Algebraic and Proof-Theoretic Foundations of the Logics for Social Behaviour

Tzimoulis, Apostolos

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ALGEBRAIC
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Algebraic and Proof-Theoretic Foundations of the Logics for Social Behaviour

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Dissertation

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by

Apostolos TZIMOULIS

Master of Science in Logic,
Universiteit van Amsterdam, The Netherlands,
born in Thessaloniki, Greece.

This dissertation has been approved by the promotors.

promotor: dr A. Palmigiano
promotor: prof. dr ir I. R. van de Poel

Composition of the doctoral committee:

Rector Magnificus,	chairperson
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Prof. dr ir C. G. Chorus,	Technische Universiteit Delft, reserve member
Other members:	
Dr G. Greco	Universiteit Utrecht

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We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.

T.S. Eliot

Contents

Summary

This thesis is part of a line of research aimed at providing a strong and modular mathematical backbone to a wide and inherently diverse class of logics, introduced to capture different facets of social behaviour.

The contributions of this thesis are rooted methodologically in duality, algebraic logic and structural proof theory, pertain to and advance three theories (*unified correspondence*, *multi-type calculi*, and *updates on algebras*) aimed at improving the semantic and proof-theoretic environment of wide classes of logics, and apply these theories to the introduction of logical frameworks specifically designed to capture concrete aspects of social behaviour, such as agents' coordination and planning concerning the transformation and use of resources, and agents' decision-making under uncertainty.

The results of this thesis include: the characterization of the axiomatic extensions of the basic DLE-logics which admit proper display calculi; an algorithm computing the analytic structural rules capturing these axiomatic extensions; the introduction of a multi-type environment to describe and reason about agents' abilities and capabilities to use and transform resources; the introduction of a proper display calculus for first-order logic; the introduction of the intuitionistic counterpart of Probabilistic Dynamic Epistemic Logic, specifically designed to address situations in which truth is socially constructed.

The results and methodologies developed in this thesis pave the way to the logical modelling of the inner workings of organizations and their dynamics, and of social phenomena such as reputational Matthew effects and bank runs.

Samenvatting

Dit proefschrift maakt deel uit van een onderzoekslijn gericht op het totstandbrengen van een sterke en modulaire wiskundige ruggengraat voor een brede en inherent diverse klasse van logica's die geïntroduceerd zijn om verschillende facetten van sociaal gedrag te kunnen beschrijven.

De bijdragen van dit proefschrift zijn methodologisch geworteld in de theorie van de dualiteit, de algebraïsche logica en de structurele bewijstheorie. Zij hebben betrekking op en dragen bij aan drie theorieën (*geünificeerde correspondentie*, *multi-type calculi*, en *updates over algebra's*) die de semantische en bewijstheoretische omgeving van brede klassen van logica's verbeteren. Bovendien worden deze theorieën toegepast om specifieke logische raamwerken te ontwerpen om concrete aspecten van sociaal gedrag vast te leggen, zoals coördinatie tussen agenten en planning met betrekking tot de transformatie en het gebruik van hulpbronnen, en de besluitvorming van agenten onder onzekerheid.

De resultaten van dit proefschrift omvatten met name: de karakterisering van de axiomatische uitbreidingen van de basis DLE-logica's die *proper display calculi* toelaten; een algoritme dat de analytische structuurregels berekent die deze axiomatische extensies vastleggen; de introductie van een multi-type omgeving om de bekwaamheden en mogelijkheden van agenten om middelen te gebruiken en te transformeren te beschrijven en daarover te redeneren; de introductie van een *proper display calculus* voor eerste-orde logica; de introductie van de intuïtionistische tegenhanger van *Probabilistic Dynamic Epistemic Logic*, speciaal ontworpen voor contexten waarin de waarheid sociaal geconstrueerd is.

De resultaten en methodologieën ontwikkeld in dit proefschrift effenen de weg naar de logische modellering van de interne werking van organisaties en hun dynamiek, en van sociale fenomenen zoals Matthew-effecten met betrekking tot reputatie en bankruns.

Chapter 1

Introduction

1.1 Main motivation and focus of this thesis

Just as many-body interaction is essential to understanding the physical world, intelligent multi-agent interaction is essential to understanding human behaviour, as it plays out in complex social situations such as the coordination of agents in organizations, or agents' strategic decision-making. In the past decades, the focus on multi-agent interaction has led to a massive expansion of the field of logic, both in its theory and its applications, to encompass a plurality of reasoning patterns specific to contexts involving e.g. dynamic changes [32], uncertainties and false beliefs [43], vagueness [18], partial information [39], which are at odds with e.g. the mathematical reasoning as is formalized in classical logic. Thanks to the development of these and other logics, collectively named *nonclassical logics*, logic as a discipline has been reaching out to new areas of applications: logical descriptions of social networks, linguistic structures where truth does not apply (such as questions or commands), information exchange in dialogue, formal models of rational behaviour. This rapid expansion has generated the need to develop *overarching theories* (cf. e.g. [1, 25, 27, 36, 40, 44]) capable to provide uniform proofs of fundamental properties–such as soundness, completeness, analiticity, decidability–for each member of vast families of logical systems, while at the same time accounting in a modular way for the specific features of each.

This thesis contributes to three such overarching theories (*unified correspondence*, *multi-type calculi*, and *updates on algebras*), and applies them to the study of social behaviour. In what follows, I will briefly describe them, and then describe the specific contributions this thesis makes to each.

1.2 Unified correspondence

Unified correspondence [11] generalizes and extends Sahlqvist theory [41] to a large family of nonclassical logics which includes intuitionistic and bi-intuitionistic normal modal logics [12], non-normal modal logics [24, 37], substructural logics [10, 13, 14], hybrid logic [16], many-valued logics [34], and logics with fixed points [8, 9]. Sahlqvist correspondence theory in modal logic originates from the very simple observation that Kripke frames can serve as models for both first-order formulas and modal formulas. A modal formula and a first-order formula correspond if they define the same class of Kripke frames. This is the starting point of a well known body of work in modal logic which has been key to its widespread success in a range of fields which includes program verification in theoretical computer science, natural language semantics in formal philosophy, multi-agent systems in AI, foundations of arithmetics, game theory in economics, categorization theory in social and management science. In particular, Sahlqvist correspondence theory provides a syntactic characterization of those modal formulas (the Sahlqvist formulas) which are equivalent to (effectively computable) first-order conditions on Kripke frames. As pointed out in [15], Sahlqvist correspondence theory can be intuitively regarded as a meta-semantic tool which makes it possible to understand the 'meaning' of a modal axiom in terms of the condition expressed by its first-order correspondent. In this way, for instance, $\Box A \to A$ can be understood as the 'reflexivity axiom', and $\Diamond\Diamond A \to \Diamond A$ as the 'transitivity axiom'. Via the duality between Kripke frames and perfect (i.e. complete and atomic) Boolean algebras with operators, Sahlqvist correspondence arguments can be translated from Kripke frames to their complex algebras, where the algebraic and order-theoretic underpinning of the arguments can be brought to light. The move from Kripke frames to algebras has made it possible to identify the core of the original result, and reproduce it in the many different contexts mentioned above. It has also made it possible to achieve the systematic connection between correspondence theory and structural proof theory which is the focus of Chapter 2.

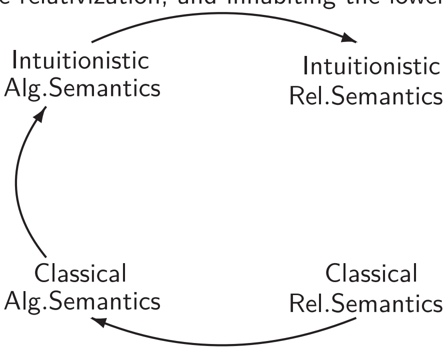
1.3 Multi-type calculi

Multi-type display calculi [21] are a natural generalization of Belnap's display calculi, aimed at capturing a wide range of logics which, as Linear Logic [26] and Semi-De Morgan logic [42], cannot be accounted for by proper display calculi [7], or which, as Dynamic Epistemic Logic (DEL) [4], can be accounted for [22], but in a way that leaves many unresolved questions, and endowing them with analytic calculi with the same excellent behaviour (concerning e.g. cut elimination cf. [19]) guaranteed by Belnap's original design. The starting point of the multi-type methodology is the insight that what makes logics such as DEL hard to treat with the standard proof-theoretic tools is the presence of certain extra-linguistic labels and devices encoding key interactions between the parameters of the language of these logics, such as agents, actions, coalitions, strategies, time, probabilities. Capturing these interactions is exactly the *raison d'être* of these logics. The core feature of multi-type calculi (from which they take their name) is the upgrade of these parameters, which become terms of the language, each of its own type. Like formulas, they thus become first-class citizens of the framework, and are endowed with their corresponding structural connectives and rules. In the multi-type environment, many features which were insurmountable hurdles to the standard treatment can be understood as symptoms of the original languages of these logics lacking the necessary expressivity to encode these key interactions within the language. By suitably providing the needed additional expressivity (in the form of e.g. heterogeneous connectives, defined between different types) these hurdles have been overcome in several significant instances, such as the original Dynamic Epistemic Logic [21], Inquisitive logic [23], PDL [20], Semi-De Morgan Logic [28], Bilattice logic [29], linear logic [30] and general lattice logic [31]. In Chapter 4, we will show how these hurdles can be overcome also in the case of first-order logic, and in Chapter 3, we will show that the additional expressivity and modularity guaranteed by the multi-type framework makes it, on the theoretical side, a powerful defining platform for *new logics* which come endowed by design with a package of excellent properties, and on the side of applications, a versatile tool for the analysis of interaction and social behaviour.

1.4 Updates on algebras

The mathematical construction of updates on algebras is a general methodology for extending the study of dynamic phenomena to settings in which classical reasoning fails. Examples of such settings are diverse and widespread, spanning from situations where truth is socially constructed, and hence admits cases in which propositions are neither true nor false, to entities (such as categories and concepts) the natural logic of which does not have negation (e.g. there is no such thing as the concept of 'non apple'). In [33, 35], this methodology has been introduced and applied to develop the intuitionistic counterparts of Public Announcements Logic (PAL) [38], and of the Logic of Epistemic Actions and Knowledge (EAK) [4] respectively (these new logics are suited to reason in contexts in which truth is based on evidence and proofs, and hence the classical law of excluded middle does not hold); in [3], it has been applied to develop a paraconsistent version of EAK, suitable to reason in settings in which agents might receive partially inconsistent information; in [6], it has been applied to develop a many-valued version of PAL, suitable to express and reason with vague statements; in [2], it has been applied to develop the algebraic semantics of refinement modal logic.

Let us illustrate updates on algebras in the simplest setting (PAL). The transformation corresponding to the simplest epistemic action (i.e. the public announcement of a formula A) is called *relativization*: a given (Kripke) model M is replaced with its relativized submodel N, obtained by deleting all the states of M on which the formula A is false. In [35], the intuitionistic counterpart of PAL is defined through the C-shaped procedure illustrated in the picture: the injection map $i : N \to M$, encoding the relativization, and inhabiting the lower-right corner of the picture, is



dually characterized across *classical* Stone duality as a pseudo-quotient map $\pi : \mathbb{A} \to \mathbb{B}$ where $\mathbb{A}$ and $\mathbb{B}$ are the algebras dually associated with M and with N, respectively. This pseudo-quotient (lower-left corner) naturally generalizes to much wider classes of algebras than those arising from the models of PAL; these include, but are not limited to, arbitrary Heyting algebras with modal operators (upper left corner). Heyting algebras canonically interpret intuitionistic logic. Hence, the dual characterization of relativization naturally leads to the definition of the intuitionistic environment for public announcements. The axiomatization of the intuitionistic counterpart of PAL is then extracted from this semantic environment. Finally, a dualization procedure across intuitionistic Stone-type duality (Esakia duality [17]) defines the Kripke models for the intuitionistic PAL. In Chapter 5, a suitable adaptation of the procedure described above will be applied to the algebras and models of Probabilistic Dynamic Epistemic Logic (PDEL) as a fundamental tool to define the intuitionistic counterpart of PDEL.

1.5 Original contributions

The contributions of this thesis are both theoretical and relative to applications. Indeed, while the formal tools presented in this thesis are rooted in and advance algebraic logic and proof theory, the nonclassical logics investigated in each chapter are motivated by and connect to issues related to multi-agent interaction. As to the theoretical contributions, we have:

1. established systematic connections between the model-theoretic and algebraic theory of unified correspondence and analytic calculi in structural proof theory. These connections hold uniformly for a wide class of logics each of which corresponds algebraically to a variety of distributive lattice expansions, and can be extended in a natural way to the environment of heterogeneous algebras [5]. Hence, these connections can be exploited in full generality and uniformity also in the framework of multi-type calculi (cf. Chapter 2);

2. concretely illustrated that the multi-type environment can serve as a *defining* tool for the introduction of new logical systems, endowed with excellent mathematical properties by design (cf. Chapter 3);

3. introduced a proper display calculus for first-order logic, paving the way to e.g. a uniform and modular study of quantified versions of nonclassical logics (cf. Chapter 4);

4. extended the construction of updates on algebras to account for probabilistic updates (cf. Chapter 5).

As to the applications, we have:

1. introduced an algebraic/proof-theoretic environment for describing and reasoning about agents' abilities, capabilities, coordination and planning motivated by the use and transformation of resources, and applied it to a variety of case studies (cf. Chapter 3);

2. introduced the intuitionistic counterpart of Probabilistic Dynamic Epistemic Logic, a logic specifically designed to describe and reason about agents' different types of uncertainty in situations in which truth is socially constructed, and used it to model a situation in which both probabilistic and strategic reasoning are called for (cf. Chapter 5).

While applications are not explicitly discussed in Chapters 2 and 4, it is worth stressing that these theoretical results are also significant *vis-à-vis* applications. Indeed:

1. the systematic connections established between unified correspondence and the theory of analytic calculi are precisely what makes it possible to guarantee three fundamental properties of multi-type calculi (namely, soundness, completeness and conservativity) in full generality and uniformity, and hence guarantee that the formal tools created by using this methodology are powerful and effective tools for real-life applications;

2. quantified logics are tailored to reason about concrete individuals, their properties, and the relations they entertain with each other. Having extended the multi-type methodology to first-order logic in a principled way opens the opportunity of building up modular environments in which different types of nonclassical reasoning can be applied to the study of the role of *individuals* within social dynamics.

Finally, perhaps the main contribution of this thesis is neither theoretical nor applied but is *conceptual*, and consists in paving a path that, from very general and abstract techniques rooted in the foundations of mathematics, leads to concrete tools for the formal analysis of social behaviour, and brings back the diversity and specificity of human reasoning as a challenge and measure of success for further improvements on foundational issues in mathematical logic.

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Chapter 2

Unified Correspondence as a Proof-Theoretic Tool

The present chapter is based on[1] *[38] and aims at establishing formal connections between correspondence phenomena, well known from the area of modal logic, and the theory of display calculi, originated by Belnap.*

These connections have been seminally observed and exploited by Marcus Kracht, in the context of his characterization of the modal axioms (which he calls primitive formulas) which can be effectively transformed into 'analytic' structural rules of display calculi. In this context, a rule is 'analytic' if adding it to a display calculus preserves Belnap's cut elimination theorem.

In recent years, the state-of-the-art in correspondence theory has been uniformly extended from classical modal logic to diverse families of nonclassical logics, ranging from (bi-)intuitionistic (modal) logics, linear, relevant and other substructural logics, to hybrid logics and mu-calculi. This generalization has given rise to a theory called unified correspondence, the most important technical tools of which are the algorithm ALBA, and the syntactic characterization of Sahlqvist-type classes of formulas and inequalities which is uniform in the setting of normal DLE-logics (logics the algebraic semantics of which is based on bounded distributive lattices).

We apply unified correspondence theory, with its tools and insights, to extend Kracht's results and prove his claims in the setting of DLE-logics. The results of the present chapter characterize the space of properly displayable DLE-logics.

[1] My specific contributions in this research have been the proof of the main results, the construction and development of examples and the draft of the first version of the paper.

2.1 Introduction

The present chapter applies the results and insights of unified correspondence theory [18] to establish formal connections between correspondence phenomena, well known from the area of modal logic, and the theory of display calculi, introduced by Belnap [2].

Sahlqvist correspondence theory. Sahlqvist theory [50] is among the most celebrated and useful results of the classical theory of modal logic, and one of the hallmarks of its success. It provides an algorithmic, syntactic identification of a class of modal formulas whose associated normal modal logics are *strongly complete* with respect to *elementary* (i.e. first-order definable) classes of frames.

Unified correspondence. In recent years, building on duality-theoretic insights [22], an encompassing perspective has emerged which has made it possible to export the state-of-the-art in Sahlqvist theory from modal logic to a wide range of logics which includes, among others, intuitionistic and distributive lattice-based (normal modal) logics [20], non-normal (regular) modal logics [46], substructural logics [21], hybrid logics [24], and mu-calculus [14–16].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as the understanding of the relationship between different methodologies for obtaining canonicity results [12, 47], or of the phenomenon of pseudo-correspondence [23]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [32] and the epistemic interpretation of modalities on RS-frames [17]. Finally, the insights of unified correspondence theory have made it possible to determine the extent to which the Sahlqvist theory of classes of normal distributive lattice expansions (DLEs) can be reduced to the Sahlqvist theory of normal Boolean expansions, by means of Gödel-type translations [13]. These and other results have given rise to a theory called *unified correspondence* [18].

Tools of unified correspondence theory. The most important technical tools in unified correspondence are: (a) a very general syntactic definition of the class of Sahlqvist formulas, which applies uniformly to each logical signature and is given purely in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives; (b) the algorithm ALBA (Ackermann Lemma Based Algorithm), which effectively computes first-order correspondents of input term-inequalities, and is guaranteed to succeed on a wide class of inequalities (the so-called *inductive* inequalities) which, like the Sahlqvist class, can be defined uniformly in each mentioned signature, and which properly and significantly extends the Sahlqvist class.

Unified correspondence and display calculi. The present chapter aims at applying the tools of unified correspondence to address the identification of the syntactic shape of axioms which can be translated into analytic structural rules[2] of a display calculus, and the definition of an effective procedure for transforming axioms into such rules. In recent years, these questions have been intensely investigated in the context of various proof-theoretic formalisms (cf. [7, 8, 11, 37, 40–42, 44, 45]). Perhaps the first paper in this line of research is [39], which addresses these questions in the setting of display calculi for basic normal modal and tense logic. Interestingly, in [39], the connections between Sahlqvist theory and display calculi started to be observed, but have not been systematically explored there nor (to the knowledge of the authors) in subsequent papers in the same research line.

Contributions. The two tools of unified correspondence can be put to use to generalize Kracht's transformation procedure from axioms into analytic rules. This generalization concerns more than one aspect. Firstly, in the same way in which the definitions of Sahlqvist and inductive inequalities can be given uniformly in each logical signature, the definition of primitive formulas/inequalities is introduced for any logical framework the algebraic semantics of which is based on distributive lattices with operators (these will be referred to as *DLE-logics*, (cf. Definition 10 and Footnote 14 for terminology). Secondly, in the context of each such logical framework, we introduce a hierarchy of subclasses of inductive inequalities, progressively extending the primitive inequalities, the largest of which is the class of so-called *analytic inductive inequalities*. This is a syntactic generalization of the class of primitive formulas/inequalities. We provide an effective procedure, based on ALBA, which transforms each analytic inductive inequality into an equivalent set of analytic rules. Moreover, we show that any analytic rule can be effectively and equivalently transformed into some analytic inductive inequality. Finally, we show that any analytic rule can be effectively and equivalently transformed into one of a particularly nice shape, collectively referred to as *special* rules.

Structure of the chapter. In Section 2.2, preliminaries on display calculi are collected. In Section 2.3, the setting of basic DLE-logics is introduced, and the algorithm ALBA for them. In Section 2.4, the display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ for DLE-logics are introduced, and their basic properties are proven. In Section 2.5, Kracht's notion of primitive formulas is generalized to primitive inequalities in each DLE-language, as well as their connection with special structural rules for display calculi (cf. Definition 6). It is also shown that, for any language $\mathcal{L}_{\mathrm{DLE}}$, each primitive $\mathcal{L}_{\mathrm{DLE}}$-inequality is equivalent on perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras to a set of special structural rules in the language of the associated display calculus $\mathbf{DL}$, and that the validity of each such special structural rule is equivalent to the validity of some primitive $\mathcal{L}_{\mathrm{DLE}}$-inequality. In Section 2.6 we extend the algorithm generating special structural rules in the language of $\mathbf{DL}$ from input primitive $\mathcal{L}_{\mathrm{DLE}}$-inequalities to a hierarchy of classes of non-primitive $\mathcal{L}_{\mathrm{DLE}}$-inequalities, the most general of which is referred to as *restricted analytic inductive inequalities* (cf. Definition 51). Our procedure for obtaining this extension makes use of ALBA to equivalently transform any restricted analytic inductive $\mathcal{L}_{\mathrm{DLE}}$-inequality into one or more primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequalities. In Section 2.7, the class of restricted analytic

[2] *Analytic* rules (cf. Definition 4) are those which can be added to a *proper display calculus* (cf. Section 2.2.2) obtaining another proper display calculus.

inductive inequalities is further extended to the *analytic inductive inequalities* (cf. Definition 55). Each analytic inductive inequality can be equivalently transformed into some analytic rule of a restricted shape, captured in the notion of *quasi-special* structural rule (cf. Definition 8) in the language of $\mathbf{DL}$. Once again, the key step of the latter procedure makes use of ALBA, this time to equivalently transform any analytic inductive inequality into one or more suitable quasi-inequalities in $\mathcal{L}^*_{\mathrm{DLE}}$. We also show that each analytic rule is equivalent to some analytic inductive inequality. This back-and-forth correspondence between analytic rules and analytic inductive inequalities characterizes the space of properly displayable DLE-logics as the axiomatic extensions of the basic DLE-logic obtained by means of analytic inductive inequalities. In Section 2.8, we show that for any language $\mathcal{L}_{\mathrm{DLE}}$, any properly displayable DLE-logic is specially displayable, which implies that any properly displayable $\mathcal{L}^*_{\mathrm{DLE}}$-logic can be axiomatized by means of primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequalities. This last result generalizes an analogous statement made by Kracht in the setting of properly displayable tense modal logics, which was proven in [9, 10] in the same setting. In Section 2.9, we outline a comparison between the present treatment and that of [9, 10]. In Section 2.10 we present our conclusions. Various proofs are collected in Sections 2.11–2.14.

2.2 Preliminaries on display calculi

In the present section, we provide an informal introduction to the main features of display calculi without any attempt at being self-contained. We refer the reader to [57] for an expanded treatment. Our presentation follows [30, Section 2.2].

Display calculi are among the approaches in structural proof theory aimed at the uniform development of an inferential theory of meaning of logical constants (logical connectives) aligned with the principles of proof-theoretic semantics [51, 52]. Display calculi have been successful in giving adequate proof-theoretic semantic accounts of logics—such as certain modal and substructural logics [35], and more recently also Dynamic Epistemic Logic [29] and PDL [28]—which have notoriously been difficult to treat with other approaches. Here we mainly report and elaborate on the work of Belnap [2], Wansing [57], Goré [34, 35], and Restall [49].

2.2.1 Belnap's display logic

Nuel Belnap introduced the first display calculus, which he calls *Display Logic* [2], as a sequent system augmenting and refining Gentzen's basic observations on structural rules. Belnap's refinement is based on the introduction of a special syntax for the constituents of each sequent. Indeed, his calculus treats sequents $X \vdash Y$ where X and Y are so-called *structures*, i.e. syntactic objects inductively defined from formulas using an array of special meta-logical connectives. Belnap's basic idea is that, in the standard Gentzen formulation, the comma symbol ',' separating formulas in the precedent and in the succedent of sequents can be recognized as a metalinguistic connective, the behaviour of which is defined by the structural rules.

Belnap took this idea further by admitting not only the comma, but also other meta-logical connectives to build up structures out of formulas, and called them *structural* *connectives*. Just like the comma in standard Gentzen sequents is interpreted contextually (that is, as conjunction when occurring on the left-hand side and as disjunction when occurring on the right-hand side), each structural connective typically corresponds to a pair of logical connectives, and is interpreted as one or the other of them contextually (more of this in Section 2.4.2). Structural connectives maintain relations with one another, the most fundamental of which take the form of adjunctions and residuations. These relations make it possible for the calculus to enjoy the powerful property which gives it its name, namely, the *display property*. Before introducing it formally, let us agree on some auxiliary definitions and nomenclature: *structures* are defined much in the same way as formulas, taking formulas as atomic components and closing under the given structural connectives; therefore, each structure can be uniquely associated with a generation tree. Every node of such a generation tree defines a *substructure*. A *sequent* $X \vdash Y$ is a pair of structures X, Y. The display property, stated similarly to the one below, appears in [2, Theorem 3.2]:

Definition 1. A proof system enjoys the *display property* iff for every sequent $X \vdash Y$ and every substructure Z of either X or Y, the sequent $X \vdash Y$ can be equivalently transformed, using the rules of the system, into a sequent which is either of the form $Z \vdash W$ or of the form $W \vdash Z$, for some structure W. In the first case, Z is *displayed in precedent position*, and in the second case, Z is *displayed in succedent position*.[3] The rules enabling this equivalent rewriting are called *display postulates*.

Thanks to the fact that display postulates are semantically based on adjunction and residuation, exactly one of the two alternatives mentioned in the definition above can soundly occur. In other words, in a calculus enjoying the display property, any substructure of any sequent $X \vdash Y$ is always displayed either only in precedent position or only in succedent position. This is why we can talk about occurrences of substructures in *precedent* or in *succedent* position, even if they are nested deep within a given sequent, as illustrated in the following example which is based on the display postulates between the structural connectives ; and $>$:

$$\dfrac{\dfrac{\dfrac{Y \vdash X > Z}{X \,; Y \vdash Z}}{Y \,; X \vdash Z}}{X \vdash Y > Z}$$

In the derivation above, the structure X is on the right side of the turnstile, but it is displayable on the left, and therefore is in precedent position. The display property is a crucial technical ingredient for Belnap's cut elimination metatheorem: for instance, it provides the core mechanism for the satisfaction of the crucial condition C_8, discussed in the following subsection.

[3] In the following sections, we will find it useful to differentiate between the full and the relativized display property (cf. discussion before Proposition 22).

2.2.2 Proper display calculi and canonical cut elimination

In [2], a metatheorem is proven, which gives sufficient conditions in order for a sequent calculus to enjoy cut elimination.[4] This metatheorem captures the essentials of the Gentzen-style cut elimination procedure, and is the main technical motivation for the design of Display Logic. Belnap's metatheorem gives a set of eight conditions on sequent calculi, which are relatively easy to check, since most of them are verified by inspection on the shape of the rules. Together, these conditions guarantee that the cut is eliminable in the given sequent calculus, and that the calculus enjoys the subformula property. When Belnap's metatheorem can be applied, it provides a much smoother and more modular route to cut elimination than the Gentzen-style proofs. Moreover, as we will see later, a Belnap style cut elimination theorem is robust with respect to adding a general class of structural rules, and with respect to adding new logical connectives, whereas a Gentzen-style cut elimination proof for the modified system cannot be deduced from the old one, but must be proved from scratch.

In a slogan, we could say that Belnap-style cut elimination is to ordinary cut elimination what canonicity is to completeness: indeed, canonicity provides a *uniform strategy* to achieve completeness. In the same way, the conditions required by Belnap's metatheorem ensure that *one and the same* given set of transformation steps is enough to achieve Gentzen-style cut elimination for any system satisfying them.

In what follows, we review and discuss eight conditions which are stronger in certain respects than those in [2],[5] and which define the notion of *proper display calculus* in [57].[6]

C$_1$: Preservation of formulas. This condition requires each formula occurring in a premise of a given inference to be the subformula of some formula in the conclusion of that inference. That is, structures may disappear, but not formulas. This condition is not included in the list of sufficient conditions of the cut elimination metatheorem, but, in the presence of cut elimination, it guarantees the subformula property of a system. Condition C_1 can be verified by inspection on the shape of the rules. In practice, condition C$_1$ bans rules in which structure variables occurring in some premise to not occur also in the conclusion, since in concrete derivations these are typically instantiated with (structures containing) formulas which would then disappear in the application of the rule.

C$_2$: Shape-alikeness of parameters. This condition is based on the relation of *congruence* between *parameters* (i.e., non-active parts) in inferences; the congruence relation is an equivalence relation which is meant to identify the different occurrences of the same formula or substructure along the branches of a derivation [2, Section 4], [49, Definition 6.5]. Condition C$_2$ requires that congruent parameters be occurrences of the same structure. This can be understood as a condition on the *design* of the rules of

[4]As Belnap observed on page 389 in [2]: 'The eight conditions are supposed to be a reminiscent of those of Curry' in [25].
[5]See also [3, 49] and the 'second formulation' of condition C6/7 in Section 4.4 of [57].
[6]See the 'first formulation' of conditions C6, C7 in Section 4.1 of [57].

the system if the congruence relation is understood as part of the specification of each given rule; that is, each schematic rule of the system comes with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). In this respect, C_2 is nothing but a sanity check, requiring that the congruence is defined in such a way that indeed identifies the occurrences which are intuitively “the same”.[7]

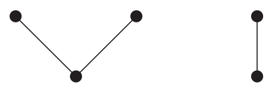
C_3: Non-proliferation of parameters. Like the previous one, also this condition is actually about the definition of the congruence relation on parameters. Condition C_3 requires that, for every inference (i.e. rule application), each of its parameters is congruent to at most one parameter in the conclusion of that inference. Hence, the condition stipulates that for a rule such as the following,

$$\frac{X \vdash Y}{X\,,X \vdash Y}$$

the structure X from the premise is congruent to *only one* occurrence of X in the conclusion sequent. Indeed, the introduced occurrence of X should be considered congruent only to itself. Moreover, given that the congruence is an equivalence relation, condition C_3 implies that, within a given sequent, any substructure is congruent only to itself. In practice, in the general schematic formulation of rules, we will use the same structure variable for two different parametric occurrences if and only if they are congruent, so a rule such as the one above is de facto banned.

Remark 2. Conditions C_2 and C_3 make it possible to follow the history of a formula along the branches of any given derivation. In particular, C_3 implies that the the history of any formula within a given derivation has the shape of a tree, which we refer to as the *history-tree* of that formula in the given derivation. Notice, however, that the history-tree of a formula might have a different shape than the portion of the underlying derivation corresponding to it; for instance, the following application of the Contraction rule gives rise to a bifurcation of the history-tree of A which is absesent in the underlying branch of the derivation tree, given that Contraction is a unary rule.

$$\frac{\begin{array}{c}\vdots\\ A\,,A \vdash X\end{array}}{A \vdash X}$$



C_4: Position-alikeness of parameters. This condition bans any rule in which a (sub)structure in precedent (resp. succedent) position in a premise is congruent to a (sub)structure in succedent (resp. precedent) position in the conclusion.

[7] Our convention throughout this chapter is that congruent parameters are denoted by the same letter. For instance, in the rule

$$\frac{X ; Y \vdash Z}{Y ; X \vdash Z}$$

the structures X, Y and Z are parametric and the occurrences of X (resp. Y, Z) in the premise and the conclusion are congruent.

C_5: Display of principal constituents. This condition requires that any principal occurrence (that is, a non-parametric formula occurring in the conclusion of a rule application, cf. [2, Condition C5]) be always either the entire antecedent or the entire consequent part of the sequent in which it occurs. In the following section, a generalization of this condition will be discussed, in view of its application to the main focus of interest of the present chapter.

The following conditions C_6 and C_7 are not reported below as they are stated in the original paper [2], but as they appear in [57, Section 4.1].

C_6: Closure under substitution for succedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in succedent position. Condition C_6 ensures, for instance, that if the following inference is an application of the rule R:

$$\frac{(X \vdash Y)\big([A]^{suc}_i \mid i \in I\big)}{(X' \vdash Y')[A]^{suc}}\ R$$

and $\big([A]^{suc}_i \mid i \in I\big)$ represents all and only the occurrences of A in the premiss which are congruent to the occurrence of A in the conclusion[8], then also the following inference is an application of the same rule R:

$$\frac{(X \vdash Y)\big([Z/A]^{suc}_i \mid i \in I\big)}{(X' \vdash Y')[Z/A]^{suc}}\ R$$

where the structure Z is substituted for A.
This condition caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in succedent position is parametric. Indeed, condition C_6 guarantees that, in the picture below, a well-formed subtree $\pi_1[Y/A]$ can be obtained from π_1 by replacing any occurrence of A corresponding to a node in the history tree of the cut-formula A by Y, and hence the following transformation step is guaranteed go through uniformly and "canonically":

$$\begin{array}{c} \vdots\ \pi'_1 \\ X' \vdash A \\ \vdots\ \pi_1 \end{array}$$
$$\frac{X \vdash A \qquad \overset{\vdots\ \pi_2}{A \vdash Y}}{X \vdash Y} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\overset{\vdots\ \pi'_1}{X' \vdash A} \qquad \overset{\vdots\ \pi_2}{A \vdash Y}}{X' \vdash Y} \\ \vdots\ \pi_1[Y/A] \\ X \vdash Y \end{array}$$

if each rule in π_1 verifies condition C_6.

[8]Clearly, if $I = \varnothing$, then the occurrence of A in the conclusion is congruent to itself.

C$_7$: Closure under substitution for precedent parameters. This condition requires each rule to be closed under simultaneous substitution of arbitrary structures for congruent formulas which occur in precedent position. Condition C$_7$ can be understood analogously to C$_6$, relative to formulas in precedent position. Therefore, for instance, if the following inference is an application of the rule R:

$$\frac{(X \vdash Y)\big([A]^{pre}_i \mid i \in I\big)}{(X' \vdash Y')[A]^{pre}}\; R$$

then also the following inference is an instance of R:

$$\frac{(X \vdash Y)\big([Z/A]^{pre}_i \mid i \in I\big)}{(X' \vdash Y')[Z/A]^{pre}}\; R$$

Similarly to what has been discussed for condition C$_6$, condition C$_7$ caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in precedent position is parametric.

C$_8$: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C$_8$ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of cut involving proper subformulas of the original cut-formulas.

Theorem 3. *(cf. [58, Section 3.3, Appendix A]) Any calculus satisfying conditions C$_2$, C$_3$, C$_4$, C$_5$, C$_6$, C$_7$, C$_8$ enjoys cut elimination. If C$_1$ is also satisfied, then the calculus enjoys the subformula property.*

Rules introducing logical connectives. In display calculi, these rules, sometimes referred to as *operational* or *logical rules* as opposed to structural rules, typically occur in two flavors: operational rules which translate one structural connective in the premises in the corresponding connective in the conclusion, and operational rules in which both the operational connective and its structural counterpart are introduced in the conclusion. An example of this pattern is provided below for the case of the modal operator 'diamond':

$$\frac{\circ A \vdash X}{\Diamond A \vdash X}\; \Diamond_L \qquad \frac{X \vdash A}{\circ X \vdash \Diamond A}\; \Diamond_R$$

In Section 2.4, this introduction pattern will be justified from a semantic viewpoint and generalized to logical connectives of arbitrary arity and polarity of their coordinates. From this example, it is clear that the introduction rules capture the rock bottom behavior of the logical connective in question; additional properties (for instance, normality, in the case in point), which might vary depending on the logical system, are to be captured at the level of additional (purely structural) rules. This enforces a clear-cut division of labour between operational rules, which only encode the basic proof-theoretic meaning of logical connectives, and structural rules, which account for all extra relations and properties, and which can be modularly added or removed, thus accounting for the space of axiomatic extensions of a given base logic. Besides being important from the viewpoint of a proof-theoretic semantic account of logical connectives, this neat division of labour is also key to the research program in proof theory aimed at developing more robust versions of Gentzen's cut elimination theory. Indeed, as we have seen, Belnap's strategy in this respect precisely pivots on the identification of conditions (mainly on the structural rules of a display calculus) which guarantee that structural rules satisfying them can be safely added in a modular fashion to proper display calculi without disturbing the canonical cut elimination. In the following subsection, we will expand on the consequences of these conditions on the design of structural rules. Specifically, we report on three general shapes of structural rules. Identifying axioms or formulas which can be effectively translated into rules of one of these shapes is the main goal of the present chapter.

2.2.3 Analytic, special and quasi-special structural rules

In the remainder of the chapter, we will adopt the following convention regarding structural variables and terms: variables X, Y, Z, W denote structures, and so do S, T, U, V. However, when describing rule schemas in abstract terms, we will often write e.g. $X \vdash S$, and in this context we understand that X, Y, Z, W denote structure variables actually occurring in the given rule scheme, whereas S, T, U, V are used as meta-variables for (possibly) compound structural terms such as $X \,;\, Y$.

Definition 4 (Analytic structural rules)**.** (cf. [10, Definition 3.13]) A structural rule which satisfies conditions C_1-C_7 is an *analytic structural rule*.

Clearly, adding analytic structural rules to a proper display calculus (cf. Section 2.2.2) yields a proper display calculus.

Remark 5. In the setting of calculi with the relativized display property[9], if a given analytic structural rule ρ can be applied in concrete derivations of the calculus then ρ is interderivable, modulo applications of display postulates, with a rule of the following form:

$$\frac{(S^i_j \vdash Y^i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i) \quad (X^k \vdash T^k_\ell \mid 1 \leq k \leq m \text{ and } 1 \leq \ell \leq m_k)}{(S \vdash T)[Y^i]^{suc}[X^k]^{pre}}$$

where X^k (resp. Y^i) might occur in S^i_j or in T^k_ℓ in precedent (resp. succedent) position for some i, j, k, ℓ and moreover, X^k and Y^j occur exactly once in $S \vdash T$ in precedent and succedent position respectively for all j, k.

The most common analytic rules occur in the following proper subclass:

[9]cf. discussion before Proposition 22

Definition 6 (Special structural rules)**.** (cf. [39, Section 5, discussion after Theorem 15]) *Special structural rules* are analytic structural rules of one of the following forms:

$$\frac{(X \vdash T_i \mid 1 \leq i \leq n)}{X \vdash T} \qquad \frac{(S_i \vdash Y \mid 1 \leq i \leq n)}{S \vdash Y}$$

where X (resp. Y) does not occur in any T_i (resp. S_i) for $1 \leq i \leq n$ nor in T (resp. S).

In [39], Kracht establishes a correspondence between special rules and primitive formulas in the setting of tense modal logic, which will be generalized in Section 2.5.1 below.

Remark 7. An alternative way to define special rules, which would also be perhaps more in line with the spirit of display calculi, would be as those rules

$$\frac{(S_i \vdash T_i \mid 1 \leq i \leq n)}{S \vdash T}$$

such that some variable X occurs exactly once in each premise and in the conclusion, and always in the same (antecedent or consequent) position. In this way, the class of special rules would be closed under under application of display postulates. Applying the general procedure described in Section 2.7.1 to primitive inequalities (cf. Definition 28) always yields special rules in the less restrictive sense here specified, but not in the sense of Definition 6 above. This fact might be taken as a motivation for adopting the less restrictive definition. However, the more restrictive definition can be immediately verified of a concrete rule, which is the reason why we prefer it over the less restricted one.

In [39], Kracht states without proof that any analytic structural rules in the language of classical tense logic Kt is equivalent to some special structural rule. Kracht's claim has been proved with model-theoretic techniques in [10], [48]. In Section 2.8, we generalize these results using ALBA from classical tense logic to arbitrary DLE-logics. The following definition is instrumental in achieving this generalization:

Definition 8 (Quasi-special structural rules)**.** *Quasi-special structural rules* are analytic structural rules of the following form:

$$\frac{(S^i_j \vdash Y^i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i) \quad (X^k \vdash T^k_\ell \mid 1 \leq k \leq m \text{ and } 1 \leq \ell \leq m_k)}{(S \vdash T)[Y^i]^{suc}[X^k]^{pre}}$$

where X^k and Y^i *do not* occur in any S^i_j, T^k_ℓ (and occur in $S \vdash T$ exactly once).

2.3 Preliminaries on DLE-logics and ALBA

In the present section, we collect preliminaries on logics for distributive lattice expansions (or *DLE-logics*), reporting in particular on their language, axiomatization and algebraic semantics. Then we report on the definition of inductive DLE-inequalities, and outline, without any attempt at being self-contained, the algorithm ALBA[10] (cf. [18, 20]) for each DLE-language.

[10]ALBA is the acronym of Ackermann Lemma Based Algorithm.

2.3.1 Syntax and semantics for DLE-logics

Our base language is an unspecified but fixed language $\mathcal{L}_{\mathrm{DLE}}$, to be interpreted over distributive lattice expansions of compatible similarity type. This setting uniformly accounts for many well known logical systems, such as distributive and positive modal logic, intuitionistic and bi-intuitionistic (modal) logic, tense logic, and (distributive) full Lambek calculus.

In our treatment, we will make heavy use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$[11] is an n-tuple $\varepsilon \in \{1, \partial\}^n$. For every order type ε, we denote its *opposite* order type by ε^∂, that is, $\varepsilon^\partial_i = 1$ iff $\varepsilon_i = \partial$ for every $1 \leq i \leq n$. For any lattice $\mathbb{A}$, we let $\mathbb{A}^1 := \mathbb{A}$ and $\mathbb{A}^\partial$ be the dual lattice, that is, the lattice associated with the converse partial order of $\mathbb{A}$. For any order type ε, we let $\mathbb{A}^\varepsilon := \Pi_{i=1}^n \mathbb{A}^{\varepsilon_i}$.

The language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$ (from now on abbreviated as $\mathcal{L}_{\mathrm{DLE}}$) takes as parameters: 1) a denumerable set of proposition letters AtProp, elements of which are denoted p, q, r, possibly with indexes; 2) disjoint sets of connectives $\mathcal{F}$ and $\mathcal{G}$.[12] Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) and is associated with some order-type ε_f over n_f (resp. ε_g over n_g).[13] The terms (formulas) of $\mathcal{L}_{\mathrm{DLE}}$ are defined recursively as follows:

$$\phi ::= p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid f(\overline{\phi}) \mid g(\overline{\phi})$$

where $p \in$ AtProp, $f \in \mathcal{F}$, $g \in \mathcal{G}$. Terms in $\mathcal{L}_{\mathrm{DLE}}$ will be denoted either by s, t, or by lowercase Greek letters such as φ, ψ, γ etc. In the context of sequents and prooftrees, $\mathcal{L}_{\mathrm{DLE}}$-formulas will be denoted by uppercase letters A, B, etc.

Definition 9. For any tuple $(\mathcal{F}, \mathcal{G})$ of disjoint sets of function symbols as above, a *distributive lattice expansion* (abbreviated as DLE) is a tuple $\mathbb{A} = (D, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A})$ such that D is a bounded distributive lattice, $\mathcal{F}^\mathbb{A} = \{f^\mathbb{A} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\mathbb{A} = \{g^\mathbb{A} \mid g \in \mathcal{G}\}$, such that every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) is an n_f-ary (resp. n_g-ary) operation on $\mathbb{A}$. A DLE is *normal* if every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) preserves finite joins (resp. meets) in each coordinate with $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and reverses finite meets (resp. joins) in each coordinate with $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$).[14] Let $\mathbb{DLE}$ be the class of DLEs.

[11] Throughout the chapter, order-types will be typically associated with arrays of variables $\vec{p} := (p_1, \ldots, p_n)$. When the order of the variables in $\vec{p}$ is not specified, we will sometimes abuse notation and write $\varepsilon(p) = 1$ or $\varepsilon(p) = \partial$.

[12] It will be clear from the treatment in the present and the following sections that the connectives in $\mathcal{F}$ (resp. $\mathcal{G}$) correspond to those referred to as *positive* (resp. *negative*) connectives in [7]. The reason why this terminology is not adopted in the present chapter is explained later on in Footnote 22. Our assumption that the sets $\mathcal{F}$ and $\mathcal{G}$ are disjoint is motivated by the desideratum of generality and modularity. Indeed, for instance, the order theoretic properties of Boolean negation $\neg$ guarantee that this connective belongs both to $\mathcal{F}$ and to $\mathcal{G}$. In such cases we prefer to define two copies $\neg_\mathcal{F} \in \mathcal{F}$ and $\neg_\mathcal{G} \in \mathcal{G}$, and introduce structural rules which encode the fact that these two copies coincide.

[13] Unary f (resp. g) will be sometimes denoted as $\Diamond$ (resp. $\Box$) if the order-type is 1, and $\lhd$ (resp. $\rhd$) if the order-type is ∂.

[14] Normal DLEs are sometimes referred to as *distributive lattices with operators* (DLOs). This terminology directly derives from the setting of Boolean algebras with operators, in which operators are understood as operations which preserve finite meets in each coordinate. However, this terminology results somewhat ambiguous in the lattice setting, in which primitive operations are typically maps which are operators if seen as $\mathbb{A}^\varepsilon \to \mathbb{A}^\eta$ for some order-type ε on n and some order-type $\eta \in \{1, \partial\}$. Rather than speaking of distributive lattices with (ε, η)-operators, we then speak of normal DLEs.

Sometimes we will refer to certain DLEs as $\mathcal{L}_{\mathrm{DLE}}$-algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed.

In the remainder of the chapter, we will abuse notation and write e.g. f for $f^{\mathbb{A}}$. Normal DLEs constitute the main semantic environment of the present chapter. Henceforth, every DLE is assumed to be normal; hence the adjective 'normal' will be typically dropped. The class of all DLEs is equational, and can be axiomatized by the usual distributive lattice identities and the following equations for any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (resp. for each $1 \leq j \leq n_g$):

- if $\varepsilon_f(i) = 1$, then $f(p_1, \ldots, p\vee q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$ and $f(p_1, \ldots, \bot, \ldots, p_{n_f}) = \bot$,
- if $\varepsilon_f(i) = \partial$, then $f(p_1, \ldots, p\wedge q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$ and $f(p_1, \ldots, \top, \ldots, p_{n_f}) = \bot$,
- if $\varepsilon_g(j) = 1$, then $g(p_1, \ldots, p\wedge q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$ and $g(p_1, \ldots, \top, \ldots, p_{n_g}) = \top$,
- if $\varepsilon_g(j) = \partial$, then $g(p_1, \ldots, p\vee q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$ and $g(p_1, \ldots, \bot, \ldots, p_{n_g}) = \top$.

Each language $\mathcal{L}_{\mathrm{DLE}}$ is interpreted in the appropriate class of DLEs. In particular, for every DLE $\mathbb{A}$, each operation $f^{\mathbb{A}} \in \mathcal{F}^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}^{\mathbb{A}}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \to \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \to \mathbb{A}$).

The generic DLE-logic is not equivalent to a sentential logic. Hence the consequence relation of these logics cannot be uniformly captured in terms of theorems, but rather in terms of sequents, which motivates the following definition:

Definition 10. For any language $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the *basic*, or *minimal* $\mathcal{L}_{\mathrm{DLE}}$-*logic* is a set of sequents $\phi \vdash \psi$, with $\phi, \psi \in \mathcal{L}_{\mathrm{DLE}}$, which contains the following axioms:

- Sequents for lattice operations:[15]

$$
\begin{array}{llll}
p \vdash p, & \bot \vdash p, & p \vdash \top, & p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r), \\
p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q,
\end{array}
$$

- Sequents for additional connectives:

$$
\begin{array}{l}
f(p_1, \ldots, \bot, \ldots, p_{n_f}) \vdash \bot, \ \text{for } \varepsilon_f(i) = 1, \\
f(p_1, \ldots, \top, \ldots, p_{n_f}) \vdash \bot, \ \text{for } \varepsilon_f(i) = \partial, \\
\top \vdash g(p_1, \ldots, \top, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = 1, \\
\top \vdash g(p_1, \ldots, \bot, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = \partial, \\
f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \ \text{for } \varepsilon_f(i) = 1, \\
f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) \vdash f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f}), \ \text{for } \varepsilon_f(i) = \partial, \\
g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = 1, \\
g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g}) \vdash g(p_1, \ldots, p \vee q, \ldots, p_{n_g}), \ \text{for } \varepsilon_g(i) = \partial,
\end{array}
$$

[15] In what follows we will use the turnstile symbol $\vdash$ both as sequent separator and also as the consequence relation of the logic.

and is closed under the following inference rules:

$$\frac{\phi \vdash \chi \quad \chi \vdash \psi}{\phi \vdash \psi} \qquad \frac{\phi \vdash \psi}{\phi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \phi \quad \chi \vdash \psi}{\chi \vdash \phi \wedge \psi} \qquad \frac{\phi \vdash \chi \quad \psi \vdash \chi}{\phi \vee \psi \vdash \chi}$$

$$\frac{\phi \vdash \psi}{f(p_1, \ldots, \phi, \ldots, p_n) \vdash f(p_1, \ldots, \psi, \ldots, p_n)} \ (\varepsilon_f(i) = 1)$$

$$\frac{\phi \vdash \psi}{f(p_1, \ldots, \psi, \ldots, p_n) \vdash f(p_1, \ldots, \phi, \ldots, p_n)} \ (\varepsilon_f(i) = \partial)$$

$$\frac{\phi \vdash \psi}{g(p_1, \ldots, \phi, \ldots, p_n) \vdash g(p_1, \ldots, \psi, \ldots, p_n)} \ (\varepsilon_g(i) = 1)$$

$$\frac{\phi \vdash \psi}{g(p_1, \ldots, \psi, \ldots, p_n) \vdash g(p_1, \ldots, \phi, \ldots, p_n)} \ (\varepsilon_g(i) = \partial).$$

The minimal DLE-logic is denoted by $\mathbf{L}_{\mathrm{DLE}}$. For any DLE-language $\mathcal{L}_{\mathrm{DLE}}$, by a DLE-*logic* we understand any axiomatic extension of the basic $\mathcal{L}_{\mathrm{DLE}}$-logic in $\mathcal{L}_{\mathrm{DLE}}$.

For every DLE $\mathbb{A}$, the symbol $\vdash$ is interpreted as the lattice order $\leq$. A sequent $\phi \vdash \psi$ is valid in $\mathbb{A}$ if $h(\phi) \leq h(\psi)$ for every homomorphism h from the $\mathcal{L}_{\mathrm{DLE}}$-algebra of formulas over AtProp to $\mathbb{A}$. The notation $\mathbb{DLE} \models \phi \vdash \psi$ indicates that $\phi \vdash \psi$ is valid in every DLE. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal DLE-logic $\mathbf{L}_{\mathrm{DLE}}$ is sound and complete with respect to its correspondent class of algebras $\mathbb{DLE}$, i.e. that any sequent $\phi \vdash \psi$ is provable in $\mathbf{L}_{\mathrm{DLE}}$ iff $\mathbb{DLE} \models \phi \vdash \psi$.

2.3.2 The expanded language $\mathcal{L}^*_{\mathrm{DLE}}$

Any given language $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$ can be associated with the language $\mathcal{L}^*_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}^*, \mathcal{G}^*)$, where $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{G}^* \supseteq \mathcal{G}$ are obtained by expanding $\mathcal{L}_{\mathrm{DLE}}$ with the following connectives:

1. the binary connectives $\leftarrow$ and $\rightarrow$, the intended interpretations of which are the right residuals of $\wedge$ in the first and second coordinate respectively, and $-\!\!<$ and $>\!\!-$, the intended interpretations of which are the left residuals of $\vee$ in the first and second coordinate, respectively;

2. the n_f-ary connective $f_i^\sharp$ for $0 \leq i \leq n_f$, the intended interpretation of which is the right residual of $f \in \mathcal{F}$ in its ith coordinate if $\varepsilon_f(i) = 1$ (resp. its Galois-adjoint if $\varepsilon_f(i) = \partial$);

3. the n_g-ary connective $g_i^\flat$ for $0 \leq i \leq n_g$, the intended interpretation of which is the left residual of $g \in \mathcal{G}$ in its ith coordinate if $\varepsilon_g(i) = 1$ (resp. its Galois-adjoint if $\varepsilon_g(i) = \partial$). [16]

[16] The adjoints of the unary connectives $\Box$, $\Diamond$, $\lhd$ and $\rhd$ are denoted $\blacklozenge$, $\blacksquare$, $\blacktriangleleft$ and $\blacktriangleright$, respectively.

We stipulate that $\succ\!\!-, -\!\!\prec \in \mathcal{F}^*$, that $\rightarrow, \leftarrow \in \mathcal{G}^*$, and moreover, that $f_i^\sharp \in \mathcal{G}^*$ if $\varepsilon_f(i) = 1$, and $f_i^\sharp \in \mathcal{F}^*$ if $\varepsilon_f(i) = \partial$. Dually, $g_i^\flat \in \mathcal{F}^*$ if $\varepsilon_g(i) = 1$, and $g_i^\flat \in \mathcal{G}^*$ if $\varepsilon_g(i) = \partial$. The order-type assigned to the additional connectives is predicated on the order-type of their intended interpretations. That is, for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$,

1. if $\varepsilon_f(i) = 1$, then $\varepsilon_{f_i^\sharp}(i) = 1$ and $\varepsilon_{f_i^\sharp}(j) = (\varepsilon_f(j))^\partial$ for any $j \neq i$.
2. if $\varepsilon_f(i) = \partial$, then $\varepsilon_{f_i^\sharp}(i) = \partial$ and $\varepsilon_{f_i^\sharp}(j) = \varepsilon_f(j)$ for any $j \neq i$.
3. if $\varepsilon_g(i) = 1$, then $\varepsilon_{g_i^\flat}(i) = 1$ and $\varepsilon_{g_i^\flat}(j) = (\varepsilon_g(j))^\partial$ for any $j \neq i$.
4. if $\varepsilon_g(i) = \partial$, then $\varepsilon_{g_i^\flat}(i) = \partial$ and $\varepsilon_{g_i^\flat}(j) = \varepsilon_g(j)$ for any $j \neq i$.

For instance, if f and g are binary connectives such that $\varepsilon_f = (1, \partial)$ and $\varepsilon_g = (\partial, 1)$, then $\varepsilon_{f_1^\sharp} = (1, 1)$, $\varepsilon_{f_2^\sharp} = (1, \partial)$, $\varepsilon_{g_1^\flat} = (\partial, 1)$ and $\varepsilon_{g_2^\flat} = (1, 1)$.[17]

Definition 11. For any language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the *basic bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-logic* is defined by specializing Definition 10 to the language $\mathcal{L}^*_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}^*, \mathcal{G}^*)$ and closing under the following additional rules:

1. residuation rules for lattice connectives:

$$\frac{\phi \wedge \psi \vdash \chi}{\psi \vdash \phi \rightarrow \chi} \qquad \frac{\phi \wedge \psi \vdash \chi}{\phi \vdash \chi \leftarrow \psi} \qquad \frac{\phi \vdash \psi \vee \chi}{\psi \succ\!\!- \phi \vdash \chi} \qquad \frac{\phi \vdash \psi \vee \chi}{\phi -\!\!\prec \chi \vdash \psi}$$

   Notice that the rules for $\rightarrow$ and $\leftarrow$ are interderivable, since $\wedge$ is commutative; similarly, the rules for $\succ\!\!-$ and $-\!\!\prec$ are interderivable, since $\vee$ is commutative.

2. Residuation rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$(\varepsilon_f(i) = 1)\ \frac{f(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_f}) \vdash \psi}{\phi \vdash f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f})} \qquad \frac{\phi \vdash g(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}{g_i^\flat(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_g}) \vdash \psi}\ (\varepsilon_g(i) = 1)$$

$$(\varepsilon_f(i) = \partial)\ \frac{f(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_f}) \vdash \psi}{f_i^\sharp(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_f}) \vdash \phi} \qquad \frac{\phi \vdash g(\varphi_1, \ldots, \psi, \ldots, \varphi_{n_g})}{\psi \vdash g_i^\flat(\varphi_1, \ldots, \phi, \ldots, \varphi_{n_g})}\ (\varepsilon_g(i) = \partial)$$

The double line in each rule above indicates that the rule is invertible. Let $\mathbf{L}^*_{\mathrm{DLE}}$ be the minimal bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-logic.[18] For any DLE-language $\mathcal{L}_{\mathrm{DLE}}$, by a *tense*

[17] Warning: notice that this notation heavily depends from the connective which is taken as primitive, and needs to be carefully adapted to well known cases. For instance, consider the 'fusion' connective $\circ$ (which, when denoted as f, is such that $\varepsilon_f = (1, 1)$). Its residuals $f_1^\sharp$ and $f_2^\sharp$ are commonly denoted $/$ and $\backslash$ respectively. However, if $\backslash$ is taken as the primitive connective g, then $g_2^\flat$ is $\circ = f$, and $g_1^\flat(x_1, x_2) := x_2/x_1 = f_1^\sharp(x_2, x_1)$. This example shows that, when identifying $g_1^\flat$ and $f_1^\sharp$, the conventional order of the coordinates is not preserved, and depends of which connective is taken as primitive.

[18] Hence, for any language $\mathcal{L}_{\mathrm{DLE}}$, there are in principle two logics associated with the expanded language $\mathcal{L}^*_{\mathrm{DLE}}$, namely the *minimal* $\mathcal{L}^*_{\mathrm{DLE}}$-logic, which we denote by $\mathbf{L}^{\underline{*}}_{\mathrm{DLE}}$, and which is obtained by instantiating Definition 10 to the language $\mathcal{L}^*_{\mathrm{DLE}}$, and the bi-intuitionistic 'tense' logic $\mathbf{L}^*_{\mathrm{DLE}}$, defined above. The logic $\mathbf{L}^*_{\mathrm{DLE}}$ is the natural logic on the language $\mathcal{L}^*_{\mathrm{DLE}}$, however it is useful to introduce a specific notation for $\mathbf{L}^{\underline{*}}_{\mathrm{DLE}}$, given that all the results holding for the minimal logic associated with an arbitrary DLE-language can be instantiated to the expanded language $\mathcal{L}^*_{\mathrm{DLE}}$ and will then apply to $\mathbf{L}^{\underline{*}}_{\mathrm{DLE}}$.

2

DLE-*logic* we understand any axiomatic extension of the basic tense bi-intuitionistic $\mathcal{L}_{\mathrm{DLE}}$-logic in $\mathcal{L}^*_{\mathrm{DLE}}$.

The algebraic semantics of $\mathbf{L}^*_{\mathrm{DLE}}$ is given by the class of bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-algebras, defined as tuples $\mathbb{A} = (H, \mathcal{F}^*, \mathcal{G}^*)$ such that H is a bi-Heyting algebra[19] and moreover,

1. for every $f \in \mathcal{F}$ s.t. $n_f \geq 1$, all $a_1, \ldots, a_{n_f} \in D$ and $b \in D$, and each $1 \leq i \leq n_f$,
   - if $\varepsilon_f(i) = 1$, then $f(a_1, \ldots, a_i, \ldots a_{n_f}) \leq b$ iff $a_i \leq f^\sharp_i(a_1, \ldots, b, \ldots, a_{n_f})$;
   - if $\varepsilon_f(i) = \partial$, then $f(a_1, \ldots, a_i, \ldots a_{n_f}) \leq b$ iff $a_i \leq^\partial f^\sharp_i(a_1, \ldots, b, \ldots, a_{n_f})$.
2. for every $g \in \mathcal{G}$ s.t. $n_g \geq 1$, any $a_1, \ldots, a_{n_g} \in D$ and $b \in D$, and each $1 \leq i \leq n_g$,
   - if $\varepsilon_g(i) = 1$, then $b \leq g(a_1, \ldots, a_i, \ldots a_{n_g})$ iff $g^\flat_i(a_1, \ldots, b, \ldots, a_{n_g}) \leq a_i$.
   - if $\varepsilon_g(i) = \partial$, then $b \leq g(a_1, \ldots, a_i, \ldots a_{n_g})$ iff $g^\flat_i(a_1, \ldots, b, \ldots, a_{n_g}) \leq^\partial a_i$.

It is also routine to prove using the Lindenbaum-Tarski construction that $\mathbf{L}^*_{\mathrm{DLE}}$ (as well as any of its sound axiomatic extensions) is sound and complete w.r.t. the class of bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-algebras (w.r.t. the suitably defined equational subclass, respectively).

Theorem 12. *The logic $\mathbf{L}^*_{\mathrm{DLE}}$ is a conservative extension of $\mathbf{L}_{\mathrm{DLE}}$, i.e., for every $\mathcal{L}_{\mathrm{DLE}}$-sequent $\phi \vdash \psi$, $\phi \vdash \psi$ is derivable in $\mathbf{L}_{\mathrm{DLE}}$ iff $\phi \vdash \psi$ is derivable in $\mathbf{L}^*_{\mathrm{DLE}}$. Moreover, every DLE-logic can be extended conservatively to a DLE*-logic.*

Proof. We only outline the proof. Clearly, every $\mathcal{L}_{\mathrm{DLE}}$-sequent which is $\mathbf{L}_{\mathrm{DLE}}$-derivable is also $\mathbf{L}^*_{\mathrm{DLE}}$-derivable. Conversely, if an $\mathcal{L}_{\mathrm{DLE}}$-sequent $\phi \vdash \psi$ is not $\mathbf{L}_{\mathrm{DLE}}$-derivable, then by the completeness of $\mathbf{L}_{\mathrm{DLE}}$ w.r.t. the class of $\mathcal{L}_{\mathrm{DLE}}$-algebras, there exists an $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ and a variable assignment v under which $\phi^{\mathbb{A}} \not\leq \psi^{\mathbb{A}}$. Consider the canonical extension $\mathbb{A}^\delta$ of $\mathbb{A}$.[20] Since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$, the sequent $\phi \vdash \psi$ is not

[19] That is, $H = (D, \rightarrow, -\!\!<)$ such that both $(D, \rightarrow)$ and $(D^\partial, -\!\!<)$ are Heyting algebras. In particular, setting $c \leftarrow b := b \rightarrow c$ and $b \succ\!\!- a := a \succ\!\!- b$ for all $a, b, c \in D$, the following equivalences hold

$$a \wedge b \leq c \text{ iff } b \leq a \rightarrow c \text{ iff } a \leq c \leftarrow b, \qquad a \leq b \vee c \text{ iff } b \succ\!\!- a \leq c \text{ iff } a -\!\!< c \leq b.$$

[20] The *canonical extension* of a BDL (bounded distributive lattice) D is a complete distributive lattice D^δ containing D as a sublattice, such that:

1. *(denseness)* every element of D^δ can be expressed both as a join of meets and as a meet of joins of elements from D;
2. *(compactness)* for all $S, T \subseteq D$, if $\bigwedge S \leq \bigvee T$ in D^δ, then $\bigwedge F \leq \bigvee G$ for some finite sets $F \subseteq S$ and $G \subseteq T$.

It is well known that the canonical extension of a BDL D is unique up to isomorphism fixing D (cf. e.g. [33, Section 2.2]), and that the canonical extension of a BDL is a *perfect* BDL, i.e. a complete and completely distributive lattice which is completely join-generated by its completely join-irreducible elements and completely meet-generated by its completely meet-irreducible elements (cf. e.g. [33, Definition 2.14]). The canonical extension of an $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A} = (D, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra (cf. Footnote 28) $\mathbb{A}^\delta := (D^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ such that $f^{\mathbb{A}^\delta}$ and $g^{\mathbb{A}^\delta}$ are defined as the σ-extension of $f^{\mathbb{A}}$ and as the π-extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$ (cf. [55, 56]).

satisfied in $\mathbb{A}^\delta$ under the variable assignment $\iota \circ v$ (ι denoting the canonical embedding $\mathbb{A} \hookrightarrow \mathbb{A}^\delta$). Moreover, since $\mathbb{A}^\delta$ is a perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra, it is naturally endowed with a structure of bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-algebra. Thus, by the completeness of $\mathbf{L}^*_{\mathrm{DLE}}$ w.r.t. the class of bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-algebras, the sequent $\phi \vdash \psi$ is not derivable in $\mathbf{L}^*_{\mathrm{DLE}}$, as required. □ Notice that the algebraic completeness of the logics $\mathbf{L}_{\mathrm{DLE}}$ and $\mathbf{L}^*_{\mathrm{DLE}}$ and the canonical embedding of DLEs into their canonical extensions immediately give completeness of $\mathbf{L}_{\mathrm{DLE}}$ and $\mathbf{L}^*_{\mathrm{DLE}}$ w.r.t. the appropriate class of perfect DLEs.

2

2.3.3 The algorithm ALBA, informally

The contribution of the present chapter is an application of unified correspondence theory [18, 20], of which the algorithm ALBA is one of the main tools. In the present subsection, we will guide the reader through the main principles which make it work, by means of an example. This presentation is based on analogous illustrations in [16] and [23].

Let us start with one of the best known examples in correspondence theory, namely $\Diamond\Box p \to \Box\Diamond p$. It is well known that for every Kripke frame $\mathbb{F} = (W, R)$,

$$\mathbb{F} \Vdash \Diamond\Box p \to \Box\Diamond p \quad \text{iff} \quad \mathbb{F} \models \forall xyz\,(Rxy \wedge Rxz \to \exists u(Ryu \wedge Rzu)).$$

As is discussed at length in [18, 20], every piece of argument used to prove this correspondence on frames can be translated by duality to complex algebras (cf. [5, Definition 5.21]). We will show how this is done in the case of the example above.

As is well known, complex algebras are characterized in purely algebraic terms as complete and atomic Boolean algebras with operators (BAOs) where the modal operations are completely join-preserving. These are also known as *perfect* BAOs [6, Definition 40, Chapter 6].

First of all, the condition $\mathbb{F} \Vdash \Diamond\Box p \to \Box\Diamond p$ translates to the complex algebra $\mathbb{A} = \mathbb{F}^+$ of $\mathbb{F}$ as $[\![\Diamond\Box p]\!] \subseteq [\![\Box\Diamond p]\!]$ for every assignment of p into A, so this validity clause can be rephrased as follows:

$$\mathbb{A} \models \forall p[\Diamond\Box p \leq \Box\Diamond p], \tag{2.3.1}$$

where the order $\leq$ is interpreted as set inclusion in the complex algebra. In perfect BAOs every element is both the join of the completely join-prime elements (the set of which is denoted $J^\infty(\mathbb{A})$) below it and the meet of the completely meet-prime elements (the set of which is denoted $M^\infty(\mathbb{A})$) above it[21]. Hence, taking some liberties in our use of notation, the condition above can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall p[\bigvee\{i \in J^\infty(\mathbb{A}) \mid i \leq \Box\Diamond p\} \leq \bigwedge\{m \in M^\infty(\mathbb{A}) \mid \Box\Diamond p \leq m\}].$$

[21] In BAOs the completely join-prime elements, the completely join-irreducible elements and the atoms coincide. Moreover, the completely meet-prime elements, the completely meet-irreducible elements and the co-atoms coincide.

By elementary properties of least upper bounds and greatest lower bounds in posets (cf. [26]), this condition is true if and only if every element in the join is less than or equal to every element in the meet; thus, condition (2.3.1) above can be rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond\Box p \;\;\&\;\; \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{2.3.2}$$

where the variables $\mathbf{i}$ and $\mathbf{m}$ range over $J^{\infty}(\mathbb{A})$ and $M^{\infty}(\mathbb{A})$ respectively (following the literature, we will refer to the former variables as *nominals*, and to the latter ones as *co-nominals*). Since $\mathbb{A}$ is a perfect BAO, the element of $\mathbb{A}$ interpreting $\Box p$ is the join of the completely join-prime elements below it. Hence, if $i \in J^{\infty}(\mathbb{A})$ and $i \leq \Diamond\Box p$, because $\Diamond$ is completely join-preserving on $\mathbb{A}$, we have that

$$i \leq \Diamond(\bigvee\{j \in J^{\infty}(\mathbb{A}) \mid j \leq \Box p\}) = \bigvee\{\Diamond j \mid j \in J^{\infty}(\mathbb{A}) \text{ and } j \leq \Box p\},$$

which implies that $i \leq \Diamond j_0$ for some $j_0 \in J^{\infty}(\mathbb{A})$ such that $j_0 \leq \Box p$. Hence, we can equivalently rewrite the validity clause above as follows:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m}[(\exists \mathbf{j}(\mathbf{i} \leq \Diamond \mathbf{j} \;\;\&\;\; \mathbf{j} \leq \Box p) \;\;\&\;\; \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{2.3.3}$$

and then use standard manipulations from first-order logic to pull out quantifiers:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \;\;\&\;\; \mathbf{j} \leq \Box p \;\;\&\;\; \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.3.4}$$

Now we observe that the operation $\Box$ preserves arbitrary meets in the perfect BAO $\mathbb{A}$. By the general theory of adjunction in complete lattices, this is equivalent to $\Box$ being a right adjoint (cf. [26, Proposition 7.34]). It is also well known that the left or lower adjoint (cf. [26, Definition 7.23]) of $\Box$ is the operation $\blacklozenge$, which can be recognized as the backward-looking diamond P, interpreted with the converse R^{-1} of the accessibility relation R of the frame $\mathbb{F}$ in the context of tense logic (cf. [5, Example 1.25] and [26, Exercise 7.18] modulo translating the notation). Hence the condition above can be equivalently rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \;\;\&\;\; \blacklozenge \mathbf{j} \leq p \;\;\&\;\; \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{2.3.5}$$

and then as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \;\;\&\;\; \exists p(\blacklozenge \mathbf{j} \leq p \;\;\&\;\; \Box\Diamond p \leq \mathbf{m})) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.3.6}$$

At this point we are in a position to eliminate the variable p and equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \;\;\&\;\; \Box\Diamond\blacklozenge \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{2.3.7}$$

Let us justify this equivalence: for the direction from top to bottom, fix an interpretation V of the variables $\mathbf{i}, \mathbf{j}$, and $\mathbf{m}$ such that $\mathbf{i} \leq \Diamond \mathbf{j}$ and $\Box\Diamond\blacklozenge \mathbf{j} \leq \mathbf{m}$. To prove that $\mathbf{i} \leq \mathbf{m}$ holds under V, consider the variant V^* of V such that $V^*(p) = \blacklozenge \mathbf{j}$. Then it can be easily verified that V^* witnesses the antecedent of (2.3.6) under V; hence $\mathbf{i} \leq \mathbf{m}$ holds under V. Conversely, fix an interpretation V of the variables $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{m}$ such that $\mathbf{i} \leq \Diamond \mathbf{j} \;\&\; \exists p(\blacklozenge \mathbf{j} \leq p \;\&\; \Box\Diamond p \leq \mathbf{m})$. Then, by monotonicity, the antecedent of (2.3.7) holds under V, and hence so does $\mathbf{i} \leq \mathbf{m}$, as required. This is an instance of the following result, known as *Ackermann's lemma* ([1], see also [19]):

Lemma 13. *Fix an arbitrary propositional language L. Let $\alpha, \beta(p), \gamma(p)$ be L-formulas such that α is p-free, β is positive and γ is negative in p. For any assignment V on an L-algebra $\mathbb{A}$, the following are equivalent:*

1. $\mathbb{A}, V \models \beta(\alpha/p) \leq \gamma(\alpha/p)$;
2. *there exists a p-variant V^* of V such that $\mathbb{A}, V^* \models \alpha \leq p$ and $\mathbb{A}, V^* \models \beta(p) \leq \gamma(p)$,*

where $\beta(\alpha/p)$ and $\gamma(\alpha/p)$ denote the result of uniformly substituting α for p in β and γ, respectively.

The proof is essentially the same as [20, Lemma 4.2]. Whenever, in a reduction, we reach a shape in which the lemma above (or its order-dual) can be applied, we say that the condition is in *Ackermann shape*.

Taking stock, we note that we have equivalently transformed (2.3.1) into (2.3.7), which is a condition in which all propositional variables (corresponding to monadic second-order variables) have been eliminated, and all remaining variables range over completely join- and meet-irreducible elements of the complex algebra $\mathbb{A}$. Via discrete Stone duality, these elements respectively correspond to singletons and complements of singletons of the Kripke frame from which $\mathbb{A}$ arises. Moreover, $\blacklozenge$ is interpreted on Kripke frames using the converse of the same accessibility relation used to interpret $\Box$. Hence, clause (2.3.7) translates equivalently into a condition in the first-order correspondence language of $\mathbb{F}$.

To facilitate this translation, we first rewrite (2.3.7) as follows, by reversing the reasoning that brought us from (2.3.1) to (2.3.2):

$$\mathbb{A} \models \forall \mathbf{j}[\Diamond \mathbf{j} \leq \Box \Diamond \blacklozenge \mathbf{j}]. \tag{2.3.8}$$

By again applying the fact that $\Box$ is a right adjoint we obtain

$$\mathbb{A} \models \forall \mathbf{j}[\blacklozenge \Diamond \mathbf{j} \leq \Diamond \blacklozenge \mathbf{j}]. \tag{2.3.9}$$

Recalling that $\mathbb{A}$ is the complex algebra of $\mathbb{F} = (W, R)$, we can interpret the variable $\mathbf{j}$ as an individual variable ranging in the universe W of $\mathbb{F}$, and the operations $\Diamond$ and $\blacklozenge$ as the set-theoretic operations defined on $\mathcal{P}(W)$ by the assignments $X \mapsto R^{-1}[X]$ and $X \mapsto R[X]$ respectively. Hence, clause (2.3.9) above can be equivalently rewritten on the side of the frames as

$$\mathbb{F} \models \forall w(R[R^{-1}[w]] \subseteq R^{-1}[R[w]]). \tag{2.3.10}$$

Notice that $R[R^{-1}[w]]$ is the set of all states $x \in W$ which have a predecessor z in common with w, while $R^{-1}[R[w]]$ is the set of all states $x \in W$ which have a successor in common with w. This can be spelled out as

$$\forall x \forall w(\exists z(Rzx \wedge Rzw) \rightarrow \exists y(Rxy \wedge Rwy))$$

or, equivalently,

$$\forall z \forall x \forall w((Rzx \wedge Rzw) \rightarrow \exists y(Rxy \wedge Rwy))$$

which is the familiar Church-Rosser condition.

Finally, the example above illustrates another important feature of the ALBA-based approach to the computation of first-order correspondents. Namely, ALBA-computations are neatly divided into two stages: the *reduction* stage, carried out from (2.3.1) into (2.3.7) in the example above; and the *translation* stage, in which the expressions (equalities and quasi-inequalities) obtained by eliminating all proposition variables from an input inequality are suitably translated into frame-correspondent language. Only the reduction stage will be relevant to the remainder of the present chapter.

2.3.4 Inductive inequalities

In the present subsection, we will report on the definition of *inductive* $\mathcal{L}_{\mathrm{DLE}}$-inequalities on which the algorithm ALBA is guaranteed to succeed (cf. [18, 20]).

Definition 14 (**Signed Generation Tree**)**.** The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}_{\mathrm{DLE}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows:

- For any node labelled with $\vee$ or $\wedge$, assign the same sign to its children nodes.
- For any node labelled with $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$, and for any $1 \leq i \leq n_h$, assign the same (resp. the opposite) sign to its ith child node if $\varepsilon_h(i) = 1$ (resp. if $\varepsilon_h(i) = \partial$).

Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).[22]

Signed generation trees will be mostly used in the context of term inequalities $s \leq t$. In this context we will typically consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side. We will also say that a term-inequality $s \leq t$ is *uniform* in a given variable p if all occurrences of p in both $+s$ and $-t$ have the same sign, and that $s \leq t$ is ε-*uniform* in a (sub)array $\vec{p}$ of its variables if $s \leq t$ is uniform in p, occurring with the sign indicated by ε, for every p in $\vec{p}$[23].

For any term $s(p_1, \ldots p_n)$, any order type ε over n, and any $1 \leq i \leq n$, an ε-*critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon_i = 1$ or $-p_i$ with $\varepsilon_i = \partial$. An ε-*critical branch* in the tree is a branch from an ε-critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ε-critical nodes are *to be solved for*, according to ε.

For every term $s(p_1, \ldots p_n)$ and every order type ε, we say that $+s$ (resp. $-s$) *agrees with* ε, and write $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ε-critical. In other words, $\varepsilon(+s)$ (resp. $\varepsilon(-s)$) means that all variable

[22] The terminology used in [7] regarding 'positive' and 'negative connectives' has not been adopted in the present chapter to avoid confusion with positive and negative nodes in signed generation trees.

[23] The following observation will be used at various points in the remainder of the present chapter: if a term inequality $s(\vec{p}, \vec{q}) \leq t(\vec{p}, \vec{q})$ is ε-uniform in $\vec{p}$ (cf. discussion after Definition 14), then the validity of $s \leq t$ is equivalent to the validity of $s(\overrightarrow{\top^{\varepsilon(i)}}, \vec{q}) \leq t(\overrightarrow{\top^{\varepsilon(i)}}, \vec{q})$, where $\top^{\varepsilon(i)} = \top$ if $\varepsilon(i) = 1$ and $\top^{\varepsilon(i)} = \bot$ if $\varepsilon(i) = \partial$.

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \wedge$	$+ \quad \wedge \quad g$ with $n_g = 1$
$- \quad \wedge \quad \vee$	$- \quad \vee \quad f$ with $n_f = 1$
SLR	SRR
$+ \quad \wedge \quad f$ with $n_f \geq 1$	$+ \quad \vee \quad g$ with $n_g \geq 2$
$- \quad \vee \quad g$ with $n_g \geq 1$	$- \quad \wedge \quad f$ with $n_f \geq 2$

Table 2.1: Skeleton and PIA nodes for DLE.

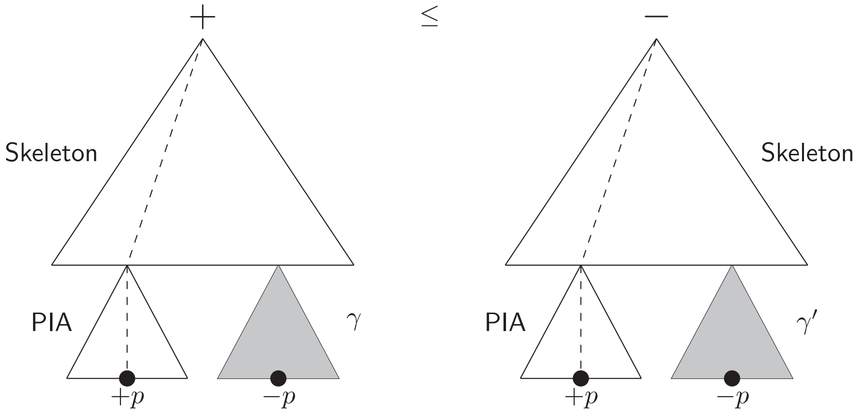


Figure 2.1: A schematic representation of inductive inequalities.

occurrences corresponding to leaves of $+s$ (resp. $-s$) are to be solved for according to ε. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(\gamma) \prec *s$ (resp. $\varepsilon^\partial(\gamma) \prec *s$) to indicate that the signed subtree γ, with the sign inherited from $*s$, agrees with ε (resp. with ε^∂).

Definition 15. Nodes in signed generation trees will be called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 2.1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes[24], and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

Definition 16 (Inductive inequalities)**.** For any order type ε and any irreflexive and transitive relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{-, +\}$) of a term $s(p_1, \ldots p_n)$ is (Ω, ε)-*inductive* if

[24]For explanations of our choice of terminologies here, we refer to [46, Remark 3.24].

1. for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good (cf. Definition 15);
2. every m-ary SRR-node that is occurring in the critical branch is of the form $\circledast(\gamma_1, \ldots, \gamma_{j-1}, \beta, \gamma_{j+1} \ldots, \gamma_m)$, where for any $h \in \{1, \ldots, m\} \setminus j$:
   (a) $\varepsilon^\partial(\gamma_h) \prec *s$ (cf. discussion before Definition 15), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in γ_h and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω, ε)-*inductive* if the signed generation trees $+s$ and $-t$ are (Ω, ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε)-inductive for some Ω and ε.

In what follows, we will find it useful to refer to formulas ϕ such that only PIA nodes occur in $+\phi$ (resp. $-\phi$) as *positive* (resp. *negative*) *PIA-formulas*, and to formulas ξ such that only Skeleton nodes occur in $+\xi$ (resp. $-\xi$) as *positive* (resp. *negative*) *Skeleton-formulas*.

The proof of the following theorem is a straightforward generalization of [20, Theorem 10.11], and hence its proof is omitted.

Theorem 17. *For any language $\mathcal{L}_{\mathrm{DLE}}$, its corresponding version of ALBA succeeds on all inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities, which are hence canonical*[25] *and their corresponding logics are complete w.r.t. the elementary classes of relational structures*[26] *defined by their first-order correspondents.*

2.3.5 The algorithm ALBA for $\mathcal{L}_{\mathrm{DLE}}$-inequalities

The present subsection reports on the rules and execution of the algorithm ALBA in the setting of $\mathcal{L}_{\mathrm{DLE}}$. ALBA manipulates inequalities and quasi-inequalities[27] in the expanded language $\mathcal{L}^{*+}_{\mathrm{DLE}}$, which is built up on the base of the lattice constants $\top, \bot$ and an enlarged set of propositional variables NOM $\cup$ CONOM $\cup$ AtProp (the variables $\mathbf{i}, \mathbf{j}$ in NOM are referred to as *nominals*, and the variables $\mathbf{m}, \mathbf{n}$ in CONOM as *conominals*), closing under the logical connectives of $\mathcal{L}^{*}_{\mathrm{DLE}}$. The natural semantic environment of $\mathcal{L}^{*+}_{\mathrm{DLE}}$ is given by *perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras*. As already mentioned in the proof of Theorem 12, these algebras are endowed with a natural structure of bi-intuitionistic 'tense' $\mathcal{L}_{\mathrm{DLE}}$-algebra. Moreover, crucially, perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras are both completely join-generated by their completely join-irreducible elements and completely meet-generated by their completely meet-irreducible elements.[28] This property plays an important part

[25] An $\mathcal{L}_{\mathrm{DLE}}$-inequality $s \leq t$ is *canonical* if the class of $\mathcal{L}_{\mathrm{DLE}}$-algebras defined by $s \leq t$ is closed under the construction of canonical extension (cf. Footnote 20).

[26] Such are those introduced in [55, 56].

[27] A *quasi-inequality* of $\mathcal{L}_{\mathrm{DLE}}$ is an expression of the form $\&_{i=1}^{n} s_i \leq t_i \Rightarrow s \leq t$, where $s_i \leq t_i$ and $s \leq t$ are $\mathcal{L}_{\mathrm{DLE}}$-inequalities for each i.

[28] A distributive lattice is *perfect* if it is complete, completely distributive and completely join-generated by the collection of its completely join-prime elements. Equivalently, a distributive lattice is perfect iff it is isomorphic to the lattice of upsets of some poset. A normal DLE is *perfect* if D is a perfect distributive lattice, and each f-operation (resp. g-operation) is completely join-preserving (resp. meet-preserving) or completely meet-reversing (resp. join-reversing) in each coordinate.

in the algebraic account of the correspondence mechanism (cf. discussion in [18, Section 1.4]). Nominals and conominals respectively range over the sets of the completely join-irreducible elements and the completely meet-irreducible elements of perfect DLEs.

The version of ALBA relative to $\mathcal{L}_{\mathrm{DLE}}$ runs as detailed in [20]. In a nutshell, $\mathcal{L}_{\mathrm{DLE}}$-inequalities are equivalently transformed into the conjunction of one or more $\mathcal{L}^{*+}_{\mathrm{DLE}}$ quasi-inequalities, with the aim of eliminating propositional variable occurrences via the application of Ackermann rules. We refer the reader to [20] for details. In what follows, we illustrate how ALBA works, while at the same time we introduce its rules. The proof of the soundness and invertibility of the general rules for the DLE-setting is similar to the one provided in [18, 20]. ALBA manipulates input inequalities $\phi \leq \psi$ and proceeds in three stages:

First stage: preprocessing and first approximation. ALBA preprocesses the input inequality $\phi \leq \psi$ by performing the following steps exhaustively in the signed generation trees $+\phi$ and $-\psi$:

1. (a) Push down, towards variables, occurrences of $+\wedge$, by distributing each of them over their children nodes labelled with $+\vee$ which are not in the scope of PIA nodes;

   (b) Push down, towards variables, occurrences of $-\vee$, by distributing each of them over their children nodes labelled with $-\wedge$ which are not in the scope of PIA nodes;

   (c) Push down, towards variables, occurrences of $+f$ for any $f \in \mathcal{F}$, by distributing each such occurrence over its ith child node whenever the child node is labelled with $+\vee$ (resp. $-\wedge$) and is not in the scope of PIA nodes, and whenever $\varepsilon_f(i) = 1$ (resp. $\varepsilon_f(i) = \partial$);

   (d) Push down, towards variables, occurrences of $-g$ for any $g \in \mathcal{G}$, by distributing each such occurrence over its ith child node whenever the child node is labelled with $-\wedge$ (resp. $+\vee$) and is not in the scope of PIA nodes, and whenever $\varepsilon_g(i) = 1$ (resp. $\varepsilon_g(i) = \partial$).

2. Apply the splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

3. Apply the monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

   for $\beta(p)$ positive in p and $\alpha(p)$ negative in p.

Remark 18. The standard ALBA preprocessing can be supplemented with the application of additional rules which replace SLR-nodes (resp. SRR-nodes) of the form $\circledast(\gamma_1, \ldots, \bot^{\varepsilon_\circledast(i)}, \ldots, \gamma_m)$ (resp. $\circledast(\gamma_1, \ldots, \top^{\varepsilon_\circledast(i)}, \ldots, \gamma_m)$) with $\bot$ (resp. $\top$). Although clearly sound, these rules have not been included in other ALBA settings such as [16, 20], since they are not strictly needed for the computation of first-order correspondents. However, in the present setting, ALBA is used for a different purpose than the one it was originally designed for. Allowing these rules to be applied during the preprocessing will address the problem of the occurrences of constants in the 'wrong' position,[29] since it allows to transform e.g. a problematic premise into a tautology and make it hence disappear. These ideas will be expanded on in Sections 2.6.2 and 2.7.1.

Another step of the preprocessing which, although sound, is not included in standard executions of ALBA concerns the exhaustive application of the following distribution rules:

(a') Push down, towards variables, occurrences of $-\wedge$ in the scope of PIA-nodes which are not Skeleton-nodes, by distributing each of them over their children nodes labelled with $-\vee$;

(b') Push down, towards variables, occurrences of $+\vee$ in the scope of PIA-nodes which are not Skeleton-nodes, by distributing each of them over their children nodes labelled with $+\wedge$;

(c') Push down, towards variables, occurrences of $-f$ for any $f \in \mathcal{F}$, by distributing each such occurrence over its ith child node whenever the child node is labelled with $-\vee$ (resp. $+\wedge$), and whenever $\varepsilon_f(i) = 1$ (resp. $\varepsilon_f(i) = \partial$);

(d') Push down, towards variables, occurrences of $+g$ for any $g \in \mathcal{G}$, by distributing each such occurrence over its ith child node whenever the child node is labelled with $+\wedge$ (resp. $-\vee$), and whenever $\varepsilon_g(i) = 1$ (resp. $\varepsilon_g(i) = \partial$).

Applied to PIA-terms, this additional step has the effect of surfacing all occurrences of $+\wedge$ and $-\vee$ up to the root of each PIA-term (so as to form a connected block of nodes including the root which are all labelled $+\wedge$ or all labelled $-\vee$). In this position, these occurrences can be all regarded as Skeleton nodes. Hence, after this step, no occurrences of $+\wedge$ and $-\vee$ will remain in the PIA subterms.[30] Notice that applying this step to (Ω, ε)-inductive terms produces (Ω, ε)-inductive terms each PIA-subterm of which contains at most one ε-critical variable occurrence.

