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Separation principle for stochastic control of continuous-time Markov jump linear systems under partial observations

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ABSTRACT

In MJLS literature the separation principle between filtering and control has been established in case the Markov mode switching process $\{\theta_t\}$ is fully observed, and the Euclidean state process $\{x_t\}$is partially observed. In case the exact $\{\theta_t\}$remains hidden, the separation principle has only been established under a linear filtering restriction. Since nonlinear filters can provide significant better estimates, the desire to extend the separation principle to MJLS with hidden $\{\theta_t\}$is a long-standing challenge. The objective of this paper is to resolve this long-standing challenge in three steps. The first step is to transform the MJLS stochastic control problem into control under a quadratic performance criterion of a linear system driven by a martingale which is influenced by the control. The certainty equivalence (CE) condition known in literature applies to stochastic control of a linear system that is driven by a control independent martingale. Therefore, the second step is to relax this known CE condition such that it allows this control influence on the martingale. The third step is to prove that the relaxed CE condition is satisfied for the general MJLS control problem considered. The overall achievement is a CE control law for a partially observed MJLS, which assures the Separation Principle between filtering and control. The paper also shows that for the case that $\{x_t\}$is fully observed and the exact$\{\theta_t\}$remains hidden, that the novel CE control law differs significantly from the in literature well-developed Averaging MJLS control policy.

1. Introduction

One of the fundamentals in feedback control theory for linear systems is that optimal control and optimal state estimation can be resolved in a decoupled way; this is known as the Separation Principle. Stochastic control of partially observed continuous-time Markov Jump Linear Systems (MJLS) constitutes a well-studied class of problems in filtering and control theory. Rich overviews of the problems and achievements in this domain are given by Mariton [1], Elliott et al. [2], Mao and Yuan [3], Costa et al. [4]. Despite these achievements, for MJLS the Separation Principle between filtering and control has only been established when the Markovian switching parameters are fully observed [5]. However, if the Markov switching is hidden, then the existing theory on the separation principle falls short. The latter even applies to the basic hidden Markov setting [6]. The lack of a general separation principle has motivated the development of sub-optimal approaches for the integration of filtering and control for MJLS [7–11]. A popular approach is to adopt the certainty equivalence (CE) control policy, which means that in the deterministic control policy the exact state is replaced by the estimated state, e.g. [8]. Another popular approach is to approximate the exact nonlinear filter by the best linear filter, and subsequently optimize the control for this linear system, e.g. [10]. The objective of this paper is to improve this situation

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fundamentally, by developing a general separation principle for stochastic control of a partially observed MJLS.

The MJLS considered is the Ito stochastic differential equation (SDE):

$$dx_t = A_{\theta_t} x_t dt + B_{\theta_t} u_t dt + C_{\theta_t} dw_t \tag{1}$$

with R^n-valued process $\{x_t\}$, feedback control $\{u_t\}$, and Brownian motion $\{w_t\}$ which is independent of (θ_0, x_0). The coefficients A_θ, B_θ, C_θ are switching according to a continuous-time Markov chain $\{\theta_t\}$ which is independent of $\{w_t\}$, has cadlag paths and assumes values in a finite set $\Theta = \{e_1, .., e_N\}$ of unity vectors $e_i \in R^N$, according to Markov transition rates $\lambda_{\theta\eta}$:

$$\mathrm{P}\{\theta_{t+\Delta} = \eta | \theta_t = \theta, (x_s, \theta_s; s < t)\} = \begin{cases} \lambda_{\theta\eta}\Delta + o(\Delta), & \eta \neq \theta \\ 1 + \lambda_{\theta\theta}\Delta + o(\Delta), & \eta = \theta \end{cases} \tag{2}$$

with $\lim_{\Delta \to 0} o(\Delta)/|\Delta| = 0$, and $\lambda_{\theta\theta} = -\sum_{\eta \neq \theta} \lambda_{\theta\eta}$.

The hybrid process $\{\theta_t, x_t\}$ is assumed to be partially observed through an m-dimensional process $\{y_t\}$. The control problem is to characterize the feedback law $\Psi : y \to u$ over a time window $[0, T]$, i.e. to map the observation process $\{y_t\}$ to the control input $\{u_t\}$ in a non-anticipatory manner, to minimize the value of the functional

$$J(\{u_t\}) = E\left\{ x'_T S_{\theta_T} x_T + \int_0^T x'_t Q_{\theta_t} x_t dt + \int_0^T u'_t R_= u_t dt \right\} \tag{3}$$

where, for all $\theta \in \Theta$, S_θ and Q_θ are positive semi-definite and $R_=$ is positive definite.

The challenge is to develop for (1-3), given partial observations, a control law that assures the Separation Principle between filtering and control to hold true.

Georgiou and Lindquist [12] extended the separation principle in stochastic control to a partially observed linear systems that is driven by a martingale process instead of Brownian motion. To follow this approach, the above MJLS will be transformed to a linear process with discontinuous martingale input. The resulting system raises two issues that are not addressed by [12]. Firstly, the "Stochastic Open Loop" reasoning [13] does not apply for a system with Markov switching coefficients. Secondly, the resulting system involves a discontinuous martingale that involves a multiplication with the process to be controlled. To overcome the first issue, the set of admissible control policies is restricted to those defining a $\{u_t\}$which is square integrable and pathwise unique. To address the second issue, the derivations of [12] are extended in this paper. These derivations show that, in contrast to [12], the discontinuous martingale plays a key role in the optimal control policy. The resulting optimal control policy is shown to add a novel coupling to the known set of coupled Riccati differential equations for MJLS control, e.g. [4].

The research is organized as follows. Section 2 introduces a transformation of (1)-(3) to stochastic control of a linear system that is driven by a discontinuous martingale. Section 3 derives the optimal control law for this transformed system given full observations of the output of this linear system only, though not about $\{\theta_t\}$.

Section 4 develops the extension of the separation principle for the transformed system from Section 2. Section 5 applies the extended separation principle to the specific cases where the optimal estimator is the Kalman filter and the Wonham filter, respectively. For the Kalman filter case, equivalence with the policy of Fragoso and Costa [5] is shown. For the Wonham filter case, it will be shown that there is a significant difference with the certainty equivalent control policy [7,8]. Section 6 elaborates the extended separation principle to the general case of linear Gaussian observations of $\{x_t\}$only. Section 7 draws conclusions.

2. Transformation to stochastic control of a linear process

Throughout the paper all processes are defined on a complete probability space (Ω, F, P), with (Ω, F) a measurable space, and with P a probability measure defined on the σ -algebra F. The σ -algebra F is equipped with a filtration $\{F_t\}_{t \in [0,\infty]}$, of increasing right-continuous sub- σ -algebra's F_t of F, with $F_\infty = F$.

Hence, in (1), the Brownian motion$\{w_t\}$is an R^m-valued martingale relative to the filtration $\{F_t\}$. Moreover, SDE (1) is assumed to have a pathwise unique $\{F_t\}$-adapted solution that evolves on Euclidean space R^n, $x_t : \Omega \to R^n$, $u_t : \Omega \to R^\ell$; and for each $\theta \in \Theta$,A_θ is an $n \times n$ matrix, B_θ is an $n \times \ell$ matrix, and C_θ is an $n \times m$ matrix.

Following [14] the Markov chain $\{\theta_t\}$ can be written as the solution of an SDE driven by $N(N-1)$ independent Poisson point processes $dp_{\theta\eta,t}$, $\eta \neq \theta$, with rates $\lambda_{\theta\eta}$:

$$d\theta_t = \sum_{\substack{\theta,\eta \in \Theta \\ \eta \neq \theta}} 1\{\theta_{t-} = \theta\}(\eta - \theta) dp_{\theta\eta,t} \tag{4}$$

where $t- = \lim_{\Delta \downarrow 0}(t - \Delta)$, and the compensated processes $\left\{dp_{\theta\eta,t} - \lambda_{\theta\eta} dt\right\}$ are $\{F_t\}$-martingales.

To ensure that the SDE pair (1,4) has a pathwise unique and square-integrable solution, we assume the following:

i. The initial condition x_0 is square integrable.
ii. All elements of $A_\theta, B_\theta, C_\theta$ are finite valued.

iii. The transition rates $\lambda_{\theta\eta}$are finite, i.e. $\lambda_{\theta\eta} < \infty$, for all $\eta \neq \theta$.
iv. The control $\{u_t\}$ is in the set of F_t-adapted, pathwise unique, square integrable processes.

Following a well-known transformation from nonlinear filtering, e.g. Bjork [15], we define for each $\theta \in \Theta$, the indicator process χ_t^θ satisfies:

$$\chi_t^\theta = \begin{cases} 1, & \text{if } \theta_t = \theta \\ 0, & \text{if } \theta_t \neq \theta \end{cases}$$

Subsequently, we also define the process $\xi_t^\theta = \chi_t^\theta x_t$. Hence: $x_t = \sum_{\theta\in\Theta} \xi_t^\theta$, and application of the differentiation rule for discontinuous semi-martingales, (see Appendix A) yields:

$$d\chi_t^\theta = \sum_{\eta\in\Theta} [\chi_{t-}^\eta dp_{\eta\theta,t}] \tag{5}$$

$$d\xi_t^\theta = A_\theta \xi_t^\theta dt + B_\theta \chi_t^\theta u_t dt + C_\theta \chi_{t-}^\theta dw_t + \sum_{\eta\in\Theta} [\xi_{t-}^\eta dp_{\eta\theta,t}] \tag{6}$$

with $dp_{\theta\theta,t} \triangleq - \sum_{\eta\neq\theta} dp_{\theta\eta,t}$. Martingale decomposition of (6) yields:

$$d\xi_t^\theta = A_\theta \xi_t^\theta dt + B_\theta \chi_t^\theta u_t dt + \sum_{\eta\in\Theta} [\lambda_{\eta\theta} \xi_{t-}^\eta] dt + dm_t^\theta \tag{7}$$

with $\{m_t^\theta\}$ satisfying:

$$dm_t^\theta = C_\theta \chi_{t-}^\theta dw_t + \sum_{\eta\in\Theta} \left[\xi_{t-}^\eta dp_{\theta\eta,t}^m\right] \tag{8}$$

where $dp_{\theta\eta,t}^m \triangleq dp_{\theta\eta,t} - \lambda_{\theta\eta} dt,\ \eta \neq \theta$. Hence $\{m_t^\theta\}$ is an $\{F_t\}$-martingale since:

$$E\{dm_t^\theta | F_{t-}\} = E\{C_\theta \chi_{t-}^\theta dw_t | F_{t-}\} + E\left\{ \sum_{\eta\in\Theta} \left[\xi_{t-}^\eta dp_{\theta\eta,t}^m\right] \middle| F_{t-} \right\} =$$

$$= C_\theta \chi_{t-}^\theta E\{dw_t | F_{t-}\} + \sum_{\eta\in\Theta} \left[\xi_{t-}^\eta E\left\{dp_{\theta\eta,t}^m \middle| F_{t-}\right\}\right] = C_\theta \chi_{t-}^\theta \cdot 0 + \sum_{\eta\in\Theta} [\xi_{t-}^\eta \cdot 0] = 0$$

where use has been made of: $E\left\{dp_{\theta\eta,t}^m \middle| F_{t-}\right\} = E\left\{dp_{\theta\eta,t} - \lambda_{\theta\eta} dt \middle| F_{t-}\right\} = [\lambda_{\theta\eta} dt - \lambda_{\theta\eta} dt] = 0$

To apply the above transformation to eq. (3), the following equalities are of use:

$$x_t' S_{\theta_t} x_t = \sum_{\eta\in\Theta} \chi_t^\eta x_t' S_\eta x_t = \sum_{\eta\in\Theta} \chi_t^\eta \chi_t^\eta x_t' S_\eta x_t = \sum_{\eta\in\Theta} \xi_t^{\eta'} S_\eta \xi_t^\eta$$

$$x_t' Q_{\theta_t} x_t = \sum_{\eta\in\Theta} \chi_t^\eta x_t' Q_\eta x_t = \sum_{\eta\in\Theta} \chi_t^\eta \chi_t^\eta x_t' Q_\eta x_t = \sum_{\eta\in\Theta} \xi_t^{\eta'} Q_\eta \xi_t^\eta$$

