Price's Theorem for Complex Variates

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Abstract—**Price's theorem is derived for complex valued variates. The derivation differs from the existing derivation in two respects. First, the normal variates are not assumed to be circularly complex. Thus the result is more general. Second, the characteristic function of the complex variates is not used.**

Index Terms—**Nonlinear transformations, normal distribution, complex stochastic variables.**

I. Introduction

Nearly all theoretical results concerning normally distributed complex valued variates have been derived for a special class: circularly complex normal variates [1]–[4]. The complex Price theorem presented in [2] is no exception. Since Price's theorem is a key result in normal distribution theory, it is worthwhile to derive it for the general complex normal distribution [5]. This derivation is the purpose of this correspondence.

The complex Price theorem may be used for applications similar to those of its real counterpart. An example is the proof of Bussgang's theorem [6]. A further example is the computation of moments illustrated in Section IV of this correspondence. Examples of normally distributed complex valued variates that are not circularly complex are samples of a carrier amplitude modulated with normally distributed noise [3].

The proof of Price's theorem presented in [2] uses the characteristic function of circularly complex variates and proceeds analogously to the proof of the original, real Price theorem [6], [7]. The latter proof involves a number of real integrations by parts. For complex variates these change into integrations by parts of functions of complex variables over their real and imaginary parts. This complicates the proof.

In this correspondence, a derivation of the complex Price theorem is presented that avoids both the restriction to circularly complex variates and the complications associated with the use of the characteristic function. This is achieved by expressing the derivatives with respect to complex variables appearing in the complex Price theorem as derivatives with respect to the real-valued real and imaginary parts. Applying the usual, real Price theorem to these real derivatives completes the proof.

In Section II derivatives with respect to complex vector variables are introduced. A proof of the general complex Price theorem based on these derivatives is given in Section III. In Section IV an example of the application of this theorem is given.

II. Complex Derivatives

In this section, partial derivatives of complex functions with respect to vector valued complex variables are introduced. They are used in the next section for the derivation of the complex Price theorem.

Suppose that $\phi\colon R^{2N\times 1} \to C$ is a function of the elements of the vector $\omega \in R^{2N\times 1}$ defined as

$$\omega = (\xi_1 \quad \eta_1 \quad \cdots \quad \xi_N \quad \eta_N)^T \tag{1}$$

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where the superscript T denotes transposition. Furthermore, suppose that the vector $v \in C^{2N\times 1}$ is defined as

$$v = (\zeta_1 \quad \zeta_1^* \quad \cdots \quad \zeta_N \quad \zeta_N^*)^T \tag{2}$$

where $\zeta_n = \xi_n + j\eta_n$, the superscript * denotes complex conjugation and $j = \sqrt{-1}$. Then

$$v = A\omega \tag{3}$$

where $A \in C^{2N\times 2N}$ is the block diagonal matrix

$$A = \mathrm{diag}\,(J \cdots J) \tag{4}$$

with blocks $J \in C^{2\times 2}$ defined as

$$J = \begin{pmatrix} 1 & j \\ 1 & -j \end{pmatrix}. \tag{5}$$

Notice that

$$J^{-1} = \frac{1}{2}J^H \quad \text{and} \quad A^{-1} = \frac{1}{2}A^H \tag{6}$$

where the superscript H denotes complex conjugate transposition. Let $\psi\colon C^{2N\times 1} \to C$ be the function of the complex variables v obtained by substituting the solution of (3) for ω in the function ϕ. Then the complex partial derivatives of ψ with respect to ζ_n and ζ_n^* are defined by

$$\begin{pmatrix} \dfrac{\partial}{\partial \zeta_n} \\ \dfrac{\partial}{\partial \zeta_n^*} \end{pmatrix} \psi = \frac{1}{2} J^* \begin{pmatrix} \dfrac{\partial}{\partial \xi_n} \\ \dfrac{\partial}{\partial \eta_n} \end{pmatrix} \phi. \tag{7}$$

This definition, which originated from complex function theory [8, pp. 49–50], was applied to optimization problems in array theory by Brandwood [9] and later to complex valued nonlinear numerical minimization by van den Bos [10]. Notice that in (7) ζ_n and ζ_n^* are considered to be separate variables as usual in complex function theory.

