q_{\text{RVE,pristine}} = \frac{7}{6}Q_{\text{applied}}; \quad (8.7)$$

$$Q_{\text{applied}} = Q_{\text{RVE,broken}} + 6Q_{\text{RVE,pristine}} = 7Q_{\text{RVE}}. \quad (8.8)$$

Consequently, the surrounding fibers experience a stress concentration factor of only  $7/6 = 1.17$ . A similar conclusion was reached by Qian et al. [101], who numerically showed that the stress concentration factor in irregular fiber packings never exceeds 1.13 in the case of axial tension.

- **Statistical independence of RVEs** Related to the assumption of no load redistribution, it is assumed that statistical properties of a single fiber governing its failure are not influenced by neighbouring RVEs. Spatial interactions and clustering effects are therefore neglected to keep the problem tractable.
- **Fiber strength variability governs damage initiation** Following Socci and Kassapoglou [7], the fiber shear failure strain is treated as a random variable; while the matrix properties are not treated statistically. Experimentally obtained standard deviations for tensile fiber failure strains will be assumed to equally represent shear fiber failure. It is based on these random values that fraction  $f$  of failed fibers will be determined, as detailed out in section 8.4.
- **Single representative degraded RVE state** Although a piecewise RVE load path containing three distinct regimes, as discussed in subsection 7.8.1, applies to individual RVEs containing either interfacial or radial matrix cracks, a further simplification is made for the evaluation of Equation 8.1.

Since fiber failure is assumed to be statistically governed by a shear failure strain  $\gamma_f$ , related to the fiber shear strength  $S_f = G_f \gamma_f$ , the displacement  $w$  at which fiber failure occurs (i.e. the first vertical load drop in chapter 7) shifts to higher or lower values, corresponding to higher or lower values of the macroscopic strain in Equation 8.4 that triggers fiber failure.

Additionally, it is assumed that the matrix crack size within an RVE is independent of the macroscopic strain at which the fiber fails. Consequently, RVEs associated with medium to high fiber strength are assumed to instantaneously obtain the matrix crack size corresponding to RVEs with lower fiber strength, where cracks have already grown progressively during loading up to the displacement that triggered failure of medium to high strength fibers. This avoids the need of tracking individual fibers and keeps the problem tractable.

Therefore, the simplification that can be introduced is the direct usage of Equation 7.71 or Equation 7.85 to sample RVE loads for a given displacement, compared to the piecewise loadpaths constructed for individual RVEs.

- **Matrix behaviour** So far the matrix has been treated as a being a linear constituent, while its shear modulus actually reduces under loading, hence behaves nonlinearly. Because of the lack of accurate shear stress-strain curves for the matrix itself, the effect on the RVE response is investigated by approximating the matrix with a Ramsberg-Osgood relation, as detailed out in section 8.3.

### 8.3. Matrix Nonlinearity

The matrix nonlinearity in this model is described using the Ramberg–Osgood relation [102], [103], which accounts for the material's softening as it approaches failure under shear loading. The shear stress–strain relationship for the matrix is given by [103]:

$$\gamma = \frac{\tau}{G_{m,0}} + K \left( \frac{\tau}{G_{m,0}} \right)^n \quad (8.9)$$

where  $\tau$  is the shear stress,  $\gamma$  is the shear strain, and  $G_{m,0}$  is the initial matrix shear modulus. The first term represents the linear elastic response, and the second term introduces the nonlinear softening behaviour.

The parameters  $K$  and  $n$  define the nonlinear material response and are derived based on two critical points: the initial nonlinear behavior and the failure point. The initial nonlinear strain, denoted as  $\gamma_0$ , and the corresponding shear stress  $\tau_0$  are related by:

$$\tau_0 = 0.99 \cdot G_{m,0} \cdot \gamma_0 \quad (8.10)$$

whereas  $\gamma_0$  is a small strain value, chosen as 0.001 in this case, whereas 0.002 would be typically chosen for metals [92]. The stress  $\tau_0$  is assumed to obtain 99% of the linear stress, as to keep Equation 8.13 well defined. At the failure point, the material reaches the shear failure strain  $\gamma_{m,u}$ , and the corresponding shear stress equals the shear strength  $S_m$ . The parameters  $K$  and  $n$  are solved for by using Equation 8.9 and these stress-strain points:

At the initial point  $(\gamma_0, \tau_0)$ , Equation 8.9 is:

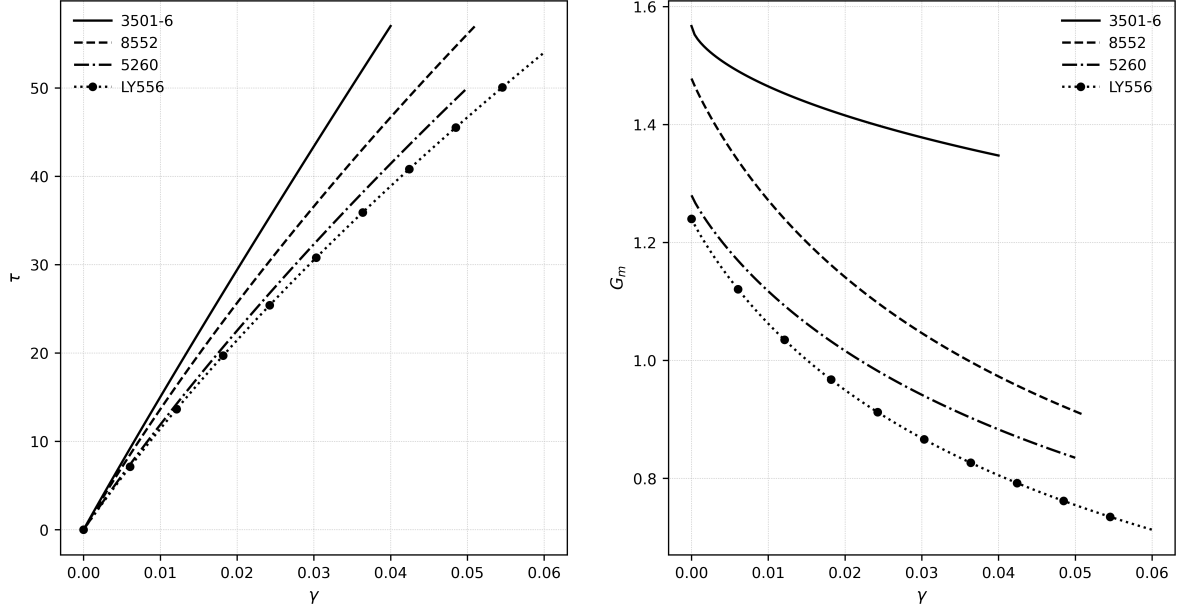
$$\gamma_0 = \frac{\tau_0}{G_{m,0}} + K \left( \frac{\tau_0}{G_{m,0}} \right)^n. \quad (8.11)$$

And at the failure point  $(\gamma_{m,u}, S_m)$ , Equation 8.9 becomes:

$$\gamma_{m,u} = \frac{S_m}{G_{m,0}} + K \left( \frac{S_m}{G_{m,0}} \right)^n \quad (8.12)$$

Solving for  $n$ :

$$n = \frac{\ln \left( \frac{\gamma_0 - \frac{\tau_0}{G_{m,0}}}{\gamma_u - \frac{\tau_u}{G_{m,0}}} \right)}{\ln \left( \frac{\tau_0}{\tau_u} \right)} \quad (8.13)$$



**Figure 8.1:** Matrix stress-strain relation and evolution of initial shear modulus  $G_{m,0}$  based on Ramberg-Osgood

And once  $n$  is known,  $K$  is given by:

$$K = \frac{\gamma_u - \frac{\tau_u}{G_{m,0}}}{\left(\frac{\tau_u}{G_{m,0}}\right)^n} \quad (8.14)$$

To obtain the matrix shear modulus  $G_m$  as a function of strain, Equation 8.9 is differentiated with respect to  $\gamma$ :

$$G_m(\gamma) = \frac{d\tau}{d\gamma} = \frac{G_{m,0}}{1 + Kn \left(\frac{\tau(\gamma)}{G_{m,0}}\right)^{n-1}} \quad (8.15)$$

where  $\tau(\gamma)$  is obtained by numerically by solving the implicit Equation 8.9.

Equation 8.15 now gives the effective shear modulus of the matrix at the given strain, reflecting the progressive softening of the matrix as the strain increases. Results for Equation 8.9 and Equation 8.15 for the considered matrix systems are shown in Figure 8.1. Comparison indicates that 3501-6 shows the least amount of nonlinearity, and thus the smallest reduction in shear modulus  $G_m$ , whereas LY556 shows the strongest nonlinearity, resulting in the largest reduction in  $G_m$ .

## 8.4. Statistical Description of Fiber Failure

Fiber failure is defined in terms of a failure strain,

$$\gamma_f = \frac{S_f}{G_f}, \quad (8.16)$$

consistent with the criterion introduced in section 7.2 and where  $G_f$  corresponds to the corrected fiber shear modulus as determined in chapter 6. To introduce variability, the fiber shear modulus  $G_f$  is assumed deterministic, while the failure strain is treated as a random variable following Socci and Kassapoglou [7],

$$\gamma_f \sim \mathcal{N}(\mu_\gamma, \sigma_\gamma^2), \quad (8.17)$$

with

$$\mu = \frac{S_f}{G_f}, \quad \sigma_\gamma = \sigma_\epsilon, \quad (8.18)$$

where  $S_f$  and the experimentally obtained standard deviation for tensile fiber failure strain  $\sigma_\epsilon$  are reported in chapter 3.

Using  $S_f = G_f \gamma_f$ , the fiber strength can be written as a scaled normal distribution,

$$S_f \sim \mathcal{N}(G_f \mu, G_f^2 \sigma_\gamma^2) = \mathcal{N}(S_f, (G_f \sigma_\epsilon)^2), \quad (8.19)$$

such that variability in fiber strength leads to a gradual failure of RVEs accross the ply. The fraction of failed RVEs at a given applied displacement is defined as

$$f(w) = P(\gamma_f \leq \gamma(w)), \quad (8.20)$$

with the applied strain given by

$$\gamma(w) = \frac{w}{d_{\text{eff}}}. \quad (8.21)$$

With this formulation,  $f(w)$  increases monotonically with applied displacement, governing the transition from pristine to progressively degraded behaviour.

## 8.5. Construction of the Ply Stress-Strain Response

Combining the homogenisation expression of Equation 8.1 with the model assumptions of section 8.2 and the statistical description of fiber failure of section 8.4, the following procedure to construct the ply stress-strain response is obtained:

1. Starting from  $w = 0$ , the applied displacement is incrementally increased;
2. For each  $w$ :
  - Compute  $f$  by using Equation 8.20;
  - Compute pristine RVE shear stress  $\tau_{\text{RVE},\text{pristine}}$  by sampling Equation 7.60, accounting for an evolving matrix modulus  $G_m$  by Equation 8.15, for the given displacement  $w$  and substitute into Equation 8.2;
  - Compute broken RVE shear stress  $\tau_{\text{RVE},\text{broken}}$  by sampling from either
    - Equation 7.71 for a ply response dominated by interfacial cracking;
    - Equation 7.85 for a ply response dominated by radial cracking;
    - Equation 7.71 or Equation 7.85 following the energy-based crack-switching mechanism of subsection 7.8.3,
 accounting for an evolving matrix modulus  $G_m$  by Equation 8.15, for the given displacement  $w$  and substitute into Equation 8.2;
3. Combine  $f$ ,  $\tau_{\text{RVE},\text{pristine}}$  and  $\tau_{\text{RVE},\text{broken}}$  into Equation 8.1.
4. Repeat until  $f = 1$ .

## 8.6. Initial Results and Model Behaviour

Comparisons between the ply stress-strain data provided by Kaddour et al. [64] and the model results from the procedure as laid out in section 8.5 are shown in Figure 8.2-8.5. Next to the stress-strain result, the failure fraction  $f$  and the evolving shear moduli  $G_{12}$  of the RVEs versus that of the ply are included. The following observations can be made:

- The accuracy of  $\tau_{\text{RVE},\text{pristine}}$  serving as an upper bound as explained in section 8.1, depends on the accuracy of the statistically informed  $G_{\text{RVE}}$  of chapter 6. Whereas  $\tau_{\text{RVE},\text{pristine}}$  provides an excellent upper bound for  $\tau_{\text{ply}}$  of IM7/8552, it underestimates  $\tau_{\text{ply}}$  for the remaining materials at low failure fractions  $f$ , as evident from Table 6.5, where  $G_{12}^{p(v_f)}$  predicts relative errors in initial modulus ranging from +4.13% for IM7/8552 to -24.7% for G40-800/5260.
- The nonlinearity introduced by the matrix by using a Ramberg-Osgood approximation, best observed in  $G_{12,\text{pristine RVE}}$ , is rather small, which is a logical result from the benign nonlinearities seen in Figure 8.1.

- Regarding the failure fraction  $f$ ; due to a rather low standard deviation  $\sigma_\epsilon$ ,  $\tau_{\text{ply}}$  for the carbon materials follows  $\tau_{\text{RVE,pristine}}$  closely with a sudden switch to  $\tau_{\text{RVE,broken,int}}$  or  $\tau_{\text{RVE,broken,rad}}$  around the mean failure strain  $S_f/G_f$ . For Glass/LY556, the high mean failure strain results in a practically zero failure fraction, such that  $\tau_{\text{ply}} = \tau_{\text{RVE,pristine}}$ . The failure fraction  $f \rightarrow 1$  does also not match up with the stress  $\tau_{\text{ply}}$  that defines the ply rupture strength.
- The lower bound of  $\tau_{\text{ply}}$  to be provided by either  $\tau_{\text{RVE,broken,int}}$  or  $\tau_{\text{RVE,broken,rad}}$  or a combination of both, only succeeds at being a lower bound for low strains and mainly so by  $\tau_{\text{RVE,broken,rad}}$ .
- The rather low bound provided by  $\tau_{\text{RVE,broken,rad}}$  is due to its initial slope  $G_{12}$  not matching up closely with that of  $\tau_{\text{ply}}$  or  $\tau_{\text{RVE,broken,int}}$ , which is a consequence of the clipping of the bounds of  $k_0$  to approximate  $\delta_{II}$  based on Nied's solution, which overestimates  $\delta_{II}$  at low load values, thereby overestimating the added compliance due to the crack.
- Overall, both  $\tau_{\text{RVE,broken,int}}$  and  $\tau_{\text{RVE,broken,rad}}$  express very little nonlinearity within the strain range upto  $f = 1$ , such that  $\tau_{\text{ply}}$  does not have the nonlinearity required to match up with the experimental data  $\tau_{\text{ply,data}}$  by Kaddour et al. [64].
- The ply response  $\tau_{\text{ply,crack preference}}$  does not show any crack switching, as explained in subsection 7.8.3, as the associated displacement  $w$  is not within the strain range up to  $f = 1$ .

