Transactions Letters

Optimized Signal Constellations for Trellis-Coded Modulation on AWGN Channels

René J. van der Vleuten and Jos H. Weber

Abstract— **In earlier publications, performance gains over Ungerboeck type trellis-coded modulation schemes were obtained by optimizing (by hand) the signal constellation. Using genetic algorithms and simulated annealing, we have found additional cases with performance gains over the Ungerboeck type schemes.**

I. Introduction

UNGERBOECK'S trellis-coded modulation (TCM) schemes [1]–[3] use the symmetries of rate $n/(n+1)$ binary convolutional codes to map the channel symbols onto the trellis. The channel symbols are selected from the pulse amplitude modulation (PAM), phase shift keying (PSK), or quadrature amplitude modulation (QAM) signal constellations that are also used for uncoded modulation. In [4] and [5], however, it was shown that a gain over the Ungerboeck type TCM schemes can be obtained, in several cases, by optimizing the signal constellation. The optimization of the signal constellation was carried out by hand and only for small trellises (up to 16 states). Here, we report on additional cases in which there is an asymptotic performance gain over the Ungerboeck type TCM schemes, which we obtained by an automatic optimization method based on genetic algorithms and simulated annealing. We applied those techniques, because, contrary to traditional optimization methods, they continue to search for a better solution, after a locally optimal solution has been found. Therefore, in the presence of many local optima, genetic algorithms and simulated annealing often produce better solutions than the traditional methods, although there is no guarantee that the global optimum is found. More information about simulated annealing and genetic algorithms can be found in [6] and [7], respectively. The convolutional codes and signal constellations are simultaneously optimized, i.e., a full search of the parity check polynomials is performed and for each individual code the signal constellation is optimized.

The figure of merit for a TCM scheme, in the presence of additive white Gaussian noise (AWGN), is the *normalized squared minimum distance*, $\rho^2_{\min}$, given by [5] and [8]

$$\rho^2_{\min} = \frac{d^2_{\text{free}} \cdot R}{S} \tag{1}$$

where d_{free} is the minimum Euclidean distance occurring between any two sequences of constellation points, R is the rate (in bits per dimension) and S is the signal energy (per dimension). To compute d^2_{free}, we used the algorithm proposed in [9]; a new algorithm was recently proposed in [10]. Ideally, the optimization would also include other parameters that are important for the performance of the TCM scheme, such as the number of nearest neighbors or the distance spectrum (see, e.g., [10]). Because of the associated additional complexity (CPU-time), however, this is not feasible, in our case.

Sections II–IV report on optimized PAM, PSK, and QAM-constellation-based TCM schemes, respectively. Section V concludes the paper.

Paper approved by S. Benedetto, the Editor for Signal Design, Modulation, and Detection of the IEEE Communications Society. Manuscript received September 1, 1994; revised August 15, 1995. This paper was presented in part at the first IEEE Symposium on Communications and Vehicular Technology in the Benelux, Delft, The Netherlands, October 27–28, 1993.

R. J. van der Vleuten was with Delft University of Technology, Department of Electrical Engineering, The Netherlands. He is now with Philips Research Laboratories, 5656 AA Eindhoven, The Netherlands.

J. H. Weber is with Delft University of Technology, Department of Electrical Engineering, 2600 GA Delft, The Netherlands.

Publisher Item Identifier S 0090-6778(96)03359-4.

II. PAM Constellations

For transmitting at rate $R = 1, 2, \cdots$, using PAM signal points and a rate 1/2 convolutional code, there are four different sets of signal points, A, $-A$, B, and $-B$ where $A = \{a_1, \cdots, a_{2^{R-1}}\}$, $B = \{b_1, \cdots, b_{2^{R-1}}\}$, and $-A$ and $-B$, respectively denote the sets $\{-a_1, \cdots, -a_{2^{R-1}}\}$ and $\{-b_1, \cdots, -b_{2^{R-1}}\}$.

Given $a_1, a_i = a_1 - (i-1)\delta$ and $b_i = a_i - \delta/2$, for $1 \le i \le 2^{R-1}$, the normalization of

$$S = 2^{-R} \sum_{i=1}^{2^{R-1}} (a_i^2 + b_i^2) = 1 \tag{2}$$

results in

$$\delta = 2 \cdot (1 - 3 \cdot 2^R + 2^{2R+1})^{-1} \cdot [3 \cdot a_1(2^R - 1) + \sqrt{3} \cdot \sqrt{2 - 3 \cdot 2^{R+1} + 2^{2R+2} + a_1^2(1 - 2^{2R})}]. \tag{3}$$

The constellation (the same as proposed in [4] and [5]) is, thus, specified by a single parameter, a_1.

