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Master thesis:

A semi-group approach to the thin-film equation with general mobility

by

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Abstract

This thesis treats the thin-film equation which models the film height h for a viscous film in the complete wetting regime. We show existence and uniqueness to the thin-film equation with mobility $m(h) = h^n$ and mobility exponent $n \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 3)$. The thin-film equation is rewritten as an abstract Cauchy problem and usage of semi-group theory yields maximal L^p-regularity for the linearized problem. With a fixed point argument, analogous to the one used by Giacomelli, Gnann, Knüpfer and Otto in [16], the nonlinear problem is treated. Under a smallness condition on the initial value to a suitably transformed version of the thin-film equation, we obtain a solution in $L^p(0, \infty; H_{k-2,\alpha-\frac{1}{2}}) \cap \dot{W}^{1,p}(0, \infty; H_{k+2,\alpha+\frac{1}{2}})$, where the H-spaces denote weighted Sobolev spaces. The novelty of this work lies in the usage of L^p-spaces in time, where the existing literature only deals with L^2-spaces. It is found that the L^p setting allows for treatment of all $n \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 3)$.

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Chapter 1

Introduction

We consider the following thin-film equation

$$h_t + (h^n h_{zzz})_z = 0 \qquad \text{for } t > 0,\ z > Z(t), \tag{1.0.1a}$$
$$h = h_z = 0 \qquad \text{for } t > 0,\ z = Z(t), \tag{1.0.1b}$$
$$\lim_{z \downarrow Z(t)} h^{n-1} h_{zzz} = Z_t(t). \tag{1.0.1c}$$

This partial differential equation (PDE) describes the change of the height $h(t,z)$ of a viscous thin film over time on a one dimensional substrate, as visualized in figure 1.1. Here, t is the time variable and z is the lateral variable. The fluid covers the interval $(Z(t), \infty)$, where $Z(t)$ is called the triple junction or contact line, this is the place where gas, liquid and solid meet. Hence, the fluid has a free boundary at $z = Z(t)$. The thin-film equation can be derived from the Navier-Stokes equations using a lubrication approximation, which is worked out in section 1.1.

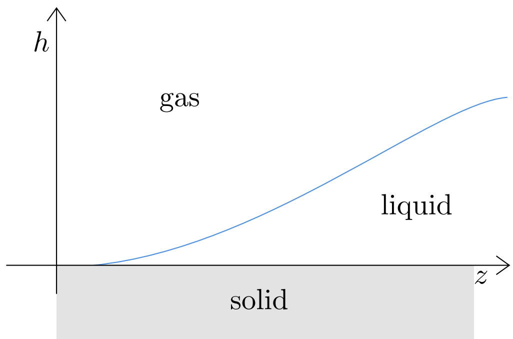


Figure 1.1: Example of a thin film as described by (1.0.1)

The power n in (1.0.1a) is called the mobility exponent with values in $(0,3)$. The value of this exponent is related to the boundary condition on the solid-liquid interface in the original Navier-Stokes problem, which is also called the slip condition. We will focus on the cases $n \in (1, \frac{3}{2})$ and $n \in (\frac{3}{2}, 3)$. This is because in the case that $n \geq 3$, the boundary of the film is unable to move [22]. If $n < 0$ the propagation speed is infinite and if $0 < n < 1$ the height of the film can become negative [8]. In the case of $n = 2$, this slip condition is called the linear Navier-slip condition [33, 34]. The case $n = \frac{3}{2}$ is not considered, since this case the analysis becomes more involved due to resonances [5].

The boundary condition $h = 0$ at $z = Z(t)$ in (1.0.1b) states that the height of the film at the contact line is equal to zero. The following condition, $h_z = 0$ at $z = Z(t)$

tells us that the angle between the solid and the liquid at the triple junction equals zero. This means that over time the fluid will cover the entire solid. This is evident from the following relation

$$\gamma_{gs} = \gamma_{ls} + \cos(\theta)\gamma_{gl}, \tag{1.0.2}$$

called Young's law [38]. The surface tensions γ_{gs}, γ_{ls} and γ_{gl} describe the tensions respectively between the gas and solid, liquid and solid and gas and liquid interfaces. This is visualized in figure 1.2. When $\gamma_{gs} < \gamma_{ls} + \gamma_{gl}$, the contact angle has to be strictly positive. In this case, an equilibrium can be obtained, hence after some time the liquid stops to spread further. This is called the partial wetting regime. In the other case, $\gamma_{gs} \geq \gamma_{ls} + \gamma_{gl}$; the contact angle must be equal to zero, and an equilibrium cannot be obtained. Hence, the liquid will continue to spread. We consider the last setting, which is also called the complete wetting regime.

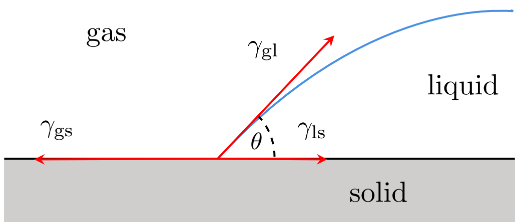


Figure 1.2: Surface tensions acting on a liquid at the triple junction.

In [16] it is shown that in the case $n = 2$, under a smallness condition for the inital value for a suitably transformed version of (1.0.1), the problem has a unique classical solution. Furthermore, for a range of $n \in (\frac{3}{17}(15 - \sqrt{21}), \frac{3}{11}(7 + \sqrt{5}))$ it is described in remark 3.4 of [16] how the same result can be obtained. The novelty of this thesis is that a semigroup approach is used to obtain the necessary maximal regularity condition. A benefit of using this method is that it is immediately clear that we get maximal L^p-regularity in time, and hence L^p-integrability in time, for $1 < p < \infty$. In the spatial variables still Hilbertian weighted Sobolev spaces are used, as is also the case in existing literature. The methods used in [16] yield maximal L^2-regularity (L^2-integrability in time), which is enough to show their results. The added bonus from the fact that we get maximal L^p-regularity is that we will be able to choose p in such a way that existence and uniqueness of a solution to (1.0.1) can be shown for all values of $n \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 3)$. Different settings where the thin-film equation has been studied include, but are not limited to, [6, 17, 18, 25, 27, 28, 35].

1.1 Lubrication Approximation

In this section, the thin-film equation will be derived, following [9, 24, 26]. For a more physical derivation, see for instance [34]. The thin-film equation (1.0.1) can be derived from the Navier-Stokes equations for the movement of fluids by means of a lubrication approximation. This means that we make use of the fact that the length scale in one direction is much larger compared to the length scale in another direction to simplify the Navier-Stokes equations. Consider a fluid with a free surface $y = h(t, z)$ that lies on a flat substrate. We assume that the fluid is uniform in the direction perpendicular to the plane (z, y), see figure 1.1. The governing equations for the velocity u, v in the z and y direction, respectively, and for the pressure p (normalized by the density) are the incompressible

Navier-Stokes equations

$$\partial_t u + u\partial_z u + v\partial_y u - \nu(\partial_z^2 u + \partial_y^2 u) + \partial_z p = 0, \tag{1.1.1a}$$
$$\partial_t v + u\partial_z v + v\partial_y v - \nu(\partial_z^2 v + \partial_y^2 v) + \partial_y p = 0, \tag{1.1.1b}$$
$$\partial_z u + \partial_y v = 0, \tag{1.1.1c}$$

where ν is the kinematic viscosity. Equations (1.1.1a)-(1.1.1b) follow from the conservation of momentum and (1.1.1c) is derived from the conservation of mass under the assumption that the fluid is Newtonian and has constant density in space and time. For a complete derivation of these equations see e.g. [3, 10, 30, 32]. In addition we need to specify boundary conditions on the solid-liquid and liquid-gas interface. On the solid-liquid interface we require first of all that the fluid cannot penetrate into the solid, i.e.,

$$v = 0 \quad \text{for } y = 0. \tag{1.1.2}$$

Furthermore, imposing a standard no-slip boundary condition (i.e., $u = 0$) gives infinite energy dissipation near the moving contact, see [12, 22]. There are multiple methods to overcome this problem, see for a review of the possibilities [7, 14, 34]. We will focus on replacing the no-slip condition by a more general condition, where a non-zero velocity is allowed on the boundary which is proportional to the normal derivative

$$u - k(h)\partial_y u = 0 \quad \text{for } y = 0. \tag{1.1.3}$$

Here, $k(h)$ is taken to be $\lambda^{3-n}h^{n-2}$ as was proposed in [19], where λ is the slip length and $n \in (0, 3)$ is, as before, the mobility exponent.

On the liquid-gas interface it is required that the tangential component of the shear stress is continuous across the interface. This leads to the condition

$$\partial_y u = 0 \quad \text{for } y = h. \tag{1.1.4}$$

The surface tension γ_{gl} (see equation (1.0.2)) causes a jump in the pressure across the liquid-gas interface known as the Young/Laplace pressure [15]

$$p - p_0 = -\gamma_{gl}\partial_z^2 h \quad \text{for } y = h, \tag{1.1.5}$$

where p_0 is the atmospheric pressure.

To derive the thin-film equation we assume that the typical thickness of the fluid H in the y-direction is small compared to the typical length scale along the solid surface L. Applying the transformations

$$z \mapsto Lz, \quad y \mapsto Hy, \quad u \mapsto Uu, \quad v \mapsto Uv, \quad p \mapsto \frac{LU}{H^2}p$$

to the Navier-Stokes equations (1.1.1) and using that $H \ll L$ gives the lubrication approximation

$$\partial_z p = \nu\partial_y^2 u \qquad \text{for } 0 < y < h, \tag{1.1.6a}$$
$$\partial_y p = 0 \qquad \text{for } 0 < y < h. \tag{1.1.6b}$$

Integrating (1.1.6b) over (y, h) and using the boundary condition (1.1.5) gives that

$$\nu\partial_y^2 u = \partial_z p = -\gamma_{gl}\partial_z^3 h \quad \text{for } 0 < y < h. \tag{1.1.7}$$

Integrating this equation again first over (y,h) and subsequently over $(0,y)$ and incorporating the boundary conditions (1.1.2) and (1.1.3) gives

$$u = \frac{\gamma_{gl}}{\nu}\big(hy - \tfrac{1}{2}y^2\big)\partial_z^3 h + k(h)(\partial_y u)\big|_{y=0}. \tag{1.1.8}$$

Then writing $(\partial_y u)\big|_{y=0} = \partial_y u - \int_0^y \partial_y^2 u dy$ and using (1.1.7) gives

$$k(h)(\partial_y u)\big|_{y=0} = \frac{\gamma_{gl}}{\nu} hk(h)\partial_z^3 h,$$

so that (1.1.8) reduces to

$$u = \frac{\gamma_{gl}}{\nu}\big(hy - \tfrac{1}{2}y^2 + hk(h)\big)\partial_z^3 h. \tag{1.1.9}$$

From this we obtain that the averaged horizontal velocity

$$\overline{u} = \frac{1}{h}\int_0^h u dy$$

is given by

$$\overline{u} = \frac{\gamma_{gl}}{\nu}\big(\tfrac{1}{3}h^3 + h^2 k(h)\big)\partial_z^3 h. \tag{1.1.10}$$

Combining the incompressibility condition (1.1.1c) and the kinematic boundary condition $\partial_t h + u\partial_z h = v$ (which ensures that the fluid stays on the free surface) gives

$$\partial_t h + \partial_z(\overline{u}h) = 0.$$

Substituting (1.1.10) into this equation leads to

$$\partial_t h + \frac{\gamma_{gl}}{\nu}\partial_z\big((\tfrac{1}{3}h^3 + \lambda^{3-n}h^n)\partial_z^3 h\big) = 0,$$

where we used the assumption $k(h) = \lambda^{3-n}h^{n-2}$ for the slip model. By a rescaling of the variables, the constants can be eliminated. Moreover, for $n \in (0,3)$ the term h^n dominates over the h^3 term, and therefore we arrive at the thin-film equation

$$\partial_t h + \partial_z\big(h^n \partial_z^3 h\big) = 0.$$

1.2 Overview of this Thesis

The rest of this thesis consists of the following four chapters:

In chapter 2 the prerequisite knowledge needed in this thesis is presented. This chapter is subdivided in the sections on functional analysis in section 2.1, semi-group theory in section 2.2 and interpolation theory in section 2.3.

In chapter 3 the setting and main result are explained. First, we derive the nonlinear Cauchy problem in section 3.1. Next, in section 3.2 the functional-analytic setting is discussed, and in section 3.3 the main result obtained in this thesis is stated.

Chapter 4 discusses maximal regularity for the linear abstract Cauchy problem. This is divided in the following sections: in section 4.1 the inhomogeneous equation of the abstract Cauchy problem is treated. Then, in section 4.2 the homogeneous equation of the abstract Cauchy problem is discussed. Sections 4.3 and 4.4 discuss parabolic maximal L^p-regularity and elliptic regularity, respectively.

Lastly, in chapter 5 the nonlinear problem is treated. This is concluded in section 5.1, where the proof of the main result is given.

Chapter 2

Prerequisites

This chapter gives an overview of the theory necessary for the analysis of the thin-film equation (1.0.1). This is divided in the sections on functional analysis in section 2.1, semigroup theory in section 2.2 and interpolation theory in section 2.3. Most of these results can also be found in the literature, an example of a book or article where it can be found is then mentioned. When the proofs may be insightful, or the statement as is is not found in the literature, the proofs are added. In other cases, the proofs can be found in the literature as mentioned and are not worked out for conciseness.

Notation
We write $a \lesssim_P b$ if there exists a constant $C \geq 1$ only depending on the parameters in the set P such that $a \leq Cb$. Similarly, we write $a \sim_P b$ if both $a \lesssim_P b$ and $b \lesssim_P a$. If P is the empty set, the subscript P is left out.

2.1 Functional Analysis

We start with defining a couple of useful spaces. We use that Ω is a domain in $\mathbb{R}^n$ and denote by $\mathcal{L}(X)$ the bounded linear operators mapping from X to itself.

Definition 2.1.1. *(L^p-spaces) For $1 \leq p < \infty$, L^p is defined as the set of all measurable functions u defined on Ω which satisfy*

$$\|u\|_{L^p(\Omega)}^p := \int_\Omega |u(x)|^p dx < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ is the set of all measurable functions u which satisfy

$$\|u\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |u(x)| < \infty.$$

The L^p-spaces, with $1 \leq p \leq \infty$ are Banach spaces with respect to the norms as defined above. For $p = 2$, $L^2(\Omega)$ is a Hilbert space, with inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)\overline{v(x)}dx \quad \text{for } u, v \in L^2(\Omega).$$

Definition 2.1.2. *(Sobolev spaces) Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define for $1 \leq p < \infty$ and $m \in \mathbb{N}_0$ the Sobolev space*

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\},$$

where $\partial^\alpha u$ is the weak derivative. This space is a Banach space with the norm

$$\|u\|^p_{W^{m,p}(\Omega)} := \sum_{0\le|\alpha|\le m} \|\partial^\alpha u\|^p_{L^p(\Omega)}.$$

Similarly as for the L^p-spaces, it holds that for $p = 2$, $W^{m,2}(\Omega)$ is a Hilbert space with inner product

$$(u, w)_{W^{m,2}(\Omega)} = \sum_{0\le|\alpha|\le k} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Omega)} \quad \text{for } u, v \in W^{m,2}(\Omega).$$

Definition 2.1.3. *(Bochner spaces $L^p(0, T; X)$)[23, definition 1.2.15] For $1 \le p < \infty$ and $T \in [0, \infty]$ define $L^p(0, T; X)$ as the space of all measurable functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that*

$$\int_0^T \|f\|^p_X dt < \infty,$$

where X is a Banach space. Endowed with the norm

$$\|f\|_{L^p(0,T;X)} := \left(\int_0^T \|f\|^p_X dt\right)^{\frac{1}{p}},$$

the spaces $L^p(0, T; X)$, $1 \le p < \infty$ are Banach spaces.

Theorem 2.1.4 (Lax-Milgram (complex version))**.** *[2, Thm 6.2] Let X be a Hilbert space over $\mathbb{C}$ and let $a : X \times X \to \mathbb{C}$ be a sesquilinear mapping. Assume that there exist constants c_0 and C_0 with $0 < c_0 \le C_0 < \infty$ such that for all $x, y \in X$*

- $|a(x, y)| \le C_0\|x\|_X\|y\|_X$, *(Continuity/Boundedness)*
- $\Re a(x, x) \ge c_0\|x\|^2$, *(Coercivity)*

Then there exists a unique map $B : X \to X$ such that

$$a(y, x) = (y, Bx)_X \quad \text{for all } x, y \in X.$$

In addition, $B \in \mathcal{L}(X)$ is an invertible operator with

$$\|B\| \le C_0 \quad \text{and} \quad \|B^{-1}\| \le \frac{1}{c_0}.$$

For showing some of the results we will need to use the Fourier transform, which is defined as follows:

Definition 2.1.5. *For functions in the Schwartz space*

$$\mathcal{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}; \mathbb{C}) : \forall k, l \in \mathbb{N}_0,\ \sup_{x\in\mathbb{R}} |x^k \partial^l_x f(x)| < \infty\},$$

we define the Fourier transform as:

$$(\mathcal{F}f)(\xi) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(x)e^{-ix\xi}dx \quad \text{for } \xi \in \mathbb{R}.$$

The inverse Fourier transform is given by

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi)e^{ix\xi}d\xi \quad \text{for } x \in \mathbb{R}.$$

On the Schwartz space, the Fourier transform is a bijection. By density of the Schwartz space in $L^p(\mathbb{R}; \mathbb{C})$, we can also define the Fourier transform on $L^p(\mathbb{R}; \mathbb{C})$.

Theorem 2.1.6. *(Plancherel) For $f, g \in L^2(\mathbb{R}, \mathbb{C})$,*

$$(\mathcal{F}f, \mathcal{F}g)_{L^2(\mathbb{R};\mathbb{C})} = (f, g)_{L^2(\mathbb{R};\mathbb{C})}.$$

2.2 Semi-group Theory

In the treatment of (1.0.1), we will often make use of the theory of semi-groups. More precisely, we will use the theory of analytic semi-groups. As such, the necessary theory is presented here. Note that both $T(t)$ and e^{tA} are used to denote the semi-group, where A is the generator which will be defined below. We start with introducing the notion of a semi-group:

Definition 2.2.1. *(C^0-semigroup)[13, definition I, 5.1; definition II, 1.2] A family $(T(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a strongly continuous, or C^0-semigroup if*

- $T(0) = I$,
- $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$,
- $t \mapsto T(t)$ *is continuous from* $\mathbb{R}^+$ *to* X, *i.e.,* $\lim_{t\downarrow 0} \|T(t)x - x\|_X = 0$ *for all* $x \in X$.

The generator of T is the linear operator A with domain $D(A)$, defined by

$$D(A) = \{x \in X : \lim_{t\downarrow 0}(T(t)x - x) \text{ exists}\},$$
$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x), \quad \text{for } x \in D(A).$$

From this definition, we can see that the notation e^{tA} for the semigroup is a natural one. This is because the first two properties show that the semigroup 'behaves in the same way' as the matrix exponential would if A where a matrix.

Definition 2.2.2. *(Analytic semigroup)[31, Def 2.0.2] Let $A : X \supset D(A) \to X$ be a sectorial operator, i.e., the resolvent set of A contains a sector*

$$S = \{\lambda \in \mathbb{C} : \ \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\},$$

with $\omega \in \mathbb{R}$, $\theta > \frac{\pi}{2}$ and there exists an $0 < M < \infty$ such that

$$\|\lambda R(\lambda, A)\| \leq M, \quad \lambda \in S \quad \text{(resolvent estimate)}.$$

Here, X and $D(A)$ are Banach spaces. The family $\{e^{tA} : t \geq 0\}$, with

$$e^{tA} = \frac{1}{2\pi i} \int_{\omega+\gamma_{r,\eta}} e^{t\lambda} R(\lambda, A) d\lambda, \quad t > 0, \tag{2.2.1}$$

where $\gamma_{r,\eta}$ is the curve $\{\lambda \in \mathbb{C} \ : \ |arg\lambda| = \eta, \ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} \ : \ |\arg\lambda| \leq \eta, \ |\lambda| = r\}$, oriented counterclockwise as in figure 2.1, is said to be the analytic semigroup generated by A in X. In figure 2.1, the spectrum is on the left of the blue lines, and the oriented contour is given by the red line.

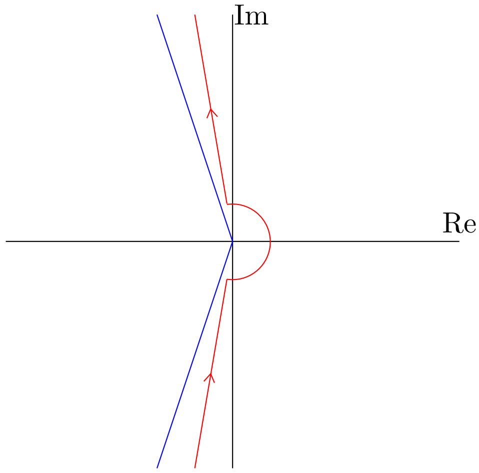


Figure 2.1: Curve around the spectrum

From this definition, we see that to show A generates an analytic semigroup, it is required that A is sectorial and that the resolvent estimate is satisfied. In the following proposition, an equivalent statement for the sectoriality property is formulated.

Proposition 2.2.3. *[31, Prop 2.1.11] Let $A : X \supset D(A) \to X$ be a linear operator such that $\rho(A)$ contains a half plane $\{\lambda \in \mathbb{C} : \Re\lambda \geq \omega\}$, and*

$$\|\lambda R(\lambda, A)\| \leq M, \quad \text{for } \Re\lambda \geq \omega,$$

with $\omega \in \mathbb{R}$ and $0 < M < \infty$. Then A is a sectorial operator.

Next, we collect some standard results for analytic semigroups.

Proposition 2.2.4. *[31, proposition 2.1.1] Let A be the generator of the analytic semigroup $T(t)$. Then, T has the following properties:*

- *$\frac{d^k}{dt^k}T(t) = A^kT(t)$ for $t > 0$ and $k \in \mathbb{N}_0$,*
- *There are constants $M_0, M_1, \ldots, M_k$ such that*

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq M_0e^{\omega t} \quad \text{for } t > 0, \tag{2.2.2}$$
$$\|t^k(A - \omega I)^ke^{tA}\|_{\mathcal{L}(X)} \leq M_ke^{\omega t} \quad \text{for } t > 0, \tag{2.2.3}$$

where ω is as in definition 2.2.2.

Now define the notion of intermediate spaces between the Banach spaces X and $D(A)$, which we need for the following lemma.

Definition 2.2.5. *[31] Let $A : D(A) \supset X \to X$ be the generator of an analytic semigroup. Define for $0 < \alpha < 1$, $1 \leq p \leq \infty$ and $(\alpha, p) = (1, \infty)$ the following notion of intermediate spaces between X and $D(A)$:*

$$\begin{cases} D_A(\alpha, p) = \{x \in X : t \mapsto v(t) := \|t^{1-\alpha-1/p}Ae^{tA}x\| \in L^p(0, 1)\} \\ \|x\|_{D_A(\alpha,p)} = \|x\| + [x]_{D_A(\alpha,p)} = \|x\| + \|v\|_{L^p(0,1)}. \end{cases}$$

Lemma 2.2.6. *Let $A : D(A) \supset X \to X$ generate an analytic semigroup, with $\rho(A) \supset \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \theta\}$, where X and $D(A)$ are Banach spaces. For*

$$\partial_tu - Au = 0, \tag{2.2.4}$$
$$u(0) = u^{(0)}, \tag{2.2.5}$$

with $u^{(0)} \in D_A(1-\frac{1}{p},p)$, the following estimate holds:

$$\|\partial_t u(t)\|^p_{L^p(0,\infty;X)} \lesssim_p [u^{(0)}]^p_{D_A(1-\frac{1}{p},p)} + \|u^{(0)}\|^p_X \lesssim \|u^{(0)}\|^p_{D_A(1-\frac{1}{p},p)}. \tag{2.2.6}$$

Here, $D_A(1-\frac{1}{p},p)$ is defined as in definition 2.2.5.