Overall, the initial predictions of the current framework are quantitatively poor, but qualitatively it contains the right building blocks. Currently the main issues with those building blocks are the degraded RVEs behaving weakly nonlinear within the strain range and the assumed statistics not being representative for the failing RVEs within the strain range. The next sections try to address both issues.

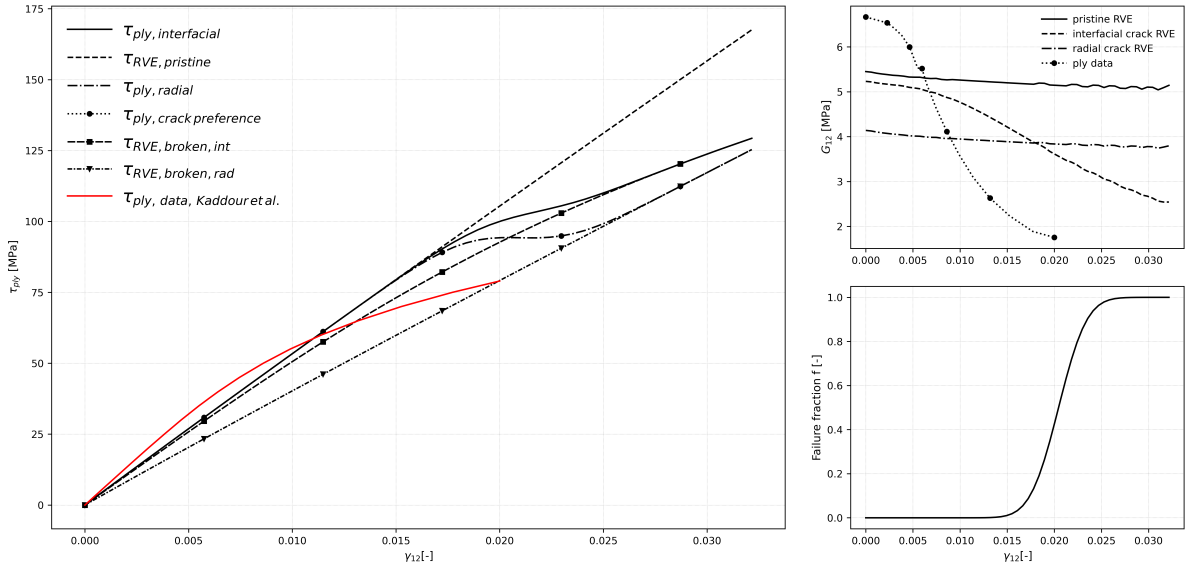


Figure 8.2: In-plane shear stress-strain prediction for AS4/3501-6.

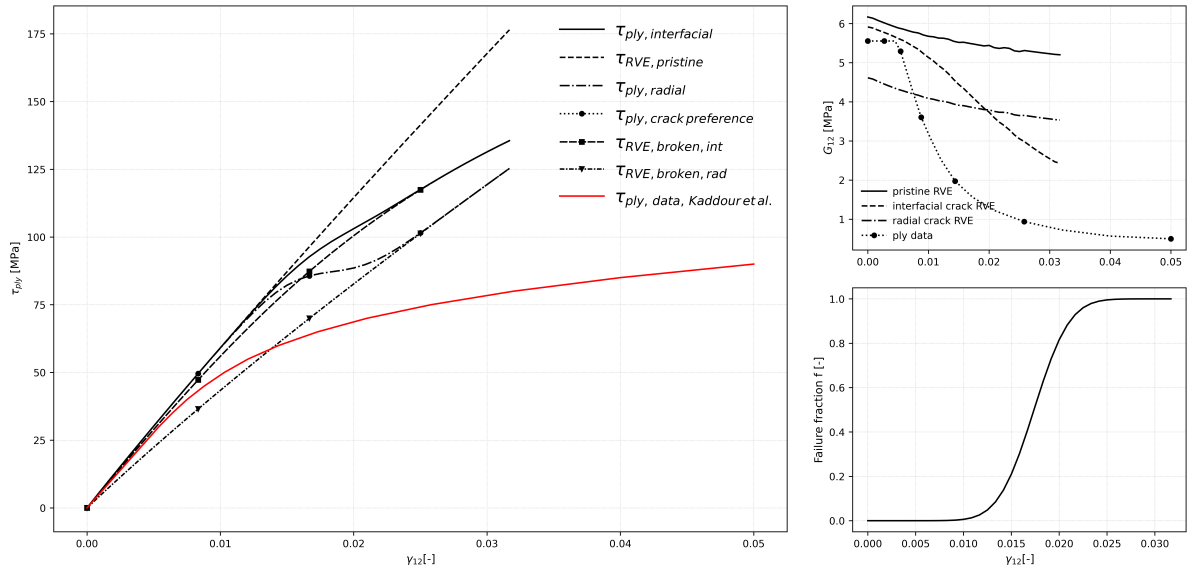


Figure 8.3: In-plane shear stress-strain prediction for IM7/8552.

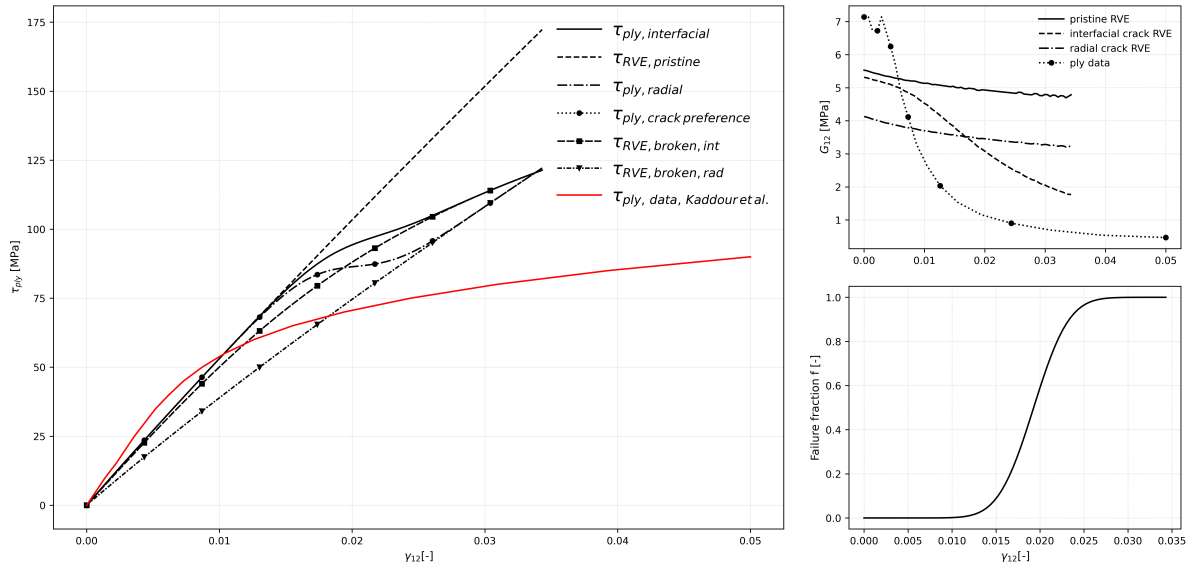


Figure 8.4: In-plane shear stress-strain prediction for G40-800/5260.

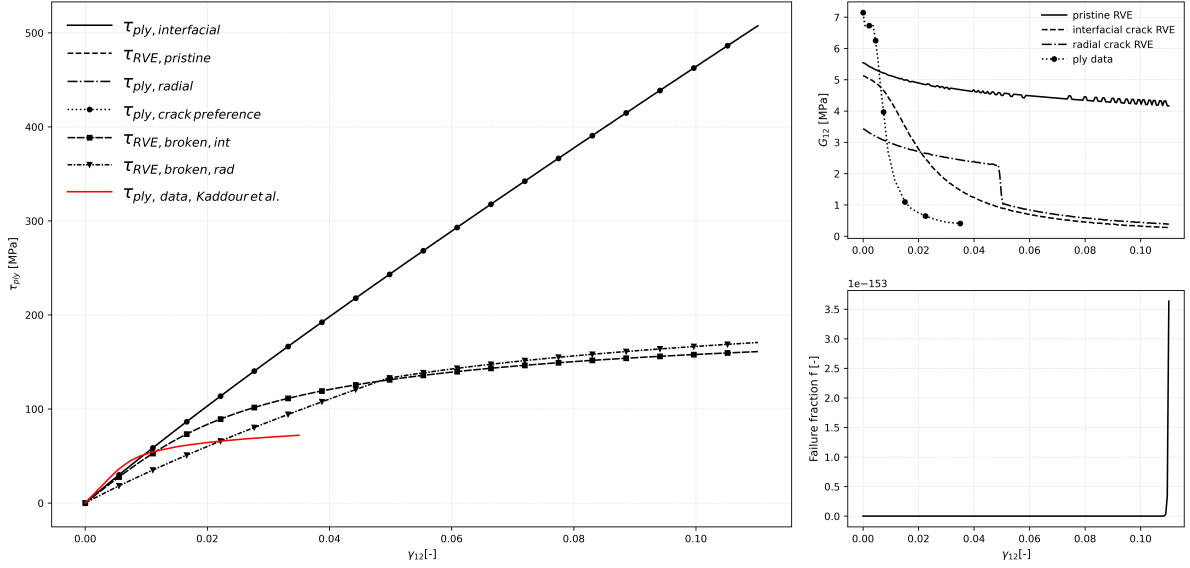


Figure 8.5: In-plane shear stress-strain prediction for Glass/LY556.

## 8.7. Addressing the Main Model Shortcomings

As identified in previous section 8.6, the current framework has two main shortcomings, i.e. a weakly nonlinear response of the RVEs that are supposed to make up the ply, and the failure statistics not being representative for the gradual progression from pristine to degraded behaviour of the ply.

### 8.7.1. Increasing the Nonlinearity of the Degraded RVE Response

Nonlinearities in the RVE response are attributed to either the growth of damage in the form of interfacial or radial cracks or the softening of the matrix. Both effects are rather benign in comparison with the observed nonlinearity of the ply from experimental data.

#### 8.7.1.1. Influence of Crack-Growth as a Function of Matrix Idealisation

The size of either interfacial or radial cracks,  $L_{int}$  and  $L_{rad}$ , is governed by the amount of excess matrix strain energy by Equation 7.55. The amount of excess matrix strain energy is in turn governed by the RVE response, as a Timoshenko beam on elastic foundation, to the appearance of the fiber crack. This response can in turn be categorized based on  $d_{eff}$ , which expresses the distance over which the effect of the fiber crack is present, as covered in subsection 7.3.5. For low values of  $d_{eff}$ , the RVE responds in a stiff manner, with a quick return to the far-field state of the RVE, while high values of  $d_{eff}$  indicate a very gradual recovery of the far-field.

This stiffness of the RVE response expressed in terms of  $d_{eff}$  is a function of the idealization of the matrix into an elastic foundation as covered in subsection 4.2.2. Within this elastic foundation, the matrix Young's modulus  $E_m$  is identified as the material property that resists the lateral motion of the fiber beam through the elastic foundation the most. The effect on  $\tau_0(x)$  by Equation 7.31 or 7.32 from varying the magnitude of  $E_m$  is shown in Figure 8.6. With respect to the matrix shear strength  $S_m$ , a decreasing magnitude of  $E_m$  allows for more excess strain energy for crack growth through a slower recovery of the far field. Note how this reduction in  $E_m$  also alleviates the issue of cascading fiber failure mentioned in subsection 7.3.4, due to decreased overshoot of the shear stress  $\tau_0(x)$ .



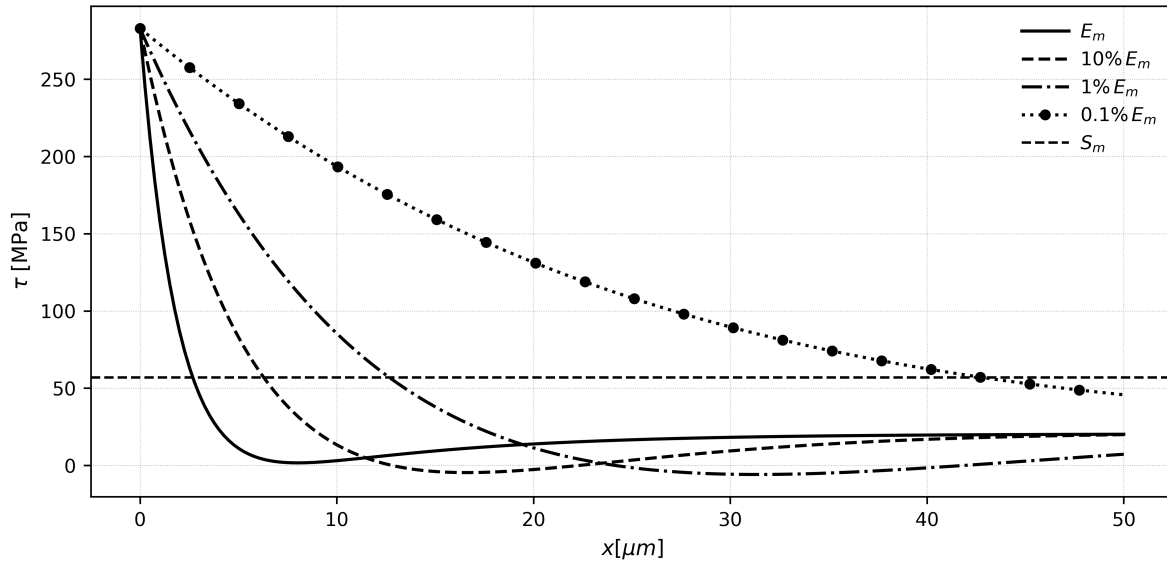


Figure 8.6: Effect of varying  $E_m$  on  $\tau_0(x)$

The effect of reducing  $E_m$  on the recovery of the far field as expressed by  $d_{\text{eff}}$  is shown in Figure 8.7, with an increase for low  $E_m$  values. Such an increased value of  $d_{\text{eff}}$  not only provides more energy for crack growth, but alters the *effective* RVE as defined in section 7.5. Consequently, for each applied displacement  $w$ , both the corresponding RVE load  $Q_{\text{RVE}}$  and stress  $\tau_{\text{RVE}}$  are associated with a smaller strain  $\gamma_{12}$ , as per Equation 8.21.

