Leavitt path algebras and their ideals

by

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Summary

The first part of this Summary will mainly address those with a minimal background in mathematics. The second part will cover the structure of this thesis in more detail.

In this thesis, a basic introduction to Leavitt path algebras will be given. Originally introduced in 2005, Leavitt path algebras are a special version of algebras, mathematical structures where addition, subtraction and multiplication are defined, but not division. Any Leavitt path algebra arises from a directed graph, a mathematical structure which abstracts the real-life concept of networks. We will prove some fundamental properties of Leavitt path algebras and see that a directed graph and the Leavitt path algebra it generates are connected in very meaningful ways. We do this by looking at specific groups of elements of the Leavitt path algebra, known as ideals. The main result of this thesis is a theorem which states that specific ideals of a Leavitt path algebra correspond directly to specific groups of elements of the directed graph it is built from.

The specific ideals mentioned above are called graded ideals. They require the algebra they belong to to be graded as well, and we will see that every Leavitt path algebra is graded. One can think of a graded algebra as an algebra where every element can be made from adding elements belonging to a different 'color' or 'grade'. The main theorem in full then says that every graded ideal of a Leavitt path algebra is generated by a hereditary saturated subset of the vertices of the graph. The definitions for a hereditary and saturated subset of vertices will be covered in this thesis, with examples to help visualize these subsets. We will also show that graded ideals belonging to a Leavitt path algebra have other natural properties. In particular, we will look at quotients of algebras by ideals and introduce quotient graphs. We will prove that for a graph and a hereditary saturated subset of the vertices of the graph, the Leavitt path algebra of the quotient graph is isomorphic to the quotient of the Leavitt path algebra of that graph by the graded ideal generated by the hereditary saturated subset. We will also introduce lattices, which generalize the structure of a power set. We will see that the hereditary saturated subsets of the vertices of a graph form a lattice, and that the graded ideals of a Leavitt path algebra form a lattice. A theorem which states that these two are isomorphic will be proven. At the end of the thesis, we will give some theorems without proof and demonstrate their utility as an outlook.

Contents

1 Introduction

This thesis serves as an introduction to Leavitt path algebras, with only a basic background in mathematics necessary. We will also look at ideals pertaining to these algebras, and concepts needed to talk about them.

The guiding work for this thesis is the book Leavitt Path Algebras [6] written by G. Abrams, P. Ara and M. S. Molina. This book, written in 2017, is the first comprehensive material about this subject, and the only one as of this thesis. The ideas behind Leavitt path algebras were first conceived by two groups of authors in 2004, who were not aware of each other's works until 2005. The first group, which included Abrams and G. A. Pino, published their article [5] in that year, introducing Leavitt path algebras as a 'finite' version of a specific C^*-algebra[1], the Cuntz-Krieger algebra. This algebra was first described in [1]. Ara's group, which consisted of him, M. A. Moreno and E. Pardo, covered Leavitt path algebras in [9] in 2007. Their article was focused on the structure of monoids of C^*-algebras, and proved the Stucture Theorem for Graded Ideals, which we will cover in Chapter 5. A historical account of the efforts made by the two groups of authors can be found in Section 1.7 of the book, or in Section 1 of Leavitt path algebras: the first decade [2], an introductory article written by G. Abrams in 2015.

We now give an overview of the following chapters of this thesis. The next chapter will cover the preliminaries we will need, like algebras and graphs. Chapter 3 describes a formal definition of Leavitt path algebras following the one presented in [5], along with some properties and examples. Chapter 4 discusses gradings of algebras and the ideals which belong to those gradings. The main result of this thesis, Theorem 4.2.4, will be stated there, which identifies specific subsets of the vertices of a graph with the graded ideals of its Leavitt path algebra. Chapter 5 looks at graded ideals in other contexts, namely quotient algebras and lattices. We will see that graded ideals have many natural properties regarding Leavitt path algebras. Chapter 6 is a conclusion, summarizing the previous chapters and giving an outlook into further results.

From Chapter 3 onwards, important statements like theorems, definitions and remarks will have their reference statement in [6] written next to the header or near the body.

[1] An introduction to C^*-algebras for those with knowledgde of basic functional analysis can be found in [4].

2 Preliminaries

This chapter will establish the notations and defintions necessary to understand Leavitt path algebras. The first section covers algebras, and the second section directed graphs and path algebras. It is assumed that the reader has a basic understanding of rings, fields and vector spaces[2].

2.1 Algebras

We begin by defining the algebraic structure of an algebra.

Definition 2.1.1. An *algebra* A over a field K is a vector space over that field with a product $\cdot : A \times A \to A$ such that the product is bilinear.

A bilinear product is, for example, a function with two arguments which is linear with respect to each argument.

Well-known examples of algebras include the $n \times n$ matrices over the real or complex numbers. The bilinear product is then standard matrix multiplication. For an arbitrary field K, we will denote these *matrix algebras* by $M_n(K)$.

Remark 2.1.2. As the underlying structure of an algebra is a vector space, that vector space can be generated by a set of elements. In the case of the 2×2 matrices, the vector space is (for example) generated by the four *standard matrix units*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a standard matrix unit with a 1 in the ith row and jth column (starting from 1), and 0 elsewhere, $f_{i,j}$ will be used to denote it.

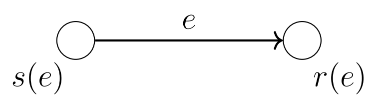
Whenever the vector space underneath an algebra is generated by a set, we will instead say that the algebra is generated by that set. Matrix algebras and standard matrix units will come back in the next chapter.

2.2 Directed graphs

Definition 2.2.1. A *directed graph* $G = (V, E, s, r)$ is made up of two sets V, E and two functions $s, r : E \to V$, which are called the *source* and the *range* respectively. In the rest of this thesis, the term directed graph will be shortened to graph.

[2] An introduction to these concepts can be found in [3].

The usual way to visualize a graph is to draw the elements of V (called *vertices*) as circles on a plane and, for every element e of E (called *edges*), to draw a connection between the two vertices $s(e), r(e)$. The directed nature of a graph is then shown by drawing the connections as arrows, going from $s(e)$ to $r(e)$. In this sense, we can see that s encodes the source of every edge, and r the range.



We can also see that for every $v \in V$, $s^{-1}(v)$ denotes the set of all edges which have v as its source, and $r^{-1}(v)$ denotes the set of all edges which have v as its range. A vertex for which $s^{-1}(v) = \emptyset$ is called a *sink*, and a vertex for which $r^{-1}(v) = \emptyset$ is called a *source*. If $|s^{-1}(v)|$ is infinite, v is called an *infinite emitter*. If v is either a sink or an infinite emitter, v is called a *singular vertex*; otherwise, v is a *regular vertex*.

For the graph shown above, $s(e)$ is a source and a regular vertex, and $r(e)$ is a sink and a singular vertex. Note that this graph has no infinte emitters. Such a graph is called *row-finite*, and we will only focus on these graphs in this thesis. To prove the results in the following chapters for graphs which are not row-finite (known as infinite graphs), more background knowledge is necessary.

A common thing to investigate with a (directed) graph is the way it represents connections between vertices. This is typically done by looking at the *paths* in a graph:

Definition 2.2.2. A path in a graph G is a sequence of edges $\mu = e_1 e_2 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i \in 1, \ldots, n$. For any path μ, $s(\mu) = s(e_1)$ denotes the source, $r(\mu) = r(e_n)$ denotes the range, and $\ell(\mu) = n$ denotes the length. We denote by $\text{Path}(G)$ the set of all paths in G and say that a path μ with $s(\mu) = v$ and $r(\mu) = w$ is a path from v to w.

Remark 2.2.3. We note that the definition for a path in graph theory is more strict than the one given above. We also deviate from graph theory by saying that vertices are paths as well, but of length 0. For a vertex v, the source and range functions are then extended as $s(v) = v$ and $r(v) = v$. These distinctions will be very important (and necessary!) when discussing paths in Leavitt path algebras.

It is possible to construct an algebra out of any graph by using the vertices and edges as elements. This is called a *path algebra*:

Definition 2.2.4. If $G = (V, E, s, r)$ is a graph and K is an arbitrary field then the path K-algebra of G (denoted by KG) is the K-algebra which is generated by $V \cup E$ and whose elements satisfy the following relations:

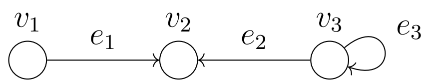
$$\text{(V)} \quad v \cdot w = \begin{cases} v & \text{if } w = v, \\ 0 & \text{else} \end{cases}$$

$$\text{(E1)} \quad s(e) \cdot e = e \cdot r(e) = e$$

for all $v, w \in V$ and $e \in E$. Note that these relations are not very strict; we do not impose any conditions on addition of elements of KG, and nothing is said about multiplying differing edges. We can, however, use these relations to show what happens when differing edges are multiplied, which will be done in the next example.