Proof. Equation (2.2.4) has the mild solution $u(t) = T(t)u^{(0)}$. Hence, we can rewrite the equation as

$$\partial_t u(t) = AT(t)u^{(0)} \text{ for } t > 0,$$

using the fact that A generates an analytic semigroup. We see in writing

$$\begin{aligned}\|\partial_t u(t)\|^p_{L^p(0,\infty;X)} &= \|AT(t)u^{(0)}\|^p_{L^p(0,\infty;X)} \\ &= \|AT(t)u^{(0)}\|^p_{L^p(0,1,X)} + \|AT(t)u^{(0)}\|^p_{L^p(1,\infty;X)},\end{aligned}$$

that the first term in the right-hand side corresponds to $[u^{(0)}]^p_{D_A(1-\frac{1}{p},p)}$. For the second term we obtain

$$\|AT(t)u^{(0)}\|^p_{L^p(1,\infty,X)} = \int_1^\infty \|AT(t)u^{(0)}\|^p_X dt \overset{(2.2.3),\, \omega=0}{\leq} \int_1^\infty (ct^{-1})^p \|u^{(0)}\|^p_X dt \lesssim_p \|u^{(0)}\|^p_X,$$

and the statement follows. □

2.2.1 Maximal Regularity

We define the notion of maximal L^p-regularity for an operator which is the generator of a bounded analytic semigroup. If an operator has this property, then we get an estimate on the unknown function of a corresponding differential equation. This estimate will be very useful in proving existence and uniqueness for solutions to a suitably transformed version of (1.0.1).

Definition 2.2.7. *[29] Consider the equation*

$$\partial_t u - Au = f, \tag{2.2.7}$$

$$u|_{t=0} = u^{(0)}. \tag{2.2.8}$$

Let A be the generator of a bounded analytic semigroup on a Banach space X. The operator A has maximal L^p-regularity for $p \in (1,\infty)$ and time in $[0,\infty)$ if for $u_0 = 0$ and $f \in L^p([0,\infty);X)$, the solution of (2.2.7) is differentiable almost everywhere, takes its values in $D(A)$ almost everywhere, and $\partial_t u$ and Au belong to $L^p([0,\infty);X)$. Then, by the closed graph theorem we have the following estimate

$$\|\partial_t u\|_{L^p([0,\infty);X)} + \|Au\|_{L^p([0,\infty);X)} \lesssim_p \|f\|_{L^p([0,\infty);X)}.$$

The next result that we state will be very helpful, because from this we get a condition on A from which it directly follows that A has maximal L^p-regularity. This condition is that A should generate a bounded analytic semigroup. We will show later on that indeed our operator A satisfies this. The original result can be found in [11] (in Italian), and this English translation is taken from [29, corollary 1.7].

Corollary 2.2.8. *[11, 29] Every generator of a bounded analytic semigroup on a Hilbert space X has maximal L^p-regularity for $1 < p < \infty$.*

For showing this result, we need the following theorem:

Theorem 2.2.9. *[29, theorem 1.6] Let X be a Hilbert space. Assume that for $m \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X))$ the sets*

$$\{m(u) : u \in \mathbb{R} \setminus \{0\}\} \quad \text{and} \quad \{um'(u) : u \in \mathbb{R} \setminus \{0\}\}$$

are bounded in $\mathcal{L}(X)$. Then the Fourier multiplier operator

$$T_m f = \mathcal{F}^{-1}(m(\cdot)\hat{f}(\cdot)), \quad f \in \mathcal{S}(\mathbb{R}, X),$$

extends to a bounded operator T_m on $L^p(\mathbb{R}, X)$.

Proof of corollary 2.2.8. This proof is taken from from [29, section 1.5]. Consider

$$\begin{aligned} \partial_t u - Au &= f, \\ u|_{t=0} &= 0, \end{aligned}$$

and let $f \in C_c^\infty(\mathbb{R}^+, D(A))$. Note that this space is dense in $L^p(\mathbb{R}^+, X)$. The above problem has a solution that is given by the mild solution formula

$$u(t) = \int_0^t T(t-s)(f(s))ds.$$

Because A generates an analytic semigroup, we have $\frac{d}{dt}T(t) = AT(t)$ and hence

$$\partial_t u(t) = \int_0^t AT(t-s)f(s)ds + f(t).$$

This implies that A has maximal L^p-regularity if and only if

$$Kf(t) := \int_0^t AT(t-s)f(s)ds, \quad f \in C_c^\infty(\mathbb{R}^+, D(A)) \tag{2.2.9}$$

extends to a bounded operator $K : L^p(\mathbb{R}^+, X) \to L^p(\mathbb{R}^+, X)$. Applying the Fourier transform to (2.2.9) gives

$$\widehat{Kf}(u) = \widehat{(AT(t))}(u) \left[\hat{f}(u)\right], \quad u \in \mathbb{R}.$$

Using that (see e.g. [31, lemma 2.1.6])

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t) dt, \text{ for } \Re(\lambda) > 0,$$

gives

$$m(u) := \widehat{(AT(t))}(u) = AR(iu, A) = iuR(iu, A) - I.$$

Because A generates a bounded analytic semigroup, $m(u)$ is bounded on $\mathbb{R} \setminus \{0\}$. The same holds for

$$um'(u) = -iuAR(iu, A)^2 = [uR(iu, A)]^2 + iuR(iu, A).$$

Applying theorem 2.2.9 gives the result.

□

2.2.2 Hardy's inequality

The following lemma is a corollary of the famous weighted Hardy inequalities (see e.g. [21]). The result can also be proved using these inequalities, but the proof given below is slightly shorter. This lemma is a technical result that is needed later on when obtaining the maximal regularity estimate.

Lemma 2.2.10. *For $g \in C_c^\infty((0,\infty))$ and $\omega \neq 0$ we have the following inequality*

$$\int_0^\infty x^{2\omega} g^2 \frac{dx}{x} \leq \frac{1}{\omega^2} \int_0^\infty x^{2\omega} (x\partial_x g)^2 \frac{dx}{x}. \tag{2.2.10}$$

Proof. Consider

$$\int_0^\infty x^{2\omega}(x\partial_x g)^2 \frac{dx}{x} = \int_0^\infty [(x\partial_x - \omega)\underbrace{x^\omega g}_{=:\tilde{g}}]^2 \frac{dx}{x} = \int_0^\infty (x\partial_x \tilde{g})^2 - 2\omega \tilde{g} x \partial_x \tilde{g} + \tilde{g}^2 \omega^2 \frac{dx}{x} =: I.$$

Noting that

$$\int_0^\infty 2\omega \tilde{g} x \partial_x \tilde{g} \frac{dx}{x} = \int_0^\infty \omega \partial_x \tilde{g}^2 dx$$

and using that $\tilde{g} \in C_c^\infty((0,\infty))$ shows that this integral vanishes. Hence, we are left with

$$I = \int_0^\infty (x\partial_x \tilde{g})^2 + \omega^2 \tilde{g}^2 \frac{dx}{x} \geq \omega^2 \int_0^\infty x^{2\omega} g^2 \frac{dx}{x}.$$

Dividing by ω^2 gives the result. $\square$

2.3 Interpolation Theory

Below we will outline several methods to construct interpolation spaces and show that their corresponding norms are equivalent.

Definition 2.3.1. *[31, Def 1.2.1] Let X and Y be Banach spaces such that $X \subset Y$. For every $x \in X$ and $t > 0$, set*

$$K(t, x, X, Y) = \inf_{x=a+b, a\in X, b\in Y} (\|a\|_X + t\|b\|_Y).$$

If there is no danger of confusion, we shall write $K(t,x)$ instead of $K(t,x,X,Y)$.

Definition 2.3.2. *(The K-method for interpolation) [31, Def 1.2.2] Let X and Y be Banach spaces such that $X \subset Y$. Let $0 < \theta \leq 1$, $1 \leq p \leq \infty$, and set*

$$\begin{cases} (X,Y)_{\theta,p} = \left\{x \in X : t \mapsto t^{-\theta-1/p} K(t,x,X,Y) \in L^p(0,\infty)\right\}, \\ \|x\|_{(X,Y)_{\theta,p}} = \|t^{-\theta-1/p} K(t,x,X,Y)\|_{L^p(0,\infty)}. \end{cases}$$

Here, the L^p norms are in the time variable.

Definition 2.3.3. *(The trace method for interpolation)[31, Def 1.2.8] For $0 \leq \theta < 1$ and $1 \leq p \leq \infty$ set*

$$\begin{aligned} V(p,\theta,Y,X) =& \{u : \mathbb{R}^+ \to X : \ t \mapsto u_\theta(t) = t^{\theta - \frac{1}{p}} u(t) \in L^p(0,\infty,Y), \\ & t \mapsto v_\theta(t) = t^{\theta-\frac{1}{p}} \partial_t u(t) \in L^p(0,\infty,X)\} \end{aligned}$$

with

$$\|u\|_{V(p,\theta,Y,X)} = \|u_\theta\|_{L^p(0,\infty;Y)} + \|v_\theta\|_{L^p(0,\infty;X)}.$$

The following corollary is necessary for showing that interpolation via the K-method and the trace method are equivalent.

Corollary 2.3.4. *[31, Corollary 1.2.9] Let u be a function such that $t \mapsto u_\theta(t) = t^{\theta-1/p}u(t)$ belongs to $L^p(0,a;X)$, with $0 < a \leq \infty$, $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then also the mean value*

$$v(t) = \frac{1}{t}\int_0^t u(s)ds, \quad t > 0$$

has the same property, and setting $v_{\theta(t)} = t^{\theta-1/p}v(t)$ we have

$$\|v_\theta\|_{L^p(0,a,X)} \leq \frac{1}{1-\theta}\|u_\theta\|_{L^p(0,a;X)}.$$

Proposition 2.3.5. *(Equivalence of K-method and trace method)[31, Proposition 1.2.10] For $(\theta,p) \in (0,1)\times[1,\infty]\cup\{(1,\infty)\}$, $(X,Y)_{\theta,p}$ is the set of traces at $t=0$ of the functions in $V(p,1-\theta,Y,X)$, and the norm*

$$\|x\|_{\theta,p}^T = \inf\{\|u\|_{V(p,1-\theta,Y,X)} : x = u(0), u \in V(p,1-\theta,Y,X)\}$$

is an equivalent norm in $(X,Y)_{\theta,p}$.

Proof. This proof has the same structure as the proof of [31, Proposition 1.2.10]. First, we show that for $x \in (X,Y)_{\theta,p}$ it holds that x is the trace at $t=0$ of a function $v \in V(p,1-\theta,Y,X)$. For this, let $x \in (X,Y)_{\theta,p}$. For all $n \in \mathbb{N}$, let a_n and b_n be so that $a_n + b_n = x$ and additionally

$$\|a_n\|_X + \frac{1}{n}\|b_n\|_Y \leq 2K(\tfrac{1}{n},x).$$

For $t > 0$, define

$$u(t) := \sum_{n=1}^{\infty} b_{n+1}\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n})}(t) = \sum_{n=1}^{\infty}(x-a_{n+1})\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n})}(t), \tag{2.3.1}$$

and

$$v(t) = \frac{1}{t}\int_0^t u(s)ds. \tag{2.3.2}$$

Because $(X,Y)_{\theta,p} \subset (X,Y)_{\theta,\infty}$ it follows that $\lim_{t\to 0} K(t,x) = 0$. Then, it is in particular also true that $x = \lim_{n\to\infty} b_n$, and hence $x = \lim_{t\to 0} u(t) = \lim_{t\to 0} v(t)$. Furthermore,

$$\|t^{1-\theta}u(t)\|_Y \leq t^{1-\theta}\sum_{n=1}^{\infty}\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n})}(t)2(n+1)K(\tfrac{1}{n},x) \leq 4t^{-\theta}K(t,x).$$

From this, it follows that $t \mapsto t^{1-\theta-\frac{1}{p}}u(t) \in L^p(0,\infty;Y)$. From corollary 2.3.4 it then follows that also $t \mapsto t^{1-\theta-\frac{1}{p}}v(t) \in L^p(0,\infty;Y)$ and

$$\|t^{1-\theta-\frac{1}{p}}v\|_{L^p(0,\infty;Y)} \leq 4\theta^{-1}\|x\|_{\theta,p}.$$

From (2.3.1) and (2.3.2) we see that

$$v(t) = x - \frac{1}{t}\int_0^t \sum_{n=1}^{\infty}\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n})}(s)a_{n+1}ds,$$

hence v is differentiable a.e. with values in X. Noting that

$$v'(t) = \frac{1}{t^2} \int_0^t g(s)ds - \frac{1}{t} g(t),$$

where $g(t) = \sum_{n=1}^{\infty} \mathbb{1}_{(\frac{1}{n+1}, \frac{1}{n})}(t) a_{n+1}$ satisfies

$$\|g(t)\|_X \le t^{-\theta} \sum_{n=1}^{\infty} \mathbb{1}_{(\frac{1}{n+1}, \frac{1}{n})}(t) 2K(\frac{1}{n+1}, x) \le 2K(t, x),$$

it follows that

$$\|t^{1-\theta} v'(t)\| \le t^{-\theta} \sup_{0<s<t} \|g(s)\| + \|t^{-\theta} g(t)\| \le 4t^{-\theta} K(t, x).$$

From this, it follows that $t \mapsto t^{1-\theta-\frac{1}{p}} v'(t) \in L^p(0, \infty; X)$ and

$$\|t^{1-\theta-\frac{1}{p}} v'\|_{L^p(0,\infty;X)} \le 4\|x\|_{\theta,p}.$$

Hence, there exists a function $v \in V(p, 1-\theta, Y, X)$ such that x is the trace of this function at $t = 0$. Additionally,

$$\|x\|_{\theta,p}^T \le 2(2 + \tfrac{1}{\theta})\|x\|_{\theta,p}.$$

Now, we show the opposite direction: assuming x is the trace of a function $u \in V(p, 1-\theta, Y, X)$ we can prove that $x \in (X, Y)_{\theta,p}$. For this, let x be the trace of a function $u \in V(p, 1-\theta, Y, X)$. Then, we have that

$$x = x - u(t) + u(t) = -\int_0^t u'(s)ds + u(t) \quad \text{for all } t > 0.$$

This implies that

$$t^{-\theta} K(t, x) \le t^{1-\theta} \|\frac{1}{t} \int_0^t u'(s)ds\|_X + t^{1-\theta} \|u(t)\|_Y.$$

From corollary 2.3.4 it follows that $t \mapsto t^{-\theta-\frac{1}{p}} K(t, x) \in L^p(0, \infty)$. Hence, $x \in (X, Y)_{\theta,p}$ and

$$\|x\|_{\theta,p} \le \frac{1}{\theta} \|x\|_{\theta,p}^T.$$

□

Now we will see that there is a connection between interpolation between X and $D(A)$, where $D(A)$ is the domain of an analytic semigroup $A : X \supset D(A) \to X$ and the intermediate spaces of X and $D(A)$ as defined in 2.2.5.

Proposition 2.3.6. *[31, Prop 2.2.2] For $0 < \alpha < 1$ and $1 \le p \le \infty$, and for $(\alpha, p) = (1, \infty)$ we have*

$$D_A(\alpha, p) = (X, D(A))_{\alpha,p},$$

with equivalence of the respective norms.

Proof. This proof is taken from [31, proposition 2.2.2]. We first show that $D_A(\alpha, p) \subset (X, D(A))_{\alpha,p}$. Let $x \in D_A(\alpha, p)$ and let $\phi : [0, \infty) \to \mathbb{R}$ be a smooth cut-off function, where $\phi(t) = 1$ for $t \le \frac{1}{3}$ and $\phi = 0$ for $t \ge 1$. Take $u(t) = \phi(t)e^{tA}x$, it then follows that $x = u(0)$, $u(t) = 0$ for $t \ge 0$, and for $0 < t \le 1$ we have that

$$\begin{cases} \|t^{1-\alpha}u(t)\|_{D(A)} \le \|t^{1-\alpha}Ae^{tA}x\| + \|t^{1-\alpha}e^{tA}x\|, \\ \|t^{1-\alpha}u'(t)\|_X \le \|t^{1-\alpha}Ae^{tA}x\| + \|\phi'\|_{L^\infty(0,1)}\|t^{1-\alpha}e^{tA}x\|. \end{cases}$$

So, we see that $u \in V(p, 1-\alpha, D(A), X)$. Additionally, we see from proposition 2.3.5 that $x \in (X, D(A))_{\alpha,p}$, and

$$\|x\|^T_{\alpha,p} \le 2[x]_{D_A(\alpha,p)} + 3c_p\|x\|, \tag{2.3.3}$$

where c_p is a constant.
Conversely, we show that $(X, D(A))_{\alpha,p} \subset D_A(\alpha, p)$. Let $x \in (X, D(A))_{\alpha,p}$. Then $x = u(0)$, where $u \in V(p, 1-\alpha, D(A), X)$, and it follows that

$$\begin{aligned} \|t^{1-\alpha}Ae^{tA}x\| &\le \|t^{1-\alpha}Ae^{tA}u(t)\| + \|t^{1-\alpha}Ae^{tA}\int_0^t u'(s)ds\| \\ &\le C_0\|t^{1-\alpha}Au(t)\| + C_1\|t^{1-\alpha}t^{-1}\int_0^\infty u'(s)ds\|. \end{aligned}$$

Here, C_0 and C_1 are constants. From corollary 2.3.4, we know that $t \mapsto \|t^{1-\alpha-\frac{1}{p}}Ae^{tA}x\|$ is in $L^p(0,1)$, so

$$\begin{aligned} \|t^{1-\alpha-\frac{1}{p}}Ae^{tA}x\|_{L^p(0,1)} &\le C_0\|t^{1-\alpha-\frac{1}{p}}Au(t)\|_{L^p(0,1)} + \alpha^{-1}C_1\|t^{1-\alpha-\frac{1}{p}}u'(t)\|_{L^p(0,1)} \\ &\le \max(C_0, \alpha^{-1}C_1)\|x\|^T_{\alpha,p}. \end{aligned}$$

This estimate also holds for $p = \infty$ when using the convention $1/\infty = 0$. Hence, $D_A(\alpha, p)$ is continuously embedded in $(X, D(A))_{\alpha,p}$, $1 \le p \le \infty$. ☐

Furthermore, we state two technical results for interpolation spaces.

Theorem 2.3.7. *[20, Thm 1.1] Let X and Y be two Banach spaces that are continuously embedded in the same linear Hausdorff space and let $0 < \theta < 1$ and $1 \le p \le \infty$. Then the following identity holds:*

$$(X, X \cap Y)_{\theta,p} = (X, Y)_{\theta,p} \cap X.$$

Theorem 2.3.8. *[36] Let X and Y be two Banach spaces. If $X \hookrightarrow Y$, then for $0 < \theta < \tilde{\theta} < 1$ and $1 \le q, \tilde{q} \le \infty$ it holds that*

$$(X, Y)_{\theta,q} \hookrightarrow (X, Y)_{\tilde{\theta},\tilde{q}}.$$

Finally, we introduce the notion of Besov spaces. These spaces are related to certain interpolation spaces, and we will need them in later chapters.

Definition 2.3.9. *[1, 7.32] Let $1 \le p < \infty$, $1 \le q \le \infty$, $0 \le \theta \le 1$ and $0 \le k < s < m$, where $s = (1-\theta)k + \theta m$. We define the Besov space $B^s_{p,q}(\Omega)$ on the domain Ω as follows*

$$B^s_{p,q}(\Omega) = (W^{k,p}(\Omega), W^{m,p}(\Omega))_{\theta,q;J} \sim (W^{k,p}(\Omega), W^{m,p}(\Omega))_{\theta,q}.$$

Here, the J denotes interpolation with respect to the J-method (see e.g. [4, section 3.2]), which is equivalent to interpolation with the K-method (see e.g. [4, theorem 3.3.1]).

Chapter 3

Setting and Main Result

3.1 Derivation of the Nonlinear Cauchy Problem

In this chapter the free-boundary problem

$$
\begin{aligned}
h_t + (h^n h_{zzz})_z &= 0 && \text{for } t > 0,\ z > Z(t), && \text{(3.1.1a)}\\
h = h_z &= 0 && \text{for } t > 0,\ z = Z(t), && \text{(3.1.1b)}\\
\lim_{z \downarrow Z(t)} h^{n-1} h_{zzz} &= Z_t(t) && \text{for } t > 0, && \text{(3.1.1c)}
\end{aligned}
$$

is rewritten as a nonlinear Cauchy problem which will be studied in the later chapters. By means of the von Mises transform and several rescalings we will reformulate the thin-film equation with general mobility. We treat the cases of the mobility exponent $n \in (1, \frac{3}{2})$ and $n \in (\frac{3}{2}, 3)$ separately. The value $n = \frac{3}{2}$ is excluded, because treating this case is more delicate due to resonances that occur [5].

3.1.1 Reformulation for $n \in (1, \frac{3}{2})$

For the mobility exponent $n \in (1, \frac{3}{2})$ the generic solution of the free boundary value problem has up to rescaling and translation a quadratic profile $h \approx z^2$. We will linearise around such a profile using the von Mises transform. The idea of this transform is to change the role of the dependent and independent variables, i.e., instead of considering h as a function of t and z we will view t and z to be dependent on $y = h$. We set

$$h(t, Z(t, y)) = y^2 \quad \text{for } t, y > 0. \tag{3.1.2}$$

Note that the profile y^2 is strictly monotone for $y > 0$ so that the transform is well-defined. Differentiating equation (3.1.2) with respect to t gives by the chain rule

$$h_t + h_z Z_t = 0 \overset{(3.1.1a)}{\iff} -(h^n h_{zzz})_z + h_z Z_t = 0 \quad \text{for } t, y > 0. \tag{3.1.3}$$

On the other hand, differentiating $h(t, Z(t, y))$ with respect to y, we see using $z = Z(t, y)$, that

$$h_y(t, Z(t, y)) = h_z(t, Z(t, y)) Z_y(t, y) \tag{3.1.4}$$

and thus

$$\partial_z = \frac{1}{Z_y} \partial_y. \tag{3.1.5}$$

Using (3.1.2) and (3.1.5) in (3.1.3) we deduce that

$$Z_t \frac{2y}{Z_y} - \frac{1}{Z_y} \partial_y y^{2n} \frac{1}{Z_y} \partial_y \frac{1}{Z_y} \partial_y \frac{1}{Z_y} 2y = 0 \quad \text{for } t, y > 0,$$

which is equivalent to

$$Z_t - \frac{1}{y}\partial_y y^{2n}\frac{1}{Z_y}\partial_y\frac{1}{Z_y}\partial_y\frac{1}{Z_y}y = 0 \quad \text{for } t, y > 0. \tag{3.1.6}$$

We now introduce the new variable $H := \frac{1}{Z_y}$ and note that $Z_{ty} = Z_{yt} = -H^{-2}H_t$. It should also be noted that the quadratic profile $h = z^2$ corresponds to $H = 1$. Using the definition of H and differentiating (3.1.6) with respect to y we get

$$-\frac{1}{H^2}H_t - \partial_y\frac{1}{y}\partial_y y^{2n}H\partial_y H\partial_y Hy = 0 \quad t, y > 0,$$

or equivalently

$$H_t + H^2\partial_y\frac{1}{y}\partial_y y^{2n}H\partial_y H\partial_y Hy = 0 \quad t, y > 0. \tag{3.1.7}$$

Note that

$$H^2\partial_y\frac{1}{y}\partial_y y^{2n}H\partial_y H\partial_y yH = y^{2n-4}H^2(y\partial_y + 2n - 3)(y\partial_y + 2n - 1)H(y\partial_y)H(y\partial_y + 1)H,$$

and this can be used to rewrite (3.1.7) as

$$y^{4-2n}H_t + H^2(y\partial_y + 2n - 3)(y\partial_y + 2n - 1)H(y\partial_y)H(y\partial_y + 1)H = 0 \quad \text{for } t, y > 0. \tag{3.1.8}$$

Finally, we apply one more change of variables $x := \frac{y^{4-2n}}{(4-2n)^4}$. It holds that $y\partial_y = (4-2n)D$, with $D = x\partial_x$. Rewriting (3.1.8) and dividing by $(4-2n)^4$ (which is allowed since $n \neq 2$ for $n \in (1, \frac{3}{2})$) gives the equation

$$xH_t + \mathcal{M}_n(H, H, H, H, H) = 0,$$

where

$$\mathcal{M}_n(H_1, H_2, H_3, H_4, H_5) = H_1H_2\left(D + \frac{2n-3}{4-2n}\right)\left(D + \frac{2n-1}{4-2n}\right)H_3DH_4\left(D + \frac{1}{4-2n}\right)H_5.$$

Linearising around the quadratic profile $u := H - 1$ now gives us the following nonlinear Cauchy problem

$$u_t + x^{-1}p_n(D)u = \mathcal{N}_n(u) \qquad t, x > 0, \tag{3.1.9a}$$

$$u|_{t=0} = u^{(0)} \qquad t = 0,\ x > 0, \tag{3.1.9b}$$

with the linear operator

$$\begin{aligned} p_n(D)u &= \mathcal{M}_n(u, 1, \ldots, 1) + \cdots + \mathcal{M}_n(1, \ldots, 1, u) \\ &= D\left(D - \frac{3-2n}{4-2n}\right)\left(D - \frac{1-2n}{4-2n}\right)\left(D - \frac{-2}{4-2n}\right)u, \end{aligned} \tag{3.1.10}$$

and the nonlinear part

$$\mathcal{N}_n(u) = -x^{-1}\mathcal{M}_n(u+1, \ldots, u+1) + x^{-1}p_n(D)u. \tag{3.1.11}$$

Hence, p_n is a fourth order polynomial and for $n \in (1, \frac{3}{2})$ the zeros are, in increasing order, equal to

$$\gamma_1 := \frac{-2}{4-2n}, \quad \gamma_2 := \frac{1-2n}{4-2n}, \quad \gamma_3 := 0, \quad \gamma_4 := \frac{3-2n}{4-2n}. \tag{3.1.12}$$

Finally, we note that that we do not have to impose boundary conditions on the Cauchy problem since the boundary conditions (3.1.1b) and (3.1.1c) are implicitly fulfilled by the von Mises transform (3.1.2).