The effect of an extreme case of only including 0.1% of  $E_m$  in the formulation of the elastic foundation on the upper and lower bounds of  $\tau_{\text{ply}}$  to be provided by  $\tau_{\text{RVE,pristine}}$  and  $\tau_{\text{RVE,broken,int}}$  or  $\tau_{\text{RVE,broken,rad}}$  respectively are shown in Figure 8.8-8.11. The maximum applied strain is still associated with  $f = 1$ , with the same statistics as in section 8.6.

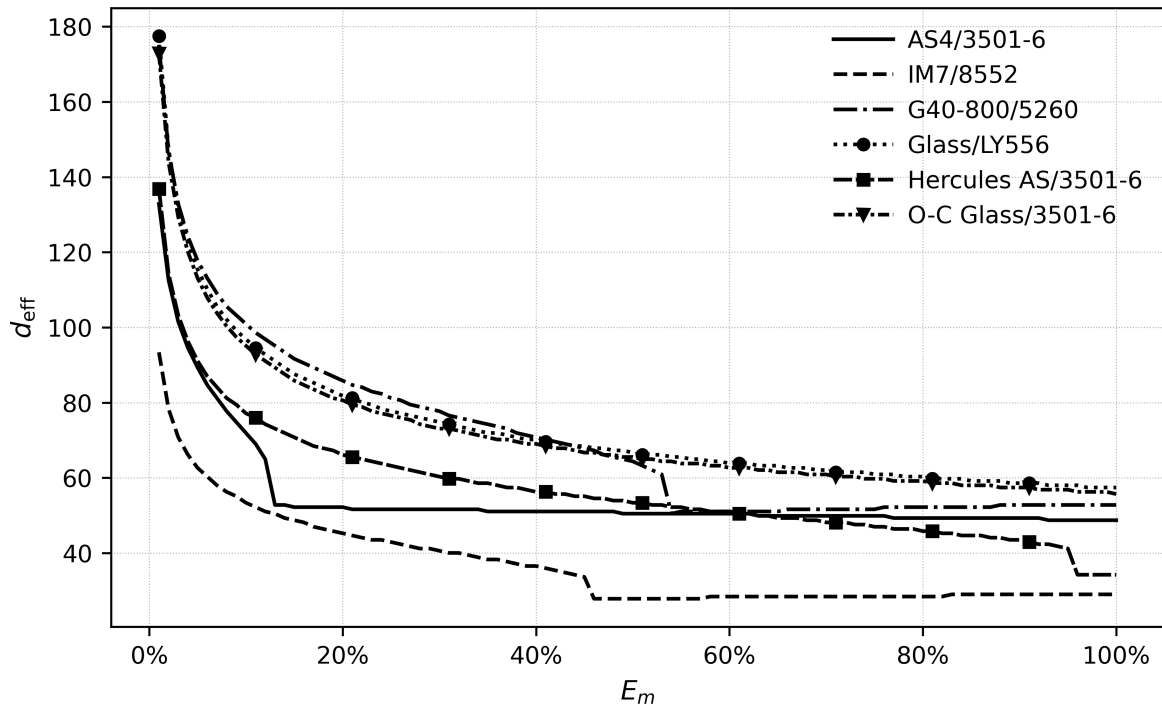


Figure 8.7:  $d_{\text{eff}}$  as a function of reduction of  $E_m$ . Discontinuities are due to numerics.

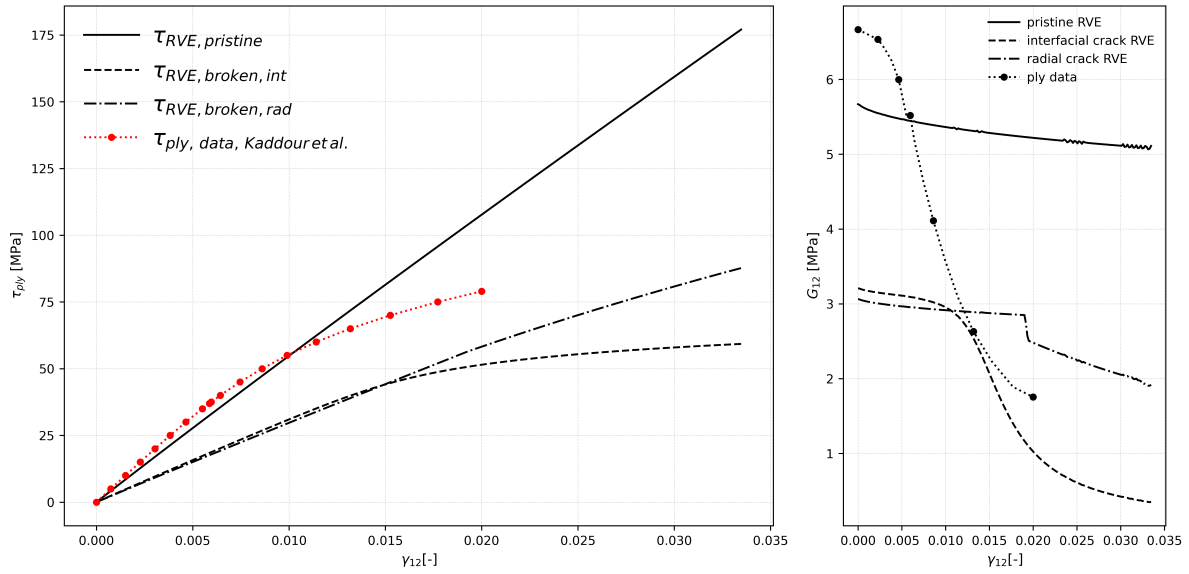


Figure 8.8: Upper and lower stress bounds to AS4/3501-6 for a reduction of  $E_m$  to 0.1%.

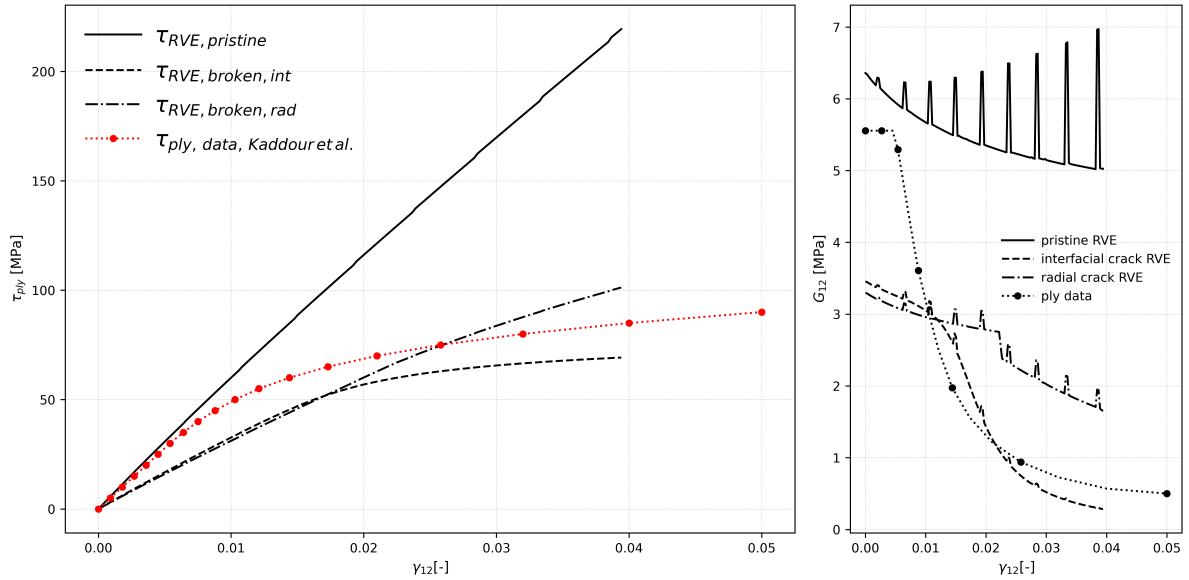
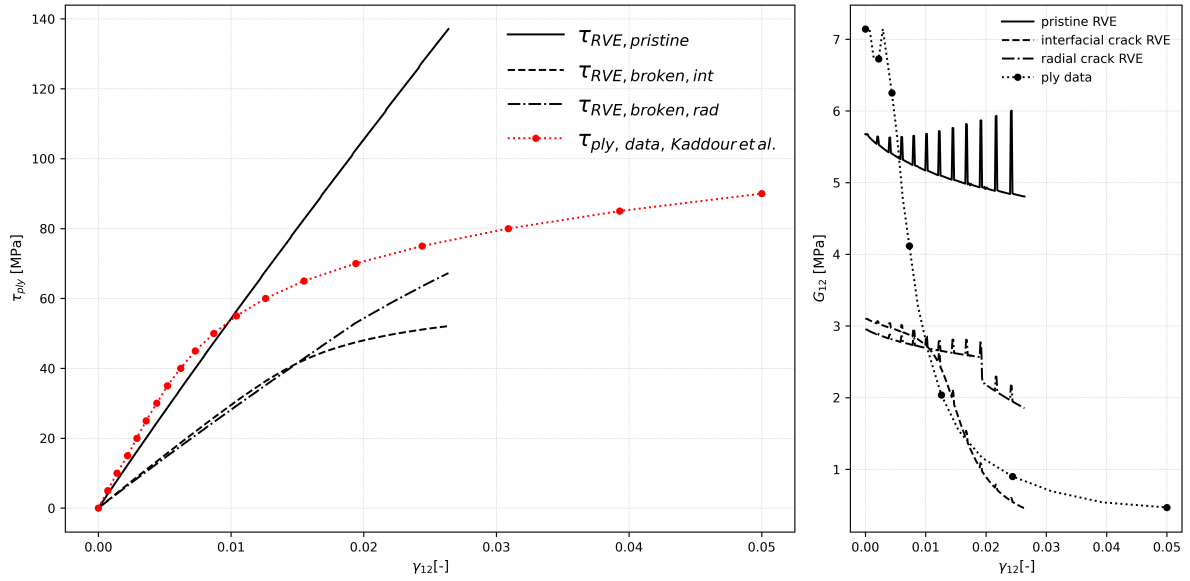
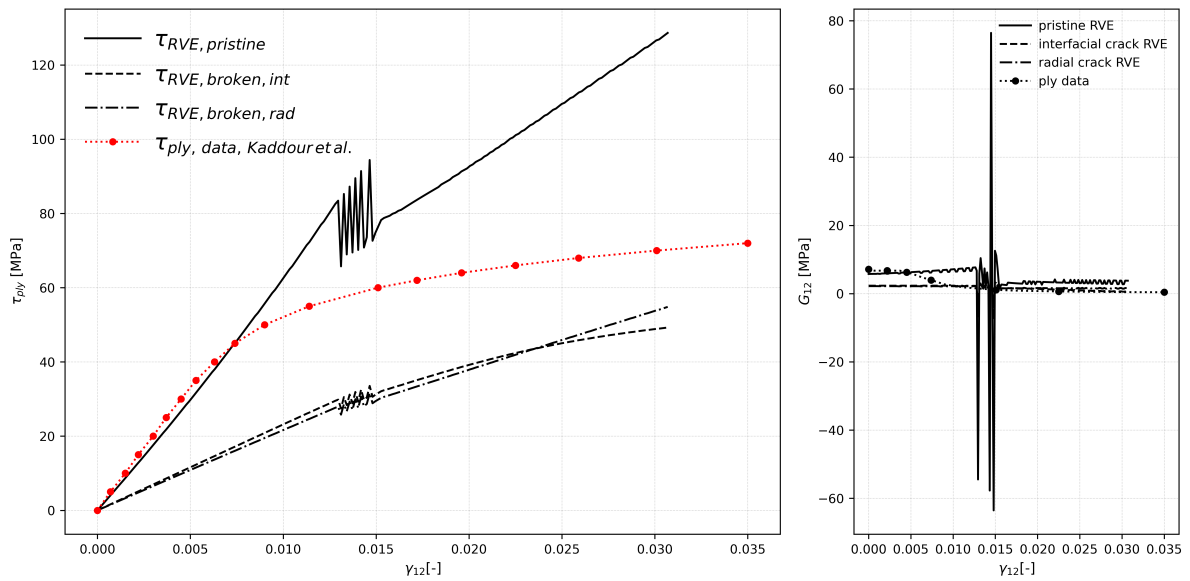


Figure 8.9: Upper and lower stress bounds to IM7/8552 for a reduction of  $E_m$  to 0.1%.

Across all material systems,  $\tau_{RVE, broken, int}$  now seems to provide proper lower bounds to  $\tau_{ply}$  for the entire strain range, while  $\tau_{RVE, broken, rad}$  does so only for lower strains and remains fairly stiff at higher strains. Qualitatively, the evolution of  $G_{12}$  for the interfacially cracked RVE matches the  $G_{12}$  evolution of the ply best, with an initial plateau followed by a smooth, monotonuous decrease. Any observed irregularities in the results are due to numerical issues, most pronounced for Glass/LY556 in Figure 8.11. The remaining issue here is that the extent to which  $E_m$  is to be reduced remains unknown. Studying the effect of reducing  $E_m$  on the initial ply stiffness predictions of chapter 4 and the correction strategy of chapter 6 shows that, although smaller in magnitude, a non-zero correction factor is still needed, as shown in Table 8.1.