Before the example is covered, one thing should be noted about the use of variables in this thesis. Throughout, v and e (with or without indices) will be used to denote elements of a graph and elements of a corresponding algebra interchangeably. It should be clear which one is being described.

Example 2.2.5. Let G be the graph shown below, where every element has been given an explicit label, and let K be a field. We construct the path algebra KG by generating a vector space from the set $\{v_1, v_2, v_3, e_1, e_2, e_3\}$ and defining multiplication according to the two relations.



We give a few representative computations:

By relation (V), we see that $v_1 \cdot v_1 = v_1$ and $v_1 \cdot v_2 = 0$.
By relation (E1), we see that $v_1 \cdot e_1 = e_1 \cdot v_2 = e_1$.
Looking at $e_1 \cdot e_2$, using (E1) and (V) gives $e_1 \cdot e_2 = e_1 \cdot v_2 \cdot v_3 \cdot e_2 = e_1 \cdot 0 \cdot e_2 = 0$.
Likewise, $e_3 \cdot e_2 = e_3 \cdot v_3 \cdot v_3 \cdot e_2 = e_3 \cdot v_3 \cdot e_2 = e_3 \cdot e_2$.

In general, multiplying edges yields zero if the edges do not form a path, and otherwise nothing more can be said.

We note that, while the definition of a path algebra given here helps to build up the Leavitt path algebras, it is non-standard. The more common definition of a path algebra sees the paths in a graph as a basis for the path algebra.[3] We will see a similar relation with Leavitt path algebras in the next chapter.

[3] A definition which is seen in [10]. Note that graphs in the context of path algebras are usually called quivers.

3 Leavitt path algebras

In this chapter, a formal definiton of Leavitt path algebras will be given, namely Defintion 1.2.3 from [6]. We then cover some properties and examples. For the rest of this chapter we assume that K is an arbitrary field.

3.1 Defintions

Before we get to Leavitt path algebras, there is one final concept we will cover: the *extended graph*.

Definition 3.1.1. For any graph $G = (V, E, s, r)$, the extended graph $\hat{G}$ is defined as $\hat{G} = (V, E \cup E^*, s', r')$, where E^* has an edge e^* for every edge e in E. The two functions s' and r' act the same on E as their original, but are defined on E^* as

$$s'(e^*) = r(e) \text{ and } r'(e^*) = s(e).$$

From these definitions, it not hard to see how to visualize the extended graph $\hat{G}$ from a graph G. For every $e \in E$, draw another edge e^* (also known as a *ghost edge*) between the same two vertices as its corresponding e, but swap the source and range.

We can now define the Leavitt path algebra of a graph. As we will shortly see, it is defined as the path algebra (Definition 2.2.4) of the extended graph, along with two extra conditions:

Definition 3.1.2. Let G be a graph. The Leavitt path algebra (LPA) of G with coefficients in K, denoted $L_K(G)$, is the path algebra of the extended graph of G, with two additional conditions, shown as (CK1) and (CK2) below. It is thus subject to five relations in total:

$$\text{(V)} \quad v \cdot w = \begin{cases} v & \text{if } w = v, \\ 0 & \text{else} \end{cases}$$

$$\text{(E1)} \quad s(e) \cdot e = e \cdot r(e) = e$$

$$\text{(E2)} \quad r(e) \cdot e^* = e^* \cdot s(e) = e^*$$

$$\text{(CK1)} \quad e^* \cdot f = \begin{cases} r(e) & \text{if } f = e, \\ 0 & \text{else} \end{cases}$$

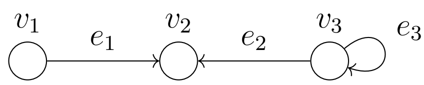
$$\text{(CK2)} \quad v = \sum_{\{e|s(e)=v\}} e \cdot e^* \quad \text{only if } v \text{ is regular}$$

for all $e, f \in E$ and $v, w \in V$. The first three relations come from the path algebra created using the extended graph. The last two relations are called the Cuntz-Krieger relations, as they are necessary to construct Cuntz-Krieger algebras.

Remark 3.1.3. We note that Leavitt path algebras and graphs are not in a one-to-one correspondence. As shown in Remark 2.6.21 in [6], it is very easy to find two non-isomorphic graphs which yield the same Leavitt path algebra.

We now give an example of a graph and its Leavitt path algebra. Note that for the rest of this thesis, the existence of ghost edges will be implied in any graph, even though they technically only exist in the extended graph.

Example 3.1.4. Let G be the graph from the previous examples.



To give an idea of how the elements of $L_K(G)$ relate to eachother, some computations are shown below.

$$v_1 \cdot e_1 = e_1 \cdot v_2 = e_1 \quad \text{by (E1)}$$
$$v_2 \cdot e_1^* = e_1^* \cdot v_1 = e_1^* \quad \text{by (E2)}$$
$$e_2^* \cdot e_2 = v_2 \;\; \text{and} \;\; e_2^* \cdot e_3 = 0 \quad \text{by (CK1)}$$
$$v_3 = e_3 \cdot e_3^* + e_2 \cdot e_2^* \quad \text{by (CK2)}$$

Note that (CK2) does not apply to vertex v_2, as it is a sink and therefore not a regular vertex.

With the addition of ghost edges in a graph, one can wonder how paths are affected. We will see that they are very fundamental to the structure of LPA's.

Notation 3.1.5. For a path $\mu = e_1 e_2 \cdots e_n$ in a graph G, we define its *ghost path* μ^* as $e_n^* \cdots e_2^* e_1^*$. For paths γ and λ, we will give $\gamma \cdot \lambda^*$ a special name. We call the product of a path and a ghost path a *monomial* in $L_K(G)$. (Like monomials in calculus, no element of a monomial in $L_K(G)$ is raised to a power greater than 1.)

Lemma 3.1.6 ([6], Lemma 1.2.12)**.** Let G be a graph and let $\gamma, \lambda, \mu, \eta$ be elements of $\text{Path}(G)$.

(i) The products of monomials in $L_K(G)$ are computed as follows:

$$(\gamma\lambda^*)(\mu\eta^*) = \begin{cases} \gamma\kappa\eta^* & \text{if } \mu = \lambda\kappa \text{ for some } \kappa \in \text{Path}(G), \\ \gamma\sigma^*\eta^* & \text{if } \lambda = \mu\sigma \text{ for some } \sigma \in \text{Path}(G), \\ 0 & \text{else.} \end{cases}$$

(ii) $L_K(G)$ is spanned as a K-vector space by elements of the form

$$\{\gamma\lambda^* \mid \gamma, \lambda \in \text{Path}(G) \text{ and } r(\gamma) = r(\lambda)\}.$$

This means that every element x of $L_K(G)$ can be written as

$$x = \sum_{i=1}^{n} k_i \gamma_i \lambda_i^*$$

for $n \geq 1$, k_i in K and γ_i, λ_i paths with $r(\gamma_i) = r(\lambda_i)$.

Proof. (i) For the first case, note that $\lambda^*\lambda = r(\lambda)$ by (CK1) and (E1)/(E2). The computation follows as $\gamma\lambda^*\lambda\kappa\eta^* = \gamma r(\lambda)\kappa\eta^* = \gamma\kappa\eta^*$. For the second case, note that $(\mu\sigma)^* = \sigma^*\mu^*$ by the definiton of a ghost path. The computation is then $\gamma\sigma^*\mu^*\mu\eta^* = \gamma\sigma r(\mu)\eta^* = \gamma\sigma^*\eta^*$. If λ and μ are not related to each other, then the product $\lambda^*\mu$ is zero by (CK1).

