Normalized Coprime Factorizations for Systems in Generalized State-Space Form

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Abstract—**This note presents a state-space algorithm for the calculation of a normalized coprime factorization of continuous-time generalized dynamical systems. It will be shown that two Riccati equations have to be solved to obtain this normalized coprime factorization.**

I. Introduction

Recent publications have shown the importance of normalized coprime factorization plant descriptions in the fields of control design [6], [1], robustness analysis [13], [15], model reduction [7], and identification for control [12].

In [9], the connection between the state-space realization of a strictly proper plant, and a coprime factorization has been established. The coprime factorization of a generalized dynamical system was presented in [17]. In [8], it has been shown that in order to calculate a normalized coprime factorization of a continuous-time strictly proper plant, one Riccati equation has to be solved. In [14], these results have been extended to proper plants. For discrete-time proper systems, the construction of a normalized coprime factorization has been formulated in [2].

In this note, we extend the results of [8] and [14] to the case of proper and nonproper systems in a generalized state-space form. It will be shown that in the calculation of a normalized coprime factorization for systems in a generalized state-space form, two Riccati equations have to be solved. An explicit algorithm to obtain this factorization will be given.

II. Preliminaries

In this note, we adopt the ring theoretic setting of [4] and [16] to study stable multivariable linear systems. That is, we consider a stable system as a transfer function matrix with all its entries belonging to the ring $\mathscr{H}$. For the application of our state-space algorithm, we will identify the ring $\mathscr{H}$ with $\mathbb{R}H_\infty$, the space of stable real rational finite-dimensional linear time-invariant continuous-time systems.

We consider the class of possibly nonproper and/or unstable multivariable systems as transfer function matrices whose entries are elements of the quotient field $\mathscr{F} := \{a/b | a \in \mathscr{H}, b \in \mathscr{H}\backslash 0\}$. The set of multiplicative units of $\mathscr{H}$ is defined as $\mathscr{J} := \{h \in \mathscr{H} | h^{-1} \in \mathscr{H}\}$. In the sequel, systems $P \in \mathscr{F}^{m\times n}$ are denoted as $P \in \mathscr{F}$ and $M^* := M^T(-s)$.

Definition 2.1 [16]: A plant $P \in \mathscr{F}$ is said to have a right (left) fractional representation if there exist $N, M(\tilde{N}, \tilde{M}) \in \mathscr{H}$ such that $P = NM^{-1} (= \tilde{M}^{-1}\tilde{N})$.

The pair M, $N(\tilde{M}, \tilde{N})$ is a right (left) coprime factorization (RCF or LCF) if it is a right (left) fractional representation and there exist $U, V(\tilde{U}, \tilde{V}) \in \mathscr{H}$ such that $UN + VM = I\ (\tilde{N}\tilde{U} + \tilde{M}\tilde{V} = I)$.

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The pair M, $N(\tilde{M}, \tilde{N})$ is called a normalized right (left) coprime factorization (NRCF or NLCF) if it is an RCF (LCF), and $M^*M + N^*N = I\ (\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I)$.

Proposition 2.2: Let $P(s) \in \mathscr{F}$ have McMillan degree r. Then $P(s)$ can be represented by $P(s) = C(sE - A)^{-1}B$, where

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2] \quad (1)$$

with $A_{12}A_{22}^T = 0$, $A_{22}^T A_{21} = 0$, and $B_2B_2^T$, $C_2^TC_2$ nonsingular, having matrix partitions assumed to be compatible with the partitioning of E.

Proof: Let $P(s) = P_{sp}(s) + P_p(s)$ with $P_{sp} = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ strictly proper and $P_p = \overline{C}(I - s\overline{J})^{-1}\overline{B}$ polynomial, $(\hat{A}, \hat{B}, \hat{C})$ and $(\overline{J}, \overline{B}, \overline{C})$ controllable and observable matrix triples, and $\overline{J}$ in Jordan form [11]. Then, operations of restricted system equivalence (RSE) [11] applied to the polynomial system matrix [10], [5] corresponding to $(\overline{J}, \overline{B}, \overline{C})$ yield

$$\left[\begin{array}{c|c} I - s\overline{J} & \overline{B} \\ \hline -\overline{C} & 0 \end{array}\right] \overset{\text{RSE}}{\rightarrow} \left[\begin{array}{cc|c} sI - \tilde{J}_{11} & -\tilde{J}_{12} & \tilde{B}_1 \\ -\tilde{J}_{21} & -\tilde{J}_{22} & B_2 \\ \hline -\tilde{C}_1 & -C_2 & 0 \end{array}\right]. \quad (2)$$