The optimized convolutional codes and constellation parameters a_1 and δ [the latter added for the reader's convenience, since it can be computed from (3)], as well as the resulting $\rho^2_{\min}$ (in dB) and the corresponding $\rho^2_{\min}$ for the Ungerboeck type TCM schemes, have been listed in Table I for $R =$ 1, 2, 3, and 4, for 2^ν-state trellises, where $2 \le \nu \le 8$. As in [3], the convolutional codes are specified by their parity-check polynomials $h^1 = (h^1_\nu, h^1_{\nu-1}, \ldots, h^1_0)$ and $h^0 = (h^0_\nu, h^0_{\nu-1}, \ldots, h^0_0)$, given in octal form. For example, $h^0 =$

TABLE I
NORMALIZED SQUARED MINIMUM DISTANCES, $\rho^2_{\min}$ (IN dB), FOR THE OPTIMIZED PAM-CONSTELLATION-BASED TCM SCHEMES (OPT), COMPARED TO THE UNGERBOECK TYPE SCHEMES (U) [3]

		Code				$\rho^2_{\min}$ [dB]		Gain
R	2^ν	h^1	h^0	a_1	δ	OPT	U	[dB]
1	4	2	7	1.21	3.88	9.03	8.57	0.46*
	8	04	13	1.31	3.69	9.62	9.03	0.59*
	16	10	23	1.27	3.78	10.17	9.44	0.73
	32	12	45	1.27	3.78	10.64	10.17	0.47
	64	010	107	1.25	3.82	10.98	10.49	0.49
	128	136	267	1.20	3.90	11.37	11.07	0.30
	256	302	433	1.24	3.84	11.71	11.34	0.37
2	4	2	7	1.50	1.76	5.59	5.35	0.24*
	8	04	13	1.50	1.76	6.19	5.81	0.38*
	16	10	23	1.50	1.76	6.71	6.22	0.49
	32	12	45	1.49	1.76	7.09	6.95	0.14
	64	032	107	1.51	1.75	7.45	7.27	0.18
	128	126	235	1.53	1.75	7.84	7.84	0.00
	256	302	433	1.46	1.77	7.99	7.84	0.15
3	4	2	7	1.61	0.87	1.23	1.04	0.19*
	8	04	13	1.61	0.87	1.82	1.50	0.32*
	16	10	23	1.61	0.87	2.33	1.91	0.42
	32	12	45	1.61	0.87	2.71	2.64	0.07
	64	032	107	1.62	0.87	3.10	2.96	0.14
	128	126	235	1.63	0.87	3.54	3.54	0.00
	256	302	433	1.59	0.87	3.57	3.54	0.03
4	4	2	5	1.69	0.43	-3.58	-3.74	0.16
	8	02	17	1.69	0.43	-3.00	-3.29	0.29
	16	02	33	1.69	0.43	-2.48	-2.87	0.39
	32	12	45	1.67	0.43	-2.10	-2.15	0.05
	64	032	135	1.68	0.43	-1.69	-1.83	0.14
	128	126	235	1.68	0.43	-1.25	-1.25	0.00
	256	302	433	1.66	0.43	-1.24	-1.25	0.01

* Same gain previously reported in [4], [5].

$(1, 0, 0, 1, 1)$ is written as $h^0 = 23$. By definition, $h^0_\nu = h^0_0 = 1$ and $h^1_\nu = h^1_0 = 0$. In all cases, the mapping function is $f(z^1z^0) = (00, 01, 10, 11) \rightarrow (A, -B, B, -A)$, where z^1 and z^0 are the output bits of the encoder, acted upon by h^1 and h^0, respectively.

It can be observed from Table I that, generally, as the rate and complexity (i.e., the number of states) increase, the gains of the optimized TCM schemes over those based on the traditional equally spaced signal constellations slowly decrease. It can also be observed from Table I that there is no gain for 128 states, for $R > 1$; the code and signal constellation found are exactly those of [3].