(ii) The proof that $\gamma\lambda^* \neq 0 \Leftrightarrow r(\gamma) = r(\lambda)$ follows similarly to computations done in Example 2.2.5. If we write out γ and λ^* as $\gamma = e_1 e_2 \cdots e_n$ and $\lambda^* = f_m^* f_{m-1}^* \cdots f_1^*$, then the multiplication within $\gamma\lambda^*$ we are interested in is $e_n f_m^*$. Expanding this using (E1) and (E2) gives $e_n \cdot r(\lambda) \cdot r(\gamma) \cdot f_m^*$, which, by (V), is 0 exactly when $r(\gamma) = r(\lambda)$. The result then follows from (i) and the fact that vertices are paths (see Remark 2.2.3). For example, an edge e for which $r(e) = v$ can be written like $e \cdot v^*$, where we define $v^* = v$.

■

We note that the representation shown in Lemma 3.1.6(ii) is almost never unique; unlike path algebras which have paths as their basis, these specific monomials do not form a basis of $L_K(G)$.

The final thing we will cover regarding LPA's is their *Universal Property*. This property will be used in the next section to establish isomorphisms between LPA's and other, more familiar algebras.

Theorem 3.1.7 (The Universal Property of $L_K(G)$)**.** Let G be a graph and A a K-algebra which contains a set $\{a_v \mid v \in V\}$ and two sets $\{a_e \mid e \in E\}, \{a_{e^*} \mid e \in E\}$. If the elements in those sets satisfy all the relations for a Leavitt path algebra (Definition 3.1.2), there exists a unique K-algebra homomorphism $\phi : L_K(G) \to A$ such that $\phi(v) = a_v, \phi(e) = a_e$ and $\phi(e^*) = a_{e^*}$ for all $v \in V$ and $e \in E$. Note that the relations of a LPA require a source and range function, whose equivalents in this case are given by $s(a_e) = a_{s(e)}$ and $r(a_e) = a_{r(e)}$.

To turn the above homomorphism into an isomorphism, we will either show it is injective and surjective, or give an inverse. For injectivity, there exist two theorems in [6] (Theorem 2.2.15 and 2.2.16) which guarantee a homomorphism is injective under certain conditions. We will state both here without proof.

Theorem 3.1.8 (The Graded Uniqueness Theorem)**.** Let G be a graph and A a so-called $\mathbb{Z}$-graded algebra. If $\phi : L_K(G) \to A$ is a $\mathbb{Z}$-graded algebra homomorphism with $\phi(v) \neq 0$ for all vertices, then ϕ is injective.

Theorem 3.1.9 (The Cuntz-Krieger Uniqueness Theorem)**.** Let G be a graph which satisfies a condition called Condition (L), and let A be a K-algebra. If $\phi : L_K(G) \to A$ is a homomorphism with $\phi(v) \neq 0$ for all vertices, then ϕ is injective.

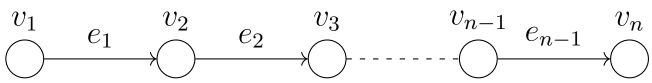
$\mathbb{Z}$-graded algebras will be introduced in Chapter 4. A definition of Condition (L) can be found in Definitions 2.2.2 in [6][4]. For the proofs in the next section, we will say without further explanation whenever either occurs.

[4] For those familiar with the terminology used in graph theory, a graph satisfies Condition (L) if every cycle in the graph has an exit.

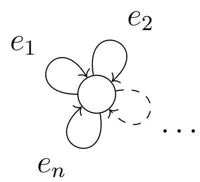
3.2 Three examples

We will now investigate three fundamental examples of LPA's, the first two of which are taken from [6]. We will let $n \in \mathbb{N}$ be arbitrary.

Notation 3.2.1. Let A_n denote the *oriented n-line graph* with n vertices and $n-1$ edges:



Let R_n denote the graph with one vertex and n edges, which all have that one vertex as source and range:



These graphs are rather simple, but their LPA's might be hard to imagine concretely. We can show that these LPA's are isomorphic as K-algebras to familiar algebras, making it easier to understand them. To demonstrate, we now prove that the LPA of A_n is isomorphic to the $n \times n$ matrix algebra:

Proposition 3.2.2 ([6], Proposition 1.3.5)**.**

$$L_K(A_n) \cong M_n(K).$$

Proof. Recall the standard matrix units $f_{i,j}$ from Remark 2.1.2. We prove this isomorphism by using the Universal Property of $L_K(A_n)$ (Theorem 3.1.7). To obtain a homomorphism between $L_K(A_n)$ and $M_n(K)$, let

$$a_{v_i} = f_{i,i},\ a_{e_i} = f_{i,i+1} \text{ and } a_{e_i^*} = f_{i+1,i}.$$

As an example, the element a_{v_2} corresponding to vertex v_2 is equal to the standard matrix unit $f_{2,2}$ with a 1 in the second row and column, and 0 elsewhere. We now show that these elements satisfy all the relations for a LPA:

$$\text{(V)} \quad a_{v_i} \cdot a_{v_j} = f_{i,i}f_{j,j} = \begin{cases} f_{i,i} = a_{v_i} & \text{if } i = j, \\ 0 \cdot f_{i,i} & \text{else} \end{cases}$$

$$\text{(E1)} \quad s(a_{e_i}) \cdot a_{e_i} = a_{v_i} \cdot a_{e_i} = f_{i,i}f_{i,i+1} = f_{i,i+1} = a_{e_i}$$
$$\text{and } a_{e_i} \cdot r(a_{e_i}) = a_{e_i} \cdot a_{v_{i+1}} = f_{i,i+1}f_{i+1,i+1} = f_{i,i+1} = a_{e_i}$$

$$\text{(E2)} \quad r(a_{e_i}) \cdot a_{e_i^*} = a_{v_{i+1}} \cdot a_{e_i^*} = f_{i+1,i+1}f_{i+1,i} = f_{i+1,i} = a_{e_i^*}$$
$$\text{and } a_{e_i^*} \cdot s(a_{e_i}) = a_{e_i^*} \cdot a_{v_i} = f_{i+1,i}f_{i,i} = f_{i+1,i} = a_{e_i^*}$$

$$\text{(CK1)} \quad a_{e_i^*} \cdot a_{e_j} = f_{i+1,i}f_{j,j+1} = \begin{cases} f_{j+1,j+1} = a_{v_{j+1}} = r(a_{e_j}) & \text{if } i = j, \\ 0 \cdot f_{j+1,j+1} & \text{else} \end{cases}$$

$$\text{(CK2)} \quad \sum_{\{j|s(a_{e_j})=a_{v_i}\}} a_{e_j} \cdot a_{e_j^*} = a_{e_i} \cdot a_{e_i^*} = f_{i,i+1}f_{i+1,i} = f_{i,i} = a_{v_i}$$

for all applicable indices i, j. As $M_n(K)$ contains the required sets, there exists a unique homomorphism $\phi : L_K(A_n) \to M_n(K)$ such that

$$\phi(v_i) = a_{v_i} = f_{i,i},\ \phi(e_i) = a_{e_i} = f_{i,i+1} \text{ and } \phi(e_i^*) = a_{e_i^*} = f_{i+1,i}.$$

To prove ϕ is an isomorphism, we show that it is injective and surjective. Injectivity follows from the fact that A_n satisfies Condition (L) and $\phi(v_i) \neq 0$ for all $i \in \{1, \dots, n\}$[5]. If $n = 2$, ϕ is surjective, and we are done. Otherwise we need to show that $f_{i,j}$ is an element of $\text{Im}(\phi)$ for all $i, j \in \{1, \dots, n\}$. In other words, we should be able to make any other standard matrix unit from the ones given in the specification of ϕ shown above. So, let $f_{i,j}$ be a standard matrix unit with $i \neq j, i \neq j+1$ and $i+1 \neq j$. The difference between i and j is thus some integer k greater than 1, which we will assume is 2 at first. We also assume that j is greater than i, which means $j = i+2$. The standard matrix unit $f_{i,j} = f_{i,i+2}$ can then be made as $f_{i,i+1}f_{i+2-1,i+2}$. If i is greater than j, then $f_{i,j} = f_{j+2,j}$ can be made as$f_{j+2,j+2-1}f_{j+2-1,j}$. For k greater than 2, one can use induction with the fact that $f_{i,i+k} = f_{i,i+k-1}f_{i+k-1,i+k}$.

■

To show the next isomorphism, we will look at another familiar algebra. The *Laurent polynomial K-algebra* is the K-algebra generated by two elements x and x^{-1}, such that $xx^{-1} = x^{-1}x = 1$. This algebra is denoted by $K[x, x^{-1}]$, and elements of this algebra can be identified with the polynomials, including those with negative powers. This brings us to the next isomorphism:

[5]Here we have used Theorem 3.1.9.