By substituting these in (3), the optimization criterion becomes:

$$J(\{u_t\}) = E\left\{ \sum_{\theta\in\Theta} \xi_T^{\theta'} S_\theta \xi_T^\theta + \int_0^T \sum_{\theta\in\Theta} \xi_t^{\theta'} Q_\theta \xi_t^\theta dt + \int_0^T u_t' R_= u_t dt \right\} \tag{9}$$

The final transformation step is to collect all θ_t -dependent process components in vectors by defining: $\xi_t = Col\{\xi_t^{e_1}, .., \xi_t^{e_N}\}$, $\chi_t = Col\{\chi_t^{e_1}, .., \chi_t^{e_N}\}$, $(\chi_t u_t) = Col\{\chi_t^{e_1} u_t, .., \chi_t^{e_N} u_t\}$, $m_t = Col\{m_t^{e_1}, .., m_t^{e_N}\}$, $S = Diag\{S_{e_1}, .., S_{e_N}\}$, $Q = Diag\{Q_{e_1}, .., Q_{e_N}\}$, $A = Diag\{A_{e_1}, .., A_{e_N}\}$, $B = Diag\{B_{e_1}, .., B_{e_N}\}$, $C = Diag\{C_{e_1}, .., C_{e_N}\}$,

$$dp_t = \begin{bmatrix} dp_{11,t} I_n & dp_{12,t} I_n & \cdot & \cdot & dp_{1N,t} I_n \\ dp_{21,t} I_n & dp_{22,t} I_n & \cdot & \cdot & dp_{2N,t} I_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ dp_{N1,t} I_n & dp_{N2,t} I_n & \cdot & \cdot & dp_{NN,t} I_n \end{bmatrix}, \text{and } \Lambda = \begin{bmatrix} \lambda_{11} I_n & \lambda_{12} I_n & \cdot & \cdot & \lambda_{1N} I_n \\ \lambda_{21} I_n & \lambda_{22} I_n & \cdot & \cdot & \lambda_{2N} I_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_{N1} I_n & \lambda_{N2} I_n & \cdot & \cdot & \lambda_{NN} I_n \end{bmatrix}$$

with I_n the nxn identity matrix.

System (7)-(9) can now be written as:

$$d\xi_t = [A + \Lambda'] \xi_t dt + B(\chi_t u_t) dt + dm_t \tag{10}$$

$$dm_t = C\chi_{t-}dw_t + \left[\xi'_{t-}dp^m_t\right]' \tag{11}$$

$$J(\{u_t\}) = E\left\{\xi'_T S\xi_T + \int_0^T \xi'_t Q\xi_t dt + \int_0^T u'_t R_= u_t dt\right\} \tag{12}$$

with $d\chi'_t = \chi'_{t-}dp_t$ and $dp^m_t = dp_t - \Lambda dt$.

Hence $\{m_t\}$ is an $\{F_t\}$-martingale.

Remark 2.1. The above transformation has been introduced by [9]. The resulting control problem (10-12) has the following characteristics:

- $\{\xi_t\}$ is the solution of an SDE that is linear in ξ_t, $(\chi_t u_t)$ and dm_t;
- Initial condition is: $\xi_0 = Col\{\chi_0^{e_1}x_0, ..., \chi_0^{e_N}x_0\}$, with $\chi_0^\theta = 1\{\theta_0 = \theta\}$;
- dm_t is a sum of dw_tmultiplied by χ_{t-}, and dp^m_t multiplied by ξ_{t-};
- Optimization criterion $J(\{u_t\})$ is quadratic in ξ_t and in u_t.

The challenge is to develop a CE route in solving control problem (10-12) under partial observations of ξ_t. As has been well explained by [12], the established CE route is to first characterize the optimal control policy given full observations of ξ_t, and subsequently to prove optimality of replacing ξ_t by the partial observation based estimator $\widehat{\xi}_t$. For system (10-12) there are two complications in following this established route. One complication is that under the nominal initial conditions $\xi_t = \chi_t x_t$ is sparse, i.e. $\xi^\theta_t = 0$ for all $\theta \neq \theta_t$, while $\widehat{\xi}_t$is not sparse. The other complication is that the established CE proof is to show that $E\{(\xi_t - \widehat{\xi}_t)(\xi_t - \widehat{\xi}_t)'\}$ is not influenced by the control, which does not hold true in our case. Section 3 addresses the first complication by considering non-sparse solutions of (10) by assuming a non-sparse initial condition for ξ_0. Section 4 addresses the second complication by proving CE under the condition that $E\{(\xi^\theta_t - \widehat{\xi}^\theta_t)(\xi^\theta_t - \widehat{\xi}^\theta_t)'\}$, $\theta \in \Theta$, is not influenced by the control. Sections 5 and 6 show that this relaxed CE condition of this novel control law hold true for specific and general MJLS cases respectively.

3. Control of system (10-12) given full observations of a non-sparse ξ_t

In this section, $\{F_t\}$-measurable partial observations are made of $\{\chi_t\}$ and full observations are made of the solution $\{\xi_t\}$ of the transformed system (10-12). At moment t, the sigma-algebra of continuous-time observations is $Y_t \supseteq \sigma\{\xi_s; s \in [0,t]\}$. Since Y_t is right-continuous and increasing with time, it defines a filtration $\{Y_t\} \subset \{F_t\}$. Rather than assuming for (10) the nominal initial condition, $\xi_0 = Col\{\xi_0^{e_1}, ..., \xi_0^{e_N}\}$, with $\chi_0^\theta = 1\{\theta_0 = 1\}$, we assume that ξ_0 is non-sparse and independent of χ_0. The consequence of this non-sparse initial condition is that ξ_t, $t > 0$, will also be non-sparse. Firstly Theorem 3.1 characterizes $J(\{u_t\})$ in terms of $(\chi_t u_t - K_t\xi_t)$, where K_t is the applicable gain matrix. Subsequently Theorem 3.2 characterizes the optimal control u^*_t. Since the non-sparse initial condition will not hold true for the original system, the resulting control law is referred to as the Non-sparse control law.

Theorem 3.1. *Let assumptions* i-iii *be satisfied for system* (10-12). *Let ξ_0 be independent of χ_0, and let $\xi_0^\theta \neq 0$, for all $\theta \in \Theta$. Let Y_t cover partial χ_t observations and full ξ_t observations, i.e. $Y_t \supseteq \sigma\{\xi_s; s \le t\}$, and let a nonlinear filter produce a pathwise unique $\widehat{\chi}_t = E\{\chi_t|Y_t\}$.*

Then $J(\{u_t\})$ satisfies:

$$J(\{u_t\}) = E\{\xi'_0 P(0)\xi_0\} + E\left\{\int_0^T [(\chi_t u_t) - K(t)\xi_t]' R[(\chi_t u_t) - K(t)\xi_t]dt\right\} + \frac{1}{2}E\{I_T^{\partial\partial}\} \tag{13}$$

with $I_T^{\partial\partial} = \int_0^T \sum_\theta \left[\chi_t^\theta C'_\theta P_{\theta\theta}(t)C_\theta\right]dt$ *and with gain matrix:*

$$K(t) = -R^{-1}B'P(t), \tag{14}$$

where $R = Diag\{R_=, .., R_=\}$, and $P(t)$, $t < T$, is the backward solution of the equation:

$$\dot{P}(t) = -[A' + \Lambda]P(t) - P(t)[A + \Lambda'] + P(t)BR^{-1}B'P(t) - Q - \sum_{\substack{\theta,\eta\in\Theta \\ \eta\neq\theta}} \left[\lambda_{\theta\eta}\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)'\right], \tag{15}$$

with $P(T) = S$, and $\Upsilon^{\theta\eta}$ is an $N\ell \times N\ell$ matrix, of which the N^2 submatrices $\Upsilon^{\theta\eta}_{ij}$ satisfy:

$$\Upsilon^{\theta\eta}_{ij} \triangleq \begin{cases} I_{\ell\times\ell}, & \text{if } i = \theta, j = \eta \\ \varnothing_{\ell\times\ell}, & \text{else} \end{cases}. \tag{16}$$

Theorem 3.2. (Non-sparse control law)

Under the assumptions of Theorem 3.1, *the control law to minimize* $J(\{u_t\})$ *satisfies:*

$$u_t^* = \left[\widehat{\chi}_t^{e_1} I_\ell, ..., \widehat{\chi}_t^{e_N} I_\ell\right] K(t)\xi_t \tag{17}$$

where I_ℓ *is the identity matrix of size* $\ell \times \ell$*, and the gain matrix* $K(t)$ *satisfies* (14-16).

Proof of Theorem 3.1: Application of the differentiation rule for discontinuous semimartingales (see Appendix A1) to $f(z_t) = f(\xi_t, P(t)) = \xi_t^{'} P(t)\xi_t$, yields: $\xi_T^{'} P(T)\xi_T = \xi_0^{'} P(0)\xi_0 + I_T^\partial + \frac{1}{2} I_T^{\partial\partial} + \Sigma_T$, (*) with I_T^∂ the contribution from the first integral in (A.1), $I_T^{\partial\partial}$ the contribution from the second integral, and Σ_T the contribution of the summation. The first integral in (A.1) yields:

$$I_T^\partial = \int_0^T \xi_{t-}^{'} dP(t)\xi_{t-} + \int_0^T \xi_{t-}^{'} P(t-) d\xi_t + \int_0^T \left[d\xi_t^{'} P(t-)\xi_{t-}\right].$$

Since (16) implies $P(t-) = P(t)$, we get:

$I_T^\partial = \int_0^T \xi_{t-}^{'} dP(t)\xi_{t-} + \int_0^T \xi_{t-}^{'} P(t) d\xi_t + \int_0^T \left[d\xi_t^{'} P(t)\xi_{t-}\right]$. Subsequent evaluation yields:

$$I_T^\partial = \int_0^T \xi_{t-}^{'} dP(t)\xi_{t-} + \int_0^T \xi_{t-}^{'} P(t)(A+\Lambda^{'})\xi_{t-} dt + \int_0^T \xi_{t-}^{'} P(t)(B(\chi_{t-}u_{t-})dt + dm_t) +$$

$$+ \int_0^T \xi_{t-}^{'} (A^{'}+\Lambda)P(t)\xi_{t-} dt + \int_0^T [(B(\chi_{t-}u_{t-})dt + dm_t)^{'} P(t)\xi_{t-}]$$

Substitution of eq. (15) for $\dot{P}(t)$ yields:

$$I_T^\partial = \int_0^T \xi_{t-}^{'}\left[-[A^{'}+\Lambda]P(t) - P(t)[A+\Lambda^{'}] + P(t)BR^{-1}B^{'}P(t) - Q - \mathrm{Z}(t)\right]\xi_{t-} dt + \int_0^T \xi_{t-}^{'} P(t)(A+\Lambda^{'})\xi_{t-} dt +$$

$$+ \int_0^T \xi_{t-}^{'} (A^{'}+\Lambda)P(t)\xi_{t-} dt + \int_0^T \xi_{t-}^{'} P(t)(B(\chi_{t-}u_{t-})dt + dm_t) + \int_0^T [(B(\chi_{t-}u_{t-})dt + dm_t)^{'} P(t)\xi_{t-}]$$

with: $\mathrm{Z}(t) \triangleq \sum_{\substack{\theta,\eta\in\Theta \\ \eta\neq\theta}} \left[\lambda_{\theta\eta}\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)^{'}\right]$. Evaluation yields:

$$I_T^\partial = \int_0^T \xi_{t-}^{'}\left[P(t)BR^{-1}B^{'}P(t) - Q - \mathrm{Z}(t)\right]\xi_{t-} dt + \int_0^T \xi_{t-}^{'} P(t)(B(\chi_{t-}u_{t-})dt + dm_t) + \int_0^T [(B(\chi_{t-}u_{t-})dt + dm_t)^{'} P(t)\xi_{t-}]$$