3.1.2 Reformulation for $n \in (\frac{3}{2}, 3)$

For the mobility exponent $n \in (\frac{3}{2}, 3)$ the generic solution of the free-boundary problem is a travelling wave $h(t,z) = H_{\mathrm{TW}}(x)$ where $x = z - Vt$ with $Z_t(t) = V < 0$ the constant velocity of the film. This change of coordinates implies

$$\partial_z h = \frac{d}{dx} H_{\mathrm{TW}} \quad \text{and} \quad \partial_t h = -V \frac{d}{dx} H_{\mathrm{TW}} \tag{3.1.13}$$

and this turns (3.1.1a) into the ODE

$$\begin{aligned} -V \frac{dH_{\mathrm{TW}}}{dx} + \frac{d}{dx}\left(H_{\mathrm{TW}}^n \frac{d^3 H_{\mathrm{TW}}}{dx^3} \right) &= 0 \qquad x > 0, & \text{(3.1.14a)} \\ H_{\mathrm{TW}} = \frac{dH_{\mathrm{TW}}}{dx} &= 0 \qquad x = 0, & \text{(3.1.14b)} \\ H_{\mathrm{TW}}^{n-1} \frac{d^3 H_{\mathrm{TW}}}{dx^3} &= -V \qquad x = 0, & \text{(3.1.14c)} \end{aligned}$$

where we assumed $Z(0) = 0$ by translation invariance. Integrating this ODE and appealing the boundary conditions gives

$$\begin{aligned} H_{\mathrm{TW}}^{n-1} \frac{d^3 H_{\mathrm{TW}}}{dx^3} &= V \qquad x > 0, & \text{(3.1.15a)} \\ H_{\mathrm{TW}} = \frac{dH_{\mathrm{TW}}}{dx} &= 0 \qquad x = 0. & \text{(3.1.15b)} \end{aligned}$$

By a rescaling of x we may assume without loss of generality that the velocity V of the travelling wave is only depending on n. In particular, we may assume that this velocity is $V = \frac{3}{n}\left(\frac{3}{n} - 1\right)\left(\frac{3}{n} - 2\right)$. This particular choice of V ensures that $H_{\mathrm{TW}} = x^{\frac{3}{n}}$ is a travelling wave solution of the ODE (3.1.15). Note that this choice reduces for the Navier-slip case ($n = 2$) to the velocity $V = -\frac{3}{8}$ of the travelling wave.

The next step is to linearise the free boundary around the travelling wave $x^{\frac{3}{n}}$ with the von Mises transform. This has been proposed for instance in [16] and has been worked out there for $n = 2$. We set

$$h(t, Z(t,x)) := x^{\frac{3}{n}}. \tag{3.1.16}$$

Then

$$h_z Z_x = \partial_x h = \frac{3}{n} x^{\frac{3}{n} - 1} \quad \text{and} \quad h_t + h_z Z_t = 0.$$

Using this in (3.1.16) together with (3.1.1a), we obtain

$$Z_t = \frac{n}{3} x^{1 - \frac{3}{n}} \partial_x \left(x^3 \frac{1}{Z_x} \partial_x \frac{1}{Z_x} \partial_x \frac{1}{Z_x} \partial_x x^{\frac{3}{n}} \right). \tag{3.1.17}$$

We again introduce the variable $H := \frac{1}{Z_x}$ and thus $\partial_t H = -H^2 Z_{xt}$. It should also be noted that the travelling wave corresponds to the constant solution $H = 1$. By writing $D = x\partial_x$ and using the commutation relation

$$D x^{\gamma} = x^{\gamma}(D + \gamma) \tag{3.1.18}$$

we finally obtain the equation

$$\partial_t H + x^{-1} \mathcal{M}_n(H, H, H, H, H) = 0,$$

where

$$\mathcal{M}_n(H_1, H_2, H_3, H_4, H_5) = H_1 H_2 D \left(D + \frac{3}{n}\right) H_3 \left(D + \frac{3}{n} - 2\right) H_4 \left(D + \frac{3}{n} - 1\right) H_5.$$

Linearising around $u := H - 1$ gives the following nonlinear Cauchy problem

$$u_t + x^{-1} p_n(D) u = \mathcal{N}_n(u) \qquad t, x > 0, \tag{3.1.19a}$$

$$u|_{t=0} = u^{(0)} \qquad t = 0,\ x > 0, \tag{3.1.19b}$$

with the linear operator

$$\begin{aligned} p_n(D)u &= \mathcal{M}_n(u, 1, \ldots, 1) + \cdots + \mathcal{M}_n(1, \ldots, 1, u) \\ &= D\left(D + \frac{3}{n}\right)(D - \omega_1)(D - \omega_2)\, u, \end{aligned} \tag{3.1.20}$$

where

$$\omega_1 := -\frac{9}{2n} + 2 - \sqrt{-\frac{27}{4n^2} + \frac{9}{n} - 2} \quad \text{and} \quad \omega_2 := -\frac{9}{2n} + 2 + \sqrt{-\frac{27}{4n^2} + \frac{9}{n} - 2}$$

and the nonlinear part

$$\mathcal{N}_n(u) = -x^{-1}\mathcal{M}_n(u+1, \ldots, u+1) + x^{-1} p_n(D) u. \tag{3.1.21}$$

Hence, p_n is a fourth order polynomial and for $n \in (\frac{3}{2}, 3)$ the zeros are ordered as follows (from small to large)

$$\gamma_1 := -\frac{3}{n}, \quad \gamma_2 := \omega_1, \quad \gamma_3 := 0, \quad \gamma_4 := \omega_2. \tag{3.1.22}$$

3.1.3 The Nonlinear Cauchy Problem

The resulting nonlinear Cauchy problem is thus

$$u_t + x^{-1} p(D) u = \mathcal{N}(u) \qquad t, x > 0, \tag{3.1.23a}$$

$$u|_{t=0} = u^{(0)} \qquad t = 0,\ x > 0, \tag{3.1.23b}$$

where

$$p(D) := p_n(D) = (D - \gamma_1)(D - \gamma_2)(D - \gamma_3)(D - \gamma_4) \tag{3.1.24}$$

with for $n \in (1, \frac{3}{2})$

$$\gamma_1 = \frac{-2}{4 - 2n}, \quad \gamma_2 = \frac{1 - 2n}{4 - 2n}, \quad \gamma_3 = 0, \quad \gamma_4 = \frac{3 - 2n}{4 - 2n}$$

and for $n \in (\frac{3}{2}, 3)$

$$\gamma_1 = -\frac{3}{n}, \quad \gamma_2 = -\frac{9}{2n} + 2 - \sqrt{-\frac{27}{4n^2} + \frac{9}{n} - 2}, \quad \gamma_3 = 0, \quad \gamma_4 = -\frac{9}{2n} + 2 + \sqrt{-\frac{27}{4n^2} + \frac{9}{n} - 2}.$$

The fact that in both the case $n \in (1, \frac{3}{2})$ and $n \in (\frac{3}{2}, 3)$ one of the roots equals zero agrees with the divergence form of (1.0.1). The nonlinear right-hand side is given by

$$\mathcal{N}(u) := \mathcal{N}_n(u) = -x^{-1}\mathcal{M}_n(u+1, \ldots, u+1) + x^{-1} p_n(D) u, \tag{3.1.25}$$

where

$$\mathcal{M}_n(H_1, H_2, H_3, H_4, H_5) = \tag{3.1.26}$$

$$\begin{cases} H_1 H_2 \left(D + \frac{2n-3}{4-2n}\right)\left(D + \frac{2n-1}{4-2n}\right) H_3 D H_4 \left(D + \frac{1}{4-2n}\right) H_5 & \text{for } n \in (1, \frac{3}{2}) \\ H_1 H_2 D \left(D + \frac{3}{n}\right) H_3 \left(D + \frac{3}{n} - 2\right) H_4 \left(D + \frac{3}{n} - 1\right) H_5 & \text{for } n \in (\frac{3}{2}, 3) \end{cases}.$$

In what follows, the subscripts n will usually be left out.

3.2 The Functional-Analytic Setting

We use the inner products

$$(\phi, \psi)_\alpha = \int_0^\infty x^{-2\alpha} \phi \overline{\psi} \frac{dx}{x}, \tag{3.2.1}$$

$$(\phi, \psi)_{k,\alpha} = \sum_{j=0}^{k} \int_0^\infty x^{-2\alpha} D^j \phi \overline{D^j \psi} \frac{dx}{x}, \tag{3.2.2}$$

where $\phi, \psi \in C_c^\infty((0,\infty);\mathbb{C})$. In (3.2.2), the weight is given by $x^{-2\alpha}$, we will call α the weight exponent. In space, we will use the weighted Sobolev spaces, which are defined as follows:

Definition 3.2.1. *Let $k \in \mathbb{N}_0,\ \alpha \in \mathbb{R}$. The space $H_{k,\alpha}$ is defined as follows:*

$$H_{k,\alpha} := \overline{C_c^\infty((0,\infty))}^{\|\cdot\|_{k,\alpha}}.$$

Here, $\|\cdot\|_{k,\alpha}$ is the norm induced by the $(\cdot,\cdot)_{k,\alpha}$ inner product.
The space $\mathscr{H}_{k,\alpha}$ is defined as

$$\mathscr{H}_{k,\alpha} := H_{k,\alpha} \cap H_{k+2,\alpha+\frac{1}{2}}.$$

The space H_α is defined as $H_{0,\alpha}$.

Note that in many of the proofs later on we use functions in $C_c^\infty((0,\infty))$ which with a density argument extends to functions in the $H_{k,\alpha}$ spaces. Also, for $\alpha = 0$ it follows that the space $H_{k,0}((0,\infty)) = W^{k,2}((0,\infty))$. For $k = \alpha = 0$, $H_{k,\alpha}((0,\infty))$ reduces to $L^2(0,\infty)$.

For treating the nonlinear problem in chapter 5, certain norms for the solution u, the right hand side function f and the initial value $u^{(0)}$ are necessary. For completeness, we gather the definitions of these norms below.

Definition 3.2.2. *Let β be the largest zero of $p(D)$, as defined in* (3.1.12) *and* (3.1.22) *for $n \in (1, \frac{3}{2})$ and $n \in (\frac{3}{2}, 3)$, respectively. Define for $\delta < \min\{\frac{1}{p}, \beta - 1\}$, $1 < p < \infty$ the norm*

$$\begin{aligned}
|||u|||^p := \sup_{t \geq 0} \Big[& \|u\|^p_{k+9,-\frac{1}{2}+\frac{1}{p}-\delta,p} + \|u - u_0\|^p_{k+9,-\frac{1}{2}+\frac{1}{p}+\delta,p} + \|u - u_0\|^p_{\tilde{k}+6,-\frac{1}{2}+\beta-\delta,p} && (3.2.3)\\
& + \|u - u_0\|^p_{\tilde{k}+6,-\frac{1}{2}+\beta+\delta,p} \Big] + \int_0^\infty \|\partial_t u\|^p_{k+7,-1-\delta+\frac{1}{p}} + \|\partial_t u\|^p_{k+7,-1+\delta+\frac{1}{p}} \\
& + \|\partial_t u\|^p_{\tilde{k}+4,-1-\delta+\beta} + \|\partial_t u\|^p_{\tilde{k}+4,-1+\delta+\beta} + \|u - u_0\|^p_{k+11,\frac{1}{p}-\delta} && (3.2.4)\\
& + \|u - u_0\|^p_{k+11,\frac{1}{p}+\delta} + \|u - u_0\|^p_{\tilde{k}+8,\beta-\delta} + \|u - u_0 - u_\beta x^\beta\|^p_{\tilde{k}+8,\beta+\delta} dt.
\end{aligned}$$

We define the following norms for the initial value and the right hand side function:

$$\begin{aligned}
\left|\left|\left| u^{(0)} \right|\right|\right|_0^p := & \|u^{(0)}\|^p_{k+9,-\frac{1}{2}+\frac{1}{p}-\delta,p} + \|u^{(0)} - u_0^{(0)}\|^p_{k+9,-\frac{1}{2}+\frac{1}{p}+\delta,p} + \|u^{(0)} - u_0^{(0)}\|^p_{\tilde{k}+6,-\frac{1}{2}+\beta-\delta,p} && (3.2.5)\\
& + \|u^{(0)} - u_0^{(0)}\|^p_{\tilde{k}+6,-\frac{1}{2}+\beta+\delta,p},
\end{aligned}$$

$$|||f|||_1^p := \int_0^\infty \|f\|^p_{k+7,-1+\frac{1}{p}-\delta} + \|f\|^p_{k+7,-1+\frac{1}{p}+\delta} + \|f\|^p_{\tilde{k}+4,-1+\beta-\delta} + \|f\|^p_{\tilde{k}+4,-1+\beta+\delta} dt, \tag{3.2.6}$$

$$|||f|||_2^p := \int_0^\infty \|f\|^p_{k+7,\frac{1}{p}-\delta} + \|f\|^p_{k+7,\frac{1}{p}+\delta} + \|f\|^p_{\tilde{k}+4,\beta-\delta} + \|f\|^p_{\tilde{k}+4,\beta+\delta} dt. \tag{3.2.7}$$

Here,

$$\tilde{k} = \begin{cases} k+1 & \text{for } p > 4, \\ k & \text{for } 2 \le p \le 4, \\ k-1 & \text{for } \frac{4}{3} \le p < 2, \\ k-2 & \text{for } 1 < p < \frac{4}{3}. \end{cases}$$

The $\|\cdot\|_{k,\alpha,p}$ *norm is defined in definition 4.2.2.*

Remark. *Note that* (3.2.6) *and* (3.2.7) *are related by a shift in weight in all of the norms with* 1*.*

3.3 The Main Result

The following theorem is the main result of this thesis:

Theorem 3.3.1. *Suppose* $k \in \mathbb{N}_0$*, and let* β *be the largest zero of* $p(D)$ *(see* (3.1.12) *and* (3.1.22)*). For every* $n \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 3)$*, choose* p *such that* $\frac{1}{p} < \beta$ *and choose* δ *such that* $0 < \delta < \min\{\beta - 1, \frac{1}{p}\}$*. Then there exists an* $\varepsilon > 0$ *such that for all locally integrable* $u^{(0)} : (0, \infty) \to \mathbb{R}$ *with* $\left|\!\left|\!\left| u^{(0)} \right|\!\right|\!\right|_0 < \varepsilon$*,*

$$\begin{aligned} u_t + x^{-1} p(D) u &= \mathcal{N}(u) \qquad && t, x > 0, \\ u|_{t=0} &= u^{(0)} && t = 0,\ x > 0, \end{aligned}$$

has a unique solution $u : (0, \infty)^2 \to \mathbb{R}$ *that is locally integrable with* $|\!|\!| u |\!|\!| < \infty$*.*

Note that this theorem is the analogue of [16, theorem 3.1]. The main differences between this theorem and the one of [16] is that here, a larger range of mobility exponents is covered. Also, in the definitions of the norms (equations (3.2.3) and (3.2.5)) it can be seen that we use L^p in time, in comparison with L^2 in time in [16]. Since close to $\frac{3}{2}$, β is small and from the fact that we need to choose $\frac{1}{p} < \beta$, we see that the value $p = 2$ is not good enough to treat all values of n. Hence, we really need to be able to choose larger and smaller values of p to get existence and uniqueness of solutions to (3.1.23) for all $n \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 3)$.

Chapter 4

Maximal Regularity for the Linear Problem

First, the linear inhomogeneous Cauchy problem

$$\partial_t u - Au = f \qquad t, x > 0, \tag{4.0.1a}$$

$$u|_{t=0} = u^{(0)} \qquad t = 0,\ x > 0, \tag{4.0.1b}$$

is studied, where $A = -x^{-1}p(D)$ and where $p(D)$ is the fourth order polynomial as introduced in section 3.1.3. Here, $A : X \supset D(A) \to X$ with $X := H_{k-2,\alpha-\frac{1}{2}}$ and $D(A) := H_{k+2,\alpha+\frac{1}{2}} \cap H_{k-2,\alpha-\frac{1}{2}}$. It is well known that this Cauchy problem has the mild solution (see e.g. [13, Def 7.2])

$$u(t) = e^{tA}u^{(0)} + \int_0^t e^{(t-s)A} f(s)ds,$$

which suggests that the problem can be rewritten as the sum of the following two problems:

$$\partial_t u_1 - Au_1 = 0, \tag{4.0.2a}$$

$$u_1(0) = u^{(0)}, \tag{4.0.2b}$$

with mild solution $u_1(t) = e^{tA}u^{(0)}$, and

$$\partial_t u_2 - Au_2 = f, \tag{4.0.3a}$$

$$u_2(0) = 0, \tag{4.0.3b}$$

with mild solution $u_2(t) = \int_0^t e^{(t-s)A} f(s)ds$.

The goal is to prove a maximal regularity estimate for the linear problem (4.0.1). To achieve this, the two problems will be dealt with separately. The corresponding resolvent problem of (4.0.3) will be treated in section 4.1 and problem (4.0.2) is treated in section 4.2 with standard semi-group and interpolation theory. Finally, in section 4.3 the results will be combined to obtain the maximal L^p-regularity estimate.

4.1 Inhomogeneous Equation

After applying the Laplace transform in time to problem (4.0.3), we obtain (after dropping the subscript, for convenience) the resolvent equation

$$\lambda u - Au = f, \tag{4.1.1}$$

with $\lambda \in \{z \in \mathbb{C} \mid \Re z \geq 0\}$ and $A = -x^{-1}p(D)$ with the fourth order polynomial p. We test (4.1.1) with the test function

$$\varphi := \sum_{j=0}^{k} c_j(-D+2\alpha)^j D^j \phi, \qquad c_j > 0, \tag{4.1.2}$$

where $\phi \in C_c^\infty((0,\infty))$ in $(\cdot,\cdot)_\alpha$. We define the bilinear form for which we will construct solutions as follows

$$\begin{aligned} B_{k,\alpha}(\phi,u) &:= (\varphi, \lambda u - Au)_\alpha \\ &= \sum_{j=0}^{k} c_j \lambda\big((-D+2\alpha)^j D^j \phi, u\big)_\alpha + \sum_{j=0}^{k} c_j\big((-D+2\alpha)^j D^j\phi, -Au\big)_\alpha, \end{aligned}$$

where $B_{k,\alpha} : \mathscr{H}_{k,\alpha} \times \mathscr{H}_{k,\alpha} \to \mathbb{C}$. Our goal in this section is to show that the resolvent equation has a unique classical solution. Moreover, we will show that $A := -x^{-1}p(D)$ generates an analytic semigroup. We start by showing that the resolvent equation (4.1.1) has a unique weak solution by using the Lax-Milgram theorem (2.1.4).

Proposition 4.1.1. *Let $w \in H_{\alpha+\frac{1}{2}}$. $(w, p(D)w)_{\alpha+\frac{1}{2}}$ is coercive with respect to $\|\cdot\|_{\alpha+\frac{1}{2}}$ (where this is the norm induced by $(\cdot,\cdot)_\alpha$), i.e., $(w,p(D)w)_{\alpha+\frac{1}{2}} \gtrsim \|w\|_{\alpha+\frac{1}{2}}$, if the following conditions hold:*

$$\alpha + \frac{1}{2} \in (-\infty, \gamma_1) \cup (\gamma_2, \gamma_3) \cup (\gamma_4, \infty), \tag{4.1.3a}$$

$$|\alpha + \tfrac{1}{2} - m(\gamma)| \leq \frac{1}{\sqrt{3}}\sigma(\gamma). \tag{4.1.3b}$$

Here, $m(\gamma)$ denotes the algebraic mean of the zeros γ_l of $p(D)$, i.e.,

$$m(\gamma) := \frac{1}{4}\sum_{l=1}^{4}\gamma_l,$$

and $\sigma(\gamma)$ the nonnegative root of the variance

$$\sigma^2(\gamma) = \frac{1}{4}\sum_{l=1}^{4}\gamma_l^2 - m^2(\gamma) = \frac{1}{4}\sum_{l=1}^{4}(\gamma_l - m(\gamma))^2.$$

Proof. See [16, Prop 5.3]. Noting that $p(-i\xi + \alpha + \frac{1}{2}) = \Pi_{l=1}^4(-i\xi - (\gamma_l - (\alpha + \frac{1}{2})))$ and that the prefactors of odd powers of ξ are imaginary, we see that

$$\Re\big(p(-i\xi + \alpha + \tfrac{1}{2})\big) = \kappa^2 - 2a\kappa + b,$$

where

$$\begin{aligned} \kappa &:= \xi^2, \\ a &:= \frac{1}{2}\sum_{1\leq j<l\leq 4}\big(\gamma_j - (\alpha+\tfrac{1}{2})\big)\big(\gamma_l - (\alpha+\tfrac{1}{2})\big), \\ b &:= \big(\gamma_1 - (\alpha+\tfrac{1}{2})\big)\big(\gamma_2 - (\alpha+\tfrac{1}{2})\big)\big(\gamma_3 - (\alpha+\tfrac{1}{2})\big)\big(\gamma_4 - (\alpha+\tfrac{1}{2})\big). \end{aligned}$$

From [16, lemma 5.2] coercivity follows for $(w,p(D)w)_{\alpha+\frac{1}{2}}$ with respect to $\|\cdot\|_{\alpha+\frac{1}{2}}$ if and only if either

$$a \leq 0 \text{ and } b > 0 \tag{4.1.4}$$

or

$$a > 0 \text{ and } b > a^2$$

holds true. Using (4.1.4), we will be able to express the condition for coercivity only in terms of $\alpha + \frac{1}{2}$. It is obvious that the condition $b > 0$ is equivalent to (4.1.3a). We now show that the condition that $a \leq 0$ is equivalent to (4.1.3b). For this, rewrite as follows

$$\begin{aligned} 2a &= 6(\alpha + \tfrac{1}{2})^2 - 3\Big(\sum_{j=1}^{4} \gamma_j\Big)(\alpha + \tfrac{1}{2}) + \sum_{1 \leq j < l \leq 4} \gamma_j \gamma_l \\ &= 6\Big(\alpha + \frac{1}{2} - \frac{1}{4}\sum_{j=1}^{4} \gamma_j\Big)^2 - \frac{3}{8}\Big(\sum_{j=1}^{4} \gamma_j\Big)^2 + \sum_{1 \leq j < l \leq 4} \gamma_j \gamma_l. \end{aligned}$$

The first term on the right hand side resembles the left hand side of (4.1.3b). We will rewrite the other two terms as:

$$\begin{aligned} \frac{3}{8}\Big(\sum_{j=1}^{4} \gamma_j\Big)^2 - \sum_{1 \leq j < l \leq 4} \gamma_j \gamma_l &= \frac{1}{2}\sum_{j=1}^{4} \gamma_j^2 - \frac{1}{8}\sum_{1 \leq j, l \leq 4} \gamma_j \gamma_l \\ &= 2\left(\frac{1}{4}\sum_{j=1}^{4} \gamma_j^2 - \Big(\frac{1}{4}\sum_{j=1}^{4} \gamma_j\Big)^2\right). \end{aligned}$$

This implies

$$2a = 6\big(\alpha + \frac{1}{2} - m(\gamma)\big)^2 - 2\sigma^2(\gamma).$$

Combining this with the fact that $a \geq 0$ gives the condition (4.1.3b). $\square$

The above proposition amounts in our case to:

Corollary 4.1.2. *Let $w \in H_{\alpha+\frac{1}{2}}$. It holds that $(w, p(D)w)_{\alpha+\frac{1}{2}}$ is coercive if*

- $\alpha + \frac{1}{2} \in \left(\frac{1-2n}{4-2n}, 0\right) \cap \left(\frac{1-2n}{8-4n} - \frac{1}{\sqrt{3}}\frac{\sqrt{\frac{13}{4} - 3n + n^2}}{4-2n}, \frac{1-2n}{8-4n} + \frac{1}{\sqrt{3}}\frac{\sqrt{\frac{13}{4} - 3n + n^2}}{4-2n}\right)$ *for* $n \in \left(1, \frac{3}{2}\right)$,
- $\alpha + \frac{1}{2} \in \left(-\frac{9}{2n} + 2 - \sqrt{-\frac{27}{4n^2} + \frac{9}{n} - 2}, 0\right) \cap \left(\frac{n-3}{n} - \frac{1}{\sqrt{2n}}, \frac{n-3}{n} + \frac{1}{\sqrt{2n}}\right)$ *for* $n \in \left(\frac{3}{2}, 3\right)$.