Let Preprocess($\phi \leq \psi$) be the finite set $\{\phi_i \leq \psi_i \mid 1 \leq i \leq n\}$ of inequalities obtained after the exhaustive application of the previous rules. We proceed separately on each of them, and hence, in what follows, we focus only on one element $\phi_i \leq \psi_i$ in Preprocess($\phi \leq \psi$), and we drop the subscript. Next, the following *first approximation rule* is applied *only once* to every inequality in Preprocess($\phi \leq \psi$):

$$\frac{\phi \leq \psi}{\mathbf{i}_0 \leq \phi \quad \psi \leq \mathbf{m}_0}$$

[29] As we will see, in the context of analytic inductive inequalities, occurrences of $+\bot$ or $-\top$ as skeleton nodes and occurrences of $-\bot$ or $+\top$ as PIA-nodes are problematic. Indeed, in the context of the procedure which transforms inequalities into equivalent structural rules (cf. Sections 2.6 and 2.7.1), these logical constants would occur within certain sequents in positions (antecedent or succedent) in which they are not the interpretation of the corresponding structural constant. This fact would block the smooth transformation of logical axioms containing them into structural rules.

[30] PIA subterms $*s$ in which no nodes $+\wedge$ and $-\vee$ occur are referred to as *definite*.

Here, $\mathbf{i}_0$ and $\mathbf{m}_0$ are a nominal and a conominal respectively. The first-approximation step gives rise to systems of inequalities $\{\mathbf{i}_0 \leq \phi_i, \psi_i \leq \mathbf{m}_0\}$ for each inequality in Preprocess($\phi \leq \psi$). Each such system is called an *initial system*, and is now passed on to the reduction-elimination cycle.

Second stage: reduction-elimination cycle. The goal of the reduction-elimination cycle is to eliminate all propositional variables from the systems received from the pre-processing phase. The elimination of each variable is effected by an application of one of the Ackermann rules given below. In order to apply an Ackermann rule, the system must have a specific shape. The adjunction, residuation, approximation, and splitting rules are used to transform systems into this shape. The rules of the reduction-elimination cycle, viz. the adjunction, residuation, approximation, splitting, and Ackermann rules, will be collectively called the *reduction* rules.

Residuation rules. Here below we provide the residuation rules relative to each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ of arity at least 1: for each $1 \leq h \leq n_f$ and each $1 \leq k \leq n_g$:

$$(\varepsilon_f(h) = 1)\ \frac{f(\psi_1, \ldots, \psi_h, \ldots, \psi_{n_f}) \leq \chi}{\psi_h \leq f^\sharp_h(\psi_1, \ldots, \chi, \ldots, \psi_{n_f})} \qquad \frac{f(\psi_1, \ldots, \psi_h, \ldots, \psi_{n_f}) \leq \chi}{f^\sharp_h(\psi_1, \ldots, \chi, \ldots, \psi_{n_f}) \leq \psi_h}\ (\varepsilon_f(h) = \partial)$$

$$(\varepsilon_g(k) = \partial)\ \frac{\chi \leq g(\psi_1, \ldots, \psi_k, \ldots, \psi_{n_g})}{\psi_k \leq g^\flat_k(\psi_1, \ldots, \chi, \ldots, \psi_{n_g})} \qquad \frac{\chi \leq g(\psi_1, \ldots, \psi_k, \ldots, \psi_{n_g})}{g^\flat_k(\psi_1, \ldots, \chi, \ldots, \psi_{n_g}) \leq \psi_k}\ (\varepsilon_g(k) = 1)$$

Approximation rules. Here below we provide the approximation rules[31] relative to each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ of arity at least 1: for each $1 \leq h \leq n_f$ and each $1 \leq k \leq n_g$,

$$(\varepsilon_f(h) = 1)\ \frac{\mathbf{i} \leq f(\psi_1, \ldots, \psi_h, \ldots, \psi_{n_f})}{\mathbf{i} \leq f(\psi_1, \ldots, \mathbf{j}, \ldots, \psi_{n_f}) \quad \mathbf{j} \leq \psi_h}$$

$$(\varepsilon_f(h) = \partial)\ \frac{\mathbf{i} \leq f(\psi_1, \ldots, \psi_h, \ldots, \psi_{n_f})}{\mathbf{i} \leq f(\psi_1, \ldots, \mathbf{n}, \ldots, \psi_{n_f}) \quad \psi_k \leq \mathbf{n}}$$

$$\frac{g(\psi_1, \ldots, \psi_k, \ldots, \psi_{n_g}) \leq \mathbf{m}}{g(\psi_1, \ldots, \mathbf{n}, \ldots, \psi_{n_g}) \leq \mathbf{m} \quad \psi_k \leq \mathbf{n}}\ (\varepsilon_g(k) = 1)$$

$$\frac{g(\psi_1, \ldots, \psi_k, \ldots, \psi_{n_g}) \leq \mathbf{m}}{g(\psi_1, \ldots, \mathbf{j}, \ldots, \psi_{n_g}) \leq \mathbf{m} \quad \mathbf{j} \leq \psi_h}\ (\varepsilon_g(k) = \partial)$$

where the variable $\mathbf{j}$ (resp. $\mathbf{n}$) is a nominal (resp. a conominal). The nominals and conominals introduced by the approximation rules must be *fresh*, i.e. must not already occur in the system before applying the rule.

[31] The version of the approximation rules given in [20, 23, 46] is slightly different from but equivalent to that of the approximation rules reported on here. That formulation is motivated by the need of enforcing the invariance of certain topological properties for the purpose of proving the canonicity of the inequalities on which ALBA succeeds. In this context, we do not need to take these constraints into account, and hence we can take this more flexible version of the approximation rules as primitive, bearing in mind that when proving canonicity one has to take a formulation analogous to that in in [20, 23, 46] as primitive.

2

Ackermann rules. These rules are the core of ALBA, since their application eliminates proposition variables. As mentioned earlier, all the preceding steps are aimed at equivalently rewriting the input system into one of a shape in which the Ackermann rules can be applied. An important feature of Ackermann rules is that they are executed on the whole set of inequalities in which a given variable occurs, and not on a single inequality.

$$\frac{\&\{\alpha_i \leq p \mid 1 \leq i \leq n\} \,\&\, \&\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\} \;\Rightarrow\; \mathbf{i} \leq \mathbf{m}}{\&\{\beta_j(\bigvee_{i=1}^{n} \alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i) \mid 1 \leq j \leq m\} \;\Rightarrow\; \mathbf{i} \leq \mathbf{m}}\ (RAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$; and $\beta_1(p), \ldots, \beta_m(p)$ are positive in p; and $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p.

$$\frac{\&\{p \leq \alpha_i \mid 1 \leq i \leq n\} \,\&\, \&\{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\} \;\Rightarrow\; \mathbf{i} \leq \mathbf{m}}{\&\{\beta_j(\bigwedge_{i=1}^{n} \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^{n} \alpha_i) \mid 1 \leq j \leq m\} \;\Rightarrow\; \mathbf{i} \leq \mathbf{m}}\ (LAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$, $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p.

Third stage: output. If there was some system in the second stage from which not all occurring propositional variables could be eliminated through the application of the reduction rules, then ALBA reports failure and terminates. Else, each system $\{\mathbf{i}_0 \leq \phi_i, \psi_i \leq \mathbf{m}_0\}$ obtained from Preprocess($\varphi \leq \psi$) has been reduced to a system, denoted Reduce($\varphi_i \leq \psi_i$), containing no propositional variables. Let ALBA($\varphi \leq \psi$) be the set of quasi-inequalities

$$\&[\text{Reduce}(\varphi_i \leq \psi_i)] \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0$$

for each $\varphi_i \leq \psi_i \in$ Preprocess($\varphi \leq \psi$).

Notice that all members of ALBA($\varphi \leq \psi$) are free of propositional variables. ALBA returns ALBA($\varphi \leq \psi$) and terminates. An inequality $\varphi \leq \psi$ on which ALBA succeeds will be called an ALBA-*inequality*.

The proof of the following theorem is a straightforward generalization of [20, Theorem 8.1], and hence its proof is omitted.

Theorem 19 (Correctness)**.** *If ALBA succeeds on a $\mathcal{L}_{\mathrm{DLE}}$-inequality $\varphi \leq \psi$, then for every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$,*

$$\mathbb{A} \models \varphi \leq \psi \quad \textit{iff} \quad \mathbb{A} \models \mathrm{ALBA}(\varphi \leq \psi).$$

2.4 Display calculi for $\mathbf{L}_{\mathrm{DLE}}$ and $\mathbf{L}^{*}_{\mathrm{DLE}}$

In the present section, we introduce the basic proof-theoretic environment of our treatment, given by the display calculi $\mathbf{DL}^*$ and $\mathbf{DL}$ for the logics $\mathbf{L}_{\mathrm{DLE}}$ and $\mathbf{L}^{*}_{\mathrm{DLE}}$ associated with any given language $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$. We also show some of their basic properties.

2.4.1 Language and rules

The present subsection is aimed at simultaneously introducing the display calculi $\mathbf{DL}^*$ and $\mathbf{DL}$ for $\mathbf{L}^*_{\mathrm{DLE}}$ and $\mathbf{L}_{\mathrm{DLE}}$, respectively. As is usual of existing logical systems which the present framework intends to capture (e.g. intuitionistic and bi-intuitionistic logics, or modal and tense logics [36]), the languages manipulated by these calculi are built up using *one and the same* set of structural terms, and differ only in the set of operational term constructors. In the tables below, each structural symbol in the upper rows corresponds to one or two logical (or operational) symbols. The idea, which will be made precise later on, is that each structural connective can be interpreted as the corresponding left-hand (resp. right-hand) side logical connective (if it exists) when occurring in antecedent (resp. consequent) position.

- Structural symbols for lattice operators:

Structural symbols	I		$;$		$>$		$<$	
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$\succ\!\!-$	$\rightarrow$	$-\!\!\prec$	$\leftarrow$

- Structural symbols for any $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

Structural symbols	H		K	
Operational symbols	f			g

- Structural symbols for any $f_i^\sharp, g_h^\flat \in (\mathcal{F}^* \cup \mathcal{G}^*) \setminus (\mathcal{F} \cup \mathcal{G})$, and any $0 \leq i \leq n_f$ and $0 \leq h \leq n_g$:

Structural symbols	H_i $(\varepsilon_f(i) = 1)$		H_i $(\varepsilon_f(i) = \partial)$	
Operational symbols		$f_i^\sharp$	$f_i^\sharp$	

Structural symbols	K_h $(\varepsilon_g(h) = 1)$		K_h $(\varepsilon_g(h) = \partial)$	
Operational symbols	$g_h^\flat$			$g_h^\flat$

Some operational symbols above appear against a gray background as a reminder that, unlike their associated structural symbols, they occur only in the language and calculus for $\mathbf{L}^*_{\mathrm{DLE}}$.

Remark 20. If $f \in \mathcal{F}$ and $g \in \mathcal{G}$ form a dual pair,[32] then $n_f = n_g$ and $\varepsilon_f = \varepsilon_g$. Then f and g can be assigned one and the same structural operator, as follows:

Structural symbols	H	
Operational symbols	f	g

Moreover, for any $1 \leq i \leq n_f = n_g$, the residuals $f_i^\sharp$ and $g_i^\flat$ are dual to one another. Hence they can also be assigned one and the same structural connective as follows:

[32]Examples of dual pairs are $(\top, \bot)$, $(\wedge, \vee)$, $(\succ\!\!-, \rightarrow)$, $(-\!\!\prec, \leftarrow)$, and $(\Diamond, \Box)$ where $\Diamond$ is defined as $\neg\Box\neg$.

Structural symbols	$H_i\ (\varepsilon_f(i) = \varepsilon_g(i) = 1)$		$H_i\ (\varepsilon_f(i) = \varepsilon_g(i) = \partial)$	
Operational symbols	$g_i^\flat$	$f_i^\sharp$	$f_i^\sharp$	$g_i^\flat$

2

Definition 21. The display calculi $\mathbf{DL}^*$ and $\mathbf{DL}$ consist of the following display postulates, structural rules, and operational rules:[33]

1. Identity and cut:

$$p \vdash p \qquad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y}$$

2. Display postulates for lattice connectives:

$$\begin{array}{c} X\,;Y \vdash Z \\ \hline\hline Y \vdash X > Z \end{array} \qquad \begin{array}{c} Z \vdash X\,;Y \\ \hline\hline X > Z \vdash Y \end{array}$$

$$\begin{array}{c} X\,;Y \vdash Z \\ \hline\hline X \vdash Z < Y \end{array} \qquad \begin{array}{c} Z \vdash X\,;Y \\ \hline\hline Z < Y \vdash X \end{array}$$

3. Display postulates for $f \in \mathcal{F}$ and $g \in \mathcal{G}$: for any $1 \leq i \leq n_f$ and $1 \leq h \leq n_g$,

$$(\varepsilon_f(i) = 1)\ \begin{array}{c} H\,(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y \\ \hline\hline X_i \vdash H_i\,(X_1, \ldots, Y, \ldots, X_{n_f}) \end{array}$$

$$(\varepsilon_f(i) = \partial)\ \begin{array}{c} H\,(X_1, \ldots, X_i, \ldots, X_{n_f}) \vdash Y \\ \hline\hline H_i\,(X_1, \ldots, Y, \ldots, X_{n_f}) \vdash X_i \end{array}$$

$$\begin{array}{c} Y \vdash K\,(X_1 \ldots, X_h, \ldots X_{n_g}) \\ \hline\hline K_h\,(X_1, \ldots, Y, \ldots, X_{n_g}) \vdash X_h \end{array}\ (\varepsilon_g(h) = 1)$$

$$\begin{array}{c} Y \vdash K\,(X_1, \ldots, X_h, \ldots, X_{n_g}) \\ \hline\hline X_h \vdash K_h\,(X_1, \ldots, Y, \ldots, X_{n_g}) \end{array}\ (\varepsilon_g(h) = \partial)$$

Notice that the display postulates for all the connectives in $\mathcal{F}^* \cup \mathcal{G}^*$ are derivable from the display postulates above. The rules for the case of connectives in the dual pairs are obtained by replacing K for H in the corresponding rules above.

4. Necessitation for $f \in \mathcal{F}$ and $g \in \mathcal{G}$: for any $1 \leq k \leq n_f$ and $1 \leq h \leq n_g$,

$$(\varepsilon_f(k) = 1)\ \frac{\Big(X_i \vdash Y_i \quad Y_j \vdash X_j \mid i \neq k, 1 \leq i, j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\Big) \qquad X_k \vdash \mathrm{I}_k}{X_k \vdash H_k(X_1, \ldots, X_{k-1}, \mathrm{I}, X_{k+1}, \ldots, X_{n_f})}$$

$$(\varepsilon_f(k) = \partial)\ \frac{\Big(X_i \vdash Y_i \quad Y_j \vdash X_j \mid j \neq k, 1 \leq i, j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\Big) \qquad \mathrm{I}_k \vdash X_k}{H_k(X_1, \ldots, X_{k-1}, \mathrm{I}, X_{k+1}, \ldots, X_{n_f}) \vdash X_k}$$

[33] The display calculus associated with the basic DLE-logic $\mathbf{L}^{\underline{*}}$ (cf. footnote 18) in the expanded language $\mathcal{L}^*_{\mathrm{DLE}}$ is denoted by $\mathbf{DL}^{\underline{*}}$, and is defined by instantiating the definition of $\mathbf{DL}$ to the expanded language $\mathcal{L}^*_{\mathrm{DLE}}$.

$$(\varepsilon_g(h) = 1)\ \frac{\Big(X_j \vdash Y_j \quad Y_i \vdash X_i \mid i \neq h, 1 \leq i, j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\Big) \qquad \mathrm{I}_h \vdash X_h}{K_h(X_1, \ldots, X_{h-1}, \mathrm{I}, X_{h+1}, \ldots, X_{n_g}) \vdash X_h}$$

$$(\varepsilon_g(h) = \partial)\ \frac{\Big(X_j \vdash Y_j \quad Y_i \vdash X_i \mid j \neq h, 1 \leq i, j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\Big) \qquad X_h \vdash \mathrm{I}_h}{X_h \vdash K_h(X_1, \ldots, X_{h-1}, \mathrm{I}, X_{h+1}, \ldots, X_{n_g})}$$

5. Structural rules encoding the distributive lattice axiomatization:

$$\mathrm{I}_L\ \frac{X \vdash Y}{\mathrm{I}\,; X \vdash Y} \qquad \frac{Y \vdash X}{Y \vdash X\,; \mathrm{I}}\ \mathrm{I}_R \qquad E_L\ \frac{Y\,; X \vdash Z}{X\,; Y \vdash Z} \qquad \frac{Z \vdash X\,; Y}{Z \vdash Y\,; X}\ E_R$$

$$W_L\ \frac{Y \vdash Z}{X\,; Y \vdash Z} \qquad \frac{Z \vdash Y}{Z \vdash Y\,; X}\ W_R \qquad C_L\ \frac{X\,; X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash X\,; X}{Y \vdash X}\ C_R$$

$$A_L\ \frac{X\,; (Y\,; Z) \vdash W}{(X\,; Y)\,; Z \vdash W} \qquad \frac{W \vdash (Z\,; Y)\,; X}{W \vdash Z\,; (Y\,; X)}\ A_R$$

6. Introduction rules for the propositional (BDL and bi-intuitionistic) connectives:

$$\bot_L\ \frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot}\ \bot_R$$

$$\top_L\ \frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top}\ \top_R$$

$$\wedge_L\ \frac{A\,; B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X\,; Y \vdash A \wedge B}\ \wedge_R$$

$$\vee_L\ \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X\,; Y} \qquad \frac{X \vdash A\,; B}{X \vdash A \vee B}\ \vee_R$$

$$\rightarrow_L\ \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B}\ \rightarrow_R$$

$$\succ_L\ \frac{A > B \vdash Z}{A \succ B \vdash Z} \qquad \frac{A \vdash X \qquad Y \vdash B}{X > Y \vdash A \succ B}\ \succ_R$$

$$\leftarrow_L\ \frac{X \vdash A \qquad B \vdash Y}{A \leftarrow B \vdash X < Y} \qquad \frac{X \vdash A < B}{X \vdash A \leftarrow B}\ \leftarrow_R$$

$$\prec_L\ \frac{A < B \vdash Z}{A \prec B \vdash Z} \qquad \frac{A \vdash X \qquad Y \vdash B}{X < Y \vdash A \prec B}\ \prec_R$$

In the presence of the exchange rules E_L and E_R, the structural connective $<$ and the corresponding operational connectives $\prec$ and $\leftarrow$ are redundant.

7. Introduction rules for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$f_L \frac{H(A_1,\ldots,A_{n_f}) \vdash X}{f(A_1,\ldots,A_{n_f}) \vdash X} \qquad \frac{X \vdash K(A_1,\ldots,A_{n_g})}{X \vdash g(A_1,\ldots,A_{n_g})} g_R$$

$$f_R \frac{\left(X_i \vdash A_i \quad A_j \vdash X_j \mid 1 \leq i,j \leq n_f, \varepsilon_f(i) = 1 \text{ and } \varepsilon_f(j) = \partial\right)}{H(X_1,\ldots,X_{n_f}) \vdash f(A_1,\ldots,A_n)}$$

$$g_L \frac{\left(A_i \vdash X_i \quad X_j \vdash A_j \mid 1 \leq i,j \leq n_g, \varepsilon_g(i) = 1 \text{ and } \varepsilon_g(j) = \partial\right)}{g(A_1,\ldots,A_{n_g}) \vdash K(X_1,\ldots,X_n)}$$

In particular, if f and g are 0-ary (i.e. they are constants), the rules f_R and g_L above reduce to the axioms (0-ary rule) $H \vdash f$ and $g \vdash K$.

8. Only for DL^*, introduction rules for each $f_i^\sharp, g_h^\flat \in (\mathcal{F}^* \cup \mathcal{G}^*) \setminus (\mathcal{F} \cup \mathcal{G})$:

(a) If $\varepsilon_f(i) = 1$ and $\varepsilon_g(h) = 1$,

$$g^\flat_{h\,L} \frac{K_h(A_1,\ldots,A_{n_g}) \vdash X}{g_h^\flat(A_1,\ldots,A_{n_f}) \vdash X} \qquad \frac{X \vdash H_i(A_1,\ldots,A_{n_g})}{X \vdash f_i^\sharp(A_1,\ldots,A_{n_g})} f^\sharp_{i\,R}$$

$$g^\flat_{h\,R} \frac{\left(X_\ell \vdash A_\ell \quad A_m \vdash X_m \mid 1 \leq \ell,m \leq n_g, \varepsilon_{g_h^\flat}(\ell) = 1 \text{ and } \varepsilon_{g_h^\flat}(m) = \partial\right)}{K_h(X_1,\ldots,X_{n_g}) \vdash g_h^\flat(A_1,\ldots,A_{n_g})}$$

$$f^\sharp_{i\,L} \frac{\left(A_\ell \vdash X_\ell \quad X_m \vdash A_m \mid 1 \leq \ell,m \leq n_g, \varepsilon_{f_i^\sharp}(\ell) = 1 \text{ and } \varepsilon_{f_i^\sharp}(m) = \partial\right)}{f_i^\sharp(A_1,\ldots,A_{n_g}) \vdash H_i(X_1,\ldots,X_{n_g})}$$

(b) If $\varepsilon_f(i) = \partial$ and $\varepsilon_g(h) = \partial$,

$$f^\sharp_{i\,L} \frac{H_i(A_1,\ldots,A_{n_f}) \vdash X}{f_i^\sharp(A_1,\ldots,A_{n_f}) \vdash X} \qquad \frac{X \vdash K_h(A_1,\ldots,A_{n_g})}{X \vdash g_h^\flat(A_1,\ldots,A_{n_g})} g^\flat_{h\,R}$$

$$f^\sharp_{i\,R} \frac{\left(X_\ell \vdash A_\ell \quad A_m \vdash X_m \mid 1 \leq \ell,m \leq n_f, \varepsilon_{f_i^\sharp}(\ell) = 1 \text{ and } \varepsilon_{f_i^\sharp}(m) = \partial\right)}{H_i(X_1,\ldots,X_{n_f}) \vdash f_i^\sharp(A_1,\ldots,A_{n_f})}$$

$$g^\flat_{h\,L} \frac{\left(A_\ell \vdash X_\ell \quad X_m \vdash A_m \mid 1 \leq \ell,m \leq n_g, \varepsilon_{g_h^\flat}(\ell) = 1 \text{ and } \varepsilon_{g_h^\flat}(m) = \partial\right)}{g_h^\flat(A_1,\ldots,A_{n_g}) \vdash K_h(X_1,\ldots,X_{n_g})}$$

A display calculus enjoys the *full display property* (resp. the *relativized display property*) if for every (derivable) sequent $X \vdash Y$ and every substructure Z of either X or Y, the sequent $X \vdash Y$ can be equivalently transformed, using the rules of the system, into a sequent which is either of the form $Z \vdash W$ or of the form $W \vdash Z$, for some structure W. A routine check will show that the display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ both enjoy the relativized display property, and moreover, if $\mathcal{F}$ and $\mathcal{G}$ are such that for every $f \in \mathcal{F}$ the dual of f is in $\mathcal{G}$ and for every $g \in \mathcal{G}$ the dual of g is in $\mathcal{F}$, then $\mathbf{DL}$ and $\mathbf{DL}^*$ both enjoy the full display property. The proof of these facts is omitted.

Proposition 22. *The display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ enjoy the relativized display property, and under the assumption above on $\mathcal{F}$ and $\mathcal{G}$ they enjoy the full display property.*

Structural connective	if in precedent position	if in succedent position	
I	$\top$	$\bot$	
$A\,;B$	$A \wedge B$	$A \vee B$	
$A > B$	$A \succ\!\!- B$	$A \rightarrow B$	
$H(\overline{A})$	$f(\overline{A})$		
$K(\overline{A})$		$g(\overline{A})$	
$H_i(\overline{A})$		$f_i^\sharp(\overline{A})$	if $\varepsilon_f(i) = 1$
$H_i(\overline{A})$	$f_i^\sharp(\overline{A})$		if $\varepsilon_f(i) = \partial$
$K_h(\overline{A})$	$g_h^\flat(\overline{A})$		if $\varepsilon_g(h) = 1$
$K_h(\overline{A})$		$g_h^\flat(\overline{A})$	if $\varepsilon_g(h) = \partial$

Table 2.2: Translation of structural connectives into logical connectives

2.4.2 Soundness, completeness, conservativity

Soundness. Let us expand on how to interpret structures and sequents in the language manipulated by the calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ in any perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ (cf. Footnote 28). Structures will be translated into formulas, and formulas will be interpreted as elements of $\mathbb{A}$. In order to translate structures as formulas, structural terms need to be translated as formulas, as is specified in Definition 24 below. To this effect, any given occurrence of a structural connective in a sequent is translated as (one or the other of) its associated logical connective(s), as reported in Table 2.2, provided its operational counterpart relative to its position (antecedent or succedent) exists. Clearly, not all structural terms will in general have a translation as formulas. This motivates the following definition:

Definition 23. A structural term S is *left-sided* (resp. *right-sided*) if in its positive (resp. negative) signed generation tree,[34] every positive node is labelled with a structural connective which is associated with a logical connective when occurring in antecedent position, and every negative node is labelled with a structural connective which is associated with a logical connective when occurring in succedent position.

Clearly, if every structural connective is associated with some logical connectives both when occurring in antecedent position and when occurring in succedent position, as is the case e.g. when $\mathcal{F}$ and $\mathcal{G}$ bijectively correspond via conjugation, every structural term is both left-sided and right-sided.

Definition 24. For every left-sided (resp. right-sided) structural term S, let $l(S)$ (resp. $r(S)$) denote the formula associated with S and defined inductively according to Table 2.2.

[34]Signed generation trees of structural terms are defined analogously to signed generation trees of logical terms. Logical formulas label the leaves of the signed generation trees of structural terms.

Structural sequents $S \vdash T$ such that S is left-sided and T is right-sided are those translatable as formula-sequents $l(S) \vdash r(T)$. These sequents in turn are interpreted in any $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ in the standard way. Hence, for any assignment $v : \mathsf{AtProp} \to \mathbb{A}$, we denote by $[\![\cdot]\!]_v$ the unique homomorphic extension of v to the formula algebra, interpret sequents $l(S) \vdash r(T)$ as inequalities $[\![l(S)]\!]_v \leq [\![r(T)]\!]_v$ and rules $(S_i \vdash T_i \mid i \in I)/S \vdash T$ as implications of the form "if $[\![l(S_i)]\!]_v \leq [\![r(T_i)]\!]_v$ for every $i \in I$, then $[\![l(S)]\!]_v \leq [\![r(T)]\!]_v$".

Under these stipulations, it is routine to check that all axioms and rules of the calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ are satisfied under any assignment. Hence, it is immediate to prove, by induction on the depth of the derivation tree, that

Proposition 25. *If $S \vdash T$ is $\mathbf{DL}$-derivable (resp. $\mathbf{DL}^*$-derivable), then S is left-sided, T is right-sided and $[\![l(S)]\!]_v \leq [\![r(T)]\!]_v$ is satisfied on every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$ and under any assignment v :* AtProp $\to \mathbb{A}$.

Completeness. At the end of Section 2.3.2, we outlined the proof of the completeness of $\mathbf{L}_{\mathrm{DLE}}$ and $\mathbf{L}^*_{\mathrm{DLE}}$ w.r.t. perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras. Hence, to show that $\mathbf{DL}$ and $\mathbf{DL}^*$ are complete w.r.t. perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras, it is enough to show that the axioms and rules of $\mathbf{L}_{\mathrm{DLE}}$ (resp. $\mathbf{L}^*_{\mathrm{DLE}}$) are derivable in $\mathbf{DL}$ (resp. $\mathbf{DL}^*$). These verifications are routine. For instance, let $f \in \mathcal{F}$ be binary and s.t. $\varepsilon_f = (1, \partial)$. Then the following sequents are derivable in $\mathbf{DL}$:

$$f_1^\sharp(A,C) \wedge f_1^\sharp(B,C) \vdash f_1^\sharp(A \wedge B, C) \qquad f_1^\sharp(A,B) \wedge f_1^\sharp(A,C) \vdash f_1^\sharp(A, B \wedge C)$$

$$f_2^\sharp(A \vee B,C) \vdash f_2^\sharp(A,C) \vee f_2^\sharp(B,C) \qquad f_2^\sharp(A,B \wedge C) \vdash f_2^\sharp(A,B) \vee f_2^\sharp(A,C).$$

By way of example, a derivation for $f_1^\sharp(A,C) \wedge f_1^\sharp(B,C) \vdash f_1^\sharp(A \wedge B, C)$ is reported below.

$$\begin{prooftree}
\AxiomC{$A \vdash A$}
\AxiomC{$C \vdash C$}
\BinaryInfC{$f_1^\sharp(A,C) \vdash H_1[A,C]$}
\UnaryInfC{$f_1^\sharp(A,C) \,;\, f_1^\sharp(B,C) \vdash H_1[A,C]$}
\UnaryInfC{$f_1^\sharp(A,C) \wedge f_1^\sharp(B,C) \vdash H_1[A,C]$}
\UnaryInfC{$H[f_1^\sharp(A,C) \wedge f_1^\sharp(B,C), C] \vdash A$}
\AxiomC{$B \vdash B$}
\AxiomC{$C \vdash C$}
\BinaryInfC{$f_1^\sharp(B,C) \vdash H_1[B,C]$}
\UnaryInfC{$f_1^\sharp(A,C) \,;\, f_1^\sharp(B,C) \vdash H_1[B,C]$}
\UnaryInfC{$f_1^\sharp(A,C) \wedge f_1^\sharp(B,C) \vdash H_1[B,C]$}
\UnaryInfC{$H[f_1^\sharp(A,C) \wedge f_1^\sharp(B,C), C] \vdash B$}
\BinaryInfC{$H[f_1^\sharp(A,C) \wedge f_1^\sharp(B,C), C] \,;\, H[f_1^\sharp(A,C) \wedge f_1^\sharp(B,C), C] \vdash A \wedge B$}
\UnaryInfC{$H[f_1^\sharp(A,C) \wedge f_1^\sharp(B,C), C] \vdash A \wedge B$}
\UnaryInfC{$f_1^\sharp(A,C) \wedge f_1^\sharp(B,C) \vdash H_1[A \wedge B, C]$}
\UnaryInfC{$f_1^\sharp(A,C) \wedge f_1^\sharp(B,C) \vdash f_1^\sharp(A \wedge B, C)$}
\end{prooftree}$$

Conservativity. Let $A \vdash B$ be a $\mathbf{DL}^*$-derivable sequent in the language of $\mathbf{DL}$ (i.e., no operational connective in $(\mathcal{F}^* \cup \mathcal{G}^*) \setminus (\mathcal{F} \cup \mathcal{G})$ occurs in the sequent). Hence, by the soundness of $\mathbf{DL}^*$ w.r.t. perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras, the inequality $A \leq B$ is valid on these algebras. By the completeness of $\mathbf{L}_{\mathrm{DLE}}$ w.r.t. perfect $\mathcal{L}_{\mathrm{DLE}}$-algebras, the inequality $A \leq B$ is derivable in $\mathbf{L}_{\mathrm{DLE}}$, which implies, by the syntactic completeness of $\mathbf{DL}$ w.r.t. $\mathbf{L}_{\mathrm{DLE}}$, that $A \vdash B$ is $\mathbf{DL}$-derivable, as required.

2.4.3 Cut elimination and subformula property

The calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ are proper display calculi, and hence, by Theorem 3, they enjoy Belnap-style cut elimination and subformula property.

Theorem 26. *The calculi* $\mathbf{DL}$ *and* $\mathbf{DL}^*$ *are proper display calculi.*

Proof. The conditions C_1–C_7 can be straightforwardly verified by inspection on the rules. As to C_8, cf. Fact 67 in the Section 2.11. □

2.4.4 Properly displayable $\mathcal{L}_{\mathrm{DLE}}$-logics

Definition 27. For any DLE-language $\mathcal{L}_{\mathrm{DLE}}$, an $\mathcal{L}_{\mathrm{DLE}}$-logic (cf. Definition 10) is *properly displayable* (resp. *specially displayable*) if it is exactly captured by a display calculus obtained by adding analytic rules (resp. special rules)—cf. Definition 4 (resp. Definition 6)—to the calculus $\mathbf{DL}$ for $\mathcal{L}_{\mathrm{DLE}}$.

2.5 Primitive inequalities and special rules

In [39, Theorem 16], Kracht showed that primitive formulas of basic normal/tense modal logic on a classical propositional base can be equivalently transformed into (a set of) special structural rules satisfying the defining conditions of proper display calculi (cf. Subsection 2.2.2). In the present section, we extend this result to any language $\mathcal{L}_{\mathrm{DLE}}$. We base this extension on the notion of primitive inequalities. Namely, in Subsection 2.5.1, we introduce the class of (left- and right-)primitive inequalities in any language $\mathcal{L}_{\mathrm{DLE}}$ (cf. Definition 28), and show (cf. Lemma 32) that these inequalities can be equivalently (and effectively) transformed into special structural rules (cf. in the restricted sense of Definition 6). We also show that special structural rules can be equivalently (and effectively) transformed into primitive inequalities. In Subsection 2.5.2, we identify the crucial order-theoretic feature induced by the syntactic shape of definite primitive inequalities (cf. Lemma 35), on the basis of which a special ALBA-type reduction for definite primitive inequalities is given (cf. Proposition 37). In Subsection 2.5.3, we take stock of the previous results and outline the way they will be further extended in Section 2.6.

2.5.1 Left- and right-primitive inequalities and special rules

In what follows, for each connective $f \in \mathcal{F}$ and $g \in \mathcal{G}$, we will write $f(\vec{p}, \vec{q})$ and $g(\vec{p}, \vec{q})$, stipulating that $\varepsilon_f(p) = \varepsilon_g(p) = 1$ for all p in $\vec{p}$, and $\varepsilon_f(q) = \varepsilon_g(q) = \partial$ for all q in $\vec{q}$. Moreover, we write e.g. $f(\vec{u}/\vec{p}, \vec{v}/\vec{q})$ to indicate that the arrays $\vec{u}$ and $\vec{p}$ (resp. $\vec{v}$ and $\vec{q}$) have the same length n (resp. m) and that, for each $1 \leq i \leq n$ (resp. for each $1 \leq j \leq m$), the formula u_i (resp. v_j) has been uniformly substituted in f for the variable p_i (resp. q_j).

Definition 28 (Primitive inequalities)**.** For any language $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, the *left-primitive* $\mathcal{L}_{\mathrm{DLE}}$-formulas ψ and *right-primitive* $\mathcal{L}_{\mathrm{DLE}}$-formulas ϕ are defined by simultaneous recursion as follows:

$$\psi := p \mid \top \mid \vee \mid \wedge \mid f(\vec{\psi}/\vec{p}, \vec{\phi}/\vec{q}),$$

$$\phi := p \mid \bot \mid \wedge \mid \vee \mid g(\vec{\phi}/\vec{p}, \vec{\psi}/\vec{q}).$$

A left-primitive (resp. right-primitive) $\mathcal{L}_{\mathrm{DLE}}$-formula is *definite* if there are no occurrences of $+\vee$ or $-\wedge$ (resp. $+\wedge$ or $-\vee$) in its positive generation tree. An $\mathcal{L}_{\mathrm{DLE}}$-inequality $s_1 \leq s_2$ is *left-primitive* (resp. *right-primitive*) if both s_1 and s_2 are left-primitive (resp. right-primitive) formulas and moreover:

1. each proposition variable in s_1 (resp. s_2) occurs at most once, in which case we say that s_1 (resp. s_2) is *scattered*.
2. s_1 and s_2 have the same order-type relative to the variables they have in common.
3. s_2 (resp. s_1) is ε-uniform w.r.t. some order-type ε on its occurring variables.

Sometimes, the scattered side of a primitive inequality will be referred to as its *head* and the other one as its *tail*.

It immediately follows from the axiomatization of the basic logic $\mathbf{L}_{\mathrm{DLE}}$ that left-primitive (resp. right-primitive) $\mathcal{L}_{\mathrm{DLE}}$-formulas can be equivalently written in disjunction (resp. conjunction) normal form of definite left-primitive (resp. right-primitive) formulas. The condition that the head of a primitive inequality is scattered implies that the head is ε-uniform for the order type ε of its occurring variables in e.g. its positive generation tree. Notice that the definition above does not exclude the possibility that some variables which do not occur in the head of a primitive inequality might occur in its tail. However, item 3 of the definition above requires the tail to be uniform in these variables. This observation will be helpful later on in the treatment of these cases (cf. proof of Lemma 32).

Remark 29. The notion of primitive terms provides the first and most basic connection of unified correspondence theory to the characterization problem of the properly displayable DLE-logics (cf. Definition 27). Indeed, it can be easily verified by direct inspection that left-primitive terms are both positive Skeleton-terms and negative PIA-terms (cf. discussion after Definition 16), and right-primitive terms are both positive PIA-terms and negative Skeleton-terms. In principle, not all positive PIA-terms (or negative Skeleton terms) are right-primitive, since $-\bot$ and $+\top$ are allowed to occur in their positive generation tree, while they are not allowed to occur in $+s$ for any right-primitive term s. Likewise, not all negative PIA-terms (or positive Skeleton-terms) are left-primitive, since $+\bot$ and $-\top$ are allowed to occur in their positive generation tree, while they are not allowed to occur in $+s$ for any left-primitive term s.

Example 30. Let $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$ be s.t. $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\rightarrow, \Box\}$. Of the following Fischer Servi inequalities (cf. [53, 54]),

$$\Diamond(q \rightarrow p) \leq \Box q \rightarrow \Diamond p \qquad \Diamond q \rightarrow \Box p \leq \Box(q \rightarrow p),$$

the second one is right-primitive, whereas the first one is neither right- nor left-primitive.

Early on, in Definition 24, left-sided and right-sided structural terms were associated with formulas. In fact, it is not difficult to show, by induction on the shape of left-sided and right-sided structural terms, that the set of definite left-primitive (resp. right-primitive) formulas (cf. Definition 28) is exactly the image of the map l (resp. r). The inverse maps of l and r are defined as follows:

Definition 31 (Structures associated with definite primitive formulas)**.** Any definite left-primitive formula s and any definite right-primitive formula t is associated with structures $S = l^{-1}(s)$ and $T = r^{-1}(t)$ respectively, by the following simultaneous induction on s and t.

if $s = p$ then $S := \zeta(p)$	if $t = p$ then $T := \zeta(p)$
if $s = \top$ then $S := \mathrm{I}$	if $t = \bot$ then $T := \mathrm{I}$
if $s = s_1 \wedge s_2$ then $S = S_1 \ ; \ S_2$	if $t = t_1 \vee t_2$ then $T = T_1 \ ; \ T_2$
if $s = f(\vec{s'}/\vec{p}, \vec{t'}/\vec{q})$ then $S := H(\vec{S'}, \vec{T'})$	if $t = g(\vec{t'}/\vec{p}, \vec{s'}/\vec{q})$ then $T := K(\vec{T'}, \vec{S'})$

where ζ is an injective map from AtProp to the set of structural variables.

Lemma 32. *Every left-primitive (resp. right-primitive) inequality $s \leq t$ is semantically equivalent to a set of special structural rules in the display calculus* $\mathbf{DL}$*.*

Proof. Assume that $s \leq t$ is right-primitive, and that both s and t are in conjunction normal form, that is, $s = \bigwedge_{i \leq n} s_i$ and $t = \bigwedge_{j \leq k} t_j$ where s_i and t_j are definite right-primitive formulas for any $i \leq n$ and any $j \leq k$. If some variables occur in s which do not occur in t, then item 3 of Definition 28 guarantees that s, and hence the whole inequality, is uniform in these variables. Hence, as discussed in Footnote 23, the inequality $s \leq t$ can be transformed into some inequality $s' \leq t$ in which each positive (resp. negative) occurrence of these variables has been suitably replaced by $\top$ (resp. $\bot$). The assumption that each term s_i is definite right-primitive implies that each term in which the substitution has been effected is equivalent to $\top$, and hence can be removed from the conjunction normal form. If the substitution has been effected on each s_i, then the inequality $s \leq t$ is equivalent to $\top \leq t$, which can be equivalently transformed into the 0-ary rule $\mathrm{I} \vdash T$, where $T := r^{-1}(t)$ as in Definition 31, which is immediately verified to be analytic. Assume now that all the variables which occur in s occur as well in t. The following chain of equivalences is sound on any $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$:

$$\begin{array}{lll} & \forall \vec{p}[s \leq t] & \\ \text{iff} & \forall \vec{p} \forall p[p \leq s \Rightarrow p \leq t] & (p \text{ fresh proposition variable}) \\ \text{iff} & \forall \vec{p} \forall p[p \leq \bigwedge_{i \leq n} s_i \Rightarrow p \leq \bigwedge_{j \leq k} t_j] & \\ \text{iff} & \forall \vec{p} \forall p[\&_{i \leq n}\, p \leq s_i \Rightarrow \&_{j \leq k}\, p \leq t_j] & \\ \text{iff} & \&_{j \leq k} \left(\forall \vec{p} \forall p[\&_{i \leq n}\, p \leq s_i \Rightarrow p \leq t_j]\right). & \end{array}$$

Recalling the definition of satisfaction of rules of $\mathbf{DL}$ on algebras (cf. Subsection 2.4.2), the chain of equivalences above proves that for every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$, the validity of $s \leq t$ on $\mathbb{A}$ is equivalent to the simultaneous validity on $\mathbb{A}$ of the following rules:

$$\left(\frac{(X \vdash S_i \mid i \leq n)}{X \vdash T_j} \quad \mid j \leq k \right)$$

where for every $i \leq n$ and $j \leq k$, the structures S_i and T_j are the ones associated with s_i and t_j respectively, as indicated in Definition 31. With a similar argument, it can be shown that if $s \leq t$ is left-primitive and both s and t are in disjunction normal form (that is, $s = \bigvee_{i\leq n} s_i$ and $t = \bigvee_{j\leq k} t_j$ where s_i and t_j are definite left-primitive formulas for any $i \leq n$ and any $j \leq k$), the validity of $s \leq t$ on $\mathbb{A}$ is equivalent to the simultaneous validity on $\mathbb{A}$ of the following rules:

$$\Bigg(\frac{(T_j \vdash Y \mid j \leq k)}{S_i \vdash Y} \quad \mid i \leq n \Bigg),$$

where for every $i \leq n$ and $j \leq k$, the structures S_i and T_j are the ones associated with s_i and t_j respectively, as indicated in Definition 31. It remains to be shown that these rules are analytic, i.e. that they satisfy conditions C_1-C_7. Condition C_1 follows from the assumption that all the variables which occur in the tail occur as well in the head. C_5 imposes restrictions on the introduction of formulas, and hence is vacuously true on structural rules. Conditions C_2, C_6, and C_7 are immediate. Condition C_3 follows from the requirement that every proposition variable occurs only once in the head of a primitive inequality. Finally, condition C_4 follows from the requirement that the formulas have the same order-type on the variables they have in common. □

Notice that the rules obtained from primitive inequalities in the way described above have the following special cases:

- if $s \leq t$ is a left-primitive (resp. right-primitive) inequality such that t (resp. s) is definite, then the corresponding set of rules consists of *unary* rules;
- if $s \leq t$ is a left-primitive (resp. right-primitive) inequality $s \leq t$ such that s (resp. t) is definite, then the corresponding set of rules consists of *one single* rule;
- if $s \leq t$ is a left-primitive (resp. right-primitive) inequality $s \leq t$ such that both s and t are definite, then the the corresponding set of rules consists of *one single unary* rule.

The other direction is also true:

Lemma 33. *Every special structural rule in the language of* $\mathbf{DL}$ *is semantically equivalent to some left-primitive or right-primitive inequality.*[35]

Proof. Let us treat the case in which the special rule is of the form

$$\frac{(X \vdash S_i \mid i \leq n)}{X \vdash T} \rho \quad ,$$

where X does not occur in any S_i nor in T. Let $l(X) = p$ and let $\vec{q}$ be the variables that appear in $r(S_i)$ and $r(T)$. As discussed in Section 2.4.2, the semantic validity of the rule above can be expressed as follows:

$$\forall p \forall \vec{q} [\mathop{\&}_{1\leq i\leq n} (p \leq r(S_i)) \implies p \leq r(T)].$$

[35] Notice that translating rules as axioms of the original DLE-language instead of as inequalities (as done e.g. in [10, Theorem 4.5]) is possible only if the basic logic has an implication-type connective with modus ponens. In the present logical setting this is not possible in general.

The fact that X does not occur in any S_i nor in T implies that p does not occur in $r(S_i)$ and $r(T)$. Then the above quasi-inequality can be equivalently rewritten as follows:

$$\forall \vec{q}[\bigwedge_{i\leq n} r(S_i) \leq r(T)].$$

The inequality between brackets is right-primitive: indeed, similarly to what has been discussed above Definition 31 it is not difficult to show that $\bigwedge_{i\leq n} r(S_i)$ and $r(T)$ are right-primitive terms. Moreover, the assumption that ρ is special implies that it is analytic, and hence ρ satisfies conditions C_1-C_7. Condition C_3 guarantees that $r(T)$ is scattered and hence item 1. of Definition 28 is satisfied. Condition C_1 guarantees Condition C_4 guarantees that $\bigwedge_{i\leq n} r(S_i)$ and $r(T)$ are uniform w.r.t. the same order-type and hence items 2. and 3. are satisfied. ☐

Example 34. Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\to, \Box\}$. The logical connectives of the display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ associated with the basic $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$-logic can be represented synoptically as follows:

Structural symbols	I		;		$>$		$\circ$		$\bullet$	
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$\succ\!\!-$	$\to$	$\Diamond$	$\Box$	$\blacklozenge$	$\blacksquare$

Below we illustrate schematically how to apply the procedure above to the Fischer Servi inequality $\Diamond q \to \Box p \leq \Box(q \to p)$, which is right-primitive (cf. Example 30):

$$\Diamond q \to \Box p \leq \Box(q \to p) \quad \rightsquigarrow \quad \frac{x \vdash \Diamond q \to \Box p}{x \vdash \Box(q \to p)} \quad \rightsquigarrow \quad \frac{X \vdash \circ Z > \circ Y}{X \vdash \circ(Z > Y)}.$$

2.5.2 Order-theoretic properties of primitive inequalities

The following lemma identifies the most important order-theoretic feature induced by the syntactic shape of primitive inequalities. Notice that, by definition, any scattered term s is monotone, hence s can be associated with an order-type on its variables, which is denoted ε_s. In these cases, we will sometimes write $s(\vec{p}, \vec{q})$ with the convention that $\varepsilon_s(p) = 1$ for any p in $\vec{p}$, and $\varepsilon_s(q) = \partial$ for any q in $\vec{q}$. Also, in what follows we will find it convenient to represent an array $\vec{s} = (s_1, \ldots, s_n)$ as $(\overrightarrow{s_{-i}}, s_i)$ for $1 \leq i \leq n$, where $\overrightarrow{s_{-i}} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Finally, we write e.g. $s(\vec{u}/\vec{p})$ to indicate that the arrays $\vec{u}$ and $\vec{p}$ have the same length n and that, for each $1 \leq i \leq n$, the formula u_i has been uniformly substituted in s for the variable p_i.

Lemma 35. *For every language $\mathcal{L}_{\mathrm{DLE}}$, any definite and scattered left-primitive (resp. right-primitive) $\mathcal{L}_{\mathrm{DLE}}$-term s and any $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$, the term function $s^{\mathbb{A}} : \mathbb{A}^{\varepsilon_s} \to \mathbb{A}$ is a (dual) operator, and if $\mathbb{A}$ is perfect, then $s^{\mathbb{A}} : \mathbb{A}^{\varepsilon_s} \to \mathbb{A}$ is a complete (dual) operator.*[36]

[36] An operation on a lattice $\mathbb{A}$ is an *operator* (resp. a *dual operator*) if it preserves finite joins (resp. meets) in each coordinate. Notice that this condition includes the preservation of the empty join $\bot$ (resp. the empty meet $\top$). An operation on a complete lattice is a *complete operator* (resp. a *complete dual operator*) if it preserves all joins (resp. meets) in each coordinate.

Proof. By induction on the complexity of s. If s is a constant or a proposition variable, the verification of the statement is immediate. Let $s = f(\vec{u}, \vec{v}) = f(\vec{u}/\vec{p}, \vec{v}/\vec{q})$. The assumptions that s is definite, left-primitive and scattered and those on the order-type of s imply that each u in $\vec{u}$ is definite, left-primitive, and scattered, and each v in $\vec{v}$ is definite, right-primitive, and scattered. Hence, by induction hypothesis, the term function $u^{\mathbb{A}} : \mathbb{A}^{\varepsilon_s} \to \mathbb{A}$ is an operator for each u, and $v^{\mathbb{A}} : \mathbb{A}^{\varepsilon_s^{\partial}} \to \mathbb{A}$ is a dual operator for each v. Let r be a variable occurring in s, and assume that the only occurrence of r belongs to a subterm u_i. If $\varepsilon_s(r) = 1$, then $\varepsilon_{u_i}(r) = 1$, and hence

$$
\begin{array}{rcll}
f(\vec{u}_{-i}, u_i[(\bigvee_{j\in I} \phi_j)/r], \vec{v}) & = & f(\vec{u}_{-i}, (\bigvee_{j\in I} u_i[\phi_j/r]), \vec{v}) & \text{(ind. hypothesis)} \\
 & = & \bigvee_{j\in I} f(\vec{u}_{-i}, u_i[\phi_j/r], \vec{v}). &
\end{array}
$$

If $\varepsilon_s(r) = \partial$, then $\varepsilon_{u_i}(r) = \partial$, hence

$$
\begin{array}{rcll}
f(\vec{u}_{-i}, u_i[(\bigwedge_{j\in I} \phi_j)/r], \vec{v}) & = & f(\vec{u}_{-i}, (\bigvee_{j\in I} u_i[\phi_j/r]), \vec{v}) & \text{(ind. hypothesis)} \\
 & = & \bigvee_{j\in I} f(\vec{u}_{-i}, u_i[\phi_j/r], \vec{v}). &
\end{array}
$$

The remaining cases can be proven with similar arguments. ☐

Corollary 36. *The following rules are sound and invertible in perfect DLEs, and derivable in ALBA for any definite scattered left-primitive term $s(\vec{p}, \vec{q})$ and definite scattered right-primitive term $t(\vec{p}, \vec{q})$:*

$$
(\textit{Approx}(s))\ \frac{\mathbf{j} \leq s(\vec{p}, \vec{q})}{\mathbf{j} \leq s(\vec{\mathbf{i}}, \vec{\mathbf{m}}) \quad \vec{\mathbf{i}} \leq \vec{p} \quad \vec{q} \leq \vec{\mathbf{m}}}
\qquad\qquad
\frac{t(\vec{p}, \vec{q}) \leq \mathbf{m}}{t(\vec{\mathbf{n}}, \vec{\mathbf{i}}) \leq \mathbf{m} \quad \vec{p} \leq \vec{\mathbf{m}} \quad \vec{\mathbf{i}} \leq \vec{q}}\ (\textit{Approx}(t))
$$

Proof. The first part of the statement is an immediate consequence of Lemma 35. The second part can be straightforwardly shown by induction on s and t. The details of the proof are omitted. ☐

Proposition 37. *For any language $\mathcal{L}_{\mathrm{DLE}}$ any left-primitive $\mathcal{L}_{\mathrm{DLE}}$-inequality $s(\vec{p}, \vec{q}) \leq s'(\vec{p}, \vec{q})$ and any right-primitive $\mathcal{L}_{\mathrm{DLE}}$-inequality $t'(\vec{p}, \vec{q}) \leq t(\vec{p}, \vec{q})$,*

1. *if $s(\vec{p}, \vec{q})$ is definite, then the following are equivalent for every perfect DLE $\mathbb{A}$:*
   (a) $\mathbb{A} \models s(\vec{p}, \vec{q}) \leq s'(\vec{p}, \vec{q})$;
   (b) $\mathbb{A} \models s(\vec{\mathbf{i}}, \vec{\mathbf{m}}) \leq s'(\vec{\mathbf{i}}, \vec{\mathbf{m}})$.
2. *If $t(\vec{p}, \vec{q})$ is definite, then the following are equivalent for every perfect DLE $\mathbb{A}$:*
   (a) $\mathbb{A} \models t'(\vec{p}, \vec{q}) \leq t(\vec{p}, \vec{q})$;
   (b) $\mathbb{A} \models t'(\vec{\mathbf{m}}, \vec{\mathbf{i}}) \leq t(\vec{\mathbf{m}}, \vec{\mathbf{i}})$.

Proof. We only prove 1, the proof of item 2 being order dual. By the assumptions and Corollary 36, the following chain of equivalences can be obtained via an ALBA reduction and hence is sound on perfect DLEs:

	$\forall\vec{p}\forall\vec{q}[s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q})]$	
iff	$\forall\vec{p}\forall\vec{q}\forall\mathbf{j}[\mathbf{j} \leq s(\vec{p},\vec{q}) \Rightarrow \mathbf{j} \leq s'(\vec{p},\vec{q})]$	
iff	$\forall\vec{p}\forall\vec{q}\forall\mathbf{j}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[(\vec{\mathbf{i}} \leq \vec{p}\ \&\ \vec{q} \leq \vec{\mathbf{m}}\ \&\ \mathbf{j} \leq s(\vec{\mathbf{i}},\vec{\mathbf{m}})) \Rightarrow \mathbf{j} \leq s'(\vec{p},\vec{q})]$	(Approx(s))
iff	$\forall\mathbf{j}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[\mathbf{j} \leq s(\vec{\mathbf{i}},\vec{\mathbf{m}}) \Rightarrow \mathbf{j} \leq s'(\vec{\mathbf{i}},\vec{\mathbf{m}})]$	(Ackermann, s, s' same order type)
iff	$\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[s(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq s'(\vec{\mathbf{i}},\vec{\mathbf{m}})]$	

□

Remark 38. Proposition 37 can be straightforwardly generalized to primitive inequalities the heads of which are not definite. For any such inequality, the preprocessing stage of ALBA produces a set of definite primitive inequalities with definite heads, to each of which Proposition 37 can then be applied separately. Notice that the preprocessing does not affect the order-type of the occurring variables. Then, one can reverse the preprocessing steps and transform the set of pure definite primitive inequalities into a substitution instance of the input primitive inequality in which proposition variables have been suitably substituted for nominals and conominals.

Example 39. Let us illustrate the reduction strategy of the proposition above by applying it to the right-primitive Fischer Servi inequality discussed in Examples 30 and 34 (cf. [43, Lemma 27]).

$$
\begin{array}{ll}
 & \forall q\forall p[\Diamond q \to \Box p \leq \Box(q \to p)] \\
\text{iff} & \forall q\forall p\forall\mathbf{i}\forall\mathbf{m}[(\mathbf{i} \leq \Diamond q \to \Box p\ \&\ \Box(q \to p) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall q\forall p\forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{n}[(\mathbf{i} \leq \Diamond q \to \Box p\ \&\ \Box(q \to \mathbf{n}) \leq \mathbf{m}\ \&\ p \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall q\forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{n}[(\mathbf{i} \leq \Diamond q \to \Box\mathbf{n}\ \&\ \Box(q \to \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall q\forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{n}\forall\mathbf{j}[(\mathbf{i} \leq \Diamond q \to \Box\mathbf{n}\ \&\ \Box(\mathbf{j} \to \mathbf{n}) \leq \mathbf{m}\ \&\ \mathbf{j} \leq q) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{n}\forall\mathbf{j}[(\mathbf{i} \leq \Diamond\mathbf{j} \to \Box\mathbf{n}\ \&\ \Box(\mathbf{j} \to \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall\mathbf{i}\forall\mathbf{n}\forall\mathbf{j}[\mathbf{i} \leq \Diamond\mathbf{j} \to \Box\mathbf{n} \Rightarrow \forall\mathbf{m}[\Box(\mathbf{j} \to \mathbf{n}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
\text{iff} & \forall\mathbf{i}\forall\mathbf{n}\forall\mathbf{j}[\mathbf{i} \leq \Diamond\mathbf{j} \to \Box\mathbf{n} \Rightarrow \mathbf{i} \leq \Box(\mathbf{j} \to \mathbf{n})] \\
\text{iff} & \forall\mathbf{n}\forall\mathbf{j}[\Diamond\mathbf{j} \to \Box\mathbf{n} \leq \Box(\mathbf{j} \to \mathbf{n})].
\end{array}
$$

2.5.3 Special rules via ALBA: main strategy

Before moving on to the next section, in the present subsection we take stock of the facts we have collected so far, and spell out their role in the context of the method we will apply in the following section. This method is to extend the class of primitive inequalities in any given language $\mathcal{L}_{\mathrm{DLE}}$ to classes of inequalities each element of which can be equivalently (and effectively) transformed into (a set of) special structural rules, hence giving rise to specially displayable DLE-logics (cf. Definition 27). This method is based on the simple but crucial observation that the languages of the display calculi $\mathbf{DL}$, $\mathbf{DL}^*$, and $\mathbf{DL}^*_-$ (cf. Definition 21 and Footnote 33) are built using *the same* set of structural connectives. For each language $\mathcal{L}_{\mathrm{DLE}}$, we are going to identify classes of *non-primitive* $\mathcal{L}_{\mathrm{DLE}}$-inequalities which can be equivalently and effectively transformed into (conjunctions of) *primitive* inequalities in the expanded language $\mathcal{L}^*_{\mathrm{DLE}}$ (cf. Section 2.3.2). By Lemma 32 applied to $\mathcal{L}^*_{\mathrm{DLE}}$, each primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequality can then be equivalently transformed into a set of special structural rules in the language of $\mathbf{DL}^*$, which, as observed above, coincides with the structural language of $\mathbf{DL}$.

Proposition 37 provides a key step in the procedure to equivalently transform input $\mathcal{L}_{\mathrm{DLE}}$-inequalities into primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequalities. Indeed, it guarantees that each definite primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequality is equivalent to a "substitution instance of itself" in which all the nominals and conominals have been uniformly substituted for proposition variables, as illustrated by the right-hand vertical equivalence in the diagram below:

$$\begin{array}{ccc}
\mathbb{A} \models s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q}) & & \mathbb{A} \models \&\left\{s^*_i(\vec{p},\vec{q}) \leq {s'}^*_i(\vec{p},\vec{q}) \mid i \in I\right\} \\
\Updownarrow \text{ Theorems 19 and 17} & & \Updownarrow \text{ Proposition 37} \\
\mathbb{A} \models \&\left\{s^*_i(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq {s'_i}^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \mid i \in I\right\} & = & \mathbb{A} \models \&\left\{s^*_i(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq {s'_i}^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \mid i \in I\right\}
\end{array}$$

Our task in the following section will be to perform ALBA-reductions aimed at equivalently transforming $\mathcal{L}_{\mathrm{DLE}}$-inequalities into sets of definite *pure* primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequalities, so as to provide the left-hand side leg of the diagram above.

2.6 Extending the class of primitive inequalities

In the present section, we introduce a hierarchy of classes of $\mathcal{L}_{\mathrm{DLE}}$-inequalities which properly extend primitive inequalities, and which can be equivalently (and effectively) transformed into sets of special structural rules (cf. Definition 6), via progressively more complex ALBA-reduction strategies. The classes of inequalities treated in the present section are all proper subclasses of the class of analytic inductive inequalities (cf. Definition 55), which is the most general, and which, in Section 2.7, will be also shown to capture analytic rules modulo equivalence. However, the procedure described in Section 2.7 does not deliver special rules in the restricted sense of Definition 6 in general, whereas the finer analysis provided in the present section is guaranteed to yield special rules in this restricted sense (cf. Remark 7) in each instance in which it is applicable. Thus, unlike the general procedure, the procedure described in the present section provides a direct and fully mechanized way[37] to obtain specially displayable DLE-logics (cf. Definition 27). Section 2.7 is independent from the present section, hence the reader is not constrained to read the present section before the next. Finally, the present chapter is intended for two very different readerships; in this respect, the present section, which is the richest in examples of the whole chapter, can be useful to the reader who wishes to become familiar with ALBA reductions.

Throughout the present section, we adopt the convention that $f(\vec{p},\vec{q})$ and $g(\vec{p},\vec{q})$ are s.t. $\varepsilon_f(p) = \varepsilon_g(p) = 1$ for every $p \in \vec{p}$ and $\varepsilon_f(q) = \varepsilon_g(q) = \partial$ for every $q \in \vec{q}$.

[37] In Section 2.8, we will show that in fact, all DLE-logics axiomatized by analytic inductive inequalities are specially displayable. However, the general procedure, derived from the results in Sections 2.7 and 2.8, to extract special rules from analytic inductive inequalities is indirect, as it consists of more than one back-and-forth toggle between inequalities and rules.

For any sequence of formulas $\vec{\psi} = (\psi_1, \ldots, \psi_n)$ and any $1 \leq i \leq n$, we let $\overrightarrow{\psi_{-i}} := (\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots, \psi_n)$.

2.6.1 Type 2: multiple occurrences of critical variables

2

By definition, each proposition letter in the head of a primitive inequality is required to occur at most once (that is, the head of primitive inequalities is required to be scattered). The present subsection is aimed at showing that this condition can be relaxed.

Definition 40 (Quasi-primitive inequalities)**.** An inequality $s_1 \leq s_2$ is *quasi left-primitive* (resp. *quasi right-primitive*) if both s_1 and s_2 are monotone (w.r.t. some order-type ε_{s_i}) and left-primitive (resp. right-primitive) formulas, and moreover s_1 and s_2 have the same order-type relative to the variables they have in common.

The definition above differs from Definition 28 in that the requirement that the head be scattered is dropped.

Remark 41. In what follows, we are going to provide an effective procedure to equivalently transform quasi-primitive inequalities into pure primitive inequalities. We will restrict our focus to quasi-primitive inequalities with *definite* head (cf. Proposition 44). Indeed, during the pre-processing stage of the execution of ALBA, each quasi-primitive inequality with non-definite head can be equivalently transformed into (the conjunction of) a set of quasi-primitive inequalities with definite head, on each of which the procedure described below can be effected in parallel. Thus, this restriction is without loss of generality.

Definition 42. For every left-primitive (resp. right-primitive) formula $s(\vec{p}, \vec{q})$, a *scattered transform* of s is a scattered left-primitive (resp. right-primitive) term $s^*(\vec{p'}, \vec{q'})$ for which there exists a substitution σ : AtProp$(s^*) \to$ AtProp(s) such that $s(\vec{p}, \vec{q}) = \sigma(s^*(\vec{p'}, \vec{q'}))$.

Clearly, we can always assume without loss of generality that $s(\vec{p}, \vec{q})$ and $s^*(\vec{p'}, \vec{q'})$ share no variables. In particular, in the following lemma, we will find it useful to consider scattered transforms which are pure, i.e. of the form $s^*(\vec{\mathbf{i}}, \vec{\mathbf{m}})$ or $s^*(\vec{\mathbf{m}}, \vec{\mathbf{i}})$, and such that their associated substitution σ maps nominals and conominals to proposition variables in a suitable way according to their polarity. This can always be done without loss of generality.

Lemma 43. *The following rules are sound and invertible in perfect DLEs and are derivable in ALBA:*

1. *for any definite quasi left-primitive term* $s(\vec{p}, \vec{q})$*,*

$$\frac{\mathbf{j} \leq s(\vec{p}, \vec{q})}{\mathbf{j} \leq s^*(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{m}}) \quad \overrightarrow{\bigvee \sigma^{-1}[p] \leq p} \quad \overrightarrow{q \leq \bigwedge \sigma^{-1}[q]}} \ (\textit{Approx}_\sigma(s))$$

where, for every p *in* $\vec{p}$ *and every* q *in* $\vec{q}$*, every variable in* $\sigma^{-1}[p]$ *is a (fresh) nominal, and every variable in* $\sigma^{-1}[q]$ *is a (fresh) conominal, and* s^* *is the scattered transform of* s *induced by* σ*.*

2. *For any definite quasi right-primitive term* $t(\vec{p},\vec{q})$*:*

$$\frac{t(\vec{p},\vec{q}) \leq \mathbf{m}}{t^*(\vec{\mathbf{n}},\vec{\mathbf{i}}) \leq \mathbf{m} \quad \vec{p} \leq \overrightarrow{\bigwedge \sigma^{-1}[p]} \quad \overrightarrow{\bigvee \sigma^{-1}[q]} \leq \vec{q}} \ (\text{Approx}_\sigma(t))$$

where, for every p *in* $\vec{p}$ *and every* q *in* $\vec{q}$*, every variable in* $\sigma^{-1}[p]$ *is a (fresh) conominal, and every variable in* $\sigma^{-1}[q]$ *is a (fresh) nominal, and* t^* *is the scattered transform of* t *induced by* σ*.*

Proof.
We only prove item 1, item 2 being order-dual.

$$\cfrac{\cfrac{\cfrac{\cfrac{\mathbf{j} \leq s(\vec{p},\vec{q})}{\mathbf{j} \leq \sigma(s^*(\vec{\mathbf{i}},\vec{\mathbf{m}}))}\ (\text{Definition 42})}{\mathbf{j} \leq s^*(\overrightarrow{\sigma(\mathbf{i})},\overrightarrow{\sigma(\mathbf{m})})}\ (\text{definition of substitution})}{\mathbf{j} \leq s^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \quad \vec{\mathbf{i}} \leq \overrightarrow{\sigma(\mathbf{i})} \quad \overrightarrow{\sigma(\mathbf{m})} \leq \vec{\mathbf{m}}}\ (\text{Approx}(s^*))}{\mathbf{j} \leq s^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \quad \overrightarrow{\bigvee \sigma^{-1}[p]} \leq \vec{p} \quad \vec{q} \leq \overrightarrow{\bigwedge \sigma^{-1}[q]}}\ (\text{reverse splitting rule})$$

□ The following proposition and its proof provide an effective procedure to equivalently transform quasi-primitive inequalities with definite head into pure primitive inequalities.

Proposition 44. *For every quasi left-primitive inequality* $s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q})$ *such that* s *is definite and every quasi right-primitive inequality* $t'(\vec{p},\vec{q}) \leq t(\vec{p},\vec{q})$ *such that* t *is definite,*

1. *the following are equivalent for every perfect* $\mathcal{L}_{\mathrm{DLE}}$ *algebra* $\mathbb{A}$*:*

   (a) $\mathbb{A} \models s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q})$*;*

   (b) $\mathbb{A} \models s^*(\vec{\mathbf{i}},\vec{\mathbf{m}}) \leq s'(\overrightarrow{\bigvee \sigma^{-1}[p]},\overrightarrow{\bigwedge \sigma^{-1}[q]})$*,*

   where s^* *is a pure scattered transform of* s *witnessed by a map* $\sigma : \mathsf{Prop}(s^*) \to \mathsf{Prop}(s)$ *such that, for every* p *in* $\vec{p}$ *and every* q *in* $\vec{q}$*, every variable in* $\sigma^{-1}[p]$ *is a nominal and every variable in* $\sigma^{-1}[q]$ *is a conominal.*

2. *The following are equivalent for every perfect* $\mathcal{L}_{\mathrm{DLE}}$ *algebra* $\mathbb{A}$*:*

   (a) $\mathbb{A} \models t'(\vec{p},\vec{q}) \leq t(\vec{p},\vec{q})$*;*

   (b) $\mathbb{A} \models t'(\overrightarrow{\bigwedge \sigma^{-1}[p]},\overrightarrow{\bigvee \sigma^{-1}[q]}) \leq t^*(\vec{\mathbf{m}},\vec{\mathbf{i}})$*,*

   where t^* *is a pure scattered transform of* t *witnessed by a map* $\sigma : \mathsf{Prop}(t^*) \to \mathsf{Prop}(t)$ *such that, for every* p *in* $\vec{p}$ *and every* q *in* $\vec{q}$*, every variable in* $\sigma^{-1}[p]$ *is a conominal and every variable in* $\sigma^{-1}[q]$ *is a nominal.*

Proof. We only prove item 1, item 2 being order-dual. The assumptions and Lemma 43 guarantee that the following ALBA reduction is sound:

$$
\begin{array}{ll}
& \forall\vec{p}\forall\vec{q}[s(\vec{p},\vec{q}) \leq s'(\vec{p},\vec{q})] \\
\text{iff} & \forall\vec{p}\forall\vec{q}\forall\mathbf{j}[\mathbf{j} \leq s(\vec{p},\vec{q}) \Rightarrow \mathbf{j} \leq s'(\vec{p},\vec{q})] \\
\text{iff} & \forall\vec{p}\forall\vec{q}\forall\mathbf{j}\forall\overrightarrow{\mathbf{i}}\forall\vec{\mathbf{m}}[(\mathbf{j} \leq s^*(\overrightarrow{\mathbf{i}},\vec{\mathbf{m}}) \ \&\ \overrightarrow{\bigvee\sigma^{-1}[p]} \leq \vec{p} \ \&\ \vec{q} \leq \overrightarrow{\bigwedge\sigma^{-1}[q]}) \Rightarrow \mathbf{j} \leq t(\vec{p},\vec{q})] \ \text{(Approx}_\sigma\text{(s))} \\
\text{iff} & \forall\mathbf{j}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[\mathbf{j} \leq s^*(\overrightarrow{\mathbf{i}},\vec{\mathbf{m}}) \Rightarrow \mathbf{j} \leq s'(\overrightarrow{\bigvee\sigma^{-1}[p]},\overrightarrow{\bigwedge\sigma^{-1}[q]})] \ \text{(Ackermann, } s, s' \text{ same order type)} \\
\text{iff} & \forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[s^*(\overrightarrow{\mathbf{i}},\vec{\mathbf{m}}) \leq s'(\overrightarrow{\bigvee\sigma^{-1}[p]},\overrightarrow{\bigwedge\sigma^{-1}[q]})].
\end{array}
$$

□

A concrete instantiation of the method. Let $\mathcal{F} = \{\cdot, \Diamond\}$ and $\mathcal{G} = \varnothing$, where $\cdot$ is binary and of order type $(1,1)$. The inequality $\Diamond\Diamond p \cdot \Diamond p \leq \Diamond p$ is quasi left-primitive and definite, and fails to be left-primitive because its head (the term on the left-hand side) is not scattered. Firstly, we run ALBA on this inequality, so as to equivalently transform it into a *pure* non-definite left-primitive inequality as follows:

$$
\begin{array}{lll}
& \forall p[\Diamond\Diamond p \cdot \Diamond p \leq \Diamond p] & \\
\text{iff} & \forall p\forall\mathbf{j}\forall\mathbf{m}[(\mathbf{j} \leq \Diamond\Diamond p \cdot \Diamond p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \\
\text{iff} & \forall p\forall\mathbf{j}\forall\mathbf{m}\forall\mathbf{i}[(\mathbf{j} \leq \Diamond\Diamond\mathbf{i} \cdot \Diamond p \ \&\ \mathbf{i} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \\
\text{iff} & \forall p\forall\mathbf{j}\forall\mathbf{m}\forall\mathbf{i}\forall\mathbf{h}[(\mathbf{j} \leq \Diamond\Diamond\mathbf{i} \cdot \Diamond\mathbf{h} \ \&\ \mathbf{i} \leq p \ \&\ \mathbf{h} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \\
\text{iff} & \forall p\forall\mathbf{j}\forall\mathbf{m}\forall\mathbf{i}\forall\mathbf{h}[(\mathbf{j} \leq \Diamond\Diamond\mathbf{i} \cdot \Diamond\mathbf{h} \ \&\ \mathbf{i} \vee \mathbf{h} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \text{(rev. split. rule)} \\
\text{iff} & \forall\mathbf{j}\forall\mathbf{m}\forall\mathbf{i}\forall\mathbf{h}[(\mathbf{j} \leq \Diamond\Diamond\mathbf{i} \cdot \Diamond\mathbf{h} \ \&\ \Diamond(\mathbf{i} \vee \mathbf{h}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] & \\
\text{iff} & \forall\mathbf{j}\forall\mathbf{i}\forall\mathbf{h}[\mathbf{j} \leq \Diamond\Diamond\mathbf{i} \cdot \Diamond\mathbf{h} \Rightarrow \forall\mathbf{m}[\Diamond(\mathbf{i} \vee \mathbf{h}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]] & \\
\text{iff} & \forall\mathbf{j}\forall\mathbf{i}\forall\mathbf{h}[\mathbf{j} \leq \Diamond\Diamond\mathbf{i} \cdot \Diamond\mathbf{h} \Rightarrow \mathbf{j} \leq \Diamond(\mathbf{i} \vee \mathbf{h})] & \\
\text{iff} & \forall\mathbf{i}\forall\mathbf{h}[\Diamond\Diamond\mathbf{i} \cdot \Diamond\mathbf{h} \leq \Diamond(\mathbf{i} \vee \mathbf{h})] &
\end{array}
$$

By Proposition 37, the pure left-primitive inequality $\Diamond\Diamond\mathbf{i}\cdot\Diamond\mathbf{h} \leq \Diamond(\mathbf{i}\vee\mathbf{h})$ is equivalent on perfect $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F},\mathcal{G})$-algebras to the left-primitive inequality $\Diamond\Diamond p_1\cdot\Diamond p_2 \leq \Diamond(p_1\vee p_2)$, which, via ALBA-distribution rule, is equivalent to the following inequality in disjunction normal form:

$$\Diamond\Diamond p_1 \cdot \Diamond p_2 \leq \Diamond p_1 \vee \Diamond p_2.$$

If we specify the non-lattice fragment of the language of the associated calculus $\mathbf{DL}$ as follows:

Structural symbols	$\circ$		$\bullet$		$\odot$		$\backslash\!\backslash_{\odot}$		$/\!/_{\odot}$	
Operational symbols	$\Diamond$			$\blacksquare$	$\cdot$			$\backslash_{\odot}$		$/_{\odot}$

then, applying the procedure indicated in the proof of Lemma 32, the inequality above can be transformed into a structural rule in the language above as follows:

$$\Diamond\Diamond p_1 \cdot \Diamond p_2 \leq \Diamond p_1 \vee \Diamond p_2 \quad \rightsquigarrow \quad \frac{\Diamond p_1 \vdash z \quad \Diamond p_2 \vdash z}{\Diamond\Diamond p_1 \cdot \Diamond p_2 \vdash z} \quad \rightsquigarrow \quad \frac{\circ X \vdash Z \quad \circ Y \vdash Z}{\circ\circ X \odot \circ Y \vdash Z}.$$

Monotone terms in quasi-primitive inequalities. The head of primitive inequalities is scattered, hence monotone (w.r.t. some order-type). In defining quasi-primitive inequalities, we have dropped the former requirement but kept the latter. Before moving on, let us illustrate why by means of an example. Let $\mathcal{F} = \{\cdot, \Diamond, \lhd\}$ and $\mathcal{G} = \varnothing$, where $\cdot$ is binary and of order type $(1,1)$, and $\lhd$ is unary and of order-type (∂). The inequality

$p \cdot \lhd p \leq \Diamond p$ is not quasi-primitive, since its head $p \cdot \lhd p$ is not monotone. Actually, this inequality behaves like a primitive inequality, in that Proposition 37 can be generalized to cover such an inequality; indeed

$$\begin{array}{ll} & \forall p[p \cdot \lhd p \leq \Diamond p] \\ \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq p \cdot \lhd p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \mathbf{j} \cdot \lhd p \ \&\ \mathbf{j} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \mathbf{j} \cdot \lhd \mathbf{j} \ \&\ \Diamond \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \mathbf{j} \cdot \lhd \mathbf{j} \Rightarrow \forall \mathbf{m}[\Diamond \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\ \text{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \mathbf{j} \cdot \lhd \mathbf{j} \Rightarrow \mathbf{i} \leq \Diamond \mathbf{j}] \\ \text{iff} & \forall \mathbf{j}[\mathbf{j} \cdot \lhd \mathbf{j} \leq \Diamond \mathbf{j}]. \end{array}$$

However, this is not good news. Indeed, this reduction does not help to solve the main problem of this inequality, namely the fact that if we apply the procedure described in the proof of Lemma 32 to this inequality, we obtain a rule which violates condition C_4 (position-alikeness of parameters).

2.6.2 Type 3: allowing PIA-subterms

In Sections 2.5.2 and 2.6.1, we have generalized Kracht's notion of primitive inequalities, first by making this notion apply uniformly to any $\mathcal{L}_{\mathrm{DLE}}$-signature, and then by dropping the requirement that the heads of inequalities be scattered. Moreover, we have identified the main order-theoretic features induced by the syntactic shape of definite scattered primitive formulas, and, thanks to this identification, we have started to see ALBA at work on primitive and quasi-primitive inequalities. However, so far we have not discussed why ALBA was guaranteed to succeed on any primitive or quasi-primitive inequality in the first place. More in general, we have not yet made use of the second tool of unified correspondence theory: the possibility of identifying Sahlqvist and inductive type of inequalities in any $\mathcal{L}_{\mathrm{DLE}}$-signature.

So let us start the present subsection by analyzing (quasi-)primitive inequalities as inductive inequalities (cf. Definition 16). Indeed, it can be easily verified by direct inspection that all primitive inequalities are a very special subclass of inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities. Specifically, as observed earlier (cf. Remark 29), all non-leaf nodes in the generation tree $+s$ (resp. $-s$) of a (quasi) left-primitive (resp. right-primitive) formula s are Skeleton nodes. This guarantees that, if $+s$ is also monotone w.r.t. some order-type ε_s, then all the variables at the leaves of such a generation tree (which are ε_s-critical) can be solved for, and moreover (together with the condition on the order-type in Definition 40), that an ALBA reduction on a (quasi-)primitive inequality is guaranteed to reach Ackermann shape using *only* approximation and splitting rules after the preprocessing stage.

A natural question arising at this point is whether or not all inductive inequalities can be transformed via ALBA into (conjunctions of) pure primitive inequalities, as outlined in Subsection 2.5.3. We can already answer this question in the negative, as the following example shows. Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\Box\}$, and consider the inequality $\Diamond p \leq \Diamond\Box p$, which is Sahlqvist for the order-type (1) and 'McKinsey-type' for the order-type (∂), and is neither left-primitive nor right-primitive. There is only one successful reduction strategy for ALBA, which consists in solving for the positive occurrence of p as follows:

$$
\begin{array}{ll}
 & \forall p[\Diamond p \leq \Diamond \Box p] \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond p \ \&\ \Diamond \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \mathbf{j} \leq p \ \&\ \Diamond \Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \Diamond \Box \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \forall \mathbf{m}[\Diamond \Box \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Diamond \Box \mathbf{j}] \\
\text{iff} & \forall \mathbf{j}[\Diamond \mathbf{j} \leq \Diamond \Box \mathbf{j}].
\end{array}
$$

Clearly, this reduction fails to improve the situation, since it leaves the troublemaking side $\Diamond \Box p$ untouched. In contrast to this example, consider the inequality $\Diamond \Box p \leq \Diamond p$, which is again neither left-primitive nor right-primitive, but is Sahlqvist for both order-types (1) and (∂). Solving for the troublemaking side we obtain:

$$
\begin{array}{ll}
 & \forall p[\Diamond \Box p \leq \Diamond p] \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond \Box p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \mathbf{j} \leq \Box p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \blacklozenge \mathbf{j} \leq p \ \&\ \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \Diamond \blacklozenge \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \forall \mathbf{m}[\Diamond \blacklozenge \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j}[\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Diamond \blacklozenge \mathbf{j}] \\
\text{iff} & \forall \mathbf{j}[\Diamond \mathbf{j} \leq \Diamond \blacklozenge \mathbf{j}],
\end{array}
$$

from which the usual steps (Proposition 37 and Lemma 32) yield the rule

$$\frac{\circ \bullet X \vdash Y}{\circ X \vdash Y}$$

These ideas motivate the following

Definition 45 (Very restricted analytic inductive inequalities)**.** For any order type ε and any irreflexive and transitive relation Ω on the variables $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{+, -\}$) of a term $s(p_1, \ldots p_n)$ is *restricted analytic* (Ω, ε)*-inductive* if

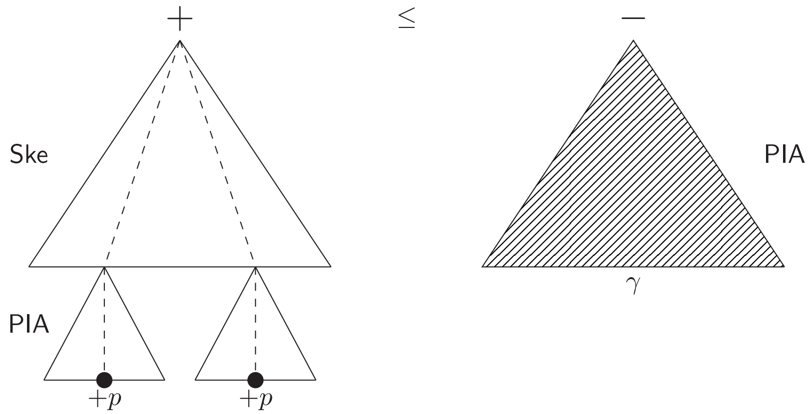
1. $*s$ is (Ω, ε)-inductive (cf. Definition 16);
2. every branch of $*s$ is good (cf. Definition 15);
3. every maximal ε^∂-uniform subtree of $*s$ occurs as an immediate subtree of an SRR node of some ε-critical branch of $*s$;

An inequality $s \leq t$ is *very restricted left-analytic* (Ω, ε)*-inductive* (resp. *very restricted right-analytic* (Ω, ε)*-inductive*) if

1. $+s$ (resp. $-t$) (which we refer to as the head of the inequality) is restricted analytic (Ω, ε)-inductive;
2. $-t$ (resp. $+s$) is ε^∂-uniform, and
3. t is left-primitive (resp. s is right-primitive) (cf. Definition 28).

An inequality $s \leq t$ is *very restricted analytic inductive* if it is very restricted (right-analytic or left-analytic) (Ω, ε)-inductive for some Ω and ε.

Remark 46. The syntactic shape specified in the definition above can be intuitively understood with the help of the following picture, which illustrates the 'left-analytic' case:



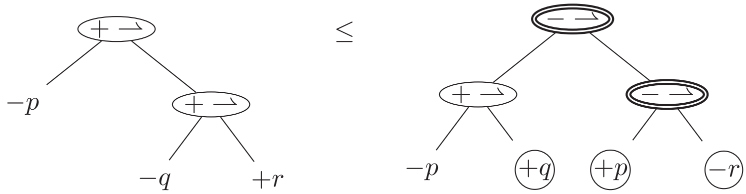
As the picture shows, this syntactic shape requires that each ε-critical occurrence is a leaf of the head of the inequality. Moreover, the definition of restricted analytic (Ω, ε)-inductive signed generation tree implies that every maximal PIA-subtree contains at least one (but possibly more) ε-critical variable occurrence. Further, the requirement that every branch be good implies that every maximal ε^∂-subtree γ of every PIA-structure consists also exclusively of PIA-nodes. Moreover, the requirement that these subtrees be attached to their main PIA-subtree by means of an SRR-node lying on a critical branch guarantees that these subtrees will be incorporated in the minimal valuation subtree of the critical occurrence at the leaf of that critical branch.

Finally, exhaustively applying the distribution rules (a')-(c') described in Remark 18 to any restricted analytic inductive term produces a restricted analytic inductive term, every maximal PIA-subterm of which is definite (cf. Footnote 30) and contains *exactly one* ε-critical variable occurrence.

Example 47. Let $\mathcal{F} = \varnothing$ and $\mathcal{G} = \{\rightharpoonup\}$, with $\rightharpoonup$ binary and of order-type $(\partial, 1)$. As observed in [20], the Frege inequality

$$p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)$$

is not Sahlqvist for any order type, but is (Ω, ε)-inductive, e.g. for $r <_\Omega p <_\Omega q$ and $\varepsilon(p, q, r) = (1, 1, \partial)$, and is also very restricted right-analytic (Ω, ε)-inductive for the same Ω and ε, as can be seen from the signed generation trees below:



In the picture above, the circled variable occurrences are the ε-critical ones, the doubly circled nodes are the Skeleton ones and the single-circle ones are PIA.

Below, we introduce an auxiliary definition which is a simplified version of [16, Definition 5.1] and is aimed at effectively calculating the residuals of definite positive and negative PIA formulas (cf. discussion after Definition 16 and Footnote 30) w.r.t. a given variable occurrence x. The intended meaning of notation such as $\phi(!x, \overline{z})$ is that the variable x occurs *exactly once* in the formula ϕ.

Definition 48. For every definite positive PIA $\mathcal{L}_{\mathrm{DLE}}$-formula $\phi = \phi(!x, \overline{z})$, and any definite negative PIA $\mathcal{L}_{DLE}$-formula $\psi = \psi(!x, \overline{z})$ such that x occurs in them exactly once, the $\mathcal{L}^{+}_{\mathrm{DLE}}$-formulas $\mathsf{LA}(\phi)(u, \overline{z})$ and $\mathsf{RA}(\psi)(u, \overline{z})$ (for $u \in Var - (x \cup \overline{z})$) are defined by simultaneous recursion as follows:

$$
\begin{array}{rcl}
\mathsf{LA}(x) & = & u; \\
\mathsf{LA}(\Box\phi(x, \overline{z})) & = & \mathsf{LA}(\phi)(\blacklozenge u, \overline{z}); \\
\mathsf{LA}(\psi(\overline{z}) \to \phi(x, \overline{z})) & = & \mathsf{LA}(\phi)(u \wedge \psi(\overline{z}), \overline{z}); \\
\mathsf{LA}(\phi_1(\overline{z}) \vee \phi_2(x, \overline{z})) & = & \mathsf{LA}(\phi_2)(u - \phi_1(\overline{z}), \overline{z}); \\
\mathsf{LA}(\psi(x, \overline{z}) \to \phi(\overline{z})) & = & \mathsf{RA}(\psi)(u \to \phi(\overline{z}), \overline{z}); \\
\mathsf{LA}(g(\overrightarrow{\phi_{-j}(\overline{z})}, \phi_j(x, \overline{z}), \overrightarrow{\psi(\overline{z})})) & = & \mathsf{LA}(\phi_j)(g^{\flat}_j(\overrightarrow{\phi_{-j}(\overline{z})}, u, \overrightarrow{\psi(\overline{z})}), \overline{z}); \\
\mathsf{LA}(g(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \psi_j(x, \overline{z}))) & = & \mathsf{RA}(\psi_j)(g^{\flat}_j(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, u), \overline{z});
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{RA}(x) & = & u; \\
\mathsf{RA}(\Diamond\psi(x, \overline{z})) & = & \mathsf{RA}(\psi)(\blacksquare u, \overline{z}); \\
\mathsf{RA}(\psi(x, \overline{z}) - \phi(\overline{z})) & = & \mathsf{RA}(\psi)(\phi(\overline{z}) \vee u, \overline{z}); \\
\mathsf{RA}(\psi_1(\overline{z}) \wedge \psi_2(x, \overline{z})) & = & \mathsf{RA}(\psi_2)(\psi_1(\overline{z}) \to u, \overline{z}); \\
\mathsf{RA}(\psi(\overline{z}) - \phi(x, \overline{z})) & = & \mathsf{LA}(\phi)(\psi(\overline{z}) - u, \overline{z}); \\
\mathsf{RA}(f(\overrightarrow{\psi_{-j}(\overline{z})}, \psi_j(x, \overline{z}), \overrightarrow{\phi(\overline{z})})) & = & \mathsf{RA}(\psi_j)(f^{\sharp}_j(\overrightarrow{\psi_{-j}(\overline{z})}, u, \overrightarrow{\phi(\overline{z})}), \overline{z}); \\
\mathsf{RA}(f(\overrightarrow{\psi(\overline{z})}, \overrightarrow{\phi_{-j}(\overline{z})}, \phi_j(x, \overline{z}))) & = & \mathsf{LA}(\phi_j)(f^{\sharp}_j(\overrightarrow{\psi(\overline{z})}, \overrightarrow{\phi_{-j}(\overline{z})}, u), \overline{z}).
\end{array}
$$

Lemma 49. *For all definite positive PIA $\mathcal{L}_{\mathrm{DLE}}$-formulas $\phi_1(!x, \overline{z})$, $\phi_2(!x, \overline{z})$, and all definite negative PIA $\mathcal{L}_{\mathrm{DLE}}$-formulas $\psi_1(!x, \overline{z})$, $\psi_2(!x, \overline{z})$ such that the variable x occurs in them exactly once,*

1. *if $+x \prec +\phi_1$, then the following rule is derivable in ALBA:*

$$
(\mathsf{LA}(\phi_1))\ \frac{\chi \leq \phi_1(x, \overline{z})}{\mathsf{LA}(\phi_1)(\chi/u, \overline{z}) \leq x}
$$

and moreover, $\mathsf{LA}(\phi_1)(u, \overline{z})$ is a definite negative PIA $\mathcal{L}^{*}_{\mathrm{DLE}}$-formula.