III. PSK CONSTELLATIONS

Codes for the nonequally spaced four-point PSK constellations proposed in [4], [5], for $R = 1/2$, were published in [11], as well as in [12] and [13]; they are therefore not discussed here. In this section, additional TCM schemes based on the nonequally spaced eight-PSK constellations proposed in [4] and [5] are presented. These constellations are used with rate 2/3 convolutional codes to transmit $R = 1$ bit per dimension. The eight signal points are $A, -A, B, -B, C, -C, D,$ and $-D$, where

$$A = (\sqrt{2}\cos(\phi/2), \sqrt{2}\sin(\phi/2))$$
$$B = (-\sqrt{2}\sin(\phi/2), \sqrt{2}\cos(\phi/2))$$
$$C = (\sqrt{2}\sin(\phi/2), \sqrt{2}\cos(\phi/2))$$
$$D = (-\sqrt{2}\cos(\phi/2), \sqrt{2}\sin(\phi/2)).$$

For the equally spaced constellation, $\phi = \pi/4$.

The optimized convolutional codes and constellation parameter $\phi/2$ as well as the resulting $\rho^2_{\min}$ (in dB) and the corresponding $\rho^2_{\min}$ for the Ungerboeck type TCM schemes, have been listed in Table II for $R = 1$. To further facilitate the comparison between the newly found and the Ungerboeck type TCM schemes, the d^2_{free} values have been listed as well. The mapping function is $f(z^2z^1z^0) = (000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (A, -A, B, -B, C, -C, D, -D)$.

Whereas, there is no gain for eight states, there is a small increasing gain for 16, 32, and 64 states.

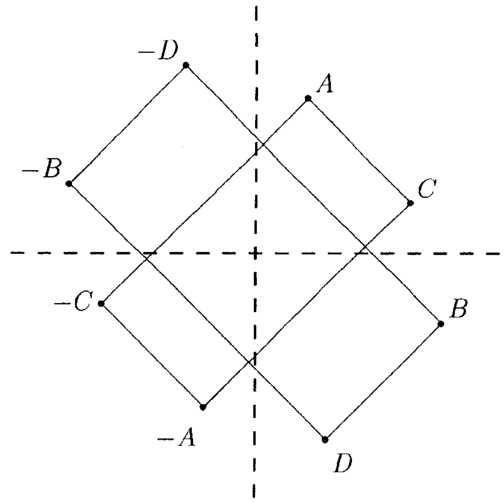

Fig. 1. Optimized QAM signal constellation [14] for the eight-state TCM scheme.

IV. QAM CONSTELLATIONS

For transmitting at $R = 1$, using a rate 2/3 convolutional code, a QAM signal constellation may outperform a PSK constellation. Although this is not the case for the Ungerboeck type schemes, for which the eight-PSK constellation outperforms the eight-QAM constellation, we have found eight- and 16-state TCM schemes with an optimized eight-QAM constellation that results in a performance gain over the equally or nonequally spaced eight-PSK constellations.

As illustrated in Fig. 1, the eight signal points, $A, -A, B, -B, C, -C, D,$ and $-D$, are specified by four

TABLE II
NORMALIZED SQUARED MINIMUM DISTANCES, $\rho^2_{\min}$ (IN dB), AND SQUARED MINIMUM DISTANCES, d^2_{free}, FOR THE OPTIMIZED PSK-CONSTELLATION-BASED TCM SCHEMES (OPT), COMPARED TO THE UNGERBOECK TYPE SCHEMES (U) [3]

		Code				$\rho^2_{\min}$ [dB]		Gain	d^2_{free} ($S = 0.5$)	
R	2^ν	h^2	h^1	h^0	$\phi/2$	OPT	U	[dB]	OPT	U
1	8	04	02	11	0.393	9.62	9.62	0.00*	4.586	4.586
	16	10	02	23	0.464	10.17	10.15	0.02*	5.200	5.172
	32	24	14	67	0.503	10.69	10.61	0.08	5.861	5.758
	64	012	074	147	0.503	11.13	11.03	0.10	6.481	6.343

* Same gain previously reported in [4], [5].