We will refer to these intervals as the coercivity range of $p(D)$.

Proof. We first consider the case $n \in (1, \frac{3}{2})$. The zeros of $p(D)$ (3.1.12) are given by $-\frac{2}{4-2n}$, $\frac{1-2n}{4-2n}$, 0, $\frac{3-2n}{4-2n}$, ordered from smallest to largest. So, condition (4.1.3a) gives

$$\alpha + \frac{1}{2} \in \Big(-\infty, -\frac{2}{4-2n}\Big) \cup \Big(\frac{1-2n}{4-2n}, 0\Big) \cup \Big(\frac{3-2n}{4-2n}, \infty\Big).$$

Furthermore, an elementary calculation shows

$$m(\gamma) = \frac{1-2n}{8-4n} \quad \text{and} \quad \sigma^2(\gamma) = \frac{\frac{13}{4} - 3n + n^2}{(4-2n)^2}.$$

To satisfy (4.1.3b), there is the requirement that

$$\left|\alpha + \frac{1}{2} - \frac{1-2n}{8-4n}\right| \leq \frac{1}{\sqrt{3}} \frac{\sqrt{\frac{13}{4} - 3n + n^2}}{(4-2n)}. \tag{4.1.5}$$

By combining the above criteria, we obtain the coercivity range

$$\alpha+\frac{1}{2}\in\left(\frac{1-2n}{4-2n},0\right)\cap\left(\frac{1-2n}{8-4n}-\frac{1}{\sqrt{3}}\frac{\sqrt{\frac{13}{4}-3n+n^2}}{4-2n},\frac{1-2n}{8-4n}+\frac{1}{\sqrt{3}}\frac{\sqrt{\frac{13}{4}-3n+n^2}}{4-2n}\right),$$

see figure 4.1a.

For the case $n\in(\frac{3}{2},3)$ we have the zeros (3.1.22)

$$\gamma_1=-\frac{3}{n},\quad \gamma_2=-\frac{9}{2n}+2-\sqrt{-\frac{27}{4n^2}+\frac{9}{n}-2},\quad \gamma_3=0,\quad \gamma_4=-\frac{9}{2n}+2+\sqrt{-\frac{27}{4n^2}+\frac{9}{n}-2}$$

ordered from smallest to largest. Hence, to satisfy (4.1.3a) we get the condition

$$\alpha+\frac{1}{2}\in\big(-\infty,-\tfrac{3}{n}\big)\cup(\gamma_2,0)\cup(\gamma_4,\infty).$$

and

$$m(\gamma)=\frac{n-3}{n}\quad\text{and}\quad\sigma^2(\gamma)=\frac{3}{2n}$$

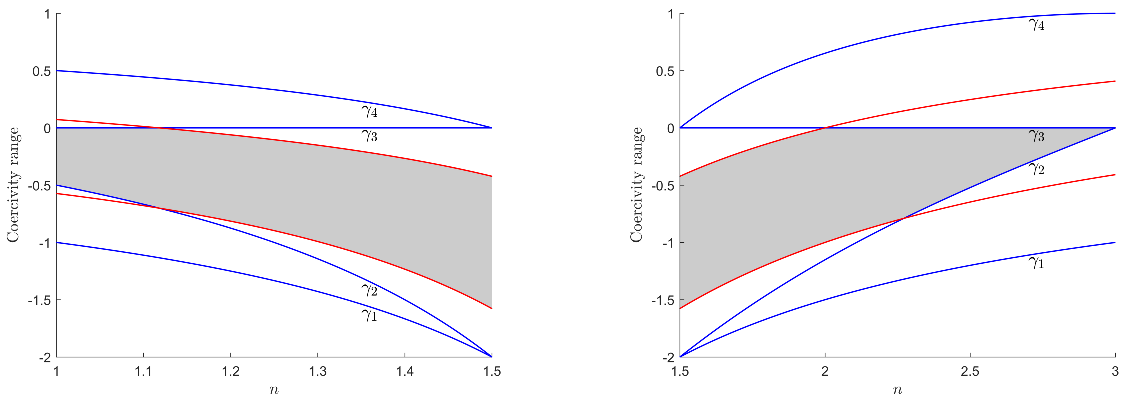
which gives the second condition (4.1.3b)

$$\left|\alpha+\frac{1}{2}-\frac{n-3}{n}\right|\le\frac{1}{\sqrt{2n}}. \tag{4.1.6}$$

We thus need to take

$$\alpha+\frac{1}{2}\in\left(-\frac{9}{2n}+2-\sqrt{-\frac{27}{4n^2}+\frac{9}{n}-2},0\right)\cap\left(\frac{n-3}{n}-\frac{1}{\sqrt{2n}},\frac{n-3}{n}+\frac{1}{\sqrt{2n}}\right)$$

to fulfill both conditions, see figure 4.1b.



(a) $n\in(1,\frac{3}{2})$ (b) $n\in(\frac{3}{2},3)$

Figure 4.1: For the two different cases of n the zeros $\gamma_1,\dots,\gamma_4$ of $p(D)$ (blue) and the upper and lower bound in (4.1.5) and (4.1.6) (red) are shown. The coercivity range for $\alpha+\frac{1}{2}$ is the shaded area.

□

Theorem 4.1.3. *There exist constants $c_0 > c_1 > c_2 > \cdots > c_k$ such that the bilinear form $B_{k,\alpha}$ is coercive and bounded for $\alpha + \frac{1}{2}$ in the coercivity range. Moreover, the following estimate holds:*

$$\Re\lambda \|u\|^2_{k,\alpha} + c\|u\|^2_{k+2,\alpha+\frac{1}{2}} \lesssim_k \|f\|^2_{k-2,\alpha-\frac{1}{2}}. \tag{4.1.7}$$

The following lemma is needed for proving this result:

Lemma 4.1.4.

$$|(x^{-1}(D-a)(D-b)u, (D-c)(D-d)v)_{k,\alpha}| \lesssim_k \|u\|_{k+2,\alpha+\frac{1}{2}} \|v\|_{k+2,\alpha+\frac{1}{2}}, \tag{4.1.8}$$

$$\left| \sum_{j=0}^{k} ((D+a)^j u, D^j v)_\alpha \right| \lesssim_k \|u\|_{k,\alpha} \|v\|_{k,\alpha} \tag{4.1.9}$$

and

$$\left| \sum_{j=0}^{k} \|(D+a)^j u)\|_\alpha \right| \lesssim_k \|u\|_{k,\alpha} \tag{4.1.10}$$

hold for $a, b, c, d \in \mathbb{R}$ constant, $k \in \mathbb{N}_0$ and $u, v \in \mathscr{H}_{k,\alpha}$.

Proof. For readability, (4.1.8) will only be proven for $b = c = d = 0$. The proof works in a similar way when a, b, c and d are all nonzero. First note that

$$\|u\|^2_{k,\alpha} = \sum_{j=0}^{k} \|D^j u\|^2_\alpha \geq \|D^j u\|^2_\alpha \tag{4.1.11}$$

holds for all $j \in \{0, \ldots, k\}$. This will be used in deriving (4.1.8):

$$\begin{aligned}
|(x^{-1}(D-a)Du, D^2 v)_{k,\alpha}| &= \left| \sum_{j=0}^{k} \int_0^\infty x^{-2\alpha-1} \left[D^j (D-a)Du\right] \left[D^j D^2 v\right] \frac{dx}{x} \right| \\
&\leq \sum_{j=0}^{k} \left| \int_0^\infty x^{-2\alpha-1} [D^{j+2}u][D^{j+2}v] \frac{dx}{x} \right| + \sum_{j=0}^{k} |a| \left| \int_0^\infty x^{-2\alpha-1} [D^{j+1}u][D^{j+2}v] \frac{dx}{x} \right| \\
&\leq \sum_{j=0}^{k} \|D^{j+2}u\|_{\alpha+\frac{1}{2}} \|D^{j+2}v\|_{\alpha+\frac{1}{2}} + \sum_{j=0}^{k} |a| \|D^{j+1}u\|_{\alpha+\frac{1}{2}} \|D^{j+2}v\|_{\alpha+\frac{1}{2}} \\
&\overset{(4.1.11)}{\leq} \|u\|_{k+2,\alpha+\frac{1}{2}} \|v\|_{k+2,\alpha+\frac{1}{2}} + |a| \sum_{j=0}^{k} \|u\|_{k+2,\alpha+\frac{1}{2}} \|v\|_{k+2,\alpha+\frac{1}{2}} \\
&= (1 + (k+1)|a|) \|u\|_{k+2,\alpha+\frac{1}{2}} \|v\|_{k+2,\alpha+\frac{1}{2}}.
\end{aligned}$$

Now for proving (4.1.9):

$$\begin{aligned}
\left| \sum_{j=0}^{k} ((D+a)^j u, D^j v)_\alpha \right| &\leq \left| \sum_{j=0}^{k} (D^j u, D^j v)_\alpha \right| + \left| \sum_{j=0}^{k} \sum_{l=0}^{j-1} (\binom{j-1}{l} a^{j-1-l} D^l u, D^j v)_\alpha \right| \\
&\leq \|u\|_{k,\alpha} \|v\|_{k,\alpha} + \sum_{j=0}^{k} \sum_{l=0}^{j-1} \binom{j-1}{l} |a|^{j-1-l} \|D^l u\|_\alpha \|D^j v\|_\alpha \\
&\overset{(4.1.11)}{\lesssim_k} \|u\|_{k,\alpha} \|v\|_{k,\alpha}.
\end{aligned}$$

Showing (4.1.10) is similar to showing (4.1.9):

$$\begin{aligned}\left|\sum_{j=0}^{k}((D+a)^j u,(D+a)^j u)_\alpha\right| \le \left|\sum_{j=0}^{k}(D^j u, D^j u)_\alpha\right| + \left|\sum_{j=0}^{k}\sum_{l=0}^{j-1}(\binom{j-1}{l}a^{j-1-l}D^l u, D^j u)_\alpha\right| \\ + \left|\sum_{j=0}^{k}\sum_{l=0}^{j-1}(D^j u, \binom{j-1}{l}a^{j-1-l}D^l u)_\alpha\right| \\ + \left|\sum_{j=0}^{k}\sum_{l=0}^{j-1}\sum_{m=0}^{j-1}(\binom{j-1}{l}a^{j-1-l}D^l u, \binom{j-1}{m}a^{j-1-m}D^j u)_\alpha\right|.\end{aligned}$$

The first three terms can be bounded by $\|u\|_{k,\alpha}^2$ in the same way as before. For the last term, note that

$$\begin{aligned}&\left|\sum_{j=0}^{k}\sum_{l=0}^{j-1}\sum_{m=0}^{j-1}(\binom{j-1}{l}a^{j-1-l}D^l u, \binom{j-1}{m}a^{j-1-m}D^j u)_\alpha\right| \\ &\le \sum_{j=0}^{k}\max_l\max_m\binom{j-1}{l}\binom{j-1}{m}\sum_{l=0}^{j-1}\sum_{m=0}^{j-1}\|D^l u\|_\alpha\|D^m u\|_\alpha.\end{aligned}$$

Using (4.1.11) we can also bound this by $\|u\|_{k,\alpha}^2$. Taking the square root gives (4.1.10). □

Proof of theorem 4.1.3. Assume that $\phi, u \in C_c^\infty((0,\infty);\mathbb{C})$.

Boundedness

We want to show that

$$|B_{k,\alpha}(\phi,u)| := \left|\sum_{j=0}^{k}c_j\lambda((-D+2\alpha)^j D^j\phi, u)_\alpha + \sum_{j=0}^{k}((-D+2\alpha)^j D^j\phi, -Au)_\alpha\right| \lesssim \|\phi\|_{\mathscr{H}_{k,\alpha}}\|u\|_{\mathscr{H}_{k,\alpha}}.$$

We have for the first part of the bilinear form

$$\left|\sum_{j=0}^{k}c_j\lambda((-D+2\alpha)^j D^j\phi, u)_\alpha\right| \le \max_j c_j|\lambda|\left|\sum_{j=0}^{k}(D^j\phi,(D+2\alpha)^j u)_\alpha\right| \overset{(4.1.9)}{\lesssim_k} \max_j c_j|\lambda|\|\phi\|_{k,\alpha}\|u\|_{k,\alpha}.$$

Now consider the second part of the bilinear form, using that $p(D) = (D-\gamma_1)(D-\gamma_2)(D-\gamma_3)(D-\gamma_4)$:

$$\begin{aligned}&|\sum_{j=0}^{k}c_j((-D+2\alpha)D^j\phi, x^{-1}p(D)u)_\alpha| \\ &\le \max_j c_j\sum_{j=0}^{k}|((-D-\gamma_1+2\alpha)(-D-\gamma_2-1+2\alpha)D^j\phi,(D-\gamma_3)(D-\gamma_4)(D-1)^j u)_{\alpha+\frac{1}{2}}|,\end{aligned}$$

where we used integration by parts and moved the factor x^{-1} into the weight. Using (4.1.8), we can rewrite this as

$$|\sum_{j=0}^{k}c_j((-D+2\alpha)D^j\phi, x^{-1}p(D)u)_\alpha| \lesssim_k \sum_{j=0}^{k}\|D^j\phi\|_{2,\alpha+\frac{1}{2}}\|(D-1)^j u\|_{2,\alpha+\frac{1}{2}}.$$

Using (4.1.11), we see that we can bound $\|D^j\phi\|_{2,\alpha+\frac{1}{2}}$ by $\|\phi\|_{k+2,\alpha+\frac{1}{2}}$. Applying (4.1.10) to $\|(D-1)^j u\|_{2,\alpha+\frac{1}{2}}$ gives us that we can bound this by $\|u\|_{k+2,\alpha+\frac{1}{2}}$. Hence,

$$|\sum_{j=0}^{k} c_j((-D+2\alpha)D^j\phi, x^{-1}p(D)u)_\alpha| \lesssim \|\phi\|_{k+2,\alpha+\frac{1}{2}}\|u\|_{k+2,\alpha+\frac{1}{2}}.$$

Now, we have that

$$|B_{k,\alpha}(u,\phi)| \lesssim_k \|\phi\|_{k,\alpha}\|u\|_{k,\alpha} + \|\phi\|_{k+2,\alpha+\frac{1}{2}}\|u\|_{k+2,\alpha+\frac{1}{2}} \lesssim \|\phi\|_{\mathscr{H}_{k,\alpha}}\|u\|_{\mathscr{H}_{k,\alpha}}.$$

Coercivity
We want to show that

$$\begin{aligned}\Re(B_{k,\alpha}(u,u)) &:= \sum_{j=0}^{k} c_j \Re(\lambda)((-D+2\alpha)^j D^j u, u)_\alpha + \sum_{j=0}^{k} c_j((-D+2\alpha)^j D^j u, -Au)_\alpha \\ &\gtrsim \|u\|^2_{\mathscr{H}_{k,\alpha}} = C(\|u\|_{k,\alpha} + \|u\|_{k+2,\alpha+\frac{1}{2}})^2.\end{aligned}$$

First consider the part of the bilinear form with the operator $-A$:

$$\begin{aligned}&\left(\sum_{j=0}^{k} c_j(-D+2\alpha)^j D^j u, x^{-1}p(D)u\right)_\alpha \\ &= \int_0^\infty \left[x^{-2\alpha}\sum_{j=0}^{k} c_j(-D+2\alpha)^j D^j u\right]\left[x^{-1}p(D)u\right]\frac{dx}{x} \\ &\overset{(3.1.18)}{=} \sum_{j=0}^{k} c_j \int_0^\infty \left[(-D)^j x^{-2\alpha}D^j u\right]\left[x^{-1}p(D)u\right]\frac{dx}{x} \\ &\overset{(3.1.18)}{=} \sum_{j=0}^{k} c_j \int_0^\infty x^{-2\alpha}\left[D^j u\right]x^{-1}\left[p(D)(D-1)^j u\right]\frac{dx}{x} \\ &= \sum_{j=0}^{k} c_j\left(D^j u, p(D)(D-1)^j u\right)_{\alpha+\frac{1}{2}}.\end{aligned}$$

We would like to apply proposition 4.1.1, so we rewrite further. For this, use the identity

$$(D-1)^j = D^j + \sum_{l=0}^{j-1}\binom{j}{l}D^l(-1)^{j-l},$$

and Young's inequality

$$ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}, \quad a,b>0.$$

We will choose the c_j iteratively. Start with $c_k = 1$. Define $A(j) := c_j(D^j u, p(D)D^j u)_{\alpha+\frac{1}{2}}$

and $B(j) := c_j(D^j u, p(D)\sum_{l=0}^{j-1}(-1)^{j-l}\binom{j}{l}D^l u)_{\alpha+\frac{1}{2}}$.

$$\begin{aligned}
&\sum_{j=0}^{k} c_j \left(D^j u, p(D)(D-1)^j u\right)_{\alpha+\frac{1}{2}} \\
=&\sum_{j=0}^{k} c_j(D^j u, p(D)D^j u)_{\alpha+\frac{1}{2}} + \sum_{j=0}^{k} c_j(D^j u, p(D)\sum_{l=0}^{j-1}(-1)^{j-l}\binom{j}{l}D^l u)_{\alpha+\frac{1}{2}} \\
=&(D^k u, p(D)D^k u)_{\alpha+\frac{1}{2}} + (D^k u, p(D)\sum_{l=0}^{k-1}(-1)^{k-l}\binom{k}{l}D^l u)_{\alpha+\frac{1}{2}} + \sum_{j=0}^{k-1} A(j) + B(j) \\
\geq& K\|D^k u\|^2_{2,\alpha+\frac{1}{2}} - \sum_{l=0}^{k}\binom{k}{l}|(D^k u, D^l u)_{2,\alpha+\frac{1}{2}}| + \sum_{j=0}^{k-1} A(j) + B(j) =: F_1,
\end{aligned}$$

where K is the coercivity constant that follows from the coercivity of $(D^k u, p(D)D^k u)_{\alpha+\frac{1}{2}}$. For the second term on the right hand side, we see by using the Cauchy-Schwarz inequality and Young's inequality that

$$|(D^k u, D^l u)_{2,\alpha+\frac{1}{2}}| \leq \frac{\varepsilon}{2}\|D^k u\|^2_{2,\alpha+\frac{1}{2}} + \frac{1}{2\varepsilon}\|D^l u\|^2_{2,\alpha+\frac{1}{2}}.$$

Defining $S_k := \sum_{l'=0}^{k-1}\binom{k}{l'}$ and choosing $\varepsilon = \frac{K}{S_k}$ gives that we can write

$$\begin{aligned}
F_1 \geq& K\|D^k u\|^2_{2,\alpha+\frac{1}{2}} - \sum_{l=0}^{k}\binom{k}{l}\left[\frac{\varepsilon}{2}\|D^k u\|^2_{2,\alpha+\frac{1}{2}} + \frac{1}{2\varepsilon}\|D^l u\|^2_{2,\alpha+\frac{1}{2}}\right] + \sum_{j=0}^{k-1}(A(j)+B(j)) \\
=& \frac{K}{2}\|D^k u\|^2_{2,\alpha+\frac{1}{2}} - \frac{S_k}{2K}\sum_{l=0}^{k-1}\binom{k}{l} + \sum_{j=0}^{k-1}(A(j)+B(j)) =: F_2.
\end{aligned}$$

Taking out the $k-1$ term out of the sum over $A(j)$ and $B(j)$ gives

$$\begin{aligned}
F_2 =& c_{k-1}(D^{k-1}u, p(D)D^{k-1}u)_{\alpha+\frac{1}{2}} + c_{k-1}(D^{k-1}u, p(D)\sum_{l=0}^{k-2}(-1)^{k-1-l}\binom{k-1}{l}D^l u)_{\alpha+\frac{1}{2}} \\
&+ \frac{K}{2}\|D^k u\|^2_{2,\alpha+\frac{1}{2}} - \frac{S_k}{2K}\sum_{l=0}^{k-1}\binom{k}{l} + \sum_{j=0}^{k-2}(A(j)+B(j)).
\end{aligned}$$

Using coercivity on the first term and Young's inequality with $\varepsilon = \frac{K}{S_{k-1}}$ on the second term gives:

$$\begin{aligned}
F_2 \leq& c_{k-1}K\|D^{k-1}u\|^2_{2,\alpha+\frac{1}{2}} - \frac{c_{k-1}K}{2}\|D^{k-1}u\|^2_{2,\alpha+\frac{1}{2}} - \frac{c_{k-1}S_{k-1}}{2K}\sum_{l=0}^{k-2}\binom{k-1}{l}\|D^l u\|^2_{2,\alpha+\frac{1}{2}} \\
&+ \frac{K}{2}\|D^k u\|^2_{2,\alpha+\frac{1}{2}} - \frac{S_k}{2K}\sum_{l=0}^{k-1}\binom{k}{l} + \sum_{j=0}^{k-2}(A(j)+B(j)).
\end{aligned}$$

We need the constant in front of $\|D^{k-1}u\|^2_{2,\alpha+\frac{1}{2}}$ term to be positive, hence we need to choose $c_{k+1} > \frac{S_k}{K^2}\binom{k}{k-1}$. Iteratively taking more terms out of the sum over $A(j)$ and $B(j)$, we can get constraints for the other c_k. From the prefactor $\frac{c_{k-1}S_{k-1}}{2K}$ in front of $\|D^l u\|^2_{2,\alpha+\frac{1}{2}}$, we see that the choice of c_{k-2} will depend on c_{k-1}. Terms like this will occur when iteratively rewriting the terms in $\sum_{j=0}^{k-1} B(j)$, hence the choice of c_m for $0 < m < k-1$ will depend on c_{m+1}. We then get

$$\sum_{j=0}^{k} c_j \left(D^j u, p(D)(D-1)^j u\right)_{\alpha+\frac{1}{2}} \geq \sum_{j=0}^{k} \kappa_j (D^j u, D^j u)_{2,\alpha+\frac{1}{2}},$$

where κ_j are positive constants. Define $m = \min_j\{\kappa_j\}$ (m is positive, since κ_j is positive for all j), then

$$\sum_{j=0}^{k} c_j \left(D^j u, p(D)(D-1)^j u\right)_{\alpha+\frac{1}{2}} \geq m(u,u)_{k+2,\alpha+\frac{1}{2}}.$$

Now for the other part:

$$\begin{aligned}
\Re(\lambda)\left(\sum_{j=0}^{k} c_j(-D+2\alpha)^j D^j u, u\right)_\alpha &= \Re(\lambda)\sum_{j=0}^{k} c_j((-D+2\alpha)^j D^j u, u)_\alpha \\
&= \Re(\lambda)\sum_{j=0}^{k} c_j \int_0^\infty (x^{-2\alpha}(-D+2\alpha)^j D^j u) u \frac{dx}{x} \\
&= \Re(\lambda)\int_0^\infty ((-D)^j x^{-2\alpha} D^j u) u \frac{dx}{x} \\
&= \Re(\lambda)\int_0^\infty (x^{-2\alpha}(D^j u)(D^j u)\frac{dx}{x} = \Re(\lambda)\sum_{j=0}^{k} c_j \|D^j u\|_\alpha^2.
\end{aligned}$$

Taking $M := \Re(\lambda)\min_j c_j$ gives that

$$\Re(\lambda)\left(\sum_{j=0}^{k} c_j(-D+2\alpha)^j D^j u, u\right)_\alpha \geq M\|u\|_{k,\alpha}^2.$$

So, we get that

$$\Re(B_{k,\alpha}(u,u)) \geq M(u,u)_{k,\alpha} + m(u,u)_{k+2,\alpha+\frac{1}{2}}.$$

Taking the minimum over m and M and using Young's inequality now gives

$$\Re(B_{k,\alpha}(u,u)) \gtrsim \|u\|_{\mathscr{H}_{k,\alpha}}^2.$$

From the above coercivity estimates it also follows that

$$\Re(B_{k,\alpha}(u,u)) \geq \Re\lambda\|u\|_{k,\alpha}^2 + m\|u\|_{k+2,\alpha+\frac{1}{2}}^2. \tag{4.1.12}$$

Furthermore, using integration by parts and the resolvent equation (4.1.1) we get the following estimate:

$$\Re(B_{k,\alpha}(u,u)) \leq K\|f\|_{k-2,\alpha-\frac{1}{2}}\|u\|_{k+2,\alpha+\frac{1}{2}} \leq \frac{K}{2\varepsilon}\|f\|_{k-2,\alpha-\frac{1}{2}}^2 + \frac{\varepsilon K}{2}\|u\|_{k+2,\alpha+\frac{1}{2}}^2, \tag{4.1.13}$$

for $\varepsilon > 0$. Combining the estimates (4.1.12), (4.1.13) and choosing ε such that $m - \frac{\varepsilon K}{2} > 0$ gives the desired estimate (4.1.7). □

Corollary 4.1.5. *Let $\alpha + \frac{1}{2}$ be in the coercivity range. Then the equation*

$$\sum_{j=0}^{k} c_j\lambda((-D+2\alpha)^j D^j\phi, u)_\alpha + \sum_{j=0}^{k} c_j((-D+2\alpha)^j D^j\phi, -Au)_\alpha = \sum_{j=0}^{k} c_j((-D+2\alpha)^j D^j\phi, f)_\alpha$$

has a unique solution $u \in \mathscr{H}_{k,\alpha}$ for every $f \in \mathscr{H}'_{k,\alpha}$ and for all $\phi \in C_c^\infty((0,\infty))$.