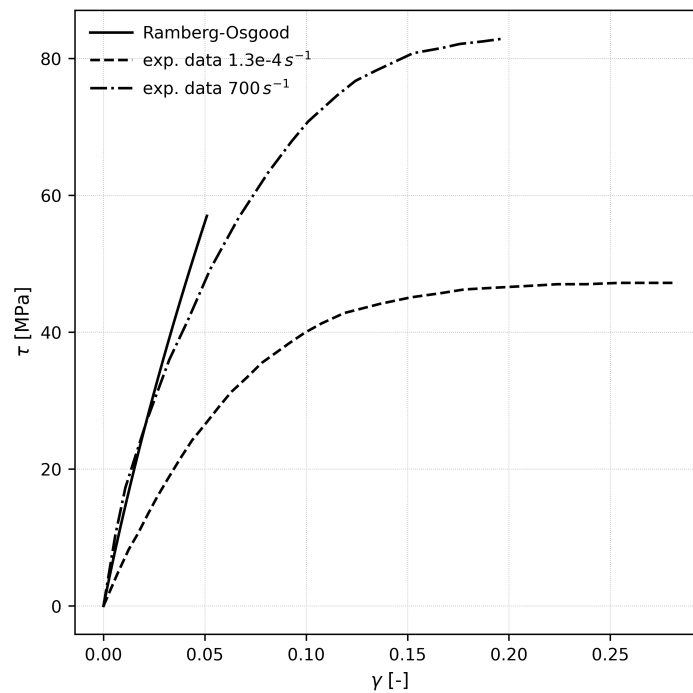
**Table 8.1:** Comparison of correction to the fiber shear modulus  $G_f$  as a function of  $E_m$ 

| Material system | $c_f$ (fiber only, 100% $E_m$ ) | $c_f$ (fiber only, 0.1% $E_m$ ) |
|-----------------|---------------------------------|---------------------------------|
| AS4/3501-6      | 0.561                           | 0.367                           |
| IM7/8552        | 0.367                           | 0.250                           |
| G40-800/5260    | 0.330                           | 0.230                           |
| Glass/LY556     | 0.287                           | 0.173                           |
| Average         | 0.386                           | 0.255                           |
| Std. dev.       | 0.121                           | 0.071                           |

**Figure 8.10:** Upper and lower stress bounds to G40-800/5260 for a reduction of  $E_m$  to 0.1%.**Figure 8.11:** Upper and lower stress bounds to Glass/LY556 for a reduction of  $E_m$  to 0.1%.

### 8.7.1.2. Influence of Matrix Nonlinearity

As shown in both Figure 8.1 and Figure 8.2-8.5, the effect of including the nonlinear Ramberg-Osgood approximation for the matrix constitutive law barely increases the nonlinearity of the RVE responses. In an attempt to validate the approximation made, a comparison is made with data from literature for the 8552 matrix system. Cheng et al. [104] reported shear stress-strain data for various strain rates. In case of high strain rates epoxy polymers tend to behave more linearly, brittle and with a higher shear modulus compared to low strain rates [104], [105]. In reporting the data for the WWFE-III, Kaddour et al. [64] did not make any mention of strain rates in relation to the matrix properties, such as the initial shear modulus  $G_{m,0}$ . Therefore, it is reasonable to assume those properties reflect a quasi-static, i.e. at very low strain rates, loading of the matrix and the plies. However, when comparing the Ramberg-Osgood approximation with the experimental data in Figure 8.12, it can be seen that the Ramberg-Osgood approximation, based on data from Kaddour et al. [64], does not match up with the quasi-static ( $1.3e-4 s^{-1}$ ) response, but rather with the high strain rate ( $700 s^{-1}$ ) response. Additionally, simply using the quasi-static response leads to another inconsistency, as its low initial modulus cannot replicate the ply initial modulus  $G_{12}$  as reported by Kaddour et al. [64]. This data inconsistency remains unresolved.



**Figure 8.12:** Comparison of a Ramberg-Osgood approximation for the 8552 matrix with experimental data [104]

### 8.7.2. Parameters Governing Failure Fraction Evolution

The fraction  $f$  of failed RVEs acts as a weighting factor in the homogenization used to obtain  $\tau_{ply}$  via Equation 8.1, with  $1 - f$  representing the fraction of intact RVEs. Due to a lack of statistical data on the shear strength and shear failure strain of individual fibers, approximate statistics were derived based on combining shear strength data with existing data related to tensile failure strains of individual fibers. This approach, as detailed out in section 8.4, resulted in

- an approximate  $\mu = \frac{S_f}{G_f}$ , with  $S_f$  from Cairns and Adams [82] and  $G_f$  based on Kaddour et al. [64] but corrected as per chapter 6;
- an approximate  $\sigma_\gamma$  based on tensile failure strain standard deviations reported by Socci and Kasapoglou [7].

As identified in section 8.6, the low assumed  $\sigma_\gamma$  resultst in a rather abrupt switch from  $\tau_{RVE,pristine}$  to either  $\tau_{RVE,broken,int}$  or  $\tau_{RVE,broke,rad}$  instead of a gradual transition. Qualitatively, a larger  $\sigma_\gamma$  resulting

in a wider spread of RVEs failing throughout loading would facilitate such a gradual transition. Quantitatively, the required increase in  $\sigma_\gamma$  remains unknown.

Since  $f$  and  $1 - f$  serve as weights in Equation 8.1, the ply data from Kaddour et al. [64] can be used to determine  $f_{\text{fit}}(\gamma_{12})$  such that the prediction matches the test results. Under the assumption that the reduction of  $E_m$  to 1% of its value, as shown in subsubsection 8.7.1.1, is representative for the RVE, a constructed  $f_{\text{fit}}(\gamma_{12})$  can be compared to the assumed failure fraction  $f$  distribution of section 8.4. Since only a proper upper bound was provided by  $\tau_{\text{RVE},\text{pristine}}$  for IM7/8552, this material is used as an example.

Let

$$\tau_{\text{ply},\text{data}} = f_{\text{fit}}(\gamma_{12}) \tau_{\text{RVE},\text{broken}} + (1 - f_{\text{fit}}(\gamma_{12})) \tau_{\text{RVE},\text{pristine}}, \quad (8.22)$$

such that solving for  $f_{\text{fit}}(\gamma_{12})$  gives

$$f_{\text{fit}}(\gamma_{12}) = \frac{\tau_{\text{ply},\text{data}} - \tau_{\text{RVE},\text{pristine}}}{\tau_{\text{RVE},\text{broken}} - \tau_{\text{RVE},\text{pristine}}}. \quad (8.23)$$

The result for  $f_{\text{fit}}(\gamma_{12})$  by fitting IM7/8552 data is shown in Figure 8.13. It can be observed that

- Initially, it is implied that the ply already contains about 28% broken RVEs and then the amount of RVEs reduces to 22%, after which it monotonically increases. Although it is not unlikely for the ply to contain broken RVEs before loading due to manufacturing defects, 28% is an unlikely high amount. This, together with the apparent reduction in broken RVEs, is a function of the initial interactions between  $\tau_{\text{ply},\text{data}}$ ,  $\tau_{\text{RVE},\text{pristine}}$  and  $\tau_{\text{RVE},\text{broken}}$  and thus subject to any previous assumptions made to obtain these stress-strain relations;
- The maximum value of  $f_{\text{fit}}(\gamma_{12})$  corresponding to the ply ultimate stress does not go above 0.9, as  $\tau_{\text{ply},\text{data}}$  and  $\tau_{\text{RVE},\text{broken}}$  do not coincide. Technically this implies that at the ply ultimate stress, only 90% of the RVEs failed via fiber failure and subsequent matrix cracking. For the remaining 10%, these RVEs either failed due to other mechanisms, or despite having declared the ply to be failed, a remainder of 10% of the RVEs are still intact;
- Fitting a scaled normal cumulative distribution function (CDF), included in Figure 8.13,

$$f_{\text{fit},\mathcal{N}}(\gamma_{12}) = A_{\mathcal{N}} \cdot \Phi\left(\frac{\gamma_{12} - \mu_{\text{fit}}}{\sigma_{\gamma,\text{fit}}}\right) \quad (8.24)$$

results in  $A_{\mathcal{N}} = 0.91$ ,  $\mu_{\text{fit}} = 0.010$ , and  $\sigma_{\gamma,\text{fit}} = 0.0094$ . Compared to  $\mu = 0.025$  and  $\sigma_\gamma = 0.00293$  as per section 8.4, this fit corresponds to a lower mean shear failure strain and a larger standard deviation, resulting in a more gradual failure progression.

- Fitting a scaled Weibull cumulative distribution function (CDF), included in Figure 8.13,

$$f_{\text{fit},\mathcal{W}}(\gamma_{12}) = A_{\mathcal{W}} \cdot \left[ 1 - \exp\left(-\left(\frac{\gamma_{12} - \gamma_0}{\lambda}\right)^k\right) \right], \quad \gamma_{12} \geq \gamma_0 \quad (8.25)$$

results in  $A_{\mathcal{W}} = 0.99$ ,  $k = 0.982$ , and  $\lambda = 0.0157$ . The location parameter  $\gamma_0$  was set to zero, as initial fits resulted in non-physical values ( $\gamma_0 < 0$ ).

The fitted shape parameter  $k \approx 1$  implies that the Weibull distribution reduces to an exponential distribution, corresponding to a constant hazard rate **<empty citation>** This suggests that the failure rate of the remaining intact RVEs is independent of the applied strain level. Its low value also suggest high scatter and thus gradual accumulation of damage, as  $k \gg 1$  would correspond to all RVEs breaking at once.

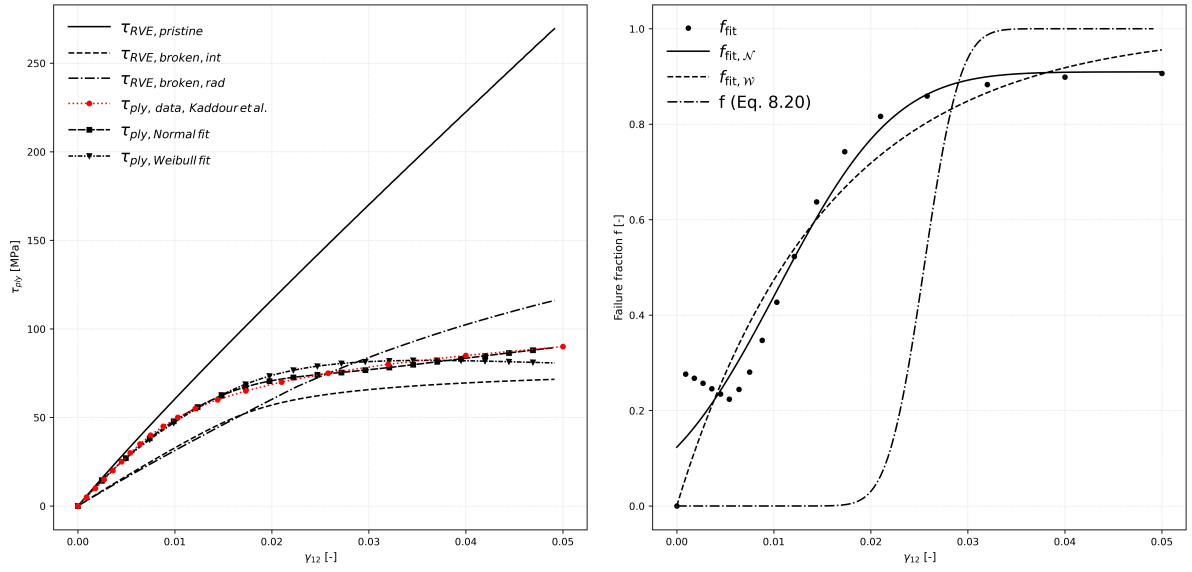
The scale parameter  $\lambda$  defines the strain at which approximately 63% of the asymptotic failure fraction is reached when  $\gamma_{12} = \lambda$ , marking the point at which the homogenisation of Equation 8.1 starts to strongly deviate from the pristine RVE response.

- Using Equation 8.24 and 8.25 under the current assumptions to obtain  $\tau_{ply}$  by substituting into Equation 8.1 results in the stress-strain curves shown in Figure 8.13. Despite low RMSE for both CDF fits, as shown in Table 8.2, the Weibull distribution does not provide a better representation of the RVE failure evolution.

**Table 8.2:** RMSE of stress prediction using different failure fraction models.

| Model       | RMSE $_{\tau_{ply}}$ [MPa] |
|-------------|----------------------------|
| Normal CDF  | 1.07                       |
| Weibull CDF | 3.08                       |

As the above observations rely on the assumption that the RVE defined by  $E_m$  being reduced to 0.1% of its value is representative, they remain qualitative rather than quantitative.



**Figure 8.13:** Comparison of assumed failure fraction  $f$  by Equation 8.20 to numerical fits based on ply data for IM7/8552

## 8.8. Conclusions

The following conclusions can be made for the developed methodology and its evaluation:

- The macroscopic shear response can be interpreted as the weighted sum of intact and degraded RVE states, provided that they form clear upper and lower bounds respectively. Upper stress-strain bounds depend on the accuracy of the statistically informed  $G_{12}^{p(v_f)}$  predictions of chapter 6. Lower stress-strain bounds are a function of the matrix cracking mechanisms of chapter 7;
- The initial RVE formulation relying on the complete idealisation of the matrix into an elastic foundation, as per chapter 4, provides insufficient excess strain energy to facilitate crack growth, resulting in a weakly nonlinear RVE response in comparison with the experimentally observed nonlinearities of the ply.
- Softening of the elastic foundation by reducing the matrix Young's modulus  $E_m$  to only 0.1% of its value, practically removing the Winkler springs of chapter 4, alleviates the issue mentioned above, increasing the nonlinearities observed;
- Studying the effect of reducing  $E_m$  showed that the initial ply stiffness predictions of chapter 4 still require a non-zero correction to the fiber shear modulus  $G_f$ , thereby showing that simply reducing  $E_m$  does not resolve issues related to both  $G_{12}$  estimation and crack-growth;

- The inclusion of the matrix material nonlinearity based on Ramberg-Osgood approximations did not increase the nonlinear response of the RVEs adequately. Additionally, in an attempt to validate the Ramberg-Osgood approximation, inconsistencies between published data on shear stress-strain curves and initial shear modulus were found;
- The procedure of relating assumed fiber failure fractions  $f$ , based on limited fiber shear strength data and tensile fiber failure strain statistics, to the failure of RVEs, such that the ply response involves the fiber failure fraction  $f$  as weights in the homogenization of the RVE responses, results in the initial ply response to be dominated by the intact RVEs for low to medium strains, with a rather abrupt transition to broken RVEs for high strains. Based on fitting to ply data, obtaining a more gradual transition requires a fiber failure strain distribution with a higher standard deviations is required. This was confirmed by fitting a scaled normal CDF to a failure fraction  $f$  derived from fitting ply stress strain data. Fitting a Weibull CDF suggests a failure rate of remaining intact RVEs to be independent of applied strain level;
- Since both the nonlinear response of the RVEs and the assumed fiber failure statistics are insufficient to reproduce the ply response, attempts are made to address these issues, as mentioned above. However, the problem is underdetermined, as multiple combinations of  $E_m$  and  $f$  can reproduce the same macroscopic response:
  - Independent calibration of  $E_m$  would allow the corresponding failure fraction distribution  $f$  to be identified.
  - Independent determination of  $f$  would allow the required calibration of  $E_m$  to be identified.