Substitution of eq. (14), and subsequent evaluation yields:

$$I_T^\partial = \int_0^T \xi_{t-}^{'}[K(t)^{'} RK(t)]\xi_{t-} dt - \int_0^T \xi_{t-}^{'} K(t)^{'} R(\chi_{t-}u_{t-}) dt - \int_0^T (\chi_{t-}u_{t-})^{'} RK(t)\xi_{t-} dt +$$

$$- \int_0^T \xi_{t-}^{'}[Q + \mathrm{Z}(t)]\xi_{t-} dt + \int_0^T \xi_{t-}^{'} P(t) dm_t + \int_0^T \left[dm_t^{'} P(t)\xi_{t-}\right]$$

By completing the squares for the terms in the first three integrals, this becomes:

$$I_T^\partial = \int_0^T \left[(\chi_{t-}u_{t-})^{'} - \xi_{t-}^{'} K(t)^{'}\right] R[(\chi_{t-}u_{t-}) - K(t)\xi_{t-}] dt - \int_0^T (\chi_{t-}u_{t-})^{'} R\chi_{t-}u_{t-} dt +$$

$$- \int_0^T \xi_{t-}^{'}[Q + \mathrm{Z}(t)]\xi_{t-} dt + \int_0^T \xi_{t-}^{'} P(t) dm_t + \int_0^T \left[dm_t^{'} P(t)\xi_{t-}\right]$$

Substituting this in eq. (*) for $\xi_T^{'} P(T)\xi_T$, and using $(\chi_{t-}u_{t-})^{'} R(\chi_{t-}u_{t-}) = u_{t-}^{'} R_{=} u_{t-}$, yields:

$$\xi_T' P(T)\xi_T = \xi_0' P(0)\xi_0 + \int_0^T \left[(\chi_{t-}u_{t-})' - \xi_{t-}' K(t)'\right] R[(\chi_{t-}u_{t-}) - K(t)\xi_{t-}]dt - \int_0^T u_{t-}' R_{=} u_{t-} dt +$$

$$-\int_0^T \xi_{t-}' Q\xi_{t-} dt - \int_0^T \xi_{t-}' Z(t)\xi_{t-} dt + \int_0^T \xi_{t-}' P(t)dm_t + \int_0^T \left[dm_t' P(t)\xi_{t-}\right] + \frac{1}{2} I_T^{\partial\partial} + \Sigma_T$$

For the integral involving $Z(t)$, by using $Z(t) = Z(t-)$ we get:

$$\int_0^T \xi_{t-}' Z(t)\xi_{t-} dt = \int_0^T \xi_{t-}' Z(t-)\xi_{t-} dt = \int_{0-}^{T-} \xi_t' Z(t)\xi_t dt = \int_0^T \xi_t' Z(t)\xi_t dt$$

Similar rewriting applies to the integral terms involving $K(t)$, R and Q. Hence:

$$\xi_T' P(T)\xi_T = \xi_0' P(0)\xi_0 + \int_0^T \left[(\chi_t u_t)' - \xi_t' K(t)'\right] R[(\chi_t u_t) - K(t)\xi_t]dt - \int_0^T u_t' R_{=} u_t dt +$$

$$-\int_0^T \xi_t' Q\xi_t dt - \int_0^T \xi_t' Z(t)\xi_t dt + \int_0^T \xi_{t-}' P(t)dm_t + \int_0^T \left[dm_t' P(t)\xi_{t-}\right] + \frac{1}{2} I_T^{\partial\partial} + \Sigma_T$$

Taking expectation, and rearranging terms, yields:

$$E\left\{\xi_T' P(T)\xi_T\right\} + E\left\{\int_0^T u_t' R_{=} u_t dt\right\} + E\left\{\int_0^T \xi_t' Q\xi_t dt\right\} =$$

$$= E\left\{\xi_0' P(0)\xi_0\right\} + E\left\{\int_0^T \left[(\chi_t u_t)' - \xi_t' K(t)'\right] R[(\chi_t u_t) - K(t)\xi_t]dt\right\} +$$

$$-E\left\{\int_0^T \xi_t' Z(t)\xi_t dt\right\} + E\left\{\int_0^T \xi_{t-}' P(t)dm_t\right\} + E\left\{\int_0^T \left[dm_t' P(t)\xi_{t-}\right]\right\} + \frac{1}{2} E\left\{I_T^{\partial\partial}\right\} + E\{\Sigma_T\}$$

Evaluation of the term involving the $\{F_t\}$-martingale $\{m_t\}$, yields:

$$E\left\{\int_0^T \xi_{t-}' P(t)dm_t\right\} = E\left\{\int_0^T E\{\xi_{t-}' P(t)dm_t | F_{t-}\right\}\right\} = E\left\{\int_0^T \xi_{t-}' P(t)E\{dm_t|F_{t-}\}\right\} = E\left\{\int_0^T \xi_{t-}' P(t)\cdot 0 \cdot dt\right\} = 0$$

Hence:

$$E\left\{\xi_T' P(T)\xi_T\right\} + E\left\{\int_0^T u_t' R_{=} u_t dt\right\} + E\left\{\int_0^T \xi_t' Q\xi_t dt\right\} =$$

$$= E\left\{\xi_0' P(0)\xi_0\right\} + E\left\{\int_0^T \left[(\chi_t u_t)' - \xi_t' K(t)'\right] R[(\chi_t u_t) - K(t)\xi_t]dt\right\} - E\left\{\int_0^T \xi_t' Z(t)\xi_t dt\right\} + \frac{1}{2} E\left\{I_T^{\partial\partial}\right\} + E\{\Sigma_T\}$$

Using $P(T) = S$, and subsequent substitution in (12) yields: $J(\{u_t\}) =$

$$= E\left\{\xi_0' P(0)\xi_0\right\} + E\left\{\int_0^T \left[(\chi_t u_t)' - \xi_t' K(t)'\right] R[(\chi_t u_t) - K(t)\xi_t]dt\right\} - E\left\{\int_0^T \xi_t' Z(T)\xi_t dt\right\} + \frac{1}{2} E\left\{I_T^{\partial\partial}\right\} + E\{\Sigma_t\}.$$

Evaluation of $I_T^{\partial\partial}$, using (A1), yields:

$$I_T^{\partial\partial} = \int_0^T \sum_{\theta,\eta} \sum_{i,j=1}^n \left[P_{\theta\eta,ij}(t) d < C_{\theta,i}\chi_t^\theta w_t, C_{\eta,j}\chi_t^\eta w_t >_t\right] = \int_0^T \sum_{\theta} \sum_{i,j=1}^n \left[P_{\theta\theta,ij}(t) d < C_{\theta,i}\chi_t^\theta w_t, C_{\theta,j}\chi_t^\theta w_t >_t\right] =$$

$$= \int_0^T \sum_{\theta} \sum_{i,j=1}^n \left[P_{\theta\theta,ij}(t) C_{\theta,i}\chi_t^\theta C_{\theta,j} dt\right] = \int_0^T \sum_{\theta} \left[\chi_t^\theta C_\theta' P_{\theta\theta}(t) C_\theta\right] dt.$$

To evaluate $E\{\Sigma_t\}$, we start with the summation term in (A.1) for $f(\xi_t, P(t)) = \xi_t' P(t)\xi_t$:

$\Sigma_T = \sum_{0<t\le T}\left[\xi_t' P(t)\xi_t - \xi_{t-}' P(t-)\xi_{t-} - \xi_{t-}'[P(t) - P(t-)]\xi_{t-} - \xi_{t-}' P(t-)\Delta_t^\xi - \Delta_t^{\xi'} P(t-)\xi_{t-}\right]$, with $\Delta_t^\xi = \xi_t - \xi_{t-}$. Using $P(t) = P(t-)$, the squares are completed as follows:

$$\Sigma_T = \sum_{0<t\le T}\left[\xi_t' P(t)\xi_t - \xi_{t-}' P(t)\xi_{t-} - \xi_{t-}' P(t)\Delta_t^\xi - \Delta_t^{\xi'} P(t)\xi_{t-}\right]$$

$$= \sum_{0<t\le T}\left[\xi_t' P(t)\xi_t - \xi_{t-}' P(t)\xi_t - \Delta_t^{\xi'} P(t)\xi_{t-}\right] =$$

$$= \sum_{0<t\le T}\left[\Delta_t^{\xi'} P(t)\xi_t - \Delta_t^{\xi'} P(t)\xi_{t-}\right] = \sum_{0<t\le T}\left[\Delta_t^{\xi'} P(t)\Delta_t^\xi\right]$$

Hence $E\{\Sigma_T\} = E\left\{\sum_{0<t\le T}[(\xi_t - \xi_{t-})' P(t)(\xi_t - \xi_{t-})]\right\}$. (**)

From (10) and (11) we know:

$$d\xi_t = [A + \Lambda']\xi_t dt + B(\chi_t u_t)dt + C\chi_{t-}dw_t + \left[\xi_t' dp_t^m\right]' = A\xi_t dt + B(\chi_t u_t)dt + C\chi_{t-}dw_t + \left[\xi_{t-}' dp_t\right]'$$

Since all terms in this SDE are square-integrable, the last term only may generate discontinuities in $\{\xi_t\}$. Hence: $(\xi_t - \xi_{t-})' = \left[\xi_{t-}' dp_t\right]' = \xi_{t-}'(p_t - p_{t-})$.

Substituting this in eq. (**) for $E\{\Sigma_T\}$yields:

$$E\{\Sigma_T\} = E\left\{\sum_{0<t\le T}\left[\xi_{t-}'(p_t - p_{t-})P(t)(p_t - p_{t-})'\xi_{t-})\right]\right\} =$$

$$= E\left\{\sum_{0<t\le T}\left[\xi_{t-}'\sum_{\substack{\theta,\eta\in\Theta\\ \eta\neq\theta}}\left[1\{p_{\theta\eta,t}\neq p_{\theta\eta,t-}\}(p_t - p_{t-})P(t)(p_t - p_{t-})'\right]\xi_{t-}\right]\right\}$$

By the definition of $\Upsilon^{\theta\eta}$ in (16) we have:

$$1\{p_{\theta\eta,t}\neq p_{\theta\eta,t-}\}(p_t - p_{t-})P(t)(p_t - p_{t-})' = 1\{p_{\theta\eta,t}\neq p_{\theta\eta,t-}\}\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)'.$$

Substitution in eq. (**) for $E\{\Sigma_t\}$, and subsequent evaluation yields:

$$E\{\Sigma_t\} = E\left\{\sum_{0<t\le T}\left[\xi_{t-}'\sum_{\substack{\theta,\eta\in\Theta\\ \eta\neq\theta}}\left[1\{p_{\theta\eta,t}\neq p_{\theta\eta,t-}\}\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)'\right]\xi_{t-}\right]\right\} =$$

$$= E\left\{\sum_{\substack{\theta,\eta\in\Theta\\ \eta\neq\theta}}\left[\int_0^T \xi_{t-}'\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)'\xi_{t-}dp_{\theta\eta,t}\right]\right\} =$$

$$= E\left\{\sum_{\substack{\theta,\eta\in\Theta\\ \eta\neq\theta}}\left[\int_0^T \xi_t'\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)'\xi_t\lambda_{\theta\eta}dt\right]\right\}$$

Hence: $E\{\Sigma_t\} = E\left\{\int_0^T \xi_t' Z(t)\xi_t dt\right\}$, which simplifies the characterization of $J(\{u_t\})$ to:

$$J(\{u_t\}) = E\{\xi_0' P(0)\xi_0\} + E\left\{\int_0^T [(\chi_t u_t) - K(t)\xi_t]R[(\chi_t u_t) - K(t)\xi_t]dt\right\} + \tfrac{1}{2}E\{I_T^{\partial\partial}\} \text{ Q.E.D.}$$

Proof of Theorem 3.2**:** *Since $E\{I_T^{\partial\partial}\}$ is not influenced by $\{u_t\}$, minimization of $J(\{u_t\})$ is achieved for*

$$u_t^* = \underset{u}{Argmin}\, E\{[(\chi_t u) - K(t)\xi_t]' R[(\chi_t u) - K(t)\xi_t]\} =$$

$$= \underset{u}{Argmin}\, E\{ [(\chi_t u) + R^{-1}B'P(t)\xi_t]' R [(\chi_t u) + R^{-1}B'P(t)\xi_t] \}$$

Using $R = Diag\{R_=, .., R_=\}$, $B = Diag\{B_{e_1}, .., B_{e_N}\}$, $\xi_t = Col\{\xi_t^{e_1}, .., \xi_t^{e_N}\}$, and subsequent evaluation yields: $R^{-1}B'P(t)\xi_t = Col\left\{R_=^{-1}B_{e_1}' \sum_{j=1}^N (P_{1j}(t)\xi_t^{e_j}), .., R_=^{-1}B_{e_N}' \sum_{j=1}^N (P_{Nj}(t)\xi_t^{e_j})\right\}$.