Proof. The statement follows from theorem 4.1.3 and the Lax-Milgram theorem (theorem 2.1.4). □

Remark. *Note that we do not characterize $\mathscr{H}'_{k,\alpha}$, as we will prove a stronger result for the resolvent equation later. Hence, we do not need details on this space.*

For showing this result, we need the following lemma.

Lemma 4.1.6. *The test functions*

$$\left\{\sum_{j=0}^{k} c_j(-D+2\alpha)^j D^j \varphi \mid \varphi \in C_c^\infty(\mathbb{R}^+)\right\}$$

are dense in $L^2(\mathbb{R}_+, x^{-2\alpha}\frac{dx}{x})$ for $c_0 > c_1 > \cdots > c_k$ suitably chosen.

Proof. Note that by the commutation relation (3.1.18) the assertion of the proposition is equivalent to

$$\left\{\sum_{j=0}^{k} c_j(-D)^j (D+2\alpha)^j \underbrace{x^{-2\alpha}\varphi}_{=\tilde{\varphi}\in C_c^\infty(\mathbb{R}^+)} \mid \varphi \in C_c^\infty(\mathbb{R}^+)\right\}$$

being dense in $L^2(\mathbb{R}_+, \frac{dx}{x})$. After applying the coordinate transformation $\tilde{\varphi}(x) = \psi(e^s)$ with $s = \log x$ this is equivalent to

$$\left\{\sum_{j=0}^{k} c_j(-\partial_s)^j (\partial_s+2\alpha)^j \psi \mid \psi \in \mathcal{S}(\mathbb{R})\right\}$$

being dense in $L^2(\mathbb{R})$ by density of the compactly supported functions in the Schwartz space. It is more convenient to shift to the Schwartz space here, since the Fourier transform is a bijection on the Schwartz space. By applying Plancherel it is thus sufficient to show that

$$\left\{\sum_{j=0}^{k} c_j(-i\xi)^j (i\xi+2\alpha)^j \psi \mid \psi \in \mathcal{S}(\mathbb{R})\right\}$$

is dense in $L^2(\mathbb{R})$. Therefore we show that the mapping

$$\psi \mapsto \sum_{j=0}^{k} c_j(-i\xi)^j (i\xi+2\alpha)^j \psi \tag{4.1.14}$$

is a surjective mapping from the Schwartz space onto itself. Let

$$\tilde{\psi}(\xi) = \sum_{j=0}^{k} c_j(-i\xi)^j (i\xi+2\alpha)^j \psi(\xi) \tag{4.1.15}$$

and assume that $\tilde{\psi}$ is a Schwartz function. We have to find a Schwartz function ψ such that (4.1.15) is satisfied. Rewriting (4.1.15) as

$$\psi(\xi) = \frac{\tilde{\psi}(\xi)}{\sum_{j=0}^{k} c_j(-i\xi)^j (i\xi+2\alpha)^j}$$

we see that we have to show that $\sum_{j=0}^{k} c_j(-i\xi)^j(i\xi+2\alpha)^j$ has no real zeros. Rewriting the real part of this polynomial gives

$$\begin{aligned}
\Re\sum_{j=0}^{k} c_j(-i\xi)^j(i\xi+2\alpha)^j &= \sum_{j=0}^{k} c_j\Re(\xi^2-2i\alpha\xi)^j \\
&= \sum_{j=0}^{k} c_j\Re\sum_{l=0}^{\lfloor\frac{j}{2}\rfloor}\binom{j}{2l}\xi^{2(j-2l)}(-1)^l2^{2l}\alpha^{2l}\xi^{2l} \\
&= \sum_{j=0}^{k} c_j\sum_{l=0}^{\lfloor\frac{j}{2}\rfloor}\binom{j}{2l}\xi^{2(j-l)}(-1)^l2^{2l}\alpha^{2l} \\
&= d_0+d_1\xi^2+d_2\xi^4+\cdots+d_k\xi^{2k}.
\end{aligned}$$

We have that $d_k = c_k$. We can now iteratively choose the other d_j, such that they are all positive. Consider the coefficient corresponding to ξ^{2j}:

$$d_j\xi^{2j} = c_j\xi^{2j} + \sum_{j<m\leq k} c_m\binom{m}{2(m-j)}(-1)^{m-j}2^{2(m-j)}\alpha^{2(m-j)}.$$

If we want this to be positive, we see that we get the condition

$$c_j > -\sum_{j<m\leq k} c_m\binom{m}{2(m-j)}(-1)^{m-j}2^{2(m-j)}\alpha^{2(m-j)}.$$

By doing this iteratively we get lower bounds on all c_j. Then, it follows that

$$d_0+d_1\xi^2+d_2\xi^4+\cdots+d_k.\xi^{2k}>0$$

for $\xi\in\mathbb{R}$. It now follows that ψ is a Schwartz function. □

Remark. *From the proofs of both theorem 4.1.3 and lemma 4.1.6 lower bounds for the constants c_j follow. Taking the maximum of $\frac{c_{j+1}}{c_j}$ for all $j=0,1,\ldots,k-1$ yields a value for these constants that works for both of the proofs.*

Proposition 4.1.7. *The resolvent equation $\lambda u - Au = f$ has a classical solution.*

Proof. Theorem 4.1.3 in combination with theorem 2.1.4 gives us that the equation has a weak solution. By lemma 4.1.6, it follows that this solution is also a classical solution. □

The next proposition will give us the tools to later on obtain maximal L^p-regularity for A.

Proposition 4.1.8. *$A := -x^{-1}p(D)$ is the generator of a bounded analytic semigroup.*

Proof. The resolvent set $\rho(A)$ contains the half plane $\{\lambda\in\mathbb{C}\mid\Re\lambda\geq 0\}$. We will show an equivalent statement to the resolvent estimate in proposition 2.2.3, which is in general true:

$$\begin{aligned}
\|R(\lambda,A)\| &\leq \frac{M}{|\lambda|} &&\iff \\
\|R(\lambda,A)f\|_{k-2,\alpha-\frac{1}{2}} &\leq \frac{M}{|\lambda|}\|f\|_{k-2,\alpha-\frac{1}{2}} &&\iff \\
\|u\|_{k-2,\alpha-\frac{1}{2}} &\leq \frac{M}{|\lambda|}\|f\|_{k-2,\alpha-\frac{1}{2}} &&\iff \\
|\lambda|\|u\|_{k-2,\alpha-\frac{1}{2}} &\leq M\|f\|_{k-2,\alpha-\frac{1}{2}}
\end{aligned} \tag{4.1.16}$$

To show (4.1.16), we start with rewriting (4.1.1) to $\lambda u = f + Au$. Then

$$\begin{aligned}|\lambda|\|u\|_{k-2,\alpha-\frac{1}{2}} &\leq \|f\|_{k-2,\alpha-\frac{1}{2}} + \|Au\|_{k-2,\alpha-\frac{1}{2}} \\ &\lesssim \|f\|_{k-2,\alpha-\frac{1}{2}} + C\|u\|_{k+2,\alpha+\frac{1}{2}} \\ &\overset{(4.1.7)}{\lesssim} \|f\|_{k-2,\alpha-\frac{1}{2}}.\end{aligned}$$

Hence, the resolvent estimate holds for $\lambda \in \{\lambda \in \mathbb{C} | \Re\lambda \geq 0\}$. By proposition 2.2.3, we know that this now also holds in a sector. Then, by definition 2.2.2, we know that A generates an analytic semigroup. The semigroup is bounded by (2.2.2). $\square$

Proposition 4.1.9. *$A := -x^{-1}p(D) : X \supset D(A) \to X$ with $X := H_{k-2,\alpha-\frac{1}{2}}$ and $D(A) := H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}}$ has maximal L^p-regularity. Moreover, the following estimate holds for* (4.0.3)*:*

$$\|\partial_t u_2\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} + \|Au_2\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \lesssim \|f\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})}.$$

Proof. By proposition 4.1.8, A generates a bounded analytic semigroup. From corollary 2.2.8 it then follows that A has maximal L^p-regularity, where it is used that the initial value of the associated Cauchy problem equals zero. The estimate follows from definition 2.2.7. $\square$

4.2 Homogeneous Equation

In this section we will show estimates for (4.0.2). For this we will often use the fact that $A := -x^{-1}p(D) : X \supset D(A) \to X$, with $X := H_{k-2,\alpha-\frac{1}{2}}$ and $D(A) := H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}}$, generates an analytic semigroup.

Lemma 4.2.1. *For* (4.0.2)*, the following estimate holds:*

$$\|\partial_t u_1(t)\|^p_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \lesssim_p [u^{(0)}]^p_{D_A(1-\frac{1}{p},p)} + \|u^{(0)}\|^p_{k-2,\alpha-\frac{1}{2}} \lesssim \|u^{(0)}\|^p_{D_A(1-\frac{1}{p},p)}. \tag{4.2.1}$$

Here, $D_A(1-\frac{1}{p},p)$ is defined as in definition 2.2.5.

Proof. The inequality follows from lemma 2.2.6 using $A = -x^{-1}p(D) : H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}} \to H_{k-2,\alpha-\frac{1}{2}}$, where by proposition 4.1.8 A generates an analytic semigroup. $\square$

Remark. *Note that the space $D_A(1-\frac{1}{p},p)$ depends on k and α, which can be seen more explicitly from the fact that this space is equivalent to the interpolation space $(X, D(A))_{1-\frac{1}{p},p}$ with $X = H_{k-2,\alpha-\frac{1}{2}}$ and $D(A) = H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}}$, see proposition 2.3.6. If it is necessary to characterize how $D_A(1-\frac{1}{p},p)$ depends on the choice of k and α, we write $D_{A,k,\alpha}(1-\frac{1}{p},p)$.*

In the next lemma, we find a characterization for the $D_A(1-\frac{1}{p},p)$ space in terms of a Besov space. Additionally, we get a lower bound of the $D_A(1-\frac{1}{p},p)$-norm. For this, first define the following norm:

Definition 4.2.2. *Define for $u \in (H_{k-2,\alpha-\frac{1}{2}}, H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p}$ the following norm*

$$\|u\|_{k,\alpha,p} := \|u\|_{(H_{k-2,\alpha-\frac{1}{2}},H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p}}.$$

Here, $(H_{k-2,\alpha-\frac{1}{2}}, H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p}$ is the interpolation space as defined in definition 2.3.2.

Lemma 4.2.3. *(Characterization of the interpolation space) It holds that*

$$\|u\|_{k,\alpha,p} := \|u\|_{(H_{k-2,\alpha-\frac{1}{2}},H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p}} \sim \|w\|_{B^{k+2-\frac{4}{p}}_{2,p}}, \tag{4.2.2}$$

where $w(s) = e^{-(\alpha+\frac{1}{2}-\frac{1}{p})s}u(e^s)$, $x = e^s$. *Additionally, the following inequalities hold true*

$$\|u\|_{\tilde{k},\alpha+\frac{1}{2}-\frac{1}{p}} + \|u\|_{k-2,\alpha-\frac{1}{2}} \lesssim \|u\|_{D_A(1-\frac{1}{p},p)}, \tag{4.2.3}$$

$$\|u\|_{\tilde{k},\alpha+\frac{1}{2}-\frac{1}{p}} \lesssim \|u\|_{k,\alpha,p}, \tag{4.2.4}$$

for $\alpha + \frac{1}{2}$ *in the coercivity range. Here,*

$$\tilde{k} = \begin{cases} k+1 & \text{for } p > 4, \\ k & \text{for } 2 \le p \le 4, \\ k-1 & \text{for } \frac{4}{3} \le p < 2, \\ k-2 & \text{for } 1 < p < \frac{4}{3}. \end{cases} \tag{4.2.5}$$

Proof. To find a lower bound for the $D_A(1-\frac{1}{p},p)$ norm, we want to find an equivalent statement for the interpolation space $(H_{k-2,\alpha-\frac{1}{2}}, H_{k+2,\alpha+\frac{1}{2}} \cap H_{k-2,\alpha-\frac{1}{2}})_{1-\frac{1}{p},p}$. Note that by theorem 2.3.7 for any pair of Banach spaces Y_1, Y_2 it holds that

$$(Y_1, Y_2)_{1-\frac{1}{p},p} \cap Y_1 = (Y_1, Y_2 \cap Y_1)_{1-\frac{1}{p},p}. \tag{4.2.6}$$

Hence, it suffices to find an upper bound for the norm of $(H_{k-2,\alpha-\frac{1}{2}}, H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p}$. We get that

$$\begin{aligned}
\|u\|^p_{(H_{k-2,\alpha-\frac{1}{2}},H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p}} &= \int_0^\infty t^{-p+1} \inf_{u=u_1+u_2} \left(\|u_1\|_{k-2,\alpha-\frac{1}{2}} + t\|u_2\|_{k+2,\alpha+\frac{1}{2}}\right)^p \frac{dt}{t} \\
&= \int_0^\infty \inf_{u=u_1+u_2} \left(t^{\frac{-p+1}{p}}\|u_1\|_{k-2,\alpha-\frac{1}{2}} + t^{\frac{1}{p}}\|u_2\|_{k+2,\alpha+\frac{1}{2}}\right)^p \frac{dt}{t} \\
&\sim \int_0^\infty \inf_{u=u_1+u_2} \left(t^{-2+\frac{2}{p}}\|u_1\|^2_{k-2,\alpha-\frac{1}{2}} + t^{\frac{2}{p}}\|u_2\|^2_{k+2,\alpha+\frac{1}{2}}\right)^{p/2} \frac{dt}{t}
\end{aligned}$$

and using the definition of the norms and substituting $t \mapsto xt$ we obtain

$$\begin{aligned}
&= \int_0^\infty \inf_{u=u_1+u_2} \left(\int_0^\infty t^{-2+\frac{2}{p}}x^{-2\alpha-1+\frac{2}{p}}\sum_{j=0}^{k-2}(D^ju_1)^2 + t^{\frac{2}{p}}x^{-2\alpha-1+\frac{2}{p}}\sum_{j=0}^{k+2}(D^ju_2)^2\frac{dx}{x}\right)^{\frac{p}{2}} \frac{dt}{t} \\
&\sim \int_0^\infty \inf_{v=v_1+v_2} \left(\int_0^\infty t^{-2+\frac{2}{p}}\sum_{j=0}^{k-2}(D^jv_1)^2 + t^{\frac{2}{p}}\sum_{j=0}^{k+2}(D^jv_2)^2\frac{dx}{x}\right)^{\frac{p}{2}} \frac{dt}{t},
\end{aligned}$$

where $v_1 := x^{-\alpha-\frac{1}{2}+\frac{1}{p}}u_1$ and $v_2 := x^{-\alpha-\frac{1}{2}+\frac{1}{p}}u_2$ and we have used the commutation relation in (3.1.18). This expression is the same as writing

$$\begin{aligned}
&\int_0^\infty \inf_{v=v_1+v_2} (t^{-2+\frac{2}{p}}\|v_1\|^2_{k-2,0} + t^{\frac{2}{p}}\|v_2\|^2_{k+2,0})^{\frac{p}{2}} \frac{dt}{t} \\
&= \int_0^\infty \inf_{w=w_1+w_2} \left(t^{-2+\frac{2}{p}}\|w_1\|^2_{W^{k-2,2}(\mathbb{R})} + t^{\frac{2}{p}}\|w_2\|^2_{W^{k+2,2}(\mathbb{R})}\right)^{\frac{p}{2}} \frac{dt}{t} \\
&\sim \int_0^\infty \left(t^{-1+\frac{1}{p}}\|w_1\|_{W^{k-2,2}(\mathbb{R})} + t^{\frac{1}{p}}\|w_2\|_{W^{k+2,2}(\mathbb{R})}\right)^p \frac{dt}{t} = \|w\|^p_{(W^{k-2,2}(\mathbb{R}),W^{k+2,2}(\mathbb{R}))_{1-\frac{1}{p},p}},
\end{aligned}$$

where $v_1(e^s) = w_1(s)$, $x = e^s$, and the same relation holds for v_2 and w_2. Combining this with the rescaling from u to v, it follows that $w(s) = e^{-(\alpha+\frac{1}{2}-\frac{1}{p})s}u(e^s)$. By definition of the Besov spaces (see definition 2.3.9) we see that now (4.2.2) holds true. By the fact that $W^{k+2,2}(\mathbb{R}) \hookrightarrow W^{k-2,2}(\mathbb{R})$ and theorem 2.3.8 it follows that

$$(W^{k-2,2}(\mathbb{R}), W^{k+2,2}(\mathbb{R}))_{1-\frac{1}{p},p} \hookrightarrow (W^{k-2,2}(\mathbb{R}), W^{k+2,2}(\mathbb{R}))_{1-\frac{1}{p}-\varepsilon,2},$$

for $\varepsilon > 0$ to be determined later. Note that the same embedding also follows from embeddings using the Besov space representation ([37, Section 3.3.1]). From standard interpolation theory (working out the minimization in the definition of the K-method, see definition 2.3.2) it is known that

$$(W^{k-2,2}(\mathbb{R}), W^{k+2,2}(\mathbb{R}))_{1-\frac{1}{p},2} = W^{\tilde{k},2}(\mathbb{R}),$$

with $\tilde{k} = k + 2 - \frac{4}{p} - 4\varepsilon$. From this, we see that indeed we can choose ε such that (4.2.5) is true. Hence,

$$\|\cdot\|_{W^{\tilde{k},2}(\mathbb{R})} \lesssim \|\cdot\|_{(W^{k-2,2}(\mathbb{R}),W^{k+2,2}(\mathbb{R}))_{1-\frac{1}{p},p}}.$$

Undoing the rescalings, we now find that

$$\|u\|_{\tilde{k},\alpha+\frac{1}{2}-\frac{1}{p}} \lesssim \|w\|_{(W^{k-2,2}(\mathbb{R}),W^{k+2,2}(\mathbb{R}))_{1-\frac{1}{p},p}}.$$

Since by (4.2.6)

$$(H_{k-2,\alpha-\frac{1}{2}}, H_{k+2,\alpha+\frac{1}{2}} \cap H_{k-2,\alpha-\frac{1}{2}})_{1-\frac{1}{p},p} = (H_{k-2,\alpha-\frac{1}{2}}, H_{k+2,\alpha+\frac{1}{2}})_{1-\frac{1}{p},p} \cap H_{k-2,\alpha-\frac{1}{2}},$$

holds true, the lower bound (4.2.3) follows.

Remark. *From lemma 4.2.3 it follows that the norms $\|\cdot\|_{D_A(1-\frac{1}{p},p)}$ and $\|\cdot\|_{k,\alpha,p} + \|\cdot\|_{k-2,\alpha-\frac{1}{2}}$ are equivalent.*

□

4.3 Parabolic Maximal L^p-Regularity Estimate

The goal of this section is to show a maximal regularity estimate for (4.0.1). For this, we use the estimates derived in sections 4.1 and 4.2.

Corollary 4.3.1. *Let $f \in L^p(0, \infty; H_{k-2,\alpha-\frac{1}{2}})$, then equation* (4.0.3) *has a solution which is differentiable almost everywhere and takes its values in $D(A)$.*

Proof. This follows by definition from the fact that $A := -x^{-1}p(D)$ has maximal L^p-regularity 2.2.7. □

The next proposition contains a technical result needed to obtain the maximal regularity estimate.

Proposition 4.3.2. *For $A = -x^{-1}p(D)$, where γ_i, $i = 1, \ldots, 4$ are the zeros of $p(D)$, we have that if $\alpha + \frac{1}{2}$ lies in the coercivity range and $2\gamma_i - 2\alpha \neq 1$ for $i = 1, \ldots, 4$, then the following estimate holds*

$$\|Au\|_{k-2,\alpha-\frac{1}{2}} \gtrsim \|u\|_{k+2,\alpha+\frac{1}{2}}, \tag{4.3.1}$$

for $u \in H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}}$.

