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On the Stability of Consensus Control Under Rotational Ambiguities

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***Abstract*—Consensus control of multiagent systems arises in various applications such as rendezvous and formation control. The input to these algorithms, e.g., the (relative) positions of neighboring agents need to be measured using various sensors. Recent works aim to reconstruct these positions, i.e., achieve localization using Euclidean distance measurements instead of displacements, for cost efficiency and scalability. However, this approach inherently introduces ambiguities, such as a rotation or a reflection, which can cause stability issues in practice without corrections by some anchors. In this letter, we conduct a thorough analysis of the stability of consensus control in the presence of localization-induced rotational ambiguities, in several scenarios including, e.g., proper and improper rotation, and the homogeneity of rotations. We give stability criteria and stability margin on the rotations, which are numerically verified with two traditional examples of consensus control.**

***Index Terms*—Cooperative control, distributed control, networked control systems.**

I. Introduction

CONSENSUS algorithms are essential in modern distributed systems across various fields [1], [2], [3], including the distributed control of mobile multiagent systems in swarm robotic applications, e.g., flocking [2], rendezvous control of dispersed robots [3], or formation control for a desired geometrical pattern [4], [5]. A large class of consensus control frameworks typically involve interagent interactions that usually require relative kinematics, e.g., relative positions, velocities, etc. The most common practice is to either measure the absolute kinematics and share the information through a communication network [6], or locally measure the relative kinematics [7]. This implies the need for global navigation satellite systems (GNSS) that are impaired in, e.g., indoor applications [8], or expensive sensing solutions that are typically not scalable with the size of the swarm.

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To improve the scalability, there has been some attention on consensus control by reconstructing global positions using a set of pairwise distance measurements [9], [10], which is sometimes referred to as relative localization. The benefits include that distance measurements can be acquired fairly easily with low-cost sensors and the solution is distributable across the agents [11]. However, the disadvantage is that global reference information is lost in scalar-valued distance encodings, and hence ambiguities emerge in reconstruction algorithms, e.g., multidimensional scaling (MDS) [12], [13]. These ambiguities are typically congruent transformations such as translations, rotations, etc, as the distances are invariant to congruent geometry [13]. An industry-standard procedure is to deploy anchors with global information to correct these ambiguities [13], which is assumed by default in application [9]. However, in anchorless networks, ambiguities must be addressed, but they are rarely studied in the literature on consensus control. General concerns about the ambiguities of consensus control include stability or convergence, invariance of the equilibrium, and convergence speed.

In this letter, we study the stability of consensus control in the presence of localization-induced ambiguities. We acknowledge that the equilibrium of such a system is invariant to these ambiguities, and we establish a stability criterion on the ambiguity modeled by rotation matrices. Note that recent works on rotational invariance or orientation alignment of distributed networked systems, e.g., [14], [15], [16], have dealt with local implementations of consensus algorithms in rotated body frames. However, we particularly focus on localization-induced ambiguities that lead to a fundamentally different formulation and analysis, which is novel to our knowledge.

The organization of this letter is as follows. In Section II, preliminaries of consensus control and a general modeling of the ambiguous system are given. In Section III, conditions of stability are proposed and proved for the general ambiguous model, and several specific scenarios. Our proposed analysis is then verified through two applications, namely classic rendezvous control and more recent affine formation control in Section IV. Finally, we summarize our conclusions and present insights for future research in Section V.

Notations. Vectors and matrices are represented by lowercase and uppercase boldface letters respectively such as $\boldsymbol{a}$ and $\boldsymbol{A}$. Sets and graphs are represented using calligraphic letters, e.g., $\mathcal{A}$. Vectors of length N of all ones and zeros are denoted by $\mathbf{1}_N$ and $\mathbf{0}_N$ respectively. An identity matrix of size N is denoted by $\boldsymbol{I}_N$. The Kronecker product is denoted by $\otimes$. A set of D-dimensional real-valued vectors and $D \times D$ real-valued matrices are denoted by $\mathbb{R}^D$ and $\mathbb{R}^{D\times D}$, respectively. The $\mathrm{diag}(\cdot)$ operator creates a diagonal matrix from a vector and $\mathrm{bdiag}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N)$ creates a block diagonal matrix from matrices $\boldsymbol{A}_i$ for $i = 1, \ldots, N$. We define the operator $\gamma(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^T)$ for a square matrix $\boldsymbol{A}$. All eigenvalues of $\boldsymbol{A}$ have strictly negative real parts if $\gamma(\boldsymbol{A})$ is negative-definite, i.e., $\gamma(\boldsymbol{A}) \prec 0$ [17].