Overall, the developed methodology captures the qualitative mechanisms governing the macroscopic shear response through a combination of intact and degraded RVE states. However, the results demonstrate that neither the current representation of RVE nonlinear response nor the assumed fiber failure statistics are sufficient to quantitatively reproduce the observed ply behaviour. In particular, modifications to the matrix idealisation and failure modelling can improve specific aspects of the response, but do not resolve all issues simultaneously. This indicates that the model is currently underdetermined, as different combinations of matrix stiffness  $E_m$  and failure fraction distribution  $f$  can lead to similar macroscopic responses. Consequently, independent characterisation of either the matrix behaviour or the failure statistics is required to resolve this and calibrate the model.

# Conclusions and Recommendations

This chapter restates the purpose of the present work and summarises its main content. The principal conclusions are subsequently presented, followed by recommendations for future improvements.

## 9.1. Summary

This work set out to develop a physics based and computationally efficient framework to predict the in-plane shear behaviour of unidirectional (UD) composite plies based on micromechanical considerations. The main contributions can be summarised as follows:

- Developed an analytical micromechanical model for the in-plane shear behaviour of UD plies based on a fiber-matrix representative volume element (RVE);
- Formulated the RVE by idealising the fiber as a Timoshenko beam on an elastic foundation representing the surrounding matrix, assuming hexagonal fiber packing;
- Solved the governing equations of the Timoshenko beam on an elastic foundation for an applied shear load both numerically and analytically;
- Derived analytical expressions for the effective RVE shear stiffness  $K_{RVE}$  and corresponding shear modulus  $G_{RVE}$ , and made comparisons against established formulations;
- Performed finite element analyses (FEA) on cylindrical and hexagonal RVEs to validate the analytical formulation and assess geometric effects;
- Introduced a statistical homogenisation framework accounting for variability in fiber volume fraction using a scaled beta distribution, and established a power-mean formulation for the ply shear modulus  $G_{12}$ ;
- Implemented stiffness corrections for the constituents to reconcile deterministic RVE predictions with the statistical homogenisation framework;
- Extended the deterministic RVE formulation to include constituent failure by formulating criteria relating far-field applied loads to local fiber and matrix failure;
- Modelled discrete fiber fracture as the primary damage mechanism and derived closed-form expressions for the resulting load redistribution between fiber and matrix over an effective length  $d_{eff}$  by superposing a local correction field onto the pre-failure deformation;
- Derived the interfacial shear stress field and extended it to a full matrix stress field, enabling assessment of matrix failure in both axial and radial directions;
- Employed an excess strain energy formulation to quantify the energy available for matrix cracking and used this to derive crack lengths for interfacial and radial matrix cracking;
- Formulated stiffness degradation mechanisms associated with interfacial and radial cracking;
- Introduced the concept of an *effective RVE*, in which local damage is homogenised over a finite length  $d_{eff}$ , allowing representation of multiple degraded RVE states;



- Constructed a piecewise RVE load path under displacement control, capturing the transition between pristine, degraded, and failure regimes;
- Extended the RVE formulation to the ply level by modelling the ply as a statistical ensemble of RVEs and defining a homogenisation framework in which the macroscopic shear stress is expressed as a weighted combination of pristine and degraded RVE responses;
- Introduced a statistical description of fiber failure by modelling the failure strain as a normally distributed variable and defining a failure fraction as its cumulative probability, used as a weighting factor in the homogenisation;
- Introduced matrix material nonlinearity in the homogenisation process through a Ramberg-Osgood formulation, allowing the matrix shear modulus to evolve with strain;
- Generated ply-level shear stress-strain predictions for multiple material systems using the developed methodology and compared these against available experimental data.

## 9.2. Conclusions

- The deterministic Timoshenko beam on elastic foundation formulation provides an analytical baseline for the fiber-matrix RVE, but significantly overpredicts the initial ply shear modulus;
- The deterministic analytical RVE exhibits a Voigt-like stiffness behaviour across the fiber volume fraction range, whereas a more Reuss-like behaviour is sought;
- Classical mixture relations underpredict the initial in-plane shear stiffness of unidirectional plies, while the deterministic RVE overpredicts;
- Finite-element analyses confirm that the effective shear stiffness depends not only on constituent properties and RVE area, but also on the local RVE geometry. However, their value remains qualitative, as the exact shear correction factor for the heterogeneous RVE for different geometries remains unknown;
- Statistical homogenisation of locally evaluated stiffnesses leads to an effective ply response that can be represented accurately by a power-mean mixture law with a volume-fraction-dependent exponent;
- The calibrated power-mean exponent shows only weak material dependence, showing that the dominant driver of the effective mixture behaviour is the fiber volume fraction rather than the constituent stiffness contrast;
- Constituent failure is governed by the relative strength-to-stiffness ratios of fiber and matrix, leading to fiber-dominated failure in carbon systems and matrix-dominated failure in glass systems;
- The effectiveness of the pristine RVE to provide an upper bound to the macroscopic ply stress is a function of the effectiveness of the calibrated power-mean mixture law in predicting the initial ply stiffness;
- The original elastic-foundation idealisation of the matrix provides insufficient excess strain energy to drive substantial crack growth, resulting in an RVE response that is too weakly nonlinear compared with the experimentally observed ply response, thereby not providing a lower bound to the macroscopic ply stress;
- Softening the elastic foundation by strongly reducing the matrix Young's modulus, which is equivalent to removing the Winkler springs, increases the nonlinearity of the degraded RVE response, but does not remove the need for non-zero constituent stiffness corrections and therefore does not simultaneously resolve both the initial-stiffness and crack-growth problems;
- The Ramberg-Osgood representation of matrix nonlinearity does not increase the nonlinear response of the RVEs sufficiently, and available published matrix data show inconsistencies between shear stress-strain curves and initial shear modulus;
- The assumed fiber-failure statistics lead to a ply response dominated by intact RVEs at low to medium strains followed by a rather abrupt transition to broken RVEs, indicating that a broader failure-strain distribution would be required for a more gradual transition;

- The full framework reproduces the qualitative mechanisms governing the ply-level shear stress-strain response, but the model remains underdetermined, since different combinations of matrix stiffness idealisation and failure-fraction evolution can reproduce similar macroscopic responses;
- Independent characterisation of matrix behaviour and failure statistics is therefore required to calibrate the framework.

Based on these conclusions, the research questions of chapter 1 can be reflected upon:

- The stress-strain response of a UD ply under in-plane shear can be constructed from constituent-level properties using a micromechanical framework, provided that appropriate homogenisation and calibration steps are introduced. The resulting framework captures the governing deformation and failure mechanisms, but does not result in a unique prediction without additional input on matrix behaviour and failure statistics;
- The formulation of the RVE as a Timoshenko beam on an elastic foundation provides a tractable and physically interpretable representation of the fiber-matrix system, capable of capturing shear deformation and load transfer, although its predictive accuracy is limited by the idealisation of the matrix and unknown parameters such as the shear correction factor;
- Local failure mechanisms, in particular fiber fracture and subsequent matrix cracking, are shown to govern the evolution of the global response through load redistribution over a finite characteristic length, and can be incorporated consistently within the analytical framework, but the quantitative impact is also limited by the idealisation of the matrix;
- The homogenisation of intact and damaged material regions through a statistical ensemble of RVEs provides a consistent means of linking local degradation to macroscopic response, but the resulting predictions depend strongly on the assumed distribution of failure strains and therefore require improved statistical characterisation.

### 9.3. Recommendations

The following recommendations concern improvements that can be implemented within the current modelling framework without requiring fundamental changes to its structure, but are not necessarily straightforward to obtain. The focus is limitations that have been identified and where refinement or removal of approximations is expected to directly improve the model:

- Including a solution for the shear correction factor of heterogeneous beams;
- Including a solution for the crack-opening displacement of a thick-walled cylinder under transverse shear, such that
  - a better comparison between interfacial and radial matrix cracking can be made and;
  - a phenomenological comparison against test specimens can be made.
- Including more experimental data on shear failure of fibers to improve:
  - Failure determination of the constituents;
  - Homogenisation of the macroscopic ply response;
- Improving the statistical description of fiber failure to better capture the gradual evolution of the failure fraction;
- Including improved experimental data on the shear behaviour of the matrix to improve modelling of its nonlinear response.

In addition to the immediate improvements outlined above, several other directions can be identified that extend beyond the current scope of this work. These topics aim to further develop the physical fidelity of the framework or explore alternative modelling approaches:

- Considering the anisotropy of fibers, in relation to stiffness and strength;
- Considering the inclusion of an interphase, reflecting the in-situ properties of the matrix, which are subject to material processing;

- Investigating interactions between neighbouring RVEs, including load redistribution and clustering effects;
- Including rate-dependent matrix behaviour to extend the methodology to in-plane shear loading of plies at non-zero strain rates.

# References

- [1] I. M. Daniel and O. Ishai, Eds., *Engineering mechanics of composite materials*, Second edition. New York: Oxford University Press, 2006.
- [2] A. K. Kaw, *Mechanics of composite materials* (Mechanical engineering series 29), 2. ed. Boca Raton, FL: CRC Taylor & Francis, 2006.
- [3] G. Dvorak, *Micromechanics of Composite Materials* (Solid Mechanics and Its Applications). Dordrecht: Springer Netherlands, 2013, vol. 186. DOI: 10.1007/978-94-007-4101-0.
- [4] V. A. Buryachenko, *Micromechanics of heterogeneous materials*. New York: Springer, 2007.
- [5] C. Kassapoglou, *Modeling the Effect of Damage in Composite Structures: Simplified Approaches*, 1st ed. Wiley, Mar. 2015. DOI: 10.1002/9781119013228.
- [6] C. Kassapoglou, "Determination of energy dissipation during cyclic loading and its use to predict fatigue life of metal alloys," *Fatigue & Fracture of Engineering Materials & Structures*, vol. 46, no. 10, pp. 3667–3679, Oct. 2023. DOI: 10.1111/ffe.14096.
- [7] C. A. Socci and C. Kassapoglou, "Effect of matrix and fibre cracks on static and fatigue strength of composite laminates," *Journal of Composite Materials*, vol. 57, no. 2, pp. 235–251, Jan. 2023. DOI: 10.1177/00219983221140638.
- [8] C. A. Socci and C. Kassapoglou, "Prediction of matrix crack initiation and evolution and their effect on the stiffness of laminates with off-axis plies under in-plane loading," *Composites Science and Technology*, vol. 200, p. 108427, Nov. 2020. DOI: 10.1016/j.compscitech.2020.108427.
- [9] R. Wojtuszewski, K. Farbaniec, C. Kassapoglou, and S. Ahsan, "Static and fatigue performance of highly loaded thermoplastic fittings," *Journal of Thermoplastic Composite Materials*, vol. 39, no. 4, pp. 1781–1800, Apr. 2026. DOI: 10.1177/08927057251375849.
- [10] H. L. Cox, "The elasticity and strength of paper and other fibrous materials," *British Journal of Applied Physics*, vol. 3, no. 3, pp. 72–79, Mar. 1952. DOI: 10.1088/0508-3443/3/3/302.
- [11] J. A. Nairn, "Fracture Mechanics of Unidirectional Composites Using the Shear-Lag Model I: Theory," *Journal of Composite Materials*, vol. 22, no. 6, pp. 561–588, Jun. 1988. DOI: 10.1177/002199838802200604.
- [12] R. Talreja and C. V. Singh, *Damage and failure of composite materials*. Cambridge: Cambridge university press, 2012.
- [13] F. Esrail and C. Kassapoglou, "An efficient approach for damage quantification in quasi-isotropic composite laminates under low speed impact," *Composites Part B: Engineering*, vol. 61, pp. 116–126, May 2014. DOI: 10.1016/j.compositesb.2014.01.033.
- [14] L. Huo, C. Kassapoglou, and R. C. Alderliesten, "A criterion for predicting delamination growth in composite laminates," *Materials & Design*, vol. 223, p. 111160, Nov. 2022. DOI: 10.1016/j.matdes.2022.111160.
- [15] R. M. Jones, *Mechanics of composite materials*, Second edition. Boca Raton London New York: CRC Press, Taylor & Francis Group, 2021.
- [16] A. Öchsner, *Micromechanics of Fiber-Reinforced Laminae* (SpringerBriefs in Applied Sciences and Technology). Cham: Springer International Publishing, 2022. DOI: 10.1007/978-3-030-94091-1.
- [17] S. W. Tsai and H. T. Hahn, *Introduction to Composite Materials*. Lancaster, PA, USA: Technomic Publishing Company, 1980.
- [18] G. W. Milton, *The Theory of Composites* (Cambridge Monographs on Applied and Computational Mathematics). Cambridge University Press, 2002.