Proof. We prove this by applying Hardy's inequality (see lemma 2.2.10) iteratively for $u \in C_c^\infty(0,\infty)$. By density of $C_c^\infty(0,\infty)$ in $H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}}$ we also get the result for u in this space. Note that $p(D)$ is a fourth order polynomial which we will rewrite as $p(D) = (D-\gamma)\tilde{p}(D)$, where γ is one of the zeros of $p(D)$ and $\tilde{p}(D)$ is the remaining third order polynomial. By application of Hardy's inequality we get that for $2\gamma - 2\alpha - 1 \neq 0$:

$$\begin{aligned}
\|Au\|^2_{k-2,\alpha-\frac{1}{2}} &= \|x^{-\alpha-\frac{1}{2}}p(D)u\|^2_{k-2,0} \\
&= \|x^{\gamma-\alpha-\frac{1}{2}}Dx^{-\gamma}\tilde{p}(D)u\|^2_{k-2,0} \\
&= \sum_{j=0}^{k-2}\int_0^\infty x^{2\gamma-2\alpha-1}(x\partial_x D^j x^{-\gamma}\tilde{p}(D)u)^2\frac{dx}{x} \\
&\gtrsim \sum_{j=0}^{k-2}\int_0^\infty x^{2\gamma-2\alpha-1}(D^j x^{-\gamma}\tilde{p}(D)u)^2\frac{dx}{x} \\
&= \sum_{j=0}^{k-2}\int_0^\infty x^{-2\alpha-1}((D-\gamma)^j\tilde{p}(D)u)^2\frac{dx}{x} \sim \|\tilde{p}(D)u\|^2_{k-2,\alpha+\frac{1}{2}},
\end{aligned}$$

i.e., $\|Au\|_{k-2,\alpha-\frac{1}{2}} \geq C\|\tilde{p}(D)u\|_{k-2,\alpha+\frac{1}{2}}$ for some constant C. We also have

$$\|Au\|_{k-2,\alpha-\frac{1}{2}} = \|(D-\gamma)\tilde{p}(D)\|_{k-2,\alpha+\frac{1}{2}} \geq \|D\tilde{p}(D)u\|_{k-2,\alpha+\frac{1}{2}} - |\gamma|\|\tilde{p}(D)u\|_{k-2,\alpha+\frac{1}{2}},$$

and combining the two inequalities gives

$$\begin{aligned}
\left(1+\frac{1+2|\gamma|}{C}\right)\|Au\|_{k-2,\alpha-\frac{1}{2}} &\geq \|D\tilde{p}(D)u\|_{k-2,\alpha+\frac{1}{2}} + (1+|\gamma|)\|\tilde{p}(D)u\|_{k-2,\alpha+\frac{1}{2}} \\
&\gtrsim \|\tilde{p}(D)u\|_{k-1,\alpha+\frac{1}{2}}.
\end{aligned}$$

Repeating this argument three times gives the desired estimate, and by a density argument this extends to $u \in H_{k-2,\alpha-\frac{1}{2}} \cap H_{k+2,\alpha+\frac{1}{2}}$. □

We are now in a position to prove a maximal regularity estimate.

Proposition 4.3.3. *For equation* (4.0.1), *with* $f \in L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})$ *and* $u^{(0)} \in D_A(1-\frac{1}{p},p)$, *the following estimate holds:*

$$\begin{aligned}
\sup_{t\geq 0}\|u\|^p_{k,\alpha,p} + \int_0^\infty \|\partial_t u\|^p_{k-2,\alpha-\frac{1}{2}} + \|u\|^p_{k+2,\alpha+\frac{1}{2}}dt \lesssim \|u^{(0)}\|^p_{k,\alpha,p} + \|u^{(0)}\|^p_{k-2,\alpha-\frac{1}{2}} \\
+ \int_0^\infty \|f\|^p_{k-2,\alpha-\frac{1}{2}}dt.
\end{aligned} \tag{4.3.2}$$

Proof. From proposition 4.1.9, we get the following maximal regularity estimate for u_2:

$$\|\partial_t u_2\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} + \|Au_2\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \lesssim \|f\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})}.$$

We will bound the norms of $\partial_t u$ and Au using this estimate and the estimates of lemma 4.2.1.

$$\begin{aligned}
\|\partial_t u\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} &= \|\partial_t(T(t)u^{(0)}) + \partial_t\left(\int_0^t T(t-s)f(s)ds\right)\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \\
&\leq \|\partial_t u_1\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} + \|\partial_t u_2\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \\
&\lesssim \|u^{(0)}\|_{k,\alpha,p} + \|u^{(0)}\|_{k-2,\alpha-\frac{1}{2}} + \|f\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})}.
\end{aligned}$$

Similarly

$$\begin{aligned}\|Au\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} &\leq \|Au_1\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} + \|Au_2\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \\ &\lesssim \|\partial_t u_1\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} + \|f\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \\ &\lesssim \|u^{(0)}\|_{k,\alpha,p} + \|u^{(0)}\|_{k-2,\alpha-\frac{1}{2}} + \|f\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})},\end{aligned}$$

where we use that $Au_1 = \partial_t u_1$ holds true since A generates an analytic semigroup. Combining these estimates with (4.3.1) from proposition 4.3.2 gives

$$\int_0^\infty \|\partial_t u\|^p_{k-2,\alpha-\frac{1}{2}} + \|u\|^p_{k+2,\alpha+\frac{1}{2}} dt \lesssim \|u^{(0)}\|^p_{k,\alpha,p} + \|u^{(0)}\|^p_{k-2,\alpha-\frac{1}{2}} + \int_0^\infty \|f\|^p_{k-2,\alpha-\frac{1}{2}}. \quad (4.3.3)$$

Using (4.2.2) of lemma 4.2.3 and the fact that the $D_A(1-\frac{1}{p},p)$ and interpolation norms are equivalent (see proposition 2.3.6 and 2.3.5), we see that

$$\|u\|_{k,\alpha,p} \lesssim \|u\|_{D_A(1-\frac{1}{p},p)}.$$

From this it follows, using theorem 2.3.7 that

$$\|u\|_{k,\alpha,p} \lesssim \|u\|_{L^p(0,\infty;H_{k+2,\alpha+\frac{1}{2}})} + \|\partial_t u\|_{L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})}.$$

Taking the supremum gives

$$\sup_{t\geq 0} \|u\|^p_{k,\alpha,p} \lesssim \int_0^\infty \|u\|^p_{k+2,\alpha+\frac{1}{2}} + \|\partial_t u\|^p_{k-2,\alpha-\frac{1}{2}} dt$$

and combining this estimate with (4.3.3) gives the estimate (4.3.2). □

Using a scaling argument, we can improve the estimate found in proposition 4.3.3.

Corollary 4.3.4. *For equation (4.0.1), with $f \in L^p(0,\infty;H_{k-2,\alpha-\frac{1}{2}})$ and $u^{(0)} \in D_A(1-\frac{1}{p},p)$, the following maximal regularity estimate holds:*

$$\sup_{t\geq 0} \|u\|^p_{k,\alpha,p} + \int_0^\infty \|\partial_t u\|^p_{k-2,\alpha-\frac{1}{2}} + \|u\|^p_{k+2,\alpha+\frac{1}{2}} dt \lesssim \|u^{(0)}\|^p_{k,\alpha,p} + \int_0^\infty \|f\|^p_{k-2,\alpha-\frac{1}{2}} dt. \quad (4.3.4)$$

Proof. We use the scaling $u(t,x) = \tilde{u}(\lambda^{-1}t, \lambda^{-1}x)$, with $\lambda > 0$. This gives

$$\partial_t u(t,x) = \lambda^{-1}(\partial_t \tilde{u})(\lambda^{-1}t, \lambda^{-1}x), \quad (Au)(t,x) = \lambda^{-1}(A\tilde{u})(\lambda^{-1}t, \lambda^{-1}x).$$

Inserting this into (4.0.1) gives

$$\lambda^{-1}(\partial_t \tilde{u} - A\tilde{u})(\lambda^{-1}t, \lambda^{-1}x) = f(t,x),$$

which can be rewritten as

$$(\partial_t \tilde{u} - A\tilde{u})(\lambda^{-1}t, \lambda^{-1}x) = \lambda f(t,x) =: \tilde{f}(\lambda^{-1}t, \lambda^{-1}x).$$

Note that writing $u(t,x) = \tilde{u}(\lambda^{-1}t, \lambda^{-1}x)$ is equivalent to writing $u(\lambda t, \lambda x) = \tilde{u}(t,x)$. Similarly, $\tilde{f}(\lambda^{-1}t, \lambda^{-1}x) = \lambda f(t,x)$ is equivalent to $\tilde{f}(t,x) = \lambda f(\lambda t, \lambda x)$. This gives the scaled equation

$$\begin{aligned}\partial_t u(\lambda t, \lambda x) - Au(\lambda t, \lambda x) &= \lambda f(\lambda t, \lambda x), \\ u(0, \lambda x) &= u^{(0)}(\lambda x).\end{aligned}$$

Consider the second term in the integral, using the change of variables $y = \lambda x$ and $\tau = \lambda t$:

$$\begin{aligned}
\int_0^t \|u(\lambda t, \lambda x)\|^p_{k+2,\alpha+\frac{1}{2}} dt &= \int_0^\infty \left(\sum_{j=0}^{k+2} \int_0^\infty x^{-2(\alpha+\frac{1}{2})} |D^j u(\lambda t, \lambda x)|^2 \frac{dx}{x} \right)^{\frac{p}{2}} dt \\
&= \int_0^\infty \left(\sum_{j=0}^{k+2} \int_0^\infty \left(\frac{y}{\lambda}\right)^{-2(\alpha+\frac{1}{2})} |D^j u(\tau, y)|^2 \frac{dy}{y} \right)^{\frac{p}{2}} \lambda^{-1} d\tau \\
&= \lambda^{p(\alpha+\frac{1}{2})-1} \int_0^\infty \|u(\tau, y)\|^p_{k+2,\alpha+\frac{1}{2}} d\tau.
\end{aligned}$$

The same reasoning applies in a similar fashion for the other terms in the estimate, except from the $\|\cdot\|_{k,\alpha,p}$ norm. For the $\|\cdot\|_{k,\alpha,p}$ norm, we note that letting $\lambda e^s = e^r$ is equivalent to $s = \log(\lambda^{-1}) + r$. This gives

$$e^{-(\alpha+\frac{1}{2}-\frac{1}{p})s} u(\lambda e^s) = e^{-(\alpha+\frac{1}{2}-\frac{1}{p})(\log(\lambda^{-1})+r)} u(e^r) = \lambda^{\alpha+\frac{1}{2}-\frac{1}{p}} e^{-(\alpha+\frac{1}{2}-\frac{1}{p})r} u(e^r).$$

Using lemma 4.2.3 we see that

$$\|u(\lambda t, \lambda x)\|_{k,\alpha,p} \sim \lambda^{\alpha+\frac{1}{2}-\frac{1}{p}} \|w\|_{B^{k+2-\frac{4}{p}}_{2,p}} \sim \lambda^{\alpha+\frac{1}{2}-\frac{1}{p}} \|u(\tau, y)\|,$$

where $w(s) = e^{-(\alpha+\frac{1}{2}-\frac{1}{p})s} u(e^s)$ and $y = e^r$. This gives the following estimate

$$\begin{aligned}
&\sup_{t\geq 0} \lambda^{p(\alpha+\frac{1}{2})-1} \|u\|^2_{k,\alpha,p} + \lambda^{p(\alpha+\frac{1}{2})-1} \int_0^\infty \|\partial_t u\|^2_{k-2,\alpha-\frac{1}{2}} + \|u\|^2_{k+2,\alpha+\frac{1}{2}} dt \\
&\lesssim \left(\lambda^{p(\alpha+\frac{1}{2})-1} \|u^{(0)}\|^2_{k,\alpha} + \lambda^{p(\alpha-\frac{1}{2})} \|u^{(0)}\|^2_{k-2,\alpha-\frac{1}{2}} + \lambda^{p(\alpha+\frac{1}{2})-1} \|f\|^2_{L^2(0,\infty;H_{k-2,\alpha-\frac{1}{2}})} \right).
\end{aligned}$$

Dividing by $\lambda^{p(\alpha+\frac{1}{2})-1}$ and letting $\lambda \to \infty$, using that $p > 1$, gives the result. ☐

Remark. *From the inequality* (4.2.4) *it follows that we also have the estimate*

$$\sup_{t\geq 0} \|u\|^p_{\tilde{k},\alpha+\frac{1}{2}-\frac{1}{p}} + \int_0^\infty \|\partial_t u\|^p_{k-2,\alpha-\frac{1}{2}} + \|u\|^p_{k+2,\alpha+\frac{1}{2}} dt \lesssim \|u^{(0)}\|^p_{k,\alpha,p} + \int_0^\infty \|f\|^p_{k-2,\alpha-\frac{1}{2}} dt,$$

where $\tilde{k}$ is defined as in (4.2.5).

Formulating an existence and uniqueness statement for the linear problem is possible:

Lemma 4.3.5. *Let $u^{(0)} \in D_A(1 - \frac{1}{p}, p)$ and $f \in L^p(0,\infty; H_{k-2,\alpha-\frac{1}{2}})$. Then, equation* (4.0.1) *has a unique solution $u \in \dot{W}^{1,p}(0,\infty; H_{k-2,\alpha-\frac{1}{2}}) \cap L^p(0,\infty; H_{k+2,\alpha+\frac{1}{2}})$.*

Proof. We have existence of a solution to (4.0.3) from corollary 4.3.1. We have existence of a solution to (4.0.2) from [13, Comment 3.9.iii]. Since (4.0.1) is the sum of (4.0.3) and (4.0.2), we now also have a solution to (4.0.1). The fact that this solution is in $\dot{W}^{1,p}(0,\infty; H_{k-2,\alpha-\frac{1}{2}}) \cap L^p(0,\infty; H_{k+2,\alpha+\frac{1}{2}})$ follows directly from the maximal regularity estimate (4.3.4). Uniqueness of the solution follows from the fact that

$$\begin{aligned}
\partial_t w - Aw &= 0, \\
w(0) &= 0,
\end{aligned}$$

has the unique solution $w = 0$ [13, Comment 3.9.iii] ☐

Corollary 4.3.6. *For finitely many (k_i, α_i), $i = 1, \dots, n$, where α_i lies in the coercivity range, it holds that if $u^{(0)} \in \bigcap_{i=1}^n D_{A,k_i,\alpha_i}$ and $f \in \bigcap_{i=1}^n L^p(0,\infty; H_{k_i-2,\alpha_i-\frac{1}{2}})$, there is a unique solution*

$$u \in \bigcap_{i=1}^n \left(\dot{W}^{1,p}(0,\infty; H_{k_i-2,\alpha_i-\frac{1}{2}}) \cap L^p(0,\infty; H_{k_i+2,\alpha_i+\frac{1}{2}}) \right)$$

to (4.0.1) *which satisfies*

$$\sup_{t\geq 0} \|u\|^p_{k_i,\alpha_i,p} + \int_0^\infty \|\partial_t u\|^p_{k_i-2,\alpha_i-\frac{1}{2}} + \|u\|^p_{k_i+2,\alpha_i+\frac{1}{2}} dt \lesssim \|u^{(0)}\|^p_{k_i,\alpha_i,p} + \int_0^\infty \|f\|^p_{k_i-2,\alpha_i-\frac{1}{2}} dt$$

for all $i = 1, \dots, n$.

Proof. The existence of a solution for all of the weights individually, including the estimates, follows from lemma 4.3.5. Since all these solutions share the same initial value and right hand side function f, from the uniqueness of solutions it follows that all of these solutions must be equal. □

4.4 Elliptic Regularity

In this section we derive the setting which will be used in treating the nonlinear equation later on.

Lemma 4.4.1. *It holds, for $x > 0$, that*

$$\ker(A) = \operatorname{span}\{x^\gamma | \ \gamma \textit{ zero of } p(D)\}.$$

Proof. It is easy to see that $Ax^\gamma = 0$ if and only if $\gamma \in \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. This shows that

$$\{x^\gamma | \ \gamma \text{ zero of } p(D)\} \subset ker(A).$$

We still need to determine for which u it holds that $Au = 0$. This is a fourth order linear ordinary differential equation. Since we know that on taking u equal to x^γ with $\gamma \in \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, $Au = 0$ holds and because $\gamma_1, \dots, \gamma_4$ are distinct, we have four linearly independent solutions to the differential equation $Au = 0$. Hence,

$$\ker(A) = \operatorname{span}\{x^\gamma | \ \gamma \text{ zero of } p(D)\}.$$

□

Proposition 4.4.2. *Let $n \in \left(1, \frac{3}{2}\right) \cup \left(\frac{3}{2}, 3\right)$ and suppose that $v \in \bigcap_{j=1}^n H_{k_j,\alpha_j}$ is the unique solution to*

$$\begin{aligned} \partial_t v - Av &= g, \\ v|_{t=0} &= v^{(0)}, \end{aligned}$$

where $g \in \bigcap_{j=1}^n L^p(0,\infty; H_{k_j-2,\alpha_j-\frac{1}{2}})$, $v^{(0)} = \bigcap_{j=1}^n D_{A,k_j,\alpha_j}(1-\frac{1}{p}, p)$. Then

$$Au = v, \quad Af = g, \quad Au^{(0)} = v^{(0)}$$

have a unique solutions u, f and $u^{(0)}$, respectively. Under the assumption that $\|f\|_{k+2,\alpha_j+\frac{1}{2}}$ for all j and $\|u^{(0)}\|_{k+4,\alpha_j+1,p} < \infty$ for $\alpha_j + 1 < 0$ there exist u_0, $u_0^{(0)}$, u_β and $u_\beta^{(0)}$ such that

$$\begin{aligned} u &\in \bigcap_{\gamma_2<\alpha_j+1<0} H_{k_j+4,\alpha_j+1}, & u^{(0)} &\in \bigcap_{\alpha_j+1<0} D_{A,k_j+4,\alpha_j+1}, \\ u - u_0 &\in \bigcap_{j=1,0<\alpha_j+1<\beta}^{n} H_{k_j+4,\alpha_j+1}, & u^{(0)} - u_0^{(0)} &\in \bigcap_{0<\alpha_j+1<\beta} D_{A,k_j+4,\alpha_j+1}, \\ u - u_0 - u_\beta x^\beta &\in \bigcap_{j=1,\beta<\alpha_j+1}^{n} H_{k_j+4,\alpha_j+1}, & u^{(0)} - u_0^{(0)} - u_\beta^{(0)} x^\beta &\in \bigcap_{\beta<\alpha_j+1} D_{A,k_j+4,\alpha_j+1}, \\ f &\in \bigcap_{j=1}^{n} H_{k_j+2,\alpha_j+\frac{1}{2}}. \end{aligned} \tag{4.4.1}$$

The coefficients u_0 and u_β are defined in (4.4.12) and (4.4.13). Additionally, for (u, u_0, u_β), $(u^{(0)}, u_0^{(0)}, u_\beta^{(0)})$ and f, the following equivalences hold true:

$$\begin{aligned} \|u\|_{k_j+4,\alpha_j+1} &\sim \|v\|_{k_j,\alpha_j} && \text{for } \alpha_j + 1 < 0, \\ \|u - u_0\|_{k_j+4,\alpha_j+1} &\sim \|v\|_{k_j,\alpha_j} && \text{for } 0 < \alpha_j + 1 < \beta, \\ \|u - u_0 - u_\beta x^\beta\|_{k_j+4,\alpha_j+1} &\sim \|v\|_{k_j,\alpha_j} && \text{for } \beta > \alpha_j + 1, \\ \|u^{(0)}\|_{k_j+4,\alpha_j+1} &\sim \|v\|_{k_j,\alpha_j} && \text{for } \alpha_j + 1 < 0, \\ \|u^{(0)} - u_0^{(0)}\|_{k_j+4,\alpha_j+1} &\sim \|v\|_{k_j,\alpha_j} && \text{for } 0 < \alpha_j + 1 < \beta, \\ \|u^{(0)} - u_0^{(0)} - u_\beta^{(0)} x^\beta\|_{k_j+4,\alpha_j+1} &\sim \|v\|_{k_j,\alpha_j} && \text{for } \beta > \alpha_j + 1, \\ \|f\|_{k+2,\alpha_j+\frac{1}{2}} &\sim \|g\|_{k-2,\alpha_j-\frac{1}{2}}. \end{aligned} \tag{4.4.2}$$

Before starting with the proof, let us first look at the idea on which it is based. Say that we have a solution u of (4.0.1). Formally applying the operator A to the Cauchy problem (4.0.1) gives

$$\partial_t(Au) - A(Au) = Af, \tag{4.4.3a}$$

$$(Au)|_{t=0} = Au^{(0)}. \tag{4.4.3b}$$

Then Au solves (4.4.3) and we can find u from this solution by inverting A. In the proof we will argue in the opposite direction, by finding a solution to

$$\partial_t v - Av = g \tag{4.4.4a}$$

$$v|_{t=0} = v^{(0)}, \tag{4.4.4b}$$

satisfying $v^{(0)} \in D_A(1-\frac{1}{p}, p)$ and $g \in L^p(0,\infty; H_{k-2,\alpha-\frac{1}{2}})$ where $g := Af$ and $v^{(0)} := Au^{(0)}$ and then inverting A to find u.

Proof. Consider (4.4.4). By lemma 4.3.5, we know that (4.4.4) has a solution $v \in \dot{W}^{1,p}(0,\infty; H_{k-2,\alpha-\frac{1}{2}}) \cap L^p(0,\infty; H_{k+2,\alpha+\frac{1}{2}})$. Because $Au = v$, we can find u from the solution v by inverting A and similarly for finding $u^{(0)}$ and f. Note that by doing this, the solution u found will depend on the weight. This fact will be used to define the coefficients u_0 and u_β later on in the proof. We can see this dependence in the following way: Note that the operator A is given by

$$A := -x^{-1}p(D) = -x^{-1}(D-\gamma_1)(D-\gamma_2)(D-\gamma_3)(D-\gamma_4),$$

where $\gamma_1, \ldots, \gamma_4$ are the zeros of the fourth order polynomial $p(D)$ (see (3.1.12) and (3.1.22)). To invert this expression we proceed in steps, inverting each factor of the polynomial one by one. For illustration only the first part is worked out in detail, the other steps work in a similar fashion. First, define $u^{(1)} := (D-\gamma_2)(D-\gamma_3)(D-\gamma_4)u$ and $\tilde{v} := -v$, so that we need to find $u^{(1)}$ from

$$x^{-1}(D-\gamma_1)u^{(1)} = \tilde{v}. \tag{4.4.5}$$

Note that this is equivalent to

$$x^{\gamma_1 - 1}D(x^{-\gamma_1}u^{(1)}) = \tilde{v}. \tag{4.4.6}$$

We need the solution v (or equivalently $\tilde{v}$) to be finite in some norm, say with weight ω and l derivatives, i.e., $\|v\|_{l,\omega} < \infty$. This also holds for $\tilde{v}$, so $\|\tilde{v}\|_{l,\omega} < \infty$. This gives that also $\|Au\|_{l,\omega} < \infty$ and that $\|x^{-1}(D-\gamma_1)u^{(1)}\|_{l,\omega} < \infty$. By writing out the norms, we see that we get a condition on v, namely that $v = o(x^\omega)$ as $x \downarrow 0$ (and the same condition for all derivatives of v up to and including the l^{th} derivative) for the norm to be finite. Here the dependence on the weight appears. Continuing the rewriting from (4.4.6) gives

$$D(x^{-\gamma_1}u^{(1)}) = x^{-\gamma_1+1}\tilde{v} \quad (= o(x^{\omega-\gamma_1+1}) \text{ as } x \downarrow 0).$$

This gives

$$x^{-\gamma_1}u^{(1)} = \begin{cases} c + \int_0^x \tilde{x}^{-\gamma_1+1}\tilde{v}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega > \gamma_1 - 1, \\ c - \int_x^\infty \tilde{x}^{-\gamma_1+1}\tilde{v}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega < \gamma_1 - 1. \end{cases}$$

It follows that

$$u^{(1)} = \begin{cases} cx^{\gamma_1} + x^{\gamma_1}\int_0^x \tilde{x}^{-\gamma_1+1}\tilde{v}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega > \gamma_1 - 1, \\ cx^{\gamma_1} - x^{\gamma_1}\int_x^\infty \tilde{x}^{-\gamma_1+1}\tilde{v}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega < \gamma_1 - 1. \end{cases} \tag{4.4.7}$$

Hence, we have that $u^{(1)}$ depends on the choice of the weight exponent. We show that the constants c vanish. Consider $\omega > \gamma_1 - 1$. Then,

$$x^{\gamma_1}\int_0^x \tilde{x}^{-\gamma_1+1}\tilde{v}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} = o(x^{\omega+1}) \quad \text{as } x \downarrow 0.$$

Hence, $u^{(1)}(x) = c(1+o(1))x^{\gamma_1}$ as $x \downarrow 0$. Now consider the norm of $u^{(1)}$ with weight exponent $\omega + 1$ (but only the part close to zero). Then

$$\int_0^\varepsilon (u^{(1)}(x))^2 x^{-2(\omega+1)}\frac{dx}{x} = c(1+o(\varepsilon^0))\int_0^\varepsilon x^{2(\gamma_1-\omega-1)}\frac{dx}{x}.$$

This integral diverges if and only if $c \neq 0$. So finiteness of the norm therefore entails $c = 0$. In the case that $\omega < \gamma_1 - 1$, a similar argument shows that also $c = 0$ in that case.