II. Fundamentals and Problem Formulation

We consider a multiagent system where N agents are interconnected through an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the set of vertices is $\mathcal{V} = 1, 2, \ldots, N$ and the set of edges is $\mathcal{E} = \mathcal{V} \times \mathcal{V}$. The set of neighbors of agent i is denoted as $\mathcal{N}_i = j \in \mathcal{V} \ : \ (i, j) \in \mathcal{E}$. A generalized Laplacian matrix $\boldsymbol{L} \in \mathbb{R}^{N\times N}$ of a graph is defined as

$$[\boldsymbol{L}]_{ij} = \begin{cases} \sum_{j\in\mathcal{N}_i} l_{ij} & \text{if } i = j \\ -l_{ij} & \text{if } i \neq j \text{ and } j \in \mathcal{N}_i \ , \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $l_{ij} \in \mathbb{R}, \forall (i, j) \in \mathcal{E}$ are the weights associated with the edges. Note that in the special case of $l_{ij} = 1$, $\boldsymbol{L}$ reduces to a standard Laplacian, which is the most common form seen in the literature. Note that the rank of a standard Laplacian for a connected graph is $N - 1$ while a generalized Laplacian can have a rank $P < N$. An example is a stress matrix [18], which is used in some formation control problems [5]. The eigenvalue decomposition of $\boldsymbol{L}$ is

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{U}_1 & \boldsymbol{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Lambda} & \\ & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_1^T \\ \boldsymbol{U}_2^T \end{bmatrix}, \tag{2}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}([\lambda_1, \ldots, \lambda_P])$ is positive-definite, and $\boldsymbol{U}_1 \in \mathbb{R}^{N\times P}$ and $\boldsymbol{U}_2 \in \mathbb{R}^{N\times(N-P)}$ span the range and the nullspace of $\boldsymbol{L}$, respectively.

A. Consensus Systems

A typical consensus system is a dynamical system, i.e.,

$$\dot{\boldsymbol{z}} = -(\boldsymbol{L} \otimes \boldsymbol{I}_D)\boldsymbol{z} = -\tilde{\boldsymbol{L}}\boldsymbol{z}, \tag{3}$$

where $\boldsymbol{z} = [\boldsymbol{z}_1^T, \ldots, \boldsymbol{z}_N^T]^T \in \mathbb{R}^{ND}$ is the global state, in which $\forall i \in \mathcal{V}$, $\boldsymbol{z}_i \in \mathbb{R}^D$ is the state of ith agent in D-dimensional Euclidean space and $\dot{\boldsymbol{z}}$ is the first order derivative of $\boldsymbol{z}$. We consider $D = 2, 3$ for practical applications. Note that an equilibrium point of (3) resides in the nullspace of $\tilde{\boldsymbol{L}}$ which, following (2), admits a decomposition

$$\tilde{\boldsymbol{L}} = (\boldsymbol{L} \otimes \boldsymbol{I}_D) = \begin{bmatrix} \tilde{\boldsymbol{U}}_1 & \tilde{\boldsymbol{U}}_2 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\Lambda}} & \\ & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{U}}_1^T \\ \tilde{\boldsymbol{U}}_2^T \end{bmatrix}, \tag{4}$$

where $\tilde{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \otimes \boldsymbol{I}_D$ are the eigenvalues and $\tilde{\boldsymbol{U}}_1 = \boldsymbol{U}_1 \otimes \boldsymbol{I}_D$ are valid eigenvectors. (See Lemma 2 Appendix).

B. Problem Formulation

We now motivate the consensus model under rotational ambiguities, which is illustrated using a simple example in Fig. 1. As shown, for a given configuration of 3 nodes, the pairwise distances d_{12}, d_{23}, d_{13} are measured instead of the positions $\boldsymbol{z}_{i\,i=1}^{\,3}$ in a common reference frame. As distances are invariant to congruent geometries, e.g., triangles in different line marks in Fig. 1(a), the reconstruction of the positions can be ambiguous, i.e., $\boldsymbol{z}'^{\,3}_{i\,i=1}$ or $\boldsymbol{z}''^{\,3}_{i\,i=1}$, etc. Based on whether the reconstruction is done locally or at a central node, the ambiguities can be homogeneous or heterogeneous across agents. In this letter, we consider the common localization-induced rotational ambiguities denoted by $\boldsymbol{H}_i = \mathcal{R}(\theta_i)\boldsymbol{T}_i \in \mathbb{R}^{D\times D}$ for $i \in \mathcal{V}$ where $\mathcal{R}(\theta_i)$ gives a rotation matrix using an angle $\theta_i \in (-\pi, \pi]$ and $\boldsymbol{T}_i$ is a diagonal matrix of ± 1s. $\boldsymbol{H}_i$ is a proper rotation if $\boldsymbol{T}_i = \boldsymbol{I}_D$ and otherwise an improper rotation. An example of proper and improper rotations for $D = 2$ and $D = 3$ are shown in Table I, where without the loss of generality we assume the rotation angle θ_i for $D = 3$ is around the "z-axis" [19].

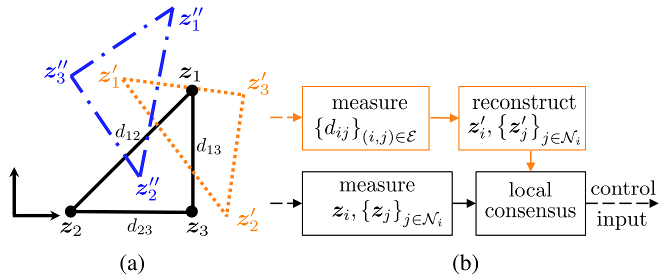


Fig. 1. (a) An illustration of the rotational ambiguity induced by relative localization. (b) A block diagram to illustrate two types of localization, i.e., distance-based (in orange) and displacement-based or absolute localization (in black), as part of the closed-loop system.