The resulting structure (2) is obtained by solely interchanging rows and columns containing an s, and by performing sign changes. Controllability and observability of systems in Jordan form imply nonsingularity of $B_2B_2^T$ and $C_2^TC_2$ [3]. The Jordan form implies $\tilde{J}_{12}\tilde{J}_{22}^T = 0$ and $\tilde{J}_{22}^T\tilde{J}_{21} = 0$. Now defining

$$A_{11} := \begin{bmatrix} \hat{A} & 0 \\ 0 & \tilde{J}_{11} \end{bmatrix}, \quad A_{12} := \begin{bmatrix} o \\ \tilde{J}_{12} \end{bmatrix}, \quad B_1 := \begin{bmatrix} \hat{B} \\ \tilde{B}_1 \end{bmatrix},$$

$$A_{21} := [0 \quad \tilde{J}_{21}] \quad A_{22} := \tilde{J}_{22}, \quad B_2 := B_2,$$

$$C_1 := \begin{bmatrix} \hat{C} & \tilde{C}_1 \end{bmatrix}, \quad C_2 := C_2 \quad (3)$$

where the partitions have consistent dimensions, leads to (1), and this proves the proposition. □

III. Main Result

The main result consists of two parts. First, we will show that an NRCF of P is a stable full-rank spectral factor of

$$\begin{bmatrix} I \\ P \end{bmatrix} (I + P^*P)^{-1} [I \quad P^*]. \quad (4)$$

Secondly, we will use this result to obtain a state-space realization of an NRCF of P. This will be presented in the form of an algorithm.

Theorem 3.1: Let $P \in \mathscr{F}$ be given. Then the following statements are equivalent:

a) (N, M) is an NRCF of P.
b) $\begin{bmatrix} M \\ N \end{bmatrix} \in \mathscr{H}$, where (N, M) right coprime is a full-rank spectral factor of

$$\begin{bmatrix} I \\ P \end{bmatrix} (I + P^*P)^{-1} [I \quad P^*].$$

Proof: a) $\rightarrow$ b). Given (N, M) as an NRCF of P. Then $\begin{bmatrix} M \\ N \end{bmatrix} \in \mathscr{H}$ is full rank and (4) can be written as

$$\begin{bmatrix} I \\ NM^{-1} \end{bmatrix}(I + M^{*-1}N^*NM^{-1})^{-1}[I \quad M^{*-1}N^*]$$
$$= \begin{bmatrix} M \\ N \end{bmatrix}(M^*M + N^*N)^{-1}[M^* \quad N^*]$$
$$= \begin{bmatrix} M \\ N \end{bmatrix}[M^* \quad N^*]$$

which shows b).

b) $\rightarrow$ a). Let $\begin{bmatrix} M \\ N \end{bmatrix} \in \mathscr{H}$ be a full-rank spectral factor of (4) with (N, M) right coprime, i.e., (4) equals $\begin{bmatrix} M \\ N \end{bmatrix}[M^* \quad N^*]$. Premultiplication by $[P \quad -I]$ yields $[P \quad -I]\begin{bmatrix} M \\ N \end{bmatrix} = 0$, which shows that (N, M) is an RCF of P. Postmultiplication of (4) by $\begin{bmatrix} M \\ N \end{bmatrix}$ yields $\begin{bmatrix} M \\ N \end{bmatrix}[M^* \quad N^*]\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix}(I + P^*P)^{-1}[I \quad P^*]\begin{bmatrix} I \\ P \end{bmatrix}M = \begin{bmatrix} M \\ N \end{bmatrix}$, which implies $[M^* \quad N^*]\begin{bmatrix} M \\ N \end{bmatrix} = I$, and this shows a). $\square$

Based on Theorem 3.1, the following algorithm has been constructed, which will lead to a state-space representation of an NRCF of a system in generalized state-space form. The proof is given in the Appendix.

Algorithm: Let $P(s)$ be a real rational (possibly nonproper) transfer function of McMillan degree r.

Step 1: Perform the construction of a system in the particular generalized state-space form (1) having the structure defined by (2) yielding (3) as formulated in Proposition 2.2 and its Proof.