Now for the second step of inverting A: define $u^{(2)} := (D-\gamma_3)(D-\gamma_4)u$. We now have to solve

$$(D-\gamma_2)u^{(2)} = u^{(1)}.$$

Note that $u^{(1)} = o(x^{\omega+1})$, which can be seen from (4.4.7). Using similar steps as before, this results in

$$u^{(2)} = \begin{cases} x^{\gamma_2}\int_0^x \tilde{x}^{-\gamma_2}u^{(1)}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega + 1 > \gamma_2, \\ -x^{\gamma_2}\int_x^\infty \tilde{x}^{-\gamma_2}u^{(1)}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega + 1 < \gamma_2. \end{cases} \tag{4.4.8}$$

For the third step, let $u^{(3)} := (D-\gamma_4)u$. Solving

$$(D-\gamma_3)u^{(3)} = u^{(2)}$$

gives

$$u^{(3)} = \begin{cases} x^{\gamma_3}\int_0^x \tilde{x}^{-\gamma_3}u^{(2)}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega+1>\gamma_3, \\ -x^{\gamma_3}\int_x^\infty \tilde{x}^{-\gamma_3}u^{(2)}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega+1<\gamma_3. \end{cases} \tag{4.4.9}$$

Finally, in the last step we solve

$$(D-\gamma_4)u = u^{(3)},$$

which gives

$$u = \begin{cases} x^{\gamma_4}\int_0^x \tilde{x}^{-\gamma_4}u^{(3)}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega+1>\gamma_4, \\ -x^{\gamma_4}\int_x^\infty \tilde{x}^{-\gamma_4}u^{(3)}(\tilde{x})\frac{d\tilde{x}}{\tilde{x}} & \text{for } \omega+1<\gamma_4. \end{cases} \tag{4.4.10}$$

Choosing specific values of ω, we can now find an ω-dependent solution to $Au = v$.

We get a maximal regularity result for equation (4.4.3):

$$\sup_{t\geq 0}\|v\|^p_{m,\alpha,p} + \int_0^\infty \|\partial_t v\|^p_{m-2,\alpha-\frac{1}{2}} + \|v\|^p_{m+2,\alpha+\frac{1}{2}}\,dt \lesssim \|v^{(0)}\|^p_{m,\alpha,p} + \int_0^\infty \|g\|^p_{m-2,\alpha-\frac{1}{2}}\,dt, \tag{4.4.11}$$

where $\alpha+\frac{1}{2}$ lies in the coercivity range. We choose the weight exponents $\alpha = -\frac{3}{2}+\frac{1}{p}\pm\delta$ and $\alpha = -\frac{3}{2}+\beta\pm\delta$. These choices will allow us to define u_0 and u_β. First consider the weight exponents $\alpha = -\frac{3}{2}+\frac{1}{p}\pm\delta$. The supremum term in the maximum regularity estimate becomes $\|v\|^p_{m,-\frac{3}{2}+\frac{1}{p}\pm\delta,p}$ and the term of v in the L^p integral in time becomes $\|v\|^p_{m+2,-1+\frac{1}{p}\pm\delta}$. From this first term we will be able to define u_0 as follows; give the solution u found using the steps from above ((4.4.10), and all integrals from before substituted) using the weight exponent $-1+\delta$ and $-1-\delta$ the name $u_{(+\delta)}$ and $u_{(-\delta)}$, respectively. It holds that

$$u_{(-\delta)} - u_{(+\delta)} \in \ker(A) = \operatorname{span}\{x^\gamma | \gamma \text{ zero of } p(D)\}.$$

By looking at the order of $u_{(+\delta)} - u_{(-\delta)}$, we can conclude that this can only contain terms that are constant in x. Hence, define

$$u_0 := u_{(-\delta)} - u_{(+\delta)}. \tag{4.4.12}$$

For the choice of these weight exponents to make sense, it is needed that $-1+\frac{1}{p}\pm\delta$ lies in the coercivity range, or equivalently, that $\frac{1}{p}\pm\delta$ lies in the coercivity range shifted up by one, see figure 4.2. This follows from the fact that the weight of the norm of v in the L^p integral in time always needs to lie in the coercivity range for the maximal regularity result to hold. This is possible for all values of p.

Now, consider the weight exponents $\alpha = -\frac{3}{2}+\beta\pm\delta$. Using these weight exponents, the term of v in the L^p integral in time becomes $\|v\|^p_{m+2,-1+\beta\pm\delta}$. We can define u_β from this as follows; let the solution found using the weight exponent $-1+\beta+\delta$ and using $-1+\beta-\delta$ be called $u_{(\beta+\delta)}$ and $u_{(\beta-\delta)}$, respectively. Note that $u_{(\beta-\delta)}$ and $u_{(+\delta)}$ are the same, since the weight exponents $\beta-\delta$ and $\beta+\delta$ both lie between γ_3 and $\gamma_4=\beta$. Similar to before, the difference between $u_{(\beta+\delta)}$ and $u_{(\beta-\delta)}$ has to lie in $\ker(A)$. Again by looking at the order of this solution, we see that

$$u_\beta x^\beta = u_{(\beta-\delta)} - u_{(\beta+\delta)},$$

and we can define u_β by dividing by x^β on both sides

$$u_\beta := \frac{u_{(\beta-\delta)} - u_{(\beta+\delta)}}{x^\beta}. \tag{4.4.13}$$

For these weight exponents we need that $-1+\beta\pm\delta$ lies in the coercivity range, or equivalently, that $\beta\pm\delta$ lies in the coercivity range shifted up by one. Looking at figure 4.1, we see that this is possible for all $n \in (1, \frac{3}{2}) \cup (\frac{3}{2}, 3)$.

Call the inversion operator as defined above with weight exponent $\omega = -\delta - 1$ the name $A^{-1}_{(-\delta)}$. We check that applying this operator to (4.4.4) gives (4.0.1). We use this weight exponent, since $\omega + 1 = \delta$ lies in the coercivity range, and hence this is the solution we are looking for (note that taking any weight that lies between γ_2 and γ_3 yields the same result). We have that

$$A^{-1}_{(-\delta)}\partial_t v = \partial_t A^{-1}_{(-\delta)} v = \partial_t u,$$

where the inverse and the time derivative commute since the inverse only contains spatial contributions. Because we have defined u as the inverse of A applied to v, the last equality follows. Similarly, since by definition f is equal to the inverse of A applied to g it follows that

$$A^{-1}_{(-\delta)} g = f.$$

Now for the last term we need to show that

$$A^{-1}_{(-\delta)} A v = A u.$$

Writing out the left hand side gives

$$A^{-1}_{(-\delta)} A v = x^{\gamma_4} \int_x^\infty x_1^{-\gamma_4} x_1^{\gamma_3} \int_{x_1}^\infty x_2^{-\gamma_3} x_2^{\gamma_2} \int_0^{x_2} x_3^{-\gamma_2} x_3^{\gamma_1} \int_0^{x_3} x_4^{-\gamma_1+1} x_4^{-1} p(D) v(x_4) \frac{dx_4}{x_4}\frac{dx_3}{x_3}\frac{dx_2}{x_2}\frac{dx_1}{x_1}.$$

The inner most integral, additionally considering the $x_3^{\gamma_1}$ in front, can be rewritten as

$$x_3^{\gamma_1} \int_0^{x_3} x_4^{-\gamma_1+1} x_4^{-1} p(D) v(x_4) \frac{dx_4}{x_4} = x_3^{\gamma_1} \int_0^{x_3} x_4^{-\gamma_1} x_4^{\gamma_1+1} \partial_x \left[x_4^{-\gamma_1} (D-\gamma_2)(D-\gamma_3)(D-\gamma_4) v \right] \frac{dx_4}{x_4},$$

using that $x\partial_x - \gamma = x^{\gamma+1}\partial_x x^{-\gamma}$. Further rewriting gives,

$$A^{-1}_{(-\delta)} A v = x_3^{\gamma_1} \int_0^{x_3} \partial_x \left[x_4^{-\gamma_1} (D-\gamma_2)(D-\gamma_3)(D-\gamma_4) v \right] dx_4 = (D-\gamma_2)(D-\gamma_3)(D-\gamma_4) v.$$

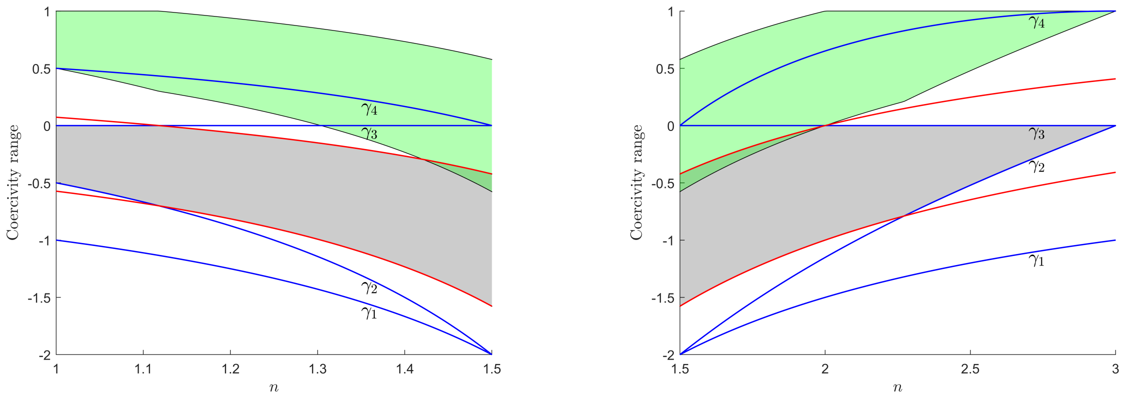
Repeating these arguments another three times then gives

$$A^{-1}_{(-\delta)} A v = v = A u,$$

since by definition $v = Au$. So, applying the inverse operator $A^{-1}_{-\delta}$ to (4.4.4) indeed gives (4.0.1). The equivalences from (4.4.2) can be seen from how the definitions of u_0 and u_β follow from inversion of v with the weight exponents as chosen above. Note that for values of the weight $\alpha_j + 1$ that lie below γ_1 or between γ_1 and γ_2 no additional terms have to be added or subtracted (like in the case when we are above zero) because we have that

$$\sup_{t\geq 0} \|u\|^p_{m_j+4,\alpha_j+1} + \int_0^\infty \|\partial_t u\|^p_{m_j+2,\alpha_j+\frac{1}{2}} + \|u\|^p_{m_j+6,\alpha_j+\frac{3}{2}} \lesssim \|u^{(0)}\|^p_{m_j+4,\alpha_j+1,p} + \int_0^\infty \|f\|^p_{m_j+2,\alpha_j+\frac{1}{2}} dt,$$

since we assumed that the right-hand side is finite.



(a) $n \in (1, \frac{3}{2})$

(b) $n \in (\frac{3}{2}, 3)$

Figure 4.2: For the two different cases of n the zeros $\gamma_1, \ldots, \gamma_4$ of $p(D)$ (blue) and the upper and lower bound in (4.1.5) and (4.1.6) (red) are shown. The coercivity range for $\alpha + \frac{1}{2}$ is the shaded area. The coercivity range shifted up by 1 is shown in green.

□

Lemma 4.4.3. *For $u, u^{(0)}$ and f as defined in proposition 4.4.2 we have the estimate*

$$|||u||| \lesssim \left|\left|\left| u^{(0)} \right|\right|\right|_0 + |||f|||_1, \tag{4.4.14}$$

where these norms are defined in definition 3.2.2 and $\tilde{k} + 8 \geq 8$.

Proof. Using the maximal regularity estimate for v in (4.4.11) with $\alpha = -\frac{3}{2} + \frac{1}{p} \pm \delta$ ($m = k + 5$) and $\alpha = -\frac{3}{2} + \beta \pm \delta$ ($m = \tilde{k} + 2$, $\tilde{k}$ as in (4.2.5)), and the equivalences in (4.4.2) we get estimates for u for all these weight exponents individually. Adding these inequalities gives the desired result. □

Lemma 4.4.4. *We have the following for the coefficients u_0 and u_β of proposition 4.4.2:*

$$u_0 \in BC^0([0, \infty)), \quad u_\beta \in L^p([0, \infty)).$$

Additionally, the following inequality holds true

$$\sup_{t \geq 0} |u_0|^p + \int_0^\infty |u_\beta|^p dt \lesssim |||u|||^p. \tag{4.4.15}$$

Proof. We have

$$|u_0(t)|^p = \int_1^2 |u_0(t)|^p dx \lesssim_\delta \int_1^2 x^{-2\delta} |u - u_0|^p \frac{dx}{x} + \int_1^2 x^{2\delta} |u|^p \frac{dx}{x} \lesssim \|u - u_0\|_\delta^p + \|u\|_{-\delta}^p \tag{4.4.16}$$

$$\leq \|u - u_0\|_{\tilde{k}+9,\delta}^p + \|u\|_{\tilde{k}+9,-\delta}^p,$$

with $\tilde{k}$ defined by (4.2.5) Taking the supremum gives

$$\sup_{t \geq 0} |u_0(t)|^p \lesssim \sup_{t \geq 0} (\|u - u_0\|_{\tilde{k}+9,\delta}^p + \|u\|_{\tilde{k}+9,-\delta}^p).$$

By (4.4.14), we know that this is bounded. Using (4.2.4) and $Au = v$, this can be rewritten as

$$\sup_{t \geq 0} |u_0(t)|^p \lesssim \sup_{t \geq 0} (\|v\|_{\tilde{k},-1+\delta}^p + \|v\|_{\tilde{k},-1-\delta}^p) \lesssim \sup_{t \geq 0} (\|v\|_{k,-\frac{3}{2}+\delta+\frac{1}{p},p}^p + \|v\|_{k,-\frac{3}{2}-\delta+\frac{1}{p},p}^p).$$

Using the trace method for interpolation (see definition 2.3.3) then gives

$$\sup_{t\geq 0}|u_0(t)|^p \lesssim \int_0^\infty \|\partial_t v\|^p_{k-2,-2+\delta+\frac{1}{p}} + \|\partial_t v\|^p_{k-2,-2-\delta+\frac{1}{p}} + \|v\|^p_{k+2,-1+\delta+\frac{1}{p}} + \|v\|^p_{k+2,-1-\delta+\frac{1}{p}} dt.$$

Applying an even reflection in time yields

$$\sup_{t\in\mathbb{R}}|u_0^r(t)|^p \lesssim \int_{-\infty}^\infty \|\partial_t v^r\|^p_{k-2,-2+\delta+\frac{1}{p}} + \|\partial_t v^r\|^p_{k-2,-2-\delta+\frac{1}{p}} + \|v^r\|^p_{k+2,-1+\delta+\frac{1}{p}} + \|v^r\|^p_{k+2,-1-\delta+\frac{1}{p}} dt.$$

Take a test function ϕ with $\int_{\mathbb{R}}\phi dx = 1$ and mollify the left and right hand side with $\phi_\varepsilon(t) := \frac{1}{\varepsilon}\phi(\frac{t}{\varepsilon})$. We can then approximate the right hand side with smooth functions in time. As $\varepsilon \to \infty$, the difference between the mollified right hand side and the original right hand side goes to zero. Hence, this should also happen to the left hand side. It follows that u_0 is continuous in time.

For u_β, we have

$$\begin{aligned}
|u_\beta|^p &\lesssim \int_1^2 |u_\beta x^\beta|^p dx \lesssim \int_1^2 |u-u_0|^p dx + \int_1^2 |u-u_0-u_\beta x^\beta|^p dx \\
&\lesssim_\delta \int_0^\infty x^{-2(\beta-\delta)}|u-u_0|^p \frac{dx}{x} + \int_0^\infty x^{-2(\beta+\delta)}|u-u_0-u_\beta x^\beta|^p \frac{dx}{x} \\
&\leq \|u-u_0\|^p_{\tilde{k}+8,\beta-\delta} + \|u-u_0-u_\beta x^\beta\|^p_{\tilde{k}+8,\beta+\delta}
\end{aligned} \tag{4.4.17}$$

Integrating both sides over t gives the result. Equation (4.4.15) follows from adding (4.4.16) and (4.4.17), using (4.2.4) and the definition of $|||u|||$ (see (3.2.3)). $\square$

Chapter 5

The Nonlinear Problem

In this chapter, the nonlinear problem (3.1.23) will be treated. Subsequently, the main result of 3.3 will be proven in section 5.1. Before this can be done, we need to show a few preliminary estimates.

Lemma 5.0.1. *For u and u_0 defined in proposition 4.4.2 it holds that*

$$\max_{l=0,\ldots,\tilde{k}+6} |D^l(u-u_0)|_{(0,1]} \lesssim \|u-u_0\|_{\tilde{k}+7,\delta} \quad \textit{and} \quad \max_{l=0,\ldots,\tilde{k}+6} |D^l u|_{[1,\infty)} \lesssim \|u\|_{\tilde{k}+7,-\delta},$$

where $|w|_I := \sup_{x\in I} |w(x)|$ for an interval $I \subset (0,\infty)$ and $\tilde{k}$ is as defined in (4.2.5).

Proof. The analogue of this proof can be found in the proof of [16, lemma 8.1, preliminaries]. We will show that these estimates hold for all necessary numbers of derivatives of $u-u_0$ or u respectively. Define $\eta_1 : (0,\infty) \to \mathbb{R}$ and $\eta_2 : (0,\infty) \to \mathbb{R}$ to be smooth cut-off functions, where $\eta_1(x) = 1$ for $x \leq 1$ and $\eta_1(x) = 0$ for $x \geq 2$. Similarly, let $\eta_2(x) = 1$ for $x \geq 1$ and $\eta_2(x) = 0$ for $x \leq \frac{1}{2}$. Applying the change of coordinates $x = e^s$, we define $\widetilde{u}(s) := u(e^s)$, $\widetilde{\eta_1}(s) = \eta_1(e^s)$ and $\widetilde{\eta_2}(s) = \eta_2(e^s)$. We then have

$$\begin{aligned}
\sup_{x\in(0,1]} |u-u_0|^2 &\leq \sup_{x\in\mathbb{R}^+} |\eta_1(x)(u-u_0)|^p = \sup_{s\in\mathbb{R}} |\widetilde{\eta_1}(s)(\widetilde{u}-u_0)|^p \\
&\lesssim \sum_{l=0}^{1} \int_{-\infty}^{\infty} (\partial_s^l(\widetilde{\eta_1}(s)(\widetilde{u}-u_0)))^2 ds \lesssim \sum_{l=0}^{1} \int_0^2 x^{2\delta} x^{-2\delta} (D^l(u-u_0))^2 \frac{dx}{x} \\
&\leq \sum_{l=0}^{1} (\sup_{x\in(0,2]} x^{2\delta}) \int_0^2 x^{-2\delta}(D^l(u-u_0))^2 \frac{dx}{x} \lesssim_\delta \sum_{l=0}^{1} \int_0^\infty x^{-2\delta}(D^l(u-u_0))^2 \frac{dx}{x} \\
&= \|u-u_0\|_{1,\delta}^2 \leq \|u-u_0\|_{\tilde{k}+7,\delta}^2.
\end{aligned}$$

Replacing u by $D^l u$ and u_0 by $D^l u_0$ shows that these arguments also hold for terms of the type $D^l(u-u_0)$ for $l = 0,\ldots,\tilde{k}+6$. Hence, for all $l = 0,\ldots,\tilde{k}+6$, the inequality $\sup_{x\in(0,1]} |D^l(u-u_0)| \lesssim \|u-u_0\|_{\tilde{k}+7,\delta}$ holds, thus it also holds for the maximum in $l \in \{0,\ldots,\tilde{k}+6\}$. The second estimate follows similarly, in this case using $\widetilde{\eta_2}$:

$$\begin{aligned}
\sup_{x\in[1,\infty)} |u|^2 \leq \sup_{x\in\mathbb{R}^+} |\eta_2(x)u|^2 = \sup_{s\in\mathbb{R}} |\widetilde{\eta_2}(s)\widetilde{u}|^2 \lesssim \sum_{l=0}^{1} \int_{-\infty}^{\infty} (\partial_s^l \widetilde{\eta_2}(s)\widetilde{u})^2 ds \\
\lesssim_\delta \|u\|_{1,-\delta}^2 \leq \|u\|_{\tilde{k}+7,-\delta}^2.
\end{aligned}$$

As before, from this calculation it follows that $\sup_{x\in[1,\infty)} |D^l u| \leq \|u\|_{\tilde{k}+7,-\delta}$ holds for all $l = 0,\ldots,\tilde{k}+6$, hence it also holds for the maximum in $l \in \{0,\ldots,\tilde{k}+6\}$. $\square$

Corollary 5.0.2. *For u, u_0 and u_β defined in proposition 4.4.2,*

$$\sup_{t\geq 0}\max_{l=0,\dots,\tilde{k}+6}\sup_{x>0}|D^l(u-u_0)|^p \lesssim |||u|||^p \tag{5.0.1}$$

and

$$\sup_{t\geq 0}(|u_0|^p+\max_{l=0,\dots,\tilde{k}+6}\sup_{x>0}|D^l(u-u_0)|^p)+\int_0^\infty |u_\beta|^p dt \lesssim |||u|||^p \tag{5.0.2}$$

hold.