TABLE I
Rotational Ambiguities $\boldsymbol{H}_i$

D	Proper rotation $\mathcal{R}(\theta_i)$	Improper rotation $\mathcal{R}(\theta_i)\boldsymbol{T}_i$
2	$\begin{bmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{bmatrix}$	$\begin{bmatrix}\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}$
3	$\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}\cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & & \\ & 1 & \\ & & -1\end{bmatrix}$

If the agents directly implement the ambiguous positions as shown in Fig. 1(b), then the consensus model (3) becomes

$$\dot{\boldsymbol{z}} = -\tilde{\boldsymbol{H}}(\boldsymbol{L} \otimes \boldsymbol{I}_D)\boldsymbol{z} = -\tilde{\boldsymbol{H}}\tilde{\boldsymbol{L}}\boldsymbol{z}, \tag{5}$$

where $\tilde{\boldsymbol{H}} = \mathrm{bdiag}(\boldsymbol{H}_1, \ldots, \boldsymbol{H}_N) \in \mathbb{R}^{ND\times ND}$ is the global ambiguity matrix. Observe that equilibrium points of (3) and (5) are the same, i.e., the kernel of $\tilde{\boldsymbol{L}}$ is preserved in terms of the zero eigenvalues, since $\tilde{\boldsymbol{H}}$ contains full-rank rotation matrices and subsequently is full-rank. Thus the steady-state solutions are unchanged. We now aim to investigate the stability of the ambiguous system (5) and give conditions on $\tilde{\boldsymbol{H}}$ to yield a stable system in various practical scenarios.

III. Stability Under Ambiguities

We now give general conditions for the ambiguous system (5) to be stable, followed by an investigation on stability under specific types of ambiguities in different scenarios based on these general conditions. We introduce the following lemma, which is later used in the proof of general stability criterion.

Lemma 1 (Eigenvalue Distributions of a Product of Matrices): Given a product of two square matrices $\boldsymbol{A} = \boldsymbol{G}\boldsymbol{Q}$ with $\boldsymbol{Q}$ being symmetric positive-definite, $\gamma(\boldsymbol{A}) \prec 0$ if and only if $\gamma(\boldsymbol{G}) \prec 0$.

Proof: Since $\boldsymbol{Q}$ is symmetric positive-definite, $\boldsymbol{A} = \boldsymbol{G}\boldsymbol{Q}$ and $\boldsymbol{G} = \boldsymbol{A}\boldsymbol{Q}^{-1}$ both satisfy the defined decomposition. Then $\gamma(\boldsymbol{A}) \prec 0$ and $\gamma(\boldsymbol{G}) \prec 0$ are equivalent using the direct conclusion from [20, Th. 3.1]. ■

A. General Conditions for Stability

We first acknowledge that the unambiguous system (3) is globally and exponentially stable, which is an extension of the conclusion in [21]. Observe that the PD nonzero eigenvalues of $-\tilde{\boldsymbol{L}}$ are strictly negative, which means that given any arbitrary initialization $\boldsymbol{z}(t)$ at $t = 0$, $\boldsymbol{z}(t)$ will exponentially converge to an equilibrium point in the nullspace of $\tilde{\boldsymbol{L}}$, i.e., $\lim_{t\to\infty}\big\|\boldsymbol{z}(t) - \boldsymbol{z}_e\big\| = 0$ where $\boldsymbol{z}_e$ lives in the nullspace of $\tilde{\boldsymbol{L}}$. Therefore, to ensure the stability of the ambiguous system (5), the nonzero eigenvalues of the product $-\tilde{\boldsymbol{H}}\tilde{\boldsymbol{L}}$ should have strictly negative real parts, which is described in the next theorem.

Theorem 1 (General Stability Criteria for Systems With Rotational Ambiguity): Let $\tilde{\boldsymbol{U}}_1$ span the range of $\tilde{\boldsymbol{L}}$, then the ambiguous system (5) is globally and exponentially stable if and only if $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$.

Proof: If we use the operator $\sigma(\cdot)$ to denote the set of eigenvalues, then using properties of eigenvalues of products of matrices and decomposition (4) it holds that

$$\begin{aligned}\sigma\Big(-\tilde{\boldsymbol{H}}\tilde{\boldsymbol{L}}\Big) &= \sigma\left(-\tilde{\boldsymbol{H}}\begin{bmatrix}\tilde{\boldsymbol{U}}_1 & \tilde{\boldsymbol{U}}_2\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{\Lambda}} & \\ & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{U}}_1^T \\ \tilde{\boldsymbol{U}}_2^T\end{bmatrix}\right)\\ &= \sigma\left(-\begin{bmatrix}\tilde{\boldsymbol{U}}_1^T \\ \tilde{\boldsymbol{U}}_2^T\end{bmatrix}\tilde{\boldsymbol{H}}\begin{bmatrix}\tilde{\boldsymbol{U}}_1 & \tilde{\boldsymbol{U}}_2\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{\Lambda}} & \\ & \boldsymbol{0}\end{bmatrix}\right).\end{aligned} \quad (6)$$