Proposition 3.2.3 ([6], Proposition 1.3.4)**.**

$$L_K(R_1) \cong K[x, x^{-1}].$$

Proof. R_1 is the graph with one vertex v and one edge e (and one ghost edge e^*). We again use the Universal Property by setting

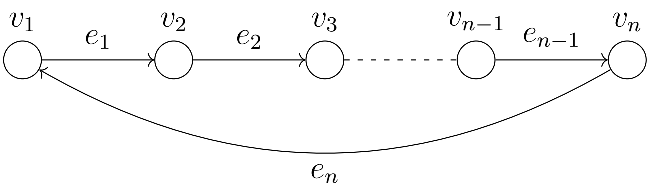
$$a_v = 1,\ a_e = x \text{ and } a_{e^*} = x^{-1}.$$

Now, the process is the same as the previous proof. To illustrate, one can compute for (CK1) that $a_e \cdot a_{e^*} = a_v$. For (CK2), because v is a regular vertex, $a_{e^*} \cdot a_e = a_v$. The unique homomorphism ϕ is then given. We describe an inverse for ϕ denoted by ϕ^{-1} as $\phi^{-1}(1) = v, \phi^{-1}(x) = e$ and $\phi^{-1}(x^{-1}) = e^*$. As $\phi \circ \phi^{-1} = K[x, x^{-1}]$ and $\phi^{-1} \circ \phi = L_K(R_1)$, ϕ is an isomorphism. ■

The general case for R_n can be found in [6]. To summarize, the LPA of R_n is for each n isomorphic to a different *Leavitt algebra* $L_K(1, n)$, introduced by W.G. Leavitt in [8]. The Leavitt path algebras are named after these algebras, as they were a motivating construction.

The third example, shown below, is not taken from the book. It appears in the Wikipedia article for Leavitt path algebras without proof. As we will see, the example combines ideas from the first and second, hence its inclusion in this thesis.

Notation 3.2.4. Let C_n denote the *cyclic graph* with n vertices and n edges:



Note that C_n is composed of A_n along with an extra edge e_n going from v_n to v_1. This creates a path from v_1 to itself, which is called a cycle in graph theory. There is also a cycle in R_1, suggesting a combination of the two isomorphisms:

Proposition 3.2.5.

$$L_K(C_n) \cong M_n(K[x, x^{-1}]).$$

Proof. The mapping used for this proof is the same on the "A_n" part of C_n as the mapping used in the proof of $L_K(A_n)$. We thus have $a_{v_i} = f_{i,i}$, $a_{e_i} = f_{i,i+1}$ and $a_{e_i^*} =$

$f_{i+1,i}$ as before. The mapping on e_n and e_n^* is now defined as $a_{e_n} = x \cdot f_{n,1}$ and $a_{e_n^*} = x^{-1} \cdot f_{1,n}$. The process again follows that of the previous proofs. For example:

(CK1) $a_{e_n^*} \cdot a_{e_n} = x^{-1} \cdot f_{1,n} \cdot x \cdot f_{n,1} = f_{1,n} f_{n,1} = f_{1,1} = a_{v_1}$.

(CK2) $\sum_{\{j | s(a_{e_j}) = a_{v_n}\}} a_{e_j} \cdot a_{e_j^*} = a_{e_n} \cdot a_{e_n^*} = x \cdot f_{n,1} \cdot x^{-1} \cdot f_{1,n} = f_{n,1} f_{1,n} = f_{n,n} = a_{v_n}$.

The rest of the relations follow as in the proof of $L_K(A_n)$. We again get a unique homomorphism ϕ by the Universal Property. For injectivity, both the algebra $M_n(K[x, x^{-1}])$ and ϕ are $\mathbb{Z}$-graded. As $\phi(v_i) \neq 0$, Theorem 3.1.8 states that ϕ is injective. From the proof of $L_K(A_n)$, we have seen that $M_n(K) \subseteq \text{Im}(\phi)$. We can thus compute $a_{e_n} \cdot f_{1,k} = x \cdot f_{n,1} f_{1,k} = x \cdot f_{n,k}$ for all $k \in \{1, \ldots, n\}$, allowing us to "transfer" the x to the standard matrix units $f_{n,k}$. In a similar way, we can bring x^{-1} to the standard matrix units $f_{1,k}$. Then, for all $i, j \in \{1, \ldots, n\}$, we can make $x \cdot f_{i,j}$ as $f_{i,n} \cdot x \cdot f_{n,j}$ and $x^{-1} \cdot f_{i,j}$ as $f_{i,1} \cdot x^{-1} \cdot f_{1,j}$. Higher or lower powers of x can be obtained by the relations (V), (E1) and (E2). This means that any element of $M_n(K[x, x^{-1}])$ can be made. The homomorphism is thus surjective and we have an isomorphism.

■

We have now built up the Leavitt path algebras as the path algebra of an extended graph under the Cuntz-Krieger relations. We have shown that paths and monomials in particular play an important role in the structure of a Leavitt path algebra, and discussed three fundemental examples using the Universal Property. We are ready to continue to Chapter 4 and discuss ideals, gradings and graded ideals.

4 Gradings and Ideals

Now that the basic concepts of Leavitt path algebras have been covered, we move on to some more advanced topics. We will discuss the $\mathbb{Z}$-grading of a LPA in Section 4.1 and go over ideals with respect to this grading in Section 4.2. The main result of Chapter 4 (and of this thesis) is Theorem 4.2.4, which relates specific subsets of vertices of a graph with specific ideals of its LPA.

To begin, we define an ideal of an algebra formally.

Definition 4.0.1. Let A be a K-algebra, and let I be a linear subspace. If I is closed under multiplication by elements from A, it is called an *ideal*. We thus assume that every ideal is *two-sided*: For $x \in I$ and $a \in A$, both ax and xa are in I.

Every K-algebra has two *trivial* ideals: $\{0\}$, the subspace with 0 as its only element, and the K-algebra itself. For a subset X of an algebra A, we use $I(X)$ to denote the ideal of A generated by X.

The specific subsets of vertices mentioned in the beginning of this chapter will now be explained in more detail. To aid in this, we define the *set of neighbors* of a vertex:

Definition 4.0.2. Let $G = (V, E, s, r)$ be a graph. For $v \in V$, the set of neighbors of v, denoted by N_v, is defined as $N_v = \{r(e) \mid e \in E, s(e) = v\}$.

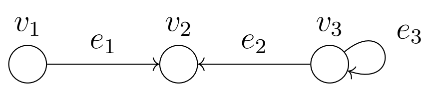
Definition 4.0.3. Let $G = (V, E, s, r)$ be a graph, and let $H \subseteq V$.

(i) H is called *hereditary* if for all $v \in H$, if there exists a $w \in V$ such that there is a path from v to w, $w \in H$.

(ii) H is called *saturated* if all regular $v \in V$ for which $N_v \subseteq H$ are included in H.

In other words, if H is hereditary, then there are no paths starting from a vertex in H ending in a vertex outside of H. If H is saturated, then for a regular vertex, if every vertex in its set of neighbors is in H, it is as well.

Example 4.0.4. Consider the graph G shown below, and sets $\{v_1\}, \{v_2\} \subseteq V$.



We see that $\{v_1\}$ is not hereditary, as there is a path from v_1 to v_2 but v_2 is not included in the set. Because v_2 has no paths to other vertices, $\{v_1, v_2\}$ is a hereditary set. For $\{v_2\}$, we see that it is not saturated, because every vertex in the set of neighbors of v_1 (so v_2) is in the set, but v_1 is not in $\{v_2\}$ (and v_1 is regular.)

For a graph G, we will use $\mathscr{H}_G$ to denote set of all hereditary saturated subsets of the vertices of G. In the example above, one can check that $\mathscr{H}_G = \{\emptyset, \{v_1, v_2\}, \{v_1, v_2, v_3\}\}$.

We will shorten our terminology by calling the sets in $\mathscr{H}_G$ hereditary saturated subsets, or just subsets. We will also call $\emptyset$ and V trivial subsets, for reasons which will become clear at the end of Section 4.2.

4.1 The $\mathbb{Z}$-Grading

This section covers gradings of algebras and ideals with respect to gradings. In particular, we will see in Theorem 4.1.6 that every Leavitt path algebra is $\mathbb{Z}$-graded, which provides the structure necessary for the results in Section 4.2. We again denote an arbitrary field by K.