2

2. *if $-x \prec +\phi_2$, then the following rule is derivable in ALBA:*

$$(LA(\phi_2))\ \frac{\chi \leq \phi_2(x,\overline{z})}{x \leq \mathsf{LA}(\phi_2)(\chi/u,\overline{z})}$$

and moreover, $\mathsf{LA}(\phi_2)(u,\overline{z})$ *is a definite positive PIA* $\mathcal{L}^*_{\mathrm{DLE}}$*-formula.*

3. *if $+x \prec +\psi_1$, then the following rule is derivable in ALBA:*

$$(RA(\psi_1))\ \frac{\psi_1(x,\overline{z}) \leq \chi}{x \leq \mathsf{RA}(\psi_1)(\chi/u,\overline{z})}$$

and moreover, $\mathsf{RA}(\psi_1)(u,\overline{z})$ *is a definite positive PIA* $\mathcal{L}^*_{\mathrm{DLE}}$*-formula.*

4. *if $-x \prec +\psi_2$, then the following rule is derivable in ALBA:*

$$(RA(\psi_2))\ \frac{\psi_2(x,\overline{z}) \leq \chi}{\mathsf{RA}(\psi_2)(\chi/u,\overline{z}) \leq x}$$

and moreover, $\mathsf{RA}(\psi_2)(u,\overline{z})$ *is a definite negative PIA* $\mathcal{L}^*_{\mathrm{DLE}}$*-formula.*

Proof. By simultaneous induction on the shapes of ϕ_1, ϕ_2, ψ_1 and ψ_2. The case in which they coincide with x immediately follows from the definitions involved. As to the inductive step, we only illustrate the case in which ϕ_1 is of the form $g(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \psi_{2,j}(x,\overline{z}))$ for some array $\overrightarrow{\phi(\overline{z})}$ of formulas which are positive PIA, some array $\overrightarrow{\psi_{-j}(\overline{z})}$ of formulas which are negative PIA, and some negative PIA formula $\psi_{2,j}(x,\overline{z})$ such that $-x \prec +\psi_{2,j}$. Then, by induction hypothesis, the following rule is derivable in ALBA for every formula χ':

$$(\mathsf{RA}(\psi_{2,j}))\ \frac{\psi_{2,j}(x,\overline{z}) \leq \chi'}{\mathsf{RA}(\psi_{2,j})(\chi'/u',\overline{z}) \leq x}$$

Moreover, by definition,

$$\mathsf{LA}(g(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \psi_{2,j}(x,\overline{z}))) = \mathsf{RA}(\psi_{2,j})(g^\flat_j(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, u)/u',\overline{z}). \tag{2.6.1}$$

Hence, we can show that $\mathsf{RA}(\phi_1)$ is a derivable ALBA-rule as follows: for every formula χ,

$$\cfrac{\cfrac{\cfrac{\chi \leq g(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \psi_{2,j}(x,\overline{z}))}{\psi_{2,j}(x,\overline{z}) \leq g^\flat_j(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \chi/u)}\ (\text{Residuation})}{\mathsf{RA}(\psi_{2,j})(g^\flat_j(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \chi/u)/u',\overline{z}) \leq x}\ \mathsf{RA}(\psi_{2,j})}{\mathsf{LA}(g(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \psi_{2,j}(x,\overline{z})))(\chi/u,\overline{z}) \leq x}\ (\text{Identity (2.6.1)})$$

To see that $\mathsf{LA}(\phi_1)(u, \overline{z})$ is a definite negative PIA-formula, one needs to show that every positive (resp. negative) node in $+\mathsf{LA}(\phi_1)(u, \overline{z})$ is labeled with a connective from $\mathcal{F}^*$ (resp. $\mathcal{G}^*$). This follows from the identity (2.6.1), the second part of the induction hypothesis (stating that $\mathsf{RA}(\psi_{2,j})(u', \overline{z})$ is definite negative PIA), the fact that $\mathsf{RA}(\psi_{2,j})(u', \overline{z})$ is negative in u', the fact that $g_j^\flat \in \mathcal{G}^*$ (and its corresponding node in $+\mathsf{LA}(\phi_1)(u, \overline{z})$ is signed $-$, as we have just remarked), the fact that the order-type of $g_j^\flat$ is the same as the order-type of g, the fact that every formula in $\overrightarrow{\phi(\overline{z})}$ is positive PIA, and for each ϕ in $\overrightarrow{\phi(\overline{z})}$,

$$-\phi \prec -g_j^\flat(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \chi/u) \prec +\mathsf{LA}(\phi_1)(u, \overline{z}),$$

and finally, the fact that every formula in $\overrightarrow{\psi_{-j}(\overline{z})}$ is negative PIA, and for each ψ in $\overrightarrow{\psi_{-j}(\overline{z})}$,

$$+\psi \prec -g_j^\flat(\overrightarrow{\phi(\overline{z})}, \overrightarrow{\psi_{-j}(\overline{z})}, \chi/u) \prec +\mathsf{LA}(\phi_1)(u, \overline{z}).$$

□

Theorem 50. *Every very restricted left-analytic (resp. right-analytic) inductive $\mathcal{L}_{\mathrm{DLE}}$-inequality can be equivalently transformed, via an ALBA-reduction, into a set of pure left-primitive (resp. right-primitive) $\mathcal{L}^*_{\mathrm{DLE}}$-inequalities.*

Proof. We only consider the case of the inequality $s \leq t$ being very restricted left-analytic (Ω, ε)-inductive, since the proof of the right-analytic case is dual. By assumption, t is a negative PIA formula (cf. page 30). Observe preliminarily that we can assume w.l.o.g. that there are no occurrences of $+\bot$ and $-\top$ in $+t$. Indeed, modulo exhaustive application of distribution rules, t can be equivalently written as the disjunction of definite negative PIA terms t_i. If $+\bot$ or $-\top$ occurred in $+t_i$ for some i, the exhaustive application of the rules which identify each $+f'(\phi_1, \ldots, \bot^{\varepsilon_{f'}}(i), \ldots, \phi_{n_{f'}})$ with $+\bot$ for every $f' \in \mathcal{F} \cup \{\wedge\}$ and each $-g'(\phi_1, \ldots, \top^{\varepsilon_{g'}}(i), \ldots, \phi_{n_{g'}})$ with $-\top$ for every $g' \in \mathcal{G} \cup \{\vee\}$ would identify t_i with $\bot$. Hence the offending subterm can be removed from the disjunction. Hence (cf. Remark 29), we can assume w.l.o.g. that t is left-primitive.

By assumption, $s := \xi(\vec{\phi}/\vec{x}, \vec{\psi}/\vec{y})$, where $\xi(!\vec{x}, !\vec{y})$ is a positive Skeleton-formula—cf. page 30—which is scattered, monotone in $\vec{x}$ and antitone in $\vec{y}$. Moreover, the formulas in $\vec{\phi}$ are positive PIA, and the formulas in $\vec{\psi}$ are negative PIA. Modulo exhaustive application of distribution and splitting rules of the standard ALBA preprocessing,[38] we can assume w.l.o.g. that the scattered positive Skeleton formula ξ is also definite. Modulo exhaustive application of the additional rules which identify $+f'(\phi_1, \ldots, \bot^{\varepsilon_{f'}}(i), \ldots, \phi_{n_{f'}})$ with $+\bot$ for every $f' \in \mathcal{F} \cup \{\wedge\}$ and $-g'(\phi_1, \ldots, \top^{\varepsilon_{g'}}(i), \ldots, \phi_{n_{g'}})$ with $-\top$ for every $g' \in \mathcal{G} \cup \{\vee\}$, which would reduce $s \leq t$ to a tautology, we can assume w.l.o.g. that there are no occurrences of $+\bot$ and $-\top$ in $+\xi$. Hence (cf. Remark 29) we can assume w.l.o.g. that ξ is scattered, definite and left-primitive. Therefore, the derived rule Approx(ξ) (cf. Corollary 36) is applicable, which justifies the last equivalence in the following chain:

[38] The applications of splitting rules at this stage give rise to a set of inequalities, each of which can be treated separately. In the remainder of the proof, we focus on one of them.

$$
\begin{aligned}
&\forall\vec{p}[\xi(\vec{\phi}/\vec{x},\vec{\psi}/\vec{y}) \leq t(\vec{p})]\\
\text{iff}\quad &\forall\vec{p}\forall\mathbf{j}\forall\mathbf{n}[(\mathbf{j} \leq \xi(\vec{\phi}/\vec{x},\vec{\psi}/\vec{y}) \ \&\ t(\vec{p}) \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{n}]\\
\text{iff}\quad &\forall\vec{p}\forall\mathbf{j}\forall\mathbf{n}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[(\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x},\vec{\mathbf{m}}/\vec{y}) \ \&\ \vec{\mathbf{i}} \leq \vec{\phi} \ \&\ \vec{\psi} \leq \vec{\mathbf{m}} \ \&\ t(\vec{p}) \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{n}]. \qquad (\mathsf{Approx}(\xi))
\end{aligned}
$$

By assumption, in each inequality $\vec{\mathbf{i}} \leq \vec{\phi}$ and $\vec{\psi} \leq \vec{\mathbf{m}}$ there is at least one ε-critical variable occurrence. Modulo exhaustive application of distribution rules (a')-(c') of Remark 18 and splitting rules, we can assume w.l.o.g. that each ϕ in $\vec{\phi}$ (resp. ψ in $\vec{\psi}$) is a definite positive (resp. negative) PIA-formula, which has exactly one ε-critical variable occurrence. That is, if $\vec{p_1}$ and $\vec{p_2}$ respectively denote the subarrays of $\vec{p}$ such that $\varepsilon(p_1) = 1$ for each p_1 in $\vec{p_1}$ and $\varepsilon(p_2) = \partial$ for each p_2 in $\vec{p_2}$, then each ϕ in $\vec{\phi}$ is either of the form $\phi_1(p_1/!x, \vec{p'}/\vec{z})$ with $+x \prec +\phi_1$, or of the form $\phi_2(p_2/!x, \vec{p'}/\vec{z})$ with $-x \prec +\phi_2$. Similarly, each ψ in $\vec{\psi}$ is either of the form $\psi_1(p_2/!x, \vec{p'}/\vec{z})$ with $+x \prec -\psi_1$, or of the form $\psi_2(p_1/!x, \vec{p'}/\vec{z})$ with $-x \prec -\psi_2$. Recall that each ϕ in $\vec{\phi}$ is definite positive PIA and each ψ in $\vec{\psi}$ is definite negative PIA. Hence, we can assume w.l.o.g. that there are no occurrences of $-\bot$ and $+\top$ in $+\phi$. Indeed, otherwise, the exhaustive application of the additional rules which identify $-f'(\phi_1, \ldots, \bot^{\varepsilon_{f'}}(i), \ldots, \phi_{n_{f'}})$ with $-\bot$ for every $f' \in \mathcal{F} \cup \{\wedge\}$ and $+g'(\phi_1, \ldots, \top^{\varepsilon_{g'}}(i), \ldots, \phi_{n_{g'}})$ with $+\top$ for every $g' \in \mathcal{G} \cup \{\vee\}$, would reduce all offending inequalities to tautological inequalities of the form $\mathbf{i} \leq \top$ which can then be removed. Likewise, we can assume w.l.o.g. that there are no occurrences of $+\bot$ and $-\top$ in $+\psi$. This shows (cf. Remark 29) that we can assume w.l.o.g. that each ϕ in $\vec{\phi}$ (resp. ψ in $\vec{\psi}$) is a right-primitive (resp. left-primitive) term. Moreover, by Lemma 49, the suitable derived adjunction rule among LA(ϕ_1), LA(ϕ_2), RA(ψ_1), RA(ψ_2) is applicable to each formula, yielding:

$$
\begin{aligned}
&\forall\vec{p_1}\forall\vec{p_2}\forall\mathbf{j}\forall\mathbf{n}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[(\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x},\vec{\mathbf{m}}/\vec{y}) \ \&\ \vec{\mathbf{i}} \leq \vec{\phi} \ \&\ \vec{\psi} \leq \vec{\mathbf{m}} \ \&\ t(\vec{p_1},\vec{p_2}) \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{n}]\\
\text{iff}\quad &\forall\vec{p_1}\forall\vec{p_2}\forall\mathbf{j}\forall\mathbf{n}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[(\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x},\vec{\mathbf{m}}/\vec{y}) \ \&\ \overrightarrow{\mathsf{LA}(\phi_1)(\mathbf{i}/u)} \leq \vec{p_1} \ \&\ \vec{p_2} \leq \overrightarrow{\mathsf{LA}(\phi_2)(\mathbf{i}/u)} \ \&\\
&\overrightarrow{\mathsf{RA}(\psi_2)(\mathbf{m}/u)} \leq \vec{p_1} \ \&\ \vec{p_2} \leq \overrightarrow{\mathsf{RA}(\psi_1)(\mathbf{m}/u)} \ \&\ t(\vec{p_1},\vec{p_2}) \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{n}].
\end{aligned}
$$

The assumptions made above imply that $t(\vec{p_1},\vec{p_2})$ is monotone in each variable in $\vec{p_1}$ and antitone in each variable $\vec{p_2}$. Hence, the quasi-inequality above is simultaneously in Ackermann shape w.r.t. all variables.[39] Applying the Ackermann rule repeatedly in the order indicated by Ω yields the following pure quasi-inequality:

$$\forall\mathbf{j}\forall\mathbf{n}\forall\vec{\mathbf{i}}\forall\vec{\mathbf{m}}[(\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x},\vec{\mathbf{m}}/\vec{y}) \ \&\ t(\vec{P_1}/\vec{p_1},\vec{P_2}/\vec{p_2}) \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{n}],$$

where P_1 and P_2 denote the pure $\mathcal{L}^*_{\mathrm{DLE}}$-terms obtained by applying the Ackermann-substitution. For instance, for every Ω-minimal p_1 in $\vec{p_1}$,

$$P_1 := \bigvee_i \mathsf{LA}(\phi_1^{(i)})(\mathbf{i}/u) \vee \bigvee_j \mathsf{RA}(\psi_2^{(j)})(\mathbf{m}/u),$$

[39] The formulas LA(ϕ_1)($\mathbf{i}/u$), LA(ϕ_2)($\mathbf{i}/u$), RA(ψ_1)($\mathbf{m}/u$), and RA(ψ_2)($\mathbf{m}/u$) do not need to be pure, and in general they are not. However, the assumptions and the general theory of ALBA guarantee that they are ε^∂-uniform and free of the variable the 'minimal valuation' of which they are part of. The reader is referred to [20] for an expanded treatment of this point.

and for every Ω-minimal p_2 in $\vec{p_2}$,

$$P_2 := \bigwedge_i \mathsf{LA}(\phi_2^{(i)})(\mathbf{i}/u) \wedge \bigwedge_j \mathsf{RA}(\psi_1^{(j)})(\mathbf{m}/u).$$

In the clauses above, the indexes i and j count the number of critical occurrences of the given variable p_1 (resp. p_2) in PIA-subterms of type ϕ_1 and ψ_2 (resp. ϕ_2 and ψ_1). The pure quasi-inequality above can be equivalently transformed into one pure inequality as follows:

$$\begin{array}{ll} & \forall \mathbf{j}\forall \mathbf{n}\forall \vec{\mathbf{i}}\forall \vec{\mathbf{m}}[(\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x}, \vec{\mathbf{m}}/\vec{y}) \ \&\ t(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2}) \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{n}] \\ \text{iff} & \forall \mathbf{j}\forall \vec{\mathbf{i}}\forall \vec{\mathbf{m}}[(\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x}, \vec{\mathbf{m}}/\vec{y}) \Rightarrow \forall \mathbf{n}[t(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2}) \leq \mathbf{n} \Rightarrow \mathbf{j} \leq \mathbf{n}]] \\ \text{iff} & \forall \mathbf{j}\forall \vec{\mathbf{i}}\forall \vec{\mathbf{m}}[\mathbf{j} \leq \xi(\vec{\mathbf{i}}/\vec{x}, \vec{\mathbf{m}}/\vec{y}) \Rightarrow \mathbf{j} \leq t(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2})] \\ \text{iff} & \forall \vec{\mathbf{i}}\forall \vec{\mathbf{m}}[\xi(\vec{\mathbf{i}}/\vec{x}, \vec{\mathbf{m}}/\vec{y}) \leq t(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2})]. \end{array}$$

To finish the proof, it remains to be shown that the inequality in the last clause above is left-primitive. This is a rather simple proof by induction on the maximum length of chains in Ω. The base case, when Ω is the discrete order (hence P_1 and P_2 are of the form displayed above), immediately follows from the polarity of ξ and t in $\vec{p_1}$ and $\vec{p_2}$, and by Lemma 49. The inductive step is routine. ☐

The Frege axiom in a pre-Heyting algebra setting. Let $\mathcal{F} = \varnothing$ and $\mathcal{G} = \{\rightharpoonup\}$, with $\rightharpoonup$ binary and of order-type $(\partial, 1)$. The logical connectives of the display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ arising from the basic $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$-logic can be represented synoptically as follows:

Structural symbols	I		;		>		≻		⊙	
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$\succ\!\!-$	$\rightarrow$		$\rightharpoonup$	$\bullet$	

As mentioned in Example 47,

$$p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)$$

is strictly right-primitive (Ω, ε)-inductive for $r <_\Omega p <_\Omega q$ and $\varepsilon(p, q, r) = (1, 1, \partial)$. Executing ALBA according to this choice of Ω and ε, we obtain:

$$\begin{array}{ll} & \forall p\forall q\forall r[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)] \\ \text{iff} & \forall p\forall q\forall r\forall \mathbf{j}\forall \mathbf{m}[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup r) \ \&\ (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall p\forall q\forall r\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup r) \ \&\ (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup \mathbf{n}) \leq \mathbf{m} \ \&\ r \leq \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall p\forall q\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup \mathbf{n}) \ \&\ (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall p\forall q\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}\forall \mathbf{i}[(\mathbf{j} \leq p \rightharpoonup (q \rightharpoonup \mathbf{n}) \ \&\ (p \rightharpoonup q) \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \ \&\ \mathbf{i} \leq p) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall q\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}\forall \mathbf{i}[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \ \&\ (\mathbf{i} \rightharpoonup q) \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall q\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}\forall \mathbf{i}\forall \mathbf{h}[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \ \&\ \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \ \&\ \mathbf{h} \leq \mathbf{i} \rightharpoonup q) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall q\forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}\forall \mathbf{i}\forall \mathbf{h}[(\mathbf{j} \leq \mathbf{i} \rightharpoonup (q \rightharpoonup \mathbf{n}) \ \&\ \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \ \&\ \mathbf{i} \bullet \mathbf{h} \leq q) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{j}\forall \mathbf{m}\forall \mathbf{n}\forall \mathbf{i}\forall \mathbf{h}[(\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \ \&\ \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}] \\ \text{iff} & \forall \mathbf{j}\forall \mathbf{n}\forall \mathbf{i}\forall \mathbf{h}[\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \Rightarrow \forall \mathbf{m}[\mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]] \\ \text{iff} & \forall \mathbf{j}\forall \mathbf{n}\forall \mathbf{i}\forall \mathbf{h}[\mathbf{j} \leq \mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \Rightarrow \mathbf{j} \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})] \\ \text{iff} & \forall \mathbf{n}\forall \mathbf{i}\forall \mathbf{h}[\mathbf{i} \rightharpoonup ((\mathbf{i} \bullet \mathbf{h}) \rightharpoonup \mathbf{n}) \leq \mathbf{h} \rightharpoonup (\mathbf{i} \rightharpoonup \mathbf{n})], \end{array}$$

The last inequality above is a pure right-primitive $\mathcal{L}^*_{DLE}$-inequality, and by Proposition 37 is equivalent on perfect DLE-algebras to

$$p \rightharpoonup ((q \bullet p) \rightharpoonup r) \leq p \rightharpoonup (q \rightharpoonup r).$$

By applying the usual procedure, we obtain the following rule:

$$p \rightharpoonup ((q \bullet p) \rightharpoonup r) \leq p \rightharpoonup (q \rightharpoonup r) \quad \rightsquigarrow \quad \frac{x \vdash p \rightharpoonup ((q \bullet p) \rightharpoonup r)}{x \vdash p \rightharpoonup (q \rightharpoonup r)} \quad \rightsquigarrow \quad \frac{X \vdash W \succ ((Y \odot W) \succ Z)}{X \vdash W \succ (Y \succ Z)}$$

2.6.3 Type 4: non-primitive terms on both sides

In all syntactic shapes of inequalities treated so far, the tail has been required to be primitive. This requirement is dropped in the syntactic shape treated in the present subsection. Let us start with a motivating example:

The Church-Rosser inequality. Let $\mathcal{F} = \{\Diamond\}$ and $\mathcal{G} = \{\Box\}$. The $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$-inequality $\Diamond\Box p \leq \Box\Diamond p$ is neither very restricted left-analytic inductive nor very restricted right-analytic inductive, given that neither side is primitive. However, the following ALBA reduction succeeds in transforming it into a pure left-primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequality:

	$\forall p[\Diamond\Box p \leq \Box\Diamond p]$	
iff	$\forall p[\blacklozenge\Diamond\Box p \leq \Diamond p]$	(Adjunction)
iff	$\forall p \forall \mathbf{i} \forall \mathbf{m}[\mathbf{i} \leq \blacklozenge\Diamond\Box p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(First approximation)
iff	$\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\mathbf{i} \leq \blacklozenge\Diamond \mathbf{j} \ \&\ \mathbf{j} \leq \Box p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Approx. rules for $\Diamond$, $\blacklozenge$)
iff	$\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\mathbf{i} \leq \blacklozenge\Diamond \mathbf{j} \ \&\ \blacklozenge \mathbf{j} \leq p \ \&\ \Diamond p \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Adjunction)
iff	$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[\mathbf{i} \leq \blacklozenge\Diamond \mathbf{j} \ \&\ \Diamond\blacklozenge \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Ackermann lemma)
iff	$\forall \mathbf{j}[\blacklozenge\Diamond \mathbf{j} \leq \Diamond\blacklozenge \mathbf{j}]$.	

Notice that this reduction departs in significant ways from the standard ALBA executions as described in Section 2.3.5, in that we have applied an adjunction rule other than a splitting rule *before* the first approximation step, that is, as part of the preprocessing, and to a *Skeleton* node. This rule application is sound, but would be redundant if our goal was restricted to calculating first-order correspondents of input formulas. Notice that this rule application succeeded in transforming the input inequality into the inequality $\blacklozenge\Diamond\Box p \leq \Diamond p$, which is very restricted left-analytic inductive (cf. Definition 45), and thus can be treated as indicated in the previous subsection. This example illustrates the ideas on which the treatment of the following class of inequalities is based:

Definition 51 (Restricted analytic inductive inequalities)**.** For any order type ε and any irreflexive and transitive relation Ω on the variables $\vec{p}$, the signed generation tree $*s$ ($* \in \{+, -\}$) of a term $s(p_1, \ldots p_n)$ is *analytic* (Ω, ε)*-inductive* if

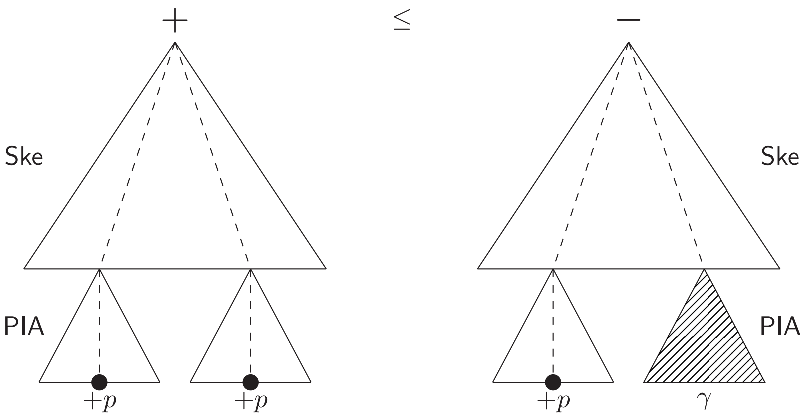
1. $*s$ is (Ω, ε)-inductive (cf. Definition 16);
2. every branch of $*s$ is good (cf. Definition 15).

An inequality $s \leq t$ in $\vec{p}$ is *restricted left-analytic* (resp. *right-analytic*) (Ω, ε)*-inductive* if

1. $+s$ (resp. $-t$) is restricted analytic (Ω, ε)-inductive (cf. Definition 45) and $-t$ (resp. $+s$) is analytic (Ω, ε)-inductive;
2. there exists exactly one ε^∂-uniform PIA subtree in $-t$ (resp. in $+s$) the root of which is attached to the Skeleton of $-t$ (resp. $+s$).

An inequality $s \leq t$ is *restricted analytic inductive* if it is restricted (right-analytic or left-analytic) (Ω, ε)-inductive for some Ω and ε.

Remark 52. The syntactic shape specified in the definition above can be intuitively understood with the help of the following picture, which illustrates the 'left-analytic' case:



As the picture shows, similarly to the very restricted analytic inductive inequalities, this syntactic shape forbids the root of any ε^∂-uniform subtree to be attached directly to the skeleton of the head of the inequality. However, in contrast to the very restricted analytic inductive inequalities, critical branches can appear now in the tail of the inequality. Finally, there exists a unique ε^∂-uniform subtree whose root is attached to the skeleton of the tail of the inequality. In the lemma below, we will denote the tail of a restricted left-analytic (resp. right-analytic) inductive inequality by $\xi(\gamma/!x, \vec{\psi}/\overline{z})$, where $\xi(!x, \overline{z})$ is a negative (resp. positive) skeleton term, and γ denotes the unique ε^∂-uniform PIA subtree attached to the skeleton, and for each $\psi \in \vec{\psi}$ with $*\psi \prec -\xi$ is a PIA subtree that contains a critical branch.

BNF presentation of analytic (Ω, ε)-inductive terms. In what follows, we adopt the following conventions: when writing e.g. $g(\vec{x}, \vec{y}, \vec{z}, \vec{w})$, we understand that the arrays of variables are of different lengths, which can be possibly 0, and moreover g is monotone in $\vec{x}$ and $\vec{z}$ and is antitone in $\vec{y}$ and $\vec{w}$. Let us first introduce the BNF (Backus-Naur Form) presentation of the ε^∂-uniform PIA terms γ and χ, which are substituted for positive and negative placeholder variables in the skeleton of analytic inductive terms respectively. This implies that we will only be interested in the signed generation trees $+\gamma$ and $-\chi$. Moreover, we will use the letter p (or $\vec{p}$) to indicate those variables which are assigned to 1 by ε, and the letter q (or $\vec{q}$) for those which are assigned to ∂.

$$\gamma := q \mid \bot \mid \top \mid \gamma \vee \gamma \mid \gamma \wedge \gamma \mid g(\vec{\gamma}/\vec{x}, \vec{\chi}/\vec{y}),$$

$$\chi := p \mid \top \mid \bot \mid \chi \wedge \chi \mid \chi \vee \chi \mid f(\vec{\chi}/\vec{x}, \vec{\gamma}/\vec{y}).$$

Next, let us introduce the BNF presentation of the non ε^∂-uniform PIA terms ϕ and ψ, which are substituted for positive and negative placeholder variables in the skeleton of analytic inductive terms respectively. This implies that we will only be interested in the signed generation trees $+\phi$ and $-\psi$. Let PosPIA and NegPIA respectively denote the sets of the ϕ- and ψ-terms. In addition, we will need—and define by simultaneous induction—the function

$$CVar : \text{PosPIA} \cup \text{NegPIA} \to \mathcal{P}(Var)$$

which maps each ϕ and ψ to the set of variables of which there are critical occurrences in ϕ and ψ.

$$\phi := p \mid \top \wedge \phi \mid \bot \wedge \phi \mid \phi \wedge \phi \mid \gamma \vee \phi \mid g(\vec{\gamma}/\vec{x}, \vec{\chi}/\vec{y}, \phi/z) \mid g(\vec{\gamma}/\vec{x}, \vec{\chi}/\vec{y}, \psi/w)$$

$$\psi := q \mid \bot \vee \psi \mid \top \wedge \psi \mid \psi \vee \psi \mid \chi \wedge \psi \mid f(\vec{\chi}/\vec{x}, \vec{\gamma}/\vec{y}, \psi/z) \mid f(\vec{\chi}/\vec{x}, \vec{\gamma}/\vec{y}, \phi/w).$$

In the two presentations above, the construction of the terms which have g or f as their main connectives is subject to the condition that all the variables in $CVar(\phi)$ (resp. $CVar(\psi)$)—where ϕ and ψ denote the immediate subformulas as indicated above—are common upper bounds of the variables occurring in $\vec{\gamma}$ and $\vec{\chi}$ w.r.t. Ω.

$$\begin{array}{rcl}
CVar(p) & = & \{p\} \\
CVar(q) & = & \{q\} \\
CVar(\phi_1 \wedge \phi_2) & = & CVar(\phi_1) \cup CVar(\phi_2) \\
CVar(\psi_1 \vee \psi_2) & = & CVar(\psi_1) \cup CVar(\psi_2) \\
CVar(g(\vec{\gamma}/\vec{x}, \vec{\chi}/\vec{y}, \phi/z)) & = & CVar(\phi) \\
CVar(f(\vec{\chi}/\vec{x}, \vec{\gamma}/\vec{y}, \psi/z)) & = & CVar(\psi) \\
CVar(g(\vec{\gamma}/\vec{x}, \vec{\chi}/\vec{y}, \psi/w)) & = & CVar(\psi) \\
CVar(f(\vec{\chi}/\vec{x}, \vec{\gamma}/\vec{y}, \phi/w)) & = & CVar(\phi)
\end{array}$$

Finally, let us introduce the BNF presentation of the analytic inductive terms s and t, which are to occur on the left-hand side and right-hand side of inequalities respectively. This implies that we will only be interested in the sign generation trees $+s$ and $-t$.

$$s := \gamma \mid \phi \mid s \vee s \mid s \wedge s \mid f(\vec{s}/\vec{x}, \vec{t}/\vec{y}),$$

$$t := \chi \mid \psi \mid t \wedge t \mid t \vee t \mid g(\vec{t}/\vec{x}, \vec{s}/\vec{y}).$$

Lemma 53. *For any $\mathcal{L}_{\mathrm{DLE}}$-inequality $s \leq t$,*

1. *if* $s \leq t = \xi(\gamma/!x, \vec{\psi}/\overline{z})$ *is restricted left-analytic* (Ω, ε)*-inductive such that* ξ *is definite and* $-x \prec -\xi$ *(resp.* $+x \prec -\xi$*), the adjunction rule LA(*ξ*) is applicable and yields the equivalent inequality* $\mathsf{LA}(\xi)(s/u, \vec{\psi}) \leq \gamma$ *(resp.* $\gamma \leq \mathsf{LA}(\xi)(s/u, \vec{\psi})$*), which is very restricted left-analytic (resp. right-analytic)* (Ω, ε)*-inductive.*

2. *if* $\xi(\gamma/!x, \vec{\psi}/\overline{z}) = s \leq t$ *is restricted right-analytic* (Ω, ε)*-inductive such that* ξ *is definite and* $+x \prec +\xi$ *(resp.* $-x \prec +\xi$*), the adjunction rule RA(*ξ*) is applicable and yields the equivalent inequality* $\gamma \leq \mathsf{RA}(\xi)(t/u, \vec{\psi})$ *(resp.* $\mathsf{RA}(\xi)(t/u, \vec{\psi}) \leq \gamma$*), which is very restricted right-analytic (resp. left-analytic)* (Ω, ε)*-inductive.*

Proof. We only show the first item in the case $-x \prec -\xi$, the remaining cases being similar. The assumptions imply (cf. Lemma 49) that the rule $LA(\xi)$ is applicable to $s \leq t$ so as to obtain the inequality $\mathsf{LA}(\xi)(s/u, \vec{\psi}) \leq \gamma$, and that $\mathsf{LA}(\xi)(s/u, \overline{z})$ is a definite negative PIA formula. Since the polarities of $\overline{z}$ do not change under the application of adjunction rules and the polarity of u is positive, in $\mathsf{LA}(\xi)(s/u, \vec{\psi})$ the subtree of each $\psi \in \vec{\psi}$ remains a PIA subtree with at least one critical branch, and the branches running through s remain good. Hence, $+\mathsf{LA}(\xi)(s/u, \vec{\psi})$ is (Ω, ε)-inductive, all of its branches are good, and all of its maximal ε^∂-uniform PIA subtrees occur as immediate subtrees of SRR nodes of some ε-critical branches. That is, $\mathsf{LA}(\xi)(s/u, \vec{\psi})$ is a restricted analytic inductive term. Furthermore, γ is negative PIA, and $-\gamma$ is ε^∂-uniform by assumption. From the above observations it follows that $\mathsf{LA}(\xi)(s/u, \vec{\psi}) \leq \gamma$ is a very restricted left-analytic (Ω, ε)-inductive inequality. □

Corollary 54. *Every restricted left-analytic (resp. right-analytic) inductive* $\mathcal{L}_{\mathrm{DLE}}$*-inequality can be equivalently transformed, via an ALBA-reduction, into a set of pure left-primitive (resp. right-primitive)* $\mathcal{L}^*_{\mathrm{DLE}}$*-inequalities.*

Proof. We only consider the case of the inequality $s \leq t = \xi(\gamma/!x, \vec{\psi}/\overline{z})$ being restricted left-analytic (Ω, ε)-inductive, and $+x \prec -\xi$, the remaining cases being similar. Modulo exhaustive application of distribution and splitting rules of the standard ALBA preprocessing,[40] we can assume w.l.o.g. that the negative Skeleton formula ξ is also definite. By Lemma 53, the adjunction rule $\mathsf{LA}(\xi)$ is applicable and yields the equivalent inequality $\gamma \leq \mathsf{LA}(\xi)(s/u, \vec{\psi})$, which is very restricted right-analytic (Ω, ε)-inductive. Hence the statement follows by Theorem 50. □

The Frege inequality, again. Early on (cf. page 59), we have discussed the Frege inequality as an example of very restricted right-analytic (Ω, ε)-inductive inequality for $r <_\Omega p <_\Omega q$ and $\varepsilon(p, q, r) = (1, 1, \partial)$. Here below, we provide an alternative solving strategy based on the fact that the Frege inequality is also a restricted left-analytic (Ω, ε)-inductive inequality for $\varepsilon(p, q, r) = (1, 1, 1)$.

[40] The applications of splitting rules at this stage give rise to a set of inequalities, each of which can be treated separately. In the remainder of the proof, we focus on one of them.

$$
\begin{array}{lll}
& \forall p \forall q \forall r[p \rightharpoonup (q \rightharpoonup r) \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)] & \\
\text{iff} & \forall p \forall q \forall r[(p \rightharpoonup q) \bullet (p \rightharpoonup (q \rightharpoonup r)) \leq p \rightharpoonup r] & \text{(Residuation)} \\
\text{iff} & \forall p \forall q \forall r[p \bullet ((p \rightharpoonup q) \bullet (p \rightharpoonup (q \rightharpoonup r))) \leq r] & \text{(Residuation)} \\
\text{iff} & \forall p \forall q \forall r \forall \mathbf{i} \forall \mathbf{m}[\mathbf{i} \leq p \bullet ((p \rightharpoonup q) \bullet (p \rightharpoonup (q \rightharpoonup r))) \ \&\ r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(First approx.)} \\
\text{iff} & \forall p \forall q \forall r \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{h} \forall \mathbf{m}[\mathbf{i} \leq \mathbf{j} \bullet (\mathbf{k} \bullet \mathbf{h}) \ \& & \\
& \mathbf{j} \leq p \ \&\ \mathbf{k} \leq p \rightharpoonup q \ \&\ \mathbf{h} \leq p \rightharpoonup (q \rightharpoonup r) \ \&\ r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Approx.)} \\
\text{iff} & \forall p \forall q \forall r \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{h} \forall \mathbf{m}[\mathbf{i} \leq \mathbf{j} \bullet (\mathbf{k} \bullet \mathbf{h}) \ \& & \\
& \mathbf{j} \leq p \ \&\ p \bullet \mathbf{k} \leq q \ \&\ q \bullet (p \bullet \mathbf{h}) \leq r \ \&\ r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Residuation)} \\
\text{iff} & \forall q \forall r \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{h} \forall \mathbf{m}[\mathbf{i} \leq \mathbf{j} \bullet (\mathbf{k} \bullet \mathbf{h}) \ \& & \\
& \mathbf{j} \bullet \mathbf{k} \leq q \ \&\ q \bullet (\mathbf{j} \bullet \mathbf{h}) \leq r \ \&\ r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Ackermann)} \\
\text{iff} & \forall r \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{h} \forall \mathbf{m}[\mathbf{i} \leq \mathbf{j} \bullet (\mathbf{k} \bullet \mathbf{h}) \ \&\ (\mathbf{j} \bullet \mathbf{k}) \bullet (\mathbf{j} \bullet \mathbf{h}) \leq r \ \&\ r \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Ackermann)} \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{h} \forall \mathbf{m}[\mathbf{i} \leq \mathbf{j} \bullet (\mathbf{k} \bullet \mathbf{h}) \ \&\ (\mathbf{j} \bullet \mathbf{k}) \bullet (\mathbf{j} \bullet \mathbf{h}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Ackermann)} \\
\text{iff} & \forall \mathbf{j} \forall \mathbf{k} \forall \mathbf{h}[\mathbf{j} \bullet (\mathbf{k} \bullet \mathbf{h}) \leq (\mathbf{j} \bullet \mathbf{k}) \bullet (\mathbf{j} \bullet \mathbf{h})]. &
\end{array}
$$

The last inequality above is a pure left-primitive $\mathcal{L}^*_{DLE}$-inequality, and by Proposition 37 is equivalent on perfect DLE-algebras to

$$p \bullet (q \bullet r) \leq (p \bullet q) \bullet (p \bullet r).$$

By applying the usual procedure, we obtain the following rule:

$$p \bullet (q \bullet r) \leq (p \bullet q) \bullet (p \bullet r) \quad \rightsquigarrow \quad \frac{(p \bullet q) \bullet (p \bullet r) \vdash y}{p \bullet (q \bullet r) \vdash y} \quad \rightsquigarrow \quad \frac{(X \odot Y) \odot (X \odot Z) \vdash W}{X \odot (Y \odot Z) \vdash W}$$

The non-primitive Fischer Servi inequality. For the $\mathcal{L}_{\mathrm{DLE}}$-setting specified as in Example 34, the Fischer Servi inequality $\Diamond(p \to q) \leq \Box p \to \Diamond q$ is restricted right-analytic (Ω, ε)-inductive w.r.t. the discrete order Ω and $\varepsilon(p, q) = (1, \partial)$. Let us apply the procedure indicated in the proof of Corollary 54 to it:

$$
\begin{array}{lll}
& \forall p \forall q[\Diamond(p \to q) \leq \Box p \to \Diamond q] & \\
\text{iff} & \forall p \forall q[p \to q \leq \blacksquare(\Box p \to \Diamond q)] & \text{(Adjunction)} \\
\text{iff} & \forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq p \to q \ \&\ \blacksquare(\Box p \to \Diamond q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(First approx.)} \\
\text{iff} & \forall p \forall q \forall \mathbf{i} \forall m \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq p \to q \ \&\ \blacksquare(\mathbf{j} \to \mathbf{n}) \leq \mathbf{m} \ \&\ \mathbf{j} \leq \Box p \ \&\ \Diamond q \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Approx.)} \\
\text{iff} & \forall p \forall q \forall \mathbf{i} \forall m \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq p \to q \ \&\ \blacksquare(\mathbf{j} \to \mathbf{n}) \leq \mathbf{m} \ \&\ \blacklozenge \mathbf{j} \leq p \ \&\ q \leq \blacksquare \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Adjunction)} \\
\text{iff} & \forall \mathbf{i} \forall m \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq \blacklozenge \mathbf{j} \to \blacksquare \mathbf{n} \ \&\ \blacksquare(\mathbf{j} \to \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] & \text{(Ackermann)} \\
\text{iff} & \forall \mathbf{j} \forall \mathbf{n}[\blacklozenge \mathbf{j} \to \blacksquare \mathbf{n} \leq \blacksquare(\mathbf{j} \to \mathbf{n})]. &
\end{array}
$$

The last inequality above is a pure right-primitive $\mathcal{L}^*_{DLE}$-inequality, and by Proposition 37 is equivalent on perfect DLE-algebras to

$$\blacklozenge p \to \blacksquare q \leq \blacksquare(p \to q).$$

By applying the usual procedure, we obtain the following rule:

$$\blacklozenge p \to \blacksquare q \leq \blacksquare(p \to q) \quad \rightsquigarrow \quad \frac{x \vdash \blacklozenge p \to \blacksquare q}{x \vdash \blacksquare(p \to q)} \quad \rightsquigarrow \quad \frac{X \vdash \bullet Y > \bullet Z}{X \vdash \bullet(Y > Z)}$$

The 'transitivity' axiom. For the $\mathcal{L}_{\mathrm{DLE}}$-setting in which we discussed the Frege inequality (cf. page 59), the inequality $(p \rightharpoonup q) \bullet (q \rightharpoonup r) \leq p \rightharpoonup r$ is restricted right-analytic (Ω, ε)-inductive w.r.t. the order $p <_\Omega q$ and $\varepsilon(p, q, r) = (1, \partial, \partial)$. Let us apply the procedure indicated in the proof of Corollary 54 to it:

	$\forall p\forall q\forall r[(p \rightharpoonup q) \bullet (q \rightharpoonup r) \leq p \rightharpoonup r]$	
iff	$\forall p\forall q\forall r[q \rightharpoonup r \leq (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r)]$	(Adjunction)
iff	$\forall pqr\forall \mathbf{i}\forall \mathbf{m}[(\mathbf{i} \leq q \rightharpoonup r \ \&\ (p \rightharpoonup q) \rightharpoonup (p \rightharpoonup r) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(First approx.)
iff	$\forall pqr\forall \mathbf{imhjn}[(\mathbf{i} \leq q \rightharpoonup r \ \&\ \mathbf{j} \rightharpoonup (\mathbf{h} \rightharpoonup \mathbf{n}) \leq \mathbf{m}\ \&$ $\mathbf{h} \leq p \ \&\ r \leq \mathbf{n} \ \&\ \mathbf{j} \leq p \rightharpoonup q) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Approx.)
iff	$\forall q\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{h}\forall \mathbf{j}\forall \mathbf{n}[(\mathbf{i} \leq q \rightharpoonup \mathbf{n} \ \&\ \mathbf{j} \rightharpoonup (\mathbf{h} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \ \&\ \mathbf{j} \leq \mathbf{h} \rightharpoonup q) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Ackermann)
iff	$\forall q\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{h}\forall \mathbf{j}\forall \mathbf{n}[(\mathbf{i} \leq q \rightharpoonup \mathbf{n} \ \&\ \mathbf{j} \rightharpoonup (\mathbf{h} \rightharpoonup \mathbf{n}) \leq \mathbf{m} \ \&\ \mathbf{h} \bullet \mathbf{j} \leq q) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Adjunction)
iff	$\forall \mathbf{i}\forall \mathbf{m}\forall \mathbf{h}\forall \mathbf{j}\forall \mathbf{n}[(\mathbf{i} \leq (\mathbf{h} \bullet \mathbf{j}) \rightharpoonup \mathbf{n} \ \&\ \mathbf{j} \rightharpoonup (\mathbf{h} \rightharpoonup \mathbf{n}) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$	(Ackermann)
iff	$\forall \mathbf{h}\forall \mathbf{j}\forall \mathbf{n}[(\mathbf{h} \bullet \mathbf{j}) \rightharpoonup \mathbf{n} \leq \mathbf{j} \rightharpoonup (\mathbf{h} \rightharpoonup \mathbf{n})]$.	

The last inequality above is a pure right-primitive $\mathcal{L}^*_{DLE}$-inequality, and by Proposition 37 is equivalent on perfect DLE-algebras to

$$(p \bullet q) \rightharpoonup r \leq q \rightharpoonup (p \rightharpoonup r).$$

By applying the usual procedure, we obtain the following rule:

$$(p \bullet q) \rightharpoonup r \leq q \rightharpoonup (p \rightharpoonup r) \quad \rightsquigarrow \quad \frac{x \vdash (p \bullet q) \rightharpoonup r}{x \vdash q \rightharpoonup (p \rightharpoonup r)} \quad \rightsquigarrow \quad \frac{X \vdash (Y \odot Z) \succ W}{X \vdash Z \succ (Y \succ W)}$$

2.7 Analytic inductive inequalities and analytic rules

In the present section, we address the most general syntactic shape considered in the chapter: in the following subsection we define the class of analytic inductive inequalities, and show that each of them can be equivalently transformed into (a set of) analytic structural rules (which are in fact quasi-special). In Subsection 2.7.2, we also show that any analytic rule is semantically equivalent to some analytic inductive inequality. Thus, the DLE-logics axiomatized by means of analytic inductive inequalities are exactly the properly displayable ones.

2.7.1 From analytic inductive inequalities to quasi-special rules

Let us start with a motivating example:

The pre-linearity axiom. Let $\mathcal{F} = \varnothing$, $\mathcal{G} = \{\rightharpoonup\}$ where $\rightharpoonup$ is binary and of order-type $(\partial, 1)$.

I		;		≻	
$\top$	$\bot$	$\wedge$	$\vee$		$\rightharpoonup$

The following inequality

$$\top \leq (p \rightharpoonup q) \vee (q \rightharpoonup p)$$

is not restricted analytic inductive for any order-type: indeed, all the non-leaf nodes of the right-hand are Skeleton, and the PIA subterms are reduced to the variables. The inequality above is not restricted right-analytic for any order-type ε, since the right-hand side contains ε^∂-uniform PIA-subterms attached to the skeleton. It is not restricted left-analytic for any order-type ε, since the right-hand side contains more than one ε^∂-uniform PIA-subterm.

We have not found an ALBA-reduction suitable to extend the strategy of the previous section so as to equivalently transform the inequality above into one or more primitive inequalities. However, the following ALBA reduction, exclusively based on applications of a modified (inverted) Ackermann rule (the soundness of which is proved in Lemma 57 below) and adjunction rules, transforms the inequality above into a quasi-inequality which gives rise to an analytic (in fact quasi-special, cf. Definition 8) structural rule.

$$
\begin{array}{l}
\forall p \forall q[\top \leq (p \rightharpoonup q) \vee (q \rightharpoonup p)] \\
\forall p \forall q \forall \vec{r}[(r_1 \leq p \ \&\ q \leq r_2 \ \&\ r_3 \leq q \ \&\ p \leq r_4) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \\
\forall q \forall \vec{r}[(r_1 \leq r_4 \ \&\ q \leq r_2 \ \&\ r_3 \leq q) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)] \\
\forall \vec{r}[(r_1 \leq r_4 \ \&\ r_3 \leq r_2) \Rightarrow \top \leq (r_1 \rightharpoonup r_2) \vee (r_3 \rightharpoonup r_4)].
\end{array}
$$

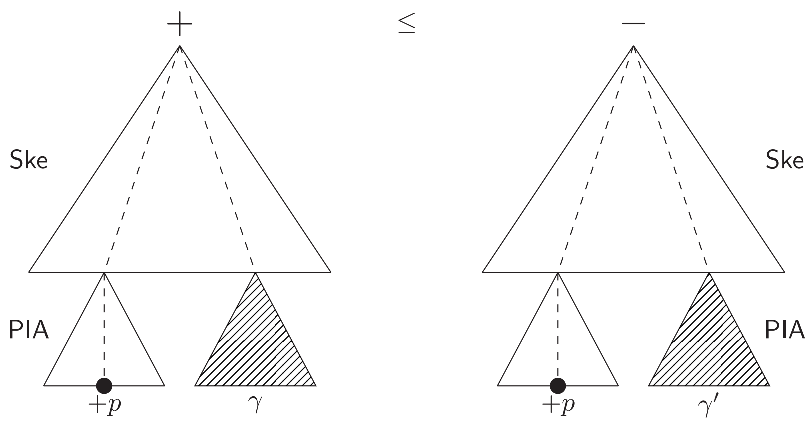
The last quasi-inequality above expresses the validity of the following quasi-special structural rule on perfect DLEs:

$$
\frac{X \vdash W \qquad Z \vdash Y}{\mathrm{I} \vdash (X \succ Y)\,;\,(Z \succ W)}
$$

We will see that the solving strategy applied to the example above can be applied to the following class of inequalities:

Definition 55 (Analytic inductive inequalities)**.** For every order type ε and every irreflexive and transitive relation Ω on the variables $p_1, \ldots p_n$, an inequality $s \leq t$ is *analytic* (Ω, ε)*-inductive* if $+s$ and $-t$ are both (Ω, ε)-analytic inductive (cf. Definition 51). An inequality $s \leq t$ is *analytic inductive* if is (Ω, ε)-analytic inductive for some Ω and ε.

Remark 56. The syntactic shape specified in the definition above can be intuitively understood with the help of the following picture:



As the picture shows, the difference between analytic inductive inequalities and restricted analytic inductive inequalities is that, in the latter, there can be exactly one ε^∂-uniform subterm attached to the skeleton of the inequality, while in the former this requirement is dropped.

Below, we discuss a slightly modified version of the Ackermann rule, which will be used in the proof of Proposition 59.

Lemma 57. *Let $s(!\vec{q})$ and $t(!\vec{r})$ be $\mathcal{L}_{\mathrm{DLE}}$-terms such that the propositional variables in the arrays $!\vec{q}$ and $!\vec{r}$ are disjoint. Let ε be the unique order-type on the concatenation $!\vec{q} \oplus !\vec{r}$ with respect to which $s \leq t$ is ε-uniform. Let $\vec{\alpha}$ and $\vec{\beta}$ be arrays of DLE-terms of the same length as $\vec{q}$ and $\vec{r}$ respectively, and such that no variable in $\vec{q} \oplus \vec{r}$ occurs in any α or β. Then the following are equivalent for any perfect DLE $\mathbb{A}$:*

1. $\mathbb{A} \models s(\vec{\alpha}/\vec{q}) \leq t(\vec{\beta}/\vec{r})$;
2. $\mathbb{A} \models \forall\vec{q}\forall\vec{r}[\&_{q\in\vec{q},r\in\vec{r}}(q \leq^{\varepsilon(q)} \alpha \;\&\; \beta \leq^{\varepsilon(r)} r) \Rightarrow s(!\vec{q}) \leq t(!\vec{r})]$,

where $\leq^1:=\leq$ and $\leq^\partial:=\geq$.

Proof. Let us assume 1. To show 2., fix an interpretation v of the variables in $\mathbb{A}$ such that $(\mathbb{A}, v) \models q \leq^{\varepsilon(q)} \alpha$ and $(\mathbb{A}, v) \models \beta \leq^{\varepsilon(r)} r$ for each q in $\vec{q}$ and r in $\vec{r}$. Hence, assumption 1. and the ε-uniformity of $s \leq t$ imply that $[\![s(\vec{q})]\!]_v \leq [\![s(\vec{\alpha}/\vec{q})]\!]_v \leq [\![t(\vec{\beta}/\vec{r})]\!]_v \leq [\![t(\vec{r})]\!]_v$, which proves that $(\mathbb{A}, v) \models s(\vec{q}) \leq t(\vec{r})$, as required. Conversely, fix a valuation v, and notice that the truth of the required condition $(\mathbb{A}, v) \models s(\vec{\alpha}/\vec{q}) \leq t(\vec{\beta}/\vec{r})$ does not depend on where v maps the variables in $\vec{q} \oplus \vec{r}$, since none of these variables occurs in $s(\vec{\alpha}/\vec{q}) \leq t(\vec{\beta}/\vec{r})$. Hence, it is enough to show that $(\mathbb{A}, v') \models s(\vec{\alpha}/\vec{q}) \leq t(\vec{\beta}/\vec{r})$ for some $\vec{q}\oplus\vec{r}$-variant v' of v. Let v' be the $\vec{q}\oplus\vec{r}$-variant of v such that $[\![q_i]\!]_{v'} := [\![\alpha_i]\!]_v = [\![\alpha_i]\!]_{v'}$ and $[\![r_j]\!]_{v'} := [\![\beta_j]\!]_v = [\![\beta_j]\!]_{v'}$. Then clearly, $(\mathbb{A}, v') \models q \leq^{\varepsilon(q)} \alpha$ and $(\mathbb{A}, v') \models \beta \leq^{\varepsilon(r)} r$. By assumption 2, this implies that $(\mathbb{A}, v') \models s(\vec{q}) \leq t(\vec{r})$, which is equivalent by construction to the required $(\mathbb{A}, v') \models s(\vec{\alpha}/\vec{q}) \leq t(\vec{\beta}/\vec{r})$. □

Remark 58. Notice that, in the quasi-inequality in item 2 of the statement of the lemma above, each variable q in $\vec{q}$ and r in $\vec{r}$ occurs twice, i.e. once in exactly one inequality in the antecedent and once in the conclusion of the quasi-inequality. These two occurrences have the same polarity in the two inequalities. For example, if q is in $\vec{q_1}$ and $\varepsilon(q) = \partial$, then q occurs negatively in the conclusion of the quasi-inequality, and also negatively in the inequality $q \leq^{\varepsilon(q)} \phi$, which can be rewritten as $\phi \leq q$. The remaining cases are analogous and left to the reader.

Proposition 59. *Every analytic (Ω, ε)-inductive inequality can be equivalently transformed, via an ALBA-reduction, into a set of quasi-special structural rules.*

Proof. By assumption, $s \leq t$ is of the form

$$\xi_1(\vec{\phi}_1/\vec{x}_1, \vec{\psi}_1/\vec{y}_1, \vec{\gamma}_1/\vec{z}_1, \vec{\chi}_1/\vec{w}_1) \leq \xi_2(\vec{\psi}_2/\vec{x}_2, \vec{\phi}_2/\vec{y}_2, \vec{\chi}_2/\vec{z}_2, \vec{\gamma}_2/\vec{w}_2),$$

where $\xi_1(!\vec{x}_1, !\vec{y}_1, !\vec{z}_1, !\vec{w}_1)$ and $\xi_2(!\vec{x}_2, !\vec{y}_2, !\vec{z}_2, !\vec{w}_2)$ respectively are a positive and a negative Skeleton-formula (cf. page 30) which are scattered, monotone in $\vec{x}$ and $\vec{z}$ and antitone in $\vec{y}$ and $\vec{w}$. Moreover, the formulas in $\vec{\phi}$ and $\vec{\gamma}$ are positive PIA, and the formulas in $\vec{\psi}$ and $\vec{\chi}$ are negative PIA. Finally, every ϕ and ψ contains at least one ε-critical variable, whereas all $+\gamma$ and $-\chi$ are ε^∂-uniform. Modulo exhaustive application

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of distribution and splitting rules of the standard ALBA preprocessing,[41] we can assume w.l.o.g. that the scattered Skeleton formulas ξ_1 and ξ_2 are also definite. Modulo exhaustive application of the additional rules which identify $+f'(\phi_1, \ldots, \bot^{\varepsilon_{f'}}(i), \ldots, \phi_{n_{f'}})$ with $+\bot$ for every $f' \in \mathcal{F} \cup \{\wedge\}$ and $-g'(\phi_1, \ldots, \top^{\varepsilon_{g'}}(i), \ldots, \phi_{n_{g'}})$ with $-\top$ for every $g' \in \mathcal{G} \cup \{\vee\}$, which would reduce $s \leq t$ to a tautology, we can assume w.l.o.g. that there are no occurrences of $+\bot$ and $-\top$ in $+\xi_1$ and $-\xi_2$. Hence (cf. Remark 29) we can assume w.l.o.g. that ξ_1 (resp. ξ_2) is scattered, definite and left-primitive (resp. right-primitive). The following equivalence is justified by Lemma 57 (in what follows, we write e.g. $\vec{q}_{1,6} \leq \vec{\phi}$ to represent concisely both $\vec{q_1} \leq \vec{\phi_1}$ and $\vec{q_6} \leq \vec{\phi_2}$):

$$\begin{array}{ll} & \forall \vec{p}[s \leq t] \\ \text{iff} & \forall \vec{p}\forall \vec{q}[(\vec{q}_{1,6} \leq \vec{\phi} \ \&\ \vec{\psi} \leq \vec{q}_{2,5} \ \&\ \vec{q}_{3,8} \leq \vec{\gamma} \ \&\ \vec{\chi} \leq \vec{q}_{4,7}) \Rightarrow \xi_1(\vec{q_1}, \vec{q_2}, \vec{q_3}, \vec{q_4}) \leq \xi_2(\vec{q_5}, \vec{q_6}, \vec{q_7}, \vec{q_8})]. \end{array}$$

By assumption, in each inequality $q \leq \phi$ (resp. $\psi \leq q$) in $\vec{q}_{1,6} \leq \vec{\phi}$ (resp. $\vec{\psi} \leq \vec{q}_{2,5}$) there is at least one ε-critical occurrence of some variable in $\vec{p}$. Modulo exhaustive application of distribution rules (a')-(c') of Remark 18 and splitting rules, we can assume w.l.o.g. that each ϕ in $\vec{\phi}$ (resp. ψ in $\vec{\psi}$) is a definite positive (resp. negative) PIA-formula, which has exactly one ε-critical variable occurrence. That is, if $\vec{p_1}$ and $\vec{p_2}$ respectively denote the subarrays of $\vec{p}$ such that $\varepsilon(p_1) = 1$ for each p_1 in $\vec{p_1}$ and $\varepsilon(p_2) = \partial$ for each p_2 in $\vec{p_2}$, then each ϕ in $\vec{\phi}$ is either of the form $\phi_+(p_1/!x, \vec{p'}/\overline{z})$ with $+x \prec +\phi_+$, or of the form $\phi_-(p_2/!x, \vec{p'}/\overline{z})$ with $-x \prec +\phi_-$. Similarly, each ψ in $\vec{\psi}$ is either of the form $\psi_+(p_1/!x, \vec{p'}/\overline{z})$ with $+x \prec -\psi_+$, or of the form $\psi_-(p_2/!x, \vec{p'}/\overline{z})$ with $-x \prec -\psi_-$. Recall that each ϕ in $\vec{\phi}$ is definite positive PIA and each ψ in $\vec{\psi}$ is definite negative PIA. Hence, we can assume w.l.o.g. that there are no occurrences of $-\bot$ and $+\top$ in each $+\phi$. Indeed, otherwise, the exhaustive application of the additional rules which identify $-f'(\phi_1, \ldots, \bot^{\varepsilon_{f'}}(i), \ldots, \phi_{n_{f'}})$ with $-\bot$ for every $f' \in \mathcal{F} \cup \{\wedge\}$ and $+g'(\phi_1, \ldots, \top^{\varepsilon_{g'}}(i), \ldots, \phi_{n_{g'}})$ with $+\top$ for every $g' \in \mathcal{G} \cup \{\vee\}$, would reduce all offending inequalities to tautological inequalities of the form $q \leq \top$ or $\bot \leq q$ which can then be removed. Likewise, we can assume w.l.o.g. that there are no occurrences of $+\bot$ and $-\top$ in each $+\psi$. This shows (cf. Remark 29) that we can assume w.l.o.g. that each ϕ in $\vec{\phi}$ (resp. ψ in $\vec{\psi}$) is a right-primitive (resp. left-primitive) term. Likewise, we can assume w.l.o.g. that each γ in $\vec{\gamma}$ (resp. χ in $\vec{\chi}$) is right-primitive (resp. left-primitive). By Lemma 49, the suitable derived adjunction rule among LA(ϕ_+), LA(ϕ_-), RA(ψ_+), RA(ψ_-) is applicable to each ϕ and ψ, yielding:

$$\begin{array}{ll} & \forall \vec{p}\forall \vec{q}[(\vec{q}_{1,6} \leq \vec{\phi} \ \&\ \vec{\psi} \leq \vec{q}_{2,5} \ \&\ \vec{q}_{3,8} \leq \vec{\gamma} \ \&\ \vec{\chi} \leq \vec{q}_{4,7}) \Rightarrow \xi_1(\vec{q_1}, \vec{q_2}, \vec{q_3}, \vec{q_4}) \leq \xi_2(\vec{q_5}, \vec{q_6}, \vec{q_7}, \vec{q_8})] \\ \text{iff} & \forall \vec{p_1}\forall \vec{p_2}\forall \vec{q}[(\overrightarrow{\mathsf{LA}(\phi_+)(q/u)} \leq \vec{p_1} \ \&\ \vec{p_2} \leq \overrightarrow{\mathsf{LA}(\phi_-)(q/u)} \ \&\ \overrightarrow{\mathsf{RA}(\psi_+)(q/u)} \leq \vec{p_1} \ \& \\ & \vec{p_2} \leq \overrightarrow{\mathsf{RA}(\psi_-)(q/u)} \ \&\ \vec{q}_{3,8} \leq \vec{\gamma} \ \&\ \vec{\chi} \leq \vec{q}_{4,7}) \Rightarrow \xi_1(\vec{q_1}, \vec{q_2}, \vec{q_3}, \vec{q_4}) \leq \xi_2(\vec{q_5}, \vec{q_6}, \vec{q_7}, \vec{q_8})]. \end{array}$$

Notice that, when applying the adjunction/residuation rules, the polarity of subterms which are parametric in the rule application remains unchanged. Hence, the assumption that there are no occurrences of $-\bot$ and $+\top$ in each $+\phi$ and there are no occurrences of $+\bot$ and $-\top$ in each $+\psi$ implies that that there are no occurrences of $-\bot$ and $+\top$ in each $+$LA(ϕ_-)(q/u) and $+$RA(ψ_-)(q/u), which are then shown to be right-primitive, and there are no occurrences of $+\bot$ and $-\top$ in each $+$LA(ϕ_+)(q/u) and $+$RA(ψ_+)(q/u), which are then shown to be left-primitive. The assumptions made above imply that each γ is antitone in each variable in $\vec{p_1}$ and monotone in each variable

[41] The applications of splitting rules at this stage give rise to a set of inequalities, each of which can be treated separately. In the remainder of the proof, we focus on one of them.

in $\vec{p_2}$, while each χ is monotone in each variable in $\vec{p_1}$ and antitone in each variable in $\vec{p_2}$. Hence, the quasi-inequality above is simultaneously in Ackermann shape w.r.t. all variables in $\vec{p}$.[42] Applying the Ackermann rule repeatedly in the order indicated by Ω yields the following quasi-inequality, free of variables in $\vec{p}$:

$$\forall\vec{q}[(\vec{q}_{3,8} \leq \vec{\gamma}(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2}) \ \&\ \vec{\chi}(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2}) \leq \vec{q}_{4,7}) \Rightarrow \xi_1(\vec{q_1}, \vec{q_2}, \vec{q_3}, \vec{q_4}) \leq \xi_2(\vec{q_5}, \vec{q_6}, \vec{q_7}, \vec{q_8})], \tag{2.7.1}$$

where P_1 and P_2 denote the $\mathcal{L}^*_{\mathrm{DLE}}$-terms, with variables in $\vec{q_1}, \vec{q_2}, \vec{q_5}, \vec{q_6}$, obtained by applying the Ackermann-substitution. For instance, for every Ω-minimal p_1 in $\vec{p_1}$,

$$P_1 := \bigvee_i \mathsf{LA}(\phi_+^{(i)})(q/u) \vee \bigvee_j \mathsf{RA}(\psi_+^{(j)})(q/u),$$

and for every Ω-minimal p_2 in $\vec{p_2}$,

$$P_2 := \bigwedge_i \mathsf{LA}(\phi_-^{(i)})(q/u) \wedge \bigwedge_j \mathsf{RA}(\psi_-^{(j)})(q/u).$$

In the clauses above, the indexes i and j count the number of critical occurrences of the given variable p_1 (resp. p_2) in PIA-subterms of type ϕ_+ and ψ_+ (resp. ϕ_- and ψ_-).

Let us show that the quasi-inequality (2.7.1) represents the validity in perfect DLEs of some analytic (in fact quasi-special, cf. Definition 8) structural rule of the calculus $\mathbf{DL}$. We have already observed above that ξ_1 is left-primitive and ξ_2 is right-primitive. Hence, the conclusion of the quasi-inequality (2.7.1) can be understood as the semantic interpretation of some structural sequent (cf. Definition 31). To see that each inequality in the antecedent of (2.7.1) is also the interpretation of some structural sequent, it is enough to show that every $\gamma(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2})$ is right-primitive, and every $\chi(\vec{P_1}/\vec{p_1}, \vec{P_2}/\vec{p_2})$ is left-primitive. Indeed, if this is the case, then we can apply distribution rules exhaustively so as to surface the $+\vee$ and $-\wedge$, and then apply splitting rules to obtain definite left-primitive and right-primitive inequalities. By Definition 31 each of these inequalities will be the interpretation of some structural sequent.

This is a rather simple proof by induction on the maximum length of chains in Ω. The base case, when Ω is the discrete order (hence P_1 and P_2 are of the form displayed above), immediately follows from the observation, made above, that each γ is right-primitive, antitone in each variable in $\vec{p_1}$ and monotone in each variable $\vec{p_2}$, while each χ is left-primitive, monotone in each variable in $\vec{p_1}$ and antitone in each variable $\vec{p_2}$, and by Lemma 49. The inductive step is routine.

Let us show that the rule so obtained is analytic (cf. Definition 4), that is, it satisfies conditions C_1-C_7. As to C_1, notice that each variable q in $\vec{q_i}$ for $1 \leq i \leq 8$ appears in some inequality in the antecedent of the initial quasi-inequality, and has not been eliminated in any ensuing transformations. This implies that each q gives rise to a parametric structural variable X which occurs in some premise and in the conclusion. Condition C_2 is guaranteed by construction: indeed, the congruence relation is defined as the transitive closure of the relation identifying only the occurrences of the structural variable

[42] The formulas $\mathsf{LA}(\phi_+)(q/u)$, $\mathsf{LA}(\phi_-)(q/u)$, $\mathsf{RA}(\psi_+)(q/u)$, and $\mathsf{RA}(\psi_-)(q/u)$ do not need to be free of all variables in $\vec{p}$, and in general they are not. However, the assumptions and the general theory of ALBA guarantee that they are ε^∂-uniform and free of the specific p-variable the 'minimal valuation' of which they are part of. The reader is referred to [20] for an expanded treatment of this point.

X corresponding to one variable q. Condition C_3 is also guaranteed by construction, given that each variable q occurs exactly once in $\xi_1(\vec{q_1}, \vec{q_2}, \vec{q_3}, \vec{q_4}) \leq \xi_2(\vec{q_5}, \vec{q_6}, \vec{q_7}, \vec{q_8})$. Condition C_4 follows from Remark 58, and the fact that adjunction rules and usual Ackermann rule preserve the polarity of the variables. Condition C_5 vacuously holds, since all constituents of structural rules are parametric. Conditions C_6 and C_7 are immediate.

Finally, observe that the rule we have obtained is in fact quasi-special. Indeed, the variables $\vec{q_3}, \vec{q_4}, \vec{q_7}, \vec{q_8}$ are fresh, and each of them occurs only once in the premises. □

2.7.2 From analytic rules to analytic inductive inequalities

In the previous section, we introduced the syntactic shape of analytic inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities for any language $\mathcal{L}_{\mathrm{DLE}}$, and showed that these inequalities can be effectively transformed via ALBA into a set of analytic structural rules of the associated display calculus $\mathbf{DL}_{\mathrm{DLE}}$. In the present section, we show that having this shape is also a necessary condition.

Lemma 60. *Let $s(\vec{q}, \vec{r})$, $t(\vec{q}, \vec{r})$, $\overrightarrow{\alpha(\vec{q}, \vec{r})}$ and $\overrightarrow{\beta(\vec{q}, \vec{r})}$ be $\mathcal{L}_{\mathrm{DLE}}$-terms such that t and each α are monotone in $\vec{r}$ and antitone in $\vec{q}$, and s and each β are monotone in $\vec{q}$ and antitone in $\vec{r}$. Then the following are equivalent for any DLE $\mathbb{A}$:*

1. $\mathbb{A} \models s(\vec{q} \wedge \vec{\alpha}/\vec{q}, \vec{r} \vee \vec{\beta}/\vec{r}) \leq t(\vec{q} \wedge \vec{\alpha}/\vec{q}, \vec{r} \vee \vec{\beta}/\vec{r})$;
2. $\mathbb{A} \models \forall \vec{q} \forall \vec{r}[(\vec{q} \leq \vec{\alpha} \ \& \ \vec{\beta} \leq \vec{r}) \Rightarrow s(\vec{q}, \vec{r}) \leq t(\vec{q}, \vec{r})]$

Proof. Assume item 1. To show item 2, fix a valuation v such that $(\mathbb{A}, v) \models \vec{q} \leq \vec{\alpha}$ and $(\mathbb{A}, v) \models \vec{\beta} \leq \vec{r}$. Hence, $(\mathbb{A}, v) \models \vec{q} \wedge \vec{\alpha} = \vec{q}$ and $(\mathbb{A}, v) \models \vec{r} \vee \vec{\beta} = \vec{r}$. By item 1, $(\mathbb{A}, v) \models s(\vec{q} \wedge \vec{\alpha}/\vec{q}, \vec{r} \vee \vec{\beta}/\vec{r}) \leq t(\vec{q} \wedge \vec{\alpha}/\vec{q}, \vec{r} \vee \vec{\beta}/\vec{r})$, which is equivalent to $(\mathbb{A}, v) \models s(\vec{q}, \vec{r}) \leq t(\vec{q}, \vec{r})$, as required.

Conversely, assume item 2 and fix a valuation v. Clearly, $(\mathbb{A}, v) \models \vec{q} \wedge \vec{\alpha} \leq \vec{\alpha}$ and $(\mathbb{A}, v) \models \vec{\beta} \leq \vec{r} \vee \vec{\beta}$. Since each α (resp. β) is monotone (resp. antitone) in $\vec{r}$ and antitone (resp. monotone) in $\vec{q}$, this implies that

$$(\mathbb{A}, v) \models \overrightarrow{\alpha(\vec{q}, \vec{r})} \leq \overrightarrow{\alpha((\vec{q} \wedge \vec{\alpha})/\vec{q}, (\vec{r} \vee \vec{\beta})/\vec{r})} \qquad (\mathbb{A}, v) \models \overrightarrow{\beta((\vec{q} \wedge \vec{\alpha})/\vec{q}, (\vec{r} \vee \vec{\beta})/\vec{r})} \leq \overrightarrow{\beta(\vec{q}, \vec{r})},$$

which immediately entail that

$$(\mathbb{A}, v) \models \vec{q} \wedge \overrightarrow{\alpha(\vec{q}, \vec{r})} \leq \overrightarrow{\alpha((\vec{q} \wedge \vec{\alpha})/\vec{q}, (\vec{r} \vee \vec{\beta})/\vec{r})} \qquad (\mathbb{A}, v) \models \overrightarrow{\beta((\vec{q} \wedge \vec{\alpha})/\vec{q}, (\vec{r} \vee \vec{\beta})/\vec{r})} \leq \overrightarrow{\beta(\vec{q}, \vec{r})} \vee \vec{r}.$$

Let v' be the $\vec{q} \oplus \vec{r}$-variant of v such that $\overrightarrow{v'(q)} := \overrightarrow{v(q)} \wedge \overrightarrow{[\![\alpha]\!]_v}$ and $\overrightarrow{v'(r)} := \overrightarrow{v(r)} \vee \overrightarrow{[\![\beta]\!]_v}$. By definition, the conditions above are equivalent to

$$(\mathbb{A}, v') \models \vec{q} \leq \overrightarrow{\alpha(\vec{q}, \vec{r})} \qquad (\mathbb{A}, v') \models \overrightarrow{\beta(\vec{q}, \vec{r})} \leq \vec{r}.$$

Hence, by assumption 2, we can conclude that $(\mathbb{A}, v') \models s \leq t$, which is equivalent to $(\mathbb{A}, v) \models s(\vec{q} \wedge \vec{\alpha}/\vec{q}, \vec{r} \vee \vec{\beta}/\vec{r}) \leq t(\vec{q} \wedge \vec{\alpha}/\vec{q}, \vec{r} \vee \vec{\beta}/\vec{r})$, as required. □

Proposition 61. *For any language $\mathcal{L}_{\mathrm{DLE}}$, every analytic rule in the language of the corresponding calculus $\mathbf{DL}$ is semantically equivalent to some analytic inductive $\mathcal{L}^*_{\mathrm{DLE}}$-inequality.*

Proof. Modulo application of display postulates, any analytic rule can be represented as follows:

$$\frac{(S^i_j \vdash Y^i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i) \quad (X^k \vdash T^k_\ell \mid 1 \leq k \leq m \text{ and } 1 \leq \ell \leq m_k)}{(S \vdash T)[Y^i]^{suc}[X^k]^{pre}}$$

where Y^i and X^k are structural variables and S^i_j, T^k_ℓ, S and T are structural terms. As discussed in Section 2.4.2, for every perfect $\mathcal{L}_{\mathrm{DLE}}$-algebra $\mathbb{A}$, the validity of the rule above on $\mathbb{A}$ is equivalent to the validity on $\mathbb{A}$ of the following quasi-inequality

$$\mathop{\&}_{\substack{1\leq i\leq n\\ 1\leq k\leq m}} \left(\bigvee_{1\leq j\leq n_i} s^i_j \leq p'_i \quad \& \quad q'_k \leq \bigwedge_{1\leq \ell\leq m_k} t^k_\ell \right) \Rightarrow s \leq t$$

where $p'_i := r(Y^i)$, and $q'_k := l(X^k)$, for each i and k, and $s := l(S)$, $t := r(T)$, and $s^i_j := l(S^i_j)$, and $t^k_\ell := r(T^k_\ell)$ for each j and ℓ. Let $s_i := \bigvee_{1\leq j\leq n_i} s^i_j$ and $t_k := \bigwedge_{1\leq \ell\leq m_k} t^k_\ell$.

By Lemma 60, the validity of the quasi-inequality above is equivalent to the validity of the following inequality, where $\vec{p'} := (p'_i)_i$, $\vec{s} := (s_i)_i$, $\vec{q'} := (q'_k)_k$ and $\vec{t} := (t_k)_k$:

$$s((\vec{p'} \vee \vec{s})/\vec{p'}, (\vec{q'} \wedge \vec{t})/\vec{q'}) \leq t((\vec{p'} \vee \vec{s})/\vec{p'}, (\vec{q'} \wedge \vec{t})/\vec{q'}). \tag{2.7.2}$$

To finish the proof, we need to show that the inequality above is analytic (Ω, ε)-inductive for some Ω and ε. Let $\vec{p}$, $\vec{q}$ be the variables in the inequality $s \leq t$, different from the variables in $\vec{p'}$ and $\vec{q'}$, and occurring in $s \vdash t$ in antecedent and succedent position respectively (by C$_4$ they are disjoint). Clearly, $s(\vec{p}, \vec{q}, \vec{p'}, \vec{q'})$ is left-primitive, and hence is positive skeleton, and $t(\vec{p}, \vec{q}, \vec{p'}, \vec{q'})$ is right-primitive, and hence is negative skeleton. Condition C$_3$ implies that X^k and Y^i are in antecedent and succedent position respectively in $S \vdash T$, and hence s (resp. t) is monotone in $\vec{q'}$ (resp. in $\vec{p'}$) and antitone in $\vec{p'}$ (resp. $\vec{q'}$). Moreover, s_i is left-primitive, and hence is negative PIA for every i, and t_k is right-primitive, and hence is positive PIA for every k. These observations immediately yield that every branch in the inequality (2.7.2) is good, and in particular, $\vec{s}$ and $\vec{t}$ are the PIA-parts.

Next, let ε be the order-type which assigns all p in $\vec{p}$ and q' in $\vec{q'}$ to 1 and all q in $\vec{q}$ and q in $\vec{p'}$ to ∂. Let Ω be the discrete order. To show that the inequality (2.7.2) is analytic (Ω, ε)-inductive, it is enough to show that all terms in $\vec{s}$ and $\vec{t}$ are ε^∂-uniform.

Since any p in $\vec{p}$ corresponds to a structural variable antecedent position, $+p \prec +s_i$ and $+p \prec -t_k$ for all i and k, hence $-p \prec -s_i$ and $-p \prec +t_k$ for all i and k. This shows that $\vec{s}$ and $\vec{t}$ are ε^∂-uniform in any p in $\vec{p}$. Similar arguments relative to the variables in $\vec{q}$, $\vec{p'}$ and $\vec{q'}$ complete the proof. □

Remark 62. If the rule

$$\frac{(S^i_j \vdash Y^i \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n_i) \quad (X^k \vdash T^k_\ell \mid 1 \leq k \leq m \text{ and } 1 \leq \ell \leq m_k)}{(S \vdash T)[Y^i]^{suc}[X^k]^{pre}}$$

is quasi-special, then, in order to transform it into an analytic inequality as in the proof of the proposition above, we can use Lemma 57 rather than Lemma 60, which yields the inequality

$$s(\vec{s}/\vec{p'}, \vec{t}/\vec{q'}) \leq t(\vec{s}/\vec{p'}, \vec{t}/\vec{q'}), \tag{2.7.3}$$

which is equivalent to (2.7.2). Indeed, all variables in $\vec{p'}$ occur only in positive position and can hence be equivalently replaced by $\bot$ and all variables in $\vec{q'}$ occur only in negative position and can be equivalently replaced by $\top$, yielding

$$s((\vec{\bot} \vee \vec{s})/\vec{p'}, (\vec{\top} \wedge \vec{t})/\vec{q'}) \leq t((\vec{\bot} \vee \vec{s})/\vec{p'}, (\vec{\top} \wedge \vec{t})/\vec{q'}), \tag{2.7.4}$$

which is equivalent to (2.7.3). We will come back to this observation in the following section.

2.8 Special rules are as expressive as analytic rules

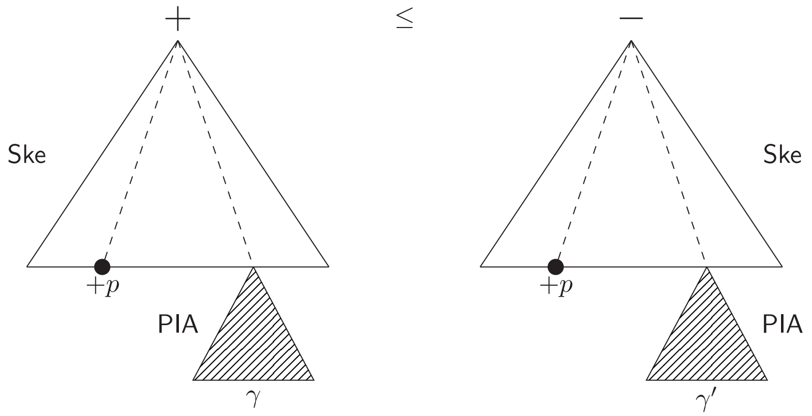
In [39], Kracht states without proof that every analytic rule in the display calculus for the classical basic tense logic Kt is equivalent to a special rule (see also the discussion in [10, Section 5.1]). A proof of this fact is presented in [10], where it is shown that, in classical tense logic, every axiom which is obtained from an analytic rule of the display calculus is equivalent to a primitive axiom. In the present section, we extend this result from classical tense logic to any DLE-logic. Namely, we show, using ALBA, that every analytic inductive inequality in any DLE-language is equivalent to some primitive inequality in the corresponding DLE*-language. We will proceed in two steps: in Section 2.8.1, we will present an intermediate subclass of analytic inductive inequalities, referred to as quasi-primitive inequalities, and show that any analytic inductive inequality can be equivalently transformed into some quasi-primitive inequality. Then, in Section 2.8.2, we will prove that every quasi-primitive inequality is equivalent to some primitive inequality.

These results imply that special structural rules (cf. Definition 6) are as expressive as analytic rules (cf. Definition 4). Hence, for any language $\mathcal{L}_{\mathrm{DLE}}$, any properly displayable DLE-logic is specially displayable. Notice that this fact does not imply that any properly displayable $\mathcal{L}_{\mathrm{DLE}}$-logic can be axiomatized by means of primitive $\mathcal{L}_{\mathrm{DLE}}$-inequalities, since the required primitive inequalities pertain to the language $\mathcal{L}^*_{\mathrm{DLE}}$. However, this fact does imply that any properly displayable $\mathcal{L}^*_{\mathrm{DLE}}$-logic can be axiomatized by means of primitive $\mathcal{L}^*_{\mathrm{DLE}}$-inequalities.

2.8.1 Quasi-special rules and inductive inequalities

Let us take stock of what was presented in Sections 2.7.1 and 2.7.2. Taken together, Proposition 61 and 59 immediately imply that every analytic rule is equivalent to a quasi-special rule. Furthermore, any analytic inductive inequality derived from an analytic rule has a special shape: every critical branch consists only of Skeleton nodes, leaving all PIA subtrees to be ε^∂-uniform. This motivates the following definition:

Definition 63. For every analytic (Ω, ε)-inductive inequality $s \leq t$, if every ε-critical branch of the signed generation trees $+s$ and $-t$ consists solely of skeleton nodes, then $s \leq t$ is a *quasi-special inductive inequality*. Such an inequality is *definite* if none of its Skeleton nodes is $+\vee$ or $-\wedge$.



Definite quasi-special inductive inequalities and quasi-special rules entertain the same privileged relation with each other as the one entertained by definite primitive inequalities and special rules. Indeed, translating into an inequality the rule obtained from a definite quasi-special inductive inequality leads to the original inequality (cf. Remark 62). Notice that these are exactly the inequalities that have this property, since the inequality that is obtained by Proposition 61 is always definite quasi-special inductive. Since every analytic inductive inequality is equivalent to a set of analytic rules (in fact quasi-special rules) and every analytic rule is equivalent to a definite quasi-special inductive inequality, is it clear that every analytic inductive inequality is equivalent to a set of definite quasi-special inductive inequalities.

2.8.2 Quasi-special inductive and primitive inequalities

The following propositions generalize [10, Lemma 5.12].

Proposition 64. *Let $\xi_1(!\vec{x}, !\vec{y}, !\vec{z}, !\vec{w})$ be a definite positive Skeleton formula and $\xi_2(\vec{x}, \vec{y})$ be a positive Skeleton formula such that $+\vec{x}, +\vec{z} \prec +\xi_1$, $-\vec{y}, -\vec{w} \prec +\xi_1$ and $-\vec{x}, +\vec{y} \prec -\xi_2$. Let $\overrightarrow{\gamma(\vec{p}, \vec{q})}$ be an array of positive PIA-formulas such that $-\vec{p}, +\vec{q} \prec +\gamma$ and let $\overrightarrow{\chi(\vec{p}, \vec{q})}$ be an array of negative PIA formulas such that $-\vec{p}, +\vec{q} \prec -\chi$. Then the following are equivalent:*

1. $\forall\vec{p}\forall\vec{q}[\xi_1(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{\gamma}/\vec{z}, \vec{\chi}/\vec{w}) \leq \xi_2(\vec{p}/\vec{x}, \vec{q}/\vec{y})]$;
2. $\forall\vec{p}\forall\vec{q}\forall\vec{p'}\forall\vec{q'}$

$$\left[\begin{array}{lcl} \xi_1(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{p'}/\vec{z}, \vec{q'}/\vec{w}) & \leq & \xi_2(\vec{p}/\vec{x}, \vec{q}/\vec{y}) \vee \\ & & (\bigvee_{z_k \in \vec{z}} \xi_1(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{p'}_{-k}/\vec{z}_{-k}, p'_k \prec\!\!- \gamma_k/z_k, \vec{q'}/\vec{w})) \vee \\ & & (\bigvee_{w_\ell \in \vec{w}} \xi_1(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{p'}/\vec{z}, \vec{q'}_{-\ell}/\vec{w}_{-\ell}, \chi_\ell \to q'_\ell/w_\ell)) \end{array}\right].$$

Proof. The inequality in item 1 of the statement can be equivalently transformed via ALBA into the following quasi-inequality:

$$\forall \mathbf{i} \forall \vec{\mathbf{j}} \forall \vec{\mathbf{j}}' \forall \vec{\mathbf{n}} \forall \vec{\mathbf{n}}' \forall \mathbf{m}[(\overrightarrow{\mathbf{j}'} \leq \overrightarrow{\gamma(\vec{\mathbf{j}}, \vec{\mathbf{n}})} \ \&\ \overrightarrow{\chi(\vec{\mathbf{j}}, \vec{\mathbf{n}})} \leq \overrightarrow{\mathbf{n}'} \ \&\ \xi_2(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{m}) \Rightarrow \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}') \leq \mathbf{m}]. \tag{2.8.1}$$

Likewise, the inequality in item 2 can be equivalently transformed via ALBA into the following quasi-inequality:

$$\forall \mathbf{i} \forall \vec{\mathbf{j}} \forall \vec{\mathbf{j}}' \forall \vec{\mathbf{n}} \forall \vec{\mathbf{n}}' \forall \mathbf{m} \left[\left(\begin{array}{c} \&_k\, \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}'_{-k}, \mathbf{j}'_k \prec \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}), \vec{\mathbf{n}}') \leq \mathbf{m} \\ \&_\ell\, \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}'_{-\ell}, \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \to \mathbf{n}'_\ell) \leq \mathbf{m} \\ \xi_2(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{m} \end{array} \right) \Rightarrow \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}') \leq \mathbf{m} \right]. \tag{2.8.2}$$

To finish the proof, it is enough to show that conditions (2.8.1) and (2.8.2) are equivalent. Assume condition (2.8.2) and let $\vec{\mathbf{j}}$ $\vec{\mathbf{n}}$, $\mathbf{m}$, $\mathbf{j}'_k$ and $\mathbf{n}'_\ell$ be such that the following inequalities hold for any γ_k in $\vec{\gamma}$ and χ_ℓ in $\vec{\chi}$:

$$\mathbf{j}'_k \leq \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \quad \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{n}'_\ell \quad \xi_2(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{m}. \tag{2.8.3}$$

By applying adjunction all inequalities above but the last one become

$$\mathbf{j}'_k \prec \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}) = \bot \quad \top = \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \to \mathbf{n}'_\ell.$$

These equalities imply that

$$\underset{k}{\&}\, \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}'_{-k}, \mathbf{j}'_k \prec \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}), \vec{\mathbf{n}}') = \bot \quad \underset{\ell}{\&}\, \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}'_{-\ell}, \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \to \mathbf{n}'_\ell) = \bot.$$

Indeed, by assumption, ξ_1 is a definite positive Skeleton formula such that any variable in it occurs at most once. Hence, ξ_1 is a definite and scattered left-primitive formula. By Lemma 35, the term function induced by ξ_1 is an operator, and hence ξ_1 preserves $\bot$ in its positive coordinates and reverses $\top$ in its negative coordinates. This finishes the proof that all $\vec{\mathbf{j}}$ $\vec{\mathbf{n}}$, $\mathbf{m}$, $\mathbf{j}'_k$ and $\mathbf{n}'_\ell$ satisfying conditions (2.8.3) satisfy also the premises of the quasi-inequality (2.8.2), namely:

$$\underset{k}{\&}\, \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}'_{-k}, \mathbf{j}'_k \prec \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}), \vec{\mathbf{n}}') \leq \mathbf{m} \quad \underset{\ell}{\&}\, \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}'_{-\ell}, \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \to \mathbf{n}'_\ell) \leq \mathbf{m} \quad \xi_2(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{m}.$$

By assumption (2.8.2), we conclude that $\xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}') \leq \mathbf{m}$, as required.