Observe that $-\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1\tilde{\boldsymbol{\Lambda}}$ corresponds to the nonzero eigenvalues. As such, given $\tilde{\boldsymbol{\Lambda}}$ is positive definite, the system is stable if and only if $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$, according to Lemma 1. Moreover, note that $\tilde{\boldsymbol{U}}_1$ might be up to an orthogonal transformation $\tilde{\boldsymbol{P}}$ than assumed for (4), which yields $\gamma(\tilde{\boldsymbol{P}}^T\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1\tilde{\boldsymbol{P}}) \succ 0$, which is the same as $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$ due to congruence. ■

Corollary 1 (Sufficient Condition for Stability): The ambiguous system (5) is globally and exponentially stable if $\gamma(\tilde{\boldsymbol{H}}) \succ 0$.

Proof: It can be verified that $\tilde{\boldsymbol{U}}_1^T\gamma(\tilde{\boldsymbol{H}})\tilde{\boldsymbol{U}}_1 = \gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1)$. As such, it holds that $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$ given that $\gamma(\tilde{\boldsymbol{H}}) \succ 0$ and a full-rank $\tilde{\boldsymbol{U}}_1$ [19], which leads to a stable system according to Theorem 1. ■

Corollary 1 further narrows down the conditions for stability directly regarding the ambiguity matrix $\tilde{\boldsymbol{H}}$. Note that this condition is sufficient but not necessary as $\tilde{\boldsymbol{U}}_1$ is a tall matrix.

B. Stability Under Homogeneous Ambiguities

In some scenarios, all agents can share homogeneous local ambiguities, i.e., $\boldsymbol{H}_i = \boldsymbol{H}, \forall i \in \mathcal{V}$, and subsequently (5) is

$$\dot{\boldsymbol{z}} = -(\boldsymbol{I}_N \otimes \boldsymbol{H})(\boldsymbol{L} \otimes \boldsymbol{I}_D)\boldsymbol{z} = -\tilde{\boldsymbol{H}}\tilde{\boldsymbol{L}}\boldsymbol{z}. \quad (7)$$

We discuss the stability conditions for both proper and improper rotations $\boldsymbol{H}$, in Theorems 2 and 3, respectively. Alternative proofs are presented in the Appendix.

Theorem 2 (Stability Under Homogeneous and Proper Rotations): The ambiguous system (7) is globally and exponentially stable if and only if the rotation angle θ of a proper rotation $\boldsymbol{H} = \mathcal{R}(\theta)$ lies within the range $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof: Recollect from Theorem 1 that $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$ is required for stability. In case of $D = 2$, using the rotation matrix from Table I, we can verify that

$$\gamma\Big(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1\Big) = \tilde{\boldsymbol{U}}_1^T\cos\theta\boldsymbol{I}_{ND}\tilde{\boldsymbol{U}}_1 = \cos\theta\boldsymbol{I}_{PD}, \quad (8)$$

and for $D = 3$, substituting for $\tilde{\boldsymbol{U}}_1$ and $\tilde{\boldsymbol{H}}$, we have

$$\gamma\Big(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1\Big) = \gamma(\boldsymbol{I}_P \otimes \boldsymbol{H}) = \boldsymbol{I}_P \otimes \mathrm{diag}\Big(\begin{bmatrix}\cos\theta & \cos\theta & 1\end{bmatrix}\Big) \quad (9)$$

Both (8) and (9) are block diagonal matrices, which are positive-definite if and only if $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. ■

A numerical example is shown in Fig. 2, where we present the eigenvalue distribution using the graph in Fig. 5 (a) and its standard Laplacian as defined in (1). Fig. 2 (a) shows the rotations of eigenvalues from lemma 3, where the eigenvalues of $-\boldsymbol{L}$ originally lie on the real axis but then are rotated by $\boldsymbol{H}$. Then $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ensures that they stay on the left complex plane. Fig. 2 (b) shows all the eigenvalues of $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1)$ across a spectrum of θ. The region $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (shaded in purple) guarantees positive eigenvalues meaning $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$, which verifies Theorem 2.

Theorem 3 (Stability Under Homogeneous and Improper Rotations): The ambiguous system (7) is unstable under improper rotations $\boldsymbol{H}$.