- [19] V. A. Buryachenko, *Local and Nonlocal Micromechanics of Heterogeneous Materials*. Cham: Springer International Publishing, 2022. DOI: 10.1007/978-3-030-81784-8.
- [20] N. An, Q. Jia, H. Jin, X. Ma, and J. Zhou, "Multiscale modeling of viscoelastic behavior of unidirectional composite laminates and deployable structures," *Materials & Design*, vol. 219, p. 110754, Jul. 2022. DOI: 10.1016/j.matdes.2022.110754.
- [21] S. H. R. Sanei and R. S. Fertig, "Uncorrelated volume element for stochastic modeling of microstructures based on local fiber volume fraction variation," *Composites Science and Technology*, vol. 117, pp. 191–198, Sep. 2015. DOI: 10.1016/j.compscitech.2015.06.010.
- [22] A. Endruweit, F. Gommer, and A. Long, "Stochastic analysis of fibre volume fraction and permeability in fibre bundles with random filament arrangement," *Composites Part A: Applied Science and Manufacturing*, vol. 49, pp. 109–118, Jun. 2013. DOI: 10.1016/j.compositesa.2013.02.012.
- [23] P. Hubert and A. Poursartip, "A review of flow and compaction modelling relevant to thermoset matrix laminate processing," *Journal of Reinforced Plastics and Composites*, vol. 17, no. 4, pp. 286–318, 1998. DOI: 10.1177/073168449801700402.
- [24] C.-h. Tsai, C. Zhang, D. A. Jack, R. Liang, and B. Wang, "The effect of inclusion waviness and waviness distribution on elastic properties of fiber-reinforced composites," *Composites Part B: Engineering*, vol. 42, no. 1, pp. 62–70, Jan. 2011. DOI: 10.1016/j.compositesb.2010.09.004.
- [25] F. Zhaohua and R. D. Cook, "Beam Elements on Two-Parameter Elastic Foundations," *Journal of Engineering Mechanics*, vol. 109, no. 6, pp. 1390–1402, Dec. 1983. DOI: 10.1061/(asce)0733-9399(1983)109:6(1390).
- [26] M. Ostoja-Starzewski, "Material spatial randomness: From statistical to representative volume element," *Probabilistic Engineering Mechanics*, vol. 21, no. 2, pp. 112–132, Apr. 2006. DOI: 10.1016/j.probenmech.2005.07.007.
- [27] S. H. R. Sanei and R. S. Fertig, "Length-scale dependence of variability in epoxy modulus extracted from composite prepreg," *Polymer Testing*, vol. 50, pp. 297–300, Apr. 2016. DOI: 10.1016/j.polymertesting.2015.12.015.
- [28] F. H. Bhuiyan, S. H. R. Sanei, and R. S. Fertig, "Predicting variability in transverse effective elastic moduli and failure initiation strengths in UD composite microstructures due to randomness in fiber location and morphology," *Composite Structures*, vol. 237, p. 111887, Apr. 2020. DOI: 10.1016/j.compstruct.2020.111887.
- [29] L. Wan et al., "Computational micromechanics-based prediction of the failure of unidirectional composite lamina subjected to transverse and in-plane shear stress states," *Journal of Composite Materials*, vol. 54, no. 24, pp. 3637–3654, Oct. 2020. DOI: 10.1177/0021998320918015.
- [30] L. Yang, Z. Wu, Y. Cao, and Y. Yan, "Micromechanical modelling and simulation of unidirectional fibre-reinforced composite under shear loading," *Journal of Reinforced Plastics and Composites*, vol. 34, no. 1, pp. 72–83, Jan. 2015. DOI: 10.1177/0731684414562873.
- [31] Yuanchen Huang, Kyo Kook Jin, and Sung Kyu Ha, "Effects of Fiber Arrangement on Mechanical Behavior of Unidirectional Composites," *Journal of Composite Materials*, vol. 42, no. 18, pp. 1851–1871, Sep. 2008. DOI: 10.1177/0021998308093910.
- [32] V. A. Buryachenko, "New RVE concept and FFT methods in micromechanics of composites subjected to body force with compact support - Valeriy A Buryachenko, 2026," *Mathematics and Mechanics of Solids*, Mar. 2026.
- [33] W. Yu, "An Introduction to Micromechanics," *Applied Mechanics and Materials*, vol. 828, pp. 3–24, Mar. 2016. DOI: 10.4028/www.scientific.net/AMM.828.3.
- [34] W. Voigt, "Ueber die Beziehung zwischen den beiden Elasticitätsconstanten isotroper Körper," *Annalen der Physik*, vol. 274, no. 12, pp. 573–587, 1889. DOI: 10.1002/andp.18892741206.
- [35] A. Reuss, "Berechnung der Fließgrenze von Mischkristallen auf Grund der Plastizitätsbedingung für Einkristalle .," *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 9, no. 1, pp. 49–58, 1929. DOI: 10.1002/zamm.19290090104.

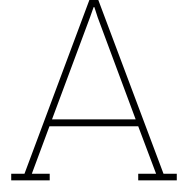
- [36] R. Hill, "The Elastic Behaviour of a Crystalline Aggregate," *Proceedings of the Physical Society. Section A*, vol. 65, no. 5, p. 349, May 1952. DOI: 10.1088/0370-1298/65/5/307.
- [37] Z. Hashin and S. Shtrikman, "A variational approach to the theory of the elastic behaviour of multiphase materials," *Journal of the Mechanics and Physics of Solids*, vol. 11, no. 2, pp. 127–140, Mar. 1963. DOI: 10.1016/0022-5096(63)90060-7.
- [38] G. J. Weng, "The theoretical connection between Mori-Tanaka's theory and the Hashin-Shtrikman-Walpole bounds," *International Journal of Engineering Science*, vol. 28, no. 11, pp. 1111–1120, Jan. 1990. DOI: 10.1016/0020-7225(90)90111-U.
- [39] T. Mori and K. Tanaka, "Average stress in matrix and average elastic energy of materials with misfitting inclusions," *Acta Metallurgica*, vol. 21, no. 5, pp. 571–574, May 1973. DOI: 10.1016/0001-6160(73)90064-3.
- [40] J. C. H. Afdl and J. L. Kardos, "The Halpin-Tsai equations: A review," *Polymer Engineering & Science*, vol. 16, no. 5, pp. 344–352, May 1976. DOI: 10.1002/pen.760160512.
- [41] C. C. Chamis, "Simplified Composite Micromechanics For Predicting Microstresses," *Journal of Reinforced Plastics and Composites*, vol. 6, no. 3, pp. 268–289, Jul. 1987. DOI: 10.1177/073168448700600305.
- [42] N. Hu, Ed., *Composites and Their Properties*. InTech, Aug. 2012. DOI: 10.5772/2816.
- [43] C. L. Dym and I. H. Shames, *Solid Mechanics: A Variational Approach, Augmented Edition*. New York, NY: Springer New York, 2013. DOI: 10.1007/978-1-4614-6034-3.
- [44] G. Xia, "Generalized foundation Timoshenko beam and its calculating methods," *Archive of Applied Mechanics*, vol. 92, no. 3, pp. 1015–1036, Mar. 2022. DOI: 10.1007/s00419-021-02090-1.
- [45] C. W. Bert, "Mathematical modeling and micromechanics of fiber-reinforced bimodulus composite materials," University of Oklahoma, Norman, Oklahoma, Tech. Rep. AD-A071389, Jun. 1979.
- [46] S. Pimenta, R. Gutkin, S. T. Pinho, and P. Robinson, "A micromechanical model for kink-band formation: Part II—Analytical modelling," *Composites Science and Technology*, vol. 69, no. 7, pp. 956–964, Jun. 2009. DOI: 10.1016/j.compscitech.2009.02.003.
- [47] A. M. Waas, C. D. Babcock, and W. G. Knauss, "A Mechanical Model for Elastic Fiber Microbuckling," *Journal of Applied Mechanics*, vol. 57, no. 1, pp. 138–149, Mar. 1990. DOI: 10.1115/1.2888295.
- [48] S. Timoshenko, "Lxvi. on the correction for shear of the differential equation for transverse vibrations of prismatic bars," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 41, no. 245, pp. 744–746, 1921. DOI: 10.1080/14786442108636264.
- [49] G. R. Cowper, "The Shear Coefficient in Timoshenko's Beam Theory," *Journal of Applied Mechanics*, vol. 33, no. 2, pp. 335–340, Jun. 1966. DOI: 10.1115/1.3625046.
- [50] J. R. Hutchinson, "Shear Coefficients for Timoshenko Beam Theory," *Journal of Applied Mechanics*, vol. 68, no. 1, pp. 87–92, Jan. 2001. DOI: 10.1115/1.1349417.
- [51] S. Dong, C. Alpdogan, and E. Taciroglu, "Much ado about shear correction factors in Timoshenko beam theory," *International Journal of Solids and Structures*, vol. 47, no. 13, pp. 1651–1665, Jun. 2010. DOI: 10.1016/j.ijsolstr.2010.02.018.
- [52] V. Birman and C. W. Bert, "On the Choice of Shear Correction Factor in Sandwich Structures," *Journal of Sandwich Structures & Materials*, vol. 4, no. 1, pp. 83–95, Jan. 2002. DOI: 10.1177/1099636202004001180.
- [53] H. Hadavinia, K. Gordnian, J. Karwatzki, and A. Aboutorabi, "Deriving Shear Correction Factor for Thick Laminated Plates Using the Energy Equivalence Method," *Structural Durability & Health Monitoring*, vol. 2, no. 04, 2006. DOI: 10.3970/sdhm.2006.002.197.
- [54] E. Suhir, "Lateral compliance of a compressed rod supported by an elastic foundation, with application to micro-electronic and fibre-optic structures," *Journal of Physics D: Applied Physics*, vol. 41, no. 1, p. 015 502, Jan. 2008. DOI: 10.1088/0022-3727/41/1/015502.

- [55] E. Suhir, "Elastic Stability of a Cantilever Beam (Rod) Supported by an Elastic Foundation, With Application to Nano-Composites," *Journal of Applied Mechanics*, vol. 79, no. 1, Jan. 2012. DOI: 10.1115/1.4005190.
- [56] Emil Winkler, *Die Lehre von der Elastizität und Festigkeit mit besonderer Rücksicht auf ihre Anwendung in der Technik für polytechnische Schulen, Bauakademien, Ingenieure, Maschinenbauer, Architekten, etc.* Dominicius, 1868.
- [57] A. D. Kerr, "A study of a new foundation model," *Acta Mechanica*, vol. 1, no. 2, pp. 135–147, 1965. DOI: 10.1007/BF01379766.
- [58] A. D. Kerr and W. J. Rhines, "A further study of elastic foundation models," New York University, School of Engineering and Science, Bronx, NY, USA, S-67-1, 1967.
- [59] A. P. S. Selvadurai, *Elastic analysis of soil-foundation interaction* (Developments in geotechnical engineering v. 17). Amsterdam New York: Elsevier Scientific Pub. Co. Distributors for the United States and Canada, Elsevier-North Holland, 1979.
- [60] C. Kassapoglou, *Design and analysis of composite structures: with applications to aerospace structures* (Aerospace series), Second ed. West Sussex, United Kingdom: John Wiley & Sons Inc, 2013.
- [61] B. J. Fregly, Y. Bei, and M. E. Sylvester, "Experimental evaluation of an elastic foundation model to predict contact pressures in knee replacements," *Journal of Biomechanics*, vol. 36, no. 11, pp. 1659–1668, Nov. 2003. DOI: 10.1016/S0021-9290(03)00176-3.
- [62] H. T. Hahn and S. W. Tsai, "Nonlinear Elastic Behavior of Unidirectional Composite Laminae," *Journal of Composite Materials*, vol. 7, no. 1, pp. 102–118, Jan. 1973. DOI: 10.1177/002199837300700108.
- [63] R. S. Sandhu, "Nonlinear Behavior of Unidirectional and Angle Ply Laminates," *Journal of Aircraft*, vol. 13, no. 2, pp. 104–111, Feb. 1976. DOI: 10.2514/3.58638.
- [64] A. Kaddour, M. Hinton, P. Smith, and S. Li, "Mechanical properties and details of composite laminates for the test cases used in the third world-wide failure exercise," *Journal of Composite Materials*, vol. 47, no. 20-21, pp. 2427–2442, Sep. 2013. DOI: 10.1177/0021998313499477.
- [65] W. Van Paeppegem, I. De Baere, and J. Degrieck, "Modelling the nonlinear shear stress–strain response of glass fibre-reinforced composites. Part I: Experimental results," *Composites Science and Technology*, vol. 66, no. 10, pp. 1455–1464, Aug. 2006. DOI: 10.1016/j.compscitech.2005.04.014.
- [66] R. Talreja, "Fatigue damage mechanisms," in *Modeling Damage, Fatigue and Failure of Composite Materials*, Elsevier, 2016, pp. 25–40. DOI: 10.1016/B978-1-78242-286-0.00002-9.
- [67] F. Campbell, *Structural Composite Materials*. ASM International, 2010.
- [68] I. Baran, K. Cinar, N. Ersoy, R. Akkerman, and J. H. Hattel, "A Review on the Mechanical Modeling of Composite Manufacturing Processes," *Archives of Computational Methods in Engineering*, vol. 24, no. 2, pp. 365–395, Apr. 2017. DOI: 10.1007/s11831-016-9167-2.
- [69] P. Hubert and A. Poursartip, "Aspects of the Compaction of Composite Angle Laminates: An Experimental Investigation," *Journal of Composite Materials*, vol. 35, no. 1, pp. 2–26, Jan. 2001. DOI: 10.1177/002199801772661849.
- [70] R. Talreja, "Incorporating manufacturing defects in damage and failure analysis," in *Modeling Damage, Fatigue and Failure of Composite Materials*, Elsevier, 2016, pp. 377–390. DOI: 10.1016/B978-1-78242-286-0.00017-0.
- [71] W. Chili, S. Galea, and R. Jones, "The role of material nonlinearities in composite structures," *Composite Structures*, vol. 38, no. 1-4, pp. 71–81, May 1997. DOI: 10.1016/S0263-8223(97)00043-3.
- [72] R. Schapery, "On the characterization of nonlinear viscoelastic materials," *Polymer Engineering Science*, vol. 9, pp. 295–310, Jul. 1969. DOI: 10.1002/pen.760090410.
- [73] S. M. Mohseni Shakib and S. Li, "A nonlinear in-plane shear model for ud composites," in *Proceedings of the 16th International Conference on Composite Materials (ICCM-16)*, Kyoto, Japan, 2007.