Substituting the latter together with $(\chi_t u_t) = Col\{\chi_t^{e_1}u_t, .., \chi_t^{e_N}u_t\}$ yields:

$$u_t^* = \underset{u}{Argmin} E\left\{ \sum_{i=1}^N \left[\left[\chi_t^{e_i} u + R_=^{-1}B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' R_= \left[\chi_t^{e_i} u + R_=^{-1}B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]\right]\right\} =$$

$$= \underset{u}{Argmin} E\left\{ \sum_{i=1}^N \left[\chi_t^{e_i} u' R_= \chi_t^{e_i} u + \chi_t^{e_i} u' B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \chi_t^{e_i} u + f_i(\chi_t^{e_i}, \xi_t)\right]\right\} =$$

$$= \underset{u}{Argmin} E\left\{ \sum_{i=1}^N \left[\chi_t^{e_i} u' R_= u + \chi_t^{e_i} u' B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \chi_t^{e_i} u + f_i(\chi_t^{e_i}, \xi_t)\right]\right\} =$$

$$= \underset{u}{Argmin} E\left\{ u' R_= u + \sum_{i=1}^N \left[\chi_t^{e_i} u' B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \chi_t^{e_i} u + f_i(\chi_t^{e_i}, \xi_t)\right]\right\} =$$

$$= \underset{u}{Argmin} \left\{ u' R_= u + E\left\{ \sum_{i=1}^N \left[\chi_t^{e_i} u' B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \chi_t^{e_i} u + f_i(\chi_t^{e_i}, \xi_t)\right] | Y_t \right\}\right\} =$$

$$= \underset{u}{Argmin} \left\{ u' R_= u + \sum_{i=1}^N \left[\widehat{\chi}_t^{e_i} u' B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \widehat{\chi}_t^{e_i} u + \widehat{f}_i(\chi_t^{e_i}, \xi_t)\right]\right\}$$

where $f_i(\chi_t^{e_i}, \xi_t)$ collects the u -invariant terms. To characterize the minimum u_t^*, we can assume a zero value for the partial derivative of the u -invariant terms at moment t:

$$\frac{\partial}{\partial u}\left[u' R_= u + \sum_{i=1}^N \left[\widehat{\chi}_t^{e_i} u' B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \widehat{\chi}_t^{e_i} u\right]\right] |u = u_t^* = 0$$

This yields: $u_t^{*\prime} R_= + R_= u_t^* + \sum_{i=1}^N \left[\widehat{\chi}_t^{e_i} B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j}) + \left[B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right]' \widehat{\chi}_t^{e_i}\right] = 0$

Hence: $u_t^* = R_=^{-1} \sum_{i=1}^N \left[\widehat{\chi}_t^{e_i} B_{e_i}' \sum_{j=1}^N (P_{ij}(t)\xi_t^{e_j})\right] = [\widehat{\chi}_t^{e_1} I_{\ell}, ..., \widehat{\chi}_t^{e_N} I_{\ell}] K(t)\xi_t$.. Q.E.D.

Remark 3.3. An important part of the proof of Theorem 3.1 involves the characterization of the novel terms $\lambda_{\theta\eta}(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta})P(t)(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta})'$ in eq. (15) for $P(t)$. These terms cover interaction between $N(N-1)$ off-diagonal matrices in $P(t)$. To verify that these novel terms are due to the multiplication of dp_t with ξ_{t-} in the martingale term of eqs. (10-11), let's see what happens if $\xi_t' - dp_t$ is replaced by $\varsigma_{t-}' dp_t$, with $\{\varsigma_t\}$ a process that is not influenced by $\{u_t\}$.

Following similar steps as in the proof of Theorem 3.1, then eq. (**) would yield:

$$E\{\Sigma_t\} = E\left\{ \sum_{0<t\le T} [(\varsigma_t - \varsigma_{t-})' P(t)(\varsigma_t - \varsigma_{t-})] \right\} =$$

$$= E\left\{ \sum_{0<t\le T} [\varsigma_{t-}'(p_t - p_{t-})P(t)(p_t - p_{t-})'\varsigma_{t-})] \right\} =$$

$$= E\left\{ \sum_{\substack{\theta,\eta\in\Theta \\ \eta\neq\theta}} \left[\int_0^T \varsigma_{t-}'(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta})P(t)(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta})'\varsigma_{t-}\lambda_{\theta\eta} dt \right] \right\}$$

Hence $E\{\Sigma_t\}$ would no longer be influenced by $\{u_t\}$, and the Υ -involving summation in eq. (15) would disappear. Hence the off-diagonal sub-matrices of $P(t)$ would be zero.

Remark 3.4. A relevant question is if the Non-sparse control law of Theorem 3.2 also holds true if ξ_0 is sparse. It can easily be verified that nowhere in the derivation of Theorems 3.1 and 3.2 use is made of non-sparse ξ_0. This means that eqs. (13)-(17) also hold true if ξ_0

is sparse. The only consequence of a sparse ξ_0 is that ξ_t and $\widehat{\chi}_t$ will be sparse, as result of which the off-diagonal matrices of $P(t)$ do not play a role in solving eq. (14).

4. Separation of control and estimation for (10-12) given partial observations

In this section, $\{F_t\}$-measurable partial observations $\{y_t\}$ are made of $\{\chi_t, \xi_t\}$ of the transformed system (10-12). At moment t, the sigma-algebra of continuous-time observations is $Y_t = \sigma\{y_s; s \in [0,t]\}$. Since Y_t is right-continuous and increasing with time, it defines another filtration $\{Y_t\} \subset \{F_t\}$. Throughout this section we assume that there is a nonlinear filter for the estimators $\widehat{\chi}_t = E\{\chi_t | Y_t\}$ and $\widehat{\xi}_t = E\{\xi_t | Y_t\}$, which have pathwise unique and square integrable solutions.

For the setting of (10) and (12), though with an $\{m_t\}$ that is not influenced by $\{\xi_t\}$, [12] has proven that, under certain assumptions, the separation principle between control and nonlinear filtering applies under the following CE condition:

C0. $E\{(\xi_t - \widehat{\xi}_t)(\xi_t - \widehat{\xi}_t)'\}$ is not influenced by $\{u_t\}$.

To cover the more demanding setting of (10)-(12), we will extend the proof of [12] under the following relaxed CE condition C0*, where $\widehat{\xi}_t^\theta = E\{\xi_t^\theta | Y_t\}$:

C0*. For each $\theta \in \Theta$, $E\left\{\left(\xi_t^\theta - \widehat{\xi}_t^\theta\right)\left(\xi_t^\theta - \widehat{\xi}_t^\theta\right)'\right\}$ is not influenced by $\{u_t\}$.

First Theorem 4.1 characterizes $J(\{u_t\})$ in terms of $(\chi_t u_t - K_t \widehat{\xi}_t)$. Subsequently Theorem 4.2 characterizes the optimal control u_t^*.

Theorem 4.1. *Let assumptions* i-iii *be satisfied for system* (10-12). *Let Y_t cover partial observations of (χ_t, ξ_t), and a nonlinear filter produces pathwise unique $\widehat{\chi}_t = E\{\chi_t | Y_t\}$ and $\widehat{\xi}_t = E\{\xi_t | Y_t\}$.*

Then $J(\{u_t\})$ satisfies:

$$J(\{u_t\}) = E\{\xi_0' P(0)\xi_0\} + E\left\{\int_0^T [(\chi_t u_t) - K(t)\widehat{\xi}_t]' R[(\chi_t u_t) - K(t)\widehat{\xi}_t] dt\right\} + \frac{1}{2}E\{I_T^{\partial\partial}\} + tr\{K(t)RK(t)'Q_t\}$$

with the $K(t)$ of eqs. (14-16), *$Q_t \triangleq E\{(\xi_t - \widehat{\xi}_t)(\xi_t - \widehat{\xi}_t)'\}$, and where neither $I_T^{\partial\partial}$ nor Q_t are influenced by u_t.*

Theorem 4.2. *Given the assumptions of* Theorem 4.1. *If condition* C0* *holds true, then the optimal control law to minimize $J(\{u_t\})$, satisfies:*

$$u_t^* = [\widehat{\chi}_t^{e_1} I_{\ell}, ..., \widehat{\chi}_t^{e_N} I_{\ell}] K(t)\widehat{\xi}_t. \quad (18)$$

Proof of Theorem 4.1: By defining $\widetilde{\xi}_t = \xi_t - \widehat{\xi}_t$ we have $E\{[(\chi_t u_t) - K(t)\widehat{\xi}_t]\widetilde{\xi}_t'\} = 0$. Hence:

$$E\left\{\int_0^T [(\chi_t u_t) - K(t)\xi_t]' R[(\chi_t u_t) - K(t)\xi_t] dt\right\} =$$

$$= E\left\{\int_0^T [(\chi_t u_t) - K(t)\widehat{\xi}_t]' R[(\chi_t u_t) - K(t)\widehat{\xi}] dt\right\} + tr\{K(t)RK(t)'Q_t\}.$$

Substituting this in eq. (13) yields:

$$J(\{u_t\}) = E\{\xi_0' P(0)\xi_0\} + E\left\{\int_0^T [(\chi_t u_t) - K(t)\widehat{\xi}_t]' R[(\chi_t u_t) - K(t)\widehat{\xi}_t] dt\right\} + \frac{1}{2}E\{I_T^{\partial\partial}\} + tr\{K(t)RK(t)'Q_t\}$$

The proof of Theorem 3.1 showed that $u_t$does not influence$E\{I_T^{\partial\partial}\}$. Evaluation of Q_t yields:

$$\Xi_t = E\{(\xi_t - \widehat{\xi}_t)(\xi_t - \widehat{\xi}_t)'\} = \sum_\theta P\{\theta_t = \theta\} E\{(\xi_t - \widehat{\xi}_t)(\xi_t - \widehat{\xi}_t)' | \theta_t = \theta\} =$$

$$= \sum_\theta P\{\theta_t = \theta\} E\{\chi_t^\theta (\xi_t - \widehat{\xi}_t)(\xi_t - \widehat{\xi}_t)' | \theta_t = \theta\} =$$

$$= \sum_\theta P\{\theta_t = \theta\} E\left\{Diag\left\{0,..,0, \chi_t^\theta \left(\xi_t^\theta - \widehat{\xi}_t^\theta\right)\left(\xi_t^\theta - \widehat{\xi}_t^\theta\right)', 0,..,0\right\} | \theta_t = \theta\right\} =$$

$$= E\left\{Diag\left\{\chi_t^{e_1}\left(\xi_t^{e_1} - \widehat{\xi}_t^{e_1}\right)\left(\xi_t^{e_1} - \widehat{\xi}_t^{e_1}\right)', ..., \chi_t^{e_N}\left(\xi_t^{e_N} - \widehat{\xi}_t^{e_N}\right)\left(\xi_t^{e_N} - \widehat{\xi}_t^{e_N}\right)'\right\}\right\}$$

This means that condition C0* assures that Q_t is neither influenced by u_t. Q.E.D.

Proof of Theorem 4.2: Since u_t only influences the integral term in the $J(\{u_t\})$ characterization of Theorem 4.1:

$$u_t^* = \underset{u}{Argmin}\{E\{[(\chi_t u) - K(t)\widehat{\xi}_t]' R[(\chi_t u) - K(t)\widehat{\xi}_t]\}\}.$$

To resolve this optimization, we follow the steps in the proof of Theorem 3.2.