Proof: We show that $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$ is impossible. Recall from (9) that $\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1$ can be simplified to $\boldsymbol{I}_P \otimes \boldsymbol{H}$. Due to the negative determinant of an improper rotation, $\gamma(\boldsymbol{H})$ is never positive-definite. As such, $\gamma(\boldsymbol{I}_P \otimes \boldsymbol{H})$ is not positive-definite either and hence the system is unstable under improper rotations independent of dimension D. ■

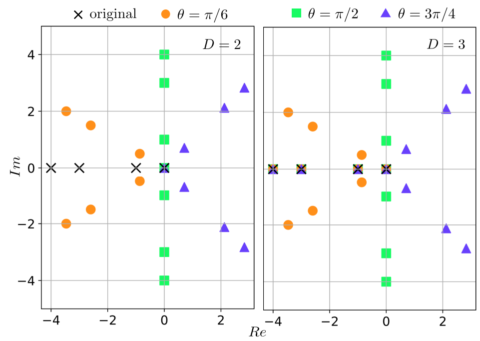
Fig. 3 shows the same example as Fig. 2 but with improper rotations, where there are always positive eigenvalues present in (a) and there is no region in (b) where $\gamma(\tilde{\boldsymbol{U}}_1^T\tilde{\boldsymbol{H}}\tilde{\boldsymbol{U}}_1) \succ 0$. Hence, system (7) is not stable under improper rotations.

C. Stability Under Heterogeneous Ambiguities

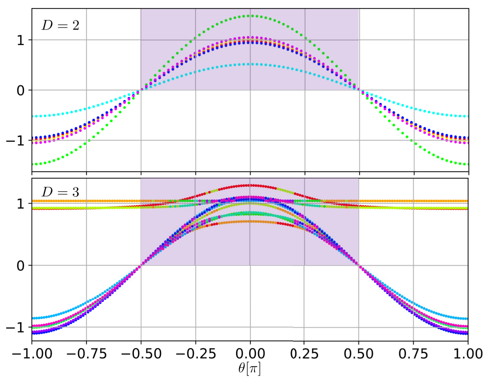
In some distributed cases, heterogeneous ambiguities occur across the agents, which specify the general ambiguous system (5) to

$$\dot{\boldsymbol{z}} = -\mathrm{bdiag}(\boldsymbol{H}_1, \ldots, \boldsymbol{H}_N)(\boldsymbol{L} \otimes \boldsymbol{I}_D)\boldsymbol{z} = -\tilde{\boldsymbol{H}}\tilde{\boldsymbol{L}}\boldsymbol{z}. \quad (10)$$

There are three potential cases under this model: (a) all proper rotations, (b) all improper rotations, or (c) a mixture of proper and improper rotations across the agents.

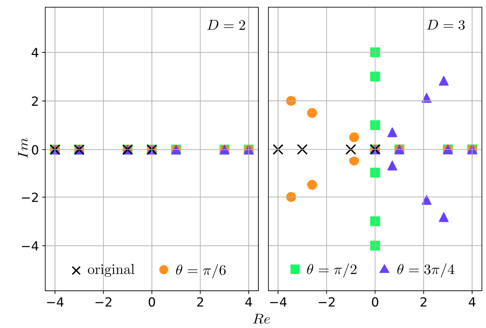


(a) Eigenvalues of $-(\boldsymbol{I}_N \otimes \boldsymbol{H})(\boldsymbol{L} \otimes \boldsymbol{I}_D) = -(\boldsymbol{L} \otimes \boldsymbol{H})$

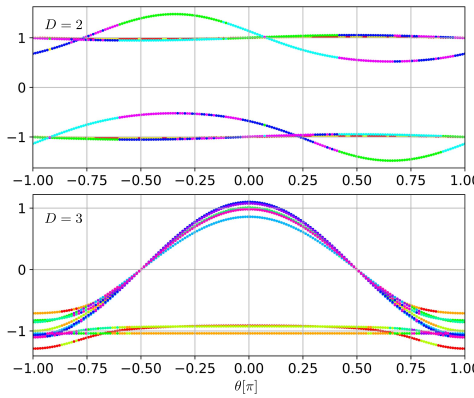


(b) Eigenvalues of $\gamma\left(\tilde{\boldsymbol{U}}_1^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{U}}_1\right)$

Fig. 2. A numerical example of the eigenvalues under homogeneous and proper rotations.



(a) Eigenvalues of $-(\boldsymbol{I}_N \otimes \boldsymbol{H})(\boldsymbol{L} \otimes \boldsymbol{I}_D) = -(\boldsymbol{L} \otimes \boldsymbol{H})$



(b) Eigenvalues of $\gamma\left(\tilde{\boldsymbol{U}}_1^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{U}}_1\right)$

Fig. 3. A numerical example of the eigenvalues under homogeneous and improper rotations.

We show that, unlike the homogeneous scenario, in the case of heterogeneous proper rotations, $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})\ \forall i \in \mathcal{V}$ is a sufficient but not necessary condition for a stable system under proper rotations $\boldsymbol{H}_i\ \forall i \in \mathcal{V}$.

Theorem 4 (Stability Under Heterogeneous and Proper Rotations): The ambiguous system (10) is globally and exponentially stable if the corresponding rotation angles θ_i of a local proper rotation $\boldsymbol{H}_i = \mathcal{R}(\theta_i)$ lie within range $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})\ \forall i \in \mathcal{V}$.