Conversely, assume condition (2.8.1) and let $\vec{\mathbf{j}}$ $\vec{\mathbf{n}}$, $\mathbf{m}$, $\mathbf{j}'_k$ and $\mathbf{n}'_\ell$ be such that the following inequalities hold for any k and ℓ as above:

$$\xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}'_{-k}, \mathbf{j}'_k \prec \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}), \vec{\mathbf{n}}') \leq \mathbf{m} \quad \xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}'_{-\ell}, \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \to \mathbf{n}'_\ell) \leq \mathbf{m} \quad \xi_2(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{m}. \tag{2.8.4}$$

By applying the appropriate residuation rules, all but the last inequality above can be equivalently written as follows:

$$\mathbf{j}'_k \prec \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathsf{RA}(\xi_1)(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}'_{-k}, \mathbf{m}/u, \vec{\mathbf{n}}') \quad \mathsf{RA}(\xi_1)(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}'_{-\ell}, \mathbf{m}/u) \leq \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \to \mathbf{n}'_\ell$$

and by applying residuation once again we obtain for every k and ℓ:

$$\mathbf{j}'_k \leq \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \vee \mathsf{RA}(\xi_1)(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}'_{-k}, \mathbf{m}/u, \vec{\mathbf{n}}') \quad \mathsf{RA}(\xi_1)(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}}', \vec{\mathbf{n}}'_{-\ell}, \mathbf{m}/u) \wedge \chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{n}'_\ell.$$

Since each $\mathbf{j}'_k$ and each $\mathbf{n}'_\ell$ is a nominal and a conominal respectively, they are interpreted as join-prime and meet-prime elements respectively. If $\mathbf{j}'_k \leq \gamma_k(\vec{\mathbf{j}}, \vec{\mathbf{n}})$ and $\chi_\ell(\vec{\mathbf{j}}, \vec{\mathbf{n}}) \leq \mathbf{n}'_\ell$ for all k and ℓ, then the antecedent of 2.8.1 is satisfied and hence we conclude $\xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}'}, \vec{\mathbf{n}'}) \leq \mathbf{m}$. Finally, if $\mathbf{j}'_k \leq \mathsf{RA}(\xi_1)(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}'}_{-k}, \mathbf{m}/u, \vec{\mathbf{n}'})$ or $\mathsf{RA}(\xi_1)(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}'}, \vec{\mathbf{n}'}_{-\ell}, \mathbf{m}/u) \leq \mathbf{n}'_\ell$ for some $\mathbf{j}'_k$ or $\mathbf{n}'_\ell$, then by applying the appropriate residuation rule we immediately obtain that $\xi_1(\vec{\mathbf{j}}, \vec{\mathbf{n}}, \vec{\mathbf{j}'}, \vec{\mathbf{n}'}) \leq \mathbf{m}$. □

2

The following proposition is order-dual to the previous one, hence its proof is omitted.

Proposition 65. *Let $\xi_1(\vec{x}, \vec{y})$ be a negative Skeleton formula and $\xi_2(!\vec{x}, !\vec{y}, !\vec{z}, !\vec{w})$ be a definite negative Skeleton formula such that $-\vec{x}, +\vec{y} \prec +\xi_1$, $+\vec{x}, +\vec{z} \prec -\xi_2$ and $-\vec{y}, -\vec{w} \prec -\xi_2$. Let $\overrightarrow{\gamma(\vec{p}, \vec{q})}$ be an array of positive PIA-formulas such that $-\vec{p}, +\vec{q} \prec +\gamma$ and let $\overrightarrow{\chi(\vec{p}, \vec{q})}$ be an array of negative PIA formulas such that $-\vec{p}, +\vec{q} \prec -\chi$. Then the following are equivalent:*

1. $\forall \vec{p} \forall \vec{q} [\xi_1(\vec{p}/\vec{x}, \vec{q}/\vec{y}) \leq \xi_2(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{\gamma}/\vec{z}, \vec{\chi}/\vec{w})]$

2. $\forall \vec{p} \forall \vec{q} \forall \vec{p'} \forall \vec{q'}$

$$\left[\begin{array}{lcl} \xi_1(\vec{p}/\vec{x}, \vec{q}/\vec{y}) \wedge & & \\ (\bigwedge_{z_k \in \vec{z}} \xi_2(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{p'}_{-k}/\vec{z}_{-k}, p'_k \prec \gamma_k/z_k, \vec{q'}/\vec{w})) \wedge & & \\ (\bigwedge_{w_\ell \in \vec{w}} \xi_2(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{p'}/\vec{z}, \vec{q'}_{-\ell}/\vec{w}_{-\ell}, \chi_\ell \to q'_\ell/w_\ell)) & \leq & \xi_2(\vec{p}/\vec{x}, \vec{q}/\vec{y}, \vec{p'}/\vec{z}, \vec{q'}/\vec{w}) \end{array}\right].$$

Corollary 66. *For any language $\mathcal{L}_{\mathrm{DLE}}$, every analytic structural rule in the language of the corresponding display calculus $\mathbf{DL}$ can be equivalently transformed into some special structural rule in the same language.*

Proof. As discussed at the beginning of Section 2.8.1, any analytic structural rule in $\mathbf{DL}$ is equivalent to a definite quasi-special inequality in $\mathcal{L}^*_{\mathrm{DLE}}$. It is easy to see that every definite quasi-special inequality can be transformed into one inequality of the form of item (1) in Propositions 64 or 65. This transformation is effected by applying suitable residuation rules so as to reduce one side of the given inequality to an ε^∂-uniform PIA subterm (analogously to the treatment of the Type 4 inequalities discussed in Section 2.6.3). Hence, either Propositions 64 or 65 is applicable, yielding an equivalent inequality as in item 2 of the propositions mentioned above. Finally, the inequality in item 2 of Proposition 64 (resp. 65) is definite left-primitive (resp. right-primitive). Hence, the statement follows by Lemma 32. □

2.9 Two methodologies: a sketch of a comparison

The generalizations of Kracht's results presented in Sections 2.5–2.8 are alternative to those proposed in [9, 10], and the aim of the present section is connecting and comparing these two generalizations. Such a comparison is not straightforward, since the methodologies the two generalizations rely on are different: while the treatment in [9, 10] relies purely on proof-theoretic notions and is therefore *internal* to proof theory, the present is *external*, in that is based on a theory (unified correspondence) originating in the model theory of modal logic, developed independently of proof theory, and whose connections with proof theory have not been systematically explored before. As to the basic settings for the two generalizations, the basic setting of the treatment in [9, 10] is given by the so called *amenable calculi* (the definition of which is reported on in Definition 70 below), which are defined for an arbitrary logical signature by means of conditions concerning the *performances* of the calculus (requiring e.g. that sequents of certain shapes be derivable) rather than the specific shape of the rules of the calculus. For any logical language, and any amenable calculus $\mathcal{C}$, the class of axioms which is proven to give rise to analytic structural rules is defined parametrically in $\mathcal{C}$, as a certain subcollection of the set $\mathcal{I}_2(\mathcal{C})$ of those "formulae A whose logical connectives can be eliminated by applying the invertible logical rules [of $\mathcal{C}$] to the premises of those rules obtained by applying some invertible rules to $\mathbf{I} \vdash A$ followed by [the Ackermann lemma]". The subcollection just mentioned is the one of *acyclic* formulas, which is defined again taking $\mathcal{C}$ as a parameter. In the present chapter, the basic environment is given by the class of perfect DLEs, which provides the common semantic environment for both the language of ALBA and for display calculi. In this setting, the logical connectives pertaining to the 'expansion' of the lattice signature are classified into two sets $\mathcal{F}$ and $\mathcal{G}$, according to the order-theoretic properties of their algebraic interpretations. Hence, any DLE-signature is uniquely determined by the sets of logical connectives/function symbols $\mathcal{F}$ and $\mathcal{G}$, which are taken as parameters of the language $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$. The display calculus $\mathbf{DL}$, the language and rules of the appropriate version of ALBA, and the inductive $\mathcal{L}_{\mathrm{DLE}}$-inequalities are then defined parametrically in $\mathcal{F}$ and $\mathcal{G}$ and are hence unique for each choice of $\mathcal{F}$ and $\mathcal{G}$. In Section 2.13, we sketch the proof that, for each $\mathcal{F}$ and $\mathcal{G}$, the associated display calculus $\mathbf{DL}$ is amenable, and in Section 2.14, we show that acyclic inequalities in $\mathcal{I}_2(\mathbf{DL})$ can be identified with analytic inductive inequalities.

Notwithstanding their different and mutually independent starting points, once a concrete setting is defined which provides a common ground for the application of the two methodologies, it is not difficult to recognize striking similarities between the algorithm defined in [9, 10] for computing analytic structural rules from input analytic inductive inequalities and the ALBA-based procedure illustrated in Section 2.7.1. In what follows, we are not giving a formal proof establishing systematic connections between the two procedures, and limit ourselves to illustrating them by means of an example.

Generalized Church-Rosser inequality. Let $\mathcal{F} = \{\cdot, \Diamond, \lhd\}$, $\mathcal{G} = \{\star, \Box, \rhd\}$, where $\cdot$ and $\star$ are binary and of order-type $(1,1)$, $\Diamond$ and $\Box$ are unary and of order-type (1), and $\lhd$ and $\rhd$ are unary and of order-type (∂). The logical and structural connectives of the display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ associated with the basic $\mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$-logic can be represented synoptically as follows (we omit the residuals of the binary connectives since they are not relevant to the present discussion):

I		$;$		$\odot$		$\circledast$		$\circledDiamond$		$\boxdot$	
$\top$	$\bot$	$\wedge$	$\vee$	$\cdot$			$\star$	$\Diamond$			$\Box$

$\circled{\lhd}$		$\circled{\rhd}$		$\circled{\blacklozenge}$		$\circled{\blacksquare}$		$\circled{\blacktriangleleft}$		$\circled{\blacktriangleright}$	
$\lhd$			$\rhd$	$\blacklozenge$			$\blacksquare$	$\blacktriangleleft$			$\blacktriangleright$

Consider the following analytic inductive inequality:

$$\Box p \cdot \rhd p \leq \Diamond p \star \lhd p.$$

Let us implement the procedure illustrated in Section 2.7.1 on the inequality above:

$$\begin{array}{ll}
 & \forall p[\Box p \cdot \rhd p \leq \Diamond p \star \lhd p] \\
\text{iff} & \forall p \forall \vec{q}[(q_1 \leq \Box p \ \&\ q_2 \leq \rhd p \ \&\ \Diamond p \leq q_3 \ \&\ \lhd p \leq q_4) \Rightarrow q_1 \cdot q_2 \leq q_3 \star q_4] \\
\text{iff} & \forall p \forall \vec{q}[(\blacklozenge q_1 \leq p \ \&\ q_2 \leq \rhd p \ \&\ \Diamond p \leq q_3 \ \&\ \blacktriangleleft q_4 \leq p) \Rightarrow q_1 \cdot q_2 \leq q_3 \star q_4] \\
\text{iff} & \forall p \forall \vec{q}[(\blacklozenge q_1 \vee \blacktriangleleft q_4 \leq p \ \&\ q_2 \leq \rhd p \ \&\ \Diamond p \leq q_3) \Rightarrow q_1 \cdot q_2 \leq q_3 \star q_4] \\
\text{iff} & \forall \vec{q}[(q_2 \leq \rhd(\blacklozenge q_1 \vee \blacktriangleleft q_4) \ \&\ \Diamond(\blacklozenge q_1 \vee \blacktriangleleft q_4) \leq q_3) \Rightarrow q_1 \cdot q_2 \leq q_3 \star q_4].
\end{array}$$

The last quasi-inequality above expresses the validity on perfect DLEs of the following quasi-special structural rule:

$$\frac{Y \vdash \circled{\rhd}(\circled{\blacklozenge} X \ ;\ \circled{\blacktriangleleft} W) \qquad \circled{\Diamond}(\circled{\blacklozenge} X \ ;\ \circled{\blacktriangleleft} W) \vdash Z}{X \odot Y \vdash Z \circledast W} \tag{2.9.1}$$

Let us apply the procedure described in [9, 10] to the calculus $\mathbf{DL}$ and the sequent

$$\Box p \cdot \rhd p \vdash \Diamond p \star \lhd p.$$

We start by exhaustively applying in reverse all invertible rules of $\mathbf{DL}$ which are applicable to the sequent. These rules are:

$$\frac{A \odot B \vdash Z}{A \cdot B \vdash Z} \qquad \frac{X \vdash A \circledast B}{X \vdash A \star B}.$$

This yields the following sequent:

$$\Box p \odot \rhd p \vdash \Diamond p \circledast \lhd p.$$

At this point, the procedure in [9, 10] calls for the display of the subformulas on which it is not possible to apply invertible rules as a-parts or s-parts of the premises of the rule-to be. The equivalence of the rule below to the sequent above is guaranteed by the Ackermann lemma:

$$\frac{X \vdash \Box p \qquad Y \vdash \rhd p \qquad \Diamond p \vdash Z \qquad \lhd p \vdash W}{X \odot Y \vdash Z \circledast W}.$$

On each of the premises of the rule above, more invertible rules of $\mathbf{DL}$ can be applied in reverse, namely the following ones:

$$\frac{X \vdash \circled{\Box} A}{X \vdash \Box A} \qquad \frac{X \vdash \circled{\rhd} A}{X \vdash \rhd A} \qquad \frac{\circled{\Diamond} A \vdash Y}{\Diamond A \vdash Y} \qquad \frac{\circled{\lhd} A \vdash Y}{\lhd A \vdash Y}.$$

Applying them exhaustively yields

$$\frac{X \vdash \circled{\Box} p \qquad Y \vdash \circled{\rhd} p \qquad \circled{\Diamond} p \vdash Z \qquad \circled{\lhd} p \vdash W}{X \odot Y \vdash Z \circledast W}.$$

Modulo replacing p with a fresh structural variable V, the rule above satisfies conditions C_2-C_7 but fails to satisfy C_1. To transform it into an analytic rule, one needs to first display all occurrences of the variable p, by suitably applying the following display postulates:

$$\frac{X \vdash \Box Y}{\blacklozenge X \vdash Y} \qquad \frac{X \vdash \vartriangleright Y}{Y \vdash \blacktriangleright X} \qquad \frac{\Diamond X \vdash Y}{X \vdash \blacksquare Y} \qquad \frac{\vartriangleleft X \vdash Y}{\blacktriangleleft Y \vdash X}.$$

This step yields the following rule:

$$\frac{\blacklozenge X \vdash p \quad p \vdash \blacktriangleright Y \quad p \vdash \blacksquare Z \quad \blacktriangleleft W \vdash p}{X \odot Y \vdash Z \circledast W}.$$

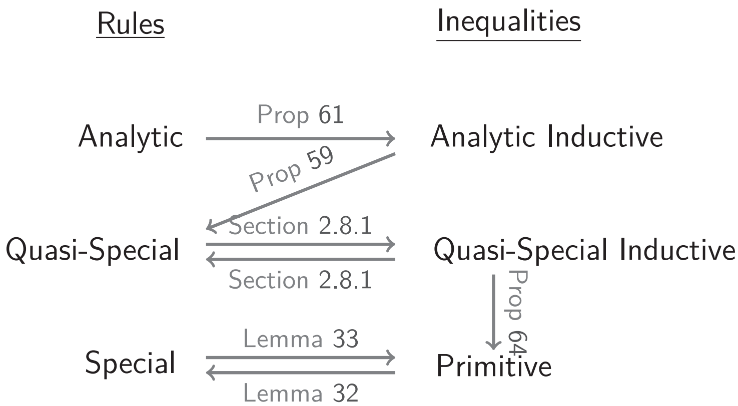
Eliminating p by means of all the possible applications of cut on the premises yields:

$$\frac{\blacklozenge X \vdash \blacktriangleright Y \quad \blacklozenge X \vdash \blacksquare Z \quad \blacktriangleleft W \vdash \blacktriangleright Y \quad \blacktriangleleft W \vdash \blacksquare Z}{X \odot Y \vdash Z \circledast W}.$$

The rule above is analytic and both semantically and $\mathbf{DL}$-equivalent to (2.9.1). Running the two procedures in parallel shows that they have the same essentials, namely adjunction and Ackermann lemma. Indeed, the cut rules applied on the premises can be assimilated to instances of the Ackermann lemma. Moreover, introduction rules for any given connective are invertible exactly on the side in which the connective is an adjoint/residual. Notice that disjunction (resp. conjunction) is no exception since in the distributive environment it is both a left (resp. right) adjoint and a right (resp. left) residual.

2.10 Power and limits of display calculi: Conclusion

The present work addresses the question of which axiomatic extensions of a basic DLE-logic admit a proper display calculus obtained by modularly adding structural rules to the proper display calculus of the basic logic. Such axiomatic extensions are referred to as *properly displayable* (cf. Definition 27). Our starting point was Kracht's paper [39], which characterizes properly displayable axiomatic extensions of the basic modal/tense logic as those associated with the *primitive axioms* of the language of classical tense logic. In the present chapter, we extend Kracht's notion of primitive axiom to primitive inequalities, uniformly defined in any DLE-languages, and prove that Kracht's characterization holds up to semantic equivalence. Specifically, we introduce the class of analytic inductive inequalities as a syntactic extension of primitive inequalities. We show that each analytic inductive inequality can be effectively translated via ALBA into (a set of) analytic rules. In fact, in Section 2.7, we show that each analytic inductive inequality can be transformed into an analytic rule which is *quasi-special* (cf. Definition 8). Moreover, in Section 2.8, we characterize the subclass of analytic inductive inequalities which exactly corresponds to quasi-special rules (cf. Definition 63), and show that each such inequality is in fact frame-equivalent to a primitive inequality. These results, taken together, characterize up to semantic equivalence the properly displayable axiomatic extensions of any basic DLE-logic as as those associated with the *primitive inequalities* of its associated DLE*-language.



Further applications. The order-theoretic approach to analyticity developed in the present chapter is applicable also to the display environments for Dynamic Epistemic Logic and PDL developed in [28–30], since the order-theoretic properties at the base of the definition of analytic inductive inequalities are available also in those settings. Notice that the settings of [28, 29] are *multi-type*, that is, their main feature are logical connectives taking in input terms of possibly different types, which semantically correspond to operations between different algebras. However, the crucial order-theoretic principles are straightforwardly applicable also to multi-type connectives. To a more limited extent, this approach is also applicable to the settings [4, 27, 31], which do not enjoy the relativized display property. It is worth noticing that the design choices of the calculus introduced in [31] depart from the standard design choices we adopt in the present chapter. The justification for this non-standard design lies precisely in the fact that, once the axioms of inquisitive logic have been translated into the multi-type environment, one of the axioms can be recognized as not analytic inductive.

2.11 Cut elimination for $\mathbf{DL}$ and $\mathbf{DL}^*$

The present section focuses on the proof that the calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ defined in Section 2.4.

Fact 67. *The display calculi $\mathbf{DL}$ and $\mathbf{DL}^*$ verify condition C_8 (cf. Section 2.2.2).*

The reduction step for axioms goes as usual:

$$\frac{p \vdash p \qquad p \vdash p}{p \vdash p} \qquad \rightsquigarrow \qquad p \vdash p$$

Now we treat the introductions of the connectives of the propositional base (we also treat here the cases relative to the two additional arrows $\leftarrow$ and $\succ\!\!-$ added to our presentation):

$$\frac{\mathrm{I} \vdash \top \qquad \dfrac{\begin{array}{c}\vdots\, \pi \\ \mathrm{I} \vdash X\end{array}}{\top \vdash X}}{\mathrm{I} \vdash X} \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\, \pi \\ \mathrm{I} \vdash X\end{array}$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi\\ X \vdash \mathrm{I}\end{array}}{X \vdash \bot} \qquad \bot \vdash \mathrm{I}}{X \vdash \mathrm{I}} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi\\ X \vdash \mathrm{I}\end{array}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ Y \vdash B\end{array}}{X ; Y \vdash A \wedge B} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ A ; B \vdash Z\end{array}}{A \wedge B \vdash Z}}{X ; Y \vdash Z}
$$

$$\rightsquigarrow$$

$$
\dfrac{\begin{array}{c}\vdots\,\pi_2\\ Y \vdash B\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ A ; B \vdash Z\end{array}}{B ; A \vdash Z}}{A \vdash B > Z}}{X \vdash B > Z}}{B ; X \vdash Z}}{X ; B \vdash Z}}{B \vdash X > Z}}{\dfrac{Y \vdash X > Z}{X ; Y \vdash Z}}
$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ Z \vdash B ; A\end{array}}{Z \vdash B \vee A} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ B \vdash Y\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ A \vdash X\end{array}}{B \vee A \vdash Y ; X}}{Z \vdash Y ; X}
$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ Z \vdash B ; A\end{array}}{Z \vdash A ; B}}{A > Z \vdash B} \qquad \begin{array}{c}\vdots\,\pi_1\\ B \vdash Y\end{array}}{A > Z \vdash Y}}{Z \vdash A ; Y}}{Z \vdash Y ; A}}{Y > Z \vdash A} \qquad \begin{array}{c}\vdots\,\pi_2\\ A \vdash X\end{array}}{\dfrac{Y > Z \vdash X}{Z \vdash Y ; X}}
$$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{Y \vdash A > B}}{Y \vdash A \to B} \qquad \dfrac{\overset{\vdots\,\pi_2}{X \vdash A} \qquad \overset{\vdots\,\pi_3}{B \vdash Z}}{A \to B \vdash X > Z}}{Y \vdash X > Z}$$

↯

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_2}{X \vdash A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{Y \vdash A > B}}{A\,;Y \vdash B} \qquad \overset{\vdots\,\pi_3}{B \vdash Z}}{A\,;Y \vdash Z}}{Y\,;A \vdash Z}}{A \vdash Y > Z}}{X \vdash Y > Z}}{\dfrac{Y\,;X \vdash Z}{X\,;Y \vdash Z}}}{Y \vdash X > Z}$$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{Y \vdash B < A}}{Y \vdash B \leftarrow A} \qquad \dfrac{\overset{\vdots\,\pi_2}{B \vdash Z} \qquad \overset{\vdots\,\pi_3}{X \vdash A}}{B \leftarrow A \vdash Z < X}}{Y \vdash Z < X}$$

↯

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_2}{X \vdash A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1}{Y \vdash B < A}}{Y;A \vdash B} \qquad \overset{\vdots\,\pi_3}{B \vdash Z}}{Y;A \vdash Z}}{A;Y \vdash Z}}{A \vdash Z < Y}}{X \vdash Z < Y}}{\dfrac{X;Y \vdash Z}{Y;X \vdash Z}}}{Y \vdash Z < X}$$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_2}{A \vdash Y} \qquad \overset{\vdots\,\pi_3}{Z \vdash B}}{Y > Z \vdash A \succ\!\!- B} \qquad \dfrac{\overset{\vdots\,\pi_1}{A > B \vdash X}}{A \succ\!\!- B \vdash X}}{Y > Z \vdash X}$$

↯

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_3}{Z \vdash B} \qquad \dfrac{\overset{\vdots\,\pi_1}{A > B \vdash X}}{B \vdash A\,;X}}{Z \vdash A\,;X}}{Z \vdash X\,;A}}{X > Z \vdash A} \qquad \overset{\vdots\,\pi_2}{A \vdash Y}}{X > Z \vdash Y}}{\dfrac{Z \vdash X\,;Y}{Z \vdash Y\,;X}}}{Y > Z \vdash X}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_2\\ Y \vdash B\end{array} \qquad \begin{array}{c}\vdots\,\pi_3\\ A \vdash Z\end{array}}{Y < Z \vdash B \prec A} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ B < A \vdash X\end{array}}{B \prec A \vdash X}}{Y < Z \vdash X}$$

$$\lightning$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_3\\ Y \vdash B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ B < A \vdash X\end{array}}{B \vdash X ; A}}{\dfrac{Y \vdash X ; A}{\dfrac{Y \vdash A ; X}{Y < X \vdash A}}} \qquad \begin{array}{c}\vdots\,\pi_2\\ A \vdash Z\end{array}}{Y < X \vdash Z}}{\dfrac{Y \vdash Z ; X}{Y \vdash X ; Z}}}{Y < Z \vdash Y}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi\\ Y \vdash K(\vec{A}_I, \vec{A}_J)\end{array}}{Y \vdash g(\vec{A}_I, \vec{A}_J)} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_i\\ A_i \vdash X_i\end{array} \quad \cdots \quad \begin{array}{c}\vdots\,\pi_j\\ X_j \vdash A_j\end{array}}{g(\vec{A}_I, \vec{A}_J) \vdash K(\vec{X}_I, \vec{X}_J)}}{Y \vdash K(\vec{X}_I, \vec{X}_J)}$$

$$\lightning$$

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi\\ Y \vdash K(\vec{A}_I, \vec{A}_J)\end{array}}{K_i(\vec{A}_I[Y/A_i], \vec{A}_J) \vdash A_{i \in I}} \qquad \begin{array}{c}\vdots\,\pi_i\\ A_i \vdash X_i\end{array}}{K_i(\vec{A}_I[Y/A_i], \vec{A}_J) \vdash X_i}}{Y \vdash K(\vec{A}_I[X_i/A_i], \vec{A}_J)}$$

$$\vdots$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_j\\ X_j \vdash A_j\end{array} \qquad \dfrac{Y \vdash K(\vec{X}_I, \vec{A}_J)}{A_{j \in J} \vdash K_j(\vec{X}_I, \vec{A}_J[Y/A_j])}}{X_j \vdash K_j(\vec{X}_I, \vec{A}_J[Y/A_j])}}{Y \vdash K(\vec{X}_I, \vec{A}_J[X_j/A_j])}$$

$$\vdots$$

$$Y \vdash K(\vec{X}_I, \vec{X}_J)$$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_i}{X_i \vdash A_i} \qquad \cdots \qquad \overset{\vdots\,\pi_j}{A_j \vdash X_j}}{H(\vec{X}_I, \vec{X}_J) \vdash f(\vec{A}_I, \vec{A}_J)} \qquad \dfrac{\overset{\vdots\,\pi}{H(\vec{A}_I, \vec{A}_J) \vdash Y}}{f(\vec{A}_I, \vec{A}_J) \vdash Y}}{H(\vec{X}_I, \vec{X}_J) \vdash Y}$$

$$\lightning$$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_i}{X_i \vdash A_i} \qquad \dfrac{\overset{\vdots\,\pi}{H(\vec{A}_I, \vec{A}_J) \vdash Y}}{A_{i\in I} \vdash H_i(\vec{A}_I[Y/A_i], \vec{A}_J)}}{X_i \vdash H_i(\vec{A}_I[Y/A_i], \vec{A}_J)}}{H(\vec{A}_I[X_i/A_i], \vec{A}_J) \vdash Y}$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{H(\vec{X}_I, \vec{A}_J) \vdash Y}{H_j(\vec{X}_I, \vec{A}_J[Y/A_j]) \vdash A_{j\in J}} \qquad \overset{\vdots\,\pi_j}{A_j \vdash X_j}}{H_j(\vec{X}_I, \vec{A}_J[Y/A_j]) \vdash X_j}}{H(\vec{X}_I, \vec{A}_J[X_j/A_j]) \vdash Y}$$

$$\vdots$$

$$H(\vec{X}_I, \vec{X}_J) \vdash Y$$

2.12 Invertible rules of $\mathbf{DL}$

The present section characterizes the invertible rules of the calculi $\mathbf{DL}$ defined in Section 2.4. Throughout the present section, fix a language $\mathcal{L}_{\mathrm{DLE}} = \mathcal{L}_{\mathrm{DLE}}(\mathcal{F}, \mathcal{G})$, and let $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

Notice that the following rules are derivable in $\mathbf{DL}$:

$$\vee_{L'}\ \dfrac{A \vdash X \qquad B \vdash X}{A \vee B \vdash X} \qquad\qquad \dfrac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}\ \wedge_{R'}$$

Hence, for the sake of the comparison of the two settings, we can add them to $\mathbf{DL}$ as primitive rules.

Lemma 68. *The rules* $\wedge_L$, $\wedge_{R'}$, $\vee_R$, $\vee_{L'}$, f_L, g_R *are invertible.*

Proof. We only show the cases of f_L and $\vee_{L'}$, the remaining cases being similar. Assume that $f(A_1, \ldots, A_{n_f}) \vdash X$. Then we can derive the premise of f_L via the following derivation:

$$\dfrac{\dfrac{A_1 \vdash A_1 \quad \ldots \quad A_{n_f} \vdash A_{n_f}}{H(A_1, \ldots, A_{n_f}) \vdash f(A_1, \ldots, A_{n_f})} \qquad f(A_1, \ldots, A_{n_f}) \vdash X}{H(A_1, \ldots, A_{n_f}) \vdash X.}$$

Assume $A \vee B \vdash X$. Then we can derive the premises of $\vee_{L'}$ via the following derivation:

$$\cfrac{\cfrac{\cfrac{A \vdash A}{A \vdash A ; B}}{A \vdash A \vee B} \quad A \vee B \vdash X}{A \vdash X.} \qquad \cfrac{\cfrac{\cfrac{\cfrac{B \vdash B}{B \vdash B ; A}}{B \vdash A ; B}}{A \vdash A \vee B} \quad A \vee B \vdash X}{B \vdash X.}$$

□

Lemma 69. *The rules $\wedge_R$, $\vee_L$, f_R, g_L are not invertible.*

Proof. Notice that for a rule to be invertible, for each instance of the rule it must be the case that the logical interpretation of each premise is valid in the class of models for $\mathbf{DL}$ in which the corresponding conclusion is valid. Hence to disprove the invertibility of a rule it is enough to find a instance of the rule for which there exists a model satisfying the conclusion but not the premises. We only show this for f_R, and $\wedge_R$, the remaining cases being similar. To show that f_R is not invertible, consider the conclusion $H(p_1, \ldots, p_{n_f}) \vdash f(q_1, \ldots, q_{n_f})$. Let $\mathbb{A}$ be any Heyting algebra with two incomparable elements b c, and let $f^{\mathbb{A}}$ be the n-ary operation such that $f(\vec{a}) = \bot$ for all $\vec{a} \in \mathbb{A}^n$ (notice that this operation is both join-preserving and meet-reversing in each coordinate). Then by letting $v(p_1) = b$ and $v(q_1) = c$, we have that $\bot \leq \bot$ but $b \nleq c$.

As for $\wedge_R$, notice preliminarily that the following instance of the conclusion is derivable:

$$\cfrac{\cfrac{A \vdash A \quad B \vdash B}{A ; B \vdash A \wedge B}}{B ; A \vdash A \wedge B}$$

Suppose for contradiction that $\wedge_R$ was invertible. Then, from $B; A \vdash A \wedge B$ we would be able to derive both $A \vdash B$ and $B \vdash A$. But since $B; A \vdash A \wedge B$ is derivable in $\mathbf{DL}$, this would imply that we can also derive $A \vdash B$ for any A and B, which contradicts the soundness of the calculus. □

2.13 The display calculi $\mathbf{DL}$ are amenable

The present section sketches the proof that the calculi $\mathbf{DL}$ defined in Section 2.4 are amenable.

Definition 70 (Amenable calculus, cf. [10], Definition 3.1)**.** Let $\mathcal{C}$ be a display calculus containing an a-structure constant $\mathbf{I}$ and an s-structure constant $\mathbf{I}'$ and satisfying C1-C8. Let $\mathfrak{S}(a)$ and $\mathfrak{S}(s)$ denote the class of a- and s-structures of $\mathcal{C}$, and let $\mathcal{L}$ be the language of $L_{\mathbf{I}}(\mathcal{C})$. A display calculus satisfying the following conditions is said to be *amenable*.

1. (interpretation functions) There are functions $l : \mathfrak{S} \mapsto For\mathcal{L}$ and $r : \mathfrak{S} \mapsto For\mathcal{L}$ such that $l(A) = A = r(A)$ for $A \in For\mathcal{L}$, and for arbitrary $X \in \mathfrak{S}(a)$ and $Y \in \mathfrak{S}(s)$:

    (a) $X \vdash l(X)$ and $Y \vdash l(Y)$ are derivable in $\mathcal{C}$.

(b) if $X \vdash Y$ is derivable in $\mathcal{C}$ then so is $l(X) \vdash r(Y)$.

2. (logical constants) There are logical constants $c_a, c_s \in For(\mathcal{L})$ such that the following sequents are derivable for arbitrary $X \in \mathfrak{S}(a)$ and $Y \in \mathfrak{S}(s)$:

$$c_a \vdash Y \qquad\qquad X \vdash c_s$$

3. (logical connectives) There are binary connectives $\wedge, \vee \in \mathcal{L}$ such that the following sequents are derivable for $\star \in \{\vee, \wedge\}$:

   (a) commutativity: $A \star B \vdash B \star A$

   (b) associativity: $A \star (B \star C) \vdash (A \star B) \star C$ and $(A \star B) \star C \vdash A \star (B \star C)$

   Also, for $A, B \in For\mathcal{L}$, $X \in \mathfrak{S}(a)$ and $Y \in \mathfrak{S}(s)$:

   $(a)_\vee$ $A \vdash Y$ and $B \vdash Y$ implies $\vee(A, B) \vdash Y$

   $(b)_\vee$ $X \vdash A$ implies $X \vdash \vee(A, B)$ for any formula B.

   $(a)_\wedge$ $X \vdash A$ and $X \vdash B$ implies $X \vdash \wedge(A, B)$

   $(b)_\wedge$ $A \vdash Y$ implies $\wedge(A, B) \vdash Y$ for any formula B.

Fact 71. *For any $\mathcal{L}_{\mathrm{DLE}}$-language, the corresponding calculus $\mathbf{DL}$ is amenable.*

Proof. The interpretation functions l and r are those defined in Definition 24. The constants are $c_a := \top$ and $c_s := \bot$. Finally, the derivations requested by item 3 are straightforward and omitted. □

2.14 Analytic inductive and acyclic $\mathcal{I}_2(\mathbf{DL})$-inequalities

The following definitions are slight modifications of Definitions 3.7–3.9 in [10]. The modifications essentially amount to specializing the original inequalities from an arbitrary display calculus $\mathcal{C}$ to $\mathbf{DL}$.

Definition 72. For any sequent $X \vdash Y$ in the language of $\mathbf{DL}$, let $inv(X \vdash Y)$ denote the collection of sets of sequents obtained by applying sequences of display postulates and invertible logical rules in $\mathbf{DL}$ (cf. Section 2.12) to it.

Definition 73. An $\mathcal{L}_{\mathrm{DLE}}$-formula is *a-soluble* (resp. *s-soluble*) if there is some $\{U_i \vdash V_i \mid i \in I\} \in inv(s \vdash \mathbf{I})$ (resp. $\in inv(\mathbf{I}) \vdash s$) containing no logical connective.

Lemma 74. *Any $\mathcal{L}_{\mathrm{DLE}}$-formula s is a-soluble (resp. s-soluble) iff s is left-primitive (resp. right-primitive).*

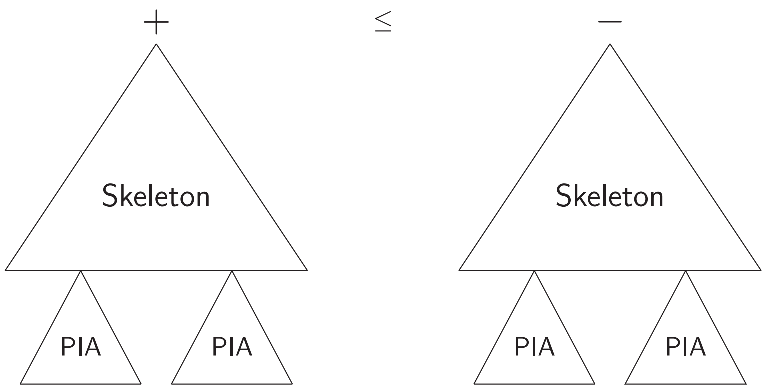
Proof. If s is left-primitive, then every non-leaf node in $+s$ is labelled in one of the following ways: $+f$, $-g$, $\pm\wedge$, or $\pm\vee$. Since the left-introduction (resp. right-introduction) rule for any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) is invertible and both introduction rules for $\wedge$ and $\vee$ are invertible, a routine induction on the shape of s shows that s is a-soluble. Conversely, if s is not left-primitive, then there exists at least one node in $+s$ which is labelled either $-f$ or $+g$ for some $f \in \mathcal{F}$ or some $g \in \mathcal{G}$. Consider one such node n, and let s' be the subterm of s rooted at n. We can assume w.l.o.g. that all the ancestors of n do not violate the left-primitive requirement. Reasoning like we did before, we can apply suitable invertible rules to all the subformulas of s rooted at the nodes in the path from the root of $+s$ to the direct ancestor of n. Then, in the set of sequents obtained as premises of the last rule application, there will be either one sequent of the form $U_i \vdash f(s_1, \ldots, s_{n_f})$ (if n is labelled $-f$) or of the form $g(s_1, \ldots, s_{n_g}) \vdash V_i$ (if n is labelled $+g$). In either case, since the right-introduction (resp. left-introduction) rule for any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) is not invertible, there is no invertible rule which can be applied to transform the main connective into a structural connective, which proves that s is not a-soluble, as required. □ The following definition slightly generalizes the original Definition 3.9 in [10] from formulas to inequalities.

Definition 75. Any $\mathcal{L}_{\mathrm{DLE}}$-inequality $s \leq t$ belongs to the class $\mathcal{I}_2(\mathbf{DL})$ iff there is some $\{U_i \vdash V_i \mid i \in I\} \in inv(s \vdash t)$ such that, for each $i \in I$, each antecedent-part (resp. succedent-part) formula in $U_i \vdash V_i$ is s-soluble (resp. a-soluble).

Proposition 76. *The following are equivalent for any $\mathcal{L}_{\mathrm{DLE}}$-inequality $s \leq t$:*

1. *$s \leq t$ belongs to $\mathcal{I}_2(\mathbf{DL})$;*
2. *every branch in $+s$ and $-t$ is good.*

Proof. By Lemma 74, a term s is left-primitive (resp. right-primitive) if and only if s is a-soluble (s-soluble). Moreover, left-primitive (resp. right-primitive) terms coincide with positive (resp. negative) Skeleton and negative (resp. positive) PIA terms (cf. discussion at the beginning of Section 2.6.2). If $s \leq t$ is such that every branch is good, then $s \leq t$ is of the form illustrated in the picture below:



Then it is clear that $s \leq t$ belongs to $\mathcal{I}_2(\mathbf{DL})$. Indeed, after applying exhaustively all possible invertible rules to the Skeleton nodes, the PIA parts are "moved to the premises" via an application of the Ackermann rule, as discussed in Section 2.9. It is straightforward but tedious to show that, when occurring in the premises, each PIA part is guaranteed to occur on the side on which it is soluble. By definition, this implies that $s \leq t$ is in $\mathcal{I}_2(\mathbf{DL})$.

As to the converse direction, notice that each step in the reasoning above can be reversed. □

To finish the comparison, we need to report on some definitions from [10]. The following one is a slight modification of [10, Definition 3.18], motivated by the purpose of highlighting its similarity with sets of inequalities in Ackermann shape:

Definition 77. A nonempty set $\mathcal{S}$ of sequents *respects multiplicities* w.r.t. a propositional variable p occurring in any of its sequents if $\mathcal{S}$ can be written in one of the following forms via application of display rules:

$$\{p \vdash U \mid p \text{ does not occur in } U\} \cup \{S \vdash T \mid p \text{ only occurs as s-part in } S \vdash T\}$$

$$\{U \vdash p \mid p \text{ does not occur in } U\} \cup \{S \vdash T \mid p \text{ only occurs as a-part in } S \vdash T\}.$$

If $\mathcal{S}$ is a set of sequents respecting multiplicities wrt p, then $\mathcal{S}$ can be equivalently transformed into the set $\mathcal{S}_p$ not containing p, and the transformation consists essentially in an application of Ackermann lemma.

Definition 78. (cf. [10, Definition 3.20]) (the set $\mathcal{S}_p$). Let $\mathcal{S}$ be a set of sequents respecting multiplicities w.r.t. p. If $\mathcal{S}$ is uniform in p, in the sense that p occurs always as an s-part or an a-part in each sequent of $\mathcal{S}$, then let $\mathcal{S}_p := \{S \vdash T \mid S \vdash T \in \mathcal{S}$ and p does not occur in $S \vdash T\}$. Otherwise, define $\mathcal{S}_p$ as the union of $\{S \vdash T \mid S \vdash T \in \mathcal{S}$ and p does not occur in $S \vdash T\}$ and the set of sequents $S' \vdash T'$ obtained by substituting p for any U such that $p \vdash U$ is in $\mathcal{S}$ (resp. $U \vdash p$ is in $\mathcal{S}$) in each sequent $S \vdash T$ in $\mathcal{S}$.

The first case of the definition above corresponds to the situation in which a given variable occurring only positively or negatively is eliminated via Ackermann by suitably replacing it by $\bot$ or $\top$. Clearly, if $\mathcal{S}$ respects multiplicities w.r.t. p, then p does not occur in $\mathcal{S}_p$ (cf. [10, Lemma 3.21]).

Definition 79. (cf. [10, Definition 3.22])(acyclic set). Let $\mathcal{C}$ display calculus. A finite set $\mathcal{S}$ of sequents built from structure variables, structure constants and propositional variables using structural connectives is acyclic if (i) the sequents in $\mathcal{S}$ do not contain any variables; or (ii) there exists a variable p such that $\mathcal{S}$ respects multiplicities w.r.t. p and $\mathcal{S}_p$ is acyclic.

Lemma 80. *Let $\mathcal{S}$ be an acyclic set of sequents in the variables $p_1, \ldots, p_n$ containing no logical connectives, such that for each variable p_i there exist $s_1, s_2 \in \mathcal{S}$ such that p_i occurs in antecedent (resp. succedent) position in s_1 (resp. in s_2). Then there exists a p such that $\mathcal{S}$ can be written in one of the following forms via application of display rules:*

$$\{p \vdash U \mid \text{ no logical variable occurs in } U\} \cup \{S \vdash T \mid p \text{ only occurs as s-part in } S \vdash T\}$$

$$\{U \vdash p \mid \text{ no logical variable occurs in } U\} \cup \{S \vdash T \mid p \text{ only occurs as a-part in } S \vdash T\}.$$

Proof. By induction on the number of variables appearing in $\mathcal{S}$. If it contains only one variable, p, then the statement immediately follows from the fact that $\mathcal{S}$ respects multiplicities w.r.t. p.

Assume that the statement holds for sets of sequents $\mathcal{S}$ on n variables, and let $\mathcal{S}$ contain $n+1$ variables. Assume for contradiction that the statement is false for each variable p such that $\mathcal{S}$ respects multiplicities w.r.t. p. This means that for every such p, there is a sequent $p \vdash U$ (or $U \vdash p$) as above such that U contains a propositional variable q.

Then for every such p, the set $\mathcal{S}_p$ inherits the same issue: Indeed substituting U for p cannot possibly create terms free from propositional variables, given that U contains q. The induction hypothesis implies that each $\mathcal{S}_p$ is not acyclic. Then $\mathcal{S}$ is not acyclic, a contradiction. □

The following definition is aimed at adapting [10, Definition 3.23] to the setting of DLE-logics.

Definition 81. (acyclic inequality). An inequality $s \leq t$ in $\mathcal{I}_2(\mathbf{DL})$ is *acyclic* if there is a set $\{\rho_i\}_{i\in I}$ of semi-structural rules[43] which is obtained by applying the procedure described in Section 2.9 to $s \leq t$ such that the set of premises of each ρ_i is acyclic.

Proposition 82. *The following are equivalent for any inequality $s \leq t$:*

1. *$s \leq t$ is acyclic and belongs to $\mathcal{I}_2(\mathbf{DL})$;*
2. *$s \leq t$ is analytic inductive.*

Proof. Let $s \leq t$ be analytic (Ω, ε)-inductive. Then by Proposition 76, $s \leq t$ is in $\mathcal{I}_2(\mathbf{DL})$. To finish the proof we need to show that it is acyclic. This amounts to proving that the set of premises obtained by applying the Ackermann rule in the procedure described in Section 2.9 is acyclic. By assumption, $s \leq t$ has the following shape:

$$\xi_1(\vec{\phi}_1/\vec{x}_1, \vec{\psi}_1/\vec{y}_1, \vec{\gamma}_1/\vec{z}_1, \vec{\theta}_1/\vec{w}_1) \leq \xi_2(\vec{\psi}_2/\vec{x}_2, \vec{\phi}_2/\vec{y}_2, \vec{\theta}_2/\vec{z}_2, \vec{\gamma}_2/\vec{w}_2),$$

where $\xi_1(!\vec{x}_1, !\vec{y}_1, !\vec{z}_1, !\vec{w}_1)$ and $\xi_2(!\vec{x}_2, !\vec{y}_2, !\vec{z}_2, !\vec{w}_2)$ respectively are a positive and a negative Skeleton-formula —cf. page 30—(hence ξ_1 is left-primitive and ξ_2 is right-primitive) which are scattered, monotone in $\vec{x}$ and $\vec{z}$ and antitone in $\vec{y}$ and $\vec{w}$. Moreover, the formulas in $\vec{\phi}$ and $\vec{\gamma}$ are positive PIA (and hence right-primitive), and the formulas in $\vec{\psi}$ and $\vec{\theta}$ are negative PIA (and hence left-primitive). Finally, every ϕ and ψ contains at least one ε-critical variable, whereas all $+\gamma$ and $-\theta$ are ε^∂-uniform. Without loss of generality we may assume that all formulas in $\vec{\phi}_i$, $\vec{\psi}_i$, $\vec{\gamma}_i$ and $\vec{\theta}_i$ for $i \in \{1,2\}$ are definite PIA (cf. Footnote 30).

Let us apply the procedure described in [9, 10] to the calculus $\mathbf{DL}$ and the inequality above, seen as a sequent. By exhaustively applying in reverse all invertible rules of $\mathbf{DL}$ which are applicable to the sequent we get the following:

$$\Xi_1(\vec{\phi}_1/\vec{x}_1, \vec{\psi}_1/\vec{y}_1, \vec{\gamma}_1/\vec{z}_1, \vec{\theta}_1/\vec{w}_1) \vdash \Xi_2(\vec{\psi}_2/\vec{x}_2, \vec{\phi}_2/\vec{y}_2, \vec{\theta}_2/\vec{z}_2, \vec{\gamma}_2/\vec{w}_2),$$

where Ξ_1 and Ξ_2 denote the structures associated with ξ_1 and ξ_2 respectively. At this point, the procedure in [9, 10] calls for the display of the subformulas on which it is not

[43] A *semi-structural* is a rule whose conclusion is constructed from structure variables and structure constants using structural connectives, and whose premises might additionally contain propositional variables.

possible to apply invertible rules as a-parts or s-parts of the premises of the rule-to be. The equivalence of the rule below to the sequent above is guaranteed by the Ackermann lemma:

$$\frac{\vec{X}_1 \vdash \vec{\phi}_1 \quad \vec{Y}_2 \vdash \vec{\phi}_2 \quad \vec{\psi}_1 \vdash \vec{Y}_1 \quad \vec{\psi}_2 \vdash \vec{X}_2 \quad \vec{Z}_1 \vdash \vec{\gamma}_1 \quad \vec{W}_2 \vdash \vec{\gamma}_2 \quad \vec{\theta}_1 \vdash \vec{W}_1 \quad \vec{\theta}_2 \vdash \vec{Z}_2}{\Xi_1(\vec{X}_1, \vec{Y}_1, \vec{Z}_1, \vec{W}_1) \vdash \Xi_2(\vec{X}_2, \vec{Y}_2, \vec{Z}_2, \vec{W}_2)}\ .$$

2

On each of the premises of the rule above, more invertible rules of $\mathbf{DL}$ can be applied in reverse. Applying them exhaustively yields

$$\frac{\vec{X}_1 \vdash \vec{\Phi}_1 \quad \vec{Y}_2 \vdash \vec{\Phi}_2 \quad \vec{\Psi}_1 \vdash \vec{Y}_1 \quad \vec{\Psi}_2 \vdash \vec{X}_2 \quad \vec{Z}_1 \vdash \vec{\Gamma}_1 \quad \vec{W}_2 \vdash \vec{\Gamma}_2 \quad \vec{\Theta}_1 \vdash \vec{W}_1 \quad \vec{\Theta}_2 \vdash \vec{Z}_2}{\Xi_1(\vec{X}_1, \vec{Y}_1, \vec{Z}_1, \vec{W}_1) \vdash \Xi_2(\vec{X}_2, \vec{Y}_2, \vec{Z}_2, \vec{W}_2)}\ .$$

By the definition of inductive inequality, if some Ω-minimal variable occurs in any Φ or Ψ subterm, then no other variable can occur in that subterm. Hence, the premises of the rule respect multiplicities w.r.t. these variables, which can then be eliminated. Likewise, one can show, by induction on Ω, that all variables can be eliminated, that is, $s \leq t$ is acyclic, as required.

For the converse direction, assume that $s \leq t$ is acyclic and belongs to $\mathcal{I}_2(\mathbf{DL})$. We may assume without loss of generality that all variables in $s \leq t$ occur both positively and negatively, since otherwise they can be eliminated by replacing them with $\top$ and $\bot$. By Proposition 76, every branch of the signed generation trees $+s$ and $-t$ is good, and by Definition 81 the following set is acyclic:

$$\vec{X}_1 \vdash \vec{\Phi}_1 \quad \vec{Y}_2 \vdash \vec{\Phi}_2 \quad \vec{\Psi}_1 \vdash \vec{Y}_1 \quad \vec{\Psi}_2 \vdash \vec{X}_2 \quad \vec{Z}_1 \vdash \vec{\Gamma}_1 \quad \vec{W}_2 \vdash \vec{\Gamma}_2 \quad \vec{\Theta}_1 \vdash \vec{W}_1 \quad \vec{\Theta}_2 \vdash \vec{Z}_2.$$

The assumption that each variable occurs both positively and negatively implies that each variable occurs both in antecedent and in consequent position in the sequents above. Hence, by Lemma 80, there exists a propositional variable p such that the above set can be written in one of the following forms via application of display rules:

$$\{p \vdash U \mid \text{ no logical variable occurs in } U\} \cup \{S \vdash T \mid p \text{ only occurs as s-part in } S \vdash T\}$$

$$\{U \vdash p \mid \text{ no logical variable occurs in } U\} \cup \{S \vdash T \mid p \text{ only occurs as a-part in } S \vdash T\}.$$

Let us define a strict partial order Ω and an order-type ε on the variables occurring in the set of premises as follows: We declare these p as Ω-minimal elements and we let $\varepsilon(p) := 1$ the set of premises is of the second form and $\varepsilon(p) := \partial$ otherwise. Clearly, the set of premises respects multiplicities w.r.t. p which can then be eliminated. In the new set of sequents produced the same reasoning applies. The new variable will be placed above all the Ω-minimal elements. Since the set is acyclic, this process is guaranteed to end after a finite number of rounds, defining an ε and Ω for all the variables present. It is routine to check that $s \leq t$ is analytic (Ω, ε)-inductive. □

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Chapter 3

The Logic of Resources and Capabilities

In this chapter, which is based on[1] *[6], we introduce the logic LRC, designed to describe and reason about agents' abilities and capabilities in using resources. The proposed framework bridges two – up to now – mutually independent strands of literature: the one on logics of abilities and capabilities, developed within the theory of agency, and the one on logics of resources, motivated by program semantics. The logic LRC is suitable to describe and reason about key aspects of social behaviour in organizations. We prove a number of properties enjoyed by LRC (soundness, completeness, canonicity, disjunction property) and its associated analytic calculus (conservativity, cut elimination and subformula property). These results lay at the intersection of the algebraic theory of unified correspondence and the theory of multi-type calculi in structural proof theory. Case studies are discussed which showcase several ways in which this framework can be extended and enriched while retaining its basic properties, so as to model an array of issues, both practically and theoretically relevant, spanning from planning problems to the logical foundations of the theory of organizations.*

[1] My specific contributions in this research have been the proof of the main results, the construction and development of examples and the draft of the first version of the paper.

3.1 Introduction

Organizations are social units of agents structured and managed to meet a need, or pursue collective goals. In economics and social science, organizations are studied in terms of agency, goals, capabilities, and inter-agent coordination [41, 67, 71]. In strategic management, the dominant approach in the study of organizational performances is the so-called *resource-based view* [2, 56, 74], which has recognized that a central role in determining the success of an organization in market competition is played by the acquisition, management, and transformation of *resources* within that organization. In order to capture this insight and create the building blocks of the logical foundations of the theory of organizations, a formal framework is needed in which it is possible to express and reason about agents' abilities and capabilities to use resources for achieving goals, to transform resources into other resources, and to coordinate the use of resources with other agents; i.e., a formal framework is needed for capturing and reasoning about the *resource flow within organizations*. The present chapter aims at introducing such a framework.

There is extensive literature in philosophical logic and formal AI accounting for agents' abilities (cf. e.g. [9, 28]) and capabilities (cf. e.g. [29, 30, 72]) and their interaction, embedding in the wider context of the logics of agency (cf. e.g. [4, 5, 10, 11, 31, 68]); some of these frameworks (viz. [29, 30]) have been used to formalize some aspects of the theory of organizations. There is also literature in theoretical computer science on the logic of resources (cf. e.g. [64, 65]), motivated by the build-up of mathematical models of computational systems. However, these two strands of research have been pursued independently, and in particular, the *interaction* between abilities, capabilities and resources has not been explored before.

The present chapter introduces a logical framework, the *logic of resources and capabilities* (LRC), designed as an environment for the logical modelling of the behaviour of agents motivated and mediated by the use and transformation of resources. In this framework, agents' capabilities are not captured via primitive actions, as is done e.g. in [72], but rather via dedicated modalities, similarly to the frameworks adopting the STIT logic approach [5, 29, 30]. However, LRC differs from these logics in two main respects; the first is the focus on *resources*, discussed above; the second is that, as a modal extension of intuitionistic logic, LRC inherits its *constructive* character: it comes equipped with a constructive proof theory which provides an explicit computational content brought out by the cut elimination theorem. This guarantees that each LRC-theorem (prediction) translates into an effective procedure, thus allowing for a greater amenability to concrete applications in planning, and paving the way for implementations in constructive programming environments. In particular, LRC enjoys the disjunction property, proof of which we have included in Section 3.2.4.

In the present chapter, the basic mathematical theory of the logic of resources and capabilities is developed in an algebraic and proof-theoretic environment. Specifically, the most important technical tool we introduce for LRC is the *proof calculus* D.LRC (cf. Section 3.3). This calculus is designed according to the *multi-type* methodology, introduced in [33–35], and further developed in [37, 44, 46, 47]. This methodology exploits facts and insights coming from various semantic theories: from the coalgebraic semantics of dynamic epistemic logics (cf. [43]), to the algebraic dual of the team semantics for inquisitive logic (cf. [37]), the representation theorems for lattices (cf. [47]), and the recently developed algebraic theory of unified correspondence [15, 18, 21, 22, 25, 26, 61], in the context of which, systematic connections have been developed (cf. [45, 55]) between Sahlqvist-type correspondence results and the theory of analytic rules for proper display calculi (cf. [73]) and Gentzen calculi.

Multi-type languages make it possible to express constituents such as actions, agents, or resources not as *parameters* in the generation of formulas, but as *terms* in their own right. They thus are regarded as first-class citizens of the multi-type framework, and are endowed with their corresponding structural connectives and rules. In this rich environment, it is possible to encode certain key interactions *within the language*, by means of structural analytic rules. This approach has made it possible to develop analytic calculi for logics notoriously impervious to the standard proof-theoretic treatment, such as Public Announcement Logic [63], Dynamic Epistemic Logic [1], their nonclassical counterparts [52, 54], and PDL [49].

One of the most important benefits of multi-type calculi is the degree of *modularity* for which they allow. When applied to the present setting, the metatheory of multi-type calculi makes it possible to add (resp. remove) analytic structural rules to (resp. from) the basic calculus D.LRC, and obtain variants endowed with a package of basic properties (soundness, completeness, cut elimination, subformula property, conservativity) as immediate consequences of general results. This feature is illustrated and exploited in Section 3.5, where we specialize D.LRC to various situations by adding certain analytic structural rules to it. More in general, an infinite class of axiomatic extensions and combinatoric variants of LRC can be captured in a systematic way within this framework. Hence, LRC can be regarded not just as one single logic, but as a *class* of interconnected logical systems. Besides being of theoretical interest, this feature is of great usefulness in practice, since this class of logics forms a coherent framework which can be adapted to very different concrete settings with minimum effort. The combined strengths of this class of logics make the resulting LRC framework into a viable proposal for capturing and reasoning about the resource flow within organizations.

Finally, LRC is the first example of a logical system designed from first principles according to the multi-type methodology. As this example shows, multi-type calculi can serve not only to provide existing logics with well-performing calculi, but also as a methodological platform for the analysis and the meta-design of new logical frameworks.

Structure of the chapter. In Section 3.2.1, the logic LRC is introduced by means of a Hilbert-style presentation, which is shown to be complete w.r.t. certain algebraic models (cf. Section 3.2.2), canonical (cf. Section 3.2.3) and to enjoy the disjunction property (cf. Section 3.2.4). Then, in Section 3.3, the multi-type calculus D.LRC is introduced, and is shown to be sound w.r.t. the algebraic models (cf. Section 3.4.1), complete (cf. Section 3.4.2), and conservative (cf. Section 3.4.4) w.r.t. the Hilbert-style presentation introduced in Section 3.2.1. In Section 3.4.3, we prove that the calculus D.LRC satisfies the assumptions of the cut elimination metatheorem proven in [34], and hence enjoys cut elimination and subformula property. In Section 3.5, we start exploring various ways in which D.LRC can be modified and adapted to different contexts so that the resulting systems retain all the properties enjoyed by the basic system. Specifically, Section 3.5.1 illustrates how *coordination* among agents helps optimizing capabilities towards a goal; Section 3.5.2 explores the solution of a planning problem which requires the suitable concatenation of *reusable* and *non-reusable* resources; Section 3.5.3 focuses on a situation in which the possibility of resources to be used in different *roles* becomes relevant; Section 3.5.4 illustrates how the *resilience* of a fragment of a system can propagate to the system as a whole.

3

3.2 The logic LRC and its algebraic semantics

In the present section, we introduce the language and Hilbert-style axiomatization for the logic of resources and capabilities. We also introduce its algebraic semantics given by heterogeneous LRC-algebras. We show the completeness of LRC with respect to the heterogeneous LRC-algebras, the canonicity of its axiomatization and the disjunction property.

3.2.1 Hilbert-style presentation of LRC

As mentioned in the introduction, the key idea is to introduce a language in which resources are not accounted for as *parameters* indexing the capability connectives, but as *logical terms* in their own right. Accordingly, we start by defining a *multi-type language* in which the different types interact via special connectives. The present setting consists of the types Res for resources and Fm for formulas (describing states of affairs). We stipulate that Res and Fm are disjoint.

Similarly to the binary connectives introduced in [35], the connectives $\rhd$, $\Diamond\!\!\!\!\mid\;$ and $\unrhd$ (referred to as *heterogeneous connectives*) facilitate the interaction between resources and formulas:[2]

$$\rhd : \mathsf{Res} \times \mathsf{Fm} \to \mathsf{Fm} \qquad \unrhd : \mathsf{Res} \times \mathsf{Res} \to \mathsf{Fm}$$
$$\Diamond : \mathsf{Fm} \to \mathsf{Fm} \qquad \Diamond\!\!\!\!\mid\; : \mathsf{Res} \to \mathsf{Fm}$$

As discussed in the next section, the mathematical environment of heterogeneous LRC-algebras provides a natural interpretation for all these connectives. Let us introduce the language of the logic of resources and capabilities. Let AtProp and AtRes be countable and disjoint sets of atomic propositions and atomic resources, respectively. The set $\mathcal{R} = \mathcal{R}(\mathsf{AtRes})$ of the *resource-terms* α *over* AtRes, and the set $\mathcal{L} = \mathcal{L}(\mathcal{R}, \mathsf{AtProp})$ of the *formula-terms* A *over* $\mathcal{R}$ and AtProp of the Logic of Resources and Capabilities (LRC) are defined as follows:

$$\alpha ::= a \in \mathsf{AtRes} \mid\ 1 \mid 0 \mid \alpha \cdot \alpha \mid \alpha \sqcup \alpha \mid \alpha \sqcap \alpha,$$

$$A ::= p \in \mathsf{AtProp} \mid \top \mid \bot \mid A \vee A \mid A \wedge A \mid A \to A \mid \alpha \rhd A \mid \Diamond A \mid \Diamond\!\!\!\!\mid\; \alpha \mid \alpha \unrhd \alpha.$$

[2]As discussed below, these modal operators intend to capture agents' abilities and capabilities vis-à-vis resources; in this section, for the sake of a simpler exposition, we present the single-agent version of LRC, where any explicit mention of the agent is omitted.

When writing formulas, we will omit brackets whenever the functional type of the connectives allows for a unique reading. Hence, for instance, we will write $\alpha \rhd (\Diamond A)$ as $\alpha \rhd \Diamond A$ and $(\alpha \cdot \beta) \rhd A$ as $\alpha \cdot \beta \rhd A$. We will also abide by the convention that $\vee$, $\wedge$, $\Diamond$, $\Diamond\!\!\!\!\mid\,$, $\rhd$ and $|\!\!\rhd$ bind more strongly than $\rightarrow$, that $\Diamond$, $\Diamond\!\!\!\!\mid\,$, $\rhd$ and $|\!\!\rhd$ bind more strongly than $\vee$ and $\wedge$, and that $\leftrightarrow$ is a weaker binder than any other connective. With this convention, for instance, $\alpha \rhd A \wedge B$ has the same reading as $(\alpha \rhd A) \wedge B$.

The (single-agent version of the) logic of resources and capabilities LRC, in its Hilbert-style presentation H.LRC, is defined as the smallest set of formulas containing the axioms and rules of intuitionistic propositional logic[3] plus the following axiom schemas:

Pure-resource entailment schemas

- R1. $\sqcup$ and $\sqcap$ are commutative, associative, idempotent, and distribute over each other;
- R2. $\cdot$ is associative with unit 1;
- R3. $\alpha \vdash 1$ and $0 \vdash \alpha$
- R4. $\alpha \cdot (\beta \sqcup \gamma) \vdash (\alpha \cdot \beta) \sqcup (\alpha \cdot \gamma)$ and $(\beta \sqcup \gamma) \cdot \alpha \vdash (\beta \cdot \alpha) \sqcup (\gamma \cdot \alpha)$.

Axiom schemas for $\Diamond$ and $\Diamond\!\!\!\!\mid\,$

D1.	$\Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$	D3.	$\Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta) \leftrightarrow \Diamond\!\!\!\!\mid\,\alpha \vee \Diamond\!\!\!\!\mid\,\beta$
D2.	$\Diamond\bot \leftrightarrow \bot$	D4.	$\Diamond\!\!\!\!\mid\,0 \leftrightarrow \bot$

Axiom schemas for $\rhd$ and $|\!\!\rhd$

B1.	$(\alpha \sqcup \beta) \rhd A \leftrightarrow \alpha \rhd A \wedge \beta \rhd A$	B4.	$(\alpha \sqcup \beta) |\!\!\rhd \gamma \leftrightarrow \alpha |\!\!\rhd \gamma \wedge \beta |\!\!\rhd \gamma$
B2.	$0 \rhd A$	B5.	$0 |\!\!\rhd \alpha$
B3.	$\alpha \rhd \beta \rhd A \rightarrow \alpha \cdot \beta \rhd A$	B6.	$\alpha |\!\!\rhd (\beta \sqcap \gamma) \leftrightarrow \alpha |\!\!\rhd \beta \wedge \alpha |\!\!\rhd \gamma$
		B7.	$\alpha |\!\!\rhd 1$

Interaction axiom schemas

- BD1. $\Diamond\!\!\!\!\mid\,\alpha \wedge \alpha \rhd A \rightarrow \Diamond A$
- BD2. $\alpha |\!\!\rhd \beta \rightarrow \alpha \rhd \Diamond\!\!\!\!\mid\,\beta$

and closed under modus ponens, uniform substitution and the following rules:

$$\frac{\alpha \vdash \beta}{\alpha \cdot \gamma \vdash \beta \cdot \gamma}\ \text{MF} \qquad \frac{A \vdash B}{\alpha \rhd A \vdash \alpha \rhd B}\ \text{MB} \qquad \frac{A \vdash B}{\Diamond A \vdash \Diamond B}\ \text{MD} \qquad \frac{\alpha \vdash \beta}{\gamma |\!\!\rhd \alpha \vdash \gamma |\!\!\rhd \beta}\ \text{MB'}$$

$$\frac{\alpha \vdash \beta}{\gamma \cdot \alpha \vdash \gamma \cdot \beta}\ \text{MF'} \qquad \frac{\alpha \vdash \beta}{\beta \rhd A \vdash \alpha \rhd A}\ \text{AB} \qquad \frac{\alpha \vdash \beta}{\Diamond\!\!\!\!\mid\,\alpha \vdash \Diamond\!\!\!\!\mid\,\beta}\ \text{MD'} \qquad \frac{\alpha \vdash \beta}{\beta |\!\!\rhd \gamma \vdash \alpha |\!\!\rhd \gamma}\ \text{AB'}$$

Finally, for all $A, B \in \mathcal{L}$, we let $A \vdash_{\mathrm{LRC}} B$ iff a proof of B exists in H.LRC which possibly uses A.

Let us expand on the intuitive meaning of the connectives, axioms and rules introduced above, and their formal properties.

[3] The classical propositional logic counterpart of LRC can be obtained as usual by adding e.g. excluded middle to the present axiomatization. Notice that classical propositional base is not needed in any of the case studies of Section 3.5.

The pure-resource fragment of LRC. The pure-resource fragment of the logic LRC is inspired by (distributive) linear logic.[4] Indeed, as is witnessed by conditions R1-R4 and rules MF and MF', the algebraic behaviour of $\sqcap$ (with unit 1), $\sqcup$ (with unit 0) and $\cdot$ (with unit 1) is that of the additive conjunction, additive disjunction and multiplicative conjunction in (distributive) linear logic, respectively. The intuitive understanding of the difference between $\alpha \cdot \beta$ and $\alpha \sqcap \beta$ is also borrowed from linear logic (cf. [42, Section 1.1.2]): indeed, $\alpha \cdot \beta$ can be intuitively understood as the resource obtained by *putting* α and β *together*. This 'putting resources together' can be interpreted in many ways in different contexts: one of them is e.g. when α (water) and β (flour) are mixed together to obtain $\alpha \cdot \beta$ (dough); another is e.g. when α (water) and β (flour), juxtaposed in separate jars, are used at the same time so to form the counterweight $\alpha \cdot \beta$ to keep something in balance. Notice that under both interpretations, $\alpha \cdot \alpha$ is distinct from α. We understand $\alpha \sqcap \beta$ as the resource which is as powerful as α and β taken *separately*. In other words, if we identify any resource γ with the (upward-closed) set of the states of affairs which can be brought about using γ (for brevity let us call such set the *power* of γ), then the resource $\alpha \sqcap \beta$ is uniquely identified by the *union* of the power of α and the power of β. Finally, we understand $\alpha \sqcup \beta$ as the resource the power of which is the intersection of the power of α and the power of β. More in general, the intended meaning of the resource-type entailment $\alpha \vdash \beta$ (namely 'α is at least as powerful a resource as β'), together with the identification of the lattice of resources with the lattice of their powers (which is a lattice of sets closed under union and intersection and hence distributive), explain intuitively the validity of resource-type entailments such as $\alpha \sqcap \alpha \dashv\vdash \alpha$, $\alpha \sqcup \alpha \dashv\vdash \alpha$, $\alpha \vdash \alpha \sqcup \beta$ and $\beta \vdash \alpha \sqcup \beta$, as well as $\alpha \sqcap (\beta \sqcup \gamma) \vdash (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma)$ and $(\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma) \vdash \alpha \sqcup (\beta \sqcap \gamma)$. Moreover, under this reading of $\vdash$, by R3, the bottom 0 and top 1 of the lattice of resources can respectively be understood as the resource that is at least as powerful as any other resource (hence 0 is impossibly powerful), and the resource any other resource, no matter how weak, is at least as powerful as (hence 1 is the resource with no power, or the *empty resource*). This intuition, together with the uniqueness of the neutral element, also justifies one of the main differences between this setting and general linear logic; namely, the fact that the unit of $\cdot$ is the unit of $\sqcap$. Indeed, it seems intuitively plausible that, under the most common interpretations of $\cdot$, putting together (e.g. mixing or juxtaposing) the empty resource and any resource α yields α as outcome.[5] Our inability to distinguish between the units of $\sqcap$ and of $\cdot$ yields as a consequence that the following entailments hold, which are also valid in linear affine logic [50, 51]

$$\alpha \cdot \beta \vdash \alpha \quad \text{and} \quad \alpha \cdot \beta \vdash \beta. \tag{3.2.1}$$

Indeed, by R3, R2 and MF', $\alpha \cdot \beta \vdash \alpha \cdot 1 \vdash \alpha$, and the second entailment goes likewise. This restricts the scope of applications of the present setting: for instance, the fact that the compound resource $\alpha \cdot \beta$ must be at least as powerful as its two components rules out the general examples of e.g. those chemical reactions in which the compound and

[4] However, the conceptual distinction is worth being stressed that, while formulas in linear logic *behave like* resources, pure-resource terms of LRC *literally denote* resources. In this respect, the pure-resource fragment of LRC is similar to the logic of resources introduced in [64, 65].

[5] This is one of the main differences between actions and resources: the idle action `skip`, represented as the identity relation, is the unit of the product operation on actions, and is clearly different from the top element in the lattice of actions (the total relation).

its components are resources of incomparable power. On the other hand, it includes the case of all resources which can be quantified: two 50 euros bills are at least as powerful a resource than each 50 euros bill; two hours of time are at least as powerful a resource than one hour time, and so on. Moreover, this restriction does not rule out the possibility that the power of $\alpha \cdot \beta$ be strictly *greater* than the union of the separate powers of α and β (which is the power of $\alpha \sqcap \beta$). This is the case for instance when a critical mass of fuel is needed for reaching a certain temperature, or a certain outcome (e.g. a nuclear chain reaction). Another difference between the pure-resource fragment of LRC and linear logic is that, in LRC, the connective $\cdot$ is not necessarily commutative.

The modal operators. The intended meaning of the formulas $\Diamond A$ and $\Diamond\!\!\!|\,\alpha$ is 'the agent is able to bring about state of affairs A' and 'the agent is in possession of resource α', respectively. By axioms D1 and D2 (resp. D3 and D4), the connective $\Diamond$ (resp. $\Diamond\!\!\!|$) is a normal diamond-type connective (i.e. its algebraic interpretation is finitely join-preserving). Axiom D1 expresses that being able to bring about $A \vee B$ is tantamount to either being able to bring about A or being able to bring about B. Axiom D2 encodes the fact that the agent can never bring about logical contradictions. Analogously, Axiom D3 says that the agent is in possession of $\alpha \sqcup \beta$ exactly in case is in possession of α or is in possession of β. Axiom D4 encodes the fact that the agent is never in possession of the 'impossibly powerful resource' 0.

The intended meaning of the formula $\alpha \rhd A$ is 'whenever resource α is in possession of the agent, using α the agent is capable to bring about A'. By axioms B1 and B2, the connective $\rhd$ is an antitone normal box-type operator in the first coordinate (i.e. its algebraic interpretation is finitely join-reversing in that coordinate). Axiom B1 says that the agent is capable of bringing about A whenever in possession of $\alpha \sqcup \beta$ iff the agent is capable of bringing about A *both* whenever in possession of α *and* whenever in possession of β. Axiom B2 means that if the agent were in possession of the impossibly powerful resource (which is never the case by D4), the agent could bring about any state of affairs. The justification of axiom B3 is connected with the constraint, encoded in (3.2.1), that the fusion $\alpha \cdot \beta$ of two resources is at least as powerful as each of its components. Taking this fact into account, let us assume that the agent is in possession of $\alpha \cdot \beta$. Hence, by (3.2.1), the resource in its possession is at least as powerful as the resources α and β taken in isolation. If $\alpha \rhd \beta \rhd A$ is the case, then by using $\alpha \cdot \beta$ up to α, the agent can bring about $\beta \rhd A$, and by using the remainder of $\alpha \cdot \beta$, the agent can bring about A, which motivates B3. However, the converse direction is arguably not valid. Indeed, let $\alpha \cdot \beta \rhd A$ express the fact that a certain temperature is reached by burning a critical mass $\alpha \cdot \beta$ of fuel. However, burning α and then β in sequence might not be enough to reach the same temperature.[6]

The intended meaning of the formula $\alpha \mathrel{\rhd\!\!\!|} \beta$ is 'the agent is capable of getting β from α, whenever in possession of α'. By axioms B4 and B5, the connective $\mathrel{\rhd\!\!\!|}$ is an antitone normal box-type operator in the first coordinate (i.e. its algebraic interpretation

[6] There is a surface similarity between B3 and Axiom Ac4 of [72, Section 4], which captures the interaction between the capabilities of agents to perform actions and composition of actions; however, as remarked in Footnote 5, composition of actions behaves differently from composition of resources, which is why B3 is an implication and not a bi-implication.

is finitely join-reversing in that coordinate). Axiom B4 says that the agent is capable of getting resource γ whenever in possession of $\alpha \sqcup \beta$ iff the agent is capable of getting resource γ *both* whenever in possession of α *and* whenever in possession of β. Axiom B5 says that if the agent were in possession of the impossibly powerful resource (which is never the case by D4), the agent could get any resource. By axioms B6 and B7, the connective $\trianglerighteq$ is a monotone normal box-type operator in the second coordinate (i.e. its algebraic interpretation is finitely meet-preserving in that coordinate). Axiom B6 says that the agent is capable of getting resource $\beta \sqcap \gamma$ whenever in possession of α iff the agent is capable of getting *both* β and γ whenever in possession of α. Axiom B7 says that any agent is capable to get the empty resource whenever in possession of any resource.

Axiom BD1 encodes the link between the agent's capabilities and abilities: indeed, it expresses the fact that if the agent is capable to bring about A whenever in possession of α ($\alpha \rhd A$), and moreover the agent is actually in possession of α ($\Diamond\!\!\!\!\mid\, \alpha$), then the agent is able to bring about A ($\Diamond A$). Notice also the analogy between this axiom and the intuitionistic axiom $A \wedge (A \rightarrow B) \leftrightarrow A \wedge B$. Axiom BD2 establishes a link between $\rhd$ and $\trianglerighteq$, via $\Diamond\!\!\!\!\mid\,$; indeed, it says that the agent's being capable to get β implies that the agent is capable to bring about a state of affairs in which the agent is in possession of β.

The rules MB and AB (resp. MB' and AB') encode the fact that $\rhd$ (resp. $\trianglerighteq$) is monotone in its second coordinate and antitone in its first. In fact, AB, MB' and AB' can be derived using B1, B4 and B6. The monotonicity of $\rhd$ in its second coordinate expresses the intuition that if the agent is capable, whenever in possession of α, to bring about A, then is capable to bring about any state of affairs which is logically implied by A. The remaining rules encode the monotonicity of $\Diamond$, $\Diamond\!\!\!\!\mid\,$ and $\cdot$.

Some additional axioms. We conclude the present discussion by mentioning some analytic axioms which might perhaps be interesting for different settings. We start mentioning $\Diamond\top$, $\Diamond\!\!\!\!\mid\, 1$, and $\alpha \rhd \top$, respectively stating that the agent is able to bring about what is always the case, such as logical tautologies; the agent is in possession of the empty resource; the agent is capable of using any resource (hence also the empty one) to bring about what is always the case. We also mention $\alpha \trianglerighteq \alpha$, stating that the agent is capable to get any resource already in the possession of the agent; $\Diamond\!\!\!\!\mid\, \alpha \wedge \alpha \trianglerighteq \beta \rightarrow \Diamond\!\!\!\!\mid\, \beta$, and $\Diamond\!\!\!\!\mid\, \alpha \wedge \alpha \trianglerighteq \beta \rightarrow \Diamond\Diamond\!\!\!\!\mid\, \beta$. The latter is a consequence of BD1 and BD2, while the former is used in the case study in Section 3.5.4. For the sake of achieving greater generality we chose not to include it in the general system. Axioms which might also be considered in special settings are $\alpha \rhd (A \vee B) \rightarrow \alpha \rhd A \vee \alpha \rhd B$, and $\alpha \rhd A \wedge \alpha \rhd B \rightarrow \alpha \rhd (A \wedge B)$. The first one would imply the distributivity of $\rhd$ over disjunction in its second coordinate. The axiom $\alpha \rhd A \wedge \alpha \rhd B \rightarrow \alpha \rhd (A \wedge B)$ is not applicable in general, given that the consequence would require the duplication of the resource α. More generally applicable variants are $\alpha \rhd A \wedge \alpha \rhd B \rightarrow \alpha \cdot \alpha \rhd (A \wedge B)$ and $\alpha \trianglerighteq \beta \wedge \alpha \trianglerighteq \gamma \rightarrow \alpha \cdot \alpha \trianglerighteq (\beta \cdot \gamma)$. The latter encodes the behaviour of scalable resources, and will be used in the case study of Sections 3.5.2 and 3.5.4. Another interesting axiom is the converse of B3, which we have discussed above.

3.2.2 Algebraic completeness

In the present section we outline the completeness of LRC w.r.t. the heterogeneous LRC-algebras[7] defined below, via a Lindenbaum-Tarski type construction.

Definition 83. A *heterogeneous* LRC-*algebra* is a tuple $F = (\mathbb{A}, \mathbb{Q}, \rhd, \Diamond, \unrhd, \Diamond\!\!\!\mid)$ such that $\mathbb{A}$ is a Heyting algebra, $\mathbb{Q} = (Q, \sqcup, \sqcap, \cdot, 0, 1)$ is a bounded distributive lattice with binary operator $\cdot$ which preserves finite joins in each coordinate and the unit of which is 1,[8] and $\rhd : \mathbb{Q} \times \mathbb{A} \to \mathbb{A}$, $\Diamond : \mathbb{A} \to \mathbb{A}$, $\unrhd : \mathbb{Q} \times \mathbb{Q} \to \mathbb{A}$, $\Diamond\!\!\!\mid : \mathbb{Q} \to \mathbb{A}$ verify the (quasi-)inequalities corresponding to the axioms and rules of LRC as presented in the previous section. A heterogeneous LRC-algebra is *perfect* if both $\mathbb{A}$ and $\mathbb{Q}$ are perfect,[9] and the operations $\rhd$, $\Diamond$, $\unrhd$, and $\Diamond\!\!\!\mid$ satisfy the infinitary versions of the join- and meet-preservation properties satisfied by definition in any heterogeneous LRC-algebra. An *algebraic* LRC-*model* is a tuple $\mathbb{M} := (F, v_{\mathsf{Fm}}, v_{\mathsf{Res}})$ such that F is a heterogeneous LRC-algebra, v_{Fm} : AtProp $\to \mathbb{A}$ and v_{Res} : AtRes $\to \mathbb{Q}$. Clearly, for every algebraic LRC-model $\mathbb{M}$, the assignments v_{Fm} and v_{Res} have unique homomorphic extensions which we identify with v_{Fm} and v_{Res} respectively. For each $\mathsf{T} \in \{\mathsf{Fm}, \mathsf{Res}\}$ and all terms a, b of type T, we let $a \models_{\mathrm{LRC}} b$ iff $v_{\mathsf{T}}(a) \leq v_{\mathsf{T}}(b)$ for every model $\mathbb{M}$.

Given AtProp and AtRes, the *Lindenbaum-Tarski heterogeneous LRC-algebra* over AtProp and AtRes is defined to be the following structure:

$$F^\star := (\mathbb{A}^\star, \mathbb{Q}^\star, \rhd^\star, \Diamond^\star, \unrhd^\star, \Diamond\!\!\!\mid^\star)$$

where:

1. $\mathbb{A}^\star$ is the quotient algebra Fm/$\dashv\vdash$, where Fm is the formula algebra corresponding to the language $\mathcal{L}$ defined in the previous subsection, and $\dashv\vdash$ is the equivalence relation on Fm defined as $A \dashv\vdash A'$ iff $A \vdash A'$ and $A' \vdash A$. Notice that the rules MD, MB,AB, MD', MB' and AB' guarantee that $\dashv\vdash$ is compatible with $\Diamond$, $\rhd$, $\Diamond\!\!\!\mid$ and $\unrhd$, hence the quotient algebra construction is well defined. The elements of $\mathbb{A}^\star$ will be typically denoted $[B]$ for some formula $B \in \mathcal{L}$;

2. $\mathbb{Q}^\star$ is the quotient algebra Res/$\dashv\vdash$, where Res is the resource algebra corresponding to the language $\mathcal{R}$ defined in the previous subsection, and $\dashv\vdash$ is the equivalence relation on Res defined as $\alpha \dashv\vdash \alpha'$ iff $\alpha \vdash \alpha'$ and $\alpha' \vdash \alpha$. Notice that the rules MF and MF' guarantee that $\dashv\vdash$ is compatible with $\cdot$, hence the quotient algebra construction is well defined. The elements of $\mathbb{Q}^\star$ will be typically denoted $[\alpha]$ for some resource $\alpha \in \mathcal{R}$;

3. $\rhd^\star : Q^\star \times \mathbb{A}^\star \to \mathbb{A}^\star$ is defined as $[\alpha] \rhd^\star [B] := [\alpha \rhd B]$;

[7] This notion specializes the more general notion of heterogeneous algebras introduced in [7] to the setting of interest of the present chapter.

[8] It immediately follows from the definition that $\alpha \cdot \beta \leq \alpha$ and $\alpha \cdot \beta \leq \beta$ for all $\alpha, \beta \in Q$.

[9] A bounded distributive lattice (BDL) is *perfect* if it is complete, completely distributive and completely join-generated by its completely join-irreducible elements. A BDL is perfect iff it is isomorphic to the lattice of the upward-closed subsets of some poset. A Heyting algebra is *perfect* if its lattice reduct is a perfect BDL. A bounded distributive lattice with operators (abbreviated DLO. *Operators* are additional operations which are finitely join-preserving in each coordinate) is *perfect* if its lattice reduct is a perfect BDL, and each operator is completely join-preserving in each coordinate.

4. $\Diamond^\star : \mathbb{A}^\star \to \mathbb{A}^\star$ is defined as $\Diamond^\star[B] := [\Diamond B]$;
5. $\trianglerighteq^\star : Q^\star \times Q^\star \to \mathbb{A}^\star$ is defined as $[\alpha_1] \trianglerighteq^\star [\alpha_2] := [\alpha_1 \trianglerighteq \alpha_2]$;
6. $\Diamond\!\!\!|^\star : Q^\star \to \mathbb{A}^\star$ is defined as $\Diamond\!\!\!|^\star[\alpha] := [\Diamond\!\!\!| \, \alpha]$;

Lemma 84. *For any* AtProp *and* AtRes*, $F^\star$ is a heterogeneous LRC-algebra.*

Proof. It is a standard verification that $\mathbb{A}^\star$ is a Heyting algebra and that $\mathbb{Q}^\star$ is a bounded distributive lattice with binary operator $\cdot$ which preserves finite joins in each coordinate and the unit of which is 1. It is also an easy verification that $\triangleright^\star$, $\Diamond^\star$, $\trianglerighteq^\star$ and $\Diamond\!\!\!|^\star$ are well-defined, and verify the additional conditions by construction. □

The canonical assignments can be defined as usual, i.e. mapping atomic propositions and resources to their canonical value in $F^\star$. Let $\mathbb{M}^*$ be the resulting LRC-algebraic model. With this definition, the proof of the following proposition is routine, and is omitted.

Proposition 85. *For all $X \subseteq \mathcal{L}$ and $A \in \mathcal{L}$, if $X \nvdash_{\mathrm{LRC}} A$, then $X \not\models_{\mathrm{LRC}} A$.*

3.2.3 Algebraic canonicity

The present subsection is aimed at showing that LRC is strongly complete w.r.t. perfect heterogeneous LRC-algebras. This will be a key ingredient in the conservativity proof of Section 3.4.4.

Definition 86. Let $F = (\mathbb{A}, \mathbb{Q}, \triangleright, \Diamond, \trianglerighteq, \Diamond\!\!\!|)$ be a heterogeneous LRC-algebra. The *canonical extension* of F is

$$F^\delta = (\mathbb{A}^\delta, \mathbb{Q}^\delta, \triangleright^\pi, \Diamond^\sigma, \trianglerighteq^\pi, \Diamond\!\!\!|^\sigma),$$

where $\mathbb{A}^\delta$ and $\mathbb{Q}^\delta$ are the canonical extensions of $\mathbb{A}$ and $\mathbb{Q}$ respectively[10], the operations $\Diamond\!\!\!|^\sigma : \mathbb{Q}^\delta \to \mathbb{A}^\delta$ and $\trianglerighteq^\pi : \mathbb{Q}^\delta \times \mathbb{Q}^\delta \to \mathbb{A}^\delta$ and $\Diamond^\sigma : \mathbb{A}^\delta \to \mathbb{A}^\delta$ and $\triangleright^\pi : \mathbb{Q}^\delta \times \mathbb{A}^\delta \to \mathbb{A}^\delta$ are defined as follows: for any $k \in K(\mathbb{A}^\delta)$, $\kappa \in K(\mathbb{Q}^\delta)$ and $o \in O(\mathbb{A}^\delta)$, $\omega \in O(\mathbb{Q}^\delta)$,[11]

[10] The *canonical extension* of a BDL L is a complete distributive lattice L^δ containing L as a sublattice, such that:

1. *(denseness)* every element of L^δ can be expressed both as a join of meets and as a meet of joins of elements from L;
2. *(compactness)* for all $S, T \subseteq L$, if $\bigwedge S \leq \bigvee T$ in L^δ, then $\bigwedge F \leq \bigvee G$ for some finite sets $F \subseteq S$ and $G \subseteq T$.