Substitution of eq. (14) yields:

$$u_t^* = \underset{u}{Argmin}\{E\{[(\chi_t u) + R^{-1}B'P(t)\hat{\xi}_t]' R[(\chi_t u) + R^{-1}B'P(t)\hat{\xi}_t]\}\}$$

Using $R = Diag\{R_=, .., R_=\}$, $B = Diag\{B_{e_1}, .., B_{e_N}\}$, $\xi_t = Col\{\xi_t^{e_1}, .., \xi_t^{e_N}\}$, and subsequent evaluation yields: $R^{-1}B'P(t)\hat{\xi}_t = Col\Big\{R_=^{-1}B'_{e_1}\sum_\theta[P_{1\theta}(t)\hat{\xi}_t^\theta], .., R_=^{-1}B'_{e_N}\sum_\theta[P_{N\theta}(t)\hat{\xi}_t^\theta]\Big\}$.

Substituting the latter together with $(\chi_t u_t) = Col\{\chi_t^{e_1}u_t, .., \chi_t^{e_N}u_t\}$ yields:

$$u_t^* = \underset{u}{Argmin}E\Big\{\textstyle\sum_{i=1}^N [[\chi_t^{e_i}u_t + R_=^{-1}B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]' R_= [\chi_t^{e_i}u_t + R_=^{-1}B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]]\Big\} =$$

$$= \underset{u}{Argmin}E\Big\{\textstyle\sum_{i=1}^N [\chi_t^{e_i}u_t' R_= \chi_t^{e_i}u_t + \chi_t^{e_i}u_t' B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta) + [B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]' \chi_t^{e_i}u_t + f_i(\chi_t^{e_i}, \hat{\xi})]\Big\} =$$

$$= \underset{u}{Argmin}E\Big\{u_t' R_= u_t + \textstyle\sum_{i=1}^N [\chi_t^{e_i}u_t' B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta) + [B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]' \chi_t^{e_i}u_t + f_i(\chi_t^{e_i}, \hat{\xi})]\Big\} =$$

$$= \underset{u}{Argmin}\ E\Big\{u_t' R_= u_t + E\Big\{\textstyle\sum_{i=1}^N [\chi_t^{e_i}u_t' B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta) + [B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]' \chi_t^{e_i}u_t + f_i(\chi_t^{e_i}, \hat{\xi})] | Y_t\Big\}\Big\} =$$

$$= \underset{u}{Argmin}\, E\Big\{u_t' R u_t + \textstyle\sum_{i=1}^N [\hat{\chi}_t^{e_i}u_t' B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta) + [B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]' \hat{\chi}_t^{e_i}u_t + \hat{f}_i(\chi_t^{e_i}, \hat{\xi})]\Big\}$$

To characterize the minimum u_t^*, we can assume zero value for the partial derivative of the u -variant terms at moment t:

$$\frac{\partial}{\partial u}\left[u' R_= u + \sum_{i=1}^N \left[\hat{\chi}_t^{e_i}u' B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta) + \left[B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)\right]' \hat{\chi}_t^{e_i}u\right]\right] |u = u_t^* = 0$$

This yields: $R_= u_t^* = \sum_{i=1}^N [\hat{\chi}_t^{e_i}B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)]$

Hence: $u_t^* = \sum_{i=1}^N [\hat{\chi}_t^{e_i}R_=^{-1}B_{e_i}\sum_\theta(P_{e_i\theta}(t)\hat{\xi}_t^\theta)] = [\hat{\chi}_t^{e_1}I_{\ell}, ..., \hat{\chi}_t^{e_N}I_{\ell}]K(t)\hat{\xi}_t$.Q.E.D.

5. MJLS cases of Kalman and Wonham filter based estimation of $\hat{\xi}_t$

This section elaborates the findings of Section 4 for two specific MJLS cases:

- Full observations of $\{\theta_t\}$, and linear Gaussian observations of $\{x_t\}$; and
- Hidden $\{\theta_t\}$, and full observations of $\{x_t\}$.

For these cases, estimation is accomplished by Kalman and Wonham filters respectively. For these two cases, it will be shown that CE condition C0 is satisfied. These two cases and the relation with existing results, are addressed in the next two subsections.

5.1. Full observations of θ_t and linear Gaussian observations of x_t

This subsection addresses the situation $Y_t = \sigma\{\theta_s, y_s; s \in [0, t]\}$, where $\{y_t\}$ satisfies:

$$dy_t = H_{\theta_t}x_t dt + G db_t \tag{19}$$

with $\{db_t\}$ a standard Brownian which is independent of $\{dw_t\}$,$\{dp_t\}$and (θ_0, x_0). For this case, the optimal control solution is well known, e.g. [Costa, Fragoso and Todorov, 2013, Theorem 4.9]. The filtering part consists of estimating $\hat{x}_t$ by a Kalman filter, the coefficients of follow the switching θ_t. The optimal control law is:

$$u_t^* = K_{\theta_t}^M(t)\hat{x}_t \tag{20}$$

where the gain matrices$K_\theta^M(t)$, $\theta \in \Theta$, satisfy:

$$K_\theta^M(t) = -R_=^{-1}B_\theta' P_\theta^M(t) \tag{21}$$

with $P_\theta^M(T) = S_\theta$, and for $t < T$, $P_\theta^M(t)$ is the solution of the coupled set of ODE's:

$$\dot{P}_\theta^M(t) = -A_\theta' P_\theta^M(t) - P_\theta^M(t)A_\theta + P_\theta^M(t)B_\theta R_=^{-1}B_\theta' P_\theta^M(t) - \sum_{\eta\in\Theta}\lambda_{\theta\eta}P_\theta^M(t) - Q_\theta. \tag{22}$$

Next we show that under full observability of $\{\theta_t\}$ the Non-sparse control policy of Theorem 4.1 yields the same u_t^* as control policy (20-22).

Theorem 5.1. *In case of full* $\{\theta_t\}$*and partial* $\{x_t\}$ *observations, i.e.* $Y_t = \sigma\{\theta_s, y_s; s \in [0, t]\}$, *with* $\{y_t\}$ *satisfying* (19), *the control policy of* Theorem 4.2 *yields the same pathwise unique* $\{u_t^*\}$ *as control policy* (20-22).

Proof. Estimation of $\widehat{x}_t$ by a Kalman filter, the coefficients of which follow the switching θ_t, also implies that $E\{(x_t - \widehat{x}_t)(x_t - \widehat{x}_t)'|Y_t\}$ is not influenced by $\{u_t\}$; hence condition C0 is satisfied. Because $\{\theta_t\}$ is observed, $\widehat{\chi}_t^\theta = \chi_t^\theta$ assumes values from {0,1} only, which implies that from the solution $P(t)$ of eq. (15) only the diagonal sub-matrices play a role in the optimal control policy. Hence it remains to be shown that the $P_{\theta\theta}(t)$ solutions of (15) are the same as the $P_\theta^M(t)$ solutions of (22). Evaluation of the last summation in eq. (15) yields: $\sum_{\theta,\eta\in\Theta}\left[\lambda_{\theta\eta}\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)P(t)\left(\Upsilon^{\theta\eta} - \Upsilon^{\theta\theta}\right)'\right] =$

$$= \sum_{\theta,\eta\in\Theta}\lambda_{\theta\eta}\left[\Upsilon^{\theta\eta}P(t)\Upsilon^{\theta\eta'} - \Upsilon^{\theta\eta}P(t)\Upsilon^{\theta\theta'} - \Upsilon^{\theta\theta}P(t)\Upsilon^{\theta\eta'} + \Upsilon^{\theta\theta}P(t)\Upsilon^{\theta\theta'}\right] =$$

$$= \sum_{\theta,\eta\in\Theta}\lambda_{\theta\eta}\left[\Upsilon^{\theta\eta}P(t)\Upsilon^{\theta\eta'} - \Upsilon^{\theta\eta}P(t)\Upsilon^{\theta\theta'} - \Upsilon^{\theta\theta}P(t)\Upsilon^{\theta\eta'} + \Upsilon^{\theta\theta}P(t)\Upsilon^{\theta\theta'}\right]$$

Using this, and eq. (15), we get for the submatrix $P_{\theta\theta}(t)$ on the diagonal of $P(t)$:

$$\dot{P}_{\theta\theta}(t) = -A_\theta{}'P_{\theta\theta}(t) - P_{\theta\theta}(t)A_\theta - 2\sum_\eta \lambda_{\theta\eta}P_{\theta\eta}(t) + P_{\theta\theta}(t)B_\theta R_{=}^{-1}B_\theta{}'P_{\theta\theta}(t) - Q_\theta$$

$$-\sum_{\eta\neq\theta}[\lambda_{\theta\eta}[P_{\eta\eta}(t) - P_{\eta\theta}(t) - P_{\theta\eta}(t) + P_{\theta\theta}(t)]] =$$

$$= -A_\theta{}'P_{\theta\theta}(t) - P_{\theta\theta}(t)A_\theta + P_{\theta\theta}(t)B_\theta R_{=}^{-1}B_\theta{}'P_{\theta\theta}(t) - Q_\theta - 2\sum_{\eta\neq\theta}\lambda_{\theta\eta}P_{\theta\eta}(t) - 2\lambda_{\theta\theta}P_{\theta\theta}(t) +$$

$$-\sum_{\eta\neq\theta}[\lambda_{\theta\eta}[P_{\eta\eta}(t) - P_{\eta\theta}(t) - P_{\theta\eta}(t) + P_{\theta\theta}(t)]]$$

$$= -A_\theta{}'P_{\theta\theta}(t) - P_{\theta\theta}(t)A_\theta + P_{\theta\theta}(t)B_\theta R_{=}^{-1}B_\theta{}'P_{\theta\theta}(t) - Q_\theta - 2\lambda_{\theta\theta}P_{\theta\theta}(t) - \sum_{\eta\neq\theta}[\lambda_{\theta\eta}[P_{\eta\eta}(t) + P_{\theta\theta}(t)]]$$

$$= -A_\theta{}'P_{\theta\theta}(t) - P_{\theta\theta}(t)A_\theta + P_{\theta\theta}(t)B_\theta R_{=}^{-1}B_\theta{}'P_{\theta\theta}(t) - Q_\theta - 2\lambda_{\theta\theta}P_{\theta\theta}(t) - \sum_{\eta\neq\theta}[\lambda_{\theta\eta}P_{\eta\eta}(t)] + \lambda_{\theta\theta}P_{\theta\theta}(t)$$

$$= -A_\theta{}'P_{\theta\theta}(t) - P_{\theta\theta}(t)A_\theta + P_{\theta\theta}(t)B_\theta R_{=}^{-1}B_\theta{}'P_{\theta\theta}(t) - Q_\theta - \lambda_{\theta\theta}P_{\theta\theta}(t) - \sum_{\eta\neq\theta}[\lambda_{\theta\eta}P_{\eta\eta}(t)]$$

$$= -A_\theta{}'P_{\theta\theta}(t) - P_{\theta\theta}(t)A_\theta + P_{\theta\theta}(t)B_\theta R_{=}^{-1}B_\theta{}'P_{\theta\theta}(t) - Q_\theta - \sum_\eta[\lambda_{\theta\eta}P_{\eta\eta}(t)]$$

The latter is equal to eq. (22) for $P_\theta^M(t)$. Q.E.D.

5.2. Hidden θ_t, and full observations of x_t

This subsection addresses the situation $Y_t = \sigma\{x_s; s\in[0,t]\}$, under the restriction C_θ is θ -invariant. For this control problem, the optimal solution is not known in literature. A well known approximation is the Averaging MJLS control policy [Fragoso, 1988]:

$$u_t^A = \sum_{\theta\in\Theta}\left[\widehat{\chi}_t^\theta K_\theta^M(t)\right]x_t, \tag{23}$$

with $K_\theta^M(t)$ the solution of (21-22), and $\widehat{\chi}_t^\theta$ the solution of the Wonham filter.