Proof: Recollect from Lemma 1 that the sufficient condition for a stable system is to have a generalized positive-definite $\tilde{\boldsymbol{H}}$. For $D = 2$ and $D = 3$, we observe

$$\gamma\left(\tilde{\boldsymbol{H}}\right) = \mathrm{bdiag}(\cos\theta_1 \boldsymbol{I}_2, \ldots, \cos\theta_N \boldsymbol{I}_2), \quad (D = 2) \quad (11)$$

$$\gamma\left(\tilde{\boldsymbol{H}}\right) = \mathrm{bdiag}\big(\gamma(\mathcal{R}(\theta_1)), \ldots, \gamma(\mathcal{R}(\theta_N))\big), \quad (D = 3)(12)$$

which are both diagonal matrices, with the property $\gamma(\tilde{\boldsymbol{H}}) \succ 0$ if $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})\ \forall i \in \mathcal{V}$, for a stable system (10). ■

We use the same settings as in Fig. 2 but with heterogeneous rotations to show another numerical example in Fig. 4 (a) where the smallest eigenvalue is shown in a heatmap across a spectrum of θ_3 and θ_4. Observe that the white bounding box, inside of which are eigenvalues greater than zero, is bigger than the area boxed by $(-\frac{\pi}{2}, \frac{\pi}{2})$ in yellow. This shows that there exists $\theta_i \notin (-\frac{\pi}{2}, \frac{\pi}{2})$ that still entails $\gamma(\tilde{\boldsymbol{U}}_1^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{U}}_1) \succ 0$, i.e., a stable system, which verifies the sufficiency but not necessity of Theorem 4.

We now discuss the cases where one or more improper rotations appear among all agents. We make a proposition and give intuitive reasoning, which is verified with numerical examples and simulations in later sections.

Proposition 1 (Instability Under Mixture of Rotations): The ambiguous system (10) is unstable if there exists i such that $\boldsymbol{H}_i$ is an improper rotation.

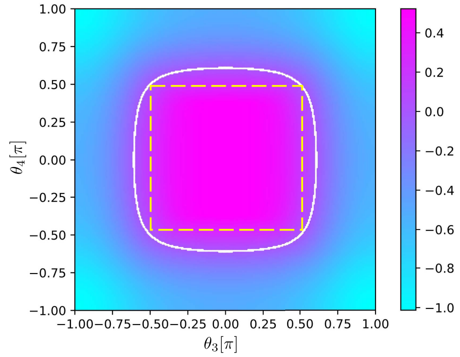
A special case of this scenario is that $\boldsymbol{H}_i\ \forall i \in \mathcal{V}$ are homogeneous improper rotations, which is proven to be unstable in Theorem 3. The more general case from (5) is

$$\tilde{\boldsymbol{H}}(\boldsymbol{L} \otimes \boldsymbol{I}_D) = \begin{bmatrix} \mathcal{R}(\theta_1) & & \\ & \ddots & \\ & & \mathcal{R}(\theta_N) \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_1 & & \\ & \ddots & \\ & & \boldsymbol{T}_N \end{bmatrix} (\boldsymbol{L} \otimes \boldsymbol{I}_D), \quad (13)$$

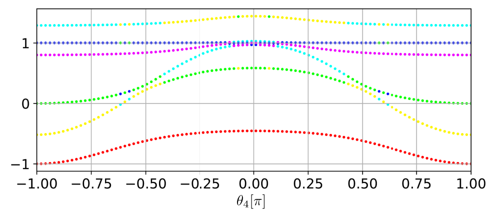
where $\boldsymbol{T}_i = \boldsymbol{I}$ if $\boldsymbol{H}_i$ for any $i \in \mathcal{V}$ is a proper rotation. We can consider $\mathrm{bdiag}(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_N)(\boldsymbol{L} \otimes \boldsymbol{I}_D)$ a new Laplacian matrix where some rows are negated if certain $\boldsymbol{H}_i$ is not proper. This new Laplacian matrix is no longer symmetric positive semi-definite in general and yields an unstable system regardless of what the proper rotation part is. The example in Fig. 4 (b) shows that as long as one $\boldsymbol{H}_i$ for any $i \in \mathcal{V}$ is improper, $\gamma(\tilde{\boldsymbol{U}}_1^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{U}}_1)$ is not positive-definite even if all the other agents are unambiguous.

IV. Examples and Simulations

In this section, we verify our theorems and proposition with two algorithms under consensus frameworks, namely, the

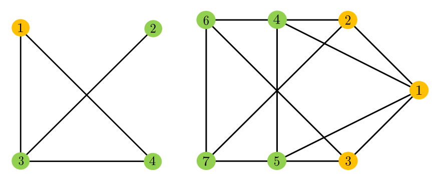


(a) The smallest eigenvalue of $\gamma(\tilde{\boldsymbol{U}}_1^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{U}}_1)$ across a spectrum of θ_3 and θ_4. In this case, we set $\theta_1 = \theta_2 = 0$. Zero values are highlighted in white, and the region between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is highlighted by a yellow dashed box.