It is well known that the canonical extension of a BDL is a *perfect* BDL (cf. Footnote 9). Completeness and complete distributivity imply that each perfect BDL is naturally endowed with a Heyting algebra structure, and hence each perfect BDL is also a perfect Heyting algebra. Moreover, if L is the lattice reduct of some Heyting algebra $\mathbb{A}$, then $\mathbb{A}$ is a subalgebra of L^δ, seen as a perfect Heyting algebra. The *canonical extension* $\mathbb{A}^\delta$ of a Heyting algebra $\mathbb{A}$ is defined as the canonical extension of the lattice reduct of $\mathbb{A}$ endowed with its natural Heyting algebra structure. The *canonical extension* $\mathbb{Q}^\delta$ of a DLO $\mathbb{Q}$ is defined as the canonical extension of the lattice reduct of $\mathbb{Q}$ endowed with the σ-extension of each additional operator. It is well known that the canonical extension of a Heyting algebra (resp. DLO) is a perfect Heyting algebra (resp. DLO).

[11] For any BDL L, an element $k \in L^\delta$ (resp. $o \in L^\delta$) is *closed* (resp. *open*) if is the meet (resp. join) of some subset of L. The set of closed (resp. open) elements of L^δ is $K(L^\delta)$ (resp. $O(L^\delta)$). We will slightly abuse notation and write $K(\mathbb{A}^\delta)$ (resp. $O(\mathbb{A}^\delta)$) and $K(\mathbb{Q}^\delta)$ (resp. $O(\mathbb{Q}^\delta)$) to refer to the sets of closed and open elements of their lattice reducts.

$$\Diamond\!\!\!\!\mid^{\sigma}\kappa := \bigwedge\{\Diamond\!\!\!\!\mid\,\alpha \mid \alpha \in \mathbb{Q} \text{ and } \kappa \leq \alpha\}$$

$$\kappa \rhd\!\!\!\!-^{\pi}\omega := \bigvee\{\alpha \rhd\!\!\!\!- \beta \mid \beta \in \mathbb{Q},\ \beta \leq \omega,\ \alpha \in \mathbb{Q} \text{ and } \kappa \leq \alpha\}$$

$$\Diamond^{\sigma} k := \bigwedge\{\Diamond a \mid a \in \mathbb{A} \text{ and } k \leq a\}$$

$$\kappa \rhd^{\pi} o := \bigvee\{\alpha \rhd a \mid a \in \mathbb{A},\ a \leq o,\ \alpha \in \mathbb{Q} \text{ and } \kappa \leq \alpha\}$$

and for any $u \in \mathbb{A}^\delta$ and $q, w \in \mathbb{Q}^\delta$,

$$\Diamond\!\!\!\!\mid^{\sigma} q := \bigvee\{\Diamond\!\!\!\!\mid^{\sigma}\kappa \mid \kappa \in K(\mathbb{Q}^\delta) \text{ and } \kappa \leq q\}$$

$$q \rhd\!\!\!\!-^{\pi} w := \bigwedge\{\kappa \rhd\!\!\!\!-^{\pi}\omega \mid \omega \in O(\mathbb{Q}^\delta),\ w \leq \omega,\ \kappa \in K(\mathbb{Q}^\delta) \text{ and } \kappa \leq q\}$$

$$\Diamond^{\sigma} u := \bigvee\{\Diamond^{\sigma} k \mid k \in K(\mathbb{A}^\delta) \text{ and } k \leq u\}$$

$$q \rhd^{\pi} u := \bigwedge\{\kappa \rhd^{\pi} o \mid o \in O(\mathbb{A}^\delta),\ u \leq o,\ \kappa \in K(\mathbb{Q}^\delta) \text{ and } \kappa \leq q\}.$$

Below we also report the definition of $\cdot^\sigma$ for the reader's convenience: For any $\kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta)$

$$\kappa_1 \cdot^\sigma \kappa_2 = \bigwedge\{\alpha \cdot \beta \mid \alpha, \beta \in \mathbb{Q} \text{ and } \kappa_1 \leq \alpha, \kappa_2 \leq \beta\},$$

and for any $q_1, q_2 \in \mathbb{Q}^\delta$

$$q_1 \cdot^\sigma q_2 = \bigvee\{\kappa_1 \cdot^\sigma \kappa_2 \mid \kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta) \text{ and } \kappa_1 \leq q_1, \kappa_2 \leq q_2\}.$$

In what follows, for the sake of readability, we will write $\cdot^\sigma$ without the superscript. This will not create ambiguities, since we use different variables to denote the elements of $\mathbb{Q}$, $K(\mathbb{Q}^\delta)$, $O(\mathbb{Q}^\delta)$ and $\mathbb{Q}^\delta$, and since $\cdot$ and $\cdot^\sigma$ coincide over $\mathbb{Q}$.

Lemma 87. *For any heterogeneous LRC-algebra F, the canonical extension F^δ is a perfect heterogeneous LRC-algebra.*

Proof. As discussed in Footnote 10, $\mathbb{A}^\delta$ is a perfect Heyting algebra and $\mathbb{Q}^\delta$ is a perfect DLO, so to finish the proof it is enough to show that the validity of all axioms and rules of LRC transfers from F to F^δ, and moreover, the join-and meet-preservation properties of the operations of F hold in their infinitary versions in F^δ. Conditions R2 hold in $\mathbb{Q}^\delta$ as consequences of the general theory of canonicity of terms purely built on operators (cf. [39, Theorem 4.6]). As to D1 and D2, by assumption the operation $\Diamond$ preserves finite joins. Hence, by a well known fact of the theory of the σ-extensions of finitely join-preserving maps, $\Diamond^\sigma$ preserves arbitrary joins (cf. [39, Theorem 3.2]). The same argument applies to D3, D4, B4, B5, B6, B7. Furthermore, by [40, Lemma 2.22] it follows that $\rhd^\pi$ turns arbitrary joins into arbitrary meets in the first coordinate, which is the infinitary version of B1.

As to axiom B2, it is enough to show that for every $u \in \mathbb{A}^\delta$,

$$0 \rhd^\pi u = \top.$$

Let us preliminarily show the identity above for $o \in O(\mathbb{A}^\delta)$. Notice that the set $\{a \mid a \in \mathbb{A}, a \leq o\}$ is always nonempty since $\bot$ belongs to it. Hence,

$$\begin{array}{rcl} 0 \rhd^\pi o & = & \bigvee\{0 \rhd a \mid a \in \mathbb{A}, a \leq o\} \\ & = & \bigvee\{\top \mid a \in \mathbb{A}, a \leq o\} \\ & = & \top. \end{array}$$

Hence, for arbitrary $u \in \mathbb{A}^\delta$

$$\begin{array}{rcl} 0 \rhd^\pi u & = & \bigwedge\{0 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o\} \\ & = & \bigwedge\{\top \mid o \in O(\mathbb{A}^\delta), u \leq o\} \\ & = & \top. \end{array}$$

As to B3, let us show that for all $q, w \in \mathbb{Q}^\delta$ and $u \in \mathbb{A}^\delta$,

$$q \rhd^\pi w \rhd^\pi u \leq q \cdot w \rhd^\pi u.$$

Let us preliminarily show that the inequality above is true for any $o \in O(\mathbb{A}^\delta)$, $\kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta)$. By definition, if $o \in O(\mathbb{A}^\delta)$ and $\kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta)$ then $\kappa_2 \rhd^\pi o \in O(\mathbb{A}^\delta)$ and $\kappa_1 \cdot \kappa_2 \in K(\mathbb{Q}^\delta)$. Therefore:

$$\begin{array}{lll} & \kappa_1 \rhd^\pi \kappa_2 \rhd^\pi o & \\ = & \bigvee\{\alpha \rhd a \mid a \in \mathbb{A}, \alpha \in \mathbb{Q}, \kappa_1 \leq \alpha, a \leq \kappa_2 \rhd^\pi o\} & \text{(definition)} \\ = & \bigvee\{\alpha \rhd a \mid a \in \mathbb{A}, \alpha \in \mathbb{Q}, \kappa_1 \leq \alpha, a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}\} & \text{(definition)} \\ = & \bigvee\{\alpha \rhd \beta \rhd a \mid a \in \mathbb{A}, a \leq o, \alpha, \beta \in \mathbb{Q}, \kappa_1 \leq \alpha, \kappa_2 \leq \beta\} & (*) \\ \leq & \bigvee\{\alpha \cdot \beta \rhd a \mid a \in \mathbb{A}, a \leq o, \alpha, \beta \in \mathbb{Q}, \kappa_1 \leq \alpha, \kappa_2 \leq \beta\} & \text{(B3 in } \mathbb{A}) \\ \leq & \bigvee\{\gamma \rhd a \mid a \in \mathbb{A}, a \leq o, \gamma \in \mathbb{Q}, \kappa_1 \cdot \kappa_2 \leq \gamma\} & (**) \\ = & \kappa_1 \cdot \kappa_2 \rhd^\pi o & \text{(definition)} \end{array}$$

Let us prove the equality marked with $(*)$. If $a \in \mathbb{A}$, $\beta \in \mathbb{Q}$, $a \leq o$ and $\kappa \leq \beta$, then $\beta \rhd a \in \mathbb{A}$ and $\beta \rhd a \in \{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}$, hence $\beta \rhd a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}$. This, in turn, implies that

$$\alpha \rhd \beta \rhd a \in \{\alpha \rhd a \mid a \in \mathbb{A}, \alpha \in \mathbb{Q}, \kappa_1 \leq \alpha, a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}\}.$$

Therefore

$$\begin{array}{ll} & \{\alpha \rhd \beta \rhd a \mid a \in \mathbb{A}, a \leq o, \alpha, \beta \in \mathbb{Q}, \kappa_1 \leq \alpha, \kappa_2 \leq \beta\} \\ \subseteq & \{\alpha \rhd a \mid a \in \mathbb{A}, \alpha \in \mathbb{Q}, \kappa_1 \leq \alpha, a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}\} \end{array}$$

and thus

$$\begin{array}{ll} & \bigvee\{\alpha \rhd \beta \rhd a \mid a \in \mathbb{A}, a \leq o, \alpha, \beta \in \mathbb{Q}, \kappa_1 \leq \alpha, \kappa_2 \leq \beta\} \\ \leq & \bigvee\{\alpha \rhd a \mid a \in \mathbb{A}, \alpha \in \mathbb{Q}, \kappa_1 \leq \alpha, a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}\}. \end{array}$$

To prove the converse inequality, it is enough to show that if $a \in \mathbb{A}$ and $a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}$, then $\alpha \rhd a \leq \alpha \rhd \beta \rhd b$ for some $b \in \mathbb{A}$ such that $b \leq o$ and some $\beta \in \mathbb{Q}$ such that $\kappa_2 \leq \beta$. By compactness (cf. Footnote 10), $a \leq \bigvee\{\beta \rhd b \mid b \in \mathbb{A}, b \leq o, \beta \in \mathbb{Q}, \kappa_2 \leq \beta\}$ implies that $a \leq \bigvee\{\beta_i \rhd b_i \mid 1 \leq i \leq n\}$ for some $b_i \in \mathbb{A}$, $\beta_i \in \mathbb{Q}$ such that $b_i \leq o$, $\kappa_2 \leq \beta_i$ for every $1 \leq i \leq n$. Since $\rhd$ is monotone in its second coordinate and antitone in its first, this implies that

$$a \leq \beta_1 \rhd b_1 \vee \ldots \vee \beta_n \rhd b_n \leq (\beta_1 \sqcap \ldots \sqcap \beta_n) \rhd (b_1 \vee \ldots \vee b_n).$$

Let $b := b_1 \vee \ldots \vee b_n$ and $\beta = \beta_1 \sqcap \ldots \sqcap \beta_n$. By definition, $b \in \mathbb{A}$, $\beta \in \mathbb{Q}$ and $b \leq o$, $\kappa_2 \leq \beta$. Moreover, again by monotonicity, the displayed inequality implies that

$\alpha \rhd a \leq \alpha \rhd \beta \rhd b$, as required. This finishes the proof of $(*)$. The inequality marked with $(**)$ holds since if $\kappa_1 \leq \alpha$ and $\kappa_2 \leq \beta$ then $\kappa_1 \cdot \kappa_2 \leq \alpha \cdot \beta$, so $\alpha \cdot \beta \rhd a \in \{\gamma \rhd a \mid a \in \mathbb{A}, a \leq o, \gamma \in \mathbb{Q}, \kappa_1 \cdot \kappa_2 \leq \gamma\}$ and therefore

$$\{\alpha{\cdot}\beta \rhd a \mid a \in \mathbb{A}, a \leq o, \alpha, \beta \in \mathbb{Q}, \kappa_1 \leq \alpha, \kappa_2 \leq \beta\} \subseteq \{\gamma \rhd a \mid a \in \mathbb{A}, a \leq o, \gamma \in \mathbb{Q}, \kappa_1{\cdot}\kappa_2 \leq \gamma\}.$$

Let us show that B3 holds for arbitrary $u \in \mathbb{A}^\delta$ and $q, w \in \mathbb{Q}^\delta$.

$$\begin{array}{cll}
& q \rhd^\pi w \rhd^\pi u & \\
= & \bigwedge\{\kappa_1 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), \kappa_1 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, w \rhd^\pi u \leq o\} & \text{(definition)} \\
= & \bigwedge\{\kappa_1 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), \kappa_1 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \bigwedge\{\kappa_2 \rhd^\pi o' \mid u \leq o', \kappa_2 \leq w\} \leq o\} & \text{(definition)} \\
\leq & \bigwedge\{\kappa_1 \rhd^\pi \kappa_2 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \kappa_2 \leq w\} & (***) \\
\leq & \bigwedge\{\kappa_1 \cdot \kappa_2 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \kappa_2 \leq w\} & (\dagger) \\
= & \bigwedge\{\left(\bigvee\{\kappa_1 \cdot \kappa_2 \mid \kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \kappa_2 \leq w\}\right) \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o\} & (\ddagger) \\
= & \bigwedge\{q \cdot w \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o\} & \text{(definition)} \\
= & q \cdot w \rhd^\pi u & \text{(definition)}
\end{array}$$

The inequality marked with $(***)$ holds since, for any $o \in O(\mathbb{A}^\delta)$ and $\kappa \in K(\mathbb{Q}^\delta)$, if $u \leq o$ and $\kappa \leq w$ then $\kappa \rhd^\pi o \in O(\mathbb{A}^\delta)$ and $\kappa \rhd^\pi o \in \{\kappa_2 \rhd^\pi o' \mid u \leq o', \kappa_2 \leq w\}$, hence $\bigwedge\{\kappa_2 \rhd^\pi o' \mid u \leq o', \kappa_2 \leq w\} \leq \kappa \rhd^\pi o$. This implies that

$$\kappa_1 \rhd^\pi \kappa_2 \rhd^\pi o \in \{\kappa_1 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), \kappa_1 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \bigwedge\{\kappa_2 \rhd^\pi o' \mid u \leq o', \kappa_2 \leq w\} \leq o\}$$

and therefore

$$\begin{array}{cl}
& \{\kappa_1 \rhd^\pi \kappa_2 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \kappa_2 \leq w\} \\
\subseteq & \{\kappa_1 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), \kappa_1 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \bigwedge\{\kappa_2 \rhd^\pi o' \mid u \leq o', \kappa_2 \leq w\} \leq o\}
\end{array}$$

which implies that

$$\begin{array}{cl}
& \bigwedge\{\kappa_1 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), \kappa_1 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \bigwedge\{\kappa_2 \rhd^\pi o' \mid u \leq o', \kappa_2 \leq w\} \leq o\} \\
\leq & \bigwedge\{\kappa_1 \rhd^\pi \kappa_2 \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta), \kappa_1 \leq q, \kappa_2 \leq w\}.
\end{array}$$

The inequality marked with $(\dagger)$ holds since as we showed above B3 holds for any $o \in O(\mathbb{A}^\delta)$, $\kappa_1, \kappa_2 \in K(\mathbb{Q}^\delta)$. The equality marked with $(\ddagger)$ holds because $\rhd^\pi$ is completely join reversing in the first coordinate.

As to axiom BD1, let us show that for any $q \in \mathbb{Q}^\delta$ and $u \in \mathbb{A}^\delta$,

$$\Phi^\sigma q \wedge q \rhd^\pi u \leq \Diamond^\sigma u.$$

Let us preliminarily show that the inequality above is true for any $o \in O(\mathbb{A}^\delta)$ and $\kappa \in K(\mathbb{Q}^\delta)$:

$$\begin{array}{cll}
& \Phi^\sigma \kappa \wedge \kappa \rhd^\pi o & \\
= & \bigwedge\{\Phi \beta \mid \beta \in \mathbb{Q}, \kappa \leq \beta\} \wedge \bigvee\{\alpha \rhd a \mid a \in \mathbb{A}, a \leq o, \alpha \in \mathbb{Q}, \kappa \leq \alpha\} & \text{(by definition)} \\
= & \bigvee\{\left(\bigwedge\{\Phi \beta \mid \beta \in \mathbb{Q}, \kappa \leq \beta\}\right) \wedge \alpha \rhd a \mid a \in \mathbb{A}, a \leq o\} & \text{(distributivity)} \\
\leq & \bigvee\{\Phi \alpha \wedge \alpha \rhd a \mid a \in \mathbb{A}, a \leq o\} & (*) \\
\leq & \bigvee\{\Diamond a \mid a \in \mathbb{A}, a \leq o\} & \text{(BD2 holds in } \mathbb{A}) \\
= & \Diamond^\sigma \bigvee\{a \mid a \in \mathbb{A}, a \leq o\} & (\Diamond^\sigma \text{ is comp. join-pres.}) \\
= & \Diamond^\sigma o. & \text{(by definition)}
\end{array}$$

The inequality marked with $(*)$ holds because if $\kappa \leq \alpha$, then $\Phi\alpha \in \{\Phi\beta \mid \beta \in \mathbb{Q}, \kappa \leq \beta\}$ and therefore $\bigwedge\{\Phi\beta \mid \beta \in \mathbb{Q}, \kappa \leq \beta\} \leq \Phi\alpha$. Let us show the inequality for arbitrary $u \in \mathbb{A}^\delta$ and $q \in \mathbb{Q}^\delta$. In what follows, let $\blacksquare$ denote the right adjoint of $\Diamond^\sigma$. It is well known (cf. [23, Lemma 10.3.3]) that $\blacksquare o \in O(\mathbb{A}^\delta)$ for any $o \in O(\mathbb{A}^\delta)$.

$$
\begin{array}{cll}
& \Phi^\sigma q \wedge q \rhd^\pi u & \\
= & \bigvee\{\Phi^\sigma\kappa \mid \kappa \in K(\mathbb{Q}^\delta), \kappa \leq q\} \wedge \bigwedge\{\kappa' \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa' \in K(\mathbb{Q}^\delta), \kappa' \leq q\} & \text{(definition)} \\
= & \bigvee\{\Phi^\sigma\kappa \wedge \bigwedge\{\kappa' \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa' \in K(\mathbb{Q}^\delta), \kappa' \leq q\} \mid \kappa \in K(\mathbb{Q}^\delta), \kappa \leq q\} & \text{(distrib.)} \\
\leq & \bigvee\{\Phi^\sigma\kappa \wedge \bigwedge\{\kappa \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o\} \mid \kappa \in K(\mathbb{Q}^\delta), \kappa \leq q\} & (*) \\
\leq & \bigvee\{\Phi^\sigma\kappa \wedge \bigwedge\{\kappa \rhd^\pi \blacksquare o \mid o \in O(\mathbb{A}^\delta), u \leq \blacksquare o\} \mid \kappa \in K(\mathbb{Q}^\delta), \kappa \leq q\} & (\blacksquare o \in O(\mathbb{A}^\delta)) \\
\leq & \bigvee\{\bigwedge\{\Phi^\sigma\kappa \wedge \kappa \rhd^\pi \blacksquare o \mid o \in O(\mathbb{A}^\delta), u \leq \blacksquare o\} \mid \kappa \in K(\mathbb{Q}^\delta), \kappa \leq q\} & \text{(distrib.)} \\
\leq & \bigvee\{\bigwedge\{\Diamond^\sigma \blacksquare o \mid o \in O(\mathbb{A}^\delta), u \leq \blacksquare o\} \mid \kappa \in K(\mathbb{Q}^\delta), \kappa \leq q\} & \text{(BD2 in } O(\mathbb{A}^\delta)) \\
= & \bigwedge\{\Diamond^\sigma \blacksquare o \mid o \in O(\mathbb{A}^\delta), u \leq \blacksquare o\} & (\Diamond^\sigma \blacksquare o \text{ does not contain } \kappa) \\
\leq & \bigwedge\{o \in O(\mathbb{A}^\delta) \mid u \leq \blacksquare o\} & (\Diamond^\sigma \blacksquare o \leq o) \\
= & \bigwedge\{o \in O(\mathbb{A}^\delta) \mid \Diamond^\sigma u \leq o\} & \text{(adjunction)} \\
= & \Diamond^\sigma u. & \text{(denseness)}
\end{array}
$$

The inequality marked with $(*)$ holds because if $\kappa \leq q$ and $u \leq o$, then

$$\kappa \rhd^\pi o \in \{\kappa' \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa' \in K(\mathbb{Q}^\delta), \kappa' \leq q\}$$

and therefore

$$\{\kappa \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o\} \subseteq \{\kappa' \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa' \in K(\mathbb{Q}^\delta), \kappa' \leq q\}$$

which yields

$$\bigwedge\{\kappa' \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o, \kappa' \in K(\mathbb{Q}^\delta), \kappa' \leq q\} \leq \bigwedge\{\kappa \rhd^\pi o \mid o \in O(\mathbb{A}^\delta), u \leq o\}.$$

Finally for axiom BD2 let us show that for any $q, w \in \mathbb{Q}^\delta$,

$$q \mathrel{|\!\!\rhd}^\pi w \leq q \rhd^\pi \Phi^\sigma w.$$

Let us preliminarily show that the inequality above is true for any $\omega \in O(\mathbb{Q}^\delta)$ and $\kappa \in K(\mathbb{Q}^\delta)$. Notice that since Φ^σ is completely join preserving, if $\omega \in O(\mathbb{Q}^\delta)$ then $\Phi^\sigma\omega \in O(\mathbb{A}^\delta)$.

$$
\begin{array}{cll}
& \kappa \mathrel{|\!\!\rhd}^\pi \omega & \\
= & \bigvee\{\alpha \mathrel{|\!\!\rhd} \beta \mid \alpha, \beta \in \mathbb{Q}, \kappa \leq \alpha, \beta \leq \omega\} & \text{(by definition)} \\
= & \bigvee\{\alpha \rhd \Phi\beta \mid \alpha, \beta \in \mathbb{Q}, \kappa \leq \alpha, \beta \leq \omega\} & \text{(BD2 holds in } \mathbb{A}) \\
\leq & \bigvee\{\alpha \rhd a \mid \alpha \in \mathbb{Q}, a \in \mathbb{A}, \kappa \leq \alpha, a \leq \Phi^\sigma\omega\} & (*) \\
= & \kappa \rhd^\pi \Phi^\sigma\omega & \text{(by definition)}
\end{array}
$$

The inequality marked with $(*)$ holds because if $\beta \leq \omega$ then $\Phi\beta \leq \Phi^\sigma\omega$, thus if $\kappa \leq \alpha$ we have

$$\alpha \rhd \Phi\beta \in \{\alpha \rhd a \mid \alpha \in \mathbb{Q}, a \in \mathbb{A}, \kappa \leq \alpha, a \leq \Phi^\sigma\omega\}$$

and therefore

$$\{\alpha \rhd \Phi\beta \mid \alpha, \beta \in \mathbb{Q}, \kappa \leq \alpha, \beta \leq \omega\} \subseteq \{\alpha \rhd a \mid \alpha \in \mathbb{Q}, a \in \mathbb{A}, \kappa \leq \alpha, a \leq \Phi^\sigma\omega\}.$$

Let us show the inequality for arbitrary $q, w \in \mathbb{Q}^\delta$. In what follows, let $\blacksquare : \mathbb{A}^\delta \to \mathbb{Q}^\delta$ denote the right adjoint of Φ^σ.

$$
\begin{array}{lll}
 & q \unrhd^{\pi} w & \\
= & \bigwedge\{\kappa \unrhd^{\pi} \omega \mid \kappa \in K(\mathbb{Q}^{\delta}), \omega \in O(\mathbb{Q}^{\delta}), \kappa \leq q, w \leq \omega\} & \text{(by definition)} \\
\leq & \bigwedge\{\kappa \unrhd^{\pi} \blacksquare o \mid \kappa \in K(\mathbb{Q}^{\delta}), o \in O(\mathbb{A}^{\delta}), \kappa \leq q, w \leq \blacksquare o\} & (\blacksquare o \in O(\mathbb{Q}^{\delta})) \\
\leq & \bigwedge\{\kappa \rhd^{\pi} \Phi^{\sigma} \blacksquare o \mid \kappa \in K(\mathbb{Q}^{\delta}), o \in O(\mathbb{A}^{\delta}), \kappa \leq q, w \leq \blacksquare o\} & \text{(BD2 for } \omega \in O(\mathbb{Q}^{\delta}),\ \kappa \in K(\mathbb{Q}^{\delta})) \\
\leq & \bigwedge\{\kappa \rhd^{\pi} o \mid \kappa \in K(\mathbb{Q}^{\delta}), o \in O(\mathbb{A}^{\delta}), \kappa \leq q, w \leq \blacksquare o\} & (\Phi^{\sigma} \blacksquare o \leq o) \\
= & \bigwedge\{\kappa \rhd^{\pi} o \mid \kappa \in K(\mathbb{Q}^{\delta}), o \in O(\mathbb{A}^{\delta}), \kappa \leq q, \Phi^{\sigma} w \leq o\} & \text{(by adjunction)} \\
= & q \rhd^{\pi} \Phi^{\sigma} w & \text{(by definition)}
\end{array}
$$

□

As an immediate consequence of Proposition 85 and Lemma 87 we get the following

Corollary 88. *The logic LRC is strongly complete w.r.t. the class of perfect heterogeneous LRC-algebras.*

3.2.4 Disjunction property

In the present section, we show that the disjunction property holds for LRC, by adapting the standard argument to the setting of heterogeneous LRC-algebras. For any heterogeneous LRC-algebra $F = (\mathbb{A}, \mathbb{Q}, \rhd, \Diamond, \unrhd, \Phi)$, let us define $F^* := (\mathbb{A}^*, \mathbb{Q}, \rhd^*, \Diamond^*, \unrhd^*, \Phi^*)$, where:

1. $\mathbb{A}^*$ is the Heyting algebra obtained by adding a new top element $\top^*$ to $\mathbb{A}$ (we let $\top_{\mathbb{A}}$ denote the top element of $\mathbb{A}$). Joins and meets in $\mathbb{A}^*$ are defined as expected. The implication $\to^*$ of $\mathbb{A}^*$ maps any $(u, w) \in \mathbb{A}^* \times \mathbb{A}^*$ to $\top^*$ if $u \leq w$, to w if $u = \top^*$, and to $u \to w$ in any other case.

2. $\rhd^* : \mathbb{Q} \times \mathbb{A}^* \to \mathbb{A}^*$ maps any (α, u) to $\top^*$ if $\alpha = 0$ or $\Phi^* 1 \leq u$, and to $\alpha \rhd u$ otherwise.

3. $\Diamond^* : \mathbb{A}^* \to \mathbb{A}^*$ maps any u to $\Diamond u$ if $u \neq \top^*$, and to $\Diamond \top_{\mathbb{A}}$ if $u = \top^*$.

4. $\unrhd^* : \mathbb{Q} \times \mathbb{Q} \to \mathbb{A}^*$ maps any (α, β) to $\top^*$ if $\alpha = 0$ or $\beta = 1$, and to $\alpha \unrhd \beta$ otherwise.

5. $\Phi^* : \mathbb{Q} \to \mathbb{A}^*$ maps any α to $\Phi\alpha$.

Lemma 89. *F^* is a heterogeneous LRC-algebra.*

Proof. It can be easily verified that the maps $\rhd^*, \Diamond^*, \unrhd^*, \Phi^*$ satisfy by definition all the monotonicity (resp. antitonicity) properties that yield the validity of the rules of LRC. Let us verify that F^* validates all the axioms of LRC. By construction, $\top^*$ is join-irreducible, i.e. if $u \vee w = \top^*$ then either $u = \top^*$ or $w = \top^*$. Hence, $\Diamond^*(u \vee w) = \Diamond^* \top^* = \Diamond \top_{\mathbb{A}} = \Diamond^* u \vee \Diamond^* w$. All the remaining cases follow from the assumptions on $\Diamond$. This finishes the verification of the validity of $D1$. The validity of axioms $D2$, $D3$ and $D4$ immediately follows from their validity in F. The validity of axiom $B1$ can be shown using the identities $\alpha \sqcup 0 = \alpha$ and $0 \sqcup \beta = \beta$. The validity of $B2$ follows immediately from the definition of $\rhd^*$. As to $B3$, if $\alpha = 0$ or $\beta = 0$, the assumption that $\cdot$ preserves finite joins in each coordinate yields $\alpha \cdot \beta = 0$, and hence $\alpha \cdot \beta \rhd^* A = \top^*$, which implies that the inequality holds. The remaining cases follow from the definition of $\rhd^*$ and the assumption that $B3$ is valid in F. Axiom $B4$ is argued similarly to $B1$.

The validity of axioms $B5$ and $B7$ follows immediately from the definition of $\trianglerighteq$, and the validity of $B6$ can be shown using the identities $\alpha \sqcap 1 = \alpha = 1 \sqcap \alpha$.

As to $BD2$, if $\alpha = 0$ or $\beta = 1$ then $\alpha \rhd^* \Diamond\!\!\!|^* \beta = \top^*$, therefore the inequality holds. All the remaining cases follow from the assumption that $BD2$ is valid in F.

As to $BD1$, if $\alpha = 0$ then $\Diamond\!\!\!|^* \alpha \wedge \alpha \rhd^* u = \bot$ for any u, therefore the inequality holds. If $\Diamond\!\!\!|^* 1 \leq u$ then, by definition, $\alpha \rhd^* u = \top^*$, hence it is enough to show that $\Diamond\!\!\!|^* \alpha \leq \Diamond^* u$. We proceed by cases: (a) if $u = \top^*$, then $\Diamond\!\!\!|^* \alpha = \Diamond\!\!\!|\, \alpha \leq \top^{\mathbb{A}} = \Diamond^* u$, as required; (b) if $u \in \mathbb{A}$, then, by the assumption that $B7$, $BD2$ and MB hold in F,

$$\top^{\mathbb{A}} \leq \alpha \trianglerighteq 1 \leq \alpha \rhd \Diamond\!\!\!|\, 1 \leq \alpha \rhd u.$$

Since $BD1$ holds in F, this implies that $\Diamond\!\!\!|^* \alpha = \Diamond\!\!\!|\, \alpha \leq \Diamond u = \Diamond^* u$, as required. All the remaining cases follow from the assumption that $BD1$ is valid in F. □

For every algebraic LRC-model $\mathbb{M} = (F, v_{\mathsf{Fm}}, v_{\mathsf{Res}})$, we let $\mathbb{M}^* := (F^*, v^*_{\mathsf{Fm}}, v_{\mathsf{Res}})$, where v^*_{Fm} is defined by composing v_{Fm} with the natural injection $\mathbb{A} \hookrightarrow \mathbb{A}^*$. Henceforth, we let $[\![a]\!]$ denote the interpretation of any T-term a in $\mathbb{M}$ and $[\![a]\!]_*$ the interpretation of a in $\mathbb{M}^*$.

Lemma 90. *For every formula A,*

1. *If $[\![A]\!]_* \neq \top^*$ then $[\![A]\!]_* = [\![A]\!]$.*
2. *If $[\![A]\!]_* = \top^*$ then $[\![A]\!] = \top_{\mathbb{A}}$.*

Proof. We prove the two statements simultaneously by induction on A. The cases of constants and atomic variables are straightforward. The case of $A = B \wedge C$ immediately follows from the induction hypothesis. The case of $A = B \vee C$ immediately follows from the induction hypothesis using the join-irreducibility of $\top^*$. If $A = B \rightarrow C$, then $[\![A]\!]_* = [\![B]\!]_* \rightarrow^* [\![C]\!]_*$. By definition of $\rightarrow^*$, if $[\![A]\!]_* \neq \top^*$ then either (a) $[\![B]\!]_* \not\leq [\![C]\!]_*$ and $[\![B]\!]_* \neq \top^*$, which implies that $[\![B]\!]_* \neq \top^* \neq [\![C]\!]_*$ in which case item 1 follows by induction hypothesis; or (b) $[\![B]\!]_* \not\leq [\![C]\!]_*$ and $[\![C]\!]_* \neq \top^*$, which implies that $[\![C]\!]_* = [\![C]\!]$ by induction hypothesis. Then either (b1) $[\![B]\!]_* = \top^*$, hence by induction hypothesis $[\![B]\!] = \top^{\mathbb{A}}$ and $[\![A]\!]_* = [\![C]\!]_* = [\![C]\!] = [\![A]\!]$, as required; or (b2) $[\![B]\!]_* \neq \top^*$, hence by induction hypothesis $[\![B]\!]_* = [\![B]\!]$ and we finish the proof as in case (a). If $[\![A]\!]_* = \top^*$, then either (c) $[\![B]\!]_* = \top^* = [\![C]\!]_*$, which implies by induction hypothesis that $[\![B]\!] = \top_{\mathbb{A}} = [\![C]\!]$, which yields $[\![A]\!] = \top^{\mathbb{A}}$, as required; or (d) $[\![B]\!]_* \leq [\![C]\!]_*$, which implies $[\![B]\!] \leq [\![C]\!]$ and hence $[\![A]\!] = \top_{\mathbb{A}}$, as required.

If $A = \Diamond B$, then $[\![A]\!]_* = \Diamond^* [\![B]\!]_*$. The definition of $\Diamond^*$ implies that $[\![A]\!]_* \neq \top^*$, hence to finish the proof of this case we need to show that $[\![A]\!]_* = [\![A]\!]$. If $[\![B]\!]_* \neq \top^*$, then by induction hypothesis $[\![B]\!]_* = [\![B]\!]$, hence $[\![A]\!]_* = \Diamond^* [\![B]\!]_* = \Diamond [\![B]\!] = [\![\Diamond B]\!] = [\![A]\!]$, as required. If $[\![B]\!]_* = \top^*$, then by induction hypothesis $[\![B]\!] = \top^{\mathbb{A}}$, hence $[\![A]\!]_* = \Diamond^* [\![B]\!]_* = \Diamond^* \top^* = \Diamond \top^{\mathbb{A}} = \Diamond [\![B]\!] = [\![\Diamond B]\!] = [\![A]\!]$, as required.

If $A = \Diamond\!\!\!|\, \alpha$, item 2 is again vacuously true, and item 1 immediately follows from the definition of $\Diamond\!\!\!|^*$.

If $A = \alpha \trianglerighteq \beta$, then $[\![A]\!]_* = [\![\alpha]\!]_* \trianglerighteq^* [\![\beta]\!]_* = [\![\alpha]\!] \trianglerighteq^* [\![\beta]\!]$. Then by definition of $\trianglerighteq^*$, if $[\![A]\!]_* \neq \top^*$, then $[\![A]\!]_* = [\![A]\!]$, as required, and if $[\![A]\!]_* = \top^*$, then either $[\![\alpha]\!] = 0$ or $[\![\beta]\!] = 1$; since axioms B5 and B7 hold in F, each case yields $[\![A]\!] = \top^{\mathbb{A}}$, as required.

Finally, if $A = \alpha \rhd B$, then $[\![A]\!]_* = [\![\alpha]\!]_* \rhd^* [\![B]\!]_* = [\![\alpha]\!] \rhd^* [\![B]\!]_*$. By definition of $\rhd^*$, if $[\![A]\!]_* \neq \top^*$, then $[\![\alpha]\!] \neq 0$, $[\![A]\!]_* = [\![\alpha]\!] \rhd [\![B]\!]_*$, and $\Phi^*1 \not\leq [\![B]\!]_*$. The latter condition implies that $[\![B]\!]_* \neq \top^*$, hence, by induction hypothesis, $[\![B]\!]_* = [\![B]\!]$, and so $[\![A]\!]_* = [\![\alpha]\!] \rhd [\![B]\!] = [\![A]\!]$, as required. If $[\![A]\!]_* = \top^*$, then either (a) $[\![\alpha]\!] = 0$, which implies by B2 that $[\![A]\!] = [\![\alpha]\!] \rhd [\![B]\!] = \top^{\mathbb{A}}$, as required; or (b) $\Phi^*1 \leq [\![B]\!]_*$, which implies by induction hypothesis that $\Phi^*1 \leq [\![B]\!]$. Hence, by BD2 and monotonicity of $\rhd$,

$$\top^{\mathbb{A}} \leq [\![\alpha]\!] \unrhd 1 \leq [\![\alpha]\!] \rhd \Phi 1 \leq [\![\alpha]\!] \rhd [\![B]\!],$$

which finishes the proof that $[\![A]\!] = [\![\alpha \rhd B]\!] = \top^{\mathbb{A}}$, as required. □

The *product* $F_1 \times F_2$ of the heterogeneous LRC-algebras F_1 and F_2 is defined in the expected way, based on the product algebras $\mathbb{A}_1 \times \mathbb{A}_2$ and $\mathbb{Q}_1 \times \mathbb{Q}_2$, and defining all (i.e. both internal and external) operations component-wise. It can be readily verified that the resulting construction is a heterogeneous LRC-algebra. The product construction can be extended to algebraic LRC-models in the expected way, i.e. by pairing the valuations. Such valuations extend as usual to T-terms, and it can be proved by a straightforward induction that $[\![a]\!]_\times = ([\![a]\!]_1, [\![a]\!]_2)$.

Proposition 91. *The disjunction property holds for the logic LRC.*

Proof. If B and C are not LRC-theorems, by completeness, algebraic LRC-models $\mathbb{M}_1$ and $\mathbb{M}_2$ exist such that $[\![B]\!]_1 \neq \top_1$ and $[\![C]\!]_2 \neq \top_2$. Consider the product model $\mathbb{M} := \mathbb{M}_1 \times \mathbb{M}_2$ as described above. Notice that $[\![B]\!] \neq (\top_1, \top_2)$ and likewise for C. The model $\mathbb{M}^*$ does not satisfy $B \vee C$. Indeed, since $\top^*$ is join-irreducible, if $[\![B \vee C]\!]_* = \top^*$ then either $[\![B]\!]_* = \top^*$ or $[\![C]\!]_* = \top^*$. By Lemma 90 this implies that either $[\![B]\!] = (\top_1, \top_2)$ or $[\![C]\!] = (\top_1, \top_2)$, contradicting the assumptions. □

3.3 The calculus D.LRC

In the present section, we introduce the multi-type calculus D.LRC for the logic of resources and capabilities. As is typical of similar existing calculi, the language manipulated by this calculus is built up from structural and operational term constructors. In the tables below, each structural symbol in the upper rows corresponds to one or two logical (aka operational) symbols in the lower rows. The idea, which will be made precise in Section 3.4.1, is that each structural connective is interpreted as the corresponding logical connective on the left-hand (resp. right-hand) side (if it exists) when occurring in antecedent (resp. consequent) position.

As discussed in the previous section, the mathematical environment of heterogeneous LRC-algebras provides natural interpretations for all connectives of the basic language of LRC. In particular, on *perfect* heterogeneous LRC-algebras, these interpretations have the following extra properties: the interpretations of $\Diamond$ and Φ are completely join-preserving, that of $\rhd$ is completely join-reversing in its first coordinate and order preserving in its second coordinate, and $\unrhd$ is completely join-reversing in its first coordinate and completely meet-preserving in its second coordinate. This implies that, in each perfect heterogeneous LRC-algebra,

- $\Diamond$ and Φ have right adjoints, denoted $\blacksquare$ and $\blacksquare\!\!\blacksquare$ respectively;

- $\triangleright$ has a Galois-adjoint $\blacktriangleright$ in its first coordinate, and ⊳ has a Galois-adjoint ▶ in its first coordinate and a left adjoint ▲ in its second coordinate.

Hence, the following connectives have a natural interpretation on perfect heterogeneous LRC-algebras:

(3.3.1) $\blacksquare$: Fm $\to$ Fm

(3.3.2) ▮▮ : Fm $\to$ Res

(3.3.3) $\blacktriangleright$: Fm $\times$ Fm $\to$ Res

(3.3.4) ▶ : Fm $\times$ Res $\to$ Res

(3.3.5) ▲ : Res $\times$ Fm $\to$ Res.

- Structural and operational symbols for pure Fm-connectives:

Structural symbols	I		;		$>$		$(<)$	
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$\succ\!\!-$	$\rightarrow$	$-\!\!\prec$	$\leftarrow$

- Structural and operational symbols for pure Res-connectives:

Structural symbols	Φ		$\odot$		,		$\dot{>}$	
Operational symbols	1	0	$\cdot$		$\sqcap$	$\sqcup$		$.\backslash$

Structural symbols	$\dot{<}$		$\sqsupset$		$(\sqsubset)$	
Operational symbols		$(/.)$	$(_\sqcup\backslash)$	$(_\sqcap\backslash)$	$(/_\sqcup)$	$(/_\sqcap)$

- Structural and operational symbols for the modal operators:

Structural symbols	$\circ$		$\triangleright$		⦶		⊳	
Operational symbols	$\Diamond$			$\triangleright$	⟐			⊳

- Structural and operational symbols for the adjoints and residuals of the modal operators:

Structural symbols	$\bullet$		$\blacktriangleright$	
Operational symbols		$\blacksquare$		$\blacktriangleright$

Structural symbols	◖		▲		▶	
Operational symbols		▮▮	▲			▶

The display-type calculus **D.LRC** consists of the following display postulates, structural rules, and operational rules:

1. Identity and cut rules:

$$p \vdash p \qquad\qquad a \vdash a$$

$$\frac{(X \vdash Y)[A]^{succ} \qquad A \vdash Z}{(X \vdash Y)[Z/A]^{succ}} \qquad\qquad \frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}$$

2. Display postulates for pure Fm-connectives:

$$\frac{X \,;Y \vdash Z}{Y \vdash X > Z} \qquad \frac{Z \vdash X \,;Y}{X > Z \vdash Y} \qquad \frac{X \,;Y \vdash Z}{X \vdash Z < Y} \qquad \frac{Z \vdash X \,;Y}{Z < Y \vdash X}$$

3. Display postulates for pure Res-connectives:

$$\frac{\Gamma \,, \Delta \vdash \Sigma}{\Delta \vdash \Gamma \sqsupset \Sigma} \qquad \frac{\Gamma \,, \Delta \vdash \Sigma}{\Gamma \vdash \Sigma \sqsubset \Delta} \qquad \frac{\Gamma \vdash \Delta \,, \Sigma}{\Delta \sqsupset \Gamma \vdash \Sigma} \qquad \frac{\Gamma \vdash \Delta \,, \Sigma}{\Gamma \sqsubset \Sigma \vdash \Delta}$$

$$\frac{\Gamma \odot \Delta \vdash \Sigma}{\Delta \vdash \Gamma \gtrdot \Sigma} \qquad\qquad \frac{\Gamma \odot \Delta \vdash \Sigma}{\Gamma \vdash \Sigma \lessdot \Delta}$$

4. Display postulates for the modal operators:

$$\frac{\circ X \vdash Y}{X \vdash \bullet Y} \qquad \frac{\obar \Gamma \vdash X}{\Gamma \vdash \LEFTcircle X} \qquad \frac{X \vdash \Gamma \rhd Y}{\Gamma \vdash X \blacktriangleright Y} \qquad \frac{X \vdash \Gamma \unrhd \Delta}{\Gamma \blacktriangle X \vdash \Delta} \qquad \frac{X \vdash \Gamma \unrhd \Delta}{\Gamma \vdash X \blacktriangleright \Delta}$$

5. Pure Fm-type structural rules:

$$\mathrm{I}_L \frac{X \vdash Y}{\mathrm{I} ; X \vdash Y} \qquad \frac{Y \vdash X}{Y \vdash X \,; \mathrm{I}} \mathrm{I}_R \qquad\qquad E_L \frac{Y \,; X \vdash Z}{X \,; Y \vdash Z} \qquad \frac{Z \vdash X \,; Y}{Z \vdash Y \,; X} E_R$$

$$W_L \frac{Y \vdash Z}{X \,; Y \vdash Z} \qquad \frac{Z \vdash Y}{Z \vdash Y \,; X} W_R \qquad\qquad C_L \frac{X \,; X \vdash Y}{X \vdash Y} \qquad \frac{Y \vdash X \,; X}{Y \vdash X} C_R$$

$$A_L \frac{X \,; (Y \,; Z) \vdash W}{(X \,; Y) \,; Z \vdash W} \qquad\qquad \frac{W \vdash (Z \,; Y) \,; X}{W \vdash Z \,; (Y \,; X)} A_R$$

6. Pure Res-type structural rules:

$$\begin{matrix}\Phi_{L1} \\ \Phi_{L2}\end{matrix} \frac{\Gamma \odot \Phi \vdash \Delta}{\frac{\Gamma \vdash \Delta}{\Phi \odot \Gamma \vdash \Delta}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \,, \Phi} \Phi_R \qquad A_L \frac{\Gamma \odot (\Delta \odot \Sigma) \vdash \Pi}{(\Gamma \odot \Delta) \odot \Sigma \vdash \Pi} \qquad W_\Phi \frac{\Phi \vdash \Delta}{\Gamma \vdash \Delta}$$

$$W_L \frac{\Gamma \vdash \Delta}{\Gamma \,, \Sigma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta \,, \Sigma} W_R \qquad C_L \frac{\Gamma \,, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta \,, \Delta}{\Gamma \vdash \Delta} C_R$$

$$E_L \frac{\Gamma \,, \Delta \vdash \Sigma}{\Delta \,, \Gamma \vdash \Sigma} \qquad \frac{\Sigma \vdash \Gamma \,, \Delta}{\Sigma \vdash \Delta \,, \Gamma} E_R \qquad A_L \frac{\Sigma \,, (\Delta \,, \Gamma) \vdash \Pi}{(\Sigma \,, \Delta) \,, \Gamma \vdash \Pi} \qquad \frac{\Pi \vdash \Sigma \,, (\Delta \,, \Gamma)}{\Pi \vdash (\Sigma \,, \Delta) \,, \Gamma} A_R$$

$$\frac{\Pi \vdash (\Gamma \gtrdot \Delta) \,, (\Gamma \gtrdot \Sigma)}{\Pi \vdash \Gamma \gtrdot (\Delta \,, \Sigma)} \, dis$$

7. Structural rules corresponding to the D-axioms:

$$\frac{X \vdash \bullet Y \,;\, \bullet Z}{X \vdash \bullet (Y \,;\, Z)}\,\mathsf{D1} \qquad \frac{\Gamma \vdash \LEFTcircle X \,,\, \LEFTcircle Y}{\Gamma \vdash \LEFTcircle (X \,;\, Y)}\,\mathsf{D3} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bullet \mathrm{I}}\,\mathsf{D2} \qquad \frac{\Gamma \vdash \Phi}{\Gamma \vdash \LEFTcircle \mathrm{I}}\,\mathsf{D4}$$

8. Structural rules corresponding to the B-axioms:

$$\frac{\Gamma \vdash (Y \RHD \Delta) \,,\, (Z \RHD \Delta)}{\Gamma \vdash (Y \,;\, Z) \RHD \Delta}\,\mathsf{B4} \qquad \frac{\Gamma \vdash (Y \blacktriangleright W) \,,\, (Z \blacktriangleright W)}{\Gamma \vdash (Y \,;\, Z) \blacktriangleright W}\,\mathsf{B1}$$

$$\mathsf{B6}\,\frac{(\Gamma \blacktriangle X) \,,\, (\Gamma \blacktriangle Y) \vdash \Delta}{\Gamma \blacktriangle (X \,;\, Y) \vdash \Delta} \qquad \frac{X \vdash \Gamma \rhd (\Delta \rhd Y)}{X \vdash \Gamma \odot \Delta \rhd Y}\,\mathsf{B3} \qquad \mathsf{B7}\,\frac{\Phi \vdash \Delta}{\Gamma \blacktriangle \mathrm{I} \vdash \Delta}$$

9. Structural rules corresponding to the BD-axioms:

$$\frac{X \vdash \Gamma \rhd \bullet Y}{X \vdash \LEFTcircle \Gamma > Y}\,\mathsf{BD1} \qquad \frac{X \vdash \Gamma \RHD \LEFTcircle Y}{X \vdash \Gamma \rhd Y}\,\mathsf{BD2}$$

10. Introduction rules for pure Fm-connectives (in the presence of the exchange rules E_L and E_R, the structural connective $<$ and the corresponding operational connectives $\prec$ and $\leftarrow$ are redundant and they are omitted):

$$\bot_L\,\frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot}\,\bot_R$$

$$\top_L\,\frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top}\,\top_R$$

$$\wedge_L\,\frac{A \,;\, B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \,;\, Y \vdash A \wedge B}\,\wedge_R$$

$$\vee_L\,\frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \,;\, Y} \qquad \frac{X \vdash A \,;\, B}{X \vdash A \vee B}\,\vee_R$$

$$\rightarrow_L\,\frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B}\,\rightarrow_R$$

11. Introduction rules for pure Res-connectives:

$$0_L\,\frac{}{0 \vdash \Phi} \qquad \frac{\Gamma \vdash \Phi}{\Gamma \vdash 0}\,0_R$$

$$1_L\,\frac{\Phi \vdash \Gamma}{1 \vdash \Gamma} \qquad \frac{}{\Phi \vdash 1}\,1_R$$

$$\cdot_L\,\frac{\alpha \odot \beta \vdash \Gamma}{\alpha \cdot \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \odot \Delta \vdash \alpha \cdot \beta}\,\cdot_R$$

$$\sqcup_L\,\frac{\alpha \vdash \Gamma \qquad \beta \vdash \Delta}{\alpha \sqcup \beta \vdash \Gamma \,,\, \Delta} \qquad \frac{\Gamma \vdash \alpha \,,\, \beta}{\Gamma \vdash \alpha \sqcup \beta}\,\sqcup_R$$

12. Introduction rules for the modal operators:

$$\Diamond_L \frac{\circ A \vdash X}{\Diamond A \vdash X} \qquad \frac{X \vdash A}{\circ X \vdash \Diamond A} \Diamond_R \qquad \frac{\Gamma \vdash \alpha \quad A \vdash X}{\alpha \rhd A \vdash \Gamma \rhd X} \rhd_L \qquad \frac{X \vdash \alpha \rhd A}{X \vdash \alpha \rhd A} \rhd_R$$

$$\Diamond\!\!\!\mid_L \frac{\Phi\alpha \vdash X}{\Diamond\!\!\!\mid \alpha \vdash X} \qquad \frac{\Gamma \vdash \alpha}{\Phi\Gamma \vdash \Diamond\!\!\!\mid \alpha} \Diamond\!\!\!\mid_R \qquad \frac{\Gamma \vdash \alpha \quad \beta \vdash \Delta}{\alpha \rhd\!\!\!\mid \alpha \vdash \Gamma \rhd\!\!\!\!\mid \Delta} \rhd\!\!\!\mid_L \qquad \frac{\Gamma \vdash \alpha \rhd\!\!\!\!\mid \alpha}{\Gamma \vdash \alpha \rhd\!\!\!\mid \alpha} \rhd\!\!\!\mid_R$$

We conclude the present section by listing some observations about D.LRC. Firstly, notice that, although very similar in spirit to a display calculus [3, 73], D.LRC does not enjoy the display property, the reason being that a display rule for displaying substructures in the scope of the second coordinate of $\rhd$ occurring in consequent position would not be sound. This is the reason why a more general form of cut rule, sometimes referred to as *surgical cut*, has been included than the standard one in display calculi where both cut formulas occur in display. However, as discussed in [34], calculi without display property can still verify the assumptions of some Belnap-style cut elimination metatheorem. In Section 3.4.3, we will verify that this is the case of D.LRC. Secondly, as usual, the version of D.LRC on a classical propositional base can be obtained by adding e.g. the following *Grishin rules*:

$$\frac{X > (Y\,; Z) \vdash W}{(X > Y)\,; Z \vdash W} \qquad \frac{X \vdash Y > (Z\,; W)}{X \vdash (Y > Z)\,; W}$$

Thirdly, the rule W_Φ encodes (and is used to derive) $\alpha \cdot \beta \vdash \alpha$, $\alpha \cdot \beta \vdash \beta$, $\alpha \vdash 1$, B2 and B5.

3.4 Basic properties of D.LRC

In the present section, we verify that the calculus D.LRC is sound w.r.t. the semantics of perfect heterogeneous LRC-algebras (cf. Definition 83), is syntactically complete w.r.t. the Hilbert calculus for LRC introduced in Section 3.2.1, enjoys cut elimination and subformula property, and conservatively extends the Hilbert calculus of Section 3.2.1.

3.4.1 Soundness

In the present subsection, we outline the verification of the soundness of the rules of D.LRC w.r.t. the semantics of perfect heterogeneous LRC-algebras (cf. Definition 83). The first step consists in interpreting structural symbols as logical symbols according to their (precedent or consequent) position,[12] as indicated in the synoptic tables at the

[12] For any (formula or resource) sequent $x \vdash y$ in the language of D.LRC, we define the signed generation trees $+x$ and $-y$ by labelling the root of the generation tree of x (resp. y) with the sign $+$ (resp. $-$), and then propagating the sign to all nodes according to the polarity of the coordinate of the connective assigned to each node. Positive (resp. negative) coordinates propagate the same (resp. opposite) sign to the corresponding child node. The only negative coordinates are the first coordinates of $>$, $\rhd$ and $\rhd\!\!\!\!\mid$. Then, a substructure z in $x \vdash y$ is in *precedent* (resp. *consequent*) *position* if the sign of its root node as a subtree of $+x$ or $-y$ is $+$ (resp. $-$).

beginning of Section 3.3. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\frac{X \vdash \Gamma \rhd \bullet Y}{X \vdash \phi\Gamma > Y}\ \text{BD1} \qquad \rightsquigarrow \qquad \forall\gamma\forall x\forall y[x \leq \gamma \rhd \blacksquare\, y \Rightarrow x \leq \Phi\,\gamma \rightarrow y]$$

$$\frac{X \vdash \Gamma \unrhd \LEFTcircle Y}{X \vdash \Gamma \rhd Y}\ \text{BD2} \qquad \rightsquigarrow \qquad \forall x\forall\gamma\forall y[x \leq \gamma \unrhd \blacksquare\!\blacksquare\, y \Rightarrow x \leq \gamma \rhd y].$$

3

The verification that the rules of D.LRC are sound on perfect LRC-algebras then consists in verifying the validity of their corresponding quasi-inequalities in perfect LRC-algebras. The validity of these quasi-inequalities follows straightforwardly from two observations. The first observation is that the quasi-inequality corresponding to each rule is obtained by running the algorithm ALBA on the axiom of the Hilbert-style presentation of Section 3.2.1 bearing the same name as the rule. Below we perform the ALBA reduction on the axiom BD1:

$$\begin{array}{ll}
 & \forall\alpha\forall p[\Phi\,\alpha \wedge \alpha \rhd p \leq \Diamond p] \\
\text{iff} & \forall\alpha\forall p\forall\gamma\forall x\forall y[(\gamma \leq \alpha \ \&\ x \leq \alpha \rhd p \ \&\ \Diamond p \leq y) \Rightarrow \Phi\,\gamma \wedge x \leq y] \\
\text{iff} & \forall\alpha\forall p\forall\gamma\forall x\forall y[(\gamma \leq \alpha \ \&\ x \leq \alpha \rhd p \ \&\ p \leq \blacksquare\, y) \Rightarrow \Phi\,\gamma \wedge x \leq y] \\
\text{iff} & \forall\gamma\forall x\forall y[x \leq \gamma \rhd \blacksquare\, y \Rightarrow \Phi\,\gamma \wedge x \leq y] \\
\text{iff} & \forall\gamma\forall x\forall y[x \leq \gamma \rhd \blacksquare\, y \Rightarrow x \leq \Phi\,\gamma \rightarrow y].
\end{array}$$

It can be readily checked that the ALBA manipulation rules applied in the computation above (adjunction rules and Ackermann rules) are sound on perfect LRC-algebras. As discussed in [45], the soundness of these rules only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference. A more substantial difference with the setting of [45] might be in principle the fact that the connective $\rhd$ is only monotone—rather than normal—in its second coordinate. However, notice that each manipulation in the chain of equivalences above involving that coordinate is an application of the Ackermann rule of ALBA, which relies on no more than monotonicity. The second observation is that the axioms of the Hilbert-style presentation of Section 3.2.1 are valid by definition on perfect LRC-algebras. We conclude the present subsection reporting the ALBA-reduction of (the condition expressing the validity of) axiom BD2.

$$\begin{array}{ll}
 & \forall\alpha\forall\beta[\alpha \unrhd \beta \leq \alpha \rhd \Phi\,\beta] \\
\text{iff} & \forall\alpha\forall\beta\forall x\forall\gamma\forall y[(x \leq \alpha \unrhd \beta \ \&\ \gamma \leq \alpha \ \&\ \Phi\,\beta \leq y) \Rightarrow x \leq \gamma \rhd y] \\
\text{iff} & \forall\alpha\forall\beta\forall x\forall\gamma\forall y[(x \leq \alpha \unrhd \beta \ \&\ \gamma \leq \alpha \ \&\ \beta \leq \blacksquare\!\blacksquare\, y) \Rightarrow x \leq \gamma \rhd y] \\
\text{iff} & \forall x\forall\gamma\forall y[x \leq \gamma \unrhd \blacksquare\!\blacksquare\, y \Rightarrow x \leq \gamma \rhd y].
\end{array}$$

3.4.2 Completeness

In the present subsection, we show that the axioms of the Hilbert-style calculus H.LRC introduced in Section 3.2.1 are derivable sequents of D.LRC, and that the rules of

H.LRC are derivable rules of D.LRC. Since H.LRC is complete w.r.t. the semantics of perfect heterogeneous LRC-algebras (cf. Definition 83), we obtain as a corollary that D.LRC is also complete w.r.t. the semantics of perfect heterogeneous LRC-algebras. The derivations of the axioms R1-R3 of H.LRC are standard and we omit them.

R4. $\alpha \cdot (\beta \sqcup \gamma) \leftrightarrow (\alpha \cdot \beta) \sqcup (\alpha \cdot \gamma)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha \odot \beta \vdash \alpha \cdot \beta}}{\beta \vdash \alpha \gtrdot \alpha \cdot \beta} \qquad \dfrac{\dfrac{\alpha \vdash \alpha \qquad \gamma \vdash \gamma}{\alpha \odot \gamma \vdash \alpha \cdot \gamma}}{\gamma \vdash \alpha \gtrdot \alpha \cdot \gamma}}{\beta \sqcup \gamma \vdash (\alpha \gtrdot \alpha \cdot \beta)\,,\,(\alpha \gtrdot \alpha \cdot \gamma)}\,\text{dis}}{\beta \sqcup \gamma \vdash \alpha \gtrdot (\alpha \cdot \beta\,,\,\alpha \cdot \gamma)}}{\alpha \odot (\beta \sqcup \gamma) \vdash \alpha \cdot \beta\,,\,\alpha \cdot \gamma}}{\alpha \cdot (\beta \sqcup \gamma) \vdash \alpha \cdot \beta\,,\,\alpha \cdot \gamma}}{\alpha \cdot (\beta \sqcup \gamma) \vdash (\alpha \cdot \beta) \sqcup (\alpha \cdot \gamma)}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \beta\,,\,\gamma}}{\beta \vdash \beta \sqcup \gamma}}{\alpha \odot \beta \vdash \alpha \cdot (\beta \sqcup \gamma)}}{\alpha \cdot \beta \vdash \alpha \cdot (\beta \sqcup \gamma)} \qquad \dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{\gamma \vdash \gamma}{\gamma \vdash \beta\,,\,\gamma}}{\gamma \vdash \beta \sqcup \gamma}}{\alpha \odot \gamma \vdash \alpha \cdot (\beta \sqcup \gamma)}}{\alpha \cdot \gamma \vdash \alpha \cdot (\beta \sqcup \gamma)}}{(\alpha \cdot \beta) \sqcup (\alpha \cdot \gamma) \vdash \alpha \cdot (\beta \sqcup \gamma)\,,\,\alpha \cdot (\beta \sqcup \gamma)}}{(\alpha \cdot \beta) \sqcup (\alpha \cdot \gamma) \vdash \alpha \cdot (\beta \sqcup \gamma)}
$$

The proof of $(\beta \sqcup \gamma) \cdot \alpha \leftrightarrow (\beta \cdot \alpha) \sqcup (\gamma \cdot \alpha)$ is analogous and we omit it.

D1. $\Diamond(A \vee B) \leftrightarrow \Diamond A \vee \Diamond B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\circ A \vdash \Diamond A}}{A \vdash \bullet\Diamond A} \qquad \dfrac{\dfrac{B \vdash B}{\circ B \vdash \Diamond B}}{B \vdash \bullet\Diamond B}}{A \vee B \vdash \bullet\Diamond A \,;\, \bullet\Diamond B}\,\text{D1}}{A \vee B \vdash \bullet(\Diamond A \,;\, \Diamond B)}}{\circ A \vee B \vdash \Diamond A \,;\, \Diamond B}}{\Diamond(A \vee B) \vdash \Diamond A \,;\, \Diamond B}}{\Diamond(A \vee B) \vdash \Diamond A \vee \Diamond B}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A \,;\, B}}{A \vdash A \vee B}}{\circ A \vdash \Diamond(A \vee B)}}{\Diamond A \vdash \Diamond(A \vee B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \vdash B}{B \vdash A \,;\, B}}{B \vdash A \vee B}}{\circ B \vdash \Diamond(A \vee B)}}{\Diamond B \vdash \Diamond(A \vee B)}}{\Diamond A \vee \Diamond B \vdash \Diamond(A \vee B) \,;\, \Diamond(A \vee B)}}{\Diamond A \vee \Diamond B \vdash \Diamond(A \vee B)}
$$

3

D3. $\Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta) \leftrightarrow \Diamond\!\!\!\!\mid\,\alpha \vee \Diamond\!\!\!\!\mid\,\beta$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\circ\!\!\!\mid\,\alpha \vdash \Diamond\!\!\!\!\mid\,\alpha}}{\alpha \vdash \bullet\!\!\!\mid\,\Diamond\!\!\!\!\mid\,\alpha} \qquad \dfrac{\dfrac{\beta \vdash \beta}{\circ\!\!\!\mid\,\beta \vdash \Diamond\!\!\!\!\mid\,\beta}}{\beta \vdash \bullet\!\!\!\mid\,\Diamond\!\!\!\!\mid\,\beta}}{\alpha \sqcup \beta \vdash \bullet\!\!\!\mid\,\Diamond\!\!\!\!\mid\,\alpha\,,\ \bullet\!\!\!\mid\,\Diamond\!\!\!\!\mid\,\beta}}{\alpha \sqcup \beta \vdash \bullet\!\!\!\mid\,(\Diamond\!\!\!\!\mid\,\alpha\,;\Diamond\!\!\!\!\mid\,\beta)}\ \text{D3}}{\circ\!\!\!\mid\,\alpha \sqcup \beta \vdash \Diamond\!\!\!\!\mid\,\alpha\,;\Diamond\!\!\!\!\mid\,\beta}}{\Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta) \vdash \Diamond\!\!\!\!\mid\,\alpha\,;\Diamond\!\!\!\!\mid\,\beta}}{\Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta) \vdash \Diamond\!\!\!\!\mid\,\alpha \vee \Diamond\!\!\!\!\mid\,\beta}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \alpha\,,\ \beta}}{\alpha \vdash \alpha \sqcup \beta}}{\circ\!\!\!\mid\,\alpha \vdash \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)}}{\Diamond\!\!\!\!\mid\,\alpha \vdash \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \alpha\,,\ \beta}}{\beta \vdash \alpha \sqcup \beta}}{\circ\!\!\!\mid\,\beta \vdash \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)}}{\Diamond\!\!\!\!\mid\,\beta \vdash \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)}}{\Diamond\!\!\!\!\mid\,\alpha \sqcup \Diamond\!\!\!\!\mid\,\beta \vdash \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)\,,\ \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)}}{\Diamond\!\!\!\!\mid\,\alpha \sqcup \Diamond\!\!\!\!\mid\,\beta \vdash \Diamond\!\!\!\!\mid\,(\alpha \sqcup \beta)}$$

D2. $\Diamond\bot \leftrightarrow \bot$

$$\dfrac{\dfrac{\dfrac{\dfrac{\bot \vdash \mathrm{I}}{\bot \vdash \bullet \mathrm{I}}\ \text{D2}}{\circ\bot \vdash \mathrm{I}}}{\circ\bot \vdash \bot}}{\Diamond\bot \vdash \bot} \qquad \dfrac{\dfrac{\bot \vdash \mathrm{I}}{\bot \vdash \Diamond\bot\,;\mathrm{I}}}{\bot \vdash \Diamond\bot}$$

D4. $\Diamond\!\!\!\!\mid\,0 \leftrightarrow \bot$

$$\dfrac{\dfrac{\dfrac{\dfrac{0 \vdash \Phi}{0 \vdash \bullet\!\!\!\mid\,\mathrm{I}}\ \text{D4}}{\circ\!\!\!\mid\,0 \vdash \mathrm{I}}}{\circ\!\!\!\mid\,0 \vdash \bot}}{\Diamond\!\!\!\!\mid\,0 \vdash \bot} \qquad \dfrac{\dfrac{0 \vdash \Phi}{0 \vdash \Diamond\!\!\!\!\mid\,0\,,\ \Phi}}{0 \vdash \Diamond\!\!\!\!\mid\,0}$$

B1. $\alpha \sqcup \beta \rhd A \leftrightarrow (\alpha \rhd A) \wedge (\beta \rhd A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \alpha\,,\ \beta}}{\alpha \vdash \alpha \sqcup \beta} \qquad A \vdash A}{\alpha \sqcup \beta \rhd A \vdash \alpha \vartriangleright A}}{\alpha \sqcup \beta \rhd A \vdash \alpha \rhd A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \alpha\,,\ \beta}}{\beta \vdash \alpha \sqcup \beta} \qquad A \vdash A}{\alpha \sqcup \beta \rhd A \vdash \beta \vartriangleright A}}{\alpha \sqcup \beta \rhd A \vdash \beta \rhd A}}{\alpha \sqcup \beta \rhd A\,; \alpha \sqcup \beta \rhd A \vdash (\alpha \rhd A) \wedge (\beta \rhd A)}}{\alpha \sqcup \beta \rhd A \vdash (\alpha \rhd A) \wedge (\beta \rhd A)}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad A \vdash A}{\alpha \rhd A \vdash \alpha \vartriangleright A}}{\alpha \vdash \alpha \rhd A \blacktriangleright A} \qquad \dfrac{\dfrac{\beta \vdash \beta \qquad A \vdash A}{\beta \rhd A \vdash \beta \vartriangleright A}}{\beta \vdash \beta \rhd A \blacktriangleright A}}{\alpha \sqcup \beta \vdash (\alpha \rhd A \blacktriangleright A) \, , \, (\beta \rhd A \blacktriangleright A)}\,\mathrm{B1}}{\alpha \sqcup \beta \vdash (\alpha \rhd A \, ; \, \beta \rhd A) \blacktriangleright A}}{\alpha \rhd A \, ; \, \beta \rhd A \vdash \alpha \sqcup \beta \vartriangleright A}}{\alpha \rhd A \, ; \, \beta \rhd A \vdash \alpha \sqcup \beta \rhd A}}{(\alpha \rhd A) \wedge (\beta \rhd A) \vdash \alpha \sqcup \beta \rhd A}
$$

B4. $\alpha \sqcup \beta \unrhd \gamma \leftrightarrow (\alpha \unrhd \gamma) \wedge (\beta \unrhd \gamma)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \alpha \, , \, \beta}}{\alpha \vdash \alpha \sqcup \beta} \qquad \gamma \vdash \gamma}{\alpha \sqcup \beta \unrhd \gamma \vdash \alpha \trianglerighteq \gamma}}{\alpha \sqcup \beta \unrhd \gamma \vdash \alpha \unrhd \gamma} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\beta \vdash \alpha \, , \, \beta}}{\beta \vdash \alpha \sqcup \beta} \qquad \gamma \vdash \gamma}{\alpha \sqcup \beta \unrhd \gamma \vdash \beta \trianglerighteq \gamma}}{\alpha \sqcup \beta \unrhd \gamma \vdash \beta \unrhd \gamma}}{\alpha \sqcup \beta \unrhd \gamma \, ; \, \alpha \sqcup \beta \unrhd \gamma \vdash (\alpha \unrhd \gamma) \wedge (\beta \unrhd \gamma)}}{\alpha \sqcup \beta \unrhd \gamma \vdash (\alpha \unrhd \gamma) \wedge (\beta \unrhd \gamma)}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \gamma \vdash \gamma}{\alpha \unrhd \gamma \vdash \alpha \trianglerighteq \gamma}}{\alpha \vdash \alpha \unrhd \gamma \blacktriangleright \gamma} \qquad \dfrac{\dfrac{\beta \vdash \beta \qquad \gamma \vdash \gamma}{\beta \unrhd \gamma \vdash \beta \trianglerighteq \gamma}}{\beta \vdash \beta \unrhd \gamma \blacktriangleright \gamma}}{\alpha \sqcup \beta \vdash (\alpha \unrhd \gamma \blacktriangleright \gamma) \, , \, (\beta \unrhd \gamma \blacktriangleright \gamma)}\,\mathrm{B4}}{\alpha \sqcup \beta \vdash (\alpha \unrhd \gamma \, ; \, \beta \unrhd \gamma) \blacktriangleright \gamma}}{\alpha \unrhd \gamma \, ; \, \beta \unrhd \gamma \vdash \alpha \sqcup \beta \trianglerighteq \gamma}}{\alpha \unrhd \gamma \, ; \, \beta \unrhd \gamma \vdash \alpha \sqcup \beta \unrhd \gamma}}{(\alpha \unrhd \gamma) \wedge (\beta \unrhd \gamma) \vdash \alpha \sqcup \beta \unrhd \gamma}
$$

B2. $0 \rhd A$

$$
\dfrac{\dfrac{\dfrac{\dfrac{0 \vdash \Phi}{0 \vdash \mathrm{I} \blacktriangleright A \, , \, \Phi}}{0 \vdash \mathrm{I} \blacktriangleright A}}{\mathrm{I} \vdash 0 \vartriangleright A}}{\mathrm{I} \vdash 0 \rhd A}
$$

B5. $0 \unrhd \alpha$

$$
\dfrac{\dfrac{\dfrac{\dfrac{0 \vdash \Phi}{0 \vdash \mathrm{I} \blacktriangleright \alpha \, , \, \Phi}}{0 \vdash \mathrm{I} \blacktriangleright \alpha}}{\mathrm{I} \vdash 0 \trianglerighteq \alpha}}{\mathrm{I} \vdash 0 \unrhd \alpha}
$$

B3. $\alpha \rhd (\beta \rhd A) \to (\alpha \cdot \beta \rhd A)$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\beta \vdash \beta \qquad A \vdash A}{\beta \rhd A \vdash \beta \mathbin{\large\rhd} A}}{\alpha \rhd (\beta \rhd A) \vdash \alpha \mathbin{\large\rhd} (\beta \mathbin{\large\rhd} A)}}{\alpha \rhd (\beta \rhd A) \vdash (\alpha \odot \beta) \mathbin{\large\rhd} A}\,\text{B3}}{\alpha \odot \beta \vdash (\alpha \rhd (\beta \rhd A)) \blacktriangleright A}}{\alpha \cdot \beta \vdash (\alpha \rhd (\beta \rhd A)) \blacktriangleright A}}{\alpha \rhd (\beta \rhd A) \vdash (\alpha \cdot \beta) \mathbin{\large\rhd} A}}{\alpha \rhd (\beta \rhd A) \vdash (\alpha \cdot \beta) \rhd A}
$$

B6. $\alpha \unrhd (\beta \sqcap \gamma) \leftrightarrow \alpha \unrhd \beta \wedge \alpha \unrhd \gamma$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{\beta \vdash \beta}{\beta\,,\gamma \vdash \beta}}{\beta \sqcap \gamma \vdash \beta}}{\alpha \unrhd \beta \sqcap \gamma \vdash \alpha \mathbin{\large\unrhd} \beta}}{\alpha \unrhd \beta \sqcap \gamma \vdash \alpha \unrhd \beta} \qquad \dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{\dfrac{\gamma \vdash \gamma}{\gamma\,,\beta \vdash \gamma}}{\beta\,,\gamma \vdash \gamma}}{\beta \sqcap \gamma \vdash \gamma}}{\alpha \unrhd \beta \sqcap \gamma \vdash \alpha \mathbin{\large\unrhd} \gamma}}{\alpha \unrhd \beta \sqcap \gamma \vdash \alpha \unrhd \gamma}}{\alpha \unrhd \beta \sqcap \gamma\,; \alpha \unrhd \beta \sqcap \gamma \vdash (\alpha \unrhd \beta) \wedge (\alpha \unrhd \gamma)}}{\alpha \unrhd \beta \sqcap \gamma \vdash (\alpha \unrhd \beta) \wedge (\alpha \unrhd \gamma)}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha \unrhd \beta \vdash \alpha \mathbin{\large\unrhd} \beta}}{\alpha \blacktriangle \alpha \unrhd \beta \vdash \beta} \qquad \dfrac{\dfrac{\alpha \vdash \alpha \qquad \gamma \vdash \gamma}{\alpha \unrhd \gamma \vdash \alpha \mathbin{\large\unrhd} \gamma}}{\alpha \blacktriangle \alpha \unrhd \gamma \vdash \gamma}}{(\alpha \blacktriangle \alpha \unrhd \beta)\,, (\alpha \blacktriangle \alpha \unrhd \gamma) \vdash \beta \sqcap \gamma}}{\alpha \blacktriangle (\alpha \unrhd \beta\,; \alpha \unrhd \gamma) \vdash \beta \sqcap \gamma}\,\text{B6}}{\alpha \unrhd \beta\,; \alpha \unrhd \gamma \vdash \alpha \mathbin{\large\unrhd} \beta \sqcap \gamma}}{\alpha \unrhd \beta\,; \alpha \unrhd \gamma \vdash \alpha \unrhd \beta \sqcap \gamma}}{(\alpha \unrhd \beta) \wedge (\alpha \unrhd \gamma) \vdash \alpha \unrhd \beta \sqcap \gamma}
$$

B7. $\alpha \unrhd 1$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Phi \vdash 1}{\alpha \blacktriangle \mathrm{I}\,, \Phi \vdash 1}}{\alpha \blacktriangle \mathrm{I} \vdash 1}}{\mathrm{I} \vdash \alpha \mathbin{\large\unrhd} 1}}{\mathrm{I} \vdash \alpha \unrhd 1}
$$

BD1. $\Diamond\!\!\!\mid\, \alpha \wedge \alpha \rhd A \rightarrow \Diamond A$

$$
\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{A \vdash A}{\circ A \vdash \Diamond A}}{A \vdash \bullet\Diamond A}}{\dfrac{\alpha \triangleright A \vdash \alpha \rhd \bullet\Diamond A}{\alpha \triangleright A \vdash \obar\alpha > \Diamond A}\;\text{BD1}}}{\obar\alpha \,;\, \alpha \triangleright A \vdash \Diamond A}}{\dfrac{\dfrac{\dfrac{\obar\alpha \vdash \Diamond A < \alpha \triangleright A}{\Diamond\!\!\mid\alpha \vdash \Diamond A < \alpha \triangleright A}}{\Diamond\!\!\mid\alpha \,;\, \alpha \triangleright A \vdash \Diamond A}}{\Diamond\!\!\mid\alpha \wedge \alpha \triangleright A \vdash \Diamond A}}
$$

BD2. $\alpha \unrhd \beta \rightarrow \alpha \triangleright \Diamond\!\!\mid\beta$

$$
\dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{\beta \vdash \beta}{\obar\beta \vdash \Diamond\!\!\mid\beta}}{\beta \vdash \varobar\Diamond\!\!\mid\beta}}{\dfrac{\alpha \unrhd \beta \vdash \alpha \rhd \varobar\Diamond\!\!\mid\beta}{\alpha \unrhd \beta \vdash \alpha \rhd \Diamond\!\!\mid\beta}\;\text{BD2}}}{\alpha \unrhd \beta \vdash \alpha \triangleright \Diamond\!\!\mid\beta}
$$

The rules of H.LRC immediately follow from applications of the introduction rules of the corresponding logical connectives in the usual way and we omit their derivations.

3.4.3 Cut elimination and subformula property

In the present subsection, we sketch the verification that the D.LRC is a proper multi-type calculus (cf. Section 3.7). By Theorem 94, this is enough to establish that the calculus enjoys cut elimination and subformula property. With the exception of C_8', all conditions are straightforwardly verified by inspecting the rules, and this verification is left to the reader.

As to the verification of condition C_8', the only interesting case is the one in which the cut formula is of the form $\alpha \triangleright A$, since the connective $\triangleright$ is monotone rather than normal in its second coordinate, which is the reason why not even a weak form of display property holds for D.LRC. This case is treated below. Notice that, since all principal formulas are in display, no surgical cuts need to be eliminated in the principal stage.

$$
\dfrac{\dfrac{\dfrac{\vdots\;\pi_1}{X \vdash \alpha \rhd A}}{X \vdash \alpha \triangleright A} \qquad \dfrac{\dfrac{\vdots\;\pi_2}{\Gamma \vdash \alpha} \quad \dfrac{\vdots\;\pi_3}{A \vdash Y}}{\alpha \triangleright A \vdash \Gamma \rhd Y}}{X \vdash \Gamma \rhd Y}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\vdots\;\pi_2}{\Gamma \vdash \alpha} \quad \dfrac{\dfrac{\vdots\;\pi_1}{X \vdash \alpha \rhd A}}{\alpha \vdash X \rhd A}}{\Gamma \vdash X \rhd A}}{X \vdash \Gamma \rhd A} \qquad \dfrac{\vdots\;\pi_3}{A \vdash Y}}{X \vdash \Gamma \rhd Y}
$$

3.4.4 Semantic conservativity

To argue that the calculus D.LRC adequately captures LRC, we follow the standard proof strategy discussed in [45]. Recall that $\vdash_{\mathrm{LRC}}$ denotes the syntactic consequence relation arising from the Hilbert system for LRC introduced in Section 3.2.1. We need to show that, for all LRC-formulas A and B, if $A \vdash B$ is a provable sequent in the calculus D.LRC, then $A \vdash_{\mathrm{LRC}} B$. This fact can be verified using the following standard argument and facts: (a) the rules of D.LRC are sound w.r.t. perfect heterogeneous LRC-algebras (cf. Section 3.4.1), and (b) LRC is strongly complete w.r.t. perfect heterogeneous LRC-algebras (cf. Corollary 88). Then, let A, B be LRC-formulas such that $A \vdash B$ is a derivable sequent in D.LRC. By (a), this implies that $A \models_{\mathrm{LRC}} B$, which implies, by (b), that $A \vdash_{\mathrm{LRC}} B$, as required.

3.5 Case studies

In this section, we present a number of case studies, with the purpose of highlighting various aspects of the basic framework and also various ways in which it can be adapted to different settings. The most common adaptations performed in the case studies below consist in adding analytic structural rules to the basic calculus. Interestingly, the resulting calculi still enjoy the same package of basic properties (soundness, completeness, cut elimination, subformula property, conservativity) which hold of D.LRC as an immediate consequence of general results. Indeed, it can be readily verified that the axioms corresponding to each of the rules introduced below are analytic inductive (cf. [45, Definition 55]), and hence are canonical (cf. [45, Theorem 19]). Therefore, the axiomatic extensions of LRC corresponding to these axioms is *sound* and *complete* w.r.t. the corresponding subclass of LRC-models. *Conservativity* can be argued by repeating verbatim the same argument given in Section 4.4 which uses the soundness of the augmented calculus w.r.t. the corresponding class of perfect LRC-models, and the completeness of the Hilbert-style presentation of the axiomatic extension which holds because the additional axioms are canonical. Finally, *cut elimination* and *subformula property* follow from the general cut elimination metatheorem.

In what follows, we will sometimes abuse terminology and speak of a formula A being derived from certain assumptions $A_1; \ldots; A_n$ meaning that the sequent $A_1; \ldots; A_n \vdash A$ is derivable in the calculus.

3.5.1 Pooling capabilities (correcting a homework assignment)

Two teaching assistants, Carl (c) and Dan (d), are assigned the task of grading a set of homework assignments consisting of two exercises, a model-theoretic one (M) and a proof-theoretic one (P). Carl is only capable of correcting exercise P, while Dan is only capable of correcting exercise M. None of the two teaching assistants can individually complete the task they have been assigned. However, they can if they *pool* their capabilities. One way in which they can complete the task is by implementing the following plan: they split the set of homework assignments into two sets α and β. Initially, Carl grades the solutions to exercise P in α and Dan those of M in β. Then they switch sets and each of them grades the solutions to the same exercise in the other set.

To capture this case study in (a multi-agent version of) D.LRC, we introduce atomic propositions such as P_α (resp. M_β), the intended meaning of which is that all solutions to exercise P (resp. M) in α (resp. β) have been graded. We also treat α and β as resources. The following table contains formulas expressing the assumptions about agents' capabilities, the initial state of affairs (which resources are initially in possession of which agent), and the plan of switching after completing the correction of one exercise in a given set:

Capabilities		initial state	planning
$\alpha \rhd_{\mathtt{c}} P_\alpha$	$\beta \rhd_{\mathtt{c}} P_\beta$	$\Phi_{\mathtt{c}}\alpha$	$M_\beta \to \Phi_{\mathtt{c}}\beta$
$\alpha \rhd_{\mathtt{d}} M_\alpha$	$\beta \rhd_{\mathtt{d}} M_\beta$	$\Phi_{\mathtt{d}}\beta$	$P_\alpha \to \Phi_{\mathtt{d}}\alpha$

In the present setting we also assume that, whenever an agent is *able* to bring about a certain state of affairs, the agent will. Formally, this corresponds to the validity of the axioms $\Diamond_{\mathtt{i}} A \to A$ for every agent $\mathtt{i}$ and formula A. This axiom does not follow from the logic H.LRC, and in many settings it would not be sound. However, for the sake of the present case study, we will assume that this axiom holds. In fact, this axiom corresponds to the following rules 'Ex$_{\mathtt{i}}$' ('Ex' stands for Execution), for each $\mathtt{i} \in \{\mathtt{c}, \mathtt{d}\}$:

$$\text{Ex}_{\mathtt{i}} \ \frac{X \vdash Y}{\circ_{\mathtt{i}} X \vdash Y}$$

Notice that these rules are analytic (cf. Section 3.7). Hence, by Theorem 94, when adding these rules to the basic calculus D.LRC, the resulting calculus (which we refer to as D.LRC + Ex) enjoys cut elimination and subformula property.

We aim at deriving the formula $(P_\alpha \wedge M_\beta) \wedge (P_\beta \wedge M_\alpha)$ from the assumptions above in the calculus D.LRC + Ex. This will provide the formal verification that executing the plan yields the completion of the task. Let us start by considering the following derivations:

3

$$\pi_1$$

$$\text{Cut}\ \dfrac{\dfrac{\vdots\ \text{proof for BD1}}{\Diamond\!\!\!\mid_{\mathtt{c}}\alpha\,;\alpha\rhd_{\mathtt{c}}P_\alpha\vdash\Diamond_{\mathtt{c}}P_\alpha}\qquad \dfrac{\text{Ex}\ \dfrac{P_\alpha\vdash P_\alpha}{\circ_{\mathtt{c}}P_\alpha\vdash P_\alpha}}{\Diamond_{\mathtt{c}}P_\alpha\vdash P_\alpha}}{\Diamond\!\!\!\mid_{\mathtt{c}}\alpha\,;\alpha\rhd_{\mathtt{c}}P_\alpha\vdash P_\alpha}$$

$$\pi_2$$

$$\text{Cut}\ \dfrac{\dfrac{\vdots\ \text{proof for BD1}}{\Diamond\!\!\!\mid_{\mathtt{d}}\beta\,;\alpha\rhd_{\mathtt{d}}M_\beta\vdash\Diamond_{\mathtt{d}}M_\beta}\qquad \dfrac{\text{Ex}\ \dfrac{M_\beta\vdash M_\beta}{\circ_{\mathtt{d}}M_\beta\vdash M_\beta}}{\Diamond_{\mathtt{d}}M_\beta\vdash M_\beta}}{\Diamond\!\!\!\mid_{\mathtt{d}}\beta\,;\beta\rhd_{\mathtt{d}}M_\beta\vdash M_\beta}$$

$$\pi_3$$

$$\text{Cut}\ \dfrac{\dfrac{\vdots\ \text{proof for BD1}}{\Diamond\!\!\!\mid_{\mathtt{c}}\beta\,;\beta\rhd_{\mathtt{c}}P_\beta\vdash\Diamond_{\mathtt{c}}P_\beta}\qquad \dfrac{\text{Ex}\ \dfrac{P_\beta\vdash P_\beta}{\circ_{\mathtt{c}}P_\beta\vdash P_\beta}}{\Diamond_{\mathtt{c}}P_\beta\vdash P_\beta}}{\Diamond\!\!\!\mid_{\mathtt{c}}\beta\,;\beta\rhd_{\mathtt{c}}P_\beta\vdash P_\beta}$$

$$\pi_4$$

$$\text{Cut}\ \dfrac{\dfrac{\vdots\ \text{proof for BD1}}{\Diamond\!\!\!\mid_{\mathtt{d}}\alpha\,;\alpha\rhd_{\mathtt{d}}M_\alpha\vdash\Diamond_{\mathtt{d}}M_\alpha}\qquad \dfrac{\text{Ex}\ \dfrac{M_\alpha\vdash M_\alpha}{\circ_{\mathtt{d}}M_\alpha\vdash M_\alpha}}{\Diamond_{\mathtt{d}}M_\alpha\vdash M_\alpha}}{\Diamond\!\!\!\mid_{\mathtt{d}}\alpha\,;\alpha\rhd_{\mathtt{d}}M_\alpha\vdash M_\alpha}$$

These derivations follow one and the same pattern, and each derives one piece of the desired conclusion. Hence, one would want to suitably prolong these derivations by applying $\wedge_R$ to reach the conclusion. However, while the conclusions of π_1 and π_2 contain only formulas which are assumptions in our case study as reported in the table above, the formulas $\Diamond\!\!\!\mid_{\mathtt{c}}\beta$ and $\Diamond\!\!\!\mid_{\mathtt{d}}\alpha$, occurring in the conclusions of π_3 and π_4 respectively, are not assumptions. However, they are provable from the assumptions. Indeed, they encode states of affairs which hold after c and d have switched the sets α and β.

Notice that the following sequents are provable (their derivations are straightforward and are omitted):

$$M_\beta; M_\beta\to\Diamond\!\!\!\mid_{\mathtt{c}}\beta\vdash\Diamond\!\!\!\mid_{\mathtt{c}}\beta\qquad P_\alpha; P_\alpha\to\Diamond\!\!\!\mid_{\mathtt{d}}\alpha\vdash\Diamond\!\!\!\mid_{\mathtt{d}}\alpha$$

These sequents say that the formulas $\Diamond\!\!\!\mid_{\mathtt{c}}\beta$ and $\Diamond\!\!\!\mid_{\mathtt{d}}\alpha$ are provable from the 'planning assumptions' (cf. table above) using the formulas M_β and P_α which have been derived purely from the assumptions by π_1 and π_2. Hence, the atoms P_β and M_α can be derived from the original assumptions via cut. Then, applying $\wedge_R$ and possibly contraction, one can derive the desired sequent.