Proposition 5.2. (Wonham filter)

Let $C_\theta = C$ for all θ. The estimator $\widehat{\chi}_t^\theta = \mathrm{P}\{\theta_t = \theta|Y_t\}$, given $Y_t = \sigma\{x_s; s\in[0,t]\}$, satisfies:

$$d\widehat{\chi}_t^\theta = \sum_{\eta\in\Theta}\left[\lambda_{\eta\theta}\widehat{\chi}_t^\eta\right]dt + \widehat{\chi}_t^\theta\left[A_\theta x_t + B_\theta u_t - \sum_{\eta\in\Theta}\left[\widehat{\chi}_t^\eta(A_\eta x_t + B_\eta u_t)\right]\right]'\left[CC'\right]^{-1}d\nu_t \tag{24}$$

with $d\nu_t = dx_t - \sum_{\eta\in\Theta}\left[\widehat{\chi}_t^\eta(A_\eta x_t + B_\eta u_t)\right]dt$.

Thanks to Theorem 4.1 we are now able to characterize the optimal control policy and compare this to the Averaging MJLS control policy (23).

Theorem 5.3. let *$C_\theta = C$ for all θ. Given $Y_t = \sigma\{x_s; s\in[0,t]\}$, the optimal control satisfies:*

$$u_t^* = \sum_\theta\left[\widehat{\chi}_t^\theta \overline{K}_\theta(t)\right]x_t \tag{25}$$

$$\text{with}: \overline{K}_\theta(t) = -\sum_\eta\left[\widehat{\chi}_t^\eta R_{=}^{-1}B_\eta^{'}P_{\eta\theta}(t)\right], \tag{26}$$

where $\widehat{\chi}_t^\theta$ is the Wonham filter solution (24), *and $P(t)$ is the solution of* (15).

Proof. To verify that Condition C0 is satisfied, we verify that $\{u_t\}$ does not influence $E\{\widetilde{\xi}_t\widetilde{\xi}_t^{'}\}$:

$$E\{\widetilde{\xi}_t\widetilde{\xi}_t^{'}|Y_t\} = E\{x_t\cdot\widetilde{\chi}_t\widetilde{\chi}_t^{'}\cdot x_t^{'}|Y_t\} = x_t\cdot E\{\widetilde{\chi}_t\widetilde{\chi}_t^{'}|Y_t\}\cdot x_t^{'} =$$

$$= x_t\cdot\sum_\theta\left[\widehat{\chi}_t^\theta\cdot E\{\widetilde{\chi}_t\widetilde{\chi}_t^{'}|Y_t,\chi_t^\theta=1\}\right]\cdot x_t^{'} = x_t\cdot\sum_\theta\left[\widehat{\chi}_t^\theta\cdot E\{(\iota^\theta-\widehat{\chi}_t)(\iota^\theta-\widehat{\chi}_t)^{'}|Y_t,\chi_t^\theta=1\}\right]\cdot x_t^{'}$$

$$= x_t\cdot\sum_\theta\left[\widehat{\chi}_t^\theta\cdot(\iota^\theta-\widehat{\chi}_t)(\iota^\theta-\widehat{\chi}_t)^{'}\right]\cdot x_t^{'}$$

where ι^θ is a column vector of length N satisfying $\iota_{\theta'}^\theta = 1$ for $\theta' = \theta$, and $\iota_{\theta'}^\theta = 0$ for $\theta' \neq \theta$.

This shows that $\{u_t\}$does not influence$E\{\widetilde{\xi}_t\widetilde{\xi}_t^{'}|Y_t\}$. Hence, $\{u_t\}$neither influences $E\{\widetilde{\xi}_t\widetilde{\xi}_t^{'}\}$. Moreover, the Wonham filter shows that $\{u_t\}$ does not influence $\{\widehat{\chi}_t\}$. This means that CE condition C0 is satisfied, and (14-18) hold true.

Substituting eq. (14) in eq. (18), and subsequent evaluation yields:

$$u_t^* = -\left[\widehat{\chi}_t^{e_1}I_{\ell}\cdots\widehat{\chi}_t^{e_N}I_{\ell}\right]R^{-1}B'P(t)\widehat{\xi}_t = -\sum_\eta\left[\widehat{\chi}_t^\eta R_{=}^{-1}B_\eta^{'}\sum_\theta\left[P_{\eta\theta}(t)\widehat{\xi}_t^\theta\right]\right] =$$

$$= -\sum_\eta\left[\widehat{\chi}_t^\eta R_{=}^{-1}B_\eta^{'}\sum_\theta\left[\widehat{\chi}_t^\theta P_{\eta\theta}(t)x_t\right]\right] = -\sum_\theta\left[\widehat{\chi}_t^\theta\left[\sum_\eta\widehat{\chi}_t^\eta R_{=}^{-1}B_\eta^{'}P_{\eta\theta}(t)\right]x_t\right] = \sum_\theta\left[\widehat{\chi}_t^\theta\overline{K}_\theta(t)x_t\right] \quad \text{Q.E.D.}$$

Remark 5.4. The difference between (25) and the Averaging MJLS feedback (23) lies in the mode-dependent feedback gains $\overline{K}_\theta(t)$ and $K_\theta^M(t)$ respectively. In contrast to $K_\theta^M(t)$, $\overline{K}_\theta(t)$ takes $\widehat{\chi}_t$ and the off-diagonal sub-matrices of $P(t)$ into account. These off-diagonal sub-matrices play an important role when $\widehat{\chi}_t$ has multiple non-zero $\widehat{\chi}_t^\theta$ components.

6. General MJLS control problem

This section addresses the more general MJLS control under partial observations $Y_t = \sigma\{y_s; s\in[0,t]\}$, where the linear Gaussian observation process $\{y_t\}$ satisfies (19).

By defining $H = Row\{H_{e_1},..,H_{e_M}\}$, (19) can be written as:

$$dy_t = H\xi_t dt + Gdb_t. \tag{27}$$

The objective is to elaborate the control policy of Theorem 4.1 for this general MJLS stochastic control problem. This elaboration involves three steps. The first step is to develop the nonlinear filtering equations for the estimation of $\widehat{\xi}_t = Col\{\widehat{\xi}_t^{e_1},...,\widehat{\xi}_t^{e_N}\}$ in eq. (17). The second step is to prove that relaxed CE condition C0* is satisfied, i.e. $E\{(\xi_t^\theta-\widehat{\xi}_t^\theta)(\xi_t^\theta-\widehat{\xi}_t^\theta)^{'}\}$,$\theta\in\Theta$, is not influenced by $\{u_t\}$. The third step is to picture how the various developments in this paper define the optimal feedback controlled system.

Proposition 6.1. *Let assumptions* i-iii *be satisfied for system* (10-12), *and H, G have finite-valued components.*

The process $\{\widehat{\xi}_t^\theta\}$, defined by $\widehat{\xi}_t^\theta = E\{\xi_t^\theta|Y_t\}$, satisfies:

$$d\widehat{\xi}_t^\theta = A_\theta\widehat{\xi}_t^\theta dt + B_\theta\widehat{\chi}_t^\theta u_t dt + \sum_{\eta\in\Theta}\left[\lambda_{\eta\theta}\widehat{\xi}_t^\eta\right]dt + \left[\widehat{q}_t^\theta H_\theta^{'} - \widehat{\xi}_t^\theta\widehat{\xi}_t^{'}H'\right](GG')^{-1}d\nu_t \tag{28}$$

with $d\nu_t = dy_t - H\widehat{\xi}_t dt$, $\widehat{\xi}_t = Col\{\widehat{\xi}_t^{e_1},...,\widehat{\xi}_t^{e_N}\}$, and where $\widehat{\chi}_t^\theta$ satisfies:

$$d\widehat{\chi}_t^\theta = \sum_{\eta\in\Theta}\left[\lambda_{\eta\theta}\widehat{\chi}_t^\eta\right]dt + \left[A_\theta\widehat{\xi}_t^\theta + \widehat{\chi}_t^\theta B_\theta u_t - \sum_{\eta\in\Theta}\left[(A_\eta\widehat{\xi}_t^\eta + \widehat{\chi}_t^\eta B_\eta u_t)\right]\right]^{'}[CC']^{-1}d\nu_t \tag{29}$$

and the matrix process $\{\widehat{q}_t^\theta\}$, defined by $\widehat{q}_t^\theta = E\{\xi_t^\theta\xi_t^{\theta'}|Y_t\}$, satisfies:

$$\begin{aligned} d\widehat{q}_t^\theta \;=\; & \left[A_\theta\widehat{q}_t^\theta + \widehat{q}_t^\theta A_\theta^{'} + B_\theta u_t\widehat{\xi}_t^{\theta'} + \widehat{\xi}_t^\theta(B_\theta u_t)^{'} + C_\theta\widehat{\chi}_t^\theta C_\theta^{'}\right]dt + \sum_{\eta\in\Theta}\left[\lambda_{\eta\theta}\widehat{q}_t^\eta\right]dt \\ & + \left[\left(\widehat{q_t^\theta\xi_t^{\theta'}}\right)H_\theta^{'} - \widehat{q}_t^\theta\widehat{\xi}_t^{'}H'\right](GG')^{-1}d\nu_t \end{aligned} \tag{30}$$

Proof. Application of the fundamental filtering theorem [Elliott [16], Th. 18.11] to eq. (7) yields:

$$d\hat{\xi}_t^\theta = A_\theta \hat{\xi}_t^\theta dt + B_\theta \hat{\chi}_t^\theta u_t dt + \sum_{\eta\in\Theta}[\lambda_{\eta\theta}\hat{\xi}_t^\eta]dt + \left[\left(\widehat{\xi_t^\theta \xi_t' H'}\right) - \hat{\xi}_t^\theta \hat{\xi}_t' H'\right](GG')^{-1}d\nu_t.$$

Since for all $\eta \neq \theta$, $\xi_t^\theta \xi_t^{\eta'} = 0$, the latter implies (28).

Application of the fundamental filtering theorem to eq. (5) yields (29). Application of the differentiation rule for discontinuous semimartingales to $q_t^\theta = \xi_t^\theta \xi_t^{\theta'}$, using eq. (6), yields:

$$dq_t^\theta = d(\xi_t^\theta \xi_t^{\theta'}) = d\xi_t^{\theta c}\xi_t^{\theta'} + \xi_t^\theta d\xi_t^{\theta c'} + C_\theta \chi_t^\theta C_\theta' dt + [\xi_t^\theta \xi_t^{\theta'} - \xi_{t-}^\theta \xi_{t-}^{\theta'}] =$$

$$= [A_\theta \xi_t^\theta dt + B_\theta \chi_t^\theta u_t dt + C_\theta \chi_t^\theta dw_t]\xi_t^{\theta'} + \xi_t^\theta [A_\theta \xi_t^\theta dt + B_\theta \chi_t^\theta u_t dt + C_\theta \chi_t^\theta dw_t]' + C_\theta \chi_t^\theta C_\theta' dt + \sum_{\eta\in\Theta}[\xi_{t-}^\eta \xi_{t-}^{\eta'} dp_{\eta\theta,t}]$$

$$= A_\theta q_t^\theta dt + q_t^\theta A' dt + B_\theta \chi_t^\theta u_t \xi_t^{\theta'} dt + \xi_t^\theta (B_\theta \chi_t^\theta u_t)' dt + C_\theta \chi_t^\theta dw_t \xi_t^{\theta'} + \xi_t^\theta (C_\theta \chi_t^\theta dw_t)' + C_\theta \chi_t^\theta C_\theta' dt + \sum_{\eta\in\Theta}[\xi_{t-}^\eta \xi_{t-}^{\eta'} dp_{\eta\theta,t}]$$

$$= [A_\theta q_t^\theta + q_t^\theta A_\theta' + B_\theta \chi_t^\theta u_t \xi_t^{\theta'} + \xi_t^\theta (B_\theta \chi_t^\theta u_t)' + C_\theta \chi_t^\theta C_\theta']dt + C_\theta \chi_t^\theta dw_t \xi_t^{\theta'} + \xi_t^\theta (C_\theta \chi_t^\theta dw_t)' + \sum_{\eta\in\Theta}[q_{t-}^\eta dp_{\eta\theta,t}]$$

Application of the fundamental filtering theorem yields:

$$d\hat{q}_t^\theta = \hat{\gamma}_t^\theta dt + \left[\left(\widehat{q_t^\theta \xi_t' H'}\right) - \hat{q}_t^\theta \hat{\xi}_t' H'\right](GG')^{-1}d\nu_t$$

with $\hat{\gamma}_t^\theta = A_\theta \hat{q}_t^\theta + \hat{q}_t^\theta A_\theta' + B_\theta u_t \hat{\xi}_t^{\theta'} + \hat{\xi}_t^\theta (B_\theta u_t)' + C_\theta \hat{\chi}_t^\theta C_\theta' + \sum_{\eta\in\Theta}[\hat{q}_{t-}^\eta \lambda_{\eta\theta}]$

This, and using $q_t^\theta \xi_t^{\eta'} = 0,\ \forall \eta \neq \theta$, yields (30). Q.E.D.