(b) Eigenvalues of $\gamma(\tilde{\boldsymbol{U}}_1^T \tilde{\boldsymbol{H}} \tilde{\boldsymbol{U}}_1)$. In this case, $\theta_1 = \theta_2 = \theta_3 = 0$ and $\boldsymbol{H}_i$ for $i = 1, 2, 3$ are proper, while only $\boldsymbol{H}_4$ is improper.

Fig. 4. A numerical example of the eigenvalues under heterogeneous rotations.



(a) (b)

Fig. 5. Graphs for (a) rendezvous control and (b) distributed formation control, where the orange nodes are leaders.

rendezvous control [22] and distributed formation control [5]. The code is available in this repository[1]. The graphs used for each case are shown in Fig. 5, where there are 4 and 7 nodes, respectively. Agents in these algorithms are assumed to adopt single-integrator dynamics $\dot{\boldsymbol{z}}_i = \boldsymbol{u}_i \ \forall i \in \mathcal{V}$. In both scenarios, the global dynamical model (3) translates to local control input $\boldsymbol{u}_i = -\sum_{j \in \mathcal{N}_i} l_{ij}(\boldsymbol{z}_i - \boldsymbol{z}_j)$, where l_{ij} are the Laplacian weights in (1).

For a consensus system (3), the equilibrium points are not unique and depend on the initialization. To better present the numerical results given a random initialization, we introduce leaders shown in Fig. 5, a small subset of nodes fixed in some positions that define a unique equilibrium. Note that the leaders do not affect the analysis and the conclusion in the theoretical part, however the discussion on the leaders' ambiguity should be excluded since they are fixed in given positions on purpose.

A. Case 1: Rendezvous Control

The rendezvous control algorithm [22], which originates from the classic average consensus algorithm [23], ensures all agents converge to a common location. It involves a standard

[1] https://github.com/asil-lab/zli-rot-ambiguity

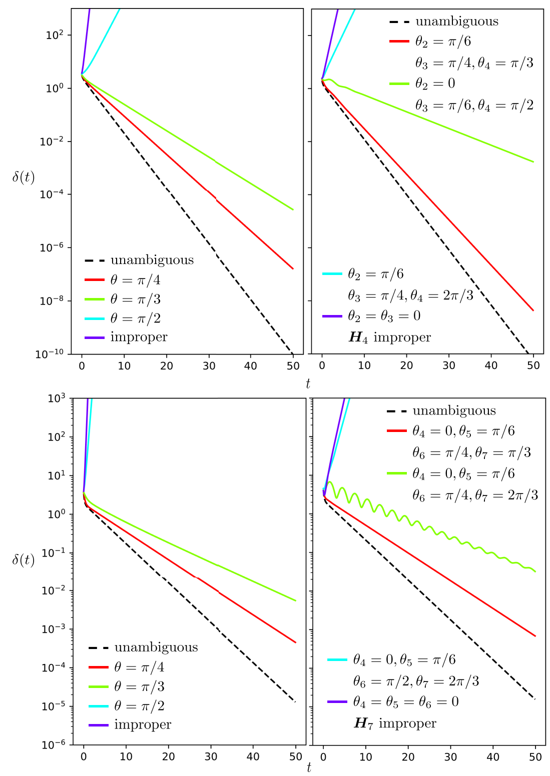


Fig. 6. The convergence in error $\delta(t)$ across time t for the rendezvous control (top) and the affine formation control algorithm (bottom) under homogeneous ambiguities (left) and heterogeneous (right).

graph Laplacian $\boldsymbol{L}$ that has rank $N-1$. For graph Fig. 5 (a), there are $P = 3$ non-zero eigenvalues for the Laplacian. We set node 1 to be the leader with a constant value $\boldsymbol{z}_1 = [0, 0]^T$. Then the equilibrium is zero, i.e., $\boldsymbol{z}_e = [0, 0]^T$. We define the error as $\delta(t) = \|\boldsymbol{z}(t) - \boldsymbol{z}_e\| = \|\boldsymbol{z}(t)\|$.

B. Case 2: Distributed Formation Control

Affine formation control [5], [24] is a type of distributed formation control method that can also fit under the consensus framework. A generalized Laplacian, called a stress matrix [18] with $P = N - D - 1$ non-zero eigenvalues, is then used instead of a standard Laplacian. Here, the desired formation is considered the equilibrium point of the system. We consider Fig. 5 (b) in $\mathbb{R}^2$ with a equilibrium $\boldsymbol{z}_e = [2, 0, 1, 1, 1, -1, 0, 1, 0, -1 - 1, 1, -1, -1]^T$. If we define the first three agents as leaders that remain at their respective target positions $[\boldsymbol{z}_1^T, \boldsymbol{z}_2^T, \boldsymbol{z}_3^T] = [2, 0, 1, 1, 1, -1]^T$, then the agents will converge to the defined equilibrium $\boldsymbol{z} \to \boldsymbol{z}_e$ as time $t \to \infty$ given any random initialization of the follower' positions. As such, we define the error $\delta(t) = \|\boldsymbol{z}(t) - \boldsymbol{z}_e\|$.