Step 2: Calculate W_2 as the stabilizing solution to the Riccati equation

$$C_2^TC_2 + W_2A_{22} + A_{22}^TW_2 - W_2B_2B_2^TW_2 = 0.$$

Step 3: Define Y, Z, $\overline{C}$, $\overline{B}$, $\overline{A}$ to be

$$Y := -(W_2A_{22} + C_2^TC_2)^{-1}(A_{12}^T - W_2B_2B_1^T)$$
$$Z := -(W_2A_{22} + C_2^TC_2)^{-1}(C_2^TC_1 + W_2A_{12})$$
$$\overline{C} := C_1 + C_2Z$$
$$\overline{A} := A_{11} + (A_{12} + Y^TC_2^TC_2)Z + Y^TC_2^TC_1$$
$$\overline{B} := B_1 - (A_{12} - B_1B_2^TW_2)(A_{22} - B_2B_2^TW_2)^{-1}B_2.$$

Step 4: Calculate W_1 as the stabilizing solution to the Riccati equation

$$\overline{C}^T\overline{C} + \overline{A}^TW_1^T + W_1\overline{A} - W_1\overline{B}\overline{B}^TW_1^T = 0.$$

Step 5: A regular state-space realization (A_n, B_n, C_N, D_n) of the NRCF $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$, having an order identical to the McMillan degree of $P(s)$, is obtained, with

$$A_n = \overline{A} - \overline{B}\overline{B}^TW_1, \qquad B_n = B_1(I - B_2^{\#}B_2) + A_{12}W_2^{-1}B_2^{\#^T}$$
$$C_n = \begin{bmatrix} (B_2^{\#}B_2 - I)B_1^TW_1 - B_2^{\#}A_{21} \\ (I - C_2C_2^{\#})C_1 - C_2^{\#^T}A_{12}^TW_1 \end{bmatrix}, \qquad D_n = \begin{bmatrix} I - B_2^{\#}B_2 \\ C_2^{\#^T}W_2B_2 \end{bmatrix}$$

using $B_2^{\#} = B_2^T(B_2B_2^T)^{-1}$ and $C_2^{\#} = (C_2^TC_2)^{-1}C_2$. $\square$

The following corollary enables the construction of both an NRCF and NLCF of a plant using the algorithm presented above.

Corollary 3.2: If (M, N) is an NRCF of system P, then (M^T, N^T) is an NLCF of P^T.

IV. Example

Assume that our nonproper system is a double differentiator $P(s) = s^2$. A generalized state-space form of $P(s)$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$
$$y = [-1 \quad 0 \quad 0]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Using Proposition 2.2, we can bring this system in the form (1), (3)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$
$$y = [0 \quad 0 \quad 1]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then, the steps outlined in the proposed algorithm yield a state-space realization (A_n, B_n, C_n, D_n) of $\begin{bmatrix} M \\ N \end{bmatrix}$ as

$$A_n = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix}, \qquad B_n = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$
$$C_n = \begin{bmatrix} 0 & -1 \\ -\sqrt{2} & 1 \end{bmatrix}, \qquad D_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, $M(s) = (1/(s^2 + \sqrt{2}s + 1))$, $N(s) = (s^2/(s^2 + \sqrt{2}s + 1))$; and $M(s)$, $N(s) \in \mathscr{H}$, $N(s)M(s)^{-1} = P(s)$, and $M^*(s)M(s) + N^*(s)N(s) = I$.

V. Conclusions

In this note, a state-space algorithm for the calculation of a normalized coprime factorization of continuous-time generalized dynamical systems has been given. It has been shown that two Riccati equations have to be solved in the calculation of this normalized coprime factorization. As shown in the Appendix, these Riccati equations are well defined.

Appendix

In this Appendix, we prove the existence of an NRCF (M, N) of $P \in \mathscr{F}$ as constructed in the algorithm.