Definition 4.1.1 ([6], Definition 2.1.1)**.** Let G be a group (not a graph) and A an algebra over K. We say A is *G-graded* if there exists a family $\{A_g\}_{g\in G}$ of vector spaces over K such that

$$A = \bigoplus_{g\in G} A_g \text{ as } K\text{-spaces, and } A_g \cdot A_f \subseteq A_{gf} \text{ for all } g, f \in G.$$

The second condition can be understood as: $a_g \cdot a_f \in A_{gf}$ for all $a_g \in A_g$ and $a_f \in A_f$.

Definition 4.1.2 ([6], Definition 2.1.1)**.** An ideal I of a G-graded K-algebra $A = \bigoplus_{g\in G} A_g$ is called a *graded ideal* if $I \subseteq \oplus_{g\in G}(I \cap A_g)$.

To see gradings in action, we return to the Laurent polynomial algebra $A = K[x, x^{-1}]$. The group we consider (and will continue to consider) for the grading is $\mathbb{Z}$, with the $\mathbb{Z}$-grading given by setting $A_i = Kx^i$ for every $i \in \mathbb{Z}$. We see that A is the direct sum of all the A_i's, and that $A_{-2} = kx^{-2}$ and $A_3 = kx^3$ for $k \in K$. Multiplying two elements from these vector spaces gives kx^1, which is an element of $A_1 = A_{-2+3}$.

Remark 4.1.3. If an algebra A is G-graded, we will call the vector space A_g a *homogeneous space of degree g*. Likewise, an element a_g of A_g is called a *homogeneous element of degree g*. Note that any element a of A can be written as a sum of homogeneous elements of different degrees:

$$a = \sum_{g\in G} a_g$$

In the case of the Laurent polynomial algebra, we can see this explicitly with the usual power series representation of polynomials.

It is known that $K[x, x^{-1}]$ has an infinite number of ideals. However, one can prove that it only has two graded ideals. In order to do this (relatively) easily, we state and prove an equivalent condition for a graded ideal.

Proposition 4.1.4 ([6], Defintions 2.1.1)**.** An ideal I of a G-graded algebra is graded if

$$a = \sum_{g \in G} a_g \in I \text{ implies } a_g \in I \text{ for all } g \in G$$

where the representation comes from Remark 4.1.3.

Proof. Starting from the original condition, we let $a \in I \subseteq \oplus_{g \in G}(I \cap A_g)$. Now assume that $a_g \notin I$ for some $g \in G$. Then for that g, $I \cap A_g = \emptyset$ and a cannot be in $\oplus_{g \in G}(I \cap A_g)$, which is a contradiction.

For the opposite direction, we let $a \in I$ and $a_g \in I$ for all $g \in G$. Then $a \in \oplus_{g \in G}(I \cap A_g)$, because for all $g \in G$, $a_g \in I \cap A_g$.

■

This is brings us to the proposition below. The statement of this proposition is mentioned in Remark 2.1.6 in [6], "proven", as it were, using only one sentence.

Proposition 4.1.5. $K[x, x^{-1}]$ has no *non-trivial* graded ideals.

Proof. Let I be a graded ideal which is not $K[x, x^{-1}]$. We will show that I must be equal to $\{0\}$. Note that 1 cannot be an element of I, as the definition of an ideal would guarantee that $I = K[x, x^{-1}]$. This implies that elements of the form $1 + \sum_i^n k_i x^i$ cannot be in I as well. (If they could, decomposing those elements into 1 and powers of x and using Proposition 4.1.4 implies $1 \in I$.) We thus consider elements of the form $\sum_i^n k_i x^i$. Let $k_j x^j$ be a summand with $k_j \neq 0$. Because K is a field, we can find an element $k_j^{-1} \in K$ such that $k_j^{-1} k_j = 1$. Multiplying the whole sum by $k_j^{-1} x^{-j}$ then yields an element of the form $1 + \sum_\ell^m k_\ell x^\ell$, which should be in I by the defintion of an ideal. Since we have shown that this is impossible, only one option remains: $k_j = 0$ for every $j \in \{i, \ldots, n\}$, which makes the sum equal to 0.

■

We finish this section with an important result from [6], namely that every LPA is $\mathbb{Z}$-graded. The proof is omitted for the sake of simplicity and can be found in [6].

Theorem 4.1.6 (Corollary 2.1.5 (iii))**.** Let G be a graph. Then $L_K(G)$ is $\mathbb{Z}$-graded as follows:

$$L_K(G) = \bigoplus_{n \in \mathbb{Z}} L_n = \bigoplus_{n \in \mathbb{Z}} \text{span}(\{\gamma\lambda^* \mid \gamma, \lambda \in \text{Path}(G) \text{ and } \ell(\gamma) - \ell(\lambda) = n\}).$$

In words, a homogeneous space of degree n is the span of monomials in $L_K(G)$ whose paths have lengths differing by n. A homogeneous element of degree n then looks like $\sum_{i=1}^m k_i \gamma_i \lambda_i^*$ for $m \geq 1, k_i \in K$ and γ and λ as described above.

This theorem will be the backbone of the results in the next section.

4.2 Graded ideals

We arrive at the main section of this thesis, covering ideals and graded ideals in more depth. The goal is to prove Theorem 4.2.4, which states that every graded ideal of a Leavitt path algebra is generated by a hereditary and saturated subset of the vertices of its underlying graph.

We begin with a description of the elements in an ideal generated by a hereditary subset.

Lemma 4.2.1 ([6], Lemma 2.2.1)**.** Let G be a graph, and H a hereditary subset of V. Then the elements of the ideal $I(H)$ of $L_K(G)$ can be viewed in the following way:

$$I(H) = \left\{ \sum_{i=1}^{n} k_i \gamma_i \lambda_i^* \mid n \geq 1,\ k_i \in K,\ \gamma_i, \lambda_i \in \mathrm{Path}(G) \text{ with } r(\gamma_i) = r(\lambda_i) \in H \right\}.$$

Proof. Let J be the set shown above, and note that it is a linear subspace. Thus, to prove that J is an ideal, we only have to consider multiplication with elements of $L_K(G)$. Let $\alpha\beta^*$ be a monomial in J and $a, b \in L_K(G)$. We will show that $a\alpha\beta^* b$ is in J as well. (Note that this is sufficient for a two-sided ideal: Setting either a or b as the multiplicative identity gives the two products shown in the beginning of this chapter).

Looking at Lemma 3.1.6(ii), we only need to prove that $\gamma\lambda^* u \mu\eta^* \in J$ for all paths $\gamma, \lambda, \mu, \eta$ and vertices $u \in H$. If the product is zero, we are done. Otherwise, by Lemma 3.1.6(i), two cases occur: $\gamma\lambda^* u\mu\eta^* = \gamma\mu'\eta^*$ if $\mu = \lambda\mu'$, or $\gamma\lambda^* u\mu\eta^* = \gamma(\lambda')^*\eta^*$ if $\lambda = \mu\lambda'$. In the first case, we should have that $r(\mu') = r(\mu) \in H$ in order to say that the product is in J. Because $\gamma\lambda^* u\mu\eta^* \neq 0$, we see that $u = s(\mu)$. As H is hereditary, $r(\mu)$ is also in H and the product is in J. In the second case, we should have that $r(\lambda') = r(\mu) \in H$. We have already proven this, so the product is also in J.

This shows that J is an ideal of $L_K(G)$. To see that it is equal to $I(H)$, note that it contains H, as vertices are paths with $r(v) = v$ and $v^* = v$ for every vertex v, and it must be contained in every ideal containing H. ■

We move on to the following lemma, which is an important step in connecting ideals and hereditary saturated subsets.

Lemma 4.2.2 ([6], Lemma 2.4.3)**.** Let G be a graph, and I an ideal of $L_K(G)$. Then $I \cap V \in \mathscr{H}_G$.

Proof. Let $v \in I$ and $w \in V$ such that there is a path from v to w. If we call this path γ, then we find that $w = r(\gamma) = \gamma^*\gamma = \gamma^* v\gamma \in I$. So $I \cap V$ is hereditary.

Now let u be a regular vertex and suppose $r(e) \in I$ for all $e \in s^{-1}(u)$. By (CK2), we find $u = \sum_{e \in s^{-1}(u)} ee^* = \sum_{e \in s^{-1}(u)} er(e)e^* \in I$. So $I \cap V$ is also saturated.

■

A last definition is given before Theorem 4.2.4 is discussed. It will be used in the proof.