3.5.2 Conjoining capabilities (the wisdom of the crow)

A BBC documentary program shows a problem-solving test conducted on a crow. In the present subsection we formalize an adapted version of this test. There is food (ϕ) positioned deep in a narrow box, out of the reach of the crow's beak. There is a short stick (σ) directly available to the crow, two stones (ρ_1, ρ_2) each inside a cage, and a long stick (λ) inside a transparent box which releases the stick if enough weight (that of two stones or more) lays inside the box. The stick σ is too short for the crow to reach the food using it. However, previous tests have shown that the crow is capable of performing the following individual steps: (a) reaching the food using the long stick; (b) retrieving the stones from the cages using the short stick; (c) retrieving the long stick by dropping stones into a slot in the box. The crow succeeded in executing these individual steps in the right order and got to the food.

An interesting feature of this case study is the interplay of different kinds of resources. Specifically, σ is a *reusable* resource (indeed, the crow uses the same stick to reach the two stones), which fact can be expressed by the sequent $\sigma \vdash \sigma \cdot \sigma$. Also, the following formula holds of all resources relevant to the present case study: $\alpha \trianglerighteq \gamma \wedge \beta \trianglerighteq \delta \rightarrow \alpha \cdot \beta \trianglerighteq \gamma \cdot \delta$. This formula implies a form of *scalability* of resources,[13] which is not a property holding in general, and hence has not been added to the general calculus. The crow passing the test shows to be able to conjoin the separate capabilities together. This is expressed by the following *transitivity*-type axiom: $\alpha \trianglerighteq \beta \wedge \beta \trianglerighteq \gamma \rightarrow \alpha \trianglerighteq \gamma$. The crow's achievement is remarkable precisely because this axiom cannot be expected to hold of any agent. These conditions translate into the following analytic rules:

$$\text{Contr}\ \frac{\Sigma \odot \Sigma \vdash \Omega}{\Sigma \vdash \Omega} \qquad \text{Scalab}\ \frac{(\Gamma \blacktriangle X) \odot (\Pi \blacktriangle Y) \vdash \Delta}{(\Gamma \odot \Pi) \blacktriangle (X\,; Y) \vdash \Delta} \qquad \text{Trans}\ \frac{(\Gamma \blacktriangle X) \blacktriangle Y \vdash \Delta}{\Gamma \blacktriangle (X\,; Y) \vdash \Delta}$$

In order for the rule Contr to satisfy C_6 and C_9, we need to work with a version of D.LRC which admits *two* types of resources: the *reusable* ones (for which the contraction rule is sound) and the general ones for which contraction is not sound. Hence, the contraction would be introduced only for the reusable type. Once the new type has been introduced, the language and calculus of LRC need to be expanded with copies of each original connective, so as to account for the fact that each copy takes in input and outputs exactly one type unambiguously. Correspondingly, copies of each original rule have to be added so that each copy accounts for exactly one reading of the original rule. This is a tedious but entirely safe procedure that guarantees that a proper multi-type calculus (cf. Definition 93) can be introduced which admits *all* the rules above. The reader is referred to [33, 35] for examples of such a disambiguation procedure.

The following table shows the assumptions of the present case study:

[13] That is, if the agent is capable of getting one (measure of) β from one (measure of) α, then is also capable to get two or n (measures of) β from two or n (measures of) α.

Initial state	Capabilities
	$\sigma \rhd \rho$
$\Phi\, \sigma$	$\rho \cdot \rho \rhd \lambda$
	$\lambda \rhd \varphi$

We aim at proving the following sequent:

$$\sigma \rhd \rho \,;\, \rho \cdot \rho \rhd \lambda \,;\, \lambda \rhd \phi \,;\, \Phi\, \sigma \vdash \Diamond \Phi\, \phi.$$

3

We do it in several steps: first, in the following derivation π_1, we prove that for any reusable resource σ, if $\sigma \rhd \rho$ then $\sigma \rhd \rho \cdot \rho$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{Contr}\;\dfrac{\dfrac{\text{Scalab}\;\dfrac{\dfrac{\dfrac{\dfrac{\sigma \vdash \sigma \qquad \rho \vdash \rho}{\sigma \rhd \rho \vdash \sigma \vartriangleright \rho}}{\sigma \blacktriangle \sigma \rhd \rho \vdash \rho} \qquad \dfrac{\dfrac{\sigma \vdash \sigma \qquad \rho \vdash \rho}{\sigma \rhd \rho \vdash \sigma \vartriangleright \rho}}{\sigma \blacktriangle \sigma \rhd \rho \vdash \rho}}{(\sigma \blacktriangle \sigma \rhd \rho) \odot (\sigma \blacktriangle \sigma \rhd \rho) \vdash \rho \cdot \rho}}{(\sigma \odot \sigma) \blacktriangle (\sigma \rhd \rho \,;\, \sigma \rhd \rho) \vdash \rho \cdot \rho}}{\sigma \rhd \rho \,;\, \sigma \rhd \rho \vdash \sigma \odot \sigma \vartriangleright \rho \cdot \rho}}{\sigma \odot \sigma \vdash (\sigma \rhd \rho \,;\, \sigma \rhd \rho) \blacktriangleright \rho \cdot \rho}}{\sigma \vdash (\sigma \rhd \rho \,;\, \sigma \rhd \rho) \blacktriangleright \rho \cdot \rho}}{\sigma \rhd \rho \,;\, \sigma \rhd \rho \vdash \sigma \vartriangleright \rho \cdot \rho}}{\sigma \rhd \rho \vdash \sigma \vartriangleright \rho \cdot \rho}}{\sigma \rhd \rho \vdash \sigma \rhd \rho \cdot \rho}$$

Second, in the following derivation π_2, we prove an instance of the transitivity axiom:

$$\dfrac{\dfrac{\dfrac{\text{Trans}\;\dfrac{\dfrac{\dfrac{\dfrac{\sigma \vdash \sigma \qquad \dfrac{\dfrac{\rho \vdash \rho \qquad \rho \vdash \rho}{\rho \odot \rho \vdash \rho \cdot \rho}}{\rho \cdot \rho \vdash \rho \cdot \rho}}{\sigma \rhd \rho \cdot \rho \vdash \sigma \vartriangleright \rho \cdot \rho}}{\sigma \blacktriangle \sigma \rhd \rho \cdot \rho \vdash \rho \cdot \rho} \qquad \lambda \vdash \lambda}{\rho \cdot \rho \rhd \lambda \vdash \sigma \blacktriangle \sigma \rhd \rho \cdot \rho \vartriangleright \lambda}}{(\sigma \blacktriangle \sigma \rhd \rho \cdot \rho) \blacktriangle \rho \cdot \rho \rhd \lambda \vdash \lambda}}{\sigma \blacktriangle (\sigma \rhd \rho \cdot \rho \,;\, \rho \cdot \rho \rhd \lambda) \vdash \lambda}}{\sigma \rhd \rho \cdot \rho \,;\, \rho \cdot \rho \rhd \lambda \vdash \sigma \vartriangleright \lambda}}{\sigma \rhd \rho \cdot \rho \,;\, \rho \cdot \rho \rhd \lambda \vdash \sigma \rhd \lambda}$$

Similarly, a derivation π_3 can be given of the following instance of the transitivity axiom:

$$\sigma \rhd \lambda \,;\, \lambda \rhd \phi \vdash \sigma \rhd \phi.$$

Finally, the following derivation π_4 is the missing piece:

$$
\dfrac{
\text{Cut}\;\dfrac{
\dfrac{\vdots\ \text{proof for BD2}}{\sigma \trianglerighteq \phi \vdash \sigma \rhd \obar\phi}
\qquad
\dfrac{\dfrac{\vdots\ \text{proof for BD1}}{\sigma \rhd \obar\phi \,;\, \obar\sigma \vdash \Diamond\obar\phi}}{\sigma \rhd \obar\phi \vdash \Diamond\obar\phi < \obar\sigma}
}{\sigma \trianglerighteq \phi \vdash \Diamond\obar\phi < \obar\sigma}
}{\sigma \trianglerighteq \phi \,;\, \obar\sigma \vdash \Diamond\obar\phi}
$$

The requested sequent can be then derived using π_1-π_4 via cuts and display postulates.

3

3.5.3 Resources having different roles (The Gift of the Magi)

The Gift of the Magi is a short story, written by O. Henry and first appeared in 1905, about a young married couple of very modest means, Jim (j) and Della (d), who have only two possessions between them which are of value (both monetarily and in the sense that they take pride in them): Della's unusually long hair (η), and Jim's family gold watch (ω). On Christmas Eve, Della sells her hair to buy a chain (γ) for Jim's watch, and Jim sells his watch to buy an ivory brush (β) for Della.

Jim and Della are materially worse off at the end of the story than at the beginning, since, while the resources ω and η could be used/enjoyed on their own, γ and β can only be used when coupled with ω and η respectively. In fact, the very choice of γ and β as presents is a direct consequence of the fact that—besides being used by their respective owners as a means to get the money to buy a present for the other—the resources ω and η are used by the *partner* of their respective owners as *beacons* guiding them in their choice of a present. For instance, their final situation would not have been as bad if Della had bought Jim a new overcoat or a pair of gloves, or if Jim had bought Della replacements for her old brown jacket or hat, the need for which is indicated in the short story. However, each wants to make their present as meaningful as possible to the other one, and hence each *targets* his/her present at the one possession the other takes pride in.

Finally, the uniqueness of the meaningful resource of each agent is the reason why "the whole affair has something of the dark inevitability of Greek tragedy" (cit. P. G. Wodehouse, *Thank you, Jeeves*): indeed, ω (resp. η) is both the only target for a meaningful present for Jim (resp. Della), and also the only means he (resp. she) has to acquire such a present for her (resp. him).

To formalize the observations above, we will need a modification of the language of LRC capturing the fact, which is sometimes relevant, that resources might have different *roles* e.g. in the generation or the acquisition of a given resource. For instance, in the production of bread, the oven has a different role as a resource than water and flour; in shooting sports, the shooter uses a shooting device, projectiles and a target in different roles, etc. Roles cannot be reduced to how resources are combined irrespective of agency (this aspect is modelled by the pure-resource connectives $\sqcap$ and $\cdot$); rather, assigning roles to resources is a facet of agency. Accordingly, we consider the following ternary connective for each agent:

$$[-,-] \trianglerighteq - : \mathsf{Res} \times \mathsf{Res} \times \mathsf{Res} \to \mathsf{Fm},$$

the intended meaning of which is 'the agent is capable of obtaining the resource in the third coordinate, whenever in possession of the resources in the first two coordinates *in their respective roles*'. Algebraically (and axiomatically), this connective is finitely join-reversing in the first two coordinates and finitely meet-preserving in the third one. Its introduction rules and display postulates are as expected:

$$\frac{\Gamma \vdash \alpha \qquad \Theta \vdash \beta \qquad \gamma \vdash \Sigma}{[\alpha,\beta] \rhd \gamma \vdash [\Gamma,\Theta] \rhd \Sigma} \qquad \frac{X \vdash [\alpha,\beta] \rhd \gamma}{X \vdash [\alpha,\beta] \rhd \gamma}$$

$$\frac{X \vdash [\Gamma,\Theta] \rhd \Sigma}{[\Gamma,\Theta] \blacktriangle X \vdash \Sigma} \qquad \frac{X \vdash [\Gamma,\Theta] \rhd \Sigma}{\Gamma \vdash [X,\Theta] \blacktriangleright^1 \Sigma} \qquad \frac{X \vdash [\Gamma,\Theta] \rhd \Sigma}{\Theta \vdash [\Gamma,X] \blacktriangleright^2 \Sigma}$$

In addition, we need two unary diamond operators $\Diamond\!\!\!\!\mid\,^1, \Diamond\!\!\!\!\mid\,^2 : \mathsf{Res} \to \mathsf{Fm}$ for each agent, the intended meaning of which is 'the agent is in possession of the resource (in the argument) *in the first* (resp. *second*) *role*'. The basic algebraic and axiomatic behaviour of $\Diamond\!\!\!\!\mid\,^1$ and $\Diamond\!\!\!\!\mid\,^2$ coincides with that of $\Diamond\!\!\!\!\mid\,$, hence the introduction and display rules relative to these connectives are like those given for $\Diamond\!\!\!\!\mid\,$. The various roles and their differences can be understood and formalized in different ways relative to different settings. In the specific situation of the short story, we stipulate that $\Diamond\!\!\!\!\mid\,^2$ has the meaning usually attributed to $\Diamond\!\!\!\!\mid\,$, and understand $\Diamond\!\!\!\!\mid\,^1\sigma$ as 'the agent has resource σ available in the role of target (or beacon)'.

The interaction of these connectives, and the difference in meaning between $\Diamond\!\!\!\!\mid\,^1$ and $\Diamond\!\!\!\!\mid\,^2$, are captured by the following axiom:

$$\Diamond\!\!\!\!\mid\,^1\sigma \wedge \Diamond\!\!\!\!\mid\,^2\xi \wedge [\sigma,\xi] \rhd \chi \to \Diamond \Diamond\!\!\!\!\mid\,^2\chi, \tag{3.5.1}$$

which is equivalent on perfect LRC-algebras to the following analytic rule:

$$\frac{\circ\varphi^2\,[\Sigma,\Xi] \blacktriangle X \vdash Y}{\varphi^1\Sigma\,;\varphi^2\,\Xi\,; X \vdash Y}\ \mathrm{RR}$$

Finally, in the specific case at hand, we will use the rules corresponding to the following slightly modified multi-agent versions of axiom (3.5.1):

$$\Diamond\!\!\!\!\mid\,^1_{\mathtt{j}}\sigma \wedge \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\xi \wedge [\sigma,\xi] \rhd_{\mathtt{j}} \chi \to \Diamond_{\mathtt{j}} \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\chi \quad \text{and} \quad \Diamond\!\!\!\!\mid\,^1_{\mathtt{d}}\sigma \wedge \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\xi \wedge [\sigma,\xi] \rhd_{\mathtt{d}} \chi \to \Diamond_{\mathtt{d}} \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\chi.$$

The following table shows the assumptions of the present case study:

		Initial state	Capabilities	Abilities
Jim	j	$\Diamond\!\!\!\!\mid\,^1_{\mathtt{j}}\eta \;\; \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\omega$	$[\eta,\omega] \rhd_{\mathtt{j}} \beta$	$\Diamond_{\mathtt{j}} \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\beta \to \Diamond_{\mathtt{j}} \neg \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\omega$
Della	d	$\Diamond\!\!\!\!\mid\,^1_{\mathtt{d}}\omega \;\; \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\eta$	$[\omega,\eta] \rhd_{\mathtt{d}} \gamma$	$\Diamond_{\mathtt{d}} \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\gamma \to \Diamond_{\mathtt{d}} \neg \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\eta$

Let H be the structural conjunction of the assumptions above. We aim at deriving the following sequent in the calculus D.LRC to which the analytic rules introduced above have been added:

$$H \vdash \Diamond_{\mathtt{j}} \neg \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\omega \wedge \Diamond_{\mathtt{j}} \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\beta \wedge \Diamond_{\mathtt{d}} \neg \Diamond\!\!\!\!\mid\,^2_{\mathtt{d}}\eta \wedge \Diamond_{\mathtt{d}} \Diamond\!\!\!\!\mid\,^2_{\mathtt{j}}\gamma.$$

We do it in several steps: first, the following derivation π_1:

$$
\begin{array}{c}
\eta \vdash \eta \qquad \omega \vdash \omega \qquad \beta \vdash \beta \\
\hline
[\eta,\omega] \rhd_{j} \beta \vdash [\eta,\omega] \unrhd_{j} \beta \\
\hline
[\eta,\omega] \blacktriangle_{j} [\eta,\omega] \rhd_{j} \beta \vdash \beta \\
\hline
\varphi^{2}_{d} \Big([\eta,\omega] \blacktriangle_{j} [\eta,\omega] \rhd_{j} \beta \Big) \vdash \Phi^{2}_{d} \beta \\
\hline
\circ_{j} \varphi^{2}_{d} \Big([\eta,\omega] \blacktriangle_{j} [\eta,\omega] \rhd_{j} \beta \Big) \vdash \Diamond_{j} \Phi^{2}_{d} \beta \\
\hline
(\varphi^{1}_{j} \eta \,;\, \varphi^{2}_{j} \omega) \,;\, [\eta,\omega] \rhd_{j} \beta \vdash \Diamond_{j} \Phi^{2}_{d} \beta \quad \mathsf{RR_{jd}} \\
\hline
(\Phi^{1}_{j} \eta \,;\, \Phi^{2}_{j} \omega) \,;\, [\eta,\omega] \rhd_{j} \beta \vdash \Diamond_{j} \Phi^{2}_{d} \beta
\end{array}
$$

With an analogous derivation π_2 we can prove that

$$\Phi^{1}_{d} \omega \,;\, \Phi^{2}_{d} \eta \,;\, [\omega,\eta] \rhd_{d} \gamma \vdash \Diamond_{d} \Phi^{2}_{j} \gamma.$$

Next, let π_3 be the following derivation:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\beta \vdash \beta}{\varphi^{2}_{d} \beta \vdash \Phi^{2}_{d} \beta}}{\circ_{j} \Phi^{2}_{d} \beta \vdash \Diamond_{j} \Phi^{2}_{d} \beta}}{\Diamond_{j} \Phi^{2}_{d} \beta \vdash \Diamond_{j} \Phi^{2}_{d} \beta}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\omega \vdash \omega}{\varphi^{2}_{j} \omega \vdash \Phi^{2}_{j} \omega}}{\Phi^{2}_{j} \omega \vdash \Phi^{2}_{j} \omega} \quad \bot \vdash \bot}{\Phi^{2}_{j} \omega \to \bot \vdash \Phi^{2}_{j} \omega > \bot}}{\Phi^{2}_{j} \omega \to \bot \vdash \Phi^{2}_{j} \omega \to \bot}\ \mathsf{def}}{\neg \Phi^{2}_{j} \omega \vdash \neg \Phi^{2}_{j} \omega}}{\circ_{j} \neg \Phi^{2}_{j} \omega \vdash \Diamond_{j} \neg \Phi^{2}_{j} \omega}}{\Diamond_{j} \neg \Phi^{2}_{j} \omega \vdash \Diamond_{j} \neg \Phi^{2}_{j} \omega}
}{
\dfrac{\Diamond_{j} \Phi^{2}_{d} \beta \to \Diamond_{j} \neg \Phi^{2}_{j} \omega \vdash \Diamond_{j} \Phi^{2}_{d} \beta > \Diamond_{j} \neg \Phi^{2}_{j} \omega}{\Diamond_{j} \Phi^{2}_{d} \beta \,;\, \Diamond_{j} \Phi^{2}_{d} \beta \to \Diamond_{j} \neg \Phi^{2}_{j} \omega \vdash \Diamond_{j} \neg \Phi^{2}_{j} \omega}
}
$$

With an analogous derivation π_4 we can prove that

$$\Diamond_{d} \Phi^{2}_{j} \gamma \,;\, \Diamond_{d} \Phi^{2}_{j} \gamma \to \Diamond_{d} \neg \Phi^{2}_{d} \eta \vdash \Diamond_{d} \neg \Phi^{2}_{d} \eta.$$

Then, by applying cut (and left weakening) on π_1 and π_3 one derives:

$$\Phi^{1}_{j} \eta \,;\, \Phi^{2}_{j} \omega \,;\, [\eta,\omega] \rhd_{j} \beta \,;\, \Diamond_{j} \Phi^{2}_{d} \beta \to \Diamond_{j} \neg \Phi^{2}_{j} \omega \vdash \Diamond_{j} \neg \Phi^{2}_{j} \omega.$$

Likewise, by applying cut (and left weakening) on π_2 and π_4 one derives:

$$\Phi^{1}_{d} \omega \,;\, \Phi^{2}_{d} \eta \,;\, [\omega,\eta] \rhd_{d} \gamma \,;\, \Diamond_{d} \Phi^{2}_{j} \gamma \to \Diamond_{d} \neg \Phi^{2}_{d} \eta \vdash \Diamond_{d} \neg \Phi^{2}_{d} \eta.$$

The derivation is concluded with applications of right-introduction of $\wedge$ and left contraction rules.

3.5.4 From local to global resilience (two production lines)

Resilience is the ability of an agent or an organization to realize their goals notwithstanding unexpected changes and disruptions. The language of LRC provides a natural way to understand resilience as the capability to realize one's goal(s) in a range of situations characterized by the reduced availability of key resources. Consider for example a factory with two production lines for products γ_1 and γ_2. Product γ_1 is of higher quality than γ_2 and can only be produced using resource α, the availability of which is subject to fluctuations. Product γ_2 can be produced using either resource α or β, and the availability of β is not subject to fluctuations. It is interesting to note that the 'local' resilience in the production of γ_2 (namely, the fact that any shortage in α can be dealt with by switching to β) results in the resilience of both production lines. Indeed, when α is available for only one of the two production lines, all of it can be employed in the production line for γ_1, and the production of γ_2 is switched to β. In the formal treatment that follows, we notice that the axioms $\Diamond\sigma \wedge \sigma \unrhd \pi \rightarrow \Diamond\pi$ and $\sigma \unrhd \chi \wedge \pi \unrhd \xi \rightarrow \sigma \cdot \pi \unrhd \chi \cdot \xi$ hold for the setting described above. These axioms are analytic and are equivalent on perfect LRC-algebras to the following rules:

$$\text{BDR}\ \frac{X \vdash \Gamma \,\boldsymbol{\unrhd}\, \boldsymbol{\obar} Y}{X \vdash \obar\Gamma > Y} \qquad\qquad \frac{(\Gamma \blacktriangle\!\blacktriangle X) \odot (\Pi \blacktriangle\!\blacktriangle Y) \vdash \Delta}{(\Gamma \odot \Pi) \blacktriangle\!\blacktriangle (X\,; Y) \vdash \Delta}\ \text{Scalab}$$

Resources	Capabilities
$\Diamond(((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta)$	$\alpha \unrhd \gamma_1$
	$\alpha \sqcup \beta \unrhd \gamma_2$

We aim at showing that the assumptions above are enough to conclude that the factory is able to realize the production of both γ_1 and γ_2:

$$\Diamond(((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta)\,; \alpha \unrhd \gamma_1\,; \alpha \sqcup \beta \unrhd \gamma_2 \vdash \Diamond(\gamma_1 \cdot \gamma_2).$$

Notice that the following is an instance of $\Diamond\sigma \wedge \sigma \unrhd \pi \rightarrow \Diamond\pi$, and hence is derivable using the rule BDR:

$$\Diamond(((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta)\,; ((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta \unrhd \gamma_1\cdot\gamma_2 \vdash \Diamond(\gamma_1\cdot\gamma_2).$$

Hence, modulo cut and left weakening, it is enough to show that

$$\alpha \unrhd \gamma_1\,; \alpha \sqcup \beta \unrhd \gamma_2 \vdash ((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta \unrhd \gamma_1\cdot\gamma_2.$$

Notice that:

$$\cfrac{\cfrac{\overset{\vdots\ \text{proof for R4}}{((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta \vdash (\alpha\cdot\alpha)\cdot\beta \sqcup \alpha\cdot\beta} \qquad \cfrac{\cfrac{\gamma_1 \vdash \gamma_1 \qquad \gamma_2 \vdash \gamma_2}{\gamma_1 \odot \gamma_2 \vdash \gamma_1\cdot\gamma_2}}{\gamma_1\cdot\gamma_2 \vdash \gamma_1\cdot\gamma_2}}{(\alpha\cdot\alpha)\cdot\beta \sqcup \alpha\cdot\beta \unrhd \gamma_1\cdot\gamma_2 \vdash ((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta \,\boldsymbol{\unrhd}\, \gamma_1\cdot\gamma_2}}{(\alpha\cdot\alpha)\cdot\beta \sqcup \alpha\cdot\beta \unrhd \gamma_1\cdot\gamma_2 \vdash ((\alpha\cdot\alpha)\sqcup\alpha)\cdot\beta \unrhd \gamma_1\cdot\gamma_2}$$

Hence, modulo cut and left weakening, it is enough to show that

$$\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \vdash (\alpha \cdot \alpha) \cdot \beta \sqcup \alpha \cdot \beta \rhd \gamma_1 \cdot \gamma_2 .$$

Indeed, a derivation for the sequent above is:

$$\dfrac{\overset{\vdots\ \pi_1}{\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \vdash \alpha \cdot (\alpha \sqcup \beta) \rhd \gamma_1 \cdot \gamma_2} \qquad \overset{\vdots\ \pi_2}{\alpha \cdot (\alpha \sqcup \beta) \rhd \gamma_1 \cdot \gamma_2 \vdash (\alpha \cdot \alpha) \cdot \beta \sqcup \alpha \cdot \beta \rhd \gamma_1 \cdot \gamma_2}}{\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \vdash (\alpha \cdot \alpha) \cdot \beta \sqcup \alpha \cdot \beta \rhd \gamma_1 \cdot \gamma_2}$$

where π_1 is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \gamma_1 \vdash \gamma_1}{\alpha \rhd \gamma_1 \vdash \alpha \vartriangleright \gamma_1}}{\alpha \blacktriangle \alpha \rhd \gamma_1 \vdash \gamma_1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha \sqcup \beta \vdash \alpha , \beta}}{\alpha \sqcup \beta \vdash \alpha \sqcup \beta} \qquad \gamma_2 \vdash \gamma_2}{\alpha \sqcup \beta \rhd \gamma_2 \vdash \alpha \sqcup \beta \vartriangleright \gamma_2}}{(\alpha \sqcup \beta) \blacktriangle \alpha \sqcup \beta \rhd \gamma_2 \vdash \gamma_2}}{}}{(\alpha \blacktriangle \alpha \rhd \gamma_1) \odot (\alpha \sqcup \beta) \blacktriangle \alpha \sqcup \beta \rhd \gamma_2 \vdash \gamma_1 \cdot \gamma_2}}{\alpha \odot (\alpha \sqcup \beta) \blacktriangle (\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2) \vdash \gamma_1 \cdot \gamma_2}\ \text{Scalab}}{\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \vdash \alpha \odot (\alpha \sqcup \beta) \vartriangleright \gamma_1 \cdot \gamma_2}}{\alpha \odot (\alpha \sqcup \beta) \vdash \alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \blacktriangleright \gamma_1 \cdot \gamma_2}}{\alpha \cdot (\alpha \sqcup \beta) \vdash \alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \blacktriangleright \gamma_1 \cdot \gamma_2}}{\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \vdash \alpha \cdot (\alpha \sqcup \beta) \vartriangleright \gamma_1 \cdot \gamma_2}}{\alpha \rhd \gamma_1 ; \alpha \sqcup \beta \rhd \gamma_2 \vdash \alpha \cdot (\alpha \sqcup \beta) \rhd \gamma_1 \cdot \gamma_2}$$

and π_2 is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \odot \Phi \vdash \alpha}}{\Phi \vdash \alpha \gtrdot \alpha}}{\alpha \vdash \alpha \gtrdot \alpha}}{\alpha \odot \alpha \vdash \alpha}}{\alpha \cdot \alpha \vdash \alpha}}{\alpha \cdot \alpha \odot \beta \vdash \alpha}}{(\alpha \cdot \alpha \cdot \beta \vdash \alpha} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \odot \Phi \vdash \alpha}}{\Phi \vdash \alpha \gtrdot \alpha}}{\alpha \vdash \alpha \gtrdot \alpha}}{\alpha \odot \alpha \vdash \alpha}}{\alpha \odot \beta \vdash \alpha}}{\alpha \cdot \beta \vdash \alpha}}{\alpha \cdot \beta \vdash \alpha , \beta}}{\alpha \cdot \beta \vdash \alpha \sqcup \beta}}{(\alpha \cdot \alpha \cdot \beta) \sqcup (\alpha \cdot \beta) \vdash \alpha \cdot (\alpha \sqcup \beta)} \qquad \dfrac{\dfrac{\gamma_1 \vdash \gamma_1 \qquad \gamma_2 \vdash \gamma_2}{\gamma_1 \odot \gamma_2 \vdash \gamma_1 \cdot \gamma_2}}{\gamma_1 \cdot \gamma_2 \vdash \gamma_1 \cdot \gamma_2}}{\dfrac{\alpha \cdot (\alpha \sqcup \beta) \rhd \gamma_1 \cdot \gamma_2 \vdash (\alpha \cdot \alpha \cdot \beta) \sqcup \alpha \cdot \beta \vartriangleright \gamma_1 \cdot \gamma_2}{\alpha \cdot (\alpha \sqcup \beta) \rhd \gamma_1 \cdot \gamma_2 \vdash (\alpha \cdot \alpha \cdot \beta) \sqcup \alpha \cdot \beta \rhd \gamma_1 \cdot \gamma_2}}$$

3.6 Conclusions and further directions

Resources and capabilities. In the present chapter, a logical framework is introduced aimed at capturing and reasoning about resource flow within organizations. This framework contributes to the line of investigation of the logics of agency (cf. e.g. [4, 11, 29–31]) by focusing specifically on the *resource*-dimension of agents' (cap)abilities (e.g. to use resources to achieve goals, to transform resources into other resources, and to coordinate the use of resources with other agents). Formally, the logic of resources and capabilities (LRC) has been introduced in a language consisting of formula-terms and resource-terms. Besides pure-formula and pure-resource connectives, the language of LRC includes connectives bridging the two types in various ways. Although action-terms are not included in LRC, perhaps the logical system of which LRC is most reminiscent is the logic of capabilities introduced in [72], which formalizes the capabilities of agents to perform actions. Indeed, looking past the differences between the two formalisms deriving from the inherent differences between actions and resources, the focus of both axiomatizations is *interaction*, between (cap)abilities and actions in [72], and between (cap)abilities and resources in the present chapter. Precisely its focus on interaction makes it worthwhile to recast the logical framework of [72] in a multi-type environment.

A study in algebraic proof theory. The main technical contribution of the chapter is the introduction of the multi-type calculus D.LRC. The definition of this calculus and the proofs of its basic properties hinge on the integration of two theories in algebraic logic and structural proof theory—namely, *unified correspondence* and *multi-type calculi*—which originated independently of each other. This integration contributes to the research program of *algebraic proof theory* [12, 14], to which the results of the present chapter pertain. Specifically, the rules of D.LRC are introduced, and their soundness proved, by applying (and adapting) the ALBA-based methodology of [45] (cf. also [13] for a purely proof-theoretic perspective on the same methodology); cut elimination is proved 'Belnap-style', by verifying that D.LRC satisfies the assumptions of the cut elimination metatheorem for multi-type calculi of [34]; conservativity is proved following the general proof strategy for conservativity illustrated in [45], to which the canonicity of the axioms of the Hilbert-style presentation of LRC is key.

It is perhaps worth stressing that the theory of proper display calculi developed in [45] cannot be applied directly to the Hilbert-style presentation of LRC, for two reasons. Firstly, the setting of [45] is a pure-formula setting, while the setting of the present chapter is multi-type. However, the results of [45] can be ported to the multi-type setting (as done also in [37, 46, 47]); indeed, the algorithm ALBA and the definition of analytic inductive inequalities are grounded in the order-theoretic properties of the algebraic interpretations of the logical connectives, and remain fundamentally unchanged when applied to maps with the required order-theoretic properties, irrespective of whether these maps are operations on one algebra or between different algebras. The second, more serious reason is that the algebraic interpretation of the capability connective $\triangleright$ is a map which reverses finite joins in its first coordinate but is only *monotone* (rather than finitely meet-preserving) in its second coordinate. Hence, (the multi-type version of) the definition of (analytic) inductive inequalities given in [45] does not apply to many axioms of the Hilbert-style presentation of LRC, and hence some results (e.g. the canonicity results of Section 3.2.3) could not be immediately inferred by directly applying the general theory. However, as we saw in Section 3.4.1, the algorithm ALBA is successful on the LRC axiomatization, which suggests the possibility of generalizing these results to arbitrary multi-type signatures in which operations are allowed to be only monotone or antitone in some coordinates. Moreover, unified correspondence theory covers various settings, from general lattice-based propositional logics [19, 20, 23, 24], to regular [62] and monotone modal logics [38], (distributive) lattice-based mu-calculi [16–18], hybrid logic [27] and many-valued logic [53]. It would be interesting to investigate whether structural proof calculi for each of these settings (or for multi-type logics based on them) could be defined by suitably extending the techniques employed in the design of D.LRC.

Proof-theoretic formalizations of social behaviour. In Section 3.5, we have discussed the formalization of situations revolving around some instances of resource flow. These situations have been captured as inferences or sequents in the language of LRC, and derived in the basic calculus D.LRC or in some of its analytic extensions. This proof-theoretic analysis makes it possible to single out the steps and assumptions which are *essential* to a given situation. For instance, thanks to this analysis, it is clear that the full power of classical logic is *not essential* to any case study we treated. In fact, as can be readily verified by inspection, many derivations treated in Section 3.5 need less than the full power of intuitionistic logic, which is the propositional base of LRC. Also, reasoning from assumptions in a given proof-theoretic environment corresponds semantically to reasoning on *all* the models of that environment satisfying those assumptions. This is a *safer* practice than e.g. starting out with an ad-hoc model, since it makes it impossible to rely on some implicit assumption or other extra feature of a chosen model.

The pure-resource fragment. In Section 3.2.1 we mentioned that the fact that 1 coincides with the weakest resource entails (and is in fact equivalent to) the validity of the sequents $\alpha \cdot \beta \vdash \alpha$ and $\alpha \cdot \beta \vdash \beta$, which in some contexts seems too restrictive. How to relax this restriction is current work in progress. However, this restriction brings also some advantages. Indeed, as discussed earlier on in Section 3.2.1, this restriction makes the pure resource fragment of LRC very similar to (the exponential-free fragment of) linear affine logic, which, unlike general linear logic, is decidable [51, 59]. Hence, this leaves open the question of the decidability of LRC (see also below).

Agents as first-class citizens. In the present chapter, we focused on the basic setting of LRC, and for the sake of not overloading notation and machinery, we have treated agents as parameters. However, a fully multi-type treatment would include terms of type Ag (agents) in the language, as done e.g. in [35]. This will be particularly relevant to the formalization of organization theory, where terms of type Ag will represent members of an organization, and Ag might be endowed with additional structure: for instance it can be a graph (capturing networks of agents), or a partial order (capturing hierarchies), or partitioned in coalitions or teams. Having agents as first-class citizens of the language will also make possible to attribute *roles* to them, analogously to the way roles are attributed to resources in Section 3.5.3. Roles in turn could provide concrete handles towards the modelling of agent coordination.

Group capabilities. Closely related to the issue of the previous paragraph is the formalization of various forms of group capabilities. This theme is particularly relevant to organization theory, since it might help to capture e.g. the contribution of leadership to the results of an organization, versus the advantages of self-organization. Another interesting notion in organization theory which could benefit from a formal theory of group capabilities is *tacit group knowledge* [69], emerging from the individual capabilities to adapt, often implicitly, to the behaviour of others.

Different types of resources. Key to the analysis of the case study of Section 3.5.2 was the interplay between reusable and non-reusable resources. The treatment of this case study suggests that analytic extensions of D.LRC can be used to develop a formal theory of resource flow that also captures other differences between resources (e.g. storable vs. non storable, scalable vs. non scalable), their interaction, direct or mediated by agents, in the production process, or in facilitating more generally the competitive success of the organization [58].

Pre-orderings on resources. In Section 3.5.3, we mentioned that the resources the agents possess at the end of the story cannot be used without those they possess at the beginning, while these can be used on their own. This observation suggest that alternative or additional orderings of resources can be considered and studied, such as the 'dependence' preorder between resources, which might be relevant to the analysis of some situations.

Comparing capabilities. The logic LRC provides a formal environment where to explore the consequences for organizations of some agent's being *more* capable than some other agent at bringing about a certain state of affairs. In this environment, we can express that agent $\mathtt{a}$ is *at least as capable* than agent $\mathtt{b}$ at bringing about A e.g. when $\alpha \rhd_{\mathtt{a}} A$ and $\beta \rhd_{\mathtt{b}} A$, and $\beta \vdash \alpha$ (i.e. to bring about the same state of affairs, $\mathtt{b}$ uses a resource which is at least as powerful as, possibly more powerful than, the resource used by $\mathtt{a}$). Ricardo's economic theory of comparative advantage with regard to the division of labour in organizations [66] can be formalized on the basis of capabilities differentials.

Algebraic canonicity and relational semantics. The theory of canonical extensions provides a way to extract relational semantics from the algebraic semantics via algebraic canonicity. In Section 3.2.3, we have shown that the logic LRC is complete w.r.t. *perfect* LRC-algebraic models. Via standard discrete Stone-type duality, perfect LRC-algebraic models can be associated with set-based structures similar to Kripke models, thus providing complete relational semantics for LRC. The specification of this relational semantics and its properties is part of future work.

Semantics of Petri nets. We are currently studying Petri nets as an alternative semantic framework for LRC. In particular, the reachability problem for finite Petri nets is equivalent to the deducibility problem for sequents in finitely axiomatized theory in the pure-tensor fragment of linear logic [57, 70]. More recently, [32] proved completeness for several versions of linear logic w.r.t. Petri nets. We are investigating similar issues in the setting of LRC.

Decidability, finite model property, complexity. The computational properties of LRC such as decidability and complexity are certainly of interest. In particular, two, in general distinct, problems are to be considered: the decidability of the set of theorems, and the decidability of the (finite) consequence relation.[14]

A standard argument establishing decidability is via the so-called finite model property (FMP), i.e. proving that any non-theorem can be refuted in a finite structure. Together with finite axiomatizability and completeness of the underlying logic, FMP entails the decidability of the set of theorems. For the second problem a stronger property is needed: the finite embeddability property, which can be seen as the finite model property for quasi-identities and, together with finite axiomatizability and completeness, entails the decidability of the finite consequence relation of the underlying logic.

We wish to stress that the decidability problems for LRC subsume the complexity and decidability of certain substructural logics. Indeed, as mentioned earlier, the pure-resource fragment of LRC is similar to (propositional, exponential-free) linear affine logic, which essentially coincides with the distributive Full Lambek calculus with weakening, a logic for which the finite consequence relation, and hence the set of theorems, are known to be decidable (see [59, 60]); FEP for integral residuated groupoids has been proved in [8], for a simple proof of FEP in the distributive setting see also [48], where a coNEXP upper bound is obtained. We hope we can use the algebraic semantics of LRC to investigate, and hopefully establish decidability of LRC and its variants using FMP or FEP.

Syntactic decidability. An alternative path towards decidability for LRC consists in adapting the techniques developed in [51], where a syntactic proof is given of the decidability of full propositional affine linear logic, by showing that it is enough to consider sequents in a suitable normal form. An encouraging hint is the fact that the full Lambek calculus with weakening is decidable [59, 60]. However, it is also known that, for certain substructural logics, distributivity is problematic for decidability.

3.7 Proper multi-type calculi and cut elimination

In the present section, we report on the Belnap-style metatheorem that we appeal to in order to show that the calculus introduced in Section 3.3 enjoys cut elimination. This metatheorem was proven in [34] for the so-called *proper multi-type calculi*. In order to make the exposition self-contained, in what follows we will report the definition of proper multi-type calculi and the statement of the metatheorem. Notice that this version is more general than the one presented in Section 2.2.2 in two main respects: firstly, the present version concerns multi-type calculi whereas the version in Section 2.2.2 concerns single-type calculi; secondly, unlike that version, the present version does not assume the display property to hold for the given calculus.

Definition 92. A sequent $x \vdash y$ is *type-uniform* if x and y are of the same type T (cf. [35, Definition 3.1]).

[14] The two problems coincide in presence of deduction theorem, which is available in intuitionistic logic and for the formula-fragment of LRC, but not for the pure-resource fragment of LRC.

Definition 93. A *proper multi-type calculus* is any calculus in a multi-type language satisfying the following list of conditions:[15]

C$_1$: Preservation of operational terms. Each operational term occurring in a premise of an inference rule *inf* is a subterm of some operational term in the conclusion of *inf*.

C$_2$: Shape-alikeness of parameters. Congruent parameters (i.e. non-active terms in the application of a rule) are occurrences of the same structure.

C$'_2$: Type-alikeness of parameters. Congruent parameters have exactly the same type. This condition bans the possibility that a parameter changes type along its history.

C$_3$: Non-proliferation of parameters. Each parameter in an inference rule *inf* is congruent to at most one constituent in the conclusion of *inf*.

C$_4$: Position-alikeness of parameters. Congruent parameters are either all precedent or all succedent parts of their respective sequents. In the case of calculi enjoying the display property, precedent and succedent parts are defined in the usual way (see [3]). Otherwise, these notions can still be defined by induction on the shape of the structures, by relying on the polarity of each coordinate of the structural connectives.

C$'_5$: Quasi-display of principal constituents. If an operational term a is principal in the conclusion sequent s of a derivation π, then a is in display, unless π consists only of its conclusion sequent s (i.e. s is an axiom).

C$''_5$: Display-invariance of axioms. If a is principal in an axiom s, then a can be isolated by applying Display Postulates and the new sequent is still an axiom.

C$'''_5$: Closure of axioms under surgical cut. If $(x \vdash y)([a]^{pre}, [a]^{suc})$, $a \vdash z[a]^{suc}$ and $v[a]^{pre} \vdash a$ are axioms, then $(x \vdash y)([a]^{pre}, [z/a]^{suc})$ and $(x \vdash y)([v/a]^{pre}, [a]^{suc})$ are again axioms.

C$'_6$: Closure under substitution for succedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position, *within each type*.

C$'_7$: Closure under substitution for precedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position, *within each type*.

Condition C$'_6$ (and likewise C$'_7$) ensures, for instance, that if the following inference is an application of the rule R:

$$\frac{(x \vdash y)\big([a]^{suc}_i \,|\, i \in I\big)}{(x' \vdash y')[a]^{suc}}\ R$$

and $\big([a]^{suc}_i \,|\, i \in I\big)$ represents all and only the occurrences of a in the premiss which are congruent to the occurrence of a in the conclusion (if $I = \varnothing$, then the occurrence of a in the conclusion is congruent to itself), then also the following inference is an application of the same rule R:

$$\frac{(x \vdash y)\big([z/a]^{suc}_i \,|\, i \in I\big)}{(x' \vdash y')[z/a]^{suc}}\ R$$

where the structure z is substituted for a.

[15] See [36] for a discussion on C$'_5$ and C$''_5$.

This condition caters for the step in the cut elimination procedure in which the cut needs to be "pushed up" over rules in which the cut-formula in succedent position is parametric (cf. [34, Section 4]).

C'_8: Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are *principal*, i.e. each of them has been introduced with the last rule application of each corresponding subdeduction. In this case, analogously to the proof Gentzen-style, condition C'_8 requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether, or is transformed in one or more applications of the cut rule, involving proper subterms of the original operational cut-term. In addition to this, specific to the multi-type setting is the requirement that the new application(s) of the cut rule be also *strongly type-uniform* (cf. condition C_{10} below).

C_9: Type-uniformity of derivable sequents. Each derivable sequent is type-uniform.

C_{10}: Preservation of type-uniformity of cut rules. All cut rules preserve type-uniformity.

In the context of proper multi-type calculi we say that a rule is *analytic* if it satisfies conditions C_1-C'_7 of the list above. Analytic rules can be added to a given proper multi-type calculus, and the resulting calculus enjoys cut elimination and subformula property.

We state the cut elimination metatheorem which we appeal to when establishing the cut elimination for the calculus introduced in Section 3.3.

Theorem 94. *Any calculus satisfying C_2, C'_2, C'_3, C_4, C'''_5, C''''_5, C'_6, C'_7, C'_8, C''_8, C_9 and C_{10} is cut-admissible. If also C_1 is satisfied, then the calculus enjoys the subformula property.*

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Chapter 4

First order logic properly displayed

In this chapter, which is based on a preliminary version of[1] *[1], we introduce a proper display calculus for first-order logic, of which we prove soundness, completeness, subformula property and cut elimination via a Belnap-style metatheorem. All inference rules are closed under uniform substitution and are without side conditions.*

[1] My specific contributions in this research have been the proof of the main results, and the draft of the first version of the paper.

4.1 Introduction

In the proof-theoretic literature, the treatment of quantifiers in first-order logic typically follows the lines of the original Gentzen's sequent calculi, and correspondingly, the introduction rules for first-order quantifiers are mostly additive:

$$\forall_L \frac{A[t/x], \Gamma \vdash \Delta}{\forall x A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A[y/x], \Delta}{\Gamma \vdash \forall x A, \Delta} \forall_R$$

$$\exists_L \frac{A[y/x], \Gamma \vdash \Delta}{\exists x A, \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash A[t/x], \Delta}{\Gamma \vdash \exists x A, \Delta} \exists_R$$

This is reflected also in the early display calculi literature, where quantifiers were not paired with structural connectives, up to Wansing [3, 33, 34], who introduces a display calculus for fragments of first-order logic based on the idea that quantifiers can be treated as modalities, and correspondingly defines introduction rules for quantifiers which involve their structural counterparts.

The key idea of this approach is that existential quantification can be viewed as a diamond-like operator of modal logic and universal quantification can be seen as a box-like operator [19, 25, 31, 32]. What underlies these similarities (observed and exploited in [19, 31, 33, 34]), and semantically supports the requirement that the resulting calculus enjoys the display property, is the order-theoretic notion of adjunction: indeed, the set-theoretic interpretations of the existential and universal quantification are the left and right adjoint of the inverse projection map respectively, and more generally, in categorical semantics, the left and right adjoint of the pullbacks along projections [20],[9, Chapter 15].

The display calculus of [34] contains rules with side conditions on the free and bound variables of formulas, similar to the ones of the original Gentzen sequent calculus. This implies that the introduction rules for first-order quantifiers are not closed under uniform substitution, i.e. the display calculus introduced in [33, 34] is not *proper* [33, Section 4.1].

The aim of this chapter is to overcome this difficulty and introduce a proper display calculus for first-order logic. The main idea is that, as was the case of other logical frameworks (cf. e.g. [6, 8, 14]), a suitable *multi-type* presentation can make it possible to encode the side conditions into analytic (structural) rules involving different types. Wansing's insight that quantifiers can be treated proof-theoretically as modal operators naturally embeds into the multi-type approach by simply regarding $(\forall x)$ and $(\exists x)$ as modal operators bridging different types (i.e. as *heterogeneous* operators). Following Lawvere [20–22] and Halmos [15], this requires to consider as many types as there are finite sets of free variables; that is, two first-order formulas have the same type if and only if they have exactly the same free variables.

Following these ideas, we introduce a proper multi-type display calculus for classical first-order logic, and show its soundness, completeness, cut elimination and subformula property. This chapter is organized as follows. In Section 4.2, we gather preliminary notions, definitions and notation for first-order logic. In Section 4.3, we recast the models of first-order logic in a framework amenable to support the semantics of the multi-type calculus for first-order logic, especially regarding the adjunction properties of quantification and substitution. From this semantic framework, we extract the defining conditions of the algebraic multi-type semantics for the calculus. In Section 4.4, we introduce the multi-type language of first-order logic. In Section 4.5, we introduce the display calculus for first-order classical logic. In Section 4.6 we prove its soundness, completeness, cut elimination and subformula property. In Section 4.7, we summarize the results of this chapter and collect further research directions.

4.2 Preliminaries on first-order logic

In this section we collect definitions, notation and basic facts about first-order logic.

Language. Let $\mathsf{Var} = \{v_1, \ldots, v_n, \ldots\}$ be a countable set of variables. A first-order logic $\mathcal{L}$ over Var consists of a set of relation symbols $(R_i)_{i\in I}$ each of finite arity n_i; a set of function symbols $(f_j)_{j\in J}$ each of finite arity n_j; and a set of constant symbols $(c_k)_{k\in K}$ (0-ary functions). The language of first-order logic is built up from *terms* defined recursively as follows:

$$\mathsf{Trm} \ni t ::= v_m \mid c_k \mid f_j(t, \ldots, t).$$

The formulas of first-order logic are defined recursively as follows:

$$\mathcal{L} \ni A ::= R_i(\bar{t}_{\bar{x}}) \mid \top \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \forall y A \mid \exists y A$$

For any term t and formula A we let $\mathsf{FV}(t) \in \mathcal{P}_\omega(\mathsf{Var})$ and $\mathsf{FV}(A) \in \mathcal{P}_\omega(\mathsf{Var})$ denote the sets of free variables of t and A respectively, recursively defined as follows:

$$\begin{array}{rcl}
\mathsf{FV}(v_m) & = & \{v_m\} \\
\mathsf{FV}(c_k) & = & \varnothing \\
\mathsf{FV}(f_j(t_1, \ldots, t_{n_j})) & = & \bigcup_{1\leq \ell \leq n_j} \mathsf{FV}(t_\ell) \\
\mathsf{FV}(\top) & = & \varnothing \\
\mathsf{FV}(\bot) & = & \varnothing \\
\mathsf{FV}(R_i(t_1, \ldots, t_{n_i})) & = & \bigcup_{1\leq \ell \leq n_i} \mathsf{FV}(t_\ell) \\
\mathsf{FV}(A \wedge B) & = & \mathsf{FV}(A) \cup \mathsf{FV}(B) \\
\mathsf{FV}(A \vee B) & = & \mathsf{FV}(A) \cup \mathsf{FV}(B) \\
\mathsf{FV}(A \rightarrow B) & = & \mathsf{FV}(A) \cup \mathsf{FV}(B) \\
\mathsf{FV}(\forall y A) & = & \mathsf{FV}(A) \setminus \{y\} \\
\mathsf{FV}(\exists y A) & = & \mathsf{FV}(A) \setminus \{y\}.
\end{array}$$

In what follows, we will often identify sequences of variables and terms with the set containing the elements of the sequence. Hence, given a sequence of terms $\bar{t}$ we define $\mathsf{FV}(\bar{t})$ as the union of the sets of free variables of the elements of the sequence. Finally, if $\bar{t}$ is a sequence of terms and s is a term, we let $s, \bar{t}$ denote the sequence obtained by prefixing s to $\bar{t}$. In what follows we will conflate notation and use $\mathcal{L}$ to denote signature, formulas and set of theorems.

Substitution. In our treatment of first-order logic we assume that substitution happens simultaneously for all free variables of a term or a formula. If $\overline{x} \supseteq \mathsf{FV}(s)$ and $\overline{x} \supseteq \mathsf{FV}(A)$ and for each $v_m \in \overline{x}$ we let t_{v_m} denote the corresponding term which will substituted for v_m, then $(\overline{t}_{\overline{x}}/\overline{x})$ denotes the simultaneous substitution of each v_m with t_{v_m} and we define $s(\overline{t}_{\overline{x}}/\overline{x})$ and $A(\overline{t}_{\overline{x}}/\overline{x})$ recursively as follows:

$$
\begin{array}{rcl}
v_n(\overline{t}_{\overline{x}}/\overline{x}) & = & t_{v_n} \\
c_k(\overline{t}_{\overline{x}}/\overline{x}) & = & c_k \\
f(s_1,\ldots,s_n)(\overline{t}/\overline{x}) & = & f(s_1(\overline{t}/\overline{x}),\ldots,s_n(\overline{t}/\overline{x})) \\
\top(\overline{t}_{\overline{x}}/\overline{x}) & = & \top \\
\bot(\overline{t}_{\overline{x}}/\overline{x}) & = & \bot \\
R(s_1,\ldots,s_n)(\overline{t}/\overline{x}) & = & R(s_1(\overline{t}/\overline{x}),\ldots,s_n(\overline{t}/\overline{x})) \\
(A \wedge B)(\overline{t}/\overline{x}) & = & A(\overline{t}/\overline{x}) \wedge B(\overline{t}/\overline{x}) \\
(A \vee B)(\overline{t}/\overline{x}) & = & A(\overline{t}/\overline{x}) \vee B(\overline{t}/\overline{x}) \\
(A \rightarrow B)(\overline{t}/\overline{x}) & = & A(\overline{t}/\overline{x}) \rightarrow B(\overline{t}/\overline{x}) \\
(\forall y A)(\overline{t}/\overline{x}) & = & \forall z(A(\overline{t}/\overline{x})), \quad z \notin \mathsf{FV}(\overline{t}) \cup \mathsf{FV}(\forall y A) \\
(\exists y A)(\overline{t}/\overline{x}) & = & \exists z(A(\overline{t}/\overline{x})), \quad z \notin \mathsf{FV}(\overline{t}) \cup \mathsf{FV}(\exists y A)
\end{array}
$$

Models. The models of a first-order logic $\mathcal{L}$ are tuples

$$M = (D, (R_i^D)_{i \in I}, (f_j^D)_{j \in J}, (c_k^D)_{k \in K})$$

where D is an arbitrary set (the domain of the model) and R_i^D, f_j^D, c_k^D are concrete n_i-ary relations over D, n_j-ary functions on D and elements of D respectively interpreting the symbols of the language in the model M. The interpretation of a formula in M is facilitated by variable interpretation maps $\iota : \mathsf{Var} \rightarrow D$ and is given recursively as follows:

$$
\begin{array}{rcl}
(v_m)^D & = & \iota(v_m) \\
(f_j(t_1,\ldots,t_{n_j}))^D & = & (f_j^D(t_1^D,\ldots,t_{n_j}^D)) \\
M,\iota \models \top & & \text{Always} \\
M,\iota \models \bot & & \text{Never} \\
M,\iota \models R_i(\overline{t}_{\overline{x}}) & \Longleftrightarrow & R_i^D(\overline{t}_{\overline{x}}^D) \\
M,\iota \models A \wedge B & \Longleftrightarrow & M,\iota \models A \text{ and } M,\iota \models A \\
M,\iota \models A \vee B & \Longleftrightarrow & M,\iota \models A \text{ or } M,\iota \models A \\
M,\iota \models A \rightarrow B & \Longleftrightarrow & M,\iota \models A \text{ implies } M,\iota \models A \\
M,\iota \models \forall y A & \Longleftrightarrow & M,\iota' \models A \text{ for all } \iota' \text{ such that } \iota'(x) = \iota(x) \text{ for all } x \neq y \\
M,\iota \models \forall y A & \Longleftrightarrow & M,\iota' \models A \text{ for some } \iota' \text{ such that } \iota'(x) = \iota(x) \text{ for all } x \neq y.
\end{array}
$$

Axiomatic system. The following deductive system, denoted with $\vdash_{\mathrm{FO}}$, is sound and complete w.r.t. the models mentioned above:

1. Propositional tautologies;
2. $\forall x(B \rightarrow C) \rightarrow (\forall x B \rightarrow \forall x C)$;
3. $B \rightarrow \forall x B$, where $x \notin \mathsf{FV}(B)$;
4. $\forall x B \rightarrow B(t/x)$;

5. All universal closures of instances of the above;

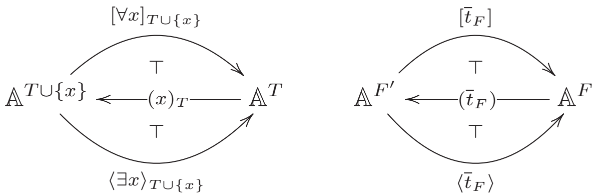
6. Modus ponens.

4.3 Semantic analysis

Let us fix a countably infinite set of variables $\mathsf{Var} = \{v_0, v_1, \ldots\}$. As discussed in the introduction, in order to develop a multi-type environment in which Wansing's insights can be carried out, we need as many types as there are finite sets of variables. Therefore, types F correspond to elements of $\mathcal{P}_\omega(\mathsf{Var})$. On the semantic side, each such type interprets formulas A such that $\mathsf{FV}(A) = F$, and hence, being closed under all propositional connectives, is naturally endowed with the structure of a Boolean (resp. Heyting) algebra. Thus, given a first-order logic $\mathcal{L}$ over Var and letting $\mathsf{Var}_x := \mathsf{Var}\backslash\{x\}$ for every $x \in \mathsf{Var}$, the semantic environment that supports the multi-type presentation of $\mathcal{L}$ is based on structures $\mathbb{H} = (\mathcal{A}, \mathcal{Q}, \mathcal{S})$, where

- $\mathcal{A} = \{\mathbb{A}_F \mid F \in \mathcal{P}_\omega(\mathsf{Var})\}$;
- $\mathcal{Q} = \{(x)_T, [\forall x]_F, \langle\exists x\rangle_F, \mid x \in F \in \mathcal{P}_\omega(\mathsf{Var}) \text{ and } T \in \mathcal{P}_\omega(\mathsf{Var}_x)\}$;
- $\mathcal{S} = \{(\bar{t}_F), [\bar{t}_F], \langle\bar{t}_F\rangle \mid F \in \mathcal{P}_\omega(\mathsf{Var}) \text{ and } t_x \in \mathsf{Trm}\}$,

where every $\mathbb{A}_F$ is a Boolean (resp. Heyting) algebra and for every $F \in \mathcal{P}_\omega(\mathsf{Var})$,



The adjunctions illustrated above justify the soundness of the following rules:

$$\frac{[\![\langle\mathsf{Q}x\rangle]\!]X \vdash_{F\backslash\{x\}} Y}{X \vdash_{F\cup\{x\}} (\!(x)\!)Y} \qquad \frac{Y \vdash_{F\backslash\{x\}} [\![\langle\mathsf{Q}x\rangle]\!]X}{(\!(x)\!)Y \vdash_{F\cup\{x\}} X}$$

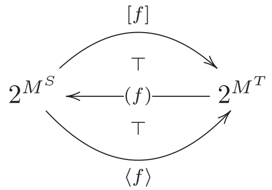
$$\frac{(\!(\bar{t}_F)\!)X \vdash_{F'} Y}{X \vdash_F [\![\langle\bar{t}_F\rangle]\!]Y} \qquad \frac{Y \vdash_{F'} (\!(\bar{t}_F)\!)X}{[\![\langle\bar{t}_F\rangle]\!]Y \vdash_F X}$$

In order to identify the relevant properties which we need to impose on the multi-type environment outlined above, in what follows, we establish a systematic connection between the standard $\mathcal{L}$-models and a subclass of the structures described above.

Let M be a model of some fixed but arbitrary first-order language $\mathcal{L}$. Let S, T be finite subsets of variables and let M^S and M^T denote the sets of functions into M with domains S and T respectively. If $f : M^S \to M^T$ is a map, then $\mathsf{graph}(f)$ induces a complete Boolean algebra homomorphism[2] $(f) : \mathcal{P}(M^T) \to \mathcal{P}(M^S)$, between the Boolean algebras $\mathcal{P}(M^T)$ and $\mathcal{P}(M^S)$ of the potential interpretations of T-*predicates*[3] and the potential interpretations of S-predicates, defined as follows: for any $B \subseteq M^T$,

$$(f)(B) = f^{-1}[B] = \{m \in M^S \mid f(m) \in B\}.$$

Hence (f) has both a right-adjoint $[f]$ and a left-adjoint $\langle f \rangle$, as in the following picture:



which are defined by the following assignments: for any $A \subseteq M^S$,

$$[f](A) = \{n \in M^T \mid \forall m \in M^S \,.\, f(m) = n \Rightarrow m \in A\}$$
$$\langle f \rangle(A) = \{n \in M^T \mid \exists m \in M^S \,.\, f(m) = n \;\&\; m \in A\}.$$

Notice in particular that if $A = \{m\} \in J^\infty(\mathcal{P}(M^S))$ for some $m : S \to M$, then $\langle f \rangle(\{m\}) = \{f(m)\}$.

There are two types of relevant instances of maps $f : M^S \to M^T$. The first is given for $S := T \cup \{x\}$ for some $x \notin T$, and $f := \pi_x : M^{T \cup \{x\}} \to M^T$ is the canonical projection along T. In this case, $(\pi_x)(A)$ is the *cylindrification* (cf. [16]) of A over the x-coordinate, and the adjoint maps $[\pi_x]$, $\langle \pi_x \rangle$ are the semantic counterparts of the usual first-order logic quantifiers $\forall x$ and $\exists x$. The second type of instance is given for f arising from a simultaneous substitution $(t_i/x_i)_{i=1}^n$. In this case, $S := \bigcup_{i=1}^n \mathsf{FV}(t_i)$ and $T := \{x_1, \ldots, x_n\}$, and $f := \bar{t}_{\overline{x}} : M^S \to M^T$ is defined by the following assignment: for any $m \in M^S$,

$$\bar{t}_{\overline{x}}(m) = (t_1(m), \ldots, t_n(m)).$$

In this case, the corresponding $(\bar{t}_{\overline{x}})$ is such that, if $B \subseteq M^T$ is the semantic interpretation of a T-predicate P, then $(\bar{t}_{\overline{x}})(B)$ is the semantic interpretation of the S-predicate $(t_i/x_i)_{i=1}^n P$ resulting from applying the simultaneous substitution $(t_i/x_i)_{i=1}^n$ to P. That is, in this case, $(\bar{t}_{\overline{x}})$ is the semantic interpretation of the substitution from which $\bar{t}_{\overline{x}}$ arises.

Proposition 95. *For every model M for $\mathcal{L}$ and for every $f : M^S \to M^T$ the following inclusions hold:*

$$[f](A) \cup [f](B) \subseteq [f](A \cup B) \qquad \langle f \rangle(A \cap B) \subseteq \langle f \rangle(A) \cap \langle f \rangle(B)$$
$$\langle f \rangle(A) \Rightarrow [f](B) \subseteq [f](A \Rightarrow B) \qquad \langle f \rangle(A \succ\!\!- B) \subseteq [f](A) \succ\!\!- \langle f \rangle(B)$$

[2] In general, every relation $R \subseteq X \times Y$ induces maps $\langle R \rangle, [R] : \mathcal{P}(Y) \to \mathcal{P}(X)$ respectively defined by the assignments $\langle R \rangle A = R^{-1}[A] = \{x \in X \mid \exists y (R(x,y)) \;\&\; y \in A\}$ and $[R]A = (R^{-1}[A^c])^c = \{x \in X \mid \forall y (R(x,y) \Rightarrow y \in A)\}$, which have a right adjoint $[R^{-1}]$ and a left adjoint $\langle R^{-1} \rangle$ respectively. When $R = \mathsf{graph}(f)$ for some function $f : X \to Y$ then $\langle R \rangle = [R]$ and we denote it (f), and abbreviate $\langle R^{-1} \rangle$ as $\langle f \rangle$ and $[R^{-1}]$ as $[f]$.

[3] By T-predicate we mean a predicate P such that $\mathsf{FV}(P) = T$.

Proof. The inclusions in the first row are a consequence of the monotonicity of $[f]$ and $\langle f\rangle$. As to proving the left hand inclusion of the second row, by adjunction and residuation it is enough to show that

$$A \cap (f)(\langle f\rangle(A) \Rightarrow [f](B)) \subseteq B.$$

Since $A \subseteq (f)(\langle f\rangle(A))$ it is enough to show that

$$(f)(\langle f\rangle(A)) \cap (f)(\langle f\rangle(A) \Rightarrow [f](B)) \subseteq B.$$

Since (f) is meet-preserving this is equivalent to

$$(f)(\langle f\rangle(A) \cap (\langle f\rangle(A) \Rightarrow [f](B))) \subseteq B$$

which, by adjunction, is equivalent to

$$\langle f\rangle(A) \cap (\langle f\rangle(A) \Rightarrow [f](B)) \subseteq [f](B)$$

which holds by the definition of implication (also when the implication is defined intuitionistically). The proof of the right hand side inclusion of the second row is dual and uses the join-preservation of (f). □

Summing up, we are justified in introducing the following

Definition 96. For every model M for $\mathcal{L}$, the *heterogeneous $\mathcal{L}$-model* associated with M is the tuple $(\mathbb{H}_M, V)$ such that $\mathbb{H}_M = (\mathcal{A}_M, \mathcal{Q}_M, \mathcal{S}_M)$ is defined as follows[4]:

- $\mathcal{A}_M = \{\mathcal{P}(M^F) \mid F \in \mathcal{P}_\omega(\mathsf{Var})\}$;
- $\mathcal{Q}_M = \{(\pi_x), [\pi_x], \langle\pi_x\rangle, \mid F \in \mathcal{P}_\omega(\mathsf{Var}_x) \text{ and } \pi_x : M^{F\cup\{x\}} \to M^F\}$;
- $\mathcal{S}_M = \{(\bar{t}_F), [\bar{t}_F], \langle\bar{t}_F\rangle \mid F \in \mathcal{P}_\omega(\mathsf{Var}) \text{ and } \bar{t}_F \in \mathsf{Trm}^F\}$,

and V is such that $V(A) \in \mathcal{P}(M^F)$ for every atomic formula A of $\mathcal{L}$ with $\mathsf{FV}(A) = F$.

Proposition 97. *For every $\mathcal{L}$-model M, the heterogeneous $\mathcal{L}$-model associated with M satisfies the following identities:*

$$\begin{array}{rclrcll}
(f)(A\cap B) & = & (f)(A)\cap(f)(B) & (f)(A\cup B) & = & (f)(A)\cup(f)(B) & (a)\\
(f)(A\Rightarrow B) & = & (f)(A)\Rightarrow(f)(B) & (f)(A \prec B) & = & (f)(A)\prec(f)(B) & (b)\\
(\pi_x)[\pi_y](A) & = & [\pi_y](\pi_x)(A) & (\pi_x)\langle\pi_y\rangle(A) & = & \langle\pi_y\rangle(\pi_x)(A) & (c)\\
(\bar{t}_F)[\pi_y](A) & = & [\pi_z](z_y,\bar{t}_F)(A) & (\bar{t}_F)\langle\pi_y\rangle(A) & = & \langle\pi_z\rangle(z_y,\bar{t}_F)(A) & (d)\\
(\pi_x)(\pi_y)(A) & = & (\pi_y)(\pi_x)(A) & (\bar{t}_F)(\bar{s}_T)(A) & = & (s(\bar{t}/\bar{x})_T)(A) & (e)\\
(s_y,\bar{t}_F)(\pi_y)(A) & = & (\pi_{z_1})\cdots(\pi_{z_k})(\bar{t}_F)(A), & & & & (f)
\end{array}$$

where $x \neq y$, $z \notin \mathsf{FV}(\bar{t}_F)\cup F$ and $\{z_1,\ldots,z_k\} = \mathsf{FV}(s)\setminus\mathsf{FV}(\bar{t}_F)$ and $f \in \{\pi_x, \bar{t}_F\}$.

[4] In category-theoretic parlance, this can be seen as a product-preserving functor from the category of terms to the category of Boolean algebras.

Proof. The identities in *(a)* and *(b)* immediately follow from (π_x) being a Boolean algebra homomorphism. Notice that the remaining identities are all Sahlqvist, so, in what follows, we will show that their correspondents computed with ALBA [4] hold in the model.

As to the identities in *(c)*, they both have the same ALBA-reduct, namely

$$\langle \pi_x \rangle (\pi_y) \mathbf{j} = (\pi_y) \langle \pi_x \rangle \mathbf{j},$$

where $\mathbf{j} \in J^\infty(\mathcal{P}(M^{F \cup \{x\}}))$, that is, $\mathbf{j}$ can be identified with a map $\mathbf{j} : F \cup \{x\} \to M$. As discussed above, (π_y) takes any such $\mathbf{j}$ in the appropriate type to the y-cylindrification of $\mathbf{j}$, and $\langle \pi_x \rangle$ forgets the x-coordinate, then it is clear that these two operations commute when $x \neq y$.

The identities in *(d)* have the following ALBA-reducts respectively:

$$\langle z_y, \bar{t}_F \rangle (\pi_z) \mathbf{j} = (\pi_y) \langle \bar{t}_F \rangle \mathbf{j} \quad \text{and} \quad (\bar{t}_F) \langle \pi_y \rangle \mathbf{j} = \langle \pi_z \rangle (z_y, \bar{t}_F) \mathbf{j},$$

where $\mathbf{j} \in J^\infty(\mathcal{P}(M^{\mathsf{FV}(\bar{t}_F)}))$, that is, we can write $\mathbf{j} : \mathsf{FV}(\bar{t}_F) \to M$. Notice that (π_z) takes $\mathbf{j}$ to its z-cylindrification in $\mathcal{P}(M^{\mathsf{FV}(\bar{t}_F) \cup \{z\}})$, and $\langle z_y, \bar{t}_F \rangle$ transforms any element $\mathbf{j}'$ of this cylindrification by sending any coordinate different from z to the corresponding $t(\mathbf{j}')$ and renaming the element in the z-coordinate by declaring it the y-coordinate; moreover $\langle \bar{t}_F \rangle$ transforms $\mathbf{j}$ into $\overline{t(\mathbf{j})}_F$, and (π_y) cylindrifies it by adding the y-coordinate. It is clear that these two compositions yield the same outcome. The second identity is argued analogously.

The left-hand identity in *(e)* has the following ALBA-reduct:

$$\langle \pi_x \rangle \langle \pi_y \rangle \mathbf{j} = \langle \pi_y \rangle \langle \pi_x \rangle \mathbf{j}.$$

Recalling that, with a bit of notational abuse, $\langle f \rangle (\mathbf{j}) = f(\mathbf{j})$, the assumption that $x \neq y$ guarantees that the order in which the projections are applied does not matter.

The right-hand identity in *(e)* has the following ALBA-reduct:

$$\langle \overline{s(\bar{t}/\bar{x})}_T \rangle \mathbf{j} = \langle \bar{s}_T \rangle \langle \bar{t}_F \rangle \mathbf{j}.$$

This identity expresses that applying $\bar{t}_F$ to $\mathbf{j}$ and then $\bar{s}_T$ to $\overline{t(\mathbf{j})}_F$ is the same as applying the compounded substitution map $\overline{s(\bar{t}/\bar{x})}_T$ to $\mathbf{j}$, which is trivially true.

Finally, the identity *(f)* has the following ALBA-reduct:

$$\langle \bar{t}_F \rangle \langle \pi_{z_n} \rangle \cdots \langle \pi_{z_n} \rangle \mathbf{j} = \langle \pi_y \rangle \langle s_y, \bar{t}_F \rangle \mathbf{j}.$$

This identity expresses that applying $\bar{t}_F$ after having forgotten the coordinates $z_1, \ldots, z_n$ is the same as applying $\bar{t}_F$ in parallel with any substitution s_y with $\{z_1, \ldots, z_n\} = \mathsf{FV}(s) \setminus \mathsf{FV}(\bar{t}_F)$ and $y \notin F$, and then forgetting the y-coordinate. This is again immediately true. □

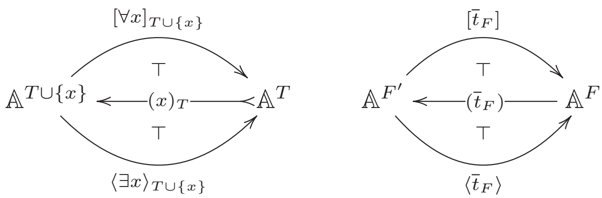
Remark 98. Notice that while the identities in *(a)* and *(b)* in the proposition above hold for any function f, the remaining identities hold true because of the particular functions involved, and hence by choosing different functions, corresponding to different notions of quantifiers and substitutions, these properties might change.

Proposition 97 provides the guidelines for completing the definition of heterogeneous $\mathcal{L}$-algebras. Namely, we are going to use the identities *(a)-(f)* as parts of the following

Definition 99. For any first-order logic $\mathcal{L}$ over a denumerable set of individual variables Var, an *heterogeneous $\mathcal{L}$-algebra* is a tuple $\mathbb{H} = (\mathcal{A}, \mathcal{Q}, \mathcal{S})$, such that

- $\mathcal{A} = \{\mathbb{A}_F \mid F \in \mathcal{P}_\omega(\mathsf{Var})\}$;
- $\mathcal{Q} = \{(x)_T, [\forall x]_F, \langle \exists x \rangle_F, \mid x \in F \in \mathcal{P}_\omega(\mathsf{Var}) \text{ and } T \in \mathcal{P}_\omega(\mathsf{Var}_x)\}$;
- $\mathcal{S} = \{(\bar{t}_F), [\bar{t}_F], \langle \bar{t}_F \rangle \mid F \in \mathcal{P}_\omega(\mathsf{Var}) \text{ and } t_x \in \mathsf{Trm}\}$,

where for every $F \in \mathcal{P}_\omega(\mathsf{Var})$, $\mathbb{A}_F$ is a Boolean algebra and



such that $(x)_T$ is an order embedding and the following equations hold (we omit mentioning the types):

$$
\begin{array}{rclrcll}
(x)(a \wedge b) & = & (x)(a) \wedge (x)(b) & (x)(a \vee b) & = & (x)(a) \vee (x)(b) & \textit{(a)} \\
(x)(a \rightarrow b) & = & (x)(a) \rightarrow (x)(b) & (x)(a \prec b) & = & (x)(a) \prec (x)(b) & \textit{(b)} \\
(x)[\forall y](a) & = & [\forall y](x)(a) & (x)\langle \exists y \rangle(a) & = & \langle \exists y \rangle(x)(a) & \textit{(c)} \\
(\bar{t}_F)[\forall y](a) & = & [\forall z](z_y, \bar{t}_F)(a) & (\bar{t}_F)\langle \exists y \rangle(a) & = & [\forall z](z_y, \bar{t}_F)(a) & \textit{(d)} \\
(x)(y)(a) & = & (y)(x)(a) & (\bar{t}_F)(\bar{s}_T)(a) & = & (\overline{s(\bar{t}/\bar{x})}_T)(a) & \textit{(e)} \\
(s_y, \bar{t}_F)(y)(a) & = & (z_1) \cdots (z_k)(\bar{t}_F)(a), & & & & \textit{(f)}
\end{array}
$$

where $x \neq y$, $z \notin \mathsf{FV}(\bar{t}_F) \cup F$ and $\{z_1, \ldots, z_k\} = \mathsf{FV}(s) \setminus \mathsf{FV}(\bar{t}_F)$. An heterogeneous $\mathcal{L}$-algebra is *perfect* if every $\mathbb{A}_F$ is.[5]

Clearly, even the class of perfect heterogeneous $\mathcal{L}$-algebras is much larger than the class of those arising from $\mathcal{L}$-models. However, as we will discuss in the next section, there is a way in which $\mathcal{L}$ is sound with respect to this larger class of models.

4.4 Multi-type presentation of first-order logic

Let $\mathcal{L}$ be a first-order language over a denumerable set of individual variables Var. The notion of heterogeneous $\mathcal{L}$-algebras (cf. Definition 99) naturally comes with the following *multi-type propositional language* $\mathcal{L}_{\mathrm{MT}}$ canonically interpreted in it. Types in this language bijectively correspond to elements $F \in \mathcal{P}_\omega(\mathsf{Var})$. The sets $\mathcal{L}_F$ of F-formulas, for each such F, are defined by simultaneous induction as follows:

$$
\begin{array}{l}
\mathcal{L}_F \ni A^F ::= R(\bar{t}_{\overline{x}}) \mid \top_F \mid \bot_F \mid A^F \wedge A^F \mid A^F \vee A^F \mid A^F \rightarrow A^F \mid A^F \prec A^F \\
\qquad\qquad \mid [\forall y]A^{F\cup\{y\}} \mid \langle \exists y \rangle A^{F\cup\{y\}} \mid (x)A^{F\setminus\{x\}} \mid (\bar{t}_{F'})A^{F'}
\end{array}
$$

[5] A Boolean algebra is *perfect* if it is complete and completely join-generated by its atoms. Equivalently, perfect Boolean algebras are isomorphic to powerset algebras.

where $\mathsf{FV}(\bar{t}_{F'}) = F$, $y \notin F$ and $x \in F$ and R is any relation symbol of $\mathcal{L}$, and $\bar{t}_{F'} : F' \to \mathsf{Trm}$. The symbol $(\bar{t}_{F'})$ denotes the simultaneous substitution of t_{v_m} for v_m for every $v_m \in F'$.

Definition 100. A *heterogeneous algebraic $\mathcal{L}$-model* is a tuple $(\mathbb{H}, V)$ such that $\mathbb{H}$ is a heterogeneous $\mathcal{L}$-algebra and V maps atomic propositions $R(\bar{t}_{F'})$ in $\mathcal{L}_F$ to elements of $\mathbb{A}_F$ so that for every $\bar{t}_{\overline{x}}$,

$$V(R(\bar{t}_{\overline{x}})) = (\bar{t}_{\overline{x}})V(R(\overline{x})).$$

The definition of the interpretation $(\mathbb{H}, V) \models A^F$ of $\mathcal{L}_{\mathrm{MT}}$-formulas into heterogeneous $\mathcal{L}$-algebras straightforwardly generalizes the definition of the interpretation of propositional languages in algebras of compatible signature.

The discussion of the previous section also motivates the definition of the following translation $(\cdot)^\tau : \mathcal{L}_{\mathrm{MT}} \to \mathcal{L}$:

$$\begin{array}{rcl@{\qquad}rcl}
R(\bar{t}_{F'})^\tau & = & R(\bar{t}_{F'}) & & & \\
\top^\tau & = & \top & \bot^\tau & = & \bot \\
(A \wedge B)^\tau & = & A^\tau \wedge B^\tau & (A \vee B)^\tau & = & A^\tau \vee B^\tau \\
(A \to B)^\tau & = & A^\tau \to B^\tau & (A \succ\!\!- B)^\tau & = & A^\tau \succ\!\!- B^\tau \\
([\forall y]A)^\tau & = & \forall y A^\tau & (\langle\exists y\rangle A)^\tau & = & \exists y A^\tau \\
((x)A)^\tau & = & A^\tau & ((\bar{t}_{F'})A)^\tau & = & A^\tau(\bar{t}_{F'}/\overline{x})
\end{array}$$

Proposition 101. *For every $\mathcal{L}$-model M and every $A \in \mathcal{L}_{\mathrm{MT}}$,*

$$M \models A^\tau \quad \textit{iff} \quad (\mathbb{H}_M, V) \models A.$$

Proof. By straightforward induction on A. □

The identities *(a)-(f)* of Proposition 97 have the following syntactic counterparts in the language $\mathcal{L}_{\mathrm{MT}}$:

$$\begin{array}{rcl@{\qquad}rcll}
(x)(A \wedge B) & \dashv\vdash & (x)A \wedge (x)B & (x)(A \vee B) & \dashv\vdash & (x)A \vee (x)B & \text{(a)} \\
(x)(A \to B) & \dashv\vdash & (x)A \to (x)B & (x)(A -\!\!\prec B) & \dashv\vdash & (x)A -\!\!\prec (x)B & \text{(b)} \\
(x)[\forall y]A & \dashv\vdash & [\forall y](x)A & (x)\langle\exists y\rangle A & \dashv\vdash & \langle\exists y\rangle(x)A & \text{(c)} \\
(\bar{t}_F)[\forall y]A & \dashv\vdash & [\forall z](z_y, \bar{t}_F)A & (\bar{t}_F)\langle\exists y\rangle A & \dashv\vdash & \langle\exists z\rangle(z_y, \bar{t}_F)A & \text{(d)} \\
(x)(y)A & \dashv\vdash & (y)(x)A & (\bar{t}_F)(\bar{s}_T)A & \dashv\vdash & (\overline{s(\bar{t}/\overline{x})}_T)A & \text{(e)} \\
(s_y, \bar{t}_F)(y)A & \dashv\vdash & (z_1)\cdots(z_k)(\bar{t}_F)A. & & & & \text{(f)}
\end{array}$$

These inequalities are all analytic inductive (cf. [13], see also Definition 55), and hence give rise to analytic structural rules. We will exploit this fact in the next section when introducing the rules of the calculus.

4.5 Multi-type calculus for first-order logic

For any first-order language $\mathcal{L}$ (cf. Section 4.2) the structural and operational connectives of the proper multi-type display calculi D.FO and D.FO* are the following:

- Logical and structural homogeneous connectives for any type F:

Structural symbols	I_F		$;$		$>$		$<$	
Operational symbols	$\top$	$\bot$	$\wedge$	$\vee$	$\succ$	$\rightarrow$	$\prec$	$\leftarrow$

- Logical and structural heterogeneous for each $x, y \in \mathsf{Var}$:

	$\mathcal{L}_{F\setminus\{x\}} \to \mathcal{L}_F$		$\mathcal{L}_{F\cup\{y\}} \to \mathcal{L}_F$		$\mathcal{L}_{F'} \to \mathcal{L}_F$		$\mathcal{L}_F \to \mathcal{L}_{F'}$	
Structural symbols	$(\!(x)\!)$		$[\langle \mathsf{Q}y \rangle]$		$(\!(\bar{t}_{F'})\!)$		$[\langle \bar{t}_{F'} \rangle]$	
Operational symbols	(x)	(x)	$\langle \exists y \rangle$	$[\forall y]$	$(\bar{t}_{F'})$	$(\bar{t}_{F'})$	$\langle \bar{t}_{F'} \rangle$	$[\bar{t}_{F'}]$

Some operational symbols above appear against a gray background to signify that, unlike their associated structural symbols, they occur only in the language and calculus for $\mathrm{D.FO}^*$.

Definition 102. The display calculi $\mathrm{D.FO}^*$ and $\mathrm{D.FO}$ consist of the following display postulates, structural rules, and operational rules:

1. Identity and cut:

$$R(\bar{t}_{F'}) \vdash R(\bar{t}_{F'}) \qquad \frac{X \vdash_F A \qquad A \vdash_F Y}{X \vdash_F Y}$$

2. Substitution axioms: This is a class of axioms which is generated by the following axioms and closed under finite applications of Cut and display postulates.

$$(\!(\bar{t}_1)\!) \cdots (\!(\bar{t}_k)\!) R(\bar{t}_{\overline{x}}) \vdash (\!(\bar{s}_1)\!) \cdots (\!(\bar{s}_m)\!) R(\bar{s}_{\overline{x}})$$

where the sequents are well-typed and[6]

$$((\bar{t}_1) \cdots (\bar{t}_k) R(\bar{t}_{\overline{x}}))^\tau = ((\bar{s}_1) \cdots (\bar{s}_m) R(\bar{s}_{\overline{x}}))^\tau.$$

3. Display postulates for homogeneous connectives:

$$\frac{X \,; Y \vdash_F Z}{Y \vdash_F X > Z} \qquad \frac{Z \vdash_F X \,; Y}{X > Z \vdash_F Y}$$

$$\frac{X \,; Y \vdash_F Z}{X \vdash_F Z < Y} \qquad \frac{Z \vdash_F X \,; Y}{Z < Y \vdash_F X}$$

4. Display postulates for quantifiers:

$$\frac{[\langle \mathsf{Q}x \rangle] X \vdash_{F\setminus\{x\}} Y}{X \vdash_{F\cup\{x\}} (\!(x)\!) Y} \qquad \frac{Y \vdash_{F\setminus\{x\}} [\langle \mathsf{Q}x \rangle] X}{(\!(x)\!) Y \vdash_{F\cup\{x\}} X}$$

5. Display postulates for substitutions:

[6] We will discuss these axioms in the conclusions.

$$\frac{(\!(\bar{t}_F)\!) X \vdash_{F'} Y}{X \vdash_F [\![\langle \bar{t}_F \rangle]\!] Y} \qquad \frac{Y \vdash_{F'} (\!(\bar{t}_F)\!) X}{[\![\langle \bar{t}_F \rangle]\!] Y \vdash_F X}$$

6. Additional adjunction-related rules

$$\frac{(\!(x)\!)[\![\langle \mathtt{Q}x \rangle]\!] X \vdash_F Y}{X \vdash_F Y} \qquad \frac{(\!(\bar{t}_{F'})\!)[\![\langle \bar{t}_{F'} \rangle]\!] X \vdash_F Y}{X \vdash_F Y}$$

7. Necessitation for quantifiers:

$$\frac{\mathrm{I}_{F\cup\{x\}} \vdash_{F\cup\{x\}} X}{(\!(x)\!)\mathrm{I}_{F\setminus\{x\}} \vdash_{F\cup\{x\}} X} \qquad \frac{X \vdash_{F\cup\{x\}} \mathrm{I}_{F\cup\{x\}}}{X \vdash_{F\cup\{x\}} (\!(x)\!)\mathrm{I}_{F\setminus\{x\}}}$$

8. Necessitation for substitution where $\mathsf{FV}(\bar{t}_{F'}) = F$:

$$\frac{\mathrm{I}_F \vdash_F X}{(\!(\bar{t}_{F'})\!)\mathrm{I}_{F'} \vdash_F X} \qquad \frac{X \vdash_F \mathrm{I}_F}{X \vdash_F (\!(\bar{t}_{F'})\!)\mathrm{I}_{F'}}$$

9. Structural rules encoding the behaviour of conjunction and disjunction:

$$\mathrm{I}_L\ \frac{X \vdash_F Y}{\mathrm{I} \,;\, X \vdash_F Y} \qquad \frac{Y \vdash_F X}{Y \vdash_F X \,;\, \mathrm{I}}\ \mathrm{I}_R$$

$$E_L\ \frac{Y \,;\, X \vdash_F Z}{X \,;\, Y \vdash_F Z} \qquad \frac{Z \vdash_F X \,;\, Y}{Z \vdash_F Y \,;\, X}\ E_R$$

$$W_L\ \frac{Y \vdash_F Z}{X \,;\, Y \vdash_F Z} \qquad \frac{Z \vdash_F Y}{Z \vdash_F Y \,;\, X}\ W_R$$

$$C_L\ \frac{X \,;\, X \vdash_F Y}{X \vdash_F Y} \qquad \frac{Y \vdash_F X \,;\, X}{Y \vdash_F X}\ C_R$$

$$A_L\ \frac{X \,;\, (Y \,;\, Z) \vdash_F W}{(X \,;\, Y) \,;\, Z \vdash_F W} \qquad \frac{W \vdash_F (Z \,;\, Y) \,;\, X}{W \vdash_F Z \,;\, (Y \,;\, X)}\ A_R$$

10. Introduction rules for the propositional connectives:

$$\bot_L \frac{}{\bot \vdash \mathrm{I}} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \bot} \bot_R$$

$$\top_L \frac{\mathrm{I} \vdash X}{\top \vdash X} \qquad \frac{}{\mathrm{I} \vdash \top} \top_R$$

$$\wedge_L \frac{A\,; B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X\,; Y \vdash A \wedge B} \wedge_R$$

$$\vee_L \frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X\,; Y} \qquad \frac{X \vdash A\,; B}{X \vdash A \vee B} \vee_R$$

$$\rightarrow_L \frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B} \rightarrow_R$$

$$(\succ_L) \frac{A > B \vdash Z}{A \succ B \vdash Z} \qquad \frac{A \vdash X \qquad Y \vdash B}{X > Y \vdash A \succ B} (\succ_R)$$

$$(\leftarrow_L) \frac{X \vdash A \qquad B \vdash Y}{A \leftarrow B \vdash X < Y} \qquad \frac{X \vdash A < B}{X \vdash A \leftarrow B} (\leftarrow_R)$$

$$(\prec_L) \frac{A < B \vdash Z}{A \prec B \vdash Z} \qquad \frac{A \vdash X \qquad Y \vdash B}{X < Y \vdash A \prec B} (\prec_R)$$

We omit the type since the rules do not move to different type. In the presence of the exchange rules E_L and E_R, the structural connective $<$ and the corresponding operational connectives $\prec$ and $\leftarrow$ are redundant.