Next define the vector $\nu \in C^{2N\times 1}$ of arbitrary complex variables as $(\nu_1 \cdots \nu_{2N})^T$. Then, by combining definition (7) with standard real differential calculus, the following relation between complex differential operators with respect to the elements of ν may be established:

$$\begin{pmatrix} \dfrac{\partial}{\partial(\nu_n + j\nu_m)} \\ \dfrac{\partial}{\partial(\nu_n - j\nu_m)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \dfrac{\partial}{\partial \nu_n} - j\dfrac{\partial}{\partial \nu_m} \\ \dfrac{\partial}{\partial \nu_n} + j\dfrac{\partial}{\partial \nu_m} \end{pmatrix} = \frac{1}{2}J^* \begin{pmatrix} \dfrac{\partial}{\partial \nu_n} \\ \dfrac{\partial}{\partial \nu_m} \end{pmatrix}. \tag{8}$$

Therefore

$$\frac{\partial}{\partial(A\nu)} = \frac{1}{2}A^* \frac{\partial}{\partial \nu}. \tag{9}$$

In this expression and in what follows, partial derivatives with respect to a row or column vector are defined as the row or column vector of partial derivatives with respect to the elements of the vector. Furthermore, since by (7) the derivative with respect to a complex variable is the conjugate of the derivative with respect to the conjugate of that variable, it follows from the transpose of (9) that

$$\frac{\partial}{\partial(\nu^H A^H)} = \frac{1}{2}\frac{\partial}{\partial \nu^H}A^T. \tag{10}$$

Equations (9) and (10) will be the main tools in the derivation of Price's theorem for complex variates described in Section III.

III. Derivation of Price's Theorem for Complex Variates

Suppose that the elements of the vector $\boldsymbol{w} \in R^{2N\times 1}$ defined as

$$\boldsymbol{w} = (x_1 \quad y_1 \quad \cdots \quad x_N \quad y_N)^T \tag{11}$$

are jointly normally distributed and that their expectation is equal to zero. Let g: $R^{2N\times 1} \rightarrow C$ be a function of these elements. Then Price's theorem for real normal variates is described by

$$\frac{\partial}{\partial \boldsymbol{C_{ww}}} E[g] = E\left[\frac{\partial}{\partial \boldsymbol{w}}\left(\frac{\partial}{\partial \boldsymbol{w}^T}\right) g\right] \tag{12}$$

where $E[\ ]$ is the expectation operator and

$$\boldsymbol{C_{ww}} = E[\boldsymbol{w}w^T] \in R^{2N\times 2N}$$

is the covariance matrix of the vector $\boldsymbol{w}$ [6], [7]. The (k, l)th element of the operator $\partial/\partial \boldsymbol{C_{ww}}$ is defined as $\partial/\partial c_{w_k w_l}$ and the corresponding element of the operator $\partial/\partial \boldsymbol{w}(\partial/\partial \boldsymbol{w}^T)$ as $\partial^2/\partial w_k \partial w_l$. Sufficient conditions to be met by the real and imaginary part of the function g are described by Papoulis [11]. Next, define the vector $\boldsymbol{v} \in C^{2N\times 1}$ by

$$\boldsymbol{v} = (z_1 \quad z_1^* \quad \cdots \quad z_N \quad z_N^*)^T \tag{13}$$

where $z_n = x_n + jy_n$. Then

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{w} \tag{14}$$

where $\boldsymbol{A}$ is defined by (4) and (5). Furthermore, the complex autocovariance matrix $\boldsymbol{C_{vv}} \in C^{2N\times 2N}$ of $\boldsymbol{v}$ is defined as

$$\boldsymbol{C_{vv}} = E[\boldsymbol{v}\boldsymbol{v}^H]. \tag{15}$$

Notice that the (k, l)th element of $\boldsymbol{C_{vv}}$ is $c_{v_k v_l} = E[v_k\ v_l^*]$. Therefore, by (14)

$$\boldsymbol{C_{vv}} = \boldsymbol{A}\boldsymbol{C_{ww}}\boldsymbol{A}^H. \tag{16}$$

This result will now be used to transform the real Price's theorem (12) into a complex counterpart. For that purpose, (12) is premultiplied by $\frac{1}{2}\boldsymbol{A}^*$ and postmultiplied by $\frac{1}{2}\boldsymbol{A}^T$, respectively

$$\frac{1}{4}\boldsymbol{A}^* \frac{\partial}{\partial \boldsymbol{C_{ww}}} \boldsymbol{A}^T E[g] = E\left[\frac{1}{4}\boldsymbol{A}^* \frac{\partial}{\partial \boldsymbol{w}}\left(\frac{\partial}{\partial \boldsymbol{w}^T}\boldsymbol{A}^T\right) g\right]. \tag{17}$$

