iii) Compute

$$C_\pi = (g_m/2\pi f_\tau) - C_\mu. \tag{12}$$

iv) Estimate C_{je}, the part of C_π attributed to depletion capacitance, at the Q-point where the unity gain frequency was measured. For strongly forward-biased pn junctions, (10) does not give correct results [6]. A suggested estimate [6, p. 40] is

$$C_{je} = 2 \times \mathbf{CJE}. \tag{13}$$

v) Compute the "charge-storage" part of C_π, C_b, using

$$C_b = C_\pi - C_{je}. \tag{14}$$

vi) The forward transit time is now found from

$$\mathbf{TF} = C_b/C_\pi. \tag{15}$$

IV. Example

Spice parameters for the BJT's of the CA3086, a general purpose NPN transistor array, were obtained as follows.

1) Curves of V_{BE} versus temperature were given for several values of I_E, from which the point $(I_E, V_{\mathrm{BE}}) = (0.5\ \mathrm{mA}, 0.68\ \mathrm{V})$ was obtained.

From Eq. (1), $\mathbf{IS} = 7.69 \times 10^{-16}$ A.

2) Given hybrid parameters, measured at $I_C = 1$ mA and $V_{\mathrm{CE}} = 3$ V, were

$$h_{\mathrm{fe}} = 100 \quad h_{\mathrm{ie}} = 3.5\ \mathrm{K}\Omega \quad h_{\mathrm{oe}} = 15.6\ \mu\mathrm{s} \quad h_{\mathrm{re}} = 1.8 \times 10^{-4}$$

From (2), $\mathbf{BF} = 100$.
From (3), $g_m = 10^{-3}/0.025 = 0.040$ S.
From (4), $r_\pi = 100/0.040 = 2.5\ \mathrm{K}\Omega$.
From (5), $r_\mu = 2.5\ \mathrm{K}/1.84 \times 10^{-4} = 13.9\ \mathrm{M}\Omega$.
From (6), $1/r_o = 15.6 \times 10^{-6} - 100/13.9 \times 10^6$, therefore $r_o = 1.19 \times 10^5\ \Omega$.
From (7), $\mathbf{VAF} = (1.190 \times 10^5) \times 1 \times 10^{-3} = 119$ V.
From (8), $\mathbf{RB} = 3.5\ \mathrm{K} - 2.5\ \mathrm{K} = 1\ \mathrm{K}\Omega$.

3) The following capacitance values and measurement conditions were listed on the data sheet.

$$C_{\mathrm{CBO}} = 0.58\ \mathrm{pF,\ at}\ V_{\mathrm{CB}} = 3\ \mathrm{V}$$

$$C_{\mathrm{EBO}} = 0.60\ \mathrm{pF,\ at}\ V_{\mathrm{EB}} = 3\ \mathrm{V}$$

$$C_{\mathrm{CI}} = 2.8\ \mathrm{pF\ pF,\ at}\ V_{\mathrm{CI}} = 3\ \mathrm{V}$$

From (9), $\mathbf{CJC} = 0.58 \times 10^{-12}[1 + 3/0.55]^{0.5}$.
From (10), $\mathbf{CJE} = 0.60 \times 10^{-12}[1 + 3/0.7]^{0.33}$
From (11), $\mathbf{CJS} = 2.8 \times 10^{-12}[1 + 3/0.52]^{0.5}$

4) The unity gain frequency and its measurement conditions were

$$f_\tau = 550\ \mathrm{MHz.\ at}\ I_C = 3\ \mathrm{mA\ and}\ V_{\mathrm{CE}} = 3\ \mathrm{V}.$$

Computing the value of $C_\mu = C_{\mathrm{CBO}}$ at the f_τ operating point gives

$$C_\mu = 1.474 \times 10^{-12}/[1 + 2.3/0.55]^{0.5} = 0.648\ \mathrm{pF}.$$

From (12), $C_\pi = (0.120/2\pi 550 \times 10^6) - 0.648 \times 10^{-12} = 34.08 \times 10^{-12}$ Fd.
From (13), $C_{je} = 2 \times \mathbf{CJE} = 2.078 \times 10^{-12}$ Fd.
From (14), $C_b = 34.08 \times 10^{-12} - 2.078 \times 10^{-12} = 32.00 \times 10^{-12}$ Fd.
From (15), $\mathbf{TF} = 32 \times 10^{-12}/0.120 = 2.667 \times 10^{-9}$ s.