Let the generalized state-space realization of the system be partitioned according to Proposition 2.2, and apply operations of restricted system equivalence [11] to a generalized state-space realization of $\begin{bmatrix} I \\ P \end{bmatrix}(I + P^*P)^{-1}[I \quad P^*]$ as follows:

$$\left[\begin{array}{cc|c} I & -W & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]\left[\begin{array}{cc|cc} C^TC & -sE^T - A^T & 0 & C^T \\ sE - A & -BB^T & B & 0 \\ \hline 0 & -B^T & I & 0 \\ -C & 0 & 0 & 0 \end{array}\right] \cdot \left[\begin{array}{cc|c} I & 0 & 0 \\ -W^T & I & 0 \\ \hline 0 & 0 & I \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} Q & -sE^T - A^T + WBB^T & -WB & C^T \\ sE - A + BB^TW^T & -BB^T & B & 0 \\ \hline B^TW^T & -B^T & I & 0 \\ -C & 0 & 0 & 0 \end{array}\right] \tag{5}$$

with

$$Q = s(E^TW^T - WE) + C^TC + A^TW^T + WA - WBB^TW^T. \tag{6}$$

The right-hand side of (5) defines a generalized state-space realization of a spectral factor of $\begin{bmatrix} I \\ P \end{bmatrix}(I + P^*P)^{-1}[I \quad P^*]$, provided that Q in (6) is equal to zero. Define $W = \begin{bmatrix} W_1 & W_{12} \\ 0 & W_2 \end{bmatrix}$ with $W_1 = W_1^T$, $W_2 = W_2^T$, W partitioned in accordance with E. Then, the first part of (6), $s(E^TW^T - WE)$, equals zero. Define $\hat{A}_{11} = A_{11} + A_{12}X$, $\hat{A}_{21} = A_{21} + A_{22}X$, $\hat{C}_1 = C_1 + C_2X$, $X = YW_1 + Z$ with

$$Y = -(W_2A_{22} + C_2^TC_2)^{-1}(A_{12}^T - W_2B_2B_1^T)$$

$$Z = -(W_2A_{22} + C_2^TC_2)^{-1}(C_2^TC_1 + W_2A_{12})$$

where W_1, W_2 are the stabilizing solutions to the Riccati equations

$$0 = C_2^TC_2 + W_2A_{22} + A_{22}^TW_2 - W_2B_2B_2^TW_2$$

$$0 = \bar{C}^T\bar{C} + \bar{A}^TW_1^T + W_1\bar{A} - W_1\bar{B}\bar{B}^TW_1^T$$

with

$$\bar{C} := C_1 + C_2Z$$

$$\bar{A} := A_{11} + (A_{12} + Y^TC_2^TC_2)Z + Y^TC_2^TC_1$$

$$\bar{B} := B_1 - (A_{12} - B_1B_2^TW_2)(A_{22} - B_2B_2^TW_2)^{-1}B_2.$$

The existence of the Riccati solutions follows directly from properties formulated in Proposition 2.2.

Using $F := [B_1^TW_1 - B_2^TW_2X \quad B_2^TW_2]$, (5) can be written as

$$\left[\begin{array}{cc|cc} 0 & -sE^T - A^T + F^TB^T & -F^T & C^T \\ sE - A + BF & -BB^T & B & 0 \\ \hline F & -B^T & I & 0 \\ -C & 0 & 0 & 0 \end{array}\right]$$

which equals a generalized state-space realization of the transfer function $\begin{bmatrix} M \\ N \end{bmatrix}[M^* \quad N^*]$ with

$$\begin{bmatrix} M \\ N \end{bmatrix} := \left[\begin{array}{c|c} sE - A + BF & B \\ \hline F & I \\ -C & 0 \end{array}\right].$$

Now it can be easily checked that $P(s) = N(s)M^{-1}(s)$. Using operations under restricted system equivalence [11], the constructed generalized state-space realization of $\begin{bmatrix} M \\ N \end{bmatrix}$ is reduced to the state-space form

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \left[\begin{array}{c|c} sI_r - \bar{A} + \bar{B}\bar{B}^TW_1 & B_1(I - B_2^{\#}B_2) + A_{12}W_2^{-1}B_2^{\#^T} \\ \hline (I - B_2^{\#}B_2)B_1^TW_1 + B_2^{\#}A_{21} & I - B_2^{\#}B_2 \\ -(I - C_2C_2^{\#})C_1 + C_2^{\#^T}A_{12}^TW_1 & C_2^{\#^T}W_2B_2 \end{array}\right]$$

with $B_2^{\#} = B_2^T(B_2B_2^T)^{-1}$ and $C_2^{\#} = (C_2^TC_2)^{-1}C_2$. Hence, $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$ is proper and asymptotically stable. This shows that the presented algorithm will lead to a state-space representation of an NRCF of a system in generalized state-space form. □

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