TABLE III
NORMALIZED SQUARED MINIMUM DISTANCES, $\rho^2_{\min}$ (IN dB), AND SQUARED MINIMUM DISTANCES, d^2_{free}, FOR THE OPTIMIZED QAM-CONSTELLATION-BASED TCM SCHEMES (OPT), COMPARED TO THE UNGERBOECK-TYPE PSK SCHEMES (U) OF [3]

		Code			Constellation			$\rho^2_{\min}$ [dB]		Gain	d^2_{free} ($S = 0.5$)	
R	2^ν	h^2	h^1	h^0	λ	μ	α	OPT	U	[dB]	OPT	U
1	8	04	02	11	0.798	0.401	1.000	9.68	9.62	0.06	4.648	4.586
	16	16	04	23	0.774	0.373	1.012	10.37	10.15	0.22	5.442	5.172

positive parameters λ, μ, α, and β

$$\begin{aligned} A &= (\lambda - \mu, \lambda + \mu) \\ B &= (\alpha + \beta, -\alpha + \beta) \\ C &= (\lambda + \mu, \lambda - \mu) \\ D &= (\alpha - \beta, -\alpha - \beta). \end{aligned}$$

Normalizing to $S = 1$ leaves only three parameters, λ, μ, and α, to be optimized, with $\beta = \sqrt{2 - \lambda^2 - \mu^2 - \alpha^2}$.

The optimized convolutional codes and constellation parameters as well as the resulting $\rho^2_{\min}$ (in dB) and the corresponding $\rho^2_{\min}$ for the Ungerboeck type PSK-constellation-based TCM schemes, are listed in Table III. The mapping function is $f(z^2z^1z^0) = (000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (A, -D, B, C, -A, D, -B, -C)$.

V. CONCLUSION

We have presented additional TCM schemes, based on nonequally spaced signal constellations, that have a larger squared minimum distance than the Ungerboeck type TCM schemes based on equally spaced signal constellations.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their helpful comments which improved the presentation of this work. In particular, one of the reviewers discovered a 3.01 dB systematic error in the $\rho^2_{\min}$ values for the PSK and QAM constellations in the tables in [11] and [14]; Table II and Table III list the correct values.

REFERENCES

[1] G. Ungerboeck, "Channel coding with multilevel/phase signals," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 55–67, Jan. 1982.

[2] ——, "Trellis-coded modulation with redundant signal sets, Part I: Introduction," *IEEE Commun. Mag.*, vol. 25, pp. 5–11, Feb. 1987.

[3] ——, "Trellis-coded modulation with redundant signal sets, Part II: State of the art," *IEEE Commun. Mag.*, vol. 25, pp. 12–21, Feb. 1987.

[4] D. Divsalar, M. K. Simon, and J. H. Yuen, "Trellis coding with asymmetric modulations," *IEEE Trans. Commun.*, vol. COM-35, pp. 130–141, Feb. 1987.

[5] E. Biglieri, D. Divsalar, P. J. McLane, and M. K. Simon, *Introduction to Trellis-Coded Modulation with Applications.* New York: Macmillan, 1991.

[6] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing.* Cambridge, MA: Cambridge University Press, 1992, 2nd ed.

[7] L. Davis, Ed., *Handbook of Genetic Algorithms.* New York: Van Nostrand Reinhold, 1991.

[8] R. E. Blahut, *Digital Transmission of Information.* Reading, MA: Addison-Wesley, 1990.

[9] M. G. Mulligan and S. G. Wilson, "An improved algorithm for evaluating trellis phase codes," *IEEE Trans. Inform. Theory*, vol. IT-30, pp. 847–851, Nov. 1984.

[10] S. Benedetto, M. Mondin, and G. Montorsi, "Performance evaluation of trellis-coded modulation schemes," *Proc. IEEE*, vol. 82, pp. 833–855, June 1994.

[11] R. J. van der Vleuten and J. H. Weber, "Trellis-coded modulation with optimized signal constellations," in *First IEEE Symp. Commun. Vehicular Technol. Benelux*, Delft, The Netherlands, Oct 27–28, 1993. ISBN 90-5326-009-9.

[12] S. Benedetto, R. Garello, and M. Mondin, "Geometrically uniform trellis codes based on multidimensional unbalanced QPSK," in *Proc. IEEE GLOBECOM'93*, Houston, TX, pp. 1444–1448, Nov. 1993.

[13] Y. Levy and D. J. Costello, "On the construction of real number trellis codes," in *Proc. IEEE Int. Symp. Inform. Theory*, Trondheim, Norway, June 27–July 1, 1994, p. 163.

[14] R. J. van der Vleuten, "Trellis-Based Source and Channel Coding," Ph.D. dissertation, Delft Univ. Technol., The Netherlands, 1994. ISBN 90-5326-013-7.