Definition 4.2.3 ([6], Definitions 2.2.9)**.** Let G be a graph. For a monomial $\gamma\lambda^*$ in $L_K(G)$, its *degree in ghost edges* (or *ghost degree*) is defined to be equal to $\ell(\lambda)$. A monomial without ghost edges has degree in ghost edges equal to 0. More generally, an element $\sum_{i=1}^{n} k_i\gamma_i\lambda_i^*$ has ghost degree equal to the maximum degree in ghost edges of the monomials in the sum. Note that because monomials do not form a basis, the notion of ghost degrees is not well-defined for every element in $L_K(G)$. We will say that the ghost degree of an element x, denoted by $\text{gdeg}(x)$, is defined as minimum ghost degree of every representation it has in $L_K(G)$.

We are now ready to cover Theorem 4.2.4. The proof shown here is an adaptation of the proof of Theorem 2.4.8 in [6], which is a general result for infinte graphs.

Theorem 4.2.4 ([6], Proposition 2.4.9)**.** Let $G = (V, E, s, r)$ be a row-finite graph. Then every graded ideal I of $L_K(G)$ is generated by $H = I \cap V \in \mathscr{H}_G$. In other words, every graded ideal of $L_K(G)$ is generated by the set of its vertices, and that set is always hereditary and saturated.

Proof. The fact that $H \in \mathscr{H}_G$ is a direct consequence of Lemma 4.2.2. We will thus prove that $I = I(H)$. The fact that $H \subseteq I$ implies that $I(H) \subseteq I(I) = I$. We now show that $I \subseteq I(H)$ by induction on the degree in ghost edges of the elements in I (introduced in Definition 4.2.3). Because I is graded, using Remark 4.1.3 we only have to consider nonzero homogeneous elements $\alpha \in I$. We will also only consider those elements such that $\alpha = \alpha v$ for some $v \in V$. Any element in a homogeneous space can then be constructed using these elements by the description given in Theorem 4.1.6.

So, suppose $\alpha \in I$ satisfies all the conditions stated above, and suppose $\text{gdeg}(\alpha) = 0$. Then α can be written as $\alpha = \sum_{i=1}^{m} k_i\gamma_i$ by Lemma 4.2.1, where our assumptions imply that $r(\gamma_i) = v$ for all $i \in \{1, \dots, n\}$. Taking one summand $k_j\gamma_j$ and multiplying $k_j^{-1}\gamma_j^*$ with α then gives $k_j^{-1}\gamma_j^* \cdot \sum_{i=1}^{m} k_i\gamma_i = k_j^{-1}\gamma_j^* k_j\gamma_j = r(\gamma_j) = v$, which is in I, and thus in $H = I \cap V$. As $v \in H$ and $I(H)$ is an ideal, $\alpha v = \alpha \in I(H)$.

We now assume the theorem is true for those homogeneous elements assumed above, but with ghost degree less than n, and prove the theorem for elements α having ghost degree equal to n. We begin by writing α as $\sum_{i=1}^{m} x_i e_i^* + \lambda$, where the x_i's are in $L_K(G)$, the e_i's are in E and λ is in $\text{Path}(G)$. We assume that this representation has the minimum ghost degree and note that this implies both the x_i's and λ have ghost degree less than n.

If $\lambda = 0$ then we can pick an edge e_j and see that $\alpha e_j = x_j$ by (CK1). Because x_j has ghost degree less n, it is in $I(H)$ by the induction hypothesis and we are done. So, we can assume that $\lambda \neq 0$.

We proceed by observing that v is not a sink, as $\alpha v = v$ implies that $e_i^* v = e_i^*$, which means that $s(e_i) = v$ for all $i \in \{1, \ldots, n\}$. We can thus choose an edge $f \in s^{-1}(v)$ to multiply with α. If f is unequal to all the e_i's, $\alpha f = \lambda f$ by (CK1) which is in $I(H)$ by the induction hypothesis. Otherwise, f is equal to some e_j, and $\alpha f = x_j e_j^* f + \lambda f = x_j + \lambda f$, which again lies in $I(H)$ by the induction hypothesis. We have thus established that $\alpha f \in I(H)$. This means that $\alpha = \alpha v = \alpha \sum_{f \in s^{-1}(v)} f f^* = \sum_{f \in s^{-1}(v)} \alpha f f^* \in I(H)$ and we are done.

■

With Theorem 4.2.4, we can now identify the graded ideals of a Leavitt path algebra with the hereditary saturated subsets of the vertices of its underlying graph. This is, of course, provided that the graph is row-finite. In the case that the graph is infinte, hereditary saturated subsets are not enough to describe the graded ideals, and a concept known as *breaking vertices* has to be introduced. This is done in Definitions 2.4.4 in [6].

If we look at Proposition 4.1.5 again, we can prove it much faster now using Theorem 4.2.4 by determining the hereditary saturated subsets of the vertices of R_1. This is trivial, as R_1 only has one vertex. The only hereditary saturated subsets are thus $\emptyset$ and $\{v\} = V$, with the only graded ideals then being $\{0\}$ and $L_K(R_1)$. Similarly, if we return to A_n and C_n from Proposition 3.2.2 and 3.2.5, we can see that neither has non-trivial hereditary saturated subsets. Either no vertices are in the subset, which corresponds to the empty set, or one vertex is picked and the hereditary and saturated conditions then imply that every vertex has to be included. Hence, the Leavitt path algebras of these graphs have no non-trivial graded ideals.

This brings us back to Example 4.0.4. We saw there that the graph G has one non-trivial hereditary saturated subset. Theorem 4.2.4 then states that the Leavitt path algebra $L_K(G)$ has a non-trivial graded ideal, which is generated by that subset.

In the next chapter, we will use this graph and others to explore more properties regarding graded ideals.

5 Naturality

The second to last chapter of this thesis covers graded ideals in two different contexts to show that they have very nice properties. The first context relates to quotients of algebras, which are similar to quotients of groups. The second context shows that graded ideals are ordered in a specific way, known as a lattice. The results presented in this chapter can be extended to infinite graphs, but require more concepts and proofs.

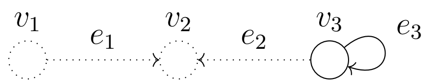
Definition 5.0.1. Let A be an algebra, and I an ideal. We define the *equivalence class* of an element a, denoted by $a+I$, as $a+I=\{a+b \mid b \in I\}$. The *quotient algebra* A/I is the algebra made from the equivalence classes of A.

5.1 Quotients

We introduce the defintion of a quotient graph here and give an example.

Definition 5.1.1 ([6], Definition 2.4.11)**.** Let $G=(V,E,s,r)$ be a graph, and let H be a hereditary subset of V. The *quotient graph of* G *by* H, denoted G/H, is made out of G by replacing V with $V \setminus H$ and E with $E \setminus \{e \mid r(e) \in H\}$. The source and range functions have their domain and codomain adjusted accordingly.

Example 5.1.2. Consider the graph G shown below, and the set $H=\{v_1,v_2\} \subseteq V$.



We saw in Example 4.0.4 that H is hereditary. The quotient graph of G by H can then be made by removing H from the vertices, and removing all the edges which have their range in H. These vertices and edges are shown as dotted. What remains is then called G/H, a graph which looks like (and is isomorphic to) R_1.

As hereditary saturated subsets are still hereditary, we can construct quotient graphs with them as well. The next theorem will show that there is a natural connection between quotient graphs made from these subsets and quotients of algebras. The proof of the theorem is adapted from the proof of Theorem 2.4.12 in [6]. That result applies, like Theorem 2.4.8 for Theorem 4.2.4, to more general graphs and requires more background knowledge.

Theorem 5.1.3 ([6], Corollary 2.4.13(i))**.** Let G be a row-finite graph and $H \in \mathscr{H}_G$. Then $L_K(G)/I(H) \cong L_K(G/H)$ as $\mathbb{Z}$-graded K-algebras. Note that the ideal $I(H)$ is automatically graded by Theorem 4.2.4.