Simulation results obtained using these parameter values agreed quite well with experimental results obtained in the laboratory.

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The Discrete Fourier Transform Data Sequence Need Not Be Circularly Defined

A. VAN DEN BOS

***Abstract*—In literature the finite discrete Fourier transform (DFT) data sequence is usually assumed to be circular. It is shown that the familiar DFT theorems can be proved without this often somewhat artificial assumption.**

I. Introduction

In the literature on the finite discrete Fourier transform (DFT) various assumptions are found with respect to the data sequence itself and/or the hypothetical sequences preceding and following it. Cooley, Lewis, and Welch [1], Oran Brigham [2], and Kay and Marple [3] assume that the finite data sequence is one period of an otherwise infinite periodic sequence. Oppenheim and Schafer [4] also represent the data sequence as one period of a periodic sequence. However, outside this period the amplitudes are assumed to be equal to zero. Moreover, the shifted version of the data sequence is represented as one period of the equally shifted periodic sequence. The purpose of this paper is not to question the correctness or the usefulness of these points of view. The purpose is to investigate whether or not it is possible to make no assumptions at all about the data sequence and the sequences preceding and following it. The motivation for this is that the DFT data representations described above may be puzzling for the student or user of the DFT. The data sequence available will often clearly not be one period of a periodic sequence nor will the sequences preceding and following it be zero-valued. For that purpose, in the next section three key DFT theorems (inversion, shift, and convolution) will be reconsidered without assumptions on the data sequence. The resulting conclusions are summarized in a final section.

II. Reconsideration of Three Key DFT Theorems

Let $x(n)$, $n = 0, \cdots, N-1$ be an otherwise unspecified and possibly complex data sequence. Define

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, \cdots, N-1 \tag{1}$$

as the discrete Fourier transform (DFT) of $x(n)$, $n = 0, \cdots, N-1$ where $W_N = \exp(-j2\pi/N)$ with $j = \sqrt{-1}$. Then the *inversion theorem* states that the inverse discrete Fourier transform (IDFT) defined by

$$\frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} \qquad n = 0, \cdots, N-1 \tag{2}$$

is equal to $x(n)$, $n = 0, \cdots, N-1$. The proof of this theorem does not require $x(n)$ to be periodic, circular, or equal to zero outside $n = 0, \cdots, N-1$; see [5].

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Next two other key theorems of the DFT, the *shift theorem* and the *convolution theorem* will be discussed. First consider the DFT $Z(k)$, $k = 0, \cdots, N-1$ of the sequence $z(n) = x(n)W_N^{ln}$, $n = 0, \cdots, N-1$, with l integer

$$Z(k) = \sum_{n=0}^{N-1} \left(x(n)W_N^{ln}\right)W_N^{kn} = \sum_{n=0}^{N-1} x(n)W_N^{(k+l)n}. \quad (3)$$

Since

$$W_N^{(k+l)n} = W_N^{((k+l) \text{ modulo } N)n} \quad (4)$$

it follows that

$$Z(k) = X\left((k+l) \text{ modulo } N\right) \qquad k = 0, \cdots, N-1 \quad (5)$$

where, by definition, $0 \leq ((k+l) \text{ modulo } N) \leq N-1$. To be absolutely clear: (5) is what would have been found in the array $Z(k)$, $k = 0, \cdots, N-1$ after computing the transformation (3). Equations (3) and (5) describe the DFT *frequency shift theorem*

$$x(n)W_N^{ln} \leftrightarrow X\left((k+l) \text{ modulo } N\right) \quad (6)$$

where $\leftrightarrow$ defines the DFT transform pair. Note that the frequency shift is circular. This is a consequence of $W_N^{(k+l)n}$ being circular, not of a supposed circularity of $X(k)$. In the same way the dual DFT *time-shift theorem* states that

$$x\left((n+l) \text{ modulo } N\right) \leftrightarrow X(k)W_N^{-lk}. \quad (7)$$

The proof of this theorem does not require $x(n)$ to be circular.