Then, applying (9) and (10) to both members of this expression yields

$$\frac{\partial}{\partial(\boldsymbol{A}\boldsymbol{C_{ww}}\boldsymbol{A}^H)} E[g] = E\left[\frac{\partial}{\partial \boldsymbol{A}\boldsymbol{w}}\left(\frac{\partial}{\partial \boldsymbol{w}^T\boldsymbol{A}^H}\right) g\right]. \tag{18}$$

Combining this with (14) and (16) yields Price's theorem for complex normal variates

$$\frac{\partial}{\partial \boldsymbol{C_{vv}}} E[h] = E\left[\frac{\partial}{\partial \boldsymbol{v}}\left(\frac{\partial}{\partial \boldsymbol{v}^H}\right) h\right] \tag{19}$$

with elements

$$\frac{\partial}{\partial c_{v_k v_l}} E[h] = E\left[\frac{\partial^2}{\partial v_k \partial v_l^*} h\right] \tag{20}$$

where $c_{v_k v_l} = E[v_k v_l^*]$ and h: $C^{2N\times 1} \rightarrow C$ is the function of the elements of $\boldsymbol{v}$ obtained by substituting $\boldsymbol{A}^{-1}\boldsymbol{v}$ for $\boldsymbol{w}$ in the function g. Equations (19) and (20) are the main result of this correspondence.

The Price theorem described by (19) and (20) has been derived without assumptions with respect to the covariances of the elements of $\boldsymbol{v}$ or, equivalently, with respect to those of the elements of $\boldsymbol{w}$. It is therefore not restricted to circularly complex variates such as the complex Price theorem described in [2]. A further difference of the derivation of the Price theorem (19) and (20) from the derivation in [2] is that no use has been made of the characteristic function of the complex variates.

IV. An Example

If the matrices in (19) are partitioned in 2×2 blocks, corresponding blocks in the left-hand and the right-hand member are described by

$$\begin{pmatrix} \dfrac{\partial}{\partial c_{z_k z_l}} & \dfrac{\partial}{\partial c_{z_k z_l^*}} \\ \dfrac{\partial}{\partial c_{z_k^* z_l}} & \dfrac{\partial}{\partial c_{z_k^* z_l^*}} \end{pmatrix} E[h] = E\left[\begin{pmatrix} \dfrac{\partial^2}{\partial z_k \partial z_l^*} & \dfrac{\partial^2}{\partial z_k \partial z_l} \\ \dfrac{\partial^2}{\partial z_k^* \partial z_l^*} & \dfrac{\partial^2}{\partial z_k^* \partial z_l} \end{pmatrix} h\right]. \tag{21}$$

This result will now be used to compute, as an example, the fourth-order moments

$$E[z_p^* z_q^* z_r z_s]. \tag{22}$$

From (21) it follows directly that (22) is a function of $c_{z_p^* z_q}$, $c_{z_p^* z_r^*}$, $c_{z_p^* z_s^*}$, $c_{z_q^* z_r^*}$, $c_{z_q^* z_s^*}$, and $c_{z_r z_s^*}$. The partial derivatives with respect to these covariances are $c_{z_r z_s^*}$, $c_{z_q^* z_s^*}$, $c_{z_q^* z_r^*}$, $c_{z_p^* z_s^*}$, $c_{z_p^* z_r^*}$, and $c_{z_p^* z_q}$, respectively. Using these results in successive integration steps, and observing the condition that (22) is equal to zero if the covariances are, yields

$$E[z_p^* z_q^* z_r z_s] = c_{z_p^* z_q} c_{z_r z_s^*} + c_{z_p^* z_r^*} c_{z_q^* z_s^*} + c_{z_p^* z_s^*} c_{z_q^* z_r^*}. \tag{23}$$

The expressions for the remaining 15 possible fourth-order moments are analogous. If (23) would be a fourth-order moment of a circularly complex normal process, the first term would be absent since, by definition, $c_{z_p^* z_q}$ and $c_{z_r z_s^*}$ would be equal to zero. This result then agrees with that of McGee [3].

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