Because the innovation term for $\hat{q}_t^\theta$, in eq. (29), involves a third order moment, a full solution of $\hat{q}_t^\theta$is complemented by a characterization of the joint conditional density of $\{\theta_t, x_t\}$, e.g. [17,18].

Proposition 6.2. *Let the conditional probability mass-density $\hat{\rho}_{\theta_t,x_t}(\theta,x)$ be in the domain of L^θ, then:*

$$d\hat{\rho}_{\theta_t,x_t}(\theta,x) = [L^\theta + J]\hat{\rho}_{\theta_t,x_t}(\theta,x)dt + \hat{\rho}_{\theta_t,x_t}(\theta,x)(H_\theta x - H\hat{\xi}_t)'(GG')^{-1}d\nu_t \tag{31}$$

where $d\nu_t = dz_t - \hat{\xi}_t H' dt$; J is the Kolmogorov operator:

$$Jf(\theta,x) = \sum_{\eta\neq\theta}[\lambda_{\eta\theta}f(\eta,x)] \tag{32}$$

and L^θ is the mode-conditional Fokker-Planck operator:

$$L^\theta f(\theta,x) = -\sum_{i=1}^n \frac{\partial}{\partial x^i}\left[(A_\theta^i x + B_\theta^i u_t)f(\theta,x)\right] + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x^i \partial x^j}\left[C_\theta^i C_\theta^{j'} f(\theta,x)\right] =$$

$$= -\sum_{i=1}^n \left[A_\theta^{ii} f(\theta,x)\right] - \sum_{i=1}^n \left[(A_\theta^i x + B_\theta^i u_t)\frac{\partial}{\partial x^i}f(\theta,x)\right] + \frac{1}{2}\sum_{i,j=1}^n \left[C_\theta^i C_\theta^{j'} \frac{\partial^2}{\partial x^i \partial x^j}f(\theta,x)\right] \tag{33}$$

Having characterized the exact nonlinear estimator of $\hat{\xi}_t^\theta$, the crucial step is to prove that relaxed CE condition C0* is satisfied, i.e. $E\{(\xi_t^\theta - \hat{\xi}_t^\theta)(\xi_t^\theta - \hat{\xi}_t^\theta)'\}, \theta \in \Theta$, is not influenced by $\{u_t\}$.

Theorem 6.3. *Define $\tilde{\xi}_t^\theta = \xi_t^\theta - \hat{\xi}_t^\theta$. Given eq. (27) observations $\{y_t\}$ of the solution $\{\xi_t\}$ of (10), then*

$$E\{\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}}\} = E\{\widetilde{\xi_0^\theta}\widetilde{\xi_0^{\theta'}}\} + \int_0^t [A_\theta E\{\widetilde{\xi_s^\theta}\widetilde{\xi_s^{\theta'}}\} + E\{\widetilde{\xi_s^\theta}\widetilde{\xi_s^{\theta'}}\}A_\theta' + \sum_{\eta\in\Theta}[\lambda_{\eta\theta}E\{\widetilde{\xi_s^\eta}\widetilde{\xi_s^{\eta'}}\}]ds \tag{34}$$

which assures C0, i.e. that the evolution of $E\{\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}}\}$ is not influenced by $\{u_t\}$, and that the optimal control policy of Theorem 4.1 holds true, with $\{\hat{\xi}_t^\theta\}$ satisfying (29).* **Proof.**

$$E\{\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}}\} = E\{(\xi_t^\theta - \hat{\xi}_t^\theta)(\xi_t^\theta - \hat{\xi}_t^\theta)'\} = E\{\xi_t^\theta \xi_t^{\theta'} + \hat{\xi}_t^\theta \hat{\xi}_t^{\theta'} - \xi_t^\theta \hat{\xi}_t^{\theta'} - \hat{\xi}_t^\theta \xi_t^{\theta'}\} =$$

$$= E\{\xi_t^\theta \xi_t^{\theta'} + \hat{\xi}_t^\theta \hat{\xi}_t^{\theta'} - E\{\xi_t^\theta \hat{\xi}_t^{\theta'} + \hat{\xi}_t^\theta \xi_t^{\theta'} | Y_t\}\} = E\{\xi_t^\theta \xi_t^{\theta'} + \hat{\xi}_t^\theta \hat{\xi}_t^{\theta'} - \hat{\xi}_t^\theta \hat{\xi}_t^{\theta'} + \hat{\xi}_t^\theta \hat{\xi}_t^{\theta'}\} =$$

$$= E\{\xi_t^\theta \xi_t^{\theta'} - \hat{\xi}_t^\theta \hat{\xi}_t^{\theta'}\} = E\{\xi_t^\theta \xi_t^{\theta'} - \hat{q}_t^\theta\}$$

Hence $E\{d(\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}})\}= E\{d(\xi_t^\theta\xi_t^{\theta'} - \widehat{q}_t^\theta) = E\{d(\xi_t^\theta \xi_t^{\theta'})\} - E\{d\widehat{q}_t^\theta\}$

Application of differentiation rule for discontinuous semimartingales to (6) yields:

$$d(\xi_t^\theta\xi_t^{\theta'}) = \xi_{t-}^\theta d\xi_t^{\theta'} + d\xi_t^\theta\xi_{t-}^{\theta'} + C_\theta\chi_{t-}^\theta\chi_{t-}^\theta C_\theta' dt + \left[\xi_t^\theta\xi_t^{\theta'} - \xi_{t-}^\theta\xi_{t-}^{\theta'} - \xi_{t-}^\theta\Delta_t^{\xi^{\theta'}} - \Delta_t^{\xi^\theta}\xi_{t-}^{\theta'}\right] = \xi_{t-}^\theta d\xi_t^{\theta c'} + d\xi_t^{c\theta}\xi_{t-}^{\theta'} + C_\theta\chi_{t-}^\theta C_\theta' dt$$
$$+\sum_\eta[\xi_{t-}^\eta\xi_{t-}^{\eta'}dp_{\eta\theta,t}] =$$
$$= \xi_t^\theta[A_\theta\xi_t^\theta dt + B_\theta\chi_t^\theta u_t dt + C_\theta\chi_t^\theta dw_t]' + [A_\theta\xi_t^\theta dt + B_\theta\chi_t^\theta u_t dt + C_\theta\chi_t^\theta dw_t]\xi_t^{\theta'} + C_\theta\chi_t^\theta C_\theta' dt + \sum_\eta[\xi_t^\eta\xi_t^{\eta'}dp_{\eta\theta,t}]$$

where $\xi_t^{\theta c}$ is the continuous part of ξ_t^θ. Together with eq. (30), this yields: $E\{d(\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}})\} = E\{d(\xi_t^\theta\xi_t^{\theta'})\} - E\{d\widehat{q}_t^\theta\} =$

$$= E\{\xi_t^\theta[A_\theta\xi_t^\theta dt + B_\theta\chi_t^\theta u_t dt]'\} + E\{[A_\theta\xi_t^\theta dt + B_\theta\chi_t^\theta u_t dt]\xi_t^{\theta'}\} + E\{C_\theta\chi_t^\theta C_\theta'\}dt + E\left\{\sum_{\eta\in\Theta}[\lambda_{\eta\theta}\xi_t^\eta\xi_t^{\eta'}]\right\}dt +$$

$$-E\{[A_\theta\widehat{q}_t^\theta + \widehat{q}_t^\theta A_\theta' + B_\theta\chi_t^\theta u_t\widehat{\xi}_t^{\theta'} + \widehat{\xi}_t^\theta(B_\theta\chi_t^\theta u_t)' + C_\theta\widehat{\chi}_t^\theta C_\theta']dt\} - E\left\{\sum_{\eta\in\Theta}[\lambda_{\eta\theta}\widehat{q}_t^\eta]dt\right\} =$$

$$= E\{\xi_t^\theta[A_\theta\xi_t^\theta]'dt\} + E\{A_\theta\xi_t^\theta\xi_t^{\theta'}dt\} + E\{C_\theta\chi_t^\theta C_\theta'\}dt + \sum_{\eta\in\Theta}[\lambda_{\eta\theta}E\{\xi_t^\eta\xi_t^{\eta'}\}]dt +$$

$$-E\{[A_\theta\widehat{q}_t^\theta + \widehat{q}_t^\theta A_\theta' + C_\theta\widehat{\chi}_t^\theta C_\theta']dt\} - \sum_{\eta\in\Theta}[\lambda_{\eta\theta}E\{\widehat{q}_t^\eta\}]dt\} +$$

$$+E\{(\xi_t^\theta - \widehat{\xi}_t^\theta)(B_\theta\chi_t^\theta u_t)'\}dt + E\{B_\theta\chi_t^\theta u_t(\xi_t^\theta - \widehat{\xi}_t^\theta)'\}dt =$$

$$\overset{(c)}{=} E\{A_\theta(\xi_t^\theta\xi_t^{\theta'} - \widehat{q}_t^\theta) + (\xi_t^\theta\xi_t^{\theta'} - \widehat{q}_t^\theta)A_\theta' + C_\theta(\chi_t^\theta - \widehat{\chi}_t^\theta)C_\theta'\}dt + \sum_{\eta\in\Theta}[\lambda_{\eta\theta}E\{\xi_t^\eta\xi_t^{\eta'} - \widehat{q}_t^\eta\}]dt =$$

$$= E\{A_\theta(\xi_t^\theta\xi_t^{\theta'} - \widehat{q}_t^\theta) + (\xi_t^\theta\xi_t^{\theta'} - \widehat{q}_t^\theta)A_\theta'\}dt + \sum_{\eta\in\Theta}[\lambda_{\eta\theta}E\{\xi_t^\eta\xi_t^{\eta'} - \widehat{q}_t^\eta\}]dt =$$

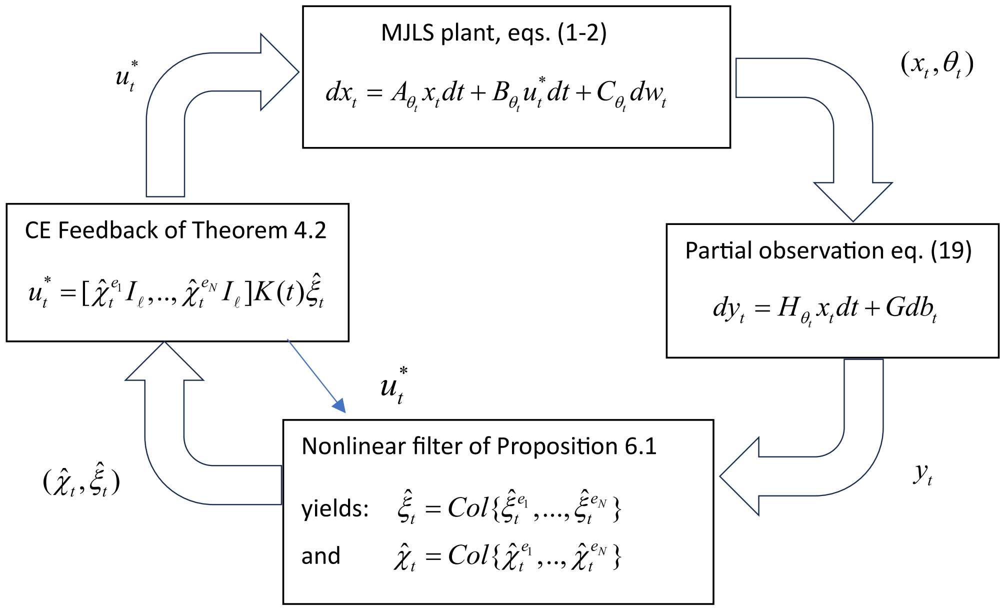

Figure 1. The optimal control loop for MJLS (1-3) given partial observations (19), where feedback gain $K(t)$ is the solution of eqs. (14-16) in the $J\{(u_t)\}$ characterization of Theorem 3.1.