Proof. The analogue of this proof can be found in the proof of [16, lemma 8.1, preliminaries]. It holds that

$$\begin{aligned}\sup_{t\geq 0}\max_{l=0,\dots,\tilde{k}+6}\sup_{x>0}|D^l(u-u_0)|^p &\lesssim \sup_{t\geq 0}\max_{l=0,\dots,\tilde{k}+6}\left[\sup_{x\in(0,1]}|D^l(u-u_0)|^p+\sup_{x\in[1,\infty)}|D^l u|^p+|u_0|^p\right]\\ &\overset{(4.4.15),\text{ lemma 5.0.1}}{\lesssim} \sup_{t\geq 0}\left[\|u-u_0\|^p_{\tilde{k}+4,\delta}+\|u\|^p_{\tilde{k}+4,-\delta}\right]+|||u|||^p \overset{(4.2.4),(3.2.3)}{\lesssim} |||u|||^p.\end{aligned}$$

This shows (5.0.1). Inequality (5.0.2) can be shown by combining (5.0.1) and (4.4.15). □

Define, for $\mathcal{N}$ as in (3.1.25)

$$\widetilde{\mathcal{N}}(u) := x\mathcal{N}(u).$$

Lemma 5.0.3. *Let $0<\delta<\min\{\beta-1,\frac{1}{p}\}$ and $k\in\mathbb{N}_0$. For $u,u_1,u_2:(0,\infty)^2\to\mathbb{R}$ smooth the following estimates hold for the nonlinearity (see* (3.1.11), (3.1.21)*)*

$$|||\mathcal{N}(u)|||_1 = \left|\left|\left|\widetilde{\mathcal{N}}(u)\right|\right|\right|_2 \lesssim_{k,\delta} \max_{j=2,5}|||u|||^j, \tag{5.0.3}$$

$$|||\mathcal{N}(u_1)-\mathcal{N}(u_2)|||_1 \sim \left|\left|\left|\widetilde{\mathcal{N}}(u_1)-\widetilde{\mathcal{N}}(u_2)\right|\right|\right|_2 \lesssim_{k,\delta} \max_{j=1,4}(|||u_1|||+|||u_2|||)^j|||u_1-u_2|||. \tag{5.0.4}$$

Proof. This is the analogue of [16, lemma 8.1]. We start with deriving (5.0.3). For both $1<n<\frac{3}{2}$ and $\frac{3}{2}<n<3$, $\mathcal{N}(u)$ has the form

$$\mathcal{N}(u) = -x^{-1}\mathcal{M}(u+1,\dots,u+1)+x^{-1}p(D)u,$$

and additionally,

$$\widetilde{\mathcal{N}}(u) = \mathcal{M}(u+1,\dots,u+1)+p(D)u,$$

where $p(D)$ and $\mathcal{M}$ are defined as in (3.1.24) and (3.1.26), respectively. Hence, $\widetilde{\mathcal{N}}(u)$ consists of the nonlinear terms of $\mathcal{M}(u+1,\dots,u+1)$. To show (5.0.3), we will describe the terms of which $\widetilde{\mathcal{N}}$ consists. This will be done in terms of $\mathcal{M}_{sym}$, which is the symmetrization of $\mathcal{M}$. Note that we now have that $\mathcal{M}_{sym}$ is multi-linear in all of its arguments. Because of this, $\widetilde{\mathcal{N}}$ can be written as a linear combination of terms of the form

$$\mathcal{M}_{sym}(u,u,w_3,w_4,w_5),\quad \text{with } w_3,w_4,w_5\in\{u,1\}. \tag{5.0.5}$$

By use of the multi-linearity of $\mathcal{M}_{sym}$, we can write each of the terms described above in the form

$$\mathcal{M}_{sym}(w_1,w_2,w_3,w_4,w_5),\quad \text{with } w_1,w_2\in\{u-u_0,u_0\} \text{ and } w_3,w_4,w_5\in\{u-u_0,u_0,1\}. \tag{5.0.6}$$

This is done since, for instance in corollary 5.0.2, we have computed bounds on differences $u-u_0$. We can see this for example by considering the term

$$\begin{aligned}\mathcal{M}_{sym}(u,u,1,1,1) =&\mathcal{M}_{sym}(u-u_0,u-u_0,1,1,1)+\mathcal{M}_{sym}(u_0,u-u_0,1,1,1)\\ &+\mathcal{M}_{sym}(u-u_0,u_0,1,1,1)+\mathcal{M}_{sym}(u_0,u_0,1,1,1),\end{aligned}$$

which exactly consists of terms of the form described in (5.0.6) (also, the second and third term are the same, since $\mathcal{M}_{sym}$ is symmetric). This works similarly for general terms of the form (5.0.5). Noting that evaluating $\mathcal{M}_{sym}$ for constants in all arguments yields zero, since

$$\mathcal{M}_{sym}(u_0, u_0, u_0, u_0, u_0) = \frac{1}{5} p(D) u_0 = 0.$$

Hence, we see that only terms that consist of at least one non constant argument of $\mathcal{M}_{sym}$ remain. Therefore, (5.0.6) can be simplified to

$$\mathcal{M}_{sym}(u - u_0, w_2, w_3, w_4, w_5), \quad \text{with } w_2 \in \{u - u_0, u_0\} \text{ and } w_3, w_4, w_5 \in \{u - u_0, u_0, 1\}. \tag{5.0.7}$$

We consider $\left\|\left|\widetilde{\mathcal{N}}(u)\right\|\right|_2$ and this consists of norms of $\widetilde{\mathcal{N}}$ with weights $\frac{1}{p} \pm \delta, \beta \pm \delta$ and $k+2$ derivatives. First consider the 'subcritical' case, which consists of the parts of the norm with weights $\frac{1}{p} \pm \delta$ and $\beta - \delta$. We will see of which terms $D^l \widetilde{\mathcal{N}}$, $l \le k+7$ or $l \le \tilde{k} + 4$, consists. Note that $\mathcal{M}_{sym}$ distributes four derivatives onto its arguments. Hence, $D^l \widetilde{\mathcal{N}}$ can be written as a product of terms $v_1 \times v_2 \times v_3 \times v_4 \times v_5$, where

$$\begin{aligned}
v_1 &= D^{l_1}(u - u_0), & l_1 &\le l + 4 \\
v_2 &\in \{D^{l_2}(u - u_0), u_0\}, & l_2 &\le \frac{1}{2}(l+4) \le \tilde{k} + 8, \\
v_3, v_4, v_5 &\in \{D^{l_3}(u - u_0), u_0, 1\}, & l_3 &\le \frac{1}{3}(l+4) \le \tilde{k} + 8.
\end{aligned}$$

From equation (5.0.2), we note that the supremum over time v_2, v_3, v_4 and v_5 can be bounded by $|||u|||$. Hence, for the part of $\left\|\left|\widetilde{\mathcal{N}}(u)\right\|\right|_2$ consisting of the norms with weights $\frac{1}{2} \pm \delta, \beta - \delta$ the following estimate holds

$$\begin{aligned}
&\int_0^\infty \|\widetilde{\mathcal{N}}(u)\|^p_{k+7, \frac{1}{p} - \delta} + \|\widetilde{\mathcal{N}}(u)\|^p_{k+7, \frac{1}{p} + \delta} + \|\widetilde{\mathcal{N}}(u)\|^p_{\tilde{k}+4, \beta - \delta} dt \qquad (5.0.8)\\
&\lesssim \int_0^\infty \|u - u_0\|^p_{k+11, \frac{1}{p} - \delta} + \|u - u_0\|^p_{k+11, \frac{1}{p} + \delta} + \|u - u_0\|^p_{\tilde{k}+8, \beta - \delta} dt \\
&\times (\sup_{t \ge 0}(\max_{l = 0, \dots, \tilde{k}+6} \sup_{x > 0} |D^l(u - u_0)|^p + |u_0|^p))(1 + (\sup_{t \ge 0}(\max_{l = 0, \dots, \tilde{k}+6} \sup_{x > 0} |D^l(u - u_0)|^p + |u_0|^p))^3) \\
&\overset{(3.2.3),(5.0.2)}{\lesssim} |||u|||^{2p}(1 + |||u|||^{3p}).
\end{aligned}$$

Here, after the first inequality sign, the L^p in time integral corresponds to v_1 and the supremum bounds correspond to v_2 until v_5.

Now, consider the 'supercritical' case, which consists of the part of the norm $\left\|\left|\widetilde{\mathcal{N}}(u)\right\|\right|_2$ with weight $\beta + \delta$. In this case, (5.0.7) turns into one of the following

$$\mathcal{M}_{sym}(u - u_0 - u_\beta x^\beta, w_2, w_3, w_4, w_5), \ w_2 \in \{u - u_0, u_0\}, \ w_3, w_4, w_5 \in \{u - u_0, u_0, 1\}, \tag{5.0.9}$$

$$\mathcal{M}_{sym}(u_\beta x^\beta, w_2, w_3, w_4, w_5), \ w_2 \in \{u - u_0, u_0\}, \ w_3, w_4, w_5 \in \{u - u_0, u_0, 1\}. \tag{5.0.10}$$

Noting that

$$\mathcal{M}_{sym}(x^\beta, u_0, u_0, u_0, u_0) = \frac{u_0^4}{5} p(D) x^\beta = 0, \tag{5.0.11}$$

since β is a root of $p(D)$, (5.0.10) can be simplified to

$$\mathcal{M}_{sym}(u_\beta x^\beta, u - u_0, w_3, w_4, w_5), \quad w_3, w_4, w_5 \in \{u - u_0, u_0, 1\}. \tag{5.0.12}$$

We will argue similarly to before in finding bounds on the part of $\left\|\left|\widetilde{\mathcal{N}}(u)\right|\right\|_2$ with weight $\beta+\delta$, which consist of an L^p in time norm of $\|\widetilde{\mathcal{N}}(u)\|_{\tilde{k}+4,\beta+\delta}$. In both cases we use that again $\mathcal{M}$ distributes four derivatives onto its arguments. First, considering the form as given in equation (5.0.9). In this case, $D^l\widetilde{\mathcal{N}}$, $l \leq \tilde{k}+4$, is a linear combination of terms of the form $v_1 \times \cdots \times v_5$ given by

$$\begin{aligned} v_1 &= D^{l_1}(u-u_0-u_\beta x^\beta), & l_1 &\leq l+4 \leq \tilde{k}+8, \\ v_2 &\in \{D^{l_2}(u-u_0), u_0\}, & l_2 &\leq l+4 \leq \tilde{k}+8, \\ v_3, v_4, v_5 &\in \{D^{l_3}(u-u_0), u_0, 1\} & l_3 &\leq \frac{1}{2}(l+4) \leq \tilde{k}+8. \end{aligned}$$

This gives

$$\|v_1 \times \cdots \times v_5\|^p_{\beta+\delta} \lesssim \|u-u_0-u_\beta x^\beta\|^p_{\tilde{k}+8,\beta+\delta} \, (\max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} |D^l(u-u_0)|^p + |u_0|^p) \tag{5.0.13}$$

$$\times (1 + (\max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} |D^l(u-u_0)|^p + |u_0|^p)^3), \tag{5.0.14}$$

where the first factor in the right hand side corresponds to v_1 and the rest of the factors correspond to v_2 until v_5.

Secondly, considering (5.0.12) we see that $D^l\widetilde{\mathcal{N}}$, $l \leq \tilde{k}+4$, is a linear combination of terms of the form $v_1 \times \cdots \times v_5$ given by

$$\begin{aligned} v_1 &= u_\beta, & & \\ v_2 &= x^\beta D^{l_1}(u-u_0), & l_1 &\leq l+4 \leq \tilde{k}+8, \\ v_3, v_4, v_5 &\in \{D^{l_2}(u-u_0), u_0, 1\}, & l_2 &\leq \frac{1}{2}(l+4) \leq \tilde{k}+8, \end{aligned}$$

since $D^l x^\beta = \beta^l x^\beta$. This gives, using that $\|x^\beta \cdot\|_{\beta+\delta} = \|\cdot\|_\delta$,

$$\|v_1 \times \cdots \times v_5\|^p_{\beta+\delta} \lesssim |u_\beta|^p \|u-u_0\|^p_{\tilde{k}+8,\delta} (1 + (\max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} |D^l(u-u_0)|^p + |u_0|^p)^3). \tag{5.0.15}$$

Here, the first factor in the right hand side corresponds to v_1, the second to v_2 and the rest to v_3, v_4 and v_5. Combining (5.0.13) and (5.0.15) gives, using L^p bounds in time for the first term and supremum bounds in time for the rest of the terms

$$\begin{aligned} \int_0^\infty \|\widetilde{\mathcal{N}}(u)\|^p_{\tilde{k}+4,\beta+\delta} dt &\lesssim \int_0^\infty \|u-u_0-u_\beta x^\beta\|^p_{\tilde{k}+8,\beta+\delta} + |u_\beta|^p dt \qquad (5.0.16) \\ &\times \sup_{t\geq 0} \Bigg[(\|u-u_0\|^p_{k+2,\delta} + \max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} |D^l(u-u_0)|^p + |u_0|^p) \\ &\times (1 + (\max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} |D^l(u-u_0)|^p + |u_0|^p)^3) \Bigg] \\ &\overset{(3.2.3),(5.0.2)}{\lesssim} \|\!|u|\!\|^{2p}(1 + \|\!|u|\!\|^{3p}). \end{aligned}$$

Equation (5.0.3) now follows by adding (5.0.8) and (5.0.16) and taking the p^{th} root.

We will derive (5.0.4). This will need similar techniques as before. Because $\mathcal{M}_{sym}$ is multi-linear, it holds that $\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2)$ is a linear combination of terms of the form

$$\mathcal{M}_{sym}(u_1 - u_2, w_2, w_3, w_4, w_5), \quad w_2 \in \{u_1, u_2\}, w_3, w_4, w_5 \in \{u_1, u_2, 1\}.$$

For the subcritical part of the norm $\left|\left|\left|\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2)\right|\right|\right|_2$, which consists of norms with weight $\frac{1}{p} \pm \delta$ and $\beta - \delta$, the terms from above are rewritten as

$$\begin{aligned}\mathcal{M}_{sym}(w_1, w_2, w_3, w_4, w_5), \text{ with } \quad & w_1 \in \{u_1 - u_{0,1} - (u_2 - u_{0,2}), u_{0,1} - u_{0,2}\},\\ & w_2 \in \{u_1 - u_{0,1}, u_2 - u_{0,2}, u_{0,1}, u_{0,2}\},\\ w_3, w_4, & w_5 \in \{u_1 - u_{0,1}, u_2 - u_{0,2}, u_{0,1}, u_{0,2}, 1\}.\end{aligned}$$

Note that before we could argue that the w_1 had to be non constant, but this reasoning does not hold here. Hence, we have to work with both the non constant and constant options for w_1. The arguments for finding bounds for $\left|\left|\left|\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2)\right|\right|\right|_2$ work in exactly the same way as before. Hence, we only give the details in the case that $w_1 = u_1 - u_{0,1} - (u_2 - u_{0,2})$ and the part of the norm with weight $\frac{1}{p} - \delta$. In this case, $D^l(\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2))$ is for $l \leq k+7$ or $l \leq \tilde{k} + 4$ a linear combination of terms of the form $v_1 \times \cdots \times v_5$ with

$$\begin{aligned}v_1 &= D^{l_1}(u_1 - u_{0,1} - (u_2 - u_{0,2})), & l_1 &\leq l + 4,\\ v_2 &\in \left\{D^{l_2}(u_1 - u_{0,1}), D^{l_2}(u_2 - u_{0,2}), u_{0,1}, u_{0,2}\right\}, & l_2 &\leq l + 4,\\ v_3, v_4, v_5 &\in \left\{D^{l_3}(u_1 - u_{0,1}), D^{l_3}(u_2 - u_{0,2}), u_{0,1}, u_{0,2}, 1\right\}, & l_3 &\leq \frac{1}{2}(l+4) \leq \tilde{k} + 8.\end{aligned}$$

Then, we see that

$$\int_0^\infty \|v_1 \times \cdots \times v_5\|^p_{\frac{1}{p}-\delta} dt \lesssim \int_0^\infty \|u_1 - u_{0,1} - (u_2 - u_{0,2})\|^p_{k+11, \frac{1}{p}-\delta} dt \times \tag{5.0.17}$$

$$\left[\sup_{t \geq 0} \left(\max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} (|D^l(u_1 - u_{0,1})|^p + |D^l(u_2 - u_{0,2})|^p) + |u_{0,1}|^p + |u_{0,2}|^p\right)\right] \times \tag{5.0.18}$$

$$\left[\sup_{t \geq 0} \left(\max_{l=0,\ldots,\tilde{k}+6} \sup_{x>0} (|D^l(u_1 - u_{0,1})|^p + |D^l(u_2 - u_{0,2})|^p) + |u_{0,1}|^p + |u_{0,2}|^p\right) + 1\right]^3 \tag{5.0.19}$$

$$\lesssim |||u_1 - u_2|||^p (|||u_1||| + |||u_2|||)^p (|||u_1||| + |||u_2||| + 1)^{3p}, \tag{5.0.20}$$

where the v_1 factor in the norm will be bounded by $|||u_1 - u_2|||$ using (3.2.3) and the other factors can be bounded by $|||u_1||| + |||u_2|||$ using (5.0.2).

In the case that $w_1 = u_{0,1} - u_{0,2}$, $D^l(\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2))$ is for $l \leq k+7$ or $l \leq \tilde{k} + 4$ a linear combination of terms of the form $v_1 \times \cdots \times v_5$ with

$$\begin{aligned}v_1 &= u_{0,1} - u_{0,2}, & &\\ v_2 &\in \left\{D^{l_2}(u_1 - u_{0,1}), D^{l_2}(u_2 - u_{0,2}), u_{0,1}, u_{0,2}\right\}, & l_2 &\leq l + 4,\\ v_3, v_4, v_5 &\in \left\{D^{l_3}(u_1 - u_{0,1}), D^{l_3}(u_2 - u_{0,2}), u_{0,1}, u_{0,2}, 1\right\}, & l_3 &\leq \frac{1}{2}(l+4) \leq \tilde{k} + 8.\end{aligned}$$

In this case, we take the v_1 factor out of the integral when considering $\left|\left|\left|\widetilde{N}(u_1) - \widetilde{N}(u_2)\right|\right|\right|$. This factor can be bounded by $|||u_1 - u_2|||$ using (5.0.2). The other factors can be bounded by $\lVert\!| u_1 |\!\rVert + \lVert\!| u_2 |\!\rVert$ also using (5.0.2). From this, we can obtain the same bound on $\left\lVert\!\left| \widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2) \right|\!\right\rVert^p$ as is constructed in (5.0.17).

For the super-critical part of the norm, the part with weight $\beta + \delta$, we decompose

$$\begin{aligned} M_{sym}(u_{0,1} - u_{0,2}, w_2, w_3, w_4, w_5), \qquad & w_2 \in \{u_1 - u_{0,1}, u_2 - u_{0,2}\}, \\ & w_3, w_4, w_5 \in \{u_1 - u_{0,1}, u_2 - u_{0,2}, 1\} \end{aligned}$$

as

$$\begin{aligned} M_{sym}(u_{0,1} - u_{0,2}, w_2, w_3, w_4, w_5), \quad \text{with} & \\ w_2 &\in \{u_1 - u_{0,1} - u_{\beta,1}x^\beta, u_2 - u_{0,2} - u_{\beta,1}x^\beta\}, \\ w_3, w_4, w_5 &\in \{u_1 - u_{0,1}, u_2 - u_{0,2}, u_{0,1}, u_{0,2}, 1\}, \end{aligned} \tag{5.0.21}$$

and

$$\begin{aligned} M_{sym}(u_{0,1} - u_{0,2}, w_2, w_3, w_4, w_5), \quad \text{with} & \\ w_2 &\in \{u_{\beta,1}x^\beta, u_{\beta,2}x^\beta\}, \\ w_3 &\in \{u_1 - u_{0,1}, u_2 - u_{0,2}\}, \\ w_4, w_5 &\in \{u_1 - u_{0,1}, u_2 - u_{0,2}, u_{0,1}, u_{0,2}, 1\}, \end{aligned} \tag{5.0.22}$$

where in the last case w_3 is nonconstant because of (5.0.11). For (5.0.21) it holds that $D^l(\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2))$, $l \leq \tilde{k} + 4$, is a linear combination of terms of the form $v_1 \times \cdots \times v_5$ with

$$\begin{aligned} v_1 &= u_{0,1} - u_{0,2}, \\ v_2 &\in \{D^{l_1}(u_1 - u_{0,1} - u_{\beta,1}x^\beta), D^{l_1}(u_2 - u_{0,2} - u_{\beta,1}x^\beta)\} \quad && l_1 \leq l + 4 \leq \tilde{k} + 8, \\ v_3, v_4, v_5 &\in \{D^{l_2}(u_1 - u_{0,1}), D^{l_2}(u_2 - u_{0,2}), u_{0,1}, u_{0,2}, 1\} && l_2 \leq l + 4 \leq \tilde{k} + 8. \end{aligned}$$

If one bounds this in the norm, the v_1 factor will be bounded by $\lVert\!| u_1 - u_2 |\!\rVert$ and the other factors by $\lVert\!| u_1 |\!\rVert + \lVert\!| u_2 |\!\rVert$.

For (5.0.22), $D^l(\widetilde{\mathcal{N}}(u_1) - \widetilde{\mathcal{N}}(u_2))$, $l \leq \tilde{k} + 4$, is a linear combination of terms of the form $v_1 \times \cdots \times v_5$ with

$$\begin{aligned} v_1 &= u_{0,1} - u_{0,2}, \\ v_2 &\in \{u_{\beta,1}, u_{\beta,2}\}, \\ v_3 &\in \{x^\beta D^{l_1}(u_1 - u_{0,1}), x^\beta D^{l_1}(u_2 - u_{0,2})\} && l_1 \leq l + 4 \leq \tilde{k} + 8, \\ v_4, v_5 &\in \{D^{l_2}(u_1 - u_{0,1}), D^{l_2}(u_2 - u_{0,2}), u_{0,1}, u_{0,2}, 1\} \quad && l_2 \leq l + 4 \leq \tilde{k} + 8. \end{aligned}$$

If one bounds this in the norm, the v_1 factor will be bounded by $\lVert\!| u_1 - u_2 |\!\rVert$ and the other factors by $\lVert\!| u_1 |\!\rVert + \lVert\!| u_2 |\!\rVert$. Combining all of these estimates and taking the p^{th} root gives (5.0.4).

$\square$

5.1 Proof of the main result

In this section, we will prove theorem 3.3.1, where we will make use of the results from lemma 5.0.3.

Proof of theorem 3.3.1. In the proof, all estimates depend on k and δ.
Existence
Let $\varepsilon > 0$. Later on in the proof an additional condition for ε will be determined. Let $u^{(0)} : (0, \infty) \to \mathbb{R}$ be locally integrable with $\left|\left|\left|u^{(0)}\right|\right|\right|_0 < \varepsilon$ and define the space

$$S := \left\{u : (0, \infty)^2 \to \mathbb{R} \text{ locally integrable } |||u||| < \eta, u|_{t=0} = u^{(0)}\right\}$$

for $\eta > 0$. We will find a constraint for η later on. We want to use Banach's fixed point theorem to show existence of a solution that is locally integrable where additionally $|||u||| < \infty$. For this, let T be the solution operator of proposition 4.4.2. With the use of this operator we can rewrite (3.1.23) in the equivalent formulation as the following fixed point equation:

$$u = \mathcal{T}(u) := T\mathcal{N}(u).$$

To be able to use the fixed point theorem, we need to show that $\mathcal{T} : S \to S$ is a contraction for $\varepsilon > 0$ sufficiently small. First, we show that indeed $\mathcal{T}$ maps S into itself. Let $u \in S$, then by the maximal regularity estimate (4.4.14) and the estimate (5.0.3) it follows that

$$|||\mathcal{T}(u)||| = |||T\mathcal{N}(u)||| \overset{(4.4.14)}{\lesssim} \left|\left|\left|u^{(0)}\right|\right|\right|_0 + |||\mathcal{N}(u)|||_1 \overset{(5.0.3)}{\lesssim} \left|\left|\left|u^{(0)}\right|\right|\right|_0 + \max_{j=2,5} |||u|||^j \lesssim \varepsilon + \eta^2 \tag{5.1.1}$$

for $\eta \leq 1$. If we take $\eta \ll 1$ and $\varepsilon \ll \eta$, (5.1.1) tells us that $\mathcal{T}$ maps S into itself. Now we want to show that $\mathcal{T}$ is a contraction. Let $u_1, u_2 \in S$. Now using (4.4.14) and (5.0.4), it follows that

$$\begin{aligned} |||\mathcal{T}(u_1) - \mathcal{T}(u_2)||| = |||T(\mathcal{N}(u_1) - \mathcal{N}(u_2))||| &\overset{(4.4.14)}{\lesssim} |||\mathcal{N}(u_1) - \mathcal{N}(u_2)|||_1 \\ &\overset{(5.0.4)}{\lesssim} \max_{j=1,4}(|||u_1||| + |||u_2|||)^j |||u_1 - u_2||| \lesssim \eta|||u_1 - u_2||| \end{aligned}$$

for $\eta \leq 1$. Hence, $\mathcal{T}$ is a contraction for $\eta \ll 1$. Now applying Banach's fixed point theorem gives existence of a solution to (3.1.23).

Uniqueness
Let u and w denote two solutions to (3.1.23), where u is the solution as constructed above and w is a different solution. We assume that there exists a time $t > 0$ such that $u(t)$ and $w(t)$ are different. The analogue of [16, lemma B.4] gives us continuity in time, and from this it follows that there exists a maximal time $T \geq 0$ such that

- $u(t, x) = w(t, x)$ for all $t \in (0, T]$ and $x > 0$,
- $u(t, x)$ and $w(t, x)$ differ for $t > T$ sufficiently small and $x > 0$.

Using $u(T) = w(T)$ as initial data, we get for $I := (T, T + \tau)$, $\tau > 0$

$$|||u - w|||_I = |||\mathcal{T}(u) - \mathcal{T}(w)|||_I \overset{(4.4.14),(5.0.4)}{\lesssim} \max_{j=1,4}(|||u|||_I + |||w|||_I)^j |||u - w|||_I. \tag{5.1.2}$$

From the analogue of [16, Lemma B.4], we obtain that $|||u|||_I \to |||u(T)|||_0$ and $|||w|||_I \to |||u(T)|||_0$ as $\tau \downarrow 0$. Additionally, $|||u(T)|||_0 \leq |||u||| \leq \eta$. So (5.1.2) gives that $|||u - w|||_I \leq 0$ for $\tau \ll 1$ and $\eta \ll 1$. Hence, $u = w$ for $t \in I = (T, T + \tau)$ and T cannot be maximal. □

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