11. Grishin rules for classical logic:

$$Gri \frac{X > (Y, Z) \vdash W}{(X > Y), Z \vdash W} \qquad \frac{W \vdash X > (Y, Z)}{W \vdash (X > Y), Z} Gri$$

12. Introduction rules for the heterogeneous connectives of D.FO:

$$\langle \exists x \rangle_L \frac{[\!\langle \mathsf{Q}x \rangle\!] A \vdash_F X}{\langle \exists x \rangle A \vdash_F X} \qquad \frac{X \vdash_F A}{[\!\langle \mathsf{Q}x \rangle\!] X \vdash_{F \setminus \{x\}} \langle \exists x \rangle A} \langle \exists x \rangle_R$$

$$[\forall x]_L \frac{A \vdash_F X}{[\forall x] A \vdash_{F \setminus \{x\}} [\!\langle \mathsf{Q}x \rangle\!] A} \qquad \frac{X \vdash_F [\!\langle \mathsf{Q}x \rangle\!] A}{X \vdash_F [\forall x] A} [\forall x]_R$$

$$(x)_L \frac{(\!(x)\!) A \vdash_F X}{(x) A \vdash_F X} \qquad \frac{X \vdash_F (\!(x)\!) A}{X \vdash_F (x) A} (x)_R$$

$$(\bar{t}_{F'})_L \frac{(\!(\bar{t}_{F'})\!) A \vdash_F X}{(\bar{t}_{F'}) A \vdash_F X} \qquad \frac{X \vdash_F (\!(\bar{t}_{F'})\!) A}{X \vdash_F (\bar{t}_{F'}) A} (\bar{t}_{F'})_R$$

13. Introduction rules for the heterogeneous connectives of D.FO*:

4

$$\langle \bar t_{F'} \rangle_L\ \frac{[\langle \bar t_{F'} \rangle] A \vdash_{F'} X}{\langle \bar t_{F'} \rangle A \vdash_{F'} X} \qquad \frac{X \vdash_F A}{[\langle \bar t_{F'} \rangle] X \vdash_{F'} \langle \bar t_{F'} \rangle A}\ \langle \bar t_{F'} \rangle_R$$

$$[\bar t_{F'}]_L\ \frac{A \vdash_F X}{[\bar t_{F'}] A \vdash_{F'} [\langle x \rangle] A} \qquad \frac{X \vdash_{F'} [\langle \bar t_{F'} \rangle] A}{X \vdash_{F'} [\bar t_{F'}] A}\ [\bar t_{F'}]_R$$

14. Monotonicity and order embedding rules

$$(\!(x)\!)_M\ \frac{X \vdash_{F \setminus \{x\}} Y}{(\!(x)\!) X \vdash_{F \cup \{x\}} (\!(x)\!) Y} \qquad (\!(\bar t_{F'})\!)_M\ \frac{X \vdash_{F'} Y}{(\!(\bar t_{F'})\!) X \vdash_F (\!(\bar t_{F'})\!) Y}$$

15. Interaction between homogeneous and heterogeneous connectives:

$$((\!(x)\!), >)_L\ \frac{(\!(x)\!) X > (\!(x)\!) Y \vdash Z}{(\!(x)\!)(X > Y) \vdash Z} \qquad \frac{Z \vdash (\!(x)\!) X > (\!(x)\!) Y}{Z \vdash (\!(x)\!)(X > Y)}\ ((\!(x)\!), >)_R$$

$$((\!(x)\!), <)_L\ \frac{(\!(x)\!) X < (\!(x)\!) Y \vdash Z}{(\!(x)\!)(X < Y) \vdash Z} \qquad \frac{Z \vdash (\!(x)\!) X < (\!(x)\!) Y}{Z \vdash (\!(x)\!)(X < Y)}\ ((\!(x)\!), <)_R$$

$$((\!(x)\!), ;)_L\ \frac{(\!(x)\!) X \,;\, (\!(x)\!) Y \vdash Z}{(\!(x)\!)(X \,;\, Y) \vdash Z} \qquad \frac{Z \vdash (\!(x)\!) X \,;\, (\!(x)\!) Y}{Z \vdash (\!(x)\!)(X \,;\, Y)}\ ((\!(x)\!), ;)_R$$

$$((\!(\bar t_F)\!), ;)_L\ \frac{(\!(\bar t_F)\!) X ; (\!(\bar t_F)\!) Y \vdash Z}{(\!(\bar t_F)\!)(X ; Y) \vdash Z} \qquad \frac{Z \vdash (\!(\bar t_F)\!) X ; (\!(\bar t_F)\!) Y}{Z \vdash (\!(\bar t_F)\!)(X ; Y)}\ ((\!(\bar t_F)\!), ;)_R$$

$$((\!(\bar t_F)\!), >)_L\ \frac{(\!(\bar t_F)\!) X > (\!(\bar t_F)\!) Y \vdash Z}{(\!(\bar t_F)\!)(X > Y) \vdash Z} \qquad \frac{Z \vdash (\!(\bar t_F)\!) X > (\!(\bar t_F)\!) Y}{Z \vdash (\!(\bar t_F)\!)(X > Y)}\ ((\!(\bar t_F)\!), >)_R$$

$$((\!(\bar t_F)\!), <)_L\ \frac{(\!(\bar t_F)\!) X < (\!(\bar t_F)\!) Y \vdash Z}{(\!(\bar t_F)\!)(X < Y) \vdash Z} \qquad \frac{Z \vdash (\!(\bar t_F)\!) X < (\!(\bar t_F)\!) Y}{Z \vdash (\!(\bar t_F)\!)(X < Y)}\ ((\!(\bar t_F)\!), <)_R$$

16. Interaction between heterogeneous connectives: In what follows, $x \neq y$, $z \notin \mathsf{FV}(\bar t_{F'}) \cup F'$ and $\{z_1, \ldots, z_k\} = \mathsf{FV}(s) \setminus \mathsf{FV}(\bar t_F)$.

$$
(\!(\!(x)\!), [\!\langle \mathtt{Q}y\rangle\!])_L\ \frac{(\!(x)\!)[\!\langle \mathtt{Q}y\rangle\!]X \vdash Y}{[\!\langle \mathtt{Q}y\rangle\!](\!(x)\!)X \vdash Y} \qquad \frac{Y \vdash (\!(x)\!)[\!\langle \mathtt{Q}y\rangle\!]X}{Y \vdash [\!\langle \mathtt{Q}y\rangle\!](\!(x)\!)X}\ (\!(\!(x)\!), [\!\langle \mathtt{Q}y\rangle\!])_R
$$

$$
(\!(\!(\bar{t}_{F'})\!), [\!\langle \mathtt{Q}y\rangle\!])_L\ \frac{(\!(\bar{t}_{F'})\!)[\!\langle \mathtt{Q}y\rangle\!]X \vdash Y}{[\!\langle \mathtt{Q}z\rangle\!](\!(z_y, \bar{t}_{F'})\!)X \vdash Y} \qquad \frac{Y \vdash (\!(\bar{t}_{F'})\!)[\!\langle \mathtt{Q}y\rangle\!]X}{Y \vdash [\!\langle \mathtt{Q}z\rangle\!](\!(z_y, \bar{t}_{F'})\!)X}\ (\!(\!(\bar{t}_{F'})\!), [\!\langle \mathtt{Q}y\rangle\!])_R
$$

$$
(\!(\!(x)\!), (\!(y)\!))_L\ \frac{(\!(x)\!)(\!(y)\!)X \vdash Y}{(\!(y)\!)(\!(x)\!)X \vdash Y} \qquad \frac{Y \vdash (\!(x)\!)(\!(y)\!)X}{Y \vdash (\!(y)\!)(\!(x)\!)X}\ (\!(\!(x)\!), (\!(y)\!))_R
$$

$$
(\!(\!(\bar{t}_F)\!), (\!(\bar{s}_{F'})\!))_L\ \frac{(\!(\bar{t}_F)\!)(\!(\bar{s}_{F'})\!)X \vdash Y}{(\!(\overline{s(\bar{t}/\bar{x})}_{F'})\!)X \vdash Y} \qquad \frac{Y \vdash (\!(\bar{t}_F)\!)(\!(\bar{s}_{F'})\!)X}{Y \vdash (\!(\overline{s(\bar{t}/\bar{x})}_{F'})\!)X}\ (\!(\!(\bar{t}_F)\!), (\!(\bar{s}_{F'})\!))_R
$$

$$
(\!(\!(\bar{t}_F)\!), (\!(y)\!))_L\ \frac{(\!(s_y, \bar{t}_F)\!)(\!(y)\!)X \vdash Y}{(\!(z_1)\!)\cdots(\!(z_k)\!)(\!(\bar{t}_F)\!)X \vdash Y} \qquad \frac{Y \vdash (\!(s_y, \bar{t}_F)\!)(\!(z)\!)X}{Y \vdash (\!(z_1)\!)\cdots(\!(z_k)\!)(\!(\bar{t}_F)\!)X}\ (\!(\!(\bar{t}_F)\!), (\!(y)\!))_R
$$

4.6 Properties

In the present section, we outline the proofs of soundness, completeness, cut elimination and subformula property of the calculus $\mathrm{D.FO}$.

4.6.1 Soundness

In the present subsection, we outline the verification of the soundness of the rules of $\mathrm{D.FO}$ w.r.t. the semantics of heterogeneous $\mathcal{L}$-algebras (cf. Definition 99). As done in analogous situations [12, 14], the first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position,[7] as indicated in the synoptic tables of Section 4.5. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, the rule on the left-hand side below is interpreted as the bi-implication on the right-hand side:

$$
\frac{(\!(\bar{t}_{F'})\!)[\!\langle \mathtt{Q}y\rangle\!]X \vdash Y}{[\!\langle \mathtt{Q}z\rangle\!](\!(z_y, \bar{t}_{F'})\!)X \vdash Y} \qquad \rightsquigarrow \qquad \forall a \forall b[(\bar{t}_{F'})\langle \exists y\rangle a \leq b \Leftrightarrow \langle \exists z\rangle(z_y, \bar{t}_{F'})a \leq b]
$$

Notice that the validity of the bi-implication is equivalent to the validity of the analytic inductive (in fact left-primitive) identity

$$
(\bar{t}_{F'})\langle \exists y\rangle a = \langle \exists z\rangle(z_y, \bar{t}_{F'})a
$$

which holds by definition on every heterogeneous $\mathcal{L}$-algebra. The soundness of the remaining unary rules is proven analogously. As to the substitution axioms (cf. Definition 102.2), since the set of these axioms is defined by closing a given set of generators under

[7] For any sequent $x \vdash y$, we define the signed generation trees $+x$ and $-y$ by labelling the root of the generation tree of x (resp. y) with the sign $+$ (resp. $-$), and then propagating the sign to all nodes according to the polarity of the coordinate of the connective assigned to each node. Positive (resp. negative) coordinates propagate the same (resp. opposite) sign to the corresponding child node. Then, a substructure z in $x \vdash y$ is in *precedent* (resp. *succedent*) *position* if the sign of its root node as a subtree of $+x$ or $-y$ is $+$ (resp. $-$).

display rules and Cut, and since these rules are sound in heterogeneous $\mathcal{L}$-algebras, it is enough to show the soundness of the generators; consider

$$((\bar{t}_1)) \cdots ((\bar{t}_k))R(\bar{t}_{\overline{x}}) \vdash ((\bar{s}_1)) \cdots ((\bar{s}_m))R(\bar{s}_{\overline{x}})$$

where by assumption

$$((\bar{t}_1) \cdots (\bar{t}_k)R(\bar{t}_{\overline{x}}))^\tau = ((\bar{s}_1) \cdots (\bar{s}_m)R(\bar{s}_{\overline{x}}))^\tau. \tag{4.6.1}$$

The translation procedure yields the following inequality

$$(\bar{t}_1) \cdots (\bar{t}_k)V(R(\bar{t}_{\overline{x}})) \leq (\bar{s}_1) \cdots (\bar{s}_m)V(R(\bar{s}_{\overline{x}}))$$

4

for any valuation V. Let us assume that $((\bar{t}_1) \cdots (\bar{t}_k)R(\bar{t}_{\overline{x}}))^\tau = R(\bar{r}_{\overline{x}})$, that is

$$((\bar{t}_1) \cdots (\bar{t}_k)(\bar{t}_{\overline{x}})R(\overline{x}))^\tau = R(\bar{r}_{\overline{x}})$$

therefore the composition of $(\bar{t}_1) \cdots (\bar{t}_k)(\bar{t}_{\overline{x}})$ is $(\bar{r}_{\overline{x}})$. By applying the right-hand equality of *(e)* of Definition 99 a finite number of times we obtain

$$(\overline{w}_\ell) \cdots (\overline{w}_1)(\bar{t}_1) \cdots (\bar{t}_k)(\bar{t}_{\overline{x}})V(R(\overline{x})) = (\bar{r}_{\overline{x}})V(R(\overline{x})) = V(R(\bar{r}_{\overline{x}})) \tag{4.6.2}$$

the second equality holding by Definition 96. By assumption (4.6.1),

$$((\bar{s}_1) \cdots (\bar{s}_m)R(\bar{s}_{\overline{x}}))^\tau = R(\bar{r}_{\overline{x}}).$$

Similar reasoning as above yields

$$(\bar{s}_1) \cdots (\bar{s}_m)(\bar{s}_{\overline{x}})V(R(\overline{x})) = V(R(\bar{r}_{\overline{x}})). \tag{4.6.3}$$

By (4.6.2) and (4.6.3) we get

$$(\bar{t}_1) \cdots (\bar{t}_k)(\bar{t}_{\overline{x}})V(R(\overline{x})) = (\bar{s}_1) \cdots (\bar{s}_m)(\bar{s}_{\overline{x}})V(R(\overline{x})).$$

Since $(\bar{t}_{\overline{x}})V(R(\overline{x})) = V(R(\bar{t}_{\overline{x}}))$ and $(\bar{s}_{\overline{x}})V(R(\overline{x})) = V(R(\bar{s}_{\overline{x}}))$ by Definition 96,

$$(\bar{t}_1) \cdots (\bar{t}_k)V(R(\bar{t}_{\overline{x}})) = (\bar{s}_1) \cdots (\bar{s}_m)V(R(\bar{s}_{\overline{x}})),$$

which completes the proof that the generators of the set of axioms of $\mathrm{D.FO}$ are sound.

This completes the proof that $\mathrm{D.FO}$ and $\mathrm{D.FO}^*$ are sound with respect to the class of heterogeneous algebraic $\mathcal{L}$-models. Since this class properly includes the structures arising from standard $\mathcal{L}$-models, by Proposition 101 $\mathrm{D.FO}$ and $\mathrm{D.FO}^*$ are sound with respect to standard $\mathcal{L}$-models.

4.6.2 Translations and completeness

The aim of this subsection is to show the following

Theorem 103. *If $A \in \mathcal{L}_{\mathrm{MT}}$ and $\vdash_{\mathrm{FO}} A^\tau$ then $\vdash_{\mathrm{D.FO}} A$.*

We will proceed as follows: For every $A \in \mathcal{L}$ and $F \in \mathcal{P}(\mathsf{Var})$, we will define, in two steps, a canonical $\mathcal{L}_F$-formula $\kappa(F, A)$, free of explicit substitutions, and show that $A^F \vdash \kappa(F, A)$ and $\kappa(F, A) \vdash A^F$ are derivable sequents in D.FO for any $A^F \in \mathcal{L}_F$ such that $(A^F)^\tau = A$. Using this, we will show that for any formulas $A, B \in \mathcal{L}_F$, if $A^\tau = B^\tau$ then $A \vdash B$ and $B \vdash A$ are derivable sequents in D.FO. Thanks to this observation, to show Theorem 103 it is enough to show that for every $\mathcal{L}$-formula A which is a theorem of first-order logic and every $F \in \mathcal{P}_\omega(\mathsf{Var})$, some $A' \in \mathcal{L}_F$ exists such that $\vdash_F A'$ is provable in D.FO.

Let us preliminarily show the following

Lemma 104. *The following rules are derivable:*

$$
\frac{\llbracket\langle\star\rangle\rrbracket X > \llbracket\langle\star\rangle\rrbracket Y \vdash Z}{\llbracket\langle\star\rangle\rrbracket (X > Y) \vdash Z}
\qquad
\frac{Z \vdash \llbracket\langle\star\rangle\rrbracket X > \llbracket\langle\star\rangle\rrbracket Y}{Z \vdash \llbracket\langle\star\rangle\rrbracket (X > Y)}
$$

$$
\frac{\llbracket\langle\star\rangle\rrbracket X < \llbracket\langle\star\rangle\rrbracket Y \vdash Z}{\llbracket\langle\star\rangle\rrbracket (X < Y) \vdash Z}
\qquad
\frac{Z \vdash \llbracket\langle\star\rangle\rrbracket X < \llbracket\langle\star\rangle\rrbracket Y}{Z \vdash \llbracket\langle\star\rangle\rrbracket (X < Y)}
$$

$$
\frac{\llbracket\langle\star\rangle\rrbracket X \,;\, \llbracket\langle\star\rangle\rrbracket Y \vdash Z}{\llbracket\langle\star\rangle\rrbracket (X \,;\, Y) \vdash Z}
\qquad
\frac{Z \vdash \llbracket\langle\star\rangle\rrbracket X \,;\, \llbracket\langle\star\rangle\rrbracket Y}{Z \vdash \llbracket\langle\star\rangle\rrbracket (X \,;\, Y)}
$$

where $\llbracket\langle\star\rangle\rrbracket \in \{\llbracket\langle \overline{s}_F\rangle\rrbracket, \llbracket\langle \mathsf{Q}x\rangle\rrbracket\}$ *(see also Proposition 95).*

Proof. The proof uses the rules of Definition 102.14. We only prove one, the others being shown similarly:

$$
\begin{array}{c}
X \vdash \llbracket\langle \mathsf{Q}x\rangle\rrbracket Y > \llbracket\langle \mathsf{Q}x\rangle\rrbracket Z \\ \hline
\llbracket\langle \mathsf{Q}x\rangle\rrbracket Y ; X \vdash \llbracket\langle \mathsf{Q}x\rangle\rrbracket Z \\ \hline
(\!(x)\!)(\llbracket\langle \mathsf{Q}x\rangle\rrbracket Y ; X) \vdash Z \\ \hline
(\!(x)\!)\llbracket\langle \mathsf{Q}x\rangle\rrbracket Y ; (\!(x)\!) X \vdash Z \\ \hline
(\!(x)\!)\llbracket\langle \mathsf{Q}x\rangle\rrbracket Y \vdash Z < (\!(x)\!) X \\ \hline
Y \vdash Z < (\!(x)\!) X \\ \hline
Y ; (\!(x)\!) X \vdash Z \\ \hline
(\!(x)\!) X \vdash Y > Z \\ \hline
X \vdash \llbracket\langle \mathsf{Q}x\rangle\rrbracket (Y > Z)
\end{array}
$$

□

Let us now define a function $\sigma : \mathcal{L}_{\mathrm{MT}} \to \mathcal{L}_{\mathrm{MT}}$ such that for any formula $A \in \mathcal{L}_{\mathrm{MT}}$ the formula $\sigma(A)$ is free of explicit substitutions. We define σ recursively as follows:

1. $\sigma(R(\bar{t}_{\overline{x}})) = R(\bar{t}_{\overline{x}})$;
2. $\sigma(B \bullet C) = \sigma(B) \bullet \sigma(C)$ where $\bullet \in \{\wedge, \vee, \rightarrow\}$;
3. $\sigma(\forall x B) = \forall x \sigma(B)$ and $\sigma(\exists x B) = \exists x \sigma(B)$;
4. $\sigma((x)B) = (x)\sigma(B)$;

5. if A is $(\overline{s}_F)B$ then $\sigma(A)$ is defined recursively on the complexity of B:
   - if B is $R(\overline{t}_{\overline{x}})$ then $\sigma(A) = R(\overline{(t(\overline{s}/\overline{x})_F)})$;
   - if B is $C \bullet D$ then $\sigma(A) = \sigma((\overline{s}_F)C) \bullet (\overline{s}_F)D)$ for $\bullet \in \{\wedge, \vee, \rightarrow\}$;
   - if B is $\langle\exists y\rangle C$ then $\sigma(A) = \langle\exists z\rangle\sigma((z_y, \overline{s}_F)C)$ where $z \notin \mathsf{FV}(B)$;
   - if B is $[\forall y]C$ then $\sigma(A) = [\forall z]\sigma((z_y, \overline{s}_F)C)$ where $z \notin \mathsf{FV}(B)$;
   - if B is $(y)C$ then $\sigma(A) = (z_1)\cdots(z_k)\sigma((\overline{s}_{F\setminus\{y\}})C)$, where $\{z_1, \ldots, z_k\} = \mathsf{FV}(s_y) \setminus \mathsf{FV}(\overline{s}_{F\setminus\{y\}})$;
   - if B is $(\overline{r}_T)C$ then $\sigma(A) = \sigma((\overline{r(\overline{s}/\overline{x})_T})C)$.

Notice that, while τ performs the substitutions and removes (x)-operators, σ simply performs the substitutions. Hence applying τ after σ is the same as applying τ. This motivates the following:

Lemma 105. *If $A \in \mathcal{L}_F$ then $\sigma(A) \in \mathcal{L}_F$. Furthermore $(\sigma(A))^\tau = A^\tau$.*

Proof. Straightforward induction on the complexity of A. □

Lemma 106. *For every $A \in \mathcal{L}_{\mathrm{MT}}$, the sequents $A \vdash \sigma(A)$ and $\sigma(A) \vdash A$ are derivable in D.FO.*

Proof. By induction on the complexity of A. If A is $R(\overline{t}_{\overline{x}})$, the identity axioms (cf. Definition 102.1) yield the desired result. If A is $B \wedge C$ then by induction hypothesis we have:

$$\dfrac{\dfrac{B \vdash \sigma(B) \qquad C \vdash \sigma(C)}{B; C \vdash \sigma(B) \wedge \sigma(C)}}{B \wedge C \vdash \sigma(B) \wedge \sigma(C)}$$

and

$$\dfrac{\dfrac{\sigma(B) \vdash B \qquad \sigma(C) \vdash C}{\sigma(B); \sigma(C) \vdash B \wedge C}}{\sigma(B) \wedge \sigma(C) \vdash B \wedge C}$$

which yields the result since $\sigma(B \wedge C) = \sigma(B) \wedge \sigma(C)$ by definition. If A is $B \vee C$ or $B \rightarrow C$ we argue similarly. If A is $(x)B$ then:

$$\dfrac{\dfrac{\dfrac{B \vdash \tau(B)}{(\!(x)\!)B \vdash (\!(x)\!)\sigma(B)}}{(x)B \vdash (\!(x)\!)\sigma(B)}}{(x)B \vdash (x)\sigma(B)}$$

which yields the result since $\sigma((x)B) = (x)\sigma(B)$. If A is $[\forall x]B$ then:

$$\dfrac{\dfrac{B \vdash \sigma(B)}{[\forall x]B \vdash [\![\langle \mathtt{Q}x\rangle]\!]\sigma(B)}}{[\forall x]B \vdash [\forall x]\sigma(B)}$$

which yields the result since $\sigma([\forall x]B) = [\forall x]\sigma(B)$, and similarly for $\langle\exists x\rangle B$. If A is $(\overline{s}_F)B$, notice that

$$\frac{(\!(\overline{s}_F)\!)B \vdash \sigma((\overline{s}_F)B)}{(\overline{s}_F)B \vdash \sigma((\overline{s}_F)B)} \qquad \frac{\sigma((\overline{s}_F)B) \vdash (\!(\overline{s}_F)\!)B}{\sigma((\overline{s}_F)B) \vdash (\overline{s}_F)B}$$

and hence it is enough to show that $(\!(\overline{s}_F)\!)B \vdash \sigma((\overline{s}_F)B)$ and $\sigma((\overline{s}_F)B) \vdash (\!(\overline{s}_F)\!)B$ are derivable. We proceed by induction on the complexity of B.

If B is $R(\overline{t}_{\overline{x}})$ then

$$(\!(\overline{s}_F)\!)R(\overline{t}_{\overline{x}}) \vdash R(\overline{t(\overline{s}/\overline{x})}_F) \quad \text{and} \quad R(\overline{t(\overline{s}/\overline{x})}_F) \vdash (\!(\overline{s}_F)\!)R(\overline{t}_{\overline{x}})$$

are instances of substitution axioms. If B is $C \wedge D$ then:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{(\!(\overline{s}_F)\!)C \vdash \sigma((\overline{s}_F)C) \qquad (\!(\overline{s}_F)\!)D \vdash \sigma((\overline{s}_F)D)}{(\!(\overline{s}_F)\!)C; (\!(\overline{s}_F)\!)D \vdash \sigma((\overline{s}_F)C) \wedge \sigma((\overline{s}_F)D)}}{(\!(\overline{s}_F)\!)(C; D) \vdash \sigma((\overline{s}_F)C) \wedge (\overline{s}_F)D)}}{(C; D) \vdash [\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)C) \wedge \sigma((\overline{s}_F)D)}}{(C \wedge D) \vdash [\!\langle\overline{s}_F\rangle\!]\sigma(\{(s_x/x)\}C) \wedge \sigma((\overline{s}_F)D)}}{(\!(\overline{s}_F)\!)(C \wedge D) \vdash \sigma((\overline{s}_F)C) \wedge \sigma((\overline{s}_F)D)}$$

and

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\sigma((\overline{s}_F)C) \vdash (\!(\overline{s}_F)\!)C}{[\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)C) \vdash C} \qquad \dfrac{\sigma((\overline{s}_F)C) \vdash (\!(\overline{s}_F)\!)C}{[\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)D) \vdash D}}{[\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)C); [\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)D) \vdash C \wedge D}\ \text{Lemma 104}}{[\!\langle\overline{s}_F\rangle\!](\sigma((\overline{s}_F)C); \sigma((\overline{s}_F)D)) \vdash C \wedge D}}{\sigma((\overline{s}_F)C); \sigma((\overline{s}_F)D) \vdash (\!(\overline{s}_F)\!)(C \wedge D)}}{\sigma((\overline{s}_F)C) \wedge \sigma((\overline{s}_F)D) \vdash (\!(\overline{s}_F)\!)(C \wedge D)}$$

and if B is $C \vee D$ then the proof is analogous.

If B is $C \to D$ then:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{(\!(\overline{s}_F)\!)C \vdash \sigma((\overline{s}_F)C) \qquad \sigma((\overline{s}_F)D) \vdash (\!(\overline{s}_F)\!)D}{\sigma((\overline{s}_F)C) \to \sigma((\overline{s}_F)D) \vdash (\!(\overline{s}_F)\!)C > (\!(\overline{s}_F)\!)D}}{\sigma((\overline{s}_F)C) \to \sigma((\overline{s}_F)D) \vdash (\!(\overline{s}_F)\!)(C > D)}}{[\!\langle\overline{s}_F\rangle\!](\sigma((\overline{s}_F)C) \to \sigma((\overline{s}_F)D)) \vdash C > D}}{[\!\langle\overline{s}_F\rangle\!](\sigma((\overline{s}_F)C) \to \sigma((\overline{s}_F)D)) \vdash C \to D}}{\sigma((\overline{s}_F)C) \to \sigma((\overline{s}_F)D) \vdash (\!(\overline{s}_F)\!)(C \to D)}$$

and

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\sigma((\overline{s}_F)C) \vdash (\!(\overline{s}_F)\!)C}{[\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)C) \vdash C} \qquad \dfrac{(\!(\overline{s}_F)\!)D \vdash \sigma((\overline{s}_F)D)}{D \vdash [\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)D)}}{C \to D \vdash [\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)C) > [\!\langle\overline{s}_F\rangle\!]\sigma((\overline{s}_F)D)}\ \text{Lemma 104}}{C \to D \vdash [\!\langle\overline{s}_F\rangle\!](\sigma((\overline{s}_F)C) > \sigma((\overline{s}_F)D))}}{(\!(\overline{s}_F)\!)(C \to D) \vdash \sigma((\overline{s}_F)C) > \sigma((\overline{s}_F)D)}}{(\!(\overline{s}_F)\!)(C \to D) \vdash \sigma((\overline{s}_F)C) \to \sigma((\overline{s}_F)D)}$$

If B is $\langle\exists x\rangle C$:

$$
\begin{array}{c}
\llparenthesis z_y,\overline{s}_F\rrparenthesis C \vdash \sigma((z_y,\overline{s}_F)C) \\ \hline
[\![\langle \mathtt{Q}z\rangle]\!]\llparenthesis z_y,\overline{s}_F\rrparenthesis C \vdash \langle\exists z\rangle\sigma((z_y,\overline{s}_F)C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis[\![\langle \mathtt{Q}y\rangle]\!] C \vdash \langle\exists z\rangle\sigma((z_y,\overline{s}_F)C) \\ \hline
[\![\langle \mathtt{Q}y\rangle]\!] C \vdash [\![\langle \overline{s}_F\rangle]\!]\langle\exists z\rangle\sigma((z_y,\overline{s}_F)C) \\ \hline
\langle\exists y\rangle C \vdash [\![\langle \overline{s}_F\rangle]\!]\langle\exists z\rangle\sigma((z_y,\overline{s}_F)C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis\langle\exists y\rangle C \vdash \langle\exists z\rangle\sigma((z_y,\overline{s}_F)C)
\end{array}
$$

and

$$
\begin{array}{c}
\sigma((z_y,\overline{s}_F)C) \vdash \llparenthesis z_y,\overline{s}_F\rrparenthesis C \\ \hline
[\![\langle z_y,\overline{s}_F\rangle]\!]\sigma((z_y,\overline{s}_F)C) \vdash C \\ \hline
[\![\langle \mathtt{Q}y\rangle]\!][\![\langle z_y,\overline{s}_F\rangle]\!]\sigma((z_y,\overline{s}_F)C) \vdash \langle\exists y\rangle C \\ \hline
\sigma((z_y,\overline{s}_F)C) \vdash \llparenthesis z_y,\overline{s}_F\rrparenthesis\llparenthesis y\rrparenthesis\langle\exists y\rangle C \\ \hline
\sigma((z_y,\overline{s}_F)C) \vdash \llparenthesis z\rrparenthesis\llparenthesis \overline{s}_F\rrparenthesis\langle\exists y\rangle C \\ \hline
[\![\langle \mathtt{Q}z\rangle]\!]\sigma((z_y,\overline{s}_F)C) \vdash \llparenthesis \overline{s}_F\rrparenthesis\langle\exists y\rangle C \\ \hline
\langle\exists z\rangle\sigma((z_y,\overline{s}_F)C) \vdash \llparenthesis \overline{s}_F\rrparenthesis\langle\exists y\rangle C
\end{array}
$$

If B is $(y)C$ then:

$$
\begin{array}{c}
\llparenthesis \overline{s}_{F\setminus\{y\}}\rrparenthesis C \vdash \sigma((\overline{s}_{F\setminus\{y\}})C) \\ \hline
\llparenthesis z_1\rrparenthesis\cdots\llparenthesis z_k\rrparenthesis\llparenthesis \overline{s}_{F\setminus\{y\}}\rrparenthesis C \vdash \llparenthesis z_1\rrparenthesis\cdots\llparenthesis z_k\rrparenthesis\sigma((\overline{s}_{F\setminus\{y\}})C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis\llparenthesis y\rrparenthesis C \vdash \llparenthesis z_1\rrparenthesis\cdots\llparenthesis z_k\rrparenthesis\sigma((\overline{s}_{F\setminus\{y\}})C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis\llparenthesis y\rrparenthesis C \vdash (z_1)\cdots(z_k)\sigma((\overline{s}_{F\setminus\{y\}})C) \\ \hline
\llparenthesis y\rrparenthesis C \vdash [\![\langle \overline{s}_F\rangle]\!](z_1)\cdots(z_k)\sigma((\overline{s}_{F\setminus\{y\}})C) \\ \hline
(y)C \vdash [\![\langle \overline{s}_F\rangle]\!](z_1)\cdots(z_k)\sigma((\overline{s}_{F\setminus\{y\}})C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis(y)C \vdash (z_1)\cdots(z_k)\sigma((\overline{s}_{F\setminus\{y\}})C)
\end{array}
$$

the other direction being symmetrical. Finally if B is $(\overline{r}_T)C$ then:

$$
\begin{array}{c}
\llparenthesis \overline{r(\overline{s}/\overline{x})}_T\rrparenthesis C \vdash \sigma((\overline{r(\overline{s}/\overline{x})}_T)C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis\llparenthesis \overline{r}_T\rrparenthesis C \vdash \sigma((\overline{r(\overline{s}/\overline{x})}_T)C) \\ \hline
\llparenthesis \overline{r}_T\rrparenthesis C \vdash [\![\langle \overline{s}_F\rangle]\!]\sigma((\overline{r(\overline{s}/\overline{x})}_T)C) \\ \hline
(\overline{r}_T)C \vdash [\![\langle \overline{s}_F\rangle]\!]\sigma((\overline{r(\overline{s}/\overline{x})}_T)C) \\ \hline
\llparenthesis \overline{s}_F\rrparenthesis(\overline{r}_T)C \vdash \sigma((\overline{r(\overline{s}/\overline{x})}_T)C)
\end{array}
$$

This concludes the proof. □

Let us define a translation $\kappa : \mathcal{P}_\omega(\mathsf{Var}) \times \mathcal{L} \to \mathcal{L}_{\mathrm{MT}}$ such that $\kappa(F, A) \in \mathcal{L}_{F\cup\mathsf{FV}(A)}$, by recursion on A as follows:

- $\kappa(F, R(\bar{t}_{\overline{x}})) = (z_1)\cdots(z_k)R(\bar{t}_{\overline{x}})$ where $\{z_1,\ldots,z_k\} = F \setminus \mathsf{FV}(\bar{t}_{\overline{x}})$;
- $\kappa(F, A \bullet B) = \kappa(F, A) \bullet \kappa(F, B))$ where $\bullet \in \{\wedge, \vee, \to\}$ and $\{z_1,\ldots,z_k\} = F \setminus \mathsf{FV}(A) \cup \mathsf{FV}(B)$;

- $\kappa(F, \exists x A) = (x)\langle \exists x\rangle \kappa(F \cup \{x\}, A)$ if $x \in F$ and $\kappa(F, \exists x A) = \langle \exists x\rangle \kappa(F \cup \{x\}, A)$ if $x \notin F$ (and similarly for $\forall x A$);

Lemma 107. *For every formula $A \in \mathcal{L}_F$ that does not contain explicit substitutions the sequents $\kappa(F, A^\tau) \vdash A$ and $A \vdash \kappa(F, A^\tau)$ are derivable in* D.FO.

Proof. We proceed by induction on the complexity of A: If A is $R(\bar{t}_{\overline{x}})$ then $\kappa(F, A^\tau) = A$ and we are done. If A is $B \wedge C$, notice that $\mathsf{FV}(A) = \mathsf{FV}(B) = F$. then:

$$\dfrac{\dfrac{\kappa(F, B^\tau) \vdash B \qquad \kappa(F, C^\tau) \vdash C}{\kappa(F, B^\tau); \kappa(F, C^\tau) \vdash B \wedge C}}{\kappa(F, B^\tau) \wedge \kappa(F, C^\tau) \vdash B \wedge C}$$

We work similarly for $\rightarrow$ and $\vee$.

If A is $\langle \exists x\rangle B$ then $\mathsf{FV}(B) = F \cup \{x\}$:

$$\dfrac{\dfrac{\kappa(F \cup \{x\}, B^\tau) \vdash B}{[\langle \mathsf{Q}x\rangle]\kappa(F \cup \{x\}, B^\tau) \vdash \langle \exists x\rangle B}}{\langle \exists x\rangle \kappa(F \cup \{x\}, B^\tau) \vdash \langle \exists x\rangle B}$$

Finally assume that A is of the form $(x_1)\cdots(x_k)B$ for some $k \in \omega$, where B is not of the form $(z)C$. We proceed by induction on the complexity of B. We wills show that $\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis B \vdash \kappa(F, ((x_1)\cdots(x_k)B)^\tau)$ and $\kappa(F, ((x_1)\cdots(x_k)B)^\tau) \vdash \llparenthesis x_k \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis B$. Notice preliminarily that $((x_1)\cdots(x_k)B)^\tau = B^\tau$ for all B.

If B is $R(\bar{t}_{\overline{x}})$ then $\kappa(F, (x)B) = (x_1)\cdots(x_k)B$:

$$\dfrac{\dfrac{R(\bar{t}_{\overline{x}}) \vdash R(\bar{t}_{\overline{x}})}{\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis R(\bar{t}_{\overline{x}}) \vdash \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis R(\bar{t}_{\overline{x}})}}{\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis R(\bar{t}_{\overline{x}}) \vdash (x_1)\cdots(x_k) R(\bar{t}_{\overline{x}})}$$

and similarly for the other direction. If B is $C \wedge D$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis C \vdash \kappa(F, C^\tau) \qquad \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis D \vdash \kappa(F, D^\tau)}{\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis C; \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis D \vdash \kappa(F, C^\tau) \wedge \kappa(F, D^\tau)}}{\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis (C; D) \vdash \kappa(F, C^\tau) \wedge \kappa(F, D^\tau)}}{(C; D) \vdash [\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, C^\tau) \wedge \kappa(F, D^\tau)}}{(C \wedge D) \vdash [\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, C^\tau) \wedge \kappa(F, D^\tau)}}{\llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis (C \wedge D) \vdash [\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, C^\tau) \wedge \kappa(F, D^\tau)}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\kappa(F, C^\tau) \vdash \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis C}{[\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, C^\tau) \vdash C} \qquad \dfrac{\kappa(F, ((x)D)^\tau) \vdash \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis D}{[\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, D^\tau) \vdash D}}{[\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, C^\tau); [\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle]\kappa(F, D^\tau) \vdash C \wedge D}}{[\langle \mathsf{Q}x_k\rangle] \cdots [\langle \mathsf{Q}x_1\rangle](\kappa(F, C^\tau); \kappa(F, D^\tau)) \vdash C \wedge D}\ \text{Lemma 104}}{\kappa(F, C^\tau); \kappa(F, D^\tau) \vdash \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis (C \wedge D)}}{\kappa(F, C^\tau) \wedge \kappa(F, D^\tau) \vdash \llparenthesis x_1 \rrparenthesis \cdots \llparenthesis x_k \rrparenthesis (C \wedge D)}$$

and if B is $C \vee D$ and $C \to D$ we argue similarly (see also proof of Lemma 106). If B is $\langle\exists x\rangle C$, let us assume without loss of generality that $x_1 = x$[8]:

$$
\begin{array}{c}
(\!(x_2)\!) \cdots (\!(x_k)\!) C \vdash \kappa(F, C^\tau) \\ \hline
[\langle \mathsf{Q}x \rangle] (\!(x_2)\!) \cdots (\!(x_k)\!) C \vdash \langle\exists x\rangle \kappa(F, C^\tau) \\ \hline
(\!(x_2)\!) \cdots (\!(x_k)\!) [\langle \mathsf{Q}x \rangle] C \vdash \langle\exists x\rangle \kappa(F, C^\tau) \\ \hline
[\langle \mathsf{Q}x \rangle] C \vdash [\langle \mathsf{Q}x_k \rangle] \cdots [\langle \mathsf{Q}x_2 \rangle] \langle\exists x\rangle \kappa(F, C^\tau) \\ \hline
\langle\exists x\rangle C \vdash [\langle \mathsf{Q}x_k \rangle] \cdots [\langle \mathsf{Q}x_2 \rangle] \langle\exists x\rangle \kappa(F, C^\tau) \\ \hline
(\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \vdash \langle\exists x\rangle \kappa(F, C^\tau) \\ \hline
(\!(x)\!)(\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \vdash (\!(x)\!) \langle\exists x\rangle \kappa(F, C^\tau) \\ \hline
(\!(x)\!)(\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \vdash (x) \langle\exists x\rangle \kappa(F, C^\tau)
\end{array}
$$

and

$$
\begin{array}{c}
\kappa(F, C^\tau) \vdash (\!(x_2)\!) \cdots (\!(x_k)\!) C \\ \hline
[\langle \mathsf{Q}x_k \rangle] \cdots [\langle \mathsf{Q}x_2 \rangle] \kappa(F, C^\tau) \vdash C \\ \hline
[\langle \mathsf{Q}x \rangle][\langle \mathsf{Q}x_k \rangle] \cdots [\langle \mathsf{Q}x_2 \rangle] \kappa(F, C^\tau) \vdash \langle\exists x\rangle C \\ \hline
\kappa(F, C^\tau) \vdash (\!(x_2)\!) \cdots (\!(x_k)\!)(\!(x)\!) \langle\exists x\rangle C \\ \hline
\kappa(F, C^\tau) \vdash (\!(x)\!)(\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \\ \hline
[\langle \mathsf{Q}x \rangle] \kappa(F, C^\tau) \vdash (\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \\ \hline
\langle\exists x\rangle \kappa(F, C^\tau) \vdash (\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \\ \hline
(\!(x)\!) \langle\exists x\rangle \kappa(F, C^\tau) \vdash (\!(x)\!)(\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C \\ \hline
(x) \langle\exists x\rangle \kappa(F, C^\tau) \vdash (\!(x)\!)(\!(x_2)\!) \cdots (\!(x_k)\!) \langle\exists x\rangle C
\end{array}
$$

For $\forall x C$ the proof is analogous. This concludes the proof. □

Corollary 108. *For any formula $A \in \mathcal{L}_F$ the sequents $A \vdash \kappa(F, A^\tau)$ and $\kappa(F, A^\tau) \vdash A$ are derivable in* D.FO.

Proof. By Lemmas 106 and 107, the sequents $A \vdash \sigma(A)$ and $\sigma(A) \vdash \kappa(F, \sigma(A)^\tau)$ are derivable in D.FO. Hence, by applying Cut, $A \vdash \kappa(F, \sigma(A)^\tau)$ is derivable in D.FO. Finally, by Lemma 105 $\sigma(A)^\tau = A^\tau$. This concludes the proof of the first part of the statement. The second part is argued analogously. □

Corollary 109. *If $A, B \in \mathcal{L}_F$ and $A^\tau = B^\tau$ then $A \vdash B$ is derivable in* D.FO.

Proof. By Corollary 108 $A \vdash \kappa(F, A^\tau)$ and $\kappa(F, B^\tau) \vdash B$ are derivable. Since $A^\tau = B^\tau$, we derive $A \vdash B$ from these two sequents by applying Cut. □

The corollary above implies that in order to prove completeness in the sense specified in Theorem 103, it is enough to prove that, for every $\mathcal{L}$-formula A which is a theorem of first-order logic and every $F \in \mathcal{P}_\omega(\mathsf{Var})$, there exists some $A' \in \mathcal{L}_F$ with $(A')^\tau = A$ such that $\vdash_F A'$ is provable in D.FO. In what follows, we will provide the required derivations.

If A is a propositional tautology then $\kappa(F, A)$ is derivable using the propositional fragment of D.FO. The derivations are omitted. If A is of the form $\forall x(B \to C) \to (\forall x B \to \forall x C)$ then let $D := \kappa(F \cup \{x\}, B)$ and $E := \kappa(F \cup \{x\}, C)$:

[8] Notice that if $x \notin \{x_1, \ldots, x_k\}$ the last two steps of the derivations are redundant.

$$
\begin{array}{c}
D \vdash D \\ \hline
[\forall x]D \vdash [\langle \mathtt{Q}x \rangle]D \\ \hline
((x))[\forall x]D \vdash D \qquad\qquad E \vdash E \\ \hline
D \to E \vdash ((x))[\forall x]D > E \\ \hline
[\forall x](D \to E) \vdash [\langle \mathtt{Q}x \rangle](((x))[\forall x]D > E) \\ \hline
((x))[\forall x](D \to E) \vdash ((x))[\forall x]D > E \\ \hline
((x))[\forall x]D\,;((x))[\forall x](D \to E) \vdash E \\ \hline
((x))([\forall x]D\,;[\forall x](D \to E)) \vdash E \\ \hline
[\forall x]D\,;[\forall x](D \to E) \vdash [\langle \mathtt{Q}x \rangle]E \\ \hline
[\forall x]D\,;[\forall x](D \to E) \vdash [\forall x]E \\ \hline
[\forall x](D \to E) \vdash [\forall x]D > [\forall x]E \\ \hline
[\forall x](D \to E) \vdash [\forall x]D \to [\forall x]E \\ \hline
\mathrm{I}_F \vdash [\forall x](D \to E) > ([\forall x]D \to [\forall x]E) \\ \hline
\mathrm{I}_F \vdash [\forall x](D \to E) \to ([\forall x]D \to [\forall x]E)
\end{array}
$$

which yields the desired result given that $([\forall x](D \to E) \to ([\forall x]D \to [\forall x]E))^\tau = A$. In case $x \in F$ we apply the introduction of $((x))$ in the penultimate step.

If A is of the form $B \to \forall x B$ where $x \notin \mathsf{FV}(B)$, then let $\kappa(F \setminus \{x\}, B) = C$:

$$
\begin{array}{c}
C \vdash C \\ \hline
((x))C \vdash ((x))C \\ \hline
((x))C \vdash (x)C \\ \hline
C \vdash [\langle \mathtt{Q}x \rangle](x)C \\ \hline
C \vdash [\forall x](x)C \\ \hline
\mathrm{I}_F \vdash C > [\forall x](x)C \\ \hline
\mathrm{I}_F \vdash C \to [\forall x](x)C
\end{array}
$$

which yields the desired result given that $(C \to [\forall x](x)C)^\tau = A$. If A is of the form $\forall x B \to B(t/x)$, where t is free for x in B then let $C = \kappa(\mathsf{FV}(B), B)$ and $(t_x, \overline{y}_{\overline{y}})$ for $\overline{y} = \mathsf{FV}(B) \setminus \{x\}$. We proceed by cases. If $x \in \mathsf{FV}(B)$:

$$
\begin{array}{c}
C \vdash C \\ \hline
((t_x, \overline{y}_{\overline{y}}))C \vdash ((t_x, \overline{y}_{\overline{y}}))C \\ \hline
C \vdash [\langle t_x, \overline{y}_{\overline{y}} \rangle]((t_x, \overline{y}_{\overline{y}}))C \\ \hline
[\forall x]C \vdash [\langle \mathtt{Q}x \rangle][\langle t_x, \overline{y}_{\overline{y}} \rangle]((t_x, \overline{y}_{\overline{y}}))C \\ \hline
((t_x, \overline{y}_{\overline{y}}))((x))[\forall x]C \vdash ((t_x, \overline{y}_{\overline{y}}))C \\ \hline
((z_1)) \cdots ((z_k))((\overline{y}_{\overline{y}}))[\forall x]C \vdash ((t_x, \overline{y}_{\overline{y}}))C \\ \hline
(z_1) \cdots (z_k)(\overline{y}_{\overline{y}})[\forall x]C \vdash (t_x, \overline{y}_{\overline{y}})C \\ \hline
\mathrm{I}_F \vdash (z_1) \cdots (x_k)(\overline{y}_{\overline{y}})[\forall x]C > (t_x, \overline{y}_{\overline{y}})C \\ \hline
\mathrm{I}_F \vdash (z_1) \cdots (x_k)(\overline{y}_{\overline{y}})[\forall x]C \to (t_x, \overline{y}_{\overline{y}})C
\end{array}
$$

which yields the desired result given that $((z_1) \cdots (x_k)(\overline{y}_{\overline{y}})[\forall x]C \to (t_x, \overline{y}_{\overline{y}})C)^\tau = A$. If $x \notin \mathsf{FV}(B)$:

$$
\begin{array}{c}
C \vdash C \\
\hline
(\!(x)\!)C \vdash (\!(x)\!)C \\
\hline
(\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)C \vdash (\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)C \\
\hline
(\!(x)\!)C \vdash [\langle t_x, \overline{y}_{\overline{y}} \rangle](\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)C \\
\hline
(x)C \vdash [\langle t_x, \overline{y}_{\overline{y}} \rangle](\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)C \\
\hline
[\forall x](x)C \vdash [\langle \mathtt{Q}x \rangle][\langle t_x, \overline{y}_{\overline{y}} \rangle](\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)C \\
\hline
(\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)[\forall x](x)C \vdash (\!(t_x, \overline{y}_{\overline{y}})\!)(\!(x)\!)C \\
\hline
(\!(z_1)\!)(\!(z_2)\!) \cdots (\!(z_k)\!)(\!(\overline{y}_{\overline{y}})\!)[\forall x](x)C \vdash (\!(z_1)\!)(\!(z_2)\!) \cdots (\!(z_k)\!)(\!(\overline{y}_{\overline{y}})\!)C \\
\hline
(\!(z_2)\!) \cdots (\!(z_k)\!)(\!(\overline{y}_{\overline{y}})\!)[\forall x](x)C \vdash (\!(z_2)\!) \cdots (\!(z_k)\!)(\!(\overline{y}_{\overline{y}})\!)C \\
\hdashline
(\!(\overline{y}_{\overline{y}})\!)[\forall x](x)C \vdash (\!(\overline{y}_{\overline{y}})\!)C \\
\hline
(\overline{y}_{\overline{y}})[\forall x](x)C \vdash (\overline{y}_{\overline{y}})C \\
\hline
\mathrm{I}_F \vdash (\overline{y}_{\overline{y}})[\forall x](x)C > (\overline{y}_{\overline{y}})C \\
\hline
\mathrm{I}_F \vdash (\overline{y}_{\overline{y}})[\forall x](x)C \rightarrow (\overline{y}_{\overline{y}})C
\end{array}
$$

which yields the desired result given that $((\overline{y}_{\overline{y}})[\forall x](x)C \rightarrow (\overline{y}_{\overline{y}})C)^\tau = A$.

Finally using necessitation we obtain the universal closure of tautologies by repeated application of the following pattern:

$$
\begin{array}{c}
\mathrm{I}_{F \cup \{x\}} \vdash A \\
\hline
(\!(x)\!)\mathrm{I}_F \vdash A \\
\hline
\mathrm{I}_F \vdash [\langle \mathtt{Q}x \rangle]A \\
\hline
\mathrm{I}_F \vdash [\forall x]A
\end{array}
$$

This concludes the proof of completeness.

4.6.3 Cut elimination and subformula property

In the present section, we outline the proof of cut elimination and subformula property for the calculus D.FO of Section 4.5. As discussed earlier on, the design of this calculus allows for its cut elimination and subformula property to be inferred from a metatheorem, following the strategy introduced by Belnap for display calculi. The metatheorem to which we will appeal is [5, Theorem 4.1] (cf. [5, Section 3] reported in Section 3.7) for the class of *multi-type calculi*, of which D.FO is a particularly well-behaved element, since it enjoys the full display property. For this reason, D.FO satisfies the following more restricted version of condition C_5''':

C_5''': Closure of axioms under cut. If $x \vdash a$ and $a \vdash y$ are axioms, then $x \vdash y$ is again an axiom.

By this metatheorem, it is enough to verify that D.FO meets the conditions listed in Section 3.7, with C_5''' modified as indicated above. All conditions except C_8 are readily satisfied by inspecting the rules. In what follows we verify C_8. We only treat the case of the heterogeneous connectives:

Axioms: If in the derivation below $X \vdash R(\bar{t}_{\overline{x}})$ and $R(\bar{t}_{\overline{x}}) \vdash Y$ are axioms then by stipulation $X \vdash Y$ is an axiom

$$\dfrac{X \vdash R(\bar{t}_{\overline{x}}) \qquad R(\bar{t}_{\overline{x}}) \vdash Y}{X \vdash Y} \quad \rightsquigarrow \quad X \vdash Y$$

Quantifiers and their adjoint:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array}}{[\langle \mathtt{Q}x\rangle]X \vdash \langle\exists x\rangle A} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ [\langle \mathtt{Q}x\rangle]A \vdash Y\end{array}}{\langle\exists x\rangle A \vdash Y}}{[\langle \mathtt{Q}x\rangle]X \vdash Y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ [\langle \mathtt{Q}x\rangle]A \vdash Y\end{array}}{A \vdash (\!(x)\!)Y}}{X \vdash (\!(x)\!)Y}}{[\langle \mathtt{Q}x\rangle]X \vdash Y}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash [\langle \mathtt{Q}x\rangle]A\end{array}}{X \vdash [\forall x]A} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ A \vdash Y\end{array}}{[\forall x]A \vdash [\langle \mathtt{Q}x\rangle]Y}}{X \vdash [\langle \mathtt{Q}x\rangle]Y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash [\langle \mathtt{Q}x\rangle]A\end{array}}{(\!(x)\!)X \vdash A} \qquad \begin{array}{c}\vdots\,\pi_2\\ A \vdash Y\end{array}}{(\!(x)\!)X \vdash Y}}{X \vdash [\langle \mathtt{Q}x\rangle]Y}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash (\!(x)\!)A\end{array}}{X \vdash (x)A} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ (\!(x)\!)A \vdash Y\end{array}}{(x)A \vdash Y}}{X \vdash Y} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash (\!(x)\!)A\end{array}}{[\langle \mathtt{Q}x\rangle]X \vdash A} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ (\!(x)\!)A \vdash Y\end{array}}{A \vdash [\langle \mathtt{Q}x\rangle]Y}}{[\langle \mathtt{Q}x\rangle]X \vdash [\langle \mathtt{Q}x\rangle]Y}}{(\!(x)\!)[\langle \mathtt{Q}x\rangle]X \vdash Y}}{X \vdash Y}$$

The cases for $(\bar{t}_F)$ is done similarly to the one above.

4.7 Conclusions and further directions

Contributions. In this chapter we have introduced a proper multi-type display calculus for classical first-order logic and shown that it is sound, complete, and enjoys cut elimination and the subformula property. We intended to capture first-order logic with a calculus in which quantifiers are represented also at the structural level and rules are closed under uniform substitution. We achieved this by developing an idea of Wansing's, that the proof-theoretic treatment of quantifiers can emulate that of modal operators, in a multi-type setting in which formulas with different sets of free variables have different types. This multi-type environment is supported semantically by certain classes of heterogeneous algebras in which the maps interpreting the existential and universal quantifiers are the left and the right adjoint respectively of one and the same injective Boolean algebra homomorphism. The same adjunction pattern accounts for the semantics of substitution.

Correspondence theory for first-order logic. The semantic analysis in Section 4.3 can be regarded as a multi-type and ALBA-powered version of the correspondence results observed in [31]. Thanks to the systematic connections established between unified correspondence theory and the theory of analytic display calculi, we are now in a position to apply the results and insights of unified correspondence to the semantic environment of Section 4.3 and systematically exploit them for proof-theoretic purposes. This might turn out to be a useful tool in the analysis of generalized quantifiers (see discussion below).

A modular environment. Thanks to the fact that, in the environment of this calculus, both substitutions and quantifiers are explicitly represented as logical and structural connectives, we can now explore systematically the space of their properties and their possible interactions. For instance, the rules $((\!(\bar{t}_{F'})\!), [\!\langle \mathtt{Q}y \rangle\!])_R$ and $((\!(\bar{t}_{F'})\!), [\!\langle \mathtt{Q}y \rangle\!])_L$ express in a transparent and explicit way the book-keeping concerning variable capturing in first-order logic. More interestingly, this environment allows for a finer-grained analysis of fundamental interactions between quantifiers and intensional connectives. For instance, in the present calculus the rules in Definition 102.15 encode the fact that the cylindrification maps are Boolean algebra homomorphisms, which in turn captures the fact that classical propositional connectives are all extensional. If we change the propositional base to e.g. intuitionistic or bi-intuitionistic logic, or if we expand classical first-order logic with modal connectives it is desirable to allow for some additional flexibility. For instance the constant domain axiom $\forall x(B(x) \vee A) \rightarrow ((\forall x B(x)) \vee A)$ can be captured by the analytic structural rule below on the left hand side, which is interderivable with the one on the right hand side:

$$\frac{X \vdash [\!\langle \mathtt{Q}x \rangle\!](Y \,;\, (\!(x)\!)Z)}{X \vdash [\!\langle \mathtt{Q}x \rangle\!]Y \,;\, Z} \qquad\qquad \frac{(\!(x)\!)X > (\!(x)\!)Y \vdash Z}{(\!(x)\!)(X > Y) \vdash Z} \qquad (*)$$

Indeed:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X \vdash [\!\langle \mathtt{Q}x \rangle\!](Y \,;\, (\!(x)\!)Z)}{(\!(x)\!)X \vdash Y \,;\, (\!(x)\!)Z}}{(\!(x)\!)Z > (\!(x)\!)X \vdash Y}}{(\!(x)\!)(Z > X) \vdash Y}}{Z > X \vdash [\!\langle \mathtt{Q}x \rangle\!]Y}}{X \vdash [\!\langle \mathtt{Q}x \rangle\!]Y \,;\, Z} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\!(x)\!)X > (\!(x)\!)Y \vdash Z}{(\!(x)\!)Y \vdash Z \,;\, (\!(x)\!)X}}{Y \vdash [\!\langle \mathtt{Q}x \rangle\!](Z \,;\, (\!(x)\!)X)}}{Y \vdash [\!\langle \mathtt{Q}x \rangle\!]Z \,;\, X}}{X > Y \vdash [\!\langle \mathtt{Q}x \rangle\!]Z}}{(\!(x)\!)(X > Y) \vdash Z}$$

Notice that in the hypersequent calculus for bi-intuitionistic first-order logic the constant domain axiom is derivable from the Mix rule, capturing the prelinearity axiom $(A \rightarrow B) \vee (B \rightarrow A)$. In the context of this calculus the prelinearity axiom corresponds to the rule

$$\frac{X \vdash Y \qquad W \vdash Z}{\mathrm{I}_F \vdash (X > Z) \,;\, (W > Y)}$$

we conjecture that in the present framework the constant domain axiom is not derivable in $\mathrm{D.FO}$ without the two rules in $(*)$ and with the prelinearity rule above.

A modular environment? Notwithstanding the increased modularity that this calculus allows, we feel that something more can be done. For instance, the heterogeneous modal operators use individual variables and terms as parameters, much in the same way in which actions and agents were used as parameters in the first versions of the display calculus for DEL (cf. [7, 11]). While this choice is unproblematic in respect to the cut elimination and actually allows to retrieve it from the metatheorem of [5], it is also practically cumbersome, since it constrains us to admit as axioms sequents which are not useful when deriving actual formulas (indeed substitution axioms in which structural adjoints $[\!\langle \bar{t}_F \rangle\!]$ occur on both sides of the sequent cannot possibly occur in derivations that conclusions of which are pure formulas), and moreover recognizing whether such a sequent is an axiom requires lengthy calculations which should be performed internally to the calculus rather than externally. This is the focus of current investigation. Indeed having axioms and rules such as the following

$$x \approx x \qquad c \approx c \qquad \frac{t_1 \approx t_2 \qquad t_2 \approx t_3}{t_1 \approx t_3}$$

$$\frac{t_1 \approx s_1 \qquad \cdots \qquad t_n \approx s_n}{f(t_1, \ldots, t_n) \approx f(s_1, \ldots, s_n)} \qquad \frac{t \approx s}{t \approx (\!(s_x)\!)x} \qquad \frac{t \approx s}{(\!(t_x)\!)x \approx s}$$

would be much neater. This choice would also be compatible with rules such as

$$\frac{(\!(x)\!)(X\,;Y) \vdash Z}{(\!(x)\!)X\,;(\!(x)\!)Y \vdash Z}$$

which are completely unproblematic when x is a parameter as in the present setting but would violate the condition C5 against proliferation when considered a term. However, the metatheorem of [5] allows proliferation limited to "flat" types, i.e. types the only rules of which are identity and cut. We are working towards a solution which allows to integrate variables and terms as types.

Logics with equality. Having variables and terms as types might also be beneficial for the treatment of equality. Indeed in the extant literature (c.f. [29]) the rules capturing the special behaviour of equality are

$$\frac{t = t, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad \frac{t = s, P(t/x), P(s/x), \Gamma \vdash \Delta}{t = s, P(t/x), \Gamma \vdash \Delta}$$

which can be equivalently rewritten in the following form, which does not violate condition C1:

$$I \vdash t = t \qquad \frac{t = s \vdash (\!(t_x, \overline{r}_F)\!)X}{t = s \vdash (\!(s_x, \overline{r}_F)\!)X}$$

For the rule on the right hand side above to be analytic, we would need the formula $t = s$ to occur at the structural level. Hence a calculus catering for terms as explicit types might be the appropriate environment to capture the rules above as a multi-type analytic rule such as

$$\frac{t \approx s}{R(t) \vdash R(s)}$$

Generalized quantifiers. The semantic analysis of Section 4.3 shows that quantifiers and substitutions correspond to a very restricted class of maps $f : M^S \to M^T$. Therefore it is natural to ask whether different notions of quantification and substitutions can be investigated in connection with larger or different classes of such functions. This study could be relevant to the analysis of generalized quantifiers in natural language semantics [2, 17, 23, 24, 35], to dependence and independence logic [10, 18, 30] and to the analysis of different notions of substitutions [26–28].

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Chapter 5

Probabilistic Epistemic Updates on Algebras

The present chapter, which is based on[1] [15], contributes to the development of the mathematical theory of epistemic updates using the tools of duality theory. Here we focus on Probabilistic Dynamic Epistemic Logic (PDEL). We dually characterize the product update construction of PDEL-models as a certain construction transforming the complex algebras associated with the given model into the complex algebra associated with the updated model. Thanks to this construction, an interpretation of the language of PDEL can be defined on algebraic models based on Heyting algebras. This justifies our proposal for the axiomatization of the intuitionistic counterpart of PDEL.

[1]My specific contributions in this research have been the proof of the main results, the construction and development of examples and case studies the draft of the first version of the paper.

5.1 Introduction

This chapter pertains to a line of research aimed at exploring the notions of agency and information flow in situations in which truth is socially constructed. Such situations are ubiquitous in the real world. A prime example is the validity of contracts. Establishing that an agreement constitutes a valid contract appeals to notions, such as legal competency and bona fide offers, which are inherently socially constructed. The ultimate way in which the validity of a contract can be ascertained is for it to be tested in a court of law. In this last instance, the validity of a contract is thus procedural, and may also admit of situations in which it is indeterminate, such as when the court declares itself incompetent. These are features at odds with standard classical logic. Accommodating these features within classical logic requires additional encoding mechanisms. The alternative is working with logics which are specifically designed to accommodate these characteristics of socially constructed truth.

Examples of situations where truth is socially constructed are certainly not confined to contract law, but are easy to find in many other contexts. These include establishing public opinion in a binding way like referendums, establishing whether a certain item of clothing is fashionable, and determining the value of products in a market.

There is a large literature on logics which very adequately capture agency and information flow (see [40] and references therein), but assume a notion of truth that is classical. There is therefore a need for a uniform methodology for transferring these logics onto nonclassical bases. In [34, 36], a uniform methodology is introduced for defining the nonclassical counterparts of dynamic epistemic logics. This methodology, further pursued in [5, 6, 38], is grounded on semantics, and is based on the dual characterizations of the transformations of models which interpret epistemic actions.

The present chapter expands on [18] and applies the methodology of [34, 36] to obtain nonclassical counterparts of probabilistic dynamic epistemic logic (PDEL) [33], [39]. We will focus specifically on the intuitionistic environment as our case study. This environment allows for a finer-grained analysis when serving as a base for more expressive formalisms such as modal and dynamic logics. Indeed, the fact that the box-type and the diamond-type modalities are no longer interdefinable makes several mutually independent choices possible which cannot be disentangled in the classical setting. Moving to the intuitionistic environment also requires the use of intuitionistic probability theory (cf. [3, 23]) as the background framework for probabilistic reasoning. From the point of view of applications this generalization is needed to account for situations in which the probability of a certain proposition p is interpreted as an agent's propensity to bet on p given some evidence for or against p. If there is little or no evidence for or against p, it should be reasonable to attribute low probability values to both p and $\neg p$, which is forbidden by classical probability theory (cf. [44]).

Finally, these mathematical developments appear in tandem with interesting analyses on the philosophical side of formal logic (e.g. [4]), exploring epistemic logic in an evidentialist key, which is congenial to the kind of social situations targeted by our research programme.

Our methodology is based on the dual characterization of the product update construction for standard PDEL-models as a certain construction transforming the complex algebras associated with a given model into the complex algebra associated with the updated model. This dual characterization naturally generalizes to much wider classes of algebras, which include arbitrary classical S5 algebras and certain monadic Heyting algebras. As an application of this dual characterization, we introduce the axiomatization of the intuitionistic analogue of PDEL semantically arising from this construction, and prove its soundness and completeness with respect to the class of so called *algebraic probabilistic epistemic models* (see Definition 177).

Structure of the chapter. In Section 5.2, we recall the definition of classical PDEL and its relational semantics. We give an alternative presentation of the product update construction which consists in two steps, as done in [34]. The two-step construction highlights the elements which will be key in the dualization. In Section 5.3, we expand on the methodology making use of Stone duality. Section 5.4 is the main section, in which the construction of the PDEL-updates on epistemic Heyting algebras is introduced. In Section 5.5, we define axiomatically the intuitionistic version of PDEL (IPDEL) and its interpretation on algebraic probabilistic epistemic models, and discuss the proof of its soundness. In Section 5.6, we prove the completeness of IPDEL with respect to algebraic probabilistic epistemic models. In Section 5.7, we introduce the relational semantics of IPDEL. In Section 5.8, we discuss the case study of a decision-making under uncertainty. In Section 5.9, we collect conclusions and further directions. The details about the proof of soundness are collected in Section 5.10.

5.2 PDEL language and updates

In the present section, we report on the language of PDEL, and give an alternative, two-step account of the product update construction on PDEL-models. This account is similar to the treatment of epistemic updates in [34, 36], and as explained in Section 5.3, it lays the ground to the dualization procedure which motivates the construction introduced in Section 5.4. The specific PDEL framework we report on shares common features with those of [2, 7] and [39].

Structure of the section. In Section 5.2.1, we recall basic facts about probability theory, we present the syntax au PDEL, the classical models and the classical event structures. In Section 5.2.2, we present the epistemic update of a PES-model by a probabilistic event structure. In Section 5.2.3 and 5.2.4 respectively, we present the semantics and the axiomatisation of classical PDEL.

5.2.1 PDEL-formulas, event structures, and PES-models

In this section, we first recall basic facts about probability distributions and probability measures, then we introduce the syntax and semantics of Probabilistic Dynamic Epistemic Logic (PDEL).

Remark 110. Given a finite set X, a *probability distribution* P over X is a map

$$P \ : \ X \to [0,1]$$

such that

$$\sum_{x \in X} P(x) = 1.$$

Recall that a *probability measure* on $\mathcal{P}X$ can be defined as a map

$$\mu \ : \ \mathcal{P}X \to [0,1]$$

satisfying the following properties:

1. $\mu(\varnothing) = 0$,
2. $\mu(X) = 1$,
3. for any $A, B \subseteq X$, we have

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

The probability measure $\mu \ : \ \mathcal{P}X \to [0,1]$ determined by the probability distribution P over X is defined as follows: for any $S \subseteq X$,

$$\mu(S) := \sum_{x \in S} P(x).$$

In the remainder of the chapter, we fix a countable set AtProp of proposition letters p, q and a non-empty finite set Ag of agents i. We let $\alpha_1, ..., \alpha_n, \beta$ denote rational numbers.

Definition 111 (PDEL syntax)**.** The set $\mathcal{L}$ of *PDEL-formulas* φ and the class of *probabilistic event structures* $\mathcal{E}$ over $\mathcal{L}$ (see Definition 116) are built by simultaneous recursion as follows:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Diamond_i \varphi \mid \Box_i \varphi \mid \langle \mathcal{E}, e \rangle \varphi \mid [\mathcal{E}, e] \varphi \mid (\sum_{k=1}^{n} \alpha_k \mathring{u} \mu_i(\varphi)) \geq \beta,$$

where $p \in$ AtProp, $i \in$ Ag, $\alpha_1, ..., \alpha_n, \beta$ are rational numbers, and the event structures $\mathcal{E}$ are such as in Definition 116.
The connectives $\top$, $\neg$, and $\leftrightarrow$ are defined by the usual abbreviations.

Definition 112 (PES-model)**.** A *probabilistic epistemic state model (PES-model)* is a structure

$$\mathbb{M} = \langle S, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, [\![\cdot]\!] \rangle$$

such that

- S is a finite set,

- each binary relation $\sim_i$ is an equivalence relation on S,
- each map $P_i : S \to]0,1]$ assigns a probability distribution over each $\sim_i$-equivalence class, (i.e. $\sum\{P_i(s') : s' \sim_i s\} = 1$), and
- the map $[\![\cdot]\!]$: AtProp $\to \mathcal{P}S$ is a valuation.

As usual, the map $[\![\cdot]\!]$ will be identified with its unique extension to $\mathcal{L}$, so that we will be able to write $[\![\varphi]\!]$ for every $\varphi \in \mathcal{L}$.

Remark 113. The assumption that the probability of each state is strictly positive is needed for the update defined in Definition 122 to be well-defined. This is also the convention followed in [42] where subjective probabilities are identified with "lotteries" assigned to each agent.

Remark 114. In the present treatment, the syntactic μ_is (introduced in Definition 111) are intended to correspond to probability measures rather than probability distributions, as is more common in the literature. Indeed, usually, in the literature formulas talking about probabilities are defined by the following syntax $\alpha P_i(\varphi) \geq \beta$. But the P_i maps are probability distributions defined over the models (i.e. in the semantics), hence the notation $P_i(\varphi)$ is ambiguous and neglects the fact that we need to use a probability measure to talk about the probability over the extension of φ.

5

Definition 115 (Substitution function)**.** A *substitution function*

$$\sigma \;:\; \mathsf{AtProp} \to \mathcal{L}$$

is a function that maps all but a finite [2] number of proposition letters to themselves.

We will call the set

$$\{p \in \mathsf{AtProp} \mid \sigma(p) \neq p\}$$

the domain of σ and denote it $dom(\sigma)$.

Let $Sub_{\mathcal{L}}$ denote the set of all substitution functions and ε the identity substitution.

Definition 116 (Probabilistic event structure over a language)**.** A *probabilistic event structure over* $\mathcal{L}$ is a tuple

$$\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub}),$$

such that

- E is a non-empty finite set,
- each $\sim_i$ is an equivalence relation on E,
- each $P_i : E \to]0,1]$ assigns a probability distribution over each $\sim_i$-equivalence class, i.e.
$$\sum\{P_i(e') \mid e' \sim_i e\} = 1,$$

[2]This assumption guarantees that events affect only a finite number of facts.

- Φ is a finite set of pairwise inconsistent $\mathcal{L}$-formulas, and
- pre assigns a probability distribution pre$(\bullet|\phi)$ over E for every $\phi \in \Phi$.
- $\mathbf{sub} : E \to Sub_{\mathcal{L}}$ assigns a substitution function to each event in E.

Remark 117. The assumption that the probability of each event is strictly positive is needed for the update defined in Definition 122 to be well-defined. This is also the convention followed in [1, 42].

Informally, elements of E encode possible events, the relations $\sim_i$ encode as usual the epistemic uncertainty of the agent i, who assigns probability $P_i(e)$ to e being the actually occurring event, formulas in Φ are intended as the preconditions of the event, and pre$(e|\phi)$ expresses the prior probability that the event $e \in E$ might occur in a(ny) state satisfying precondition ϕ. In addition, the substitution map $\mathbf{sub}(e)$ assigned to each event $e \in E$ describes how the event e changes the atomic facts of the world as represented by the proposition letters. In what follows, we will refer to the structures $\mathcal{E}$ defined above as *event structures over* $\mathcal{L}$.

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Notation 118. For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$, any probabilistic event structure $\mathcal{E}$, any $s \in S$ and $e \in E$, we let pre$(e \mid s)$ denote the value pre$(e \mid \phi)$, for the unique $\phi \in \Phi$ such that $\mathbb{M}, s \Vdash \phi$ (recall that the formulas in Φ are pairwise inconsistent). If no such ϕ exists then we let pre$(e \mid s) = 0$.

5.2.2 Epistemic updates

In this subsection, we introduce an alternative and equivalent presentation of the update construction on PES-models. This presentation is a variant of those introduced in [34, 36] for models of public announcement logic and dynamic epistemic logic, and consists in a two-step process, namely, a co-product-type construction followed by a suboject-type construction. This two-step presentation makes it possible to dualize the two steps separately, and thus obtain the construction of (probabilistic) epistemic updates on algebras as the composition of the two dualized constructions. The two steps are given in Definition 119 and Definition 122, and Lemma 123 proves that the updated model of a PES-model is a PES-model too.

Definition 119 (Intermediate structure)**.** For any probabilistic epistemic state model $\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$ and any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$, let the *intermediate structure* of $\mathbb{M}$ and $\mathcal{E}$ be the tuple

$$\coprod_{\mathcal{E}} \mathbb{M} := \left\langle \coprod_{|E|} S, (\sim_i^{\coprod})_{i\in\mathsf{Ag}}, (P_i^{\coprod})_{i\in\mathsf{Ag}}, [\![\cdot]\!]_{\coprod} \right\rangle$$

where

- $\coprod_{|E|} S \cong S \times E$ is the $|E|$-fold coproduct of S,

- each binary relation $\sim_i^{\coprod}$ on $\coprod_{|E|} S$ is defined as follows:

$$(s,e) \sim_i^{\coprod} (s',e') \quad \text{iff} \quad s \sim_i s' \text{ and } e \sim_i e',$$

- each map $P_i^{\coprod} : \coprod_{|E|} S \to [0,1]$ is defined by

$$(s,e) \mapsto P_i(s) \cdot P_i(e) \cdot \mathsf{pre}(e \mid s),$$

- and the valuation $[\![\cdot]\!]_{\coprod} : \mathsf{AtProp} \to \mathcal{P}S$ is defined by

$$[\![p]\!]_{\coprod} := \{(s,e) \mid s \in [\![p]\!]_{\mathbb{M}}\} = [\![p]\!]_{\mathbb{M}} \times E$$

for every $p \in \mathsf{AtProp}$.

Remark 120. In general, $P_i^{\coprod}$ does not induce probability distributions over the $\sim_i^{\coprod}$-equivalence classes. Hence, $\coprod_{\mathcal{E}} \mathbb{M}$ is not a PES-model. However, the second step of the construction will yield a PES-model.

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Finally, in order to define the updated model, observe that the map $\mathsf{pre} : E \times \Phi \to [0,1]$ in $\mathcal{E}$ induces the map $pre : E \to \mathcal{L}$ defined below.

Definition 121. Given $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ a probabilistic event structure

over $\mathcal{L}$, let the map pre be defined as follows:

$$\begin{aligned} pre : E &\to \mathcal{L} \\ e &\mapsto \bigvee \{\phi \in \Phi \mid \mathsf{pre}(e \mid \phi) \neq 0\}. \end{aligned}$$

Definition 122 (Updated model)**.** For any probabilistic epistemic state model $\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$ and any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$, let the *epistemic update* $\mathbb{M}^{\mathcal{E}}$ of the model $\mathbb{M}$ by the probabilistic event structure $\mathcal{E}$ be as follows:

$$\mathbb{M}^{\mathcal{E}} := \langle S^{\mathcal{E}}, (\sim_i^{\mathcal{E}})_{i\in\mathsf{Ag}}, (P_i^{\mathcal{E}})_{i\in\mathsf{Ag}}, [\![\cdot]\!]_{\mathbb{M}^{\mathcal{E}}}\rangle$$

with

1. $S^{\mathcal{E}} := \left\{(s,e) \in \coprod_{|E|} S \;\middle|\; \mathbb{M}, s \Vdash pre(e)\right\}$;
2. $\sim_i^{\mathcal{E}} = \sim_i^{\coprod} \cap (S^{\mathcal{E}} \times S^{\mathcal{E}})$ for any $i \in \mathsf{Ag}$;
3. each map $P_i^{\mathcal{E}} : S^{\mathcal{E}} \to [0,1]$ is defined by the assignment

$$(s,e) \mapsto \frac{P_i^{\coprod}(s,e)}{\sum\left\{P_i^{\coprod}(s',e') \;\middle|\; (s,e) \sim_i (s',e')\right\}};$$

4. the map $[\![\cdot]\!]_{\mathbb{M}^{\mathcal{E}}}$: AtProp $\to \mathcal{P}(S^{\mathcal{E}})$ is defined as follows:

$$[\![p]\!]_{\mathbb{M}^{\mathcal{E}}} := [\![sub(p)]\!]_{\coprod} \cap S^{\mathcal{E}}$$

where the map $sub(p) : E \to \mathcal{L}$ is given by:

$$sub(p)(e) := \begin{cases} \mathbf{sub}(e)(p) & \text{if } p \in dom(\mathbf{sub}(e)) \\ p & \text{otherwise.} \end{cases}$$

5

Lemma 123. *For any PES-model $\mathbb{M}$ and any probabilistic event structure $\mathcal{E}$ over $\mathcal{L}$, the* epistemic update $\mathbb{M}^{\mathcal{E}}$ *of the model $\mathbb{M}$ by the probabilistic event structure $\mathcal{E}$ is a PES-model.*

Proof. To prove that $\mathbb{M}^{\mathcal{E}}$ is a PES-model (Definition 112), we need to show that it satisfies the following properties:

1. the set $S^{\mathcal{E}}$ is finite,

2. each relation $\sim_i^{\mathcal{E}}$ is an equivalence relation on S,

3. each map $P_i^{\mathcal{E}} : S^{\mathcal{E}} \to]0,1]$ assigns a probability distribution over each $\sim_i^{\mathcal{E}}$-equivalence class,

4. the map $[\![\cdot]\!]$: AtProp $\to \mathcal{P}S^{\mathcal{E}}$ is a valuation map.

`Proof of item 1.` The product of finite sets is finite.

`Proof of items 2 and 4.` Trivial.

`Proof of item 3.` Since, for every $s \in S$, we have $P_i(s) > 0$, it immediately follows that $P_i^{\mathcal{E}}(s,e) > 0$. Moreover, by construction, it is a probability distribution over $\sim_i^{\mathcal{E}}$-equivalence classes. □

5.2.3 Semantics

In this subsection, we provide the semantics of PDEL over PES-models.

Definition 124 (Probability measure)**.** Given a PES-model

$$\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle,$$

let the *probability measure* $\mu_i^{\mathbb{M}} : S \times \mathcal{L} \to [0,1]$ be defined as follows: for any $\phi \in \mathcal{L}$,

$$\mu_i^{\mathbb{M}}(s,\phi) \quad := \sum_{\substack{s\sim_i s' \\ s'\in[\![\phi]\!]}} P_i(s').$$

Notice that μ_i defines a probability measure on each $\sim_i$-equivalence class.

Definition 125 (Semantics of PDEL)**.** Given a PES-model

$$\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle\,,$$

and the probability measures $\mu_i^{\mathbb{M}}$ defined as in Definition 124, the formulas of the language $\mathcal{L}$ are interpreted as follows:

$$\begin{array}{rll}
\mathbb{M}, s \models \bot & & \text{never} \\
\mathbb{M}, s \models p & \text{iff} & s \in [\![p]\!] \\
\mathbb{M}, s \models \phi \wedge \psi & \text{iff} & \mathbb{M}, s \models \phi \quad \text{and} \quad \mathbb{M}, s \models \psi \\
\mathbb{M}, s \models \phi \vee \psi & \text{iff} & \mathbb{M}, s \models \phi \quad \text{or} \quad \mathbb{M}, s \models \psi \\
\mathbb{M}, s \models \phi \to \psi & \text{iff} & \mathbb{M}, s \models \phi \quad \text{implies} \quad \mathbb{M}, s \models \psi \\
\mathbb{M}, s \models \Diamond_i \phi & \text{iff} & \text{there exists } s' \sim_i s \text{ such that } \mathbb{M}, s' \models \phi \\
\mathbb{M}, s \models \Box_i \phi & \text{iff} & \mathbb{M}, s' \models \phi \quad \text{for all } s' \sim_i s \\
\mathbb{M}, s \models \langle \mathcal{E}, e\rangle \phi & \text{iff} & \mathbb{M}, s \models pre(e) \quad \text{and} \quad \mathbb{M}^{\mathcal{E}}, (s,e) \models \phi \\
\mathbb{M}, s \models [\mathcal{E}, e] \phi & \text{iff} & \mathbb{M}, s \models pre(e) \quad \text{implies} \quad \mathbb{M}^{\mathcal{E}}, (s,e) \models \phi \\
\mathbb{M}, s \models \left(\sum_{k=1}^{n} \alpha_k \mathring{u} \mu_i(\varphi)\right) \geq \beta & \text{iff} & \sum_{k=1}^{n} \alpha_k \mathring{u} \mu_i^{\mathbb{M}}(\varphi) \geq \beta
\end{array}$$

5.2.4 Axiomatization

PDEL is a logical framework bringing together epistemics, dynamics, and probabilities. Hence its axiomatization describes the behaviour of each of these components as well as their interactions. The full axiomatization of PDEL is given in Table 5.1 on page 182 and includes the axioms of classical multi-modal logic S5, understood as the basic epistemic logic, axioms capturing the theory of linear inequalities with rational coefficients (cf. [21, Theorem 4.3]), axioms capturing basic classical probability theory (cf. [1, 20, 21, 39, 42]), and axioms encoding the interaction between the dynamic modalities and the other logical connectives [1, 39], as well as the following inference rules: modus ponens, uniform substitution (see [43]), necessitation for the static and dynamic modalities, and a substitution rule for the probabilistic operators μ_i (cf. [1, 20, 39, 42]).

Lemma 126 (Soundness and Completeness)**.** *PDEL is sound and complete w.r.t. the axiomatization given in Table 5.1.*

Proof. The statement follows from the general proof in Section 5.6 and Stone type duality. □

5.3 Methodology

In the present section, we expand on the methodology of the chapter. In the previous section, we gave a two-step account of the *product update* construction which, for any PES-model $\mathbb{M}$ and any event model $\mathcal{E}$ over $\mathcal{L}$, yields the updated model $\mathbb{M}^{\mathcal{E}}$ as a certain

Table 5.1: Axioms of PDEL

	Axioms of classical modal logic S5
	Tautologies of classical propositional logic
k.	$\Box_i(\varphi \to \psi) \to (\Box_i\varphi \to \Box_i\psi)$
dual.	$\Box_i\varphi \leftrightarrow \neg\Diamond_i\neg\varphi$
t.	$\Box_i\varphi \to \varphi$
iv.	$\Box_i\varphi \to \Box_i\Box_i\varphi$
v.	$\neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi$
	Axioms of linear inequalities with rational coefficients
n0.	$t \geq t$
n1.	$(t \geq \beta) \leftrightarrow (t + 0 \cdot \mu_i(\varphi) \geq \beta)$
n2.	$\left(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta\right) \to \left(\sum_{k=1}^n \alpha_{\sigma(k)} \cdot \mu_i(\varphi_{\sigma(k)}) \geq \beta\right)$ for any permutation σ over $\{1, ..., n\}$
n3.	$\left(\left(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta\right) \wedge \left(\sum_{k=1}^n \alpha'_k \cdot \mu_i(\varphi_k) \geq \beta'\right)\right) \to \left(\sum_{k=1}^n (\alpha_k + \alpha'_k) \cdot \mu_i(\varphi_k) \geq (\beta + \beta')\right)$
n4.	$((t \geq \beta) \wedge (d \geq 0)) \to (d \cdot t \geq d \cdot \beta)$
n5.	$(t \geq \beta) \vee (\beta \geq t)$
n6.	$((t \geq \beta) \wedge (\beta \geq \gamma)) \to (t \geq \gamma)$
	Axioms of basic classical probability theory
p1.	$\mu_i(\bot) = 0$
p2.	$\mu_i(\top) = 1$
p3.	$\mu_i(\varphi \wedge \psi) + \mu_i(\varphi \wedge \neg\psi) = \mu_i(\varphi)$
p4.	$\Box_i\varphi \leftrightarrow (\mu_i(\varphi) = 1)$
p5.	$\left(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta\right) \to \Box_i \left(\sum_{k=1}^n \alpha_k \cdot \mu_i(\varphi_k) \geq \beta\right)$
	Reduction Axioms
i1.	$[\mathcal{E}, e]p \leftrightarrow (pre(e) \to \mathbf{sub}(e)(p))$
i2.	$[\mathcal{E}, e]\neg\varphi \leftrightarrow (pre(e) \to \neg[\mathcal{E}, e]\varphi)$
i4.	$[\mathcal{E}, e](\varphi \wedge \psi) \leftrightarrow ([\mathcal{E}, e]A \wedge [\mathcal{E}, e]B)$
i5.	$[\mathcal{E}, e]\Box_i A \leftrightarrow \left(pre(e) \to \bigwedge\{\Box_i[\mathcal{E}, f]A \mid e \sim_i f\}\right)$
i6.	$[\mathcal{E}, e]\left(\sum_{k=1}^n \alpha_k \cdot \mu_i(\psi_k) \geq \beta\right) \leftrightarrow (pre(e) \to C \geq D)$ with $C = \sum_{\phi\in\Phi} \sum_{e\sim_i f} \sum_{k=1}^n \alpha_k \cdot \mathrm{pre}(f \mid \phi) \cdot \mu_i(\phi \wedge [\mathcal{E}, f]\psi_k)$ and $D = \sum_{\phi\in\Phi} \sum_{e\sim_i f} \beta \cdot \mathrm{pre}(f \mid \phi) \cdot \mu_i(\phi)$
	Inference Rules
MP	if $\vdash A \to B$ and $\vdash A$, then $\vdash B$
Nec$_i$	if $\vdash A$, then $\vdash \Box_i A$
Nec$_\alpha$	if $\vdash A$, then $\vdash [\mathcal{E}, e]A$
Sub$_\mu$	if $\vdash A \to B$, then $\vdash \mu_i(A) \leq \mu_i(B)$
SubEq	if $\vdash A \leftrightarrow B$, then $\vdash \phi \leftrightarrow \phi[A/B]$

submodel of a certain *intermediate model* $\coprod_{\mathcal{E}} \mathbb{M}$. This account is analogous to those given in [34, 36] of the product updates of models of PAL and Baltag-Moss-Solecki's dynamic epistemic logic EAK. In each instance, the original product update construction can be illustrated by the following diagram (which uses the notation introduced in the instance treated in the previous section):

$$\mathbb{M} \hookrightarrow \coprod_{\mathcal{E}} \mathbb{M} \hookleftarrow \mathbb{M}^{\mathcal{E}}.$$

As is well known (see e.g. [19]) in duality theory, coproducts can be dually characterized as products, and subobjects as quotients. In the light of this fact, the construction of *product update*, regarded as a "subobject after coproduct" concatenation, can be dually characterized on the algebras dual to the relational structures of PES-models by means of a "quotient after product" concatenation, as illustrated in the following diagram:

$$\mathbb{A} \twoheadleftarrow \prod_{\mathcal{E}} \mathbb{A} \twoheadrightarrow \mathbb{A}^{\mathcal{E}},$$

resulting in the following two-step process. First, the coproduct $\coprod_{\mathcal{E}} M$ is dually characterized as a certain *product* $\prod_{\mathcal{E}} \mathbb{A}$, indexed as well by the states of $\mathcal{E}$, and such that $\mathbb{A}$ is the algebraic dual of $\mathbb{M}$; second, an appropriate *quotient* of $\prod_{\mathcal{E}} \mathbb{A}$ is then taken, which dually characterizes the submodel step. On which algebras are we going to apply the "quotient after product" construction? The prime candidates are the algebras associated with the PES-models via standard Stone-type duality:

Definition 127 (Complex algebra)**.** For any probabilistic epistemic state model $\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$, its *complex algebra* is the tuple

$$\mathbb{M}^+ := \big(\mathcal{P}S, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}}, (P_i^+)_{i\in\mathsf{Ag}}\big)$$

where for each $i \in \mathsf{Ag}$ and $X \in \mathcal{P}S$,

$$\begin{aligned}
\Diamond_i X &= \{s \in S \mid \exists x\, (s \sim_i x \text{ and } x \in X)\}\,,\\
\Box_i X &= \{s \in S \mid \forall x\, (s \sim_i x \implies x \in X)\}\,,\\
\mathsf{dom}(P_i^+) &= \{X \in \mathcal{P}S \mid \exists y\, \forall x\, (x \in X \implies x \sim_i y)\}\,,\\
P_i^+ X &= \sum_{x\in X} P_i(x).
\end{aligned}$$

Notice that the domain of P_i^+ consists of all the subsets of the equivalence classes of $\sim_i$.