$$= A_\theta E\{\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}}\}dt + E\{\widetilde{\xi_t^\theta}\widetilde{\xi_t^{\theta'}}\}A_\theta' dt + \sum_{\eta\in\Theta}[\lambda_{\eta\theta}E\{\widetilde{\xi_t^\eta}\widetilde{\xi_t^{\eta'}}\}]dt$$

where for equality $\stackrel{(c)}{=}$ use is made of:

$$E\{B_\theta\chi_t^\theta u_t(\xi_t^\theta - \widehat{\xi}_t^\theta)'\} = B_\theta E\{\chi_t^\theta u_t(\xi_t^\theta - \widehat{\xi}_t^\theta)'\} = B_\theta E\{E\{\chi_t^\theta u_t(\xi_t^\theta - \widehat{\xi}_t^\theta)'|Y_t\}\} =$$

$$= B_\theta E\{P\{\theta_t = \theta|Y_t\}E\{\chi_t^\theta u_t(\xi_t^\theta - \widehat{\xi}_t^\theta)'|\theta_t = \theta, Y_t\}\} = B_\theta E\{\widehat{\chi}_t^\theta E\{u_t(\xi_t^\theta - \widehat{\xi}_t^\theta)'|\theta_t = \theta, Y_t\}\} =$$

$$= B_\theta E\{\widehat{\chi}_t^\theta u_t E\{(\xi_t^\theta - \widehat{\xi}_t^\theta)'|\theta_t = \theta, Y_t\}\} = B_\theta E\{\widehat{\chi}_t^\theta u_t \cdot 0\} = 0$$

Writing (b) in integral form yields (34). Q.E.D.

The final step is to picture the resulting optimal feedback control loop for MJLS (1-3) under partial observation (19) in Figure 1.

Remark 6.4. If the CE version of the Averaging MJLS control is adopted, then in the left block in Figure 1, the optimal control $u_t^* = [\widehat{\chi}_t^{e_1}I_\ell, .., \widehat{\chi}_t^{e_N}I_\ell]K(t)\widehat{\xi}_t$ has to be replaced by $u_t^A = \sum_{\theta\in\Theta} K_\theta^M(t)\widehat{\xi}_t^\theta$ with $K_\theta^M(t)$ the solution of eqs. (21-22).

Remark 6.5. For the MJLS control problem addressed by Theorem 6.3, [Everdij and Blom, 1996] have developed the Open Loop Optimal Feedback (OLOF) control u_t^{OLOF} under the OLOF assumption that observations beyond moment t are ignored during the control optimization. Thanks to Theorem 6.3 we now know that this OLOF assumption is correct, which means $u_t^{OLOF} = u_t^*$. For a simple MJLS example, [Everdij and Blom, 1996] have also conducted simulations to compare the use of u_t^{OLOF} versus the use of the CE version of the Averaging MJLS control $u_t^A = \sum_{\theta\in\Theta} K_\theta^M(t)\widehat{\xi}_t^\theta$. This comparison showed the important role played by non-sparse $\widehat{\chi}_t$ in $u_t^{OLOF} = u_t^*$. Although $u_t^{OLOF} = u_t^*$, it is relevant to be aware that the control law equations for u_t^{OLOF} are much more complicated than eq. (18) for u_t^*.

Remark 6.6. It can be noticed that there is a partial form of duality only between the $P(t)$ solution of eq. (15), and the covariance $[Diag(\widehat{q}_t^{e_1}, ..., \widehat{q}_t^{e_N}) - \widehat{\xi}_t\widehat{\xi}_t']$ from the nonlinear filter. There is a coupled Riccati type of duality between the $N = |\Theta|$ diagonal matrix components of $P(t)$ and $Diag(\widehat{q}_t^{e_1}, ..., \widehat{q}_t^{e_N})$. However, such type of duality does not apply to the $N(N-1)$ off-diagonal matrices of $P(t)$ and $[Diag(\widehat{q}_t^{e_1}, ..., \widehat{q}_t^{e_N}) - \widehat{\xi}_t\widehat{\xi}_t']$ respectively.

Remark 6.7. The innovation term in eq. (30) causes the nonlinear filter of Proposition 6.1 to be infinite-dimensional. Hence, in literature, finite-dimensional numerical approximation methods have been developed. The main methods are:

- Continuous-time Interacting Multiple Model (IMM) estimator [19–21]
- IMM Particle Filter [21]
- IMM Feedback Particle Filter [22]
- Grid-based numerical integration of eq. (31) for $\widehat{\rho}_{\theta_t,x_t}(.,.)$ [23]

Each of these methods can be used to numerically estimate $\widehat{\xi}_t$. IMM has the lowest computational load, at the cost of assuming second order density approximations. The computational load of the grid-based approach grows linearly with the number of grid points used. For many practical problems this is impractically large. The IMM Particle Filter approach makes use of a flexible grid which adapts to the evolution of the joint conditional density of (θ_t, x_t). The adaptation of this grid is further improved by the IMM Feedback Particle Filter.

7. Conclusions

This paper has derived a general separation principle for optimal control of partially observed MJLS with n-dimensional Euclidean state process $\{x_t\}$ and a finite state Markov process $\{\theta_t\}$. To accomplish this, in section 2, the MJLS system has been transformed to optimal control of an Nn-dimensional process $\{\xi_t\}$ that is a solution of a martingale driven linear system. Section 3 has derived a Non-sparse optimal control law given full observations of a non-sparse $\{\xi_t\}$ solution of the transformed system of section 2. In Section 4, for partial observations of the process $\{\xi_t\}$ a generally applicable Separation Principle has been derived, under a relaxed CE condition C0*. In section 5, two partially observed MJLS cases have been considered. The first case has full observations of $\{\theta_t\}$ and a Kalman estimator of $\widehat{x}_t$; it has been shown that the optimal control solution of section 4 is equal to the known solution [4]. The second case has full observations of $\{x_t\}$ and a Wonham filter to estimate $\{\theta_t\}$; it has been shown that the optimal control of section 4 significantly enriches the averaging MJLS control policy of Fragoso [7]. Section 6 considers the general MJLS case of partial observations of $\{x_t\}$ only; it has been proven that CE condition C0* of section 4 holds true. Subsequently, for this general MJLS case the optimal feedback-control law has been depicted in Figure 1.

There are interesting directions for follow-on research. One direction is to investigate if and how the obtained separation principle can be extended to discrete time setting of a partially observed MJLS, e.g. Costa et al. [24]. A complementary direction is to investigate if and how this separation principle can be extended to a MJLS that is enriched with hybrid jumps [25–28], i.e. jumps in the Euclidean-valued process $\{x_t\}$ that occur simultaneous with a $\{\theta_t\}$switching, and the jump size may depend on the mode values before and after the switching.

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Declaration of competing interest

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Appendix A

A1. Differentiation rule for discontinuous semimartingales

The differentiation rule for discontinuous semimartingales as well as the relation to other differentiation rules are well explained by [Protter [29], Chapter II].

Let $\{z_t\}$ be an L-vector semimartingale, and let f be a twice continuously differentiable mapping of z_t into $\mathbb{R}$. Then $f(z_t)$ is a semimartingale satisfying up to indistinguishability:

$$f(z_t) = f(z_0) + \sum_{i=1}^{L} \int_0^t \frac{\partial}{\partial z^i} f(z_{s-}) dz_s^i + \frac{1}{2} \sum_{i,j=1}^{L} \int_0^t \frac{\partial^2}{\partial z^i \partial z^j} f(z_{s-}) d < m_c^{z^i}, m_c^{z^j} >_s +$$

$$+ \sum_{0<s\le t} \left[f(z_s) - f(z_{s-}) - \sum_{i=1}^{L} \left[\frac{\partial}{\partial z^i} f(z_{s-}) \Delta_s^{z^i} \right] \right] \tag{A.1}$$

where $m_c^{z^i}$ is the i-th component of the continuous martingale part of z_t, $d < m_c^{z^i}, m_c^{z^j} >_s$ is the quadratic co-variation of $\left(m_{c,t}^{z^i}, m_{c,t}^{z^j}\right)$, $\Delta_s^{z^i} \triangleq z_s^i - z_{s-}^i$, and the summation in the last term is over all time moments $s \in (0,t]$ at which $z_s^i \neq z_{s-}^i$ for some $i \in [1, L]$.

A2. Derivation of eq. (5)

Because $\{\theta_t\}$ is purely discontinuous, we have: $d\theta_t = \Delta_t^\theta$.

By defining the mapping $f^\theta(\theta_t) = 1(\theta_t = \theta) = \chi_t^\theta$, the differentiation rule yields:

$$d\chi_t^\theta = df^\theta(\theta_t) = f^\theta(\theta_t) - f^\theta(\theta_{t-}) = \chi_t^\theta - \chi_{t-}^\theta = \sum_{\eta\neq\theta} \chi_{t-}^\eta dp_{\eta\theta,t} - \sum_{\eta\neq\theta} \chi_{t-}^\theta dp_{\theta\eta,t}$$

where the first sum covers all possible jumps from $\theta_{t-} \neq \theta$ to $\theta_t = \theta$, and the second sum covers all possible jumps from $\theta_{t-} = \theta$ to $\theta_t \neq \theta$. Using $dp_{\theta\theta,t} \triangleq -\sum_{\eta\neq\theta} dp_{\theta\eta,t}$, yields:

$d\chi_t^\theta = \sum_{\eta\neq\theta} \chi_{t-}^\eta dp_{\eta\theta,t} - \chi_{t-}^\theta dp_{\theta\theta,t} = \sum_{\eta\in\Theta} \chi_{t-}^\eta dp_{\eta\theta,t}$ Q.E.D.

A3. Derivation of eq. (6)

Because $\{x_t\}$ has no discontinuities, we have $\Delta_t^x = 0$.

By defining the mapping $F^\theta(\theta_t, x_t) = f^\theta(\theta_t) x_t = \chi_t^\theta x_t = \xi_t^\theta$, we get: $\frac{\partial^2}{\partial x^i \partial x^j} F^\theta(\theta, x) = 0$.

Together with $d\theta_t = \Delta_t^\theta$, application of the differentiation rule yields:

$$d\xi_t^\theta = \sum_{i=1}^{n} \left[\frac{\partial}{\partial x^i} F^\theta(x_{t-}, \theta_{t-}) dx_t^i \right] + \left[F^\theta(x_t, \theta_t) - F^\theta(x_{t-}, \theta_{t-}) \right] =$$

$$= \chi_{t-}^\theta dx_t + \left[\chi_t^\theta x_t - \chi_{t-}^\theta x_{t-} \right]$$

For the terms within brackets, we sum over all possible jumps from $\theta_{t-} \neq \theta$ to $\theta_t = \theta$, and distract the sum over all possible jumps from $\theta_{t-} = \theta$ to $\theta_t \neq \theta$, which yields

$$d\xi_t^\theta = \chi_{t-}^\theta dx_t + \sum_{\eta\neq\theta}\left[\chi_{t-}^\eta x_{t-} dp_{\eta\theta,t}\right] - \sum_{\eta\neq\theta}\left[\chi_{t-}^\theta x_{t-} dp_{\theta\eta,t}\right]$$

Using $dp_{\theta\theta,t} \triangleq -\sum_{\eta\neq\theta} dp_{\theta\eta,t}$, yields: $d\xi_t^\theta = \chi_{t-}^\theta dx_t + \sum_{\eta\in\Theta}\left[\chi_{t-}^\eta x_{t-} dp_{\eta\theta,t}\right] = \chi_{t-}^\theta dx_t + \sum_{\eta\in\Theta}\left[\xi_{t-}^\eta dp_{\eta\theta,t}\right]$.
Substitution of (1), and subsequent evaluation yields (6). Q.E.D.

Data availability

No data was used for the research described in the article.

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