- [74] D. Krajcinovic, B. M., and S. D., "Basic principles," in *Damage Mechanics of Composite Materials*, R. Talreja, Ed., Amsterdam: Elsevier Science, 1994, pp. 187–243.
- [75] K. Giannadakis and J. Varna, "Analysis of nonlinear shear stress–strain response of unidirectional GF/EP composite," *Composites Part A: Applied Science and Manufacturing*, vol. 62, pp. 67–76, Jul. 2014. DOI: 10.1016/j.compositesa.2014.03.009.
- [76] W. Van Paepegem, I. De Baere, and J. Degrieck, "Modelling the nonlinear shear stress–strain response of glass fibre-reinforced composites. Part II: Model development and finite element simulations," *Composites Science and Technology*, vol. 66, no. 10, pp. 1465–1478, Aug. 2006. DOI: 10.1016/j.compscitech.2005.04.018.
- [77] C. C. Chamis et al., "Micromechanics-based progressive failure analysis prediction for WWFE-III composite coupon test cases," *Journal of Composite Materials*, vol. 47, no. 20-21, pp. 2695–2712, Sep. 2013. DOI: 10.1177/0021998313499478.
- [78] H. J. Herrmann, "1 - introduction to basic notions and facts," in *Statistical Models for the Fracture of Disordered Media*, ser. Random Materials and Processes, H. J. Herrmann and S. Roux, Eds., Amsterdam: North-Holland, 1990, pp. 1–31. DOI: <https://doi.org/10.1016/B978-0-444-88551-7.50007-0>.
- [79] "Composite materials handbook: Volume 1 — polymer matrix composites: Guidelines for characterization of structural materials," Department of Defense, Tech. Rep. MIL-HDBK-17-1F, Jun. 2002.
- [80] "The fiber bundle model," in *The Fiber Bundle Model*. John Wiley Sons, Ltd, 2015, ch. 1, pp. 1–10. DOI: <https://doi.org/10.1002/9783527671960.ch1>.
- [81] M. R. Wisnom, "7.4 size effects in composites," in *Comprehensive composite materials II*, P. W. Beaumont and C. H. Zweben, Eds., Oxford: Elsevier, 2018, pp. 56–78. DOI: <https://doi.org/10.1016/B978-0-12-803581-8.03918-7>.
- [82] D. S. Cairns and D. F. Adams, "Moisture and Thermal Expansion Properties of Unidirectional Composite Materials and the Epoxy Matrix," *Journal of Reinforced Plastics and Composites*, vol. 2, no. 4, pp. 239–255, Oct. 1983. DOI: 10.1177/073168448300200403.
- [83] Hexcel Corporation, "HexTow® IM7 Carbon Fiber: Product Data Sheet," Hexcel Corporation, USA, Product Data Sheet, 2023.
- [84] A. Kaddour, M. Hinton, P. Smith, and S. Li, "A comparison between the predictive capability of matrix cracking, damage and failure criteria for fibre reinforced composite laminates: Part A of the third world-wide failure exercise," *Journal of Composite Materials*, vol. 47, no. 20-21, pp. 2749–2779, Sep. 2013. DOI: 10.1177/0021998313499476.
- [85] D. A. Dillard, B. Mukherjee, P. Karnal, R. C. Batra, and J. Frechette, "A review of Winkler's foundation and its profound influence on adhesion and soft matter applications," *Soft Matter*, vol. 14, no. 19, pp. 3669–3683, 2018. DOI: 10.1039/C7SM02062G.
- [86] S. Timoshenko and J. N. Goodier, *Theory of Elasticity*. McGraw-Hill, 1969.
- [87] A. Steinboeck, A. Kugi, and H. Mang, "Energy-consistent shear coefficients for beams with circular cross sections and radially inhomogeneous materials," *International Journal of Solids and Structures*, vol. 50, no. 11-12, pp. 1859–1868, Jun. 2013. DOI: 10.1016/j.ijsolstr.2013.01.030.
- [88] ANSYS, Inc., *Solid186 element description*, ANSYS, Inc., 2018.
- [89] *Asme boiler and pressure vessel code, section ii, division 2*, American Society of Mechanical Engineers, 1998.
- [90] P. F. Pai, "A new look at shear correction factors and warping functions of anisotropic laminates," *International Journal of Solids and Structures*, vol. 32, no. 16, pp. 2295–2313, Aug. 1995. DOI: 10.1016/0020-7683(94)00258-X.
- [91] D. Hull and T. W. Clyne, *An Introduction to Composite Materials* (Cambridge Solid State Science Series), 2nd ed. Cambridge University Press, 1996.



- [92] Department of Defense, USA, "Military handbook: Metallic materials and elements for aerospace vehicle structures (mil-hdbk-5h)," Department of Defense, Washington, DC, Handbook MIL-HDBK-5H, Dec. 1998.
- [93] A. K. Gupta and S. Nadarajah, Eds., *Handbook of Beta Distribution and Its Applications* (Statistics: A Series of Textbooks and Monographs). New York: Marcel Dekker, 2004, vol. 174.
- [94] S. Ji, "Generalized means as an approach for predicting Young's moduli of multiphase materials," *Materials Science and Engineering: A*, vol. 366, no. 1, pp. 195–201, Feb. 2004. DOI: 10.1016/j.msea.2003.09.066.
- [95] M. de Jong et al., "A Statistical Learning Framework for Materials Science: Application to Elastic Moduli of k-nary Inorganic Polycrystalline Compounds," *Scientific Reports*, vol. 6, no. 1, p. 34 256, Oct. 2016. DOI: 10.1038/srep34256.
- [96] Z.-M. Huang, "On micromechanics approach to stiffness and strength of unidirectional composites," *Journal of Reinforced Plastics and Composites*, vol. 38, no. 4, pp. 167–196, Feb. 2019. DOI: 10.1177/0731684418811938.
- [97] Z.-m. Huang, "Micromechanical prediction of ultimate strength of transversely isotropic fibrous composites," *International Journal of Solids and Structures*, vol. 38, no. 22, pp. 4147–4172, May 2001. DOI: 10.1016/S0020-7683(00)00268-7.
- [98] M. Shioya, S. Yasui, and A. Takaku, "A refined method for estimating fiber and interfacial shear strength by using a single-fiber composite," *Composite Interfaces*, vol. 4, no. 6, pp. 379–399, Jan. 1996. DOI: 10.1163/156855497X00226.
- [99] H. F. Nied and F. Erdogan, "The elasticity problem for a thick-walled cylinder containing a circumferential crack," *International Journal of Fracture*, vol. 22, no. 4, pp. 277–301, Aug. 1983. DOI: 10.1007/BF01140157.
- [100] G. C. Sih, Ed., *Methods of analysis and solutions of crack problems*. Dordrecht: Springer Netherlands, 1973. DOI: 10.1007/978-94-017-2260-5.
- [101] C. Qian, T. Westphal, C. Kassapoglou, and R. P. L. Nijssen, "Development of a multi-fibre unit cell for use in modelling of fatigue of unidirectional composites," *Composite Structures*, vol. 99, pp. 288–295, May 2013. DOI: 10.1016/j.compstruct.2012.11.010.
- [102] C. Hong, S. Lee, and W. Ji, "Shear strength determining mechanism of a +/-45 laminate under tensile loading," *Composite Structures*, vol. 326, p. 117 627, Dec. 2023. DOI: 10.1016/j.compstruct.2023.117627.
- [103] W. Ramberg and W. R. Osgood, "Description of stress-strain curves by three parameters," National Advisory Committee for Aeronautics (NACA), Washington, DC, USA, TN-902, 1943.
- [104] C. Cheng, Z. Zong, and R. Mahnken, "Enhanced mean-field modelling for impact response of composite laminates incorporating strain rate-dependent matrix behaviour and 3D failure criteria," *Archive of Applied Mechanics*, vol. 94, no. 11, pp. 3573–3595, Nov. 2024. DOI: 10.1007/s00419-024-02684-5.
- [105] A. Gilat, R. K. Goldberg, and G. D. Roberts, "Strain Rate Sensitivity of Epoxy Resin in Tensile and Shear Loading," *Journal of Aerospace Engineering*, vol. 20, no. 2, pp. 75–89, Apr. 2007. DOI: 10.1061/(ASCE)0893-1321(2007)20:2(75).



# Algebraic Derivations

Using the general solutions of subsection 4.3.2 to the governing differential equation of subsection 4.3.1, which are here repeated for clarity:

$$w(x) = \begin{cases} (Ae^{\alpha x} + Ce^{-\alpha x}) \cos \beta x + (Be^{\alpha x} + De^{-\alpha x}) \sin \beta x, & \Delta < 0, \\ (Ae^{\alpha x} + Ce^{-\alpha x}) \cosh \beta x + (Be^{\alpha x} + De^{-\alpha x}) \sinh \beta x, & \Delta > 0. \end{cases} \quad (\text{A.1})$$

$$\frac{d^4 w}{dx^4} - \frac{K_w D_b + R_i C_s + T_s C_s + R_i T_s}{D_b(C_s + T_s)} \frac{d^2 w}{dx^2} + \frac{K_w(R_i + C_s)}{D_b(C_s + T_s)} w = 0 \quad (\text{A.2})$$

$$f(x) = \begin{cases} \left[ (c_{1,f}^{(-)} A + c_{2,f}^{(-)} B) e^{\alpha x} + (c_{3,f}^{(-)} C + c_{4,f}^{(-)} D) e^{-\alpha x} \right] \cos \beta x \\ \quad + \left[ (c_{5,f}^{(-)} A + c_{6,f}^{(-)} B) e^{\alpha x} + (c_{7,f}^{(-)} C + c_{8,f}^{(-)} D) e^{-\alpha x} \right] \sin \beta x, & \Delta < 0, \\ \left[ (c_{1,f}^{(+)} A + c_{2,f}^{(+)} B) e^{\alpha x} + (c_{3,f}^{(+)} C + c_{4,f}^{(+)} D) e^{-\alpha x} \right] \cosh \beta x \\ \quad + \left[ (c_{5,f}^{(+)} A + c_{6,f}^{(+)} B) e^{\alpha x} + (c_{7,f}^{(+)} C + c_{8,f}^{(+)} D) e^{-\alpha x} \right] \sinh \beta x, & \Delta > 0. \end{cases} \quad (\text{A.3})$$

where  $f(x)$  denotes the field quantity of interest,

$$f(x) \in \{w(x), \phi(x), Q(x), M(x), |\varepsilon_{11}(x)|, \gamma_{13}(x)\}.$$

The coefficients  $c_{i,f}^{(\pm)}$  depend on the selected field variable and on the sign of the characteristic parameter  $\Delta$ . The index  $i = 1, \dots, 8$  denotes the coefficient number, the subscript  $f$  indicates the associated field variable, and the superscripts  $(\pm)$  correspond to the cases  $\Delta < 0$  and  $\Delta > 0$ , respectively. All coefficients are documented in Table A.1-A.6.

| $i$ | $c_{i,w}^{(-)}$ | $c_{i,w_x}^{(-)}$ | $c_{i,w_{xx}}^{(-)}$ | $c_{i,w_{xxx}}^{(-)}$        | $c_{i,w_{xxxx}}^{(-)}$                  |
|-----|-----------------|-------------------|----------------------|------------------------------|-----------------------------------------|
| 1   | 1               | $\alpha$          | $\alpha^2 - \beta^2$ | $\alpha^3 - 3\alpha\beta^2$  | $\alpha^4 + \beta^4 - 6\alpha^2\beta^2$ |
| 2   | 0               | $\beta$           | $2\alpha\beta$       | $-\beta^3 + 3\alpha^2\beta$  | $4\alpha^3\beta - 4\alpha\beta^3$       |
| 3   | 1               | $-\alpha$         | $\alpha^2 - \beta^2$ | $-\alpha^3 + 3\alpha\beta^2$ | $\alpha^4 + \beta^4 - 6\alpha^2\beta^2$ |
| 4   | 0               | $-\beta$          | $-2\alpha\beta$      | $-\beta^3 + 3\alpha^2\beta$  | $4\alpha\beta^3 - 4\alpha^3\beta$       |
| 5   | 0               | $\beta$           | $\alpha^2 - \beta^2$ | $\beta^3 - 3\alpha^2\beta$   | $4\alpha\beta^3 - 4\alpha^3\beta$       |
| 6   | 1               | $\alpha$          | $-2\alpha\beta$      | $\alpha^3 - 3\alpha\beta^2$  | $\alpha^4 + \beta^4 - 6\alpha^2\beta^2$ |
| 7   | 0               | $-\beta$          | $\alpha^2 - \beta^2$ | $\beta^3 - 3\alpha^2\beta$   | $4\alpha^3\beta - 4\alpha\beta^3$       |
| 8   | 1               | $-\alpha$         | $2\alpha\beta$       | $-\alpha^3 + 3\alpha\beta^2$ | $\alpha^4 + \beta^4 - 6\alpha^2\beta^2$ |

**Table A.1:** Coefficient sets associated with the displacement field and its derivatives for  $\Delta < 0$ .

| $i$ | $c_{i,\phi}^{(-)}$                          | $c_{i,Q}^{(-)}$                             | $c_{i,M}^{(-)}$             |
|-----|---------------------------------------------|---------------------------------------------|-----------------------------|
| 1   | $\alpha U + (\alpha^3 - 3\alpha\beta^2)V$   | $\alpha I - (\alpha^3 - 3\alpha\beta^2)J$   | $H - G(\alpha^2 - \beta^2)$ |
| 2   | $\beta U + (-\beta^3 + 3\alpha^2\beta)V$    | $\beta I - (-\beta^3 + 3\alpha^2\beta)J$    | $-2\alpha\beta G$           |
| 3   | $-\alpha U + (-\alpha^3 + 3\alpha\beta^2)V$ | $-\alpha I - (-\alpha^3 + 3\alpha\beta^2)J$ | $H - G(\alpha^2 - \beta^2)$ |
| 4   | $-\beta U + (-\beta^3 + 3\alpha^2\beta)V$   | $\beta I - (-\beta^3 + 3\alpha^2\beta)J$    | $2\alpha\beta G$            |
| 5   | $\beta U + (\beta^3 - 3\alpha^2\beta)V$     | $-\beta I - (\beta^3 - 3\alpha^2\beta)J$    | $2\alpha\beta G$            |
| 6   | $\alpha U + (\alpha^3 - 3\alpha\beta^2)V$   | $\alpha I - (\alpha^3 - 3\alpha\beta^2)J$   | $H - G(\alpha^2 - \beta^2)$ |
| 7   | $-\beta U + (\beta^3 - 3\alpha^2\beta)V$    | $-\beta I - (\beta^3 - 3\alpha^2\beta)J$    | $-2\alpha\beta G$           |
| 8   | $-\alpha U + (-\alpha^3 + 3\alpha\beta^2)V$ | $-\alpha I - (-\alpha^3 + 3\alpha\beta^2)J$ | $H - G(\alpha^2 - \beta^2)$ |

**Table A.2:** Coefficient sets associated with rotation, shear force and bending moment for  $\Delta < 0$ .

| $i$ | $\frac{2}{d_f} \cdot c_{i, \varepsilon_{11} }^{(-)}$                                   | $c_{i,\gamma_{13}}^{(-)}$                       |
|-----|----------------------------------------------------------------------------------------|-------------------------------------------------|
| 1   | $(\alpha^2 - \beta^2)U + (\alpha^4 + \beta^4 - 6\alpha^2\beta^2)V$                     | $\alpha(1 - U) - (\alpha^3 - 3\alpha\beta^2)V$  |
| 2   | $2\alpha\beta U + (\alpha^3\beta - \alpha\beta^3 + 3\alpha^3\beta - 3\alpha\beta^3)V$  | $\beta(1 - U) - (-\beta^3 + 3\alpha^2\beta)V$   |
| 3   | $(\alpha^2 - \beta^2)U + (\alpha^4 + \beta^4 - 6\alpha^2\beta^2)V$                     | $-\alpha(1 - U) + (\alpha^3 - 3\alpha\beta^2)V$ |
| 4   | $-2\alpha\beta U + (\alpha\beta^3 - \alpha^3\beta + 3\alpha\beta^3 - 3\alpha^3\beta)V$ | $\beta(1 - U) - (-\beta^3 + 3\alpha^2\beta)V$   |
| 5   | $-2\alpha\beta U + (\alpha\beta^3 - \alpha^3\beta + 3\alpha\beta^3 - 3\alpha^3\beta)V$ | $-\beta(1 - U) + (-\beta^3 + 3\alpha^2\beta)V$  |
| 6   | $(\alpha^2 - \beta^2)U + (\alpha^4 + \beta^4 - 6\alpha^2\beta^2)V$                     | $\alpha(1 - U) - (\alpha^3 - 3\alpha\beta^2)V$  |
| 7   | $2\alpha\beta U + (\alpha^3\beta - \alpha\beta^3 + 3\alpha^3\beta - 3\alpha\beta^3)V$  | $-\beta(1 - U) + (-\beta^3 + 3\alpha^2\beta)V$  |
| 8   | $(\alpha^2 - \beta^2)U + (\alpha^4 + \beta^4 - 6\alpha^2\beta^2)V$                     | $-\alpha(1 - U) + (\alpha^3 - 3\alpha\beta^2)V$ |