Proof. We employ the Universal Property from Theorem 3.1.7 once again. Note that we will have to consider $L_K(G)$ first to use it. We do this by describing the map from $L_K(G)$ to $L_K(G/H)$ as a function, called ψ. The map follows:

$$\psi(v) = \begin{cases} v & \text{if } v \notin H, \\ 0 & \text{else} \end{cases} \qquad \psi(e) = \begin{cases} e & \text{if } r(e) \notin H, \\ 0 & \text{else} \end{cases} \qquad \psi(e^*) = \begin{cases} e^* & \text{if } s(e^*) \notin H, \\ 0 & \text{else} \end{cases}$$

It it not hard to imagine that the relations of the original LPA still hold when looking at G/H only. By the Universal Property, we are given a unique homomorphism ϕ from $L_K(G)$ to $L_K(G/H)$ which is identical to ψ. As ϕ is clearly 0 on $I(H)$, we can induce the map on $L_K(G)/I(H)$. Now we define an inverse map for ϕ, called θ:

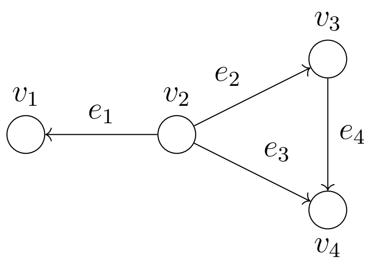
$$\theta(v) = v + I(H), \quad \theta(e) = e + I(H), \quad \theta(e^*) = e^* + I(H).$$

θ maps an element of $L_K(G/H)$ to its equivalence class in $L_K(G)/I(H)$. As operations between equivalence classes are induced by their elements, the relations of the original LPA hold. The Universal Property then gives us a another homomorphism ϕ^{-1}. We see that $\phi \circ \phi^{-1} = L_K(G/H)$ and $\phi^{-1} \circ \phi = L_K(G)/I(H)$, so ϕ is an isomorphism. ■

We return to Example 5.1.2. Though the Leavitt path algebra of the full graph G has not been determined[6], we can say something about $L_K(G)/I(H)$, the quotient of the LPA with respect to the ideal generated by H. Theorem 5.1.3 implies that this quotient is isomorphic to the Leavitt path algebra of the quotient graph $L_K(G/H)$, which was shown to be isomorphic to $K[x, x^{-1}]$ by Proposition 3.2.3.

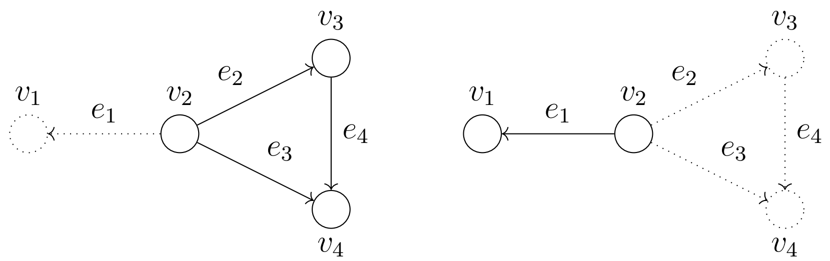
We now give another example with a bigger graph. We will determine its hereditary saturated subsets and construct quotient graphs and quotient algebras with them.

Example 5.1.4. Consider a different graph, G_2, shown below.



[6]As is traditional to do in mathematical literature, we leave this as an exercise to the reader.

The hereditary saturated subsets can be found by applying the hereditary and saturated conditions to four sets with each a different vertex.
$\{v_1\}$: As v_1 is a sink, the hereditary condition is vacuously satisfied. The saturated condition would not be satisfied if v_1 was the only vertex in the set of neighbors of v_2. This is not the case, so $\{v_1\}$ is also saturated. It is thus an element of $\mathscr{H}_{G_2}$.
$\{v_2\}$: v_2 is not a sink, so every vertex in its set of neighbors should be included to make the set hereditary, resulting in $\{v_1, v_2, v_3, v_4\}$. Because every vertex of the graph is now contained in the set, it is automatically saturated.
$\{v_3\}$: The only vertex in the range v_3 is v_4, so it is included in the set to make it hereditary. It is also saturated, for reasons similar to v_1. We have $\{v_3, v_4\} \in \mathscr{H}_{G_2}$.
$\{v_4\}$: Applying the saturated condition, we find that v_3 should be included in the set. This case is thus the same as the previous.
We see that $\mathscr{H}_{G_2} = \{\emptyset, \{v_1\}, \{v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\}$.
Similarly to Example 5.1.2, the quotients graphs corresponding to the two non-trivial subsets are shown below.



If we label the subsets by setting $H_1 = \{v_1\}$ and $H_2 = \{v_3, v_4\}$, then we see that G/H_1 resembles A_3 and G/H_2 is isomorphic to A_2. (The graphs A_n are introduced in Notation 3.2.1). Theorem 5.1.3 then states that $L_K(G)/I(H_1) \cong L_K(G/H_1)$ and $L_K(G)/I(H_1) \cong L_K(G/H_1)$.

We saw in Proposition 3.2.2 that $L_K(A_n) \cong M_n(K)$. Combining this proposition with Example 5.1.4 above gives that $L_K(G)/I(H_2) \cong L_K(G/H_2) \cong M_2(K)$. One might wonder if something similar happens with G/H_1, and if $L_K(G/H_1)$ is isomorphic to $M_3(K)$. This not the case, however. Though G/H_1 and A_3 both have three vertices, the fact that the paths in the two graphs are different means that their Leavitt path algebras are not isomorphic. Instead, $L_K(G/H_1) \cong L_K(A_4)$[7]. We find that $L_K(G)/I(H_1) \cong L_K(G/H_1) \cong M_4(K)$.

This graph will come back in the next section, where we will study the relations between the graded ideals from a different perspective. In order to achieve this, a

[7]As mentioned in Remark 3.1.3, non-isomorphic graphs can have the same Leavitt path algebra.

corollary of Theorem 5.1.3 is given below. The corollary is not stated in the book directly. For general graphs, both the statement and the proof can be found in [6] under Corollary 2.4.16(i).

Corollary 5.1.5. Let G be a row-finite graph and let $H \in \mathscr{H}_G$. Then $I(H) \cap V = H$.

Proof. As $H \subseteq V$, the inclusion $H \subseteq I(H) \cap V$ is clear. We now prove the converse by showing that there is no vertex $v \in V \setminus H$ which is in $I(H) \cap V$. For such a vertex v, observe that $\psi(v)$ is unequal to zero in $L_K(G/H)$, where ψ comes from the proof of Theorem 5.1.3. The theorem then says that v is unequal to zero in $L_K(G)/I(H)$, which implies that v is not in $I(H) \cap V$.

■

5.2 Lattices

This section will discuss lattices, and in particular the lattice of graded ideals. One may have encountered lattices before in group theory. In that sense, they are related to real coordinate spaces. There is another definition of a lattice however, which is less visual. It is built from a *partially ordered set*:

Definition 5.2.1. A partially ordered set is a set X equipped with a relation $\leq$ which satisfies three properties. For $x, y, z \in X$, they are given as:

$$\text{Reflexitivity: } x \leq x.$$
$$\text{Antisymmetry: if } x \leq y \text{ and } y \leq x, \text{ then } x = y.$$
$$\text{Transitivity: if } x \leq y \text{ and } y \leq z, \text{ then } x \leq z.$$

A standard example of a partially ordered set is a power set, where the elements are subsets and the relation is inclusion.

As the reader is aware, one can do more things in a power set than verfiy or establish inclusions. For example, we can take two subsets and create their union and intersection. These two operations make a power set a *lattice*:

Definition 5.2.2. A lattice is a partially ordered set, where every subset of two elements has a greatest lower bound (called a meet) and a lowest upper bound (called a join).

Because the relation of a power set is inclusion, the meet of two sets has to be the set which included in both, and is as big as possible. We can identify this set with the intersection. Similarly, the join of two sets is their union, as it is the smallest set which contains both.

We now bring lattices to the context of Leavitt path algebras by looking at graded ideals and hereditary saturated subsets.

Notation 5.2.3 ([6], Definition 2.5.1)**.** Let G be a graph and K a field. The set of all graded ideals of $L_K(G)$ is a lattice. The relation here is again inclusion, and the meet and join are intersection and direct sum respectively. We denote this lattice by $\mathscr{L}_{gr}(L_K(G))$.

Remark 5.2.4 ([6], Remark 2.5.2). We can also see that the set of all hereditary saturated subsets of vertices of G form a lattice. The structure of this lattice is similar to that of a power set, but the join is defined slightly differently. One can imagine that simply taking the union of two hereditary saturated subsets might not result in a hereditary saturated subset. The join used for these subsets is called the *hereditary saturated closure*, and is defined in [6] as Definition 2.0.6. We will this see closure in effect in Example 5.2.6.