C. Discussion

The numerical results for both rendezvous control and affine formation control are shown in Fig. 6, where the cases discussed in Section III are simulated as compared to unambiguous cases. As can be seen, the errors present an exponential decay (straight lines under log scale) if converging. In the homogeneous case, errors are converging for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ cases and diverging for $\theta = \frac{\pi}{2}$ and improper rotations, which agree to Theorem 2 and 3. In the heterogeneous case, errors are converging in cases where $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all followers and where one exceeds this range, which proves the sufficient but not necessary condition in Theorem 4. We observe that the error diverges with one follower under improper rotations even if the other agents are unambiguous, which supports Proposition 1.

V. Conclusion

In this letter, we conducted a theoretical analysis of the stability of consensus control where the localizations of multiagent systems are subject to rotational ambiguities. We show that the system is robust to proper rotations in both homogeneous and heterogeneous cases within certain margins, but the stability is compromised in the presence of improper rotations. This provides insightful guidance for the design of anchorless relative localization and the implementation of consensus control in various applications, which should further reduce the cost of sensing capabilities in large-scale networks. In our future work, we aim to generalize our solution to directed graphs which could benefit a broader set of applications. In addition, we aim for a rigorous proof of Proposition 1 involving a mixture of rotations.

Appendix

Alternative Proof for Stability under Homogeneous Rotations

Lemma 2 (Eigenvalues and Eigenvectors of Kronecker Products of Matrices [25]): Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of size N and M respectively and they admit $\boldsymbol{A}\boldsymbol{v}_n = \lambda_n \boldsymbol{v}_n$ for $n = 1, \dots, N$ and $\boldsymbol{B}\boldsymbol{w}_m = \mu_m \boldsymbol{w}_m$ for $m = 1, \dots, M$, then $\boldsymbol{v}_n \otimes \boldsymbol{w}_m$ is an eigenvector of $\boldsymbol{A} \otimes \boldsymbol{B}$ corresponding to the eigenvalue $\lambda_n \mu_m$. Additionally, the set of all eigenvalues of $\boldsymbol{A} \otimes \boldsymbol{B}$ is $\lambda_n \mu_m : n = 1, \dots, N, m = 1, \dots, M$.

Lemma 3 (Rotation of Eigenvalues of Proper Rotations): The eigenvalues of the rotated system (7) are the ones of the negative Laplacian $-\boldsymbol{L}$ rotated in the complex plane by $\theta, -\theta$ if $D = 2$ and $0, \theta, -\theta$ if $D = 3$, given $\boldsymbol{H} = \mathcal{R}(\theta)$.

Proof: Observe $-(\boldsymbol{I}_N \otimes \boldsymbol{H})(\boldsymbol{L} \otimes \boldsymbol{I}_D) = -(\boldsymbol{L} \otimes \boldsymbol{H})$. Let λ_n for $n = 1, \dots, N$ denote the eigenvalues of $-\boldsymbol{L}$ and μ_d for $d = 1, \dots, D$ denote those of $\boldsymbol{H}$. Based on Lemma 2, the eigenvalues of $-(\boldsymbol{L} \otimes \boldsymbol{H})$ are $\mu_d \lambda_n$ for $d = 1, \dots, D, n = 1, \dots, N$. Since $\sigma(\boldsymbol{H}) = \{e^{i\theta}, e^{-i\theta}\}$ and $\sigma(\boldsymbol{H}) = \{1, e^{i\theta}, e^{-i\theta}\}$ for $D = 2$ and 3, respectively, the resulting eigenvalues are $e^{i\theta}\lambda_n, e^{-i\theta}\lambda_n$ and $\lambda_n, e^{i\theta}\lambda_n, e^{-i\theta}\lambda_n$ for $n = 1, \dots, N$, for respective dimensions D. Thus, the eigenvalues are rotated in the complex plane by θ, $-\theta$ for $D = 2$ and θ, $-\theta$ and 0 for $D = 3$. ∎

Lemma 4 (Rotation and Mirroring of Eigenvalues of Improper Rotations): Let $\boldsymbol{H}$ be an improper rotation matrix, and $\boldsymbol{L}$ be a Laplacian (2), then there always exists positive eigenvalues for $-(\boldsymbol{I}_N \otimes \boldsymbol{H})(\boldsymbol{L} \otimes \boldsymbol{I}_D)$.

Proof: We simplify the again $-(\boldsymbol{I}_N \otimes \boldsymbol{H})(\boldsymbol{L} \otimes \boldsymbol{I}_D) = -(\boldsymbol{L} \otimes \boldsymbol{H})$, whose eigenvalues are $\mu_d \lambda_n$ for $d = 1, \dots, D, n = 1, \dots, N$. It is known that $\sigma(\boldsymbol{H}) = \{-1, 1\}$ for $D = 2$ and $\sigma(\boldsymbol{H}) = -1, e^{i\theta}, e^{-i\theta}$ for $D = 3$ for an improper $\boldsymbol{H}$. Hence, there exists a set of eigenvalues of $-\boldsymbol{L}$ mirrored from the negative part to the positive part of the real axis by the -1 eigenvalue of $\boldsymbol{H}$. ∎

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