**Table A.3:** Coefficient sets associated with strain quantities for  $\Delta < 0$ .

| $i$ | $c_{i,w}^{(+)}$ | $c_{i,w_x}^{(+)}$ | $c_{i,w_{xx}}^{(+)}$ | $c_{i,w_{xxx}}^{(+)}$        | $c_{i,w_{xxxx}}^{(+)}$                  |
|-----|-----------------|-------------------|----------------------|------------------------------|-----------------------------------------|
| 1   | 1               | $\alpha$          | $\alpha^2 + \beta^2$ | $\alpha^3 + 3\alpha\beta^2$  | $\alpha^4 + \beta^4 + 6\alpha^2\beta^2$ |
| 2   | 0               | $\beta$           | $2\alpha\beta$       | $\beta^3 + 3\alpha^2\beta$   | $4\alpha^3\beta + 4\alpha\beta^3$       |
| 3   | 1               | $-\alpha$         | $\alpha^2 + \beta^2$ | $-\alpha^3 - 3\alpha\beta^2$ | $\alpha^4 + \beta^4 + 6\alpha^2\beta^2$ |
| 4   | 0               | $\beta$           | $-2\alpha\beta$      | $\beta^3 + 3\alpha^2\beta$   | $-4\alpha^3\beta - 4\alpha\beta^3$      |
| 5   | 0               | $\beta$           | $\alpha^2 + \beta^2$ | $\beta^3 + 3\alpha^2\beta$   | $4\alpha^3\beta + 4\alpha\beta^3$       |
| 6   | 1               | $\alpha$          | $2\alpha\beta$       | $\alpha^3 + 3\alpha\beta^2$  | $\alpha^4 + \beta^4 + 6\alpha^2\beta^2$ |
| 7   | 0               | $\beta$           | $\alpha^2 + \beta^2$ | $\beta^3 + 3\alpha^2\beta$   | $-4\alpha^3\beta - 4\alpha\beta^3$      |
| 8   | 1               | $-\alpha$         | $-2\alpha\beta$      | $-\alpha^3 - 3\alpha\beta^2$ | $\alpha^4 + \beta^4 + 6\alpha^2\beta^2$ |

**Table A.4:** Coefficient sets associated with the displacement field and its derivatives for  $\Delta > 0$ .

| $i$ | $c_{i,\phi}^{(+)}$                          | $c_{i,Q}^{(+)}$                             | $c_{i,M}^{(+)}$             |
|-----|---------------------------------------------|---------------------------------------------|-----------------------------|
| 1   | $\alpha U + (\alpha^3 + 3\alpha\beta^2)V$   | $\alpha I - (\alpha^3 + 3\alpha\beta^2)J$   | $H - G(\alpha^2 + \beta^2)$ |
| 2   | $\beta U + (\beta^3 + 3\alpha^2\beta)V$     | $\beta I - (\beta^3 + 3\alpha^2\beta)J$     | $-2\alpha\beta G$           |
| 3   | $-\alpha U + (-\alpha^3 - 3\alpha\beta^2)V$ | $-\alpha I - (-\alpha^3 - 3\alpha\beta^2)J$ | $H - G(\alpha^2 + \beta^2)$ |
| 4   | $\beta U + (\beta^3 + 3\alpha^2\beta)V$     | $\beta I - (\beta^3 + 3\alpha^2\beta)J$     | $2\alpha\beta G$            |
| 5   | $\beta U + (\beta^3 + 3\alpha^2\beta)V$     | $\beta I - (\beta^3 + 3\alpha^2\beta)J$     | $-2\alpha\beta G$           |
| 6   | $\alpha U + (\alpha^3 + 3\alpha\beta^2)V$   | $\alpha I - (\alpha^3 + 3\alpha\beta^2)J$   | $H - G(\alpha^2 + \beta^2)$ |
| 7   | $\beta U + (\beta^3 + 3\alpha^2\beta)V$     | $\beta I - (\beta^3 + 3\alpha^2\beta)J$     | $2\alpha\beta G$            |
| 8   | $-\alpha U + (-\alpha^3 - 3\alpha\beta^2)V$ | $-\alpha I - (-\alpha^3 - 3\alpha\beta^2)J$ | $H - G(\alpha^2 + \beta^2)$ |

**Table A.5:** Coefficient sets associated with rotation, shear force and bending moment for  $\Delta > 0$ .

| $i$ | $\frac{2}{d_f} \cdot c_{i,\varepsilon_{11}}^{(+)}$                 | $c_{i,\gamma_{13}}^{(+)}$                        |
|-----|--------------------------------------------------------------------|--------------------------------------------------|
| 1   | $(\alpha^2 + \beta^2)U + (\alpha^4 + \beta^4 + 6\alpha^2\beta^2)V$ | $\alpha(1 - U) - (\alpha^3 + 3\alpha\beta^2)V$   |
| 2   | $2\alpha\beta U + (4\alpha\beta^3 + 4\alpha^3\beta)V$              | $\beta(1 - U) - (\beta^3 + 3\alpha^2\beta)V$     |
| 3   | $(\alpha^2 + \beta^2)U + (\alpha^4 + \beta^4 + 6\alpha^2\beta^2)V$ | $-\alpha(1 - U) - (-\alpha^3 - 3\alpha\beta^2)V$ |
| 4   | $-2\alpha\beta U - (4\alpha\beta^3 + 4\alpha^3\beta)V$             | $\beta(1 - U) - (\beta^3 + 3\alpha^2\beta)V$     |
| 5   | $2\alpha\beta U + (4\alpha\beta^3 + 4\alpha^3\beta)V$              | $\beta(1 - U) - (\beta^3 + 3\alpha^2\beta)V$     |
| 6   | $(\alpha^2 + \beta^2)U + (\alpha^4 + \beta^4 + 6\alpha^2\beta^2)V$ | $\alpha(1 - U) - (\alpha^3 + 3\alpha\beta^2)V$   |
| 7   | $-2\alpha\beta U - (4\alpha\beta^3 + 4\alpha^3\beta)V$             | $\beta(1 - U) - (\beta^3 + 3\alpha^2\beta)V$     |
| 8   | $(\alpha^2 + \beta^2)U + (\alpha^4 + \beta^4 + 6\alpha^2\beta^2)V$ | $-\alpha(1 - U) - (-\alpha^3 - 3\alpha\beta^2)V$ |

**Table A.6:** Coefficient sets associated with strain quantities for  $\Delta > 0$ .

# B

## FEA Mesh Convergence and Approach Verifications

### B.1. Mesh Convergence

A mesh convergence study was performed to assess the sensitivity of the predicted RVE shear modulus  $G_{RVE}$  to the discretisation. Two RVE geometries were considered, both under the same loading. Two baseline mesh sizes were initially evaluated, namely  $h = 0.5$  mm and  $h = 0.2$  mm. For each  $v_f$ , the relative difference in the computed shear modulus was defined as

$$\epsilon_G = \frac{G_{RVE}(h) - G_{RVE}(h_{ref})}{G_{RVE}(h_{ref})}, \quad (B.1)$$

where  $h_{ref} = 0.5$  mm is taken as the reference mesh.

To account for the changing geometry with  $v_f$ , the mesh density was expressed in terms of the ratio between a characteristic RVE dimension, taken as the fiber center-to-center spacing  $s$  from Equation 4.1, and the mesh size  $h$ :

$$\eta = \frac{s}{h}. \quad (B.2)$$

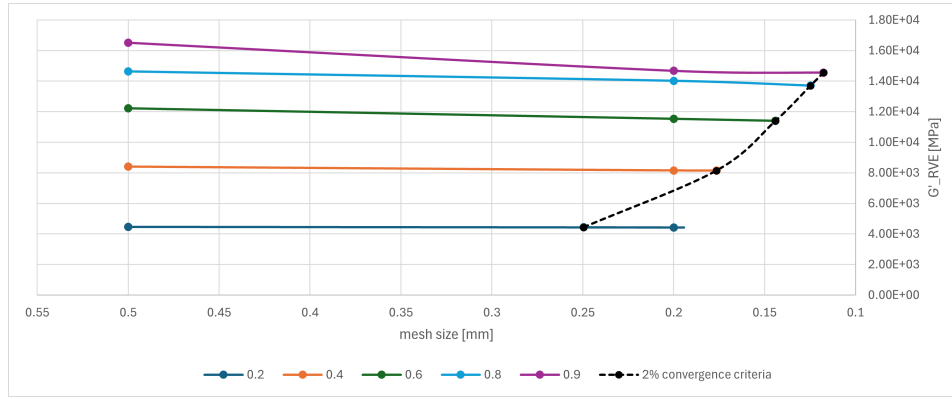
The convergence behaviour was analysed by relating  $\epsilon_G$  to  $\eta$  across the considered range of  $v_f$ . Based on this analysis, a tolerance of  $|\epsilon_G| \leq 2\%$  was selected as the convergence criterion. The minimum value of  $\eta$  satisfying this criterion was then identified and used to determine the required mesh size as a function of  $v_f$ :

$$h_{req}(v_f) = \frac{s(v_f)}{\eta_{min}}. \quad (B.3)$$

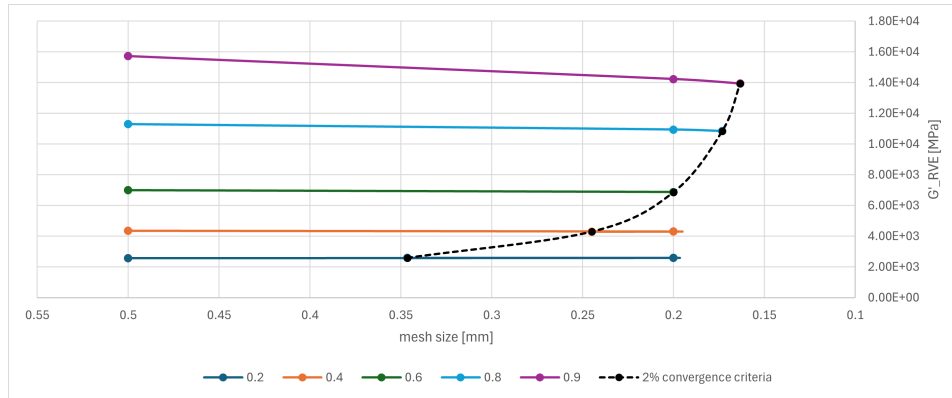
This allows for scaling the mesh size with the evolving geometry, ensuring consistent accuracy across all fiber volume fractions. Result of the above procedure for the cylindrical half model and hexagonal half model are shown in Table B.1. The resulting values for  $G_{RVE}$  from 2% convergence criteria are shown in Figure B.2-B.1.

**Table B.1:** Mesh convergence results for cylindrical and hexagonal half models.

| (a) Cylindrical half model |                                       |                  |                | (b) Hexagonal half model |                                       |                  |                |
|----------------------------|---------------------------------------|------------------|----------------|--------------------------|---------------------------------------|------------------|----------------|
| $v_f$                      | $\epsilon_G(0.5 \rightarrow 0.2)$ [%] | $\eta = s/h$ [-] | $h_{req}$ [mm] | $v_f$                    | $\epsilon_G(0.5 \rightarrow 0.2)$ [%] | $\eta = s/h$ [-] | $h_{req}$ [mm] |
| 0.2                        | -0.64%                                | 74.53            | 0.250          | 0.2                      | 0.67%                                 | 74.53            | 0.346          |
| 0.4                        | -3.09%                                | 52.70            | 0.176          | 0.4                      | -1.15%                                | 52.70            | 0.245          |
| 0.6                        | -5.63%                                | 43.03            | 0.144          | 0.6                      | -1.92%                                | 43.03            | 0.200          |
| 0.8                        | -4.21%                                | 37.27            | 0.125          | 0.8                      | -3.10%                                | 37.27            | 0.173          |
| 0.9                        | -11.1%                                | 35.13            | 0.118          | 0.9                      | -9.49%                                | 35.13            | 0.163          |



**Figure B.1:** Hexagonal model  $G_{RVE}$  as a function of mesh size, 2% convergence criteria included



**Figure B.2:** Cylindrical half model  $G_{RVE}$  as a function of mesh size, 2% convergence criteria included

## B.2. Verification of Half Model Approach

To verify the use of the cylindrical half model, a comparison of  $G_{RVE}$  for both the full and half model was made for the converged mesh sizes as reported in Table B.1(a). With a maximum difference of 6.54%, the half model was judged to represent the full model accurately enough.

**Table B.2:** Comparison of converged RVE stiffness for full and half models, including relative differences in  $G_{RVE}$ .

| $v_f$ | Full RVE           |                    | Half RVE           |                    | Difference<br>$\Delta G$ [%] |
|-------|--------------------|--------------------|--------------------|--------------------|------------------------------|
|       | $K_{RVE}$ [N]      | $G_{RVE}$ [MPa]    | $K_{RVE}$ [N]      | $G_{RVE}$ [MPa]    |                              |
| 0.2   | $8.31 \times 10^5$ | $4.32 \times 10^3$ | $8.50 \times 10^5$ | $4.44 \times 10^3$ | +2.78                        |
| 0.4   | $7.67 \times 10^5$ | $7.98 \times 10^3$ | $7.80 \times 10^5$ | $8.15 \times 10^3$ | +2.13                        |
| 0.6   | $6.89 \times 10^5$ | $1.07 \times 10^4$ | $7.27 \times 10^5$ | $1.14 \times 10^4$ | +6.54                        |
| 0.8   | $6.24 \times 10^5$ | $1.30 \times 10^4$ | $6.56 \times 10^5$ | $1.37 \times 10^4$ | +5.38                        |
| 0.9   | $6.01 \times 10^5$ | $1.41 \times 10^4$ | $6.19 \times 10^5$ | $1.46 \times 10^4$ | +3.55                        |