This brings us to Theorem 5.2.5, which connects graded ideals and hereditary saturated subsets in an important way. The proof is adapted from Theorem 2.5.8 in [6], which is a version of Theorem 5.2.5 for general graphs.

Theorem 5.2.5 (The Structure Theorem for Graded Ideals, [6] Thm 2.5.9)**.** *Let G be a row-finite graph and K a field. The map $\phi : \mathscr{L}_{gr}(L_K(G)) \to \mathscr{H}_G$ defined as $\phi(I) = I \cap V$ is a lattice isomorphism. Its inverse $\phi' : \mathscr{H}_G \to \mathscr{L}_{gr}(L_K(G))$ is defined as $\phi'(H) = I(H)$.*

Proof. We first note that Theorem 4.2.2 guarantees that for a graded ideal I, $I \cap V$ is a hereditary saturated subset of V. We also have by Theorem 4.2.4 that for a hereditary saturated subset H, $I(H)$ is a graded ideal. Both maps are thus well defined.

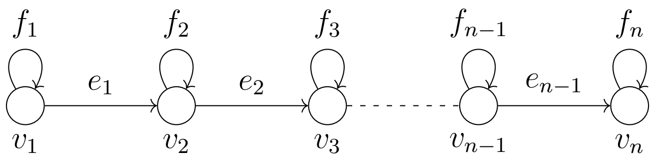
Theorem 4.2.4 also tells us that $\phi' \circ \phi = \mathrm{Id}_{\mathscr{L}_{gr}(L_K(G))}$ and Corollary 5.1.5 that $\phi \circ \phi' = \mathrm{Id}_{\mathscr{H}_G}$. The maps ϕ and ϕ' thus define an isomorphism. To show that it is a lattice isomorphism, we show that both maps preserve the order of the lattice. We start with ϕ'. Let $H_1, H_2 \in \mathscr{H}_G$ such that $H_1 \leq H_2$, or equivalently $H_1 \subseteq H_2$. Then $I(H_1) \subseteq I(H_2)$ trivially as well. Now for ϕ. Let $I_1, I_2 \in \mathscr{L}_{gr}(L_K(G))$ such that $I_1 \subseteq I_2$. Then $I_1 \cap V$ has to be included in $I_2 \cap V$ by Theorem 4.2.4, and we are done. ∎

In words, if one hereditary saturated subset is contained in another, the graded ideal of the former is contained in the graded ideal of the latter.

Example 5.2.6. We return to the graph G_2 from Example 5.1.4. We saw there that the two non-trivial hereditary saturated subsets are $\{v_1\}$ and $\{v_3, v_4\}$. As these sets have no intersection, Theorem 5.2.5 states that the graded ideals generated by these subsets have no intersection either. We can also demonstrate the difference between the join of two hereditary saturated subsets, the hereditary saturated closure, and the join of two subsets in a powerset mentioned in Remark 5.2.4. Indeed, the hereditary saturated closure of $\{v_1\}$ and $\{v_3, v_4\}$ is not equal to $\{v_1, v_3, v_4\}$, as that set is not saturated, but to $\{v_1, v_2, v_3, v_4\}$. Theorem 5.2.5 and Theorem 4.2.4 then say that v_2 has to be included in the direct sum of $I(\{v_1\})$ and $I(\{v_3, v_4\})$.

We will close this chapter by applying both Theorem 5.1.3 and Theorem 5.2.5 on a different graph, where the structure of the graded ideals is more visible. The graph is taken from Appendix A of [7], an article modeling quantum spheres (among other objects studied in physics) as Cuntz-Krieger algebras.

Example 5.2.7. Let $\tilde{L}_{2n-1}$ be the graph shown below.



Determining the hereditary saturated subsets of this graph is not a difficult task. Indeed, for any subset which contains a vertex v_i, the vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ must also be in that set in order to satisfy the hereditary property. We also see that for every vertex, the set of its neighbors always includes itself. The saturated condition is thus satisfied for any subset. If we denote by $H_k = \{v_n, v_{n-1}, \ldots, v_{n-k+1}\}$, we see that $\mathscr{H}_{\tilde{L}_{2n-1}} = \{\emptyset, H_1, H_2, \ldots, H_n\}$.

Note that the quotient graph $\tilde{L}_{2n-1}/H_k \cong \tilde{L}_{2(n-k)-1}$. Theorem 5.1.3 then states that $L_K(\tilde{L}_{2n-1})/I(H_k) \cong L_K(\tilde{L}_{2n-1}/H_k) \cong L_K(\tilde{L}_{2(n-k)-1})$ for all $k \in \{1, \ldots, n-1\}$. As $H_k \subseteq H_{k+1}$, theorem 5.2.5 states that $I(H_k) \subseteq I(H_{k+1})$ for all $k \in \{1, \ldots, n-1\}$.

6 Conclusion

In this thesis we has covered some introductory results regarding Leavitt path algebras and analyzed their ideals with respect to a grading. We saw a formal defintion of Leavitt path algebras in Chapter 3 (Definition 3.1.2) and that some LPA's are isomorphic to familiar algebras. Chapter 4 introduced hereditary and saturated subsets of a graph and graded ideals of a graded algebra. Theorem 4.1.6 stated that every Leavitt path algebra has a $\mathbb{Z}$-grading, and Theorem 4.2.4 then showed that every graded ideal of a LPA is generated by a hereditary saturated subset of the vertices. This is the main result of this thesis. Chapter 5 went over other natural properties of graded ideals, in the context of quotient graphs and lattices. Theorem 5.1.3 stated that the quotient of a Leavitt path algebra by a graded ideal is isomorphic to the LPA of the quotient graph. The Structure Theorem for graded ideals (Theorem 5.2.5) stated that the graded ideals of a Leavitt path algebra form a lattice, which is isomorphic to the lattice of the hereditary saturated subsets.

The main conlusion of this thesis is then Theorem 4.2.4: Graded ideals of a Leavitt path algebra are in a one-to-one correspondence with the hereditary saturated subsets of its underlying graph. This result connects graphs and their Leavitt path algebras in an important way, as we have seen in the theorems that followed.

We end this thesis by giving two more theorems as an outlook. Chapter 5 explored the naturality of the graded ideals regarding Leavitt path algebras. The first theorem we record illustrates this naturality more, and reinforces the idea that graded ideals and their Leavitt path algebra are very connected. The theorem holds for arbitrary graphs.

Theorem 6.0.1. Every graded ideal of a Leavitt path algebra is isomorphic to a Leavitt path algebra. Moreover, an ideal which is not graded is necessarily not isomorphic to a Leavitt path algebra.

We will omit the proof in the interest of time. The proof of the first statement can be found under Theorem 2.5.22 in [6]. The proof of the second statement can be found under Corollary 2.9.11 in the same book.

This result allows us to expand on the theorems we have discussed in this thesis. For example, we have seen in Theorem 5.1.3 that the quotient of a Leavitt path algebra by a graded ideal is isomorphic another Leavitt path algebra. Combining this with Theorem 6.0.1, we see that taking quotients of Leavitt path algebras by Leavitt path algebras can yield Leavitt path algebras. Theorem 5.2.5 states that the graded ideals of a Leavitt path algebra form a lattice. Theorem 6.0.1 then states that the Leavitt path algebras isomorphic to these graded ideals form a lattice.

The second theorem we will state, without proof, connects a graph with its Leavitt path algebra. This theorem allows us to know about the hereditary subsets of a graph (and thus the graded ideals of its Leavitt path algebra) by looking at whether its Leavitt path algebra is simple. (An algebra A is called *simple* if $A^2 \neq 0$ and the only two-sided ideals of A are $\{0\}$ and A.)

Theorem 6.0.2 (The Simplicity Theorem, [6] Thm 2.9.1)**.** Let G be an arbitrary graph and let K be a field. Then $L_K(G)$ is simple if and only if

(i) $\mathscr{H}_G = \{\emptyset, V\}$ (G has no non-trivial hereditary saturated subsets) and

(ii) G satisfies Condition (L), defined in [6] as Definitions 2.2.2.

In words, if a Leavitt path algebra is simple, then the only hereditary saturated subsets its graph has (and the only graded ideals the algebra has) are trivial. With this theorem, one can prove quickly that the graph A_n, having $M_n(K)$ as its Leavitt path algebra, has no non-trivial hereditary saturated subsets. This also works the other way around; we have seen that A_n satisfies Condition (L) (without further explanation), so if we show that the graph only has trivial hereditary saturated subsets, we can say that $L_K(A_n) \cong M_n(K)$ is simple.

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