Central in Fourier theory in general are the convolution theorems. The DFT *frequency convolution* theorem may be described as follows. Let the DFT of the sequences $x(n)$, $n = 0, \cdots, N-1$ and $y(n)$, $n = 0, \cdots, N-1$ be $X(k)$, $k = 0, \cdots, N-1$ and $Y(k)$, $k = 0, \cdots N-1$, respectively. Then

$$x(n)y(n) \leftrightarrow \frac{1}{N}\sum_{l=0}^{N-1} X(l)Y\left((k-l) \text{ modulo } N\right). \quad (8)$$

Proof:

$$\begin{aligned}\sum_{n=0}^{N-1} x(n)y(n)W_N^{kn} &= \sum_{n=0}^{N-1} y(n)\left(\frac{1}{N}\sum_{l=0}^{N-1} X(l)W_N^{-nl}\right)W_N^{kn} \\ &= \frac{1}{N}\sum_{l=0}^{N-1} X(l)\left(\sum_{n=0}^{N-1} y(n)W_N^{(k-l)n}\right) \\ &= \frac{1}{N}\sum_{l=0}^{N-1} X(l)Y\left((k-l) \text{ modulo } N\right). \quad (9)\end{aligned}$$

This completes the proof. Note that the circularity of the convolution in (8) is a consequence of the circularity of $W_N^{(k-l)n}$. The dual *time-convolution theorem* is proved analogously. It is given by

$$\sum_{l=0}^{N-1} x(l)y\left((n-l) \text{ modulo } N\right) \leftrightarrow X(k)Y(k). \quad (10)$$

For the proof of this theorem neither $x(n)$ nor $y(n)$ need be circular. From (8) and (10) follow the dual, generalized forms of *Parseval's theorem*:

$$\sum_{n=0}^{N-1} x(n)y(n) = \frac{1}{N}\sum_{l=0}^{N-1} X(l)Y\left((N-l) \text{ modulo } N\right) \quad (11)$$

and

$$\sum_{l=0}^{N-1} x(l)y\left((N-l) \text{ modulo } N\right) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y(k). \quad (12)$$

With respect to the above results the following observations can be made. In the first place, the results can easily be extended to include all further conventional DFT theorems. Furthermore, the circularity of the shift and convolution operations arises in a natural way, not as a consequence of assumptions. The corresponding theorems describe what the results would be of a computing device knowing the definitions of the DFT and the IDFT, but not aware of any assumptions concerning the data sequence. Finally, the DFT $X(k)$ in (1) is defined for $k = 0, \cdots, N-1$ only. However, no serious objection can be made to assuming $X(k)$ periodic with period N. The motivation for the definition chosen in this paper is that thus the number of complex entities concerned remains the same in both domains. Moreover, this definition preserves the symmetry of the dual theorems.

III. Conclusions

It has been shown that the familiar DFT theorems can be proved without the usual assumption that the data sequence is circular. Circularity of DFT shift and convolution is a consequence of the DFT properties, not necessarily of those of the data sequence. The advantage of this alternative viewpoint is that puzzling circularity assumptions with respect to nonperiodic data sequences are avoided.

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The Square Matrix Rule of the Convolution Integral

ZHANG ZUHAO

***Abstract*—This paper presents a square matrix rule, which can easily determine the limit and domain of the convolution integral. The rule is demonstrated theoretically. Some examples are given to explain its application.**

The convolution integral of two functions can be expressed as

$$D(t) * S(t) = \int_{-\infty}^{\infty} D(t-\xi)S(\xi)\, d\xi.$$

In the graphical approach, the mirror-image of $D(t)$ about the Y axis is translated, while the graph of $S(t)$ is at rest. For convenience of description, we call $D(t)$ the dynamic function and $S(t)$ the static function.

Two functions with step continuity are often encountered in electrical engineering, and when we are integrating, the problem is: How do we determine the limit and the domain of the convolution integral? This problem is usually solved graphically. In this paper, a square matrix rule is presented as an alternate way of finding this domain. It is rather simple and can be derived as follows:

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