In this setting, the "quotient after product" construction behaves exactly in the desired way, in the sense that one can check *a posteriori* that the following holds:

Proposition 128. *For every PES-model $\mathbb{M}$ and any event structure $\mathcal{E}$ over $\mathcal{L}$, the algebraic structures $(\mathbb{M}^+)^{\mathcal{E}}$ and $(\mathbb{M}^{\mathcal{E}})^+$ can be identified.*

Proof. This results follows from: (1) Fact 12 in [34] that states that for any (non probabilistic) Kripke model $\mathbb{N}$, the structures $(\mathbb{N}^+)^{\mathcal{E}}$ and $(\mathbb{N}^{\mathcal{E}})^+$ can be identified, and (2) Lemma 172 on page 209 that states that the probability measures on the complex algebras $(\mathbb{M}^+)^{\mathcal{E}}$ and $(\mathbb{M}^{\mathcal{E}})^+$ are the same. □

Moreover, the "quotient after product" construction holds in much greater generality than the class of complex algebras of PES-models, which is exactly its added value over the update on relational structures. In the following section, we are going to define it in detail in the setting of epistemic Heyting algebras.

5.4 Updates on finite Heyting algebras

The present section aims at introducing the algebraic counterpart of the event update construction presented in Section 5.2. For the sake of enforcing a neat separation between syntax and semantics, throughout the present section, we will disregard the logical language $\mathcal{L}$, and work on *algebraic probabilistic epistemic structures* (APE-structures, see Definition 141) rather than on APE-models (i.e. APE-structures endowed with valuations). To be able to define the update construction, we will need to base our treatment on a modified definition of event structure over an algebra, rather than over $\mathcal{L}$.

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Structure of the section. In Section 5.4.1, we introduce epistemic Heyting algebras. In Section 5.4.2, we recall the definition of intuitionistic probability from [44] and endow epistemic Heyting algebras with measures to define algebraic probabilistic epistemic structures. In Section 5.4.3, we define probabilistic event structures over epistemic algebras, as the intuitionistic algebraic counterparts of classical probabilistic event structures. In Section 5.4.4, we introduce the construction of intermediate pre-probabilistic event structure as the first step of the algebraic event update construction. Finally, in Section 5.4.5, we introduce the pseudo-quotient update construction and define the event update on algebraic probabilistic epistemic structures.

5.4.1 Epistemic Heyting algebras

In this section we introduce epistemic Heyting algebras. We start by recalling the definition of monadic Heyting algebras, which provide algebraic semantics for the logic MIPC, the intuitionistic analogue of the classical modal logic S5 (cf. [10, 11, 34]). Then, we introduce the concept of i-minimal elements of monadic Heyting algebras. Finally, we define epistemic Heyting algebras as those monadic Heyting algebras whose i-minimal elements are enough to describe certain subalgebras of interest for the developments of the next sections.

Definition 129 (Monadic Heyting algebra (cf. [10])). A *monadic Heyting algebra* is a tuple

$$\mathbb{A} := (\mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$$

such that $\mathbb{L}$ is a Heyting algebra, and each $\Diamond_i$ and $\Box_i$ is a monotone unary operation on $\mathbb{L}$ such that for all $a, b \in \mathbb{L}$,

$$a \leq \Diamond_i a \tag{M1}$$
$$\Box_i a \leq a \tag{M2}$$
$$\Diamond_i(a \vee b) \leq \Diamond_i a \vee \Diamond_i b \tag{M3}$$
$$\Box_i(a \to b) \leq \Box_i a \to \Box_i b \tag{M4}$$
$$\Diamond_i a \leq \Box_i \Diamond_i a \tag{M5}$$
$$\Diamond_i \Box_i a \leq \Box_i a \tag{M6}$$
$$\Box_i(a \to b) \leq \Diamond_i a \to \Diamond_i b \tag{M7}$$
$$\Diamond_i \bot \leq \bot \tag{M8}$$
$$\top \leq \Box_i \top \tag{M9}$$

Remark 130. The algebraic and duality theoretic treatment of monadic Heyting algebras has been developed in [10] and [11]. In particular, as mentioned in [10, Lemma 2], in the presence of (M9), axiom (M4) is equivalent to $\Box_i a \wedge \Box_i b \leq \Box_i(a \wedge b)$, so all modalities are normal, and $\Diamond_i \Diamond_i a \leq \Diamond_i a$ and $\Box_i a \leq \Box_i \Box_i a$ are derivable from the axioms. These conditions correspond also in the best known intuitionistic settings to the transitivity of the associated accessibility relations (cf. [16]). This implies in particular that $\Diamond_i$ is a closure operator for each $i \in \mathsf{Ag}$.

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The next definition intends to capture algebraically the notion of equivalence cell in the epistemic space of agents. Notice that for any equivalence relation R on a set X and any $x \in X$, the equivalence cell $R[x] = R^{-1}[x] = \langle R \rangle \{x\}$ is a minimal nonempty fixed point of $\langle R \rangle$.[3] This justifies the following definition.

Definition 131 (i-minimal elements)**.** Let $\mathbb{A}$ be a monadic Heyting algebra. An element $a \in \mathbb{A}$ is *i-minimal* if

1. $a \neq \bot$,
2. $\Diamond_i a = a$ and
3. if $b \in \mathbb{A}$, $b < a$ and $\Diamond_i b = b$, then $b = \bot$.

Let $\mathsf{Min}_i(\mathbb{A})$ denote the set of the i-minimal elements of $\mathbb{A}$.

Remark 132. Notice that, for any $b \in \mathbb{A} \setminus \{\bot\}$, there exists at most one $a \in \mathsf{Min}_i(\mathbb{A})$ such that $b \leq a$. Indeed every such a must coincide with $\Diamond_i b$.

[3] Recall that, for any binary relation $R \subseteq X \times X$, we define the maps R, R^{-1} and $\langle R \rangle$ as follows:

$$\begin{aligned} R \,:\, & X \to \mathcal{P}X \\ & x \mapsto \{x' \in X \mid (x, x') \in R\} \end{aligned} \qquad \begin{aligned} R^{-1} \,:\, & X \to \mathcal{P}X \\ & x \mapsto \{x' \in X \mid (x', x) \in R\} \end{aligned}$$

$$\begin{aligned} \langle R \rangle \,:\, & \mathcal{P}X \to \mathcal{P}X \\ & S \mapsto \{x' \in X \mid \exists x \in S,\ (x', x) \in R\}. \end{aligned}$$

Definition 133 (Epistemic Heyting algebra)**.** An *epistemic Heyting algebra* is a finite monadic Heyting algebra

$$\mathbb{A} := (\mathbb{L}, (\Diamond_i)_{i\in \mathsf{Ag}}, (\Box_i)_{i\in \mathsf{Ag}})$$

such that for every $i \in \mathsf{Ag}$ and every $a \in \mathbb{A}$ the following holds:

$$\Diamond_i a \vee \neg\Diamond_i a = \top. \tag{E}$$

Remark 134. The axiom above captures algebraically the requirement that i-minimal elements, representing cells in the partition, cover the whole space.

In the remainder of the present section, $\mathbb{A}$ will denote an epistemic Heyting algebra.

Lemma 135. *If $\mathbb{A}$ is an Epistemic Heyting algebra, then, for every agent i,*

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$$\Diamond_i\mathbb{A} := \{\Diamond_i a \in \mathbb{A} \mid a \in \mathbb{A}\}$$

is a Boolean sub-algebra of $\mathbb{A}$. Furthermore, if

$$\Box_i\mathbb{A} := \{\Box_i a \in \mathbb{A} \mid a \in \mathbb{A}\},$$

then $\Diamond_i\mathbb{A} = \Box_i\mathbb{A}$.

Proof. That $\Diamond_i\mathbb{A}$ is a subalgebra of $\mathbb{A}$ follows from the fact that the equalities

$$\Diamond_i(\Diamond_i a \wedge b) = \Diamond_i a \wedge \Diamond_i b \qquad \text{and} \qquad \Diamond_i(\Diamond_i a \to \Diamond_i b) = \Diamond_i a \to \Diamond_i b$$

hold in every monadic Heything algebra (see for example [10, Lemma 2]). That $\Diamond_i\mathbb{A}$ is a Boolean algebra follows from the axiom (E) : $\Diamond_i a \vee \neg\Diamond_i a = \top$.

Finally, we can easily prove that $\Diamond_i\mathbb{A} = \Box_i\mathbb{A}$ using the axioms (M1), (M2), (M5) and (M6). ☐

Remark 136. Given the fact that Epistemic Heyting algebras are finite and since $\Diamond_i\mathbb{A}$ is a Boolean algebra, it is not hard to see that i-minimal elements are the atoms of $\Diamond_i\mathbb{A}$ and hence $\bigvee \mathsf{Min}_i(\mathbb{A}) = \top$.

Notation 137. For any poset (partially ordered set) $\mathbb{P} = (P, \leq)$, we let

$$\begin{aligned} \downarrow_{\mathbb{P}} :\ & \mathcal{P}\mathbb{P} \to \mathcal{P}\mathbb{P} \\ & X \mapsto X{\downarrow_{\mathbb{P}}} := \{x' \in \mathbb{P} \mid x' \leq x \text{ for some } x \in X\}. \end{aligned}$$

For the sake of readability, we drop the subscript and let $X\downarrow$ denote the downset generated by X. In addition, if $X = \{x\}$, we let $x\downarrow$ denote the downset generated by $\{x\}$.

5.4.2 Algebraic probabilistic epistemic structures

In this section, we introduce i-premeasures and i-measures and define algebraic pre-probabilistic and probabilistic epistemic structures which will serve as the underlying structures of intuitionistic probabilistic epistemic logic.

The following definition is an adaptation of a proposal of Weatherson's (see [44, page 2]) in which the notion of probability is generalised and made parametric in a given consequence relation. Even though there is no consensus on what an intuitionistic probability function should be, Weatherson's proposal captures necessary conditions for such a function and establishes a systematic link between logic and probability. The definition below has also been adopted by [3, 23].

Definition 138 (Intuitionistic probability measures)**.** Let $\mathbb{H}$ be a Heyting algebra. A function $\texttt{Pr} : \mathbb{H} \to [0,1]$ is *an intuitionistic probability measure* if the following conditions are satisfied: for all $a, b \in \mathbb{H}$,

$$\begin{aligned}
&\texttt{Pr}(\bot) = 0,\\
&\texttt{Pr}(\top) = 1,\\
&\text{if } a \leq_{\mathbb{H}} b, \text{ then } \texttt{Pr}(a) \leq \texttt{Pr}(b),\\
&\texttt{Pr}(a) + \texttt{Pr}(b) = \texttt{Pr}(a \vee b) + \texttt{Pr}(a \wedge b).
\end{aligned}$$

Notice that, for intuitionistic probability measures, it does no longer hold that $\Pr(p \vee \neg p) = 1$.

Given that, in classical PDEL, the probability functions range over equivalence classes instead of the whole model, we need to mirror that fact by defining probability functions that are probability measures on the quotient algebras generated by i-minimal elements.

Definition 139 (i-premeasure & i-measure)**.** A partial function $\mu : \mathbb{A} \to \mathbb{R}^+$ is an i*-premeasure* on $\mathbb{A}$, if it satisfies the following properties:

1. $\mathrm{dom}(\mu) = \mathrm{Min}_i(\mathbb{A}){\downarrow}$;
2. μ is order-preserving;
3. for every $a \in \mathrm{Min}_i(\mathbb{A})$ and all $b, c \in a{\downarrow}$, we have $\mu(b \vee c) = \mu(b) + \mu(c) - \mu(b \wedge c)$;
4. $\mu(\bot) = 0$ if $\mathrm{dom}(\mu) \neq \varnothing$.

An i-premeasure on $\mathbb{A}$ is an i*-measure*, if it satisfies the following properties:

5. $\mu(a) = 1$ for every $a \in \mathrm{Min}_i(\mathbb{A})$.
6. for every $a \in \mathrm{Min}_i(\mathbb{A})$ and all $b, c \in a{\downarrow}$ such that $b < c$, it holds that $\mu(b) < \mu(c)$;

Condition (1) ensures that the probability measures are defined on the quotient algebras generated by i-minimal elements. Conditions (2) to (5) are imported from Wheatherson's definition of intuitionistic probabilistic functions. Condition (6) corresponds to the fact that in the classical case, the probability distributions over the elements of the equivalence classes do not take value 0 (see Definition 112, page 176)

Remark 140. In the case when $\mathsf{Min}_i(\mathbb{A})\downarrow = \varnothing$, there exists a unique i-(pre)measure, the empty function. Throughout this section, all the results regarding i-minimal elements and i-(pre)measure hold vacuously in the case when $\mathsf{Min}_i(\mathbb{A})\downarrow = \varnothing$.

Definition 141 (ApPE-structure & APE-structure)**.** An *algebraic pre-probabilistic epistemic structure (ApPE-structure)* is a tuple

$$\mathcal{F} := (\mathbb{A}, (\mu_i)_{i \in \mathsf{Ag}})$$

such that

1. $\mathbb{A}$ is an epistemic Heyting algebra (see Definition 133), and
2. each μ_i is an i-premeasure on $\mathbb{A}$.

An ApPE-structure $\mathcal{F}$ is an *algebraic probabilistic epistemic structure (APE-structure)* if each μ_i is an i-measure on $\mathbb{A}$.

We refer to $\mathbb{A}$ as the *support* of $\mathcal{F}$ and we denote it $\mathsf{support}(\mathcal{F})$.

The algebraic epistemic structure associated to a classical model.

Lemma 142. *For any PES-model $\mathbb{M}$, the i-minimal elements of its complex algebra $\mathbb{M}^+$ are exactly the equivalence classes of $\sim_i$.*

Proof. Let $\mathbb{M} = \langle S, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, [\![\cdot]\!]\rangle$ be a probabilistic epistemic state model and $\mathbb{M}^+ = \big(\mathcal{P}S, (\Diamond_i)_{i \in \mathsf{Ag}}, (\Box_i)_{i \in \mathsf{Ag}}, (P_i^+)_{i \in \mathsf{Ag}}\big)$ be its complex algebra. For any $i \in \mathsf{Ag}$ and any $s \in S$, let $[s]_i$ be the $\sim_i$-equivalence cell of s. Fix $i \in \mathsf{Ag}$.

First, let us prove that any $\sim_i$-equivalence cell corresponds to an i-minimal element of $\mathbb{M}^+$. Since $\sim_i$ is reflexive, $[s]_i \neq \varnothing$. Since $\sim_i$ is symmetric and transitive, $[s]_i = \Diamond_i\{s\} = \Diamond_i\Diamond_i\{s\} = \Diamond_i[s]_i$. This shows that $[s]_i$ is a fixed-point of $\Diamond_i$. It remains to show that $[s]_i$ is a minimal fixed-point $\Diamond_i$. Let $X \subseteq S$ be an i-minimal element of $\mathbb{M}^+$. By definition, we have that $X \subseteq [s]_i$, $X \neq \varnothing$ and $\Diamond_i X = X$. The assumption that $\Diamond_i X = X$ implies that $X = \bigcup_{x \in X} \Diamond_i\{x\} = \bigcup_{x \in X}[x]_i$. The assumption that $X \subseteq [s]_i$ implies that all $x \in X$ must be $\sim_i$-equivalent to s, and hence to each other. Therefore, X cannot be the union of more than one equivalence cell. Moreover, the assumption that $X \neq \varnothing$ implies that there exists at least one equivalence cell in $\bigcup_{x \in X}[x]_i$. This concludes the proof that, for any $s \in S$, its $\sim_i$-equivalence cell $[s]_i$ corresponds to an i-minimal element of $\mathbb{M}^+$, as required.

Now, let us prove that any i-minimal element of $\mathbb{M}^+$ correspond to the $\sim_i$-equivalence cell of an element $s \in S$. Let X be an i-minimal element of $\mathbb{M}^+$. The assumption that $X = \Diamond_i X$ implies that $X = \bigcup_{x \in X}[x]_i$. The assumption that $X \neq \varnothing$ implies that there exists at least one equivalence cell $[s]_i$ in $\bigcup_{x \in X}[x]_i$. Since $[s]_i$ is an i-minimal element of $\mathbb{M}^+$ and $[s]_i \subseteq X$, we have $X = [s]_i$ by minimality of X. □

Proposition 143. *For any PES-model $\mathbb{M}$, its complex algebra $\mathbb{M}^+$ (see Definition 127) is an APE-structure (see Definition 141).*

Proof. Let $\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$ be a PES-model (see Definition 112) and let $\mathbb{M}^+ = \big(\mathcal{P}S, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}}, (P_i^+)_{i\in\mathsf{Ag}}\big)$ be its complex algebra. $\mathbb{M}^+$ is an APE-structure if its support is an epistemic Heyting algebra and if each P_i^+ is an i-measure over $\langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}\rangle$. Clearly, $(\mathcal{P}S, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$ is an epistemic Heyting algebra (see Definition 133), since $\sim_i$ is an equivalence relation and $\mathcal{P}S$ is a boolean algebra. To finish the proof we need to show that each P_i^+ is an i-measure on support($\mathbb{M}^+$). Hence, for every $i \in \mathsf{Ag}$, we need to prove the following properties:

(a) dom(P_i^+) = Min$_i$(support$(\mathbb{M}^+))\!\downarrow$;

(b) P_i^+ is order-preserving;

(c) for every i-minimal element $X \in \mathcal{P}S$ and all $Y_1, Y_2 \in X\!\downarrow$, we have

$$P_i^+(Y_1 \cup Y_2) = P_i^+(Y_1) + P_i^+(Y_2) - P_i^+(Y_1 \cap Y_2);$$

(d) $P_i^+(\varnothing) = 0$ if dom$(P_i^+) \neq \varnothing$;

(e) for every i-minimal element $X \in \mathcal{P}S$, we have $P_i^+(X) = 1$.

(f) for every i-minimal element $X \in \mathcal{P}S$ and all $Y_1, Y_2 \in X\!\downarrow$ such that $Y_1 \subset Y_2$, it holds that

$$P_i^+(b) < P_i^+(c);$$

Fix $i \in \mathsf{Ag}$.

`Proof of (a).` By definition, dom$(P_i^+) = \{X \in \mathcal{P}S \mid \exists y\, \forall x\, (x \in X \implies x \sim_i y)\}$. Notice that

$$\{X \in \mathcal{P}S \mid \exists y\, \forall x\, (x \in X \implies x \sim_i y)\} = \{X \mid X \subseteq [s] \text{ and } s \in S\}.$$

By Lemma 142, we deduce that dom(P_i^+) = Min$_i$(support$(\mathbb{M}^+))\!\downarrow$.

`Proof of (b).` Since $P_i(s) \geq 0$ for all $s \in S$, the maps P_i^+ are monotone.

`Proof of (c).` By Lemma 142, if X is an i-minimal element of $\mathbb{M}^+$, then $X = [s]$ for some $s \in S$. If $Y_1, Y_2 \in X\!\downarrow$, then $Y_1 \cup Y_2 \subseteq [s]$. Hence,

$$\begin{aligned} P_i^+(Y_1 \cup Y_2) &= \sum_{x \in Y_1 \cup Y_2} P_i(x) && \text{(Definition of } P_i^+\text{)}\\ &= \sum_{x\in Y_1} P_i(x) + \sum_{x \in Y_2} P_i(x) - \sum_{x \in Y_1 \cap Y_2} P_i(x)\\ &= P_i^+(Y_1) + P_i^+(Y_2) - P_i^+(Y_1 \cap Y_2). && \text{(Definition of } P_i^+\text{)} \end{aligned}$$

`Proof of (d).` By definition, $P_i^+(\varnothing) = 0$.

`Proof of (e).` Let $X \in \mathcal{P}S$ be an i-minimal element. By Lemma 142, there exists an $s \in S$ such that $[s] = X$. Hence, using the definition of P_i (see Definition 127), we have:

$$P_i^+(X) = \sum_{x \in [s]} P_i(x) = 1.$$

`Proof of (f).` Let $X \in \mathcal{P}S$ be i-minimal element and $Y_1, Y_2 \in X\!\downarrow$ such that $Y_1 \subset Y_2$. By definition, we have that

$$\begin{aligned} P_i^+(Y_2) &= \sum_{x \in Y_2} P_i(x) \\ &= \sum_{x \in Y_1} P_i(x) + \sum_{x \in Y_2 \smallsetminus Y_1} P_i(x) \\ &= P_i^+(Y_1) + \sum_{x \in Y_2 \smallsetminus Y_1} P_i(x). \end{aligned}$$

Since $Y_1 \subset Y_2$, there exists $s \in Y_2 \smallsetminus Y_1$. Since $P_i : S \to]0,1]$, we have $P_i(s) > 0$ for all $s \in Y_2 \smallsetminus Y_1$. Hence $\sum_{x \in Y_2 \smallsetminus Y_1} P_i(x) > 0$ and $P_i^+(Y_1) < P_i^+(Y_2)$.

□

5.4.3 Probabilistic event structures over epistemic HAs

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In this section, we introduce intuitionistic event structures, which are needed to correctly generalise probabilistic epistemic updates to an intuitionistic metatheory.

We will find it useful to introduce the following auxiliary definitions. Recall that a *multiset* is a generalisation of the concept of set that allows multiple instances of the same element. Hence, $\{a, a, b\}$ and $\{a, b\}$ are the same set, but different multisets. However, order does not matter, so $\{a, a, b\}$ and $\{a, b, a\}$ are the same multiset. Let Φ be a multiset on the set X and $a, b \in \Phi$. We say that a and b *arise from the same element* if a and b are copies of the same element from X. We denote it $a =_X b$.

Definition 144 (Ordered multiset on a lattice)**.** Let $\mathbb{L} = (L, \leq)$ be a finite lattice. An *ordered multiset* $\mathbf{\Phi} = (\Phi, \prec)$ on $\mathbb{L}$ is a multiset Φ of elements of L equipped with a strict order $\prec$ such that, for all pairwise distinct elements $x, y, z \in \Phi$,

1. if $x \prec y$, then $x \leq_{\mathbb{L}} y$;
2. if $x \neq \bot$ and $x \leq_{\mathbb{L}} y$, then $x \prec y$ or $y \prec x$;
3. if $x \prec y$ and $x \prec z$, then $y \prec z$ or $z \prec y$.

In the present chapter, we use the membership symbol $\in$ in the context of multisets on $\mathbb{L}$ always referring to the copies of a given element of $\mathbb{L}$. For instance, the variable y in the symbol $y \in \Phi$ refers to one specific copy of some element of $\mathbb{L}$.

Remark 145. In Section 5.5, we will be working with event structures over logical languages rather than with event structures over algebras (see Definition 146). Event structures over languages (see Definition 174) are tuples where Φ is a set of formulas each pair of which is made either of incompatible formulas or of formulas one of which implies the other. However, some of these formulas might be identified with each other under some valuations. In order to define updates on algebras independently from logic, in Definition 146 the ordered multisets above will play the same role played by the sets

Φ in event structures over languages. Specifically, the multiset structure serves to keep track of the fact that some elements of the lattice might be the interpretation of more than one formula in the set Φ, and the order on the multiset $\mathbf{\Phi}$ helps to keep track of the logical structure of the set Φ. Finally, condition 3 makes sure that the order structure of $\mathbf{\Phi}$ is an upward forest, and conditions 1 and 2 together guarantee that, with the exception of formulas which are mapped to $\bot$, the logical structure of the set Φ is preserved and reflected by the order $\prec$.

Now let us introduce probabilistic event structures in the intuitionistic setting:

Definition 146 (Probabilistic event structure over an epistemic Heyting algebra)**.** For any epistemic Heyting algebra $\mathbb{A}$ (see Definition 133), a *probabilistic event structure over* $\mathbb{A}$ is a tuple

$$\mathbb{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \mathbf{\Phi}, \overline{\mathsf{pre}})$$

such that

1. E is a non-empty finite set;
2. each $\sim_i$ is an equivalence relation on E;
3. each $P_i : E \to]0, 1]$ assigns a probability distribution over each $\sim_i$-equivalence class, i.e.
$$\sum \{P_i(e') \mid e' \sim_i e\} = 1;$$
4. $\mathbf{\Phi} = (\Phi, \prec)$ is a finite ordered multiset on $\mathbb{A}$ such that, for all $a, b \in \Phi$ which arise from distinct elements in $\mathbb{A}$, either
$$a \wedge_{\mathbb{A}} b = \bot \quad \text{or} \quad a <_{\mathbb{A}} b \quad \text{or} \quad b <_{\mathbb{A}} a;$$
5. the map $\overline{\mathsf{pre}} : E \times \Phi \to [0, 1]$ assigns a probability distribution $\overline{\mathsf{pre}}(\bullet|a)$ over E for every $a \in \Phi$;
6. for all $a \in \Phi$ and $e \in E$, if $\overline{\mathsf{pre}}(e|a) = 0$ then $\overline{\mathsf{pre}}(e|b) = 0$ for all $b \in \Phi$ such that $a \prec b$.

The definition above is a proper generalization of the analogous definition given in the classical setting (Definition 116). The main generalization concerns the fact that the elements in Φ (which are the potential interpretants of formulas) are no longer required to be mutually inconsistent but may also be 'logically dependent'. In this latter case, the precondition function is required to satisfy an additional compatibility condition which is similar to the one adopted in [3]. For the sake of readability, in what follows, we will simply refer to probabilistic event structures over epistemic Heyting algebras as *event structures*.

Remark 147 (The substitution map)**.** Clearly, a purely algebraic counterpart of the substitution map which was part of the definition of *probabilistic event structures over a language* (see Definition 116) cannot be given.

Remark 148 (The order $\leq_{\mathbb{A}}$ on the set Φ)**.** The classical and the intuitionistic setting are distinguished by the fact that states are pairwise incomparable in the classical setting and (non-trivially) ordered in the intuitionistic setting. Thus, in probabilistic event structures over a language (see Definition 116) it is enough to require the set Φ to contain mutually inconsistent formulas in order to tell apart states of the Kripke model. However, due to the order between states of intuitionistic Kripke frames, mutually incompatible formulas are not enough to separate distinct but comparable states. To overcome this hurdle we require Φ to satisfy the following condition: for all $a_k, a_j \in \Phi$,

$$a_j \wedge a_k = \bot \quad \text{or} \quad a_j < a_k \quad \text{or} \quad a_k < a_j.$$

This condition makes it possible to compute the probabilities of a given non-maximal state, even if there is no proposition uniquely identifying this state (cf. Definition 152).

5.4.4 The intermediate (pre-)probabilistic epistemic structure

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In the present subsection, we define the intermediate ApPE-structure $\prod_{\mathbb{E}} \mathcal{F}$ associated with any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support of $\mathcal{F}$ (see Definition 141 for the definition of support):

$$\prod_{\mathbb{E}} \mathcal{F} := \left(\prod_{\mathbb{E}} \mathbb{A}, (\mu'_i)_{i \in \mathsf{Ag}} \right). \tag{5.4.1}$$

Structure of the subsection. First, we define the intermediate algebra $\prod_{|E|} \mathbb{A}$ which will become the support of the intermediate ApPE-structure $\prod_{\mathbb{E}} \mathcal{F}$ (see Definition 149 and Proposition 150) and we identify its i-minimal elements (see Proposition 151). Then, we introduce the i-premeasures on the intermediate algebra (see Definition 155 and Proposition 156). Finally, we show that the definition ApPE-structure is coherent with the relational semantics in the classical case (see Proposition 160).

The intermediate algebra and its i-minimal elements

Definition 149 (Intermediate algebra)**.** Given any epistemic Heyting algebra $\mathbb{A} = (\mathbb{L}, (\Diamond_i)_{i \in \mathsf{Ag}}, (\Box_i)_{i \in \mathsf{Ag}})$ and any event structure $\mathbb{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \boldsymbol{\Phi}, \overline{\mathsf{pre}})$ over $\mathbb{A}$, let *the intermediate algebra* be

$$\prod_{\mathbb{E}} \mathbb{A} := (\prod_{|E|} \mathbb{L}, \{\Diamond'_i, \Box'_i \mid i \in \mathsf{Ag}\}),$$

where

1. $\prod_{|E|} \mathbb{L}$ is the $|E|$-fold power of $\mathbb{L}$, the elements of which can be seen either as $|E|$-tuples of elements in $\mathbb{A}$, or as maps $f : E \to \mathbb{A}$;
2. for any $f : E \to \mathbb{A}$, let us define $\Diamond'_i(f)$ as follows:

$$\begin{aligned} \Diamond'_i(f) &: E \to \mathbb{A} \\ &e \mapsto \bigvee \{\Diamond_i f(e') \mid e' \sim_i e\}; \end{aligned}$$

3. for any $f : E \to \mathbb{A}$, let us define $\Box'_i(f)$ as follows:

$$\begin{aligned}\Box'_i(f) : E &\to \mathbb{A}\\ e &\mapsto \bigwedge\{\Box_i f(e') \mid e' \sim_i e\}.\end{aligned}$$

Below, the algebra $\prod_{\mathbb{E}} \mathbb{A}$ will be sometimes abbreviated as $\mathbb{A}'$.

We refer to [34, Section 3.1] for an extensive justification of the definition of the operations $\Diamond'_i$ and $\Box'_i$.

Proposition 150. *For every epistemic Heyting algebra $\mathbb{A}$ and every event structure $\mathbb{E}$ over $\mathbb{A}$, the algebra $\mathbb{A}'$ is an epistemic Heyting algebra.*

Proof. To prove that $\mathbb{A}'$ is an epistemic Heyting algebra (Definition 133), we need to show that $\mathbb{A}'$ is a monadic Heyting algebra such that for every $i \in$ Ag and every $f \in \mathbb{A}'$, we have: $\Diamond_i f \vee \neg\Diamond_i f = \top$.

The proof that $\mathbb{A}'$ is a monadic Heyting algebra can be found in [34, Proposition 8.1]. Let $i \in$ Ag, $f \in \mathbb{A}'$, and $e \in E$. We have:

$$\begin{aligned}(\Diamond'_i f \vee \neg\Diamond'_i f)(e) &= (\Diamond'_i f)(e) \vee \neg(\Diamond'_i f)(e)\\ &= \bigvee\{\Diamond_i(f(e')) \mid e' \sim e\} \vee \neg\bigvee\{\Diamond_i(f(e')) \mid e' \sim e\}\\ &\qquad\text{(by definition of } \Diamond'_i)\\ &= \Diamond_i \bigvee\{f(e') \mid e' \sim e\} \vee \neg\Diamond_i \bigvee\{f(e') \mid e' \sim e\}\\ &\qquad\text{(by the normality of } \Diamond_i)\\ &= \top. \qquad\text{(since } \Diamond_i a \vee \neg\Diamond_i a = \top)\end{aligned}$$

Hence, $(\Diamond'_i f \vee \neg\Diamond'_i f)(e) = \top$ for all $e \in E$, which by definition yields that $\Diamond'_i f \vee \neg\Diamond'_i f = \top$.

□

Proposition 151. *For every epistemic Heyting algebra $\mathbb{A}$ and every agent $i \in$ Ag,*

$$\text{Min}_i(\mathbb{A}') = \{f_{e,a} \mid e \in E \text{ and } a \in \text{Min}_i(\mathbb{A})\},$$

where for any $e \in E$ and $a \in \text{Min}_i(\mathbb{A})$, the map $f_{e,a}$ is defined as follows:

$$\begin{aligned}f_{e,a} : E &\to \mathbb{A}\\ e' &\mapsto \begin{cases} a & \text{if } e' \sim_i e\\ \bot & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. Recall that $f \in \mathbb{A}'$ is an i-minimal element (see Definition 131) if it satisfies the following conditions: (1) $f \neq \bot$, (2) $\Diamond_i f = f$ and (3) if $g \in \mathbb{A}$, $g < f$ and $\Diamond_i g = g$, then $g = \bot$.

Let us first prove that any map $f_{e,a}$ as above is an i-minimal element of $\mathbb{A}'$. By definition, $f_{e,a}(e) = a \neq \bot_{\mathbb{A}}$. Hence $f_{e,a} \neq \bot_{\mathbb{A}'}$. As to showing that $\blacklozenge'_i f_{e,a} = f_{e,a}$, fix $e' \in E$, and let us show that $(\blacklozenge'_i f_{e,a})(e') = f_{e,a}(e')$. By definition,

$$\blacklozenge'_i f_{e,a}(e') = \bigvee\{\blacklozenge_i f_{e,a}(e'') \mid e'' \sim_i e'\}.$$

We proceed by cases: (a) If $e' \sim_i e$, then:

$$\begin{aligned}
\blacklozenge'_i f_{e,a}(e') &= \bigvee\{\blacklozenge_i f_{e,a}(e'') \mid e'' \sim_i e'\} && \text{(by definition)}\\
&= \bigvee\{\blacklozenge_i a \mid e'' \sim_i e'\}\\
& (\ f_{e,a}(e'') = a, \text{ since } e \sim_i e' \text{ and } \sim_i \text{ symmetric and transitive})\\
&= \blacklozenge_i a && \text{(the join is nonempty since } \sim_i \text{ is reflexive)}\\
&= a && (a \text{ is } i\text{-minimal, hence is a fixed point of } \lozenge_i)\\
&= f_{e,a}(e'). && \text{(definition of } f_{e,a} \text{ and } e' \sim_i e)
\end{aligned}$$

(b) If $e' \nsim e$, then:

$$\begin{aligned}
\blacklozenge'_i f_{e,a}(e') &= \bigvee\{\blacklozenge_i f_{e,a}(e'') \mid e'' \sim_i e'\} && \text{(by definition)}\\
&= \bigvee\{\blacklozenge_i \bot \mid e'' \sim_i e'\} && (e \nsim_i e')\\
&= \blacklozenge_i \bot\\
&= \bot && (\lozenge_i \bot = \bot)\\
&= f_{e,a}(e').
\end{aligned}$$

Finally, we need to show that $f_{e,a}$ is a minimal non-bottom fixed-point of $\blacklozenge'_i$. Notice preliminarily that if $g : E \to \mathbb{A}$ is a fixed point for $\blacklozenge'_i$ then

$$g(e) = g(e') \text{ whenever } e \sim_i e'. \tag{5.4.2}$$

Indeed,

$$g(e) = (\blacklozenge'_i g)(e) = \bigvee\{\blacklozenge_i g(e'') \mid e'' \sim_i e\} = \bigvee\{\blacklozenge_i g(e'') \mid e'' \sim_i e'\} = (\blacklozenge'_i g)(e') = g(e').$$

Given that $\sim_i$ is reflexive, this implies in particular that, for every $e' \in E$,

$$(\blacklozenge'_i g)(e') = \blacklozenge_i g(e'). \tag{5.4.3}$$

Let g be as above, assume that $\bot \neq g \leq f_{e,a}$, and let us show that $g = f_{e,a}$. Clearly, the assumption $g \leq f_{e,a}$ implies that $g(e') = \bot$ for every $e' \in E$ such that $e' \not\sim_i e$. Let $e' \in E$ such that $g(e') \neq \bot$. Together with the assumption that $g \leq f_{e,a}$, this implies that $f_{e,a}(e') \neq \bot$, hence $e' \sim_i e$ and $\bot \neq g(e') \leq a$. To prove that $g(e) = a$, by the i-minimality of a it suffices to show that $g(e')$ is a fixed point of $\blacklozenge_i$. Indeed, by (5.4.3):

$$\blacklozenge_i g(e') = (\blacklozenge'_i g)(e') = g(e'),$$

as required. Finally, the fact above and the preliminary observation (5.4.2) imply that $g(e') = a$ for every $e' \in E$ such that $e' \sim_i e$.

This finishes the proof that $f_{e,a}$ is i-minimal.

Conversely, let $g : E \to \mathbb{A}$ be i-minimal in $\mathbb{A}'$, and let us show that $g = f_{e,a}$ for some $e \in E$ and some i-minimal element $a \in \mathbb{A}$. The assumption that $g \neq \bot$ implies that

$g(e) \neq \bot$ for some $e \in E$. Let $g(e) = a \in \mathbb{A}$. Then, the assumption that $g = \blacklozenge'_i g$ and the observation (5.4.2) imply that $g(e') = a$ for every $e' \in E$ such that $e' \sim_i e$. Then, the proof is finished if we show that a is i-minimal in $\mathbb{A}$. Indeed, then, by construction we would have $\bot \neq f_{e,a} \leq g$, hence the minimality of g would yield $f_{e,a} = g$.

By definition, we have that $a = g(e') \neq \bot$. By observation (5.4.3),

$$\blacklozenge_i a = \blacklozenge_i g(e) = (\blacklozenge'_i g)(e) = g(e) = a,$$

which shows that a is a fixed point of $\blacklozenge_i$. Finally, let $\bot \neq b \leq a$ such that $\blacklozenge_i b = b$. Then, with an argument analogous to the one given above, the map $f_{e,b} : E \to \mathbb{A}$ would be proven to be a non-bottom fixed-point of $\blacklozenge'_i$. Moreover, $f_{e,b} \leq g$, and hence the i-minimality of g would yield $f_{e,b} = g$, hence $a = b$. □

The i-premeasures on the intermediate algebra

Before providing i-premeasures for the product epistemic algebra (Definition 155 and Proposition 156), we present an auxiliary definition.

Definition 152. Let $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i\in\mathsf{Ag}})$ be an APE-structure and let $\mathbb{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \mathbf{\Phi}, \overline{\mathsf{pre}})$ be an event structure over $\mathbb{A}$. For all $a \in \Phi$ and $i \in \mathsf{Ag}$, we define the partial function $\mu_i^a : \mathbb{A} \to \mathbb{R}^+$ by

$$\mu_i^a(x) := \mu_i(x \wedge a) - \sum_{b\in\mathrm{mb}(a)} \mu_i(x \wedge b) \tag{5.4.4}$$

where $\mathrm{mb}(a)$ denotes the multiset of the $\prec$-maximal elements of $\mathbf{\Phi}$ $\prec$-below a.

We make the following observations regarding μ_i^a:

Proposition 153. *For every APE-structure $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i\in\mathsf{Ag}})$ and every event structure $\mathbb{E}$ over $\mathbb{A}$, μ_i^a is an i-premeasure over $\mathbb{A}$. Furthermore, if $a \leq y$ then $\mu_i^a(x) = \mu_i^a(x \wedge y)$.*

Proof. For every $a \in \Phi$ and every $i \in \mathsf{Ag}$, we want to prove that μ_i^a is an i-premeasure over $\mathbb{A}$, hence we need to prove that μ_i^a is a partial function $\mathbb{A} \to \mathbb{R}^+$ that satisfies items (1 - 4) of Definition 139. Fix $a \in \Phi$ and $i \in \mathsf{Ag}$.

`Proof of item 1.` We want to prove that $\mathrm{dom}(\mu) = \mathrm{Min}_i(\mathbb{A})\downarrow$. The map μ_i is an i-premeasure, hence $\mathrm{dom}(\mu_i) = \mathrm{Min}_i(\mathbb{A})\downarrow$. Therefore the map μ_i^a is only defined on $\mathrm{Min}_i(\mathbb{A})\downarrow$ and $\mathrm{dom}(\mu_i^a) = \mathrm{Min}_i(\mathbb{A})\downarrow$.

`Proof that` μ_i^a `is well-defined.` We need to prove that $\mu_i^a(x) \geq 0$ for all $x \in \mathrm{Min}_i(\mathbb{A})\downarrow$. Recall that $\mathbf{\Phi}$ is a finite ordered multiset of elements of $\mathbb{A}$ such that, for all distinct $b, c \in \Phi$, either $b \wedge c = \bot$ or $b < c$ or $c < b$ (see Definition 146 and Remark 148). Hence, for every $b, c \in \mathrm{mb}(a)$ we have $b \wedge c = \bot$. Indeed, by item 2 of Definition 144 and what was mentioned above, if $b \wedge c \neq \bot$, then either $b \prec c$ or $c \prec b$. Hence, they cannot both be maximal.

Fix $x \in \mathrm{Min}_i(\mathbb{A})\downarrow$. Let us prove by induction on the size of S that for any $S \subseteq \mathrm{mb}(a)$,

$$\mu_i\left(\bigvee_{b\in S} x \wedge b\right) = \sum_{b\in S} \mu_i(x \wedge b). \tag{5.4.5}$$

`Base case :` $|S| = 0$. Assume that $S = \varnothing$. Then, we trivially have that

$$\mu_i(\bigvee_{b \in S} x \wedge b) = \mu_i(\bot) = 0 = \sum_{b \in S} \mu_i(x \wedge b). \qquad (\mathtt{IH_0})$$

`Induction step :` $\mathtt{IH_n} \Rightarrow \mathtt{IH_{n+1}}$. Assume that, for any set S' that contains exactly n elements, we have

$$\mu_i(\bigvee_{b' \in S'} x \wedge b') = \sum_{b' \in S'} \mu_i(x \wedge b'). \qquad (\mathtt{IH_n})$$

Let S contain exactly $n+1$ elements, $S' \subset S$ contain exactly n elements, and $S = S' \cup \{c\}$. Let us prove $\mathtt{IH}_{n+1}$:

$$\begin{aligned}
&\mu_i\left(\bigvee_{b \in S} x \wedge b\right) \\
&= \mu_i\left((x \wedge c) \vee \bigvee_{b' \in S'} (x \wedge b')\right) && (S = S' \cup \{c\}) \\
&= \mu_i(x \wedge c) + \mu_i\left(\bigvee_{b' \in S'} x \wedge b'\right) - \mu_i\left((x \wedge c) \wedge \bigvee_{b' \in S'} (x \wedge b')\right) \\
& && (\mu_i \text{ is an } i\text{-premeasure}) \\
&= \mu_i(x \wedge c) + \mu_i\left(\bigvee_{b' \in S'} x \wedge b'\right) - \mu_i\left(\bigvee_{b' \in S'} x \wedge c \wedge x \wedge b'\right) && (\wedge \text{ distributes over } \vee) \\
&= \mu_i(x \wedge c) + \mu_i\left(\bigvee_{b' \in S'} x \wedge b'\right) - \mu_i(\bot) && (c \neq b' \text{ implies } c \wedge b' = \bot) \\
&= \mu_i(x \wedge c) + \sum_{b' \in S'} \mu_i(x \wedge b') && (\mu_i(\bot) = 0 \text{ and } (\mathtt{IH_n})) \\
&= \sum_{b \in S} \mu_i(x \wedge b) && (S = S' \cup \{c\})
\end{aligned}$$

By induction, for any $x \in \mathsf{Min}_i(\mathbb{A})\!\downarrow$, we have $\mu_i\left(\bigvee_{b \in \mathrm{mb}(a)} x \wedge b\right) = \sum_{b \in \mathrm{mb}(a)} \mu_i(x \wedge b)$.

Since $\mathrm{mb}(a)$ denotes the set of the $\prec$-maximal elements of $(\mathbf{\Phi} \cap {\downarrow} a) \setminus \{a\}$, we have that $\bigvee_{b \in \mathrm{mb}(a)} x \wedge b \leq x \wedge a$. By monotonicity of μ_i, we get that

$$\sum_{b \in \mathrm{mb}(a)} \mu_i(x \wedge b) = \mu_i\left(\bigvee_{b \in \mathrm{mb}(a)} x \wedge b\right) \leq \mu_i(x \wedge a).$$

Hence, $\mu_i^a(x) \geq 0$ for any $x \in \mathsf{Min}_i(\mathbb{A})\!\downarrow$ as required.

`Proof of item 2.` We want to show that μ_i^a is order-preserving. Using (5.4.5) and the fact that $\wedge$ distributes over $\vee$, we get that: for any $x \in \mathsf{Min}_i(\mathbb{A})\!\downarrow$,

$$\sum_{b\in \mathrm{mb}(a)} \mu_i\,(x\wedge b) = \mu_i\left(\bigvee_{b\in \mathrm{mb}(a)} x\wedge b\right) = \mu_i\left(x\wedge \bigvee_{b\in \mathrm{mb}(a)} b\right). \tag{5.4.6}$$

Fix $x, y \in \mathsf{Min}_i(\mathbb{A})\!\downarrow$ such that $x \leq y$. Notice that $\bigvee_{b\in\mathrm{mb}(a)} b \leq a$ and $x \wedge a \wedge y = x$. Furthermore, $x \wedge a \leq y \wedge a$ and $y \wedge (\bigvee_{b\in\mathrm{mb}(a)} b) \leq y \wedge a$. Hence $(x \wedge a) \vee (y \wedge (\bigvee_{b\in\mathrm{mb}(a)} b)) \leq y \wedge a$. From this we can deduce that:

$$
\begin{aligned}
& (x\wedge a)\vee\left(y\wedge\left(\bigvee_{b\in\mathrm{mb}(a)} b\right)\right) \leq y\wedge a \\
\Leftrightarrow\quad & \mu_i\left((x\wedge a)\vee\left(y\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right)\right) \leq \mu_i(y\wedge a) && (\mu_i \text{ is order-preserving})\\
\Leftrightarrow\quad & \mu_i(x\wedge a)+\mu_i\left(y\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right)-\mu_i\left(x\wedge a\wedge y\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right) \leq \mu_i(y\wedge a) \\
& && (\mu_i \text{ is an } i\text{-premeasure})\\
\Leftrightarrow\quad & \mu_i(x\wedge a)+\mu_i\left(y\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right)-\mu_i\left(x\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right) \leq \mu_i(y\wedge a) \\
& && (x\wedge a\wedge y = x)\\
\Leftrightarrow\quad & \mu_i(x\wedge a)-\mu_i\left(x\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right) \leq \mu_i(y\wedge a)-\mu_i\left(y\wedge \bigvee_{b\in\mathrm{mb}(a)} b\right)\\
\Leftrightarrow\quad & \mu_i(x\wedge a)-\sum_{b\in\mathrm{mb}(a)}\mu_i(x\wedge b) \leq \mu_i(y\wedge a)-\sum_{b\in\mathrm{mb}(a)}\mu_i(y\wedge b) && (\text{by } (5.4.6))\\
\Leftrightarrow\quad & \mu_i^a(x)\leq\mu_i^a(y).
\end{aligned}
$$

`Proof of item 3.` We need to show that $\mu_i^a(x \vee y) = \mu_i^a(x) + \mu_i^a(y) - \mu_i^a(x \wedge y)$ for all $x, y \in \mathsf{Min}_i(\mathbb{A})\!\downarrow$. We have:

$$
\begin{aligned}
\mu_i^a(x\vee y) &= \mu_i((x\vee y)\wedge a)-\sum_{b\in\mathrm{mb}(a)}\mu_i((x\vee y)\wedge b)\\
&= \mu_i((x\wedge a)\vee(y\wedge a))-\sum_{b\in\mathrm{mb}(a)}\mu_i((x\wedge b)\vee(y\wedge b)) && (\text{distributivity})\\
&= (\mu_i(x\wedge a)+\mu_i(y\wedge a)-\mu_i(x\wedge y\wedge a))-\sum_{b\in\mathrm{mb}(a)}(\mu_i(x\wedge b)+\mu_i(y\wedge b)-\mu_i(x\wedge y\wedge b))\\
& && (\mu_i \text{ is an } i\text{-measure})\\
&= \mu_i^a(x)+\mu_i^a(y)-\mu_i^a(x\wedge y).
\end{aligned}
$$

`Proof of item 4.` If $\mathsf{Min}_i(\mathbb{A})\downarrow \neq \varnothing$, it follows from $\mu_i(\bot) = 0$ (because μ_i is a i-premeasure) that $\mu_i^a(\bot) = 0$. □

Remark 154. Notice that if $a \leq y$, then for every $b \in \mathrm{mb}(a)$ we have $b \leq y$, thus $\mu_i(x \wedge y \wedge a) = \mu_i(x \wedge a)$ and $\mu_i(x \wedge y \wedge b) = \mu_i(x \wedge b)$, which implies that $\mu_i^a(x) = \mu_i^a(x \wedge y)$.

Definition 155 (Intermediate structure). For any APE-structure $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i\in\mathsf{Ag}})$ and any event structure $\mathbb{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \overline{\mathsf{pre}})$ over $\mathbb{A}$, let the *intermediate structure* be

$$\prod_{\mathbb{E}} \mathcal{F} := \left(\prod_{\mathbb{E}} \mathbb{A}, (\mu_i')_{i\in\mathsf{Ag}}\right)$$

where

1. $\prod_{\mathbb{E}} \mathbb{A} = \mathbb{A}'$ is defined as in Definition 149;
2. each μ_i' is defined as follows:

$$\begin{aligned} \mu_i' : \mathsf{Min}_i(\mathbb{A}')\downarrow &\to \mathbb{R}^+ \\ f &\mapsto \sum_{e\in E}\sum_{a\in\Phi} P_i(e)\cdot \mu_i^a(f(e))\cdot \overline{\mathsf{pre}}(e \mid a). \end{aligned} \tag{5.4.7}$$

Proposition 156. *For every APE-structure $\mathcal{F}$ and every event structure $\mathbb{E}$ over the support of $\mathcal{F}$, the intermediate structure $\prod_{\mathbb{E}} \mathcal{F}$ is an ApPE-structure (see Definition 141). Furthermore, if $\bigvee_{a\in\Phi} a \leq y$ then $\mu_i'(x) = \mu_i'(x \wedge y)$.*

Proof. Proposition 150 states that $\prod_{\mathbb{E}} \mathbb{A}$ is an epistemic Heyting algebra. To prove that $\prod_{\mathbb{E}} \mathcal{F}$ is an ApPE-structure, it remains to show that for every $i \in \mathsf{Ag}$, the map μ_i' is an i-premeasure (see items (1 - 4) of Definition 139). Fix $i \in \mathsf{Ag}$. The map μ_i' is clearly well-defined. Since the maps $\{\mu_i^a\}_{a\in\Phi}$ are i-premeasures, the items 1, 2, and 4 are trivially true.

`Proof of item 3.` By Proposition 151, i-minimal elements of $\mathbb{A}'$ are of the form $f_{e,b} : E \to \mathbb{A}$ for some $e \in E$ and some i-minimal element $b \in \mathsf{Min}_i(\mathbb{A})$. Fix one such element $f_{e,b} \in \mathsf{Min}_i(\mathbb{A}')$, and let $g, h : E \to \mathbb{A}$ such that $g, h \leq f_{e,b}$. By definition, $f \leq f_{e,b}$ can be rewritten as $f(e') \leq f_{e,b}(e')$ for any $e' \in E$. Since $f_{e,b}(e') = \bot$ for any $e' \nsim_i e$, we can deduce that $g(e') = h(e') = \bot$ for any $e' \nsim_i e$. Similarly, we can deduce that $g(e') \leq b$ and $h(e') \leq b$ for any $e' \sim_i e$. Hence,

$$\begin{aligned} \mu_i'(g \vee h) &= \sum_{e'\in E}\sum_{a\in\Phi} P_i(e')\cdot \mu_i^a(g(e') \vee h(e'))\cdot \overline{\mathsf{pre}}(e' \mid a) && \text{(by definition)} \\ &= \sum_{e'\in E}\sum_{a\in\Phi} P_i(e')\cdot (\mu_i^a(g(e')) + \mu_i^a(h(e')) - \mu_i^a(g(e') \wedge h(e')))\cdot \overline{\mathsf{pre}}(e' \mid a) \\ & \quad (\mu_i^a \text{ is an } i\text{-premeasure, } b \in \mathsf{Min}_i(\mathbb{A}), \text{ and } g(e') \leq b \text{ and } h(e') \leq b \text{ for any } e' \in E) \\ &= \mu_i'(g) + \mu_i'(h) - \mu_i'(g \wedge h). && \text{(by definition)} \end{aligned}$$

Finally, the fact that if $\left(\bigvee_{a\in\Phi} a\right) \leq y$ then $\mu_i'(x) = \mu_i'(x \wedge y)$ follows from Proposition 153. □

The intermediate algebra for the classical case

Here, we show that the construction described above, applied to the complex algebras of classical models, dualizes the construction of the intermediate model of Section 5.2.2. This is the first step towards the result stated in Proposition 128.

Definition 157. For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$ (see Definition 112) and any probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre})$ over $\mathcal{L}$ (see Definition 116), let the *probabilistic event structure over* $\mathbb{M}^+$ (see Definitions 127 and 146) be

$$\mathbb{E}_{\mathcal{E}} := (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \mathbf{\Phi}_{\mathbb{M}}, \overline{\mathsf{pre}}_{\mathbb{M}}),$$

where

- $\mathbf{\Phi}_{\mathbb{M}} = (\Phi_{\mathbb{M}}, \prec_{\mathbb{M}})$ is the ordered multiset such that $\Phi_{\mathbb{M}} := \{[\![\phi]\!]_{\mathbb{M}} \mid \phi \in \Phi\}$ and the strict order $\prec_{\mathbb{M}}$ is the empty relation;
- the map $\overline{\mathsf{pre}}_{\mathbb{M}} : \Phi_{\mathbb{M}} \to (E \to [0,1])$ assigns a probability distribution $\overline{\mathsf{pre}}_{\mathbb{M}}(\bullet|[\![\phi]\!]) : E \to [0,1]$ over E for every $\phi \in \Phi$ such that:

$$\begin{aligned}\overline{\mathsf{pre}}_{\mathbb{M}}(\bullet|[\![\phi]\!]) : E &\to [0,1] \\ e &\mapsto \mathsf{pre}(e \mid \phi).\end{aligned} \tag{5.4.8}$$

Fact 158. *For any PES-model $\mathbb{M}$ (see Definition 112) and any event structure $\mathcal{E}$ over $\mathcal{L}$ (see Definition 116), the tuple $\mathbb{E}_{\mathcal{E}}$ is an event structure over the epistemic Heyting algebra underlying $\mathbb{M}^+$.*

Proof. We need to verify that the tuple $\mathbb{E}_{\mathcal{E}}$ satisfies the conditions of Definition 146. Items 1 to 3 are trivially satisfied. Hence, we only need to prove that

4. $\mathbf{\Phi}_{\mathbb{M}} = (\Phi_{\mathbb{M}}, \prec_{\mathbb{M}})$ is a finite ordered multiset on $\mathbb{M}^+$ such that, for all $a, b \in \Phi_{\mathbb{M}}$ which arise from distinct elements in $\mathbb{M}^+$, either

$$a \wedge_{\mathbb{M}^+} b = \bot \quad \text{or} \quad a <_{\mathbb{M}^+} b \quad \text{or} \quad b <_{\mathbb{M}^+} a;$$

5. the map $\overline{\mathsf{pre}}_{\mathbb{M}} : E \times \Phi_{\mathbb{M}} \to [0,1]$ assigns a probability distribution $\overline{\mathsf{pre}}_{\mathbb{M}}(\bullet|a)$ over E for every $a \in \Phi_{\mathbb{M}}$;
6. for all $a \in \Phi$ and $e \in E$, if $\overline{\mathsf{pre}}_{\mathbb{M}}(e|a) = 0$ then $\overline{\mathsf{pre}}_{\mathbb{M}}(e|b) = 0$ for all $b \in \Phi$ such that $a \prec b$.

`Proof of 4.` First, we need to prove that $\mathbf{\Phi}_{\mathbb{M}}$ is an ordered multiset (Definition 144). $\Phi_{\mathbb{M}}$ is clearly a multiset, hence we only need to prove that the empty relation $\prec_{\mathbb{M}}$ satisfies the following conditions: for all pairwise distinct elements $x, y, z \in \Phi_{\mathbb{M}}$,

(i) if $x \prec_{\mathbb{M}} y$, then $x \leq_{\mathbb{M}^+} y$;

(ii) if $x \neq \bot_{\mathbb{M}^+}$ and $x \leq_{\mathbb{M}^+} y$, then $x \prec_{\mathbb{M}} y$ or $y \prec_{\mathbb{M}} x$;

(iii) if $x \prec_{\mathbb{M}} y$ and $x \prec_{\mathbb{M}} z$, then $y \prec_{\mathbb{M}} z$ or $z \prec_{\mathbb{M}} y$.

Conditions (i) and (iii) are trivially satisfied. Notice that, since $\mathcal{E}$ is a (classical) probabilistic event structure, Φ is a finite set of pairwise inconsistent $\mathcal{L}$-formulas. Assume that $[\![\phi]\!], [\![\psi]\!] \in \Phi_{\mathbb{M}}$ are pairwise distinct (i.e. $\phi \neq \psi$ in the language $\mathcal{L}$) and such that $[\![\phi]\!] \leq_{\mathbb{M}^+} [\![\psi]\!]$. One can easily verify that $\phi \wedge \psi = \bot$ implies that $[\![\phi]\!] = \bot_{\mathbb{M}^+}$. Hence, $\prec_{\mathbb{M}}$ satisfies condition (ii). This finishes the proof that the ordered multiset $\mathbf{\Phi}_{\mathbb{M}}$ is well-defined.

Let $[\![\phi]\!], [\![\psi]\!] \in \Phi_{\mathbb{M}}$ arise from distinct elements in $\mathbb{M}^+$. By definition, $\phi \wedge \psi = \bot$. Hence, $a \wedge_{\mathbb{M}^+} b = \bot$, which proves item 4.

`Proof of 5.` Since $\mathcal{E}$ is a (classical) probabilistic event structure, pre assigns a probability distribution pre$(\bullet|\phi)$ over E for every $\phi \in \Phi$. Hence, the map $\overline{\text{pre}}_{\mathbb{M}}$ is well-defined.

`Proof of 6.` Since $\prec_{\mathbb{M}}$ is the empty relation, this condition is trivially true. □

Remark 159. Notice that, in the classical case, $\mathrm{mb}(a) = \varnothing$ for all $a \in \mathbf{\Phi}$. Indeed, $\mathrm{mb}(a)$ denotes the multiset of the $\prec$-maximal elements of $\mathbf{\Phi}$ $\prec$-below a. But, since in the classical case $\prec_{\mathbb{M}}$ is the empty relation, there is no element below a in $\mathbf{\Phi}$.

Proposition 160. *For every PES-model $\mathbb{M}$ and any event structure $\mathcal{E}$ over $\mathcal{L}$,*

$$(\coprod_{\mathcal{E}} \mathbb{M})^+ \cong \prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{M}^+.$$

Proof. The proof that the supports of the two APE-structures (Definition 141) can be identified is essentially the same as that of [34, Fact 23.3], and is omitted. Recall that the basic identification between $\mathcal{P}(\coprod_{|E|} S)$ and $\prod_{|E|} \mathcal{P}(S)$ associates every subset $X \subseteq \coprod_{|E|} S$ with the map

$$\begin{aligned} g : E &\to \mathcal{P}(S) \\ e &\mapsto X_e := \{s \in S \mid (s, e) \in X\}. \end{aligned}$$

Let us prove that this identification induces an identification between the maps[4]

$$(P_i^+)' : \prod_{|E|} \mathcal{P}(S) \to [0,1] \qquad \text{and} \qquad (P_i^{\coprod})^+ : \mathcal{P}(\coprod_{|E|} S) \to [0,1].$$

In what follows, we fix a subset $X \subseteq \coprod_{|E|} S$ in the domain of $P_i^{\coprod}$ and let $g \in \prod_{|E|} \mathcal{P}(S)$ be defined as its counterpart as discussed above. Recall that for any $s \in S$ and $e \in E$, pre$(e \mid s)$ denotes the value pre$(e \mid \phi)$ for the unique $\phi \in \Phi$ such that

[4]Refer to Definitions 119 and 127 for the definitions of the intermediate structure $\coprod_{\mathcal{E}} \mathbb{M}$ and of the complex algebra associated to a model.

$\mathbb{M}, s \Vdash \phi$ (see Notation 118). Then, we have:

$$
\begin{aligned}
(P_i^{\coprod})^+(X) &= \sum_{(s,e)\in X} P_i^{\coprod}((s,e)) && \text{(Definition 127 on } P_i^{\coprod}) \\
&= \sum_{(s,e)\in X} P_i(s) \cdot P_i(e) \cdot \mathsf{pre}(e \mid s) && \text{(Definition 119)} \\
&= \sum_{e\in E} \sum_{s\in X_e} P_i(s) \cdot P_i(e) \cdot \mathsf{pre}(e \mid s) && (X_e := \{s \in S \mid (s,e) \in X\}) \\
&= \sum_{e\in E} P_i(e) \cdot \left(\sum_{s\in X_e} P_i(s) \cdot \mathsf{pre}(e \mid s) \right) \\
&= \sum_{e\in E} P_i(e) \cdot \sum_{\phi\in\Phi} \left(\sum_{s\in X_e \cap [\![\phi]\!]} P_i(s) \cdot \mathsf{pre}(e \mid s) \right) \\
& && (\Phi \text{ provides a partition of } \{s \in S \mid \mathsf{pre}(e \mid s) \neq 0\}) \\
&= \sum_{e\in E} P_i(e) \cdot \left(\sum_{\phi\in\Phi} \left(\sum_{s\in X_e \cap [\![\phi]\!]} P_i(s) \right) \cdot \mathsf{pre}(e \mid \phi) \right) && \text{(Notation 118)} \\
&= \sum_{e\in E} P_i(e) \cdot \left(\sum_{\phi\in\Phi} P_i^+(X_e \cap [\![\phi]\!]) \cdot \overline{\mathsf{pre}}_{\mathbb{M}}(e \mid [\![\phi]\!]) \right) && \text{(Definition 127)} \\
&= \sum_{e\in E} P_i(e) \cdot \left(\sum_{\phi\in\Phi} (P_i^+)^{[\![\phi]\!]}(X_e) \cdot \overline{\mathsf{pre}}_{\mathbb{M}}(e \mid [\![\phi]\!]) \right) \\
& && \text{(Remark 159 : } \mathrm{mb}([\![\phi]\!]) = \varnothing) \\
&= \sum_{e\in E} \sum_{\phi\in\Phi} P_i(e) \cdot (P_i^+)^{[\![\phi]\!]}(X_e) \cdot \overline{\mathsf{pre}}_{\mathbb{M}}(e \mid [\![\phi]\!]) \\
&= \sum_{e\in E} \sum_{\phi\in\Phi} P_i(e) \cdot (P_i^+)^{[\![\phi]\!]}(g(e)) \cdot \overline{\mathsf{pre}}_{\mathbb{M}}(e \mid [\![\phi]\!]) \\
&= (P_i^+)'(g) && \text{(Definition 155 on } \mathbb{M}^+)
\end{aligned}
$$

□

Corollary 161. *For every PES-model $\mathbb{M}$ and any event structure $\mathcal{E}$ over $\mathcal{L}$, the complex algebra $(\coprod_{\mathcal{E}} \mathbb{M})^+$ of the intermediate structure $\coprod_{\mathcal{E}} \mathbb{M}$ is an ApPE-structure.*

5.4.5 The pseudo-quotient and the updated APE-structure

In the present subsection, we define the APE-structure $\mathcal{F}^{\mathbb{E}}$, resulting from the update of the APE-structure $\mathcal{F}$ with the event structure $\mathbb{E}$ over the support of $\mathcal{F}$, by taking a suitable pseudo-quotient of the intermediate APE-structure $\prod_{\mathbb{E}} \mathcal{F}$. Some of the results which are relevant for the ensuing treatment (such as the characterization of the i-minimal elements in the pseudo-quotient) are independent of the fact that we will be working with the intermediate algebra. Therefore, in what follows, we will discuss them in the more general setting of arbitrary epistemic Heyting algebras $\mathbb{A}$.

Structure of the subsection. First, we define the pseudo-quotient algebra (Definition 162) and prove that it is an epistemic Heyting algebra (Proposition 163). Then, we characterize the i-minimal elements of the pseudo-quotient algebra (Proposition 166). Finally, we define the APE-structure $\mathcal{F}^{\mathbb{E}}$, resulting from the update of the APE-structure $\mathcal{F}$ with the event structure $\mathbb{E}$ (Definition 168 and Proposition 170) and show that this definition is compatible with the update on PES-models (Lemma 172).

Pseudo-quotient algebra.

Definition 162 (Pseudo-quotient algebra)**.** (cf. [36, Sections 3.2, 3.3]) For any epistemic Heyting algebra $\mathbb{A} := (\mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$, and any $a \in \mathbb{A}$, let the *pseudo-quotient algebra* be

$$\mathbb{A}^a := (\mathbb{L}/{\cong_a}, (\Diamond_i^a)_{i\in\mathsf{Ag}}, (\Box_i^a)_{i\in\mathsf{Ag}}),$$

where

- $\cong_a$ is defined as follows: for all $b, c \in \mathbb{L}$,

$$b \cong_a c \quad \text{iff} \quad b \wedge a = c \wedge a,$$

- for every $i \in \mathsf{Ag}$ the operations $\Diamond_i^a$ and $\Box_i^a$ are defined as follows:

$$\begin{aligned} \Diamond_i^a : \mathbb{A}^a &\to \mathbb{A}^a \\ b &\mapsto [\Diamond_i(b \wedge a)] \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Box_i^a : \mathbb{A}^a &\to \mathbb{A}^a \\ b &\mapsto [\Box_i(a \to b)], \end{aligned}$$

  where $[c]$ denotes the $\cong_a$-equivalence class of $c \in \mathbb{A}$.

Proposition 163. *(cf. [36, Fact 12]) For any epistemic Heyting algebra $\mathbb{A}$, the pseudo-quotient algebra $\mathbb{A}^a$ (see Definition 162) is an epistemic Heyting algebra.*

Proof. The proof that $\mathbb{A}^a$ is a monadic Heyting algebra can be found in [36, Fact 12]. To show that $\mathbb{A}^a$ is an epistemic Heyting algebra (see Definition 133), it remains to prove that $\Diamond_i^a[b] \vee \neg\Diamond_i^a[b] = [\top]$ for all $i \in \mathsf{Ag}$ and $b \in \mathbb{A}^a$. We have that $\Diamond_i^a[b] = [\Diamond_i(b \wedge a)]$ and that $\neg\Diamond_i^a[b] = \neg[\Diamond_i(b \wedge a)] = [\neg\Diamond_i(b \wedge a)]$. Hence,

$$\Diamond_i^a[b] \vee \neg\Diamond_i^a[b] = [\Diamond_i(b \wedge a) \vee \neg\Diamond_i(b \wedge a)] = [\top],$$

since $\mathbb{A}$ is an epistemic Heyting algebra. ☐

The i-minimal elements of the pseudo-quotient algebra.

Lemma 164. *For any epistemic Heyting algebra $\mathbb{A}$ and any $a \in \mathbb{A}$, if $b \in \mathsf{Min}_i(\mathbb{A})$ and $b \wedge a \neq \bot$, then $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$.*

Proof. Fix some $b \in \mathsf{Min}_i(\mathbb{A})$ such that $b \wedge a \neq \bot$. We need to prove that $[b] \in \mathbb{A}^a$ satisfies items 1, 2, and 3 of Definition 131.

`Proof of item 1.` By assumption, $[b] \neq \bot$, hence $[b]$ satisfies item 1.

`Proof of item 2.` To show that $\blacklozenge_i^a[b] = [b]$, it is enough to show that $\blacklozenge_i(b \wedge a) \wedge a = b \wedge a$. Clearly, $b \wedge a \leq b$ implies that $\blacklozenge_i(b \wedge a) \wedge a \leq \blacklozenge_i b \wedge a = b \wedge a$, making use that $\blacklozenge_i b = b$. Conversely, recalling that $\blacklozenge_i$ is reflexive (Definition 129, axiom (M1)), we have $b \wedge a = (b \wedge a) \wedge a \leq \blacklozenge_i(b \wedge a) \wedge a$. Hence, $\blacklozenge_i^a[b] = [b]$.

`Proof of item 3.` We need to prove that $[b]$ is a minimal fixed point of $\lozenge_i^a$. Let $[\bot] \neq [c] \leq [b]$ such that $\blacklozenge_i^a[c] = [c]$, and let us show that $[c] = [b]$. It is enough to show that $c \wedge a = b \wedge a$. The assumption that $[c] \leq [b]$ implies that $c \wedge a \leq b \wedge a \leq b$. Hence, $\blacklozenge_i(c \wedge a) \leq \blacklozenge_i b = b$. Notice that the assumption that $\blacklozenge_i$ is transitive (Definition 129, axiom (M6)) implies that $\blacklozenge_i \blacklozenge_i(c \wedge a) = \blacklozenge_i(c \wedge a)$, that is $\blacklozenge_i(c \wedge a)$ is a fixed point of $\blacklozenge_i$. Moreover, $\bot \neq c \wedge a \leq \blacklozenge_i(c \wedge a)$ implies that $\blacklozenge_i(c \wedge a) \neq \bot$. Hence, by the i-minimality of b in $\mathbb{A}$, we conclude that $\blacklozenge_i(c \wedge a) = b$, and hence $\blacklozenge_i(c \wedge a) \wedge a = b \wedge a$. Moreover, the assumption that $\blacklozenge_i^a[c] = [c]$ implies that $\blacklozenge_i(c \wedge a) \wedge a = c \wedge a$. Thus, the following chain of identities holds $c \wedge a = \blacklozenge_i(c \wedge a) \wedge a = b \wedge a$ as required. □

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Lemma 165. *For any epistemic Heyting algebra $\mathbb{A}$ and any $a \in \mathbb{A}$, if $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$, then $\blacklozenge_i(b \wedge a)$ is the unique i-minimal element of $\mathbb{A}$ which belongs to $[b]$.*

Proof. Let us first prove that $\blacklozenge_i(b \wedge a) \in [b]$. By assumption, $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$, hence $[b] = \blacklozenge_i^a[b] = b \wedge a = \blacklozenge_i(b \wedge a) \wedge a$. This implies that $\blacklozenge_i(b \wedge a) \in [b]$.

Now, we need to show that $\blacklozenge_i(b \wedge a)$ is an i-minimal element of $\mathbb{A}$. Hence, we need to prove that $\blacklozenge_i(b \wedge a)$ satisfies items 1, 2, and 3 of Definition 131.

`Proof of item 1.` By assumption, $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$, hence $[b] \neq \bot$ and $b \wedge a \neq \bot$. Since $\blacklozenge_i$ is reflexive (Definition 129, axiom (M1)), $\bot \neq b \wedge a \leq \blacklozenge_i(b \wedge a)$, which shows that $\blacklozenge_i(b \wedge a) \neq \bot$ as required.

`Proof of item 2.` Since $\blacklozenge_i$ is transitive (Definition 129, axiom (M6)), we have that $\blacklozenge_i(b \wedge a) = \blacklozenge_i \blacklozenge_i(b \wedge a)$ as required.

`Proof of item 3.` Let $c \in \mathsf{Min}_i(\mathbb{A})$ and $c \leq \blacklozenge_i(b \wedge a)$. We need to prove that $c = \blacklozenge_i(b \wedge a)$. To do so, we follow the following steps:

(i) we prove that $[b] = [c]$,

(ii) we show that $c \wedge a \neq \bot$,

(iii) we prove that $\blacklozenge_i(b \wedge a)$.

`Step (i).` From the assumptions that $c \leq \blacklozenge_i(b \wedge a)$ and that $[b] = \blacklozenge_i^a[b]$, we get that $c \wedge a \leq \blacklozenge_i(b \wedge a) \wedge a = b \wedge a$, which proves that $[c] \leq [b]$.

`Step (ii).` Since $c \leq \Diamond_i(b \wedge a)$, we have that $c \leq \Diamond_i a$, that is $c = c \wedge \Diamond_i a$. This gives the following chain of equalities:

$$c = c \wedge \Diamond_i a = \Diamond_i c \wedge \Diamond_i a = \Diamond_i(\Diamond_i c \wedge a).$$

The last equality is true in all monadic Heyting algebras (see e.g. [10, Definition 1]). Now, since $\Diamond_i c = c$, we get that $c = \Diamond_i(c \wedge a)$, which implies $\Diamond_i(c \wedge a) \neq \bot$ and $c \wedge a \neq \bot$.

`Step (iii).` By Lemma 164, $[c] \in \mathsf{Min}_i(\mathbb{A}^a)$. By the i-minimality of $[b]$, we get $[b] = [c]$, that is $b \wedge a = c \wedge a$. Hence $\blacklozenge_i(b \wedge a) = \blacklozenge_i(c \wedge a) \leq \blacklozenge_i(c) = c$, which, together with the assumption that $c \leq \blacklozenge_i(b \wedge a)$, proves that $\blacklozenge_i(b \wedge a) = c$, as required. This finishes the proof that $\blacklozenge_i(b \wedge a)$ is an i-minimal element of $\mathbb{A}$.

To show the uniqueness, let $c_1, c_2 \in [b]$ and assume that both c_1 and c_2 are i-minimal elements of $\mathbb{A}$. Then $c_1 \wedge a = c_2 \wedge a$, and hence $\blacklozenge_i(c_1 \wedge a) = \blacklozenge_i(c_2 \wedge a)$. Reasoning as above, one can show that $\bot \neq \blacklozenge_i(c_j \wedge a) \leq c_j$ and $\blacklozenge_i(c_j \wedge a)$ is a fixed point of $\blacklozenge_i$ for $1 \leq j \leq 2$. Hence, the i-minimality of c_j implies that $\blacklozenge_i(c_j \wedge a) = c_j$. Thus, the following chain of identities holds:

$$c_1 = \blacklozenge_i(c_1 \wedge a) = \blacklozenge_i(c_2 \wedge a) = c_2.$$

□

Combining the two lemmas above, we obtain the following result.

Proposition 166. *The following are equivalent for any $\mathbb{A}$ and any $a \in \mathbb{A}$:*

1. $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$;
2. $[b] = [b']$ *for a unique* $b' \in \mathsf{Min}_i(\mathbb{A})$ *such that* $b' \wedge a \neq \bot$.

Notation 167. In what follows, whenever $[b] \in \mathsf{Min}_i(\mathbb{A}^a)$, we will assume w.l.o.g. that $b \in \mathsf{Min}_i(\mathbb{A})$ is the "canonical" (in the sense of Proposition 166) representant of $[b]$.

The updated APE-structure.

For any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support $\mathbb{A}$ of $\mathcal{F}$, the map $\overline{\mathsf{pre}}$ in $\mathbb{E}$ induces the map $\overline{pre}$ defined as follows:

$$\begin{aligned} \overline{pre} &: E \to \mathbb{A} \\ e &\mapsto \bigvee_{\substack{a \in \Phi \\ \overline{\mathsf{pre}}(e|a) \neq 0}} a \end{aligned} \tag{5.4.9}$$

It immediately follows from Propositions 151 and 166 that the i-minimal elements of $\mathbb{A}^{\mathbb{E}}$ are exactly the elements $[f_{e,b}]$ for $e \in E$ and $b \in \mathsf{Min}_i(\mathbb{A})$ such that $b \wedge \overline{pre}(e') \neq \bot$ for some $e' \sim_i e$.

Definition 168 (Updated APE-structure)**.** For any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support of $\mathcal{F}$, the *updated APE-structure* is the tuple

$$\mathcal{F}^{\mathbb{E}} := (\mathbb{A}^{\mathbb{E}}, (\mu_i^{\mathbb{E}})_{i \in \mathsf{Ag}})$$

such that:

1. $\mathbb{A}^{\mathbb{E}}$ is obtained by instantiating Definition 162 to $\prod_{\mathbb{E}} \mathbb{A}$ and $\overline{pre} \in \prod_{\mathbb{E}} \mathbb{A}$, i.e.

$$\mathbb{A}^{\mathbb{E}} := (\prod_{\mathbb{E}} \mathbb{A})^{\overline{pre}};$$

2. The maps $\mu_i^{\mathbb{E}}$ are defined as follows:

$$\mu_i^{\mathbb{E}} : \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})\!\downarrow \to [0,1]$$
$$[g] \mapsto \begin{cases} 0 & \text{if } [g] = \bot, \\ \frac{\mu_i'(g)}{\mu_i'(f)} & \text{otherwise,} \end{cases}$$

where [f] is the only element in $\mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})$ such that $[g] \leq [f]$.[5]

5

Lemma 169. *For any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support of $\mathcal{F}$, the maps $(\mu_i^{\mathbb{E}})_{i \in \mathsf{Ag}}$ of the updated APE-structure $\mathcal{F}^{\mathbb{E}} := (\mathbb{A}^{\mathbb{E}}, (\mu_i^{\mathbb{E}})_{i \in \mathsf{Ag}})$ are well-defined.*

Proof. Let us first prove the following claim.
Claim. For each $[h] \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})\!\downarrow$ such that $[h] \neq \bot$, we have $\mu_i'([h]) \neq 0$.
Proof of claim. Let $e \in E$ be such that $(h \wedge \overline{pre})(e) \neq \bot$. Notice that

$$(h \wedge \overline{pre})(e) \neq \bot \quad \text{iff} \quad h(e) \wedge \bigvee_{\substack{a \in \Phi \\ \overline{\mathsf{pre}}(e|a) \neq 0}} a \neq \bot.$$

This implies that there is $a \in \mathbf{\Phi}$ such that

$$\overline{\mathsf{pre}}(e \mid a) > 0 \qquad \text{and} \qquad h(e) \wedge a \neq \bot.$$

Since μ_i is an i-measure (see Definition 139), we have $\mu_i((h \wedge a)(e)) > 0$. Then, the following set is non empty

$$\{a \in \Phi \mid \mu_i((h \wedge a)(e)) > 0 \text{ and } \overline{\mathsf{pre}}(e|a) > 0\}.$$

Since $\mathbf{\Phi}$ is finite, it is well-founded with respect to the order of the multiset $\prec$, hence it contains at least one minimal element. Let a_0 be such a minimal element. From item (6) of Definition 146, we deduce that, for every $b \in \Phi$ such that $b \prec a_0$, it is the case that $\overline{\mathsf{pre}}(e|b) > 0$. The minimality of a_0 implies that, for every $b \in \Phi$ such

[5] See Definition 155 for the definition of the maps $(\mu_i')_{i \in \mathsf{Ag}}$.

that $b \prec a_0$, we have $\mu_i((h \wedge b)(e)) = 0$. This implies that, for all $b \in \mathrm{mb}(a)$, we have $\mu_i((h \wedge b)(e)) = 0$. Hence,

$$\begin{aligned}\mu_i^{a_0}(h(e)) &= \mu_i(g(e) \wedge a_0) - \sum_{b \in \mathrm{mb}(a_0)} \mu_i(h(e) \wedge b) \qquad \text{(see Definition 152)}\\ &= \mu_i(h(e) \wedge a_0).\end{aligned}$$

Therefore $\mu_i^{a_0}(h(e)) > 0$ and $P_i(e) \cdot \mu_i^{a_0}(h(e)) \cdot \overline{\mathsf{pre}}(e|a_0) > 0$. This guarantees that $\mu_i'([h]) \neq 0$. This finishes the proof of the claim. ■

Now, let us prove that the map $\mu_i^{\mathbb{E}}$ is well-defined. Recall that, if $[g] \neq \bot$, then $[f]$ is unique (see Remark 132). From the claim above, it follows that the division $\frac{\mu_i'(g)}{\mu_i'(f)}$ is defined. Finally, let us verify that $\mu_i^{\mathbb{E}}$ assigns exactly one value to every $[g] \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})\downarrow$. Let $g_1, g_2 \in [g]$. Then we have $\mu_i'(g_1) = \mu_i'(g_1 \wedge \overline{pre}) = \mu_i'(g_2 \wedge \overline{pre}) = \mu_i'(g_2)$ (see Proposition 156). Hence, $\mu_i^{\mathbb{E}}$ is well-defined for any $i \in \mathsf{Ag}$. □

5

Proposition 170. *For any APE-structure $\mathcal{F}$ and any event structure $\mathbb{E}$ over the support of $\mathcal{F}$, the tuple $\mathcal{F}^{\mathbb{E}} = (\mathbb{A}^{\mathbb{E}}, (\mu_i^{\mathbb{E}})_{i \in \mathsf{Ag}})$ is an APE-structure.*

Proof. Let $\mathcal{F} := (\mathbb{A}, (\mu_i)_{i \in \mathsf{Ag}})$ and $\mathbb{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \mathbf{\Phi}, \overline{\mathsf{pre}})$ be an APE-structure and and an event structure over the support of $\mathcal{F}$ respectively. To prove that $\mathcal{F}^{\mathbb{E}}$ is an APE-structure (see Definition 141), we need to prove that $\mathbb{A}^{\mathbb{E}}$ is an epistemic Heyting algebra (see Definition 133), and that each map $\mu_i^{\mathbb{E}}$ is an i-measure on $\mathbb{A}^{\mathbb{E}}$. By Proposition 163, $\mathbb{A}^{\mathbb{E}}$ is an epistemic Heyting algebra. Hence, it remains to prove that, for each $i \in \mathsf{Ag}$, the map $\mu_i^{\mathbb{E}}$ is an i-measure (see Definition 139), i.e. we need to prove that:

1. $\mathsf{dom}(\mu_i^{\mathbb{E}}) = \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})\downarrow$;
2. $\mu_i^{\mathbb{E}}$ is order-preserving;
3. for every $a \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})$ and all $b, c \in a\downarrow$, it holds that $\mu_i^{\mathbb{E}}(b \vee c) = \mu_i^{\mathbb{E}}(b) + \mu_i^{\mathbb{E}}(c) - \mu_i^{\mathbb{E}}(b \wedge c)$;
4. $\mu_i^{\mathbb{E}}(\bot) = 0$ if $\mathsf{dom}(\mu_i^{\mathbb{E}}) \neq \varnothing$;
5. $\mu_i^{\mathbb{E}}(a) = 1$ for every $a \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})$;
6. for every $a \in \mathsf{Min}_i(\mathbb{A}^{\mathbb{E}})$ and all $b, c \in a\downarrow$ such that $b < c$, it holds that $\mu_i^{\mathbb{E}}(b) < \mu_i^{\mathbb{E}}(c)$.

`Proof of` (1)`.` This condition is satisfied by definition.

The remaining items, are trivially satisfied if the domain of $\mu_i^{\mathbb{E}}$ is empty. For the remaining of the proof, let us assume that the domain of $\mu_i^{\mathbb{E}}$ is non-empty.

`Proof of item` (2)`.` The definition of μ_i' (see Definition 155), the Proposition 153 and the fact that, if $\overline{\mathsf{pre}}(e \mid a) \neq 0$, then $a \leq \overline{pre}(e)$ (see Definition of $\overline{pre}$ (5.4.9)), imply that $\mu_i'(g) = \mu_i'(g \wedge \overline{pre})$. Assume that $[g_1] \leq [g_2] \leq [f_{e,a}]$. This means that

$g_1 \wedge \overline{pre} \leq g_2 \wedge \overline{pre}$. Since μ_i' is an i-premeasure (Proposition 156), it is monotone. Hence, $\mu_i'(g_1) = \mu_i'(g_1 \wedge \overline{pre}) \leq \mu_i'(g_2 \wedge \overline{pre}) = \mu_i'(g_2)$. This implies that

$$\frac{\mu_i'(g_1)}{\mu_i'(f_{e,a})} \leq \frac{\mu_i'(g_2)}{\mu_i'(f_{e,a})}$$

that is, $\mu_i^{\mathbb{E}}([g_1]) \leq \mu_i^{\mathbb{E}}([g_2])$.

`Proof of item` (3). Let $[g_1]$ and $[g_2]$ in $\mathcal{F}^{\mathbb{E}}$ such that $[g_1] \leq [f_{e,a}]$ and $[g_2] \leq [f_{e,a}]$. We have:

$$\begin{aligned}
&\mu_i^{\mathbb{E}}([g_1] \vee [g_2]) \\
&= \frac{\mu_i'((g_1 \wedge \overline{pre}) \vee (g_2 \wedge \overline{pre}))}{\mu_i'(f_{e,a})} \\
&= \frac{\mu_i'(g_1 \wedge \overline{pre}) + \mu_i'(g_2 \wedge \overline{pre}) - \mu_i'((g_1 \wedge g_2) \wedge \overline{pre})}{\mu_i'(f_{e,a})} \\
&\qquad\qquad \text{(Proposition 156. } \mu_i' \text{ is an } i\text{-premeasure)} \\
&= \frac{\mu_i'(g_1 \wedge \overline{pre})}{\mu_i'(f_{e,a})} + \frac{\mu_i'(g_2 \wedge \overline{pre})}{\mu_i'(f_{e,a})} - \frac{\mu_i'((g_1 \wedge g_2) \wedge \overline{pre})}{\mu_i'(f_{e,a})} \\
&= \frac{\mu_i'(g_1)}{\mu_i'(f_{e,a})} + \frac{\mu_i'(g_2)}{\mu_i'(f_{e,a})} - \frac{\mu_i'(g_1 \wedge g_2)}{\mu_i'(f_{e,a})} \\
&= \mu_i^{\mathbb{E}}([g_1]) + \mu_i^{\mathbb{E}}([g_2]) - \mu_i^{\mathbb{E}}([g_1 \wedge g_2]).
\end{aligned}$$

`Proof of Items` (4) `and` (5). Trivial.

`Proof of item` (6). Recall that, if $[g] \neq \bot$, then $\mu_i^{\mathbb{E}}([g]) > 0$ (see Claim in Lemma 169). Let $\bot \neq [g] < [h]$. The monotonicity of the μ_i^a guarantees that, for all $e \in E$ and $a \in \Phi$, we have

$$P_i(e) \cdot \mu_i^a(g(e)) \cdot \overline{\mathsf{pre}}(e|a) \quad \leq \quad P_i(e) \cdot \mu_i^a(h(e)) \cdot \overline{\mathsf{pre}}(e|a).$$

Furthermore, since $[g] < [h]$, there exists an $e \in E$ such that the set

$$\{\ a \in \Phi \quad | \quad \overline{\mathsf{pre}}(e|a) > 0 \text{ and } g(e) \wedge a < h(e) \wedge a\ \}$$

is non-empty. Since Φ is finite, the order $\prec$ is well-founded and the aforementioned set contains at least one minimal element. Let a_0 be such a minimal element. From Definition 146, we have that, $\overline{\mathsf{pre}}(e|b) > 0$ for all $b \in \Phi$ with $b \prec a_0$. By the minimality of a_0, we have that $g(e) \wedge b = h(e) \wedge b$ for all such $b \prec a_0$. Hence,

$$\sum_{b \in \mathrm{mb}(a_0)} \mu_i(g(e) \wedge b) = \sum_{b \in \mathrm{mb}(a_0)} \mu_i(h(e) \wedge b)$$

where $\mathrm{mb}(a)$ denotes the multiset of the $\prec$-maximal elements of $\mathbf{\Phi}$ $\prec$-below a (see Definition 152). Since $\mathcal{F}$ is an APE-structure, μ_i is strictly monotone. Hence, $g(e) \wedge a_0 <$

$h(e) \wedge a_0$ implies that

$$\begin{aligned}\mu_i^{a_0}(g(e)) &= \mu_i(g(e) \wedge a_0) - \sum_{b \in \mathrm{mb}(a_0)} \mu_i(g(e) \wedge b) \\ &< \mu_i(h(e) \wedge a_0) - \sum_{b \in \mathrm{mb}(a_0)} \mu_i(h(e) \wedge b) \\ &= \mu_i^{a_0}(h(e)).\end{aligned}$$

Hence, for some $e \in E$ and $a \in \Phi$, we have

$$P_i(e) \cdot \mu_i^a(g(e)) \cdot \overline{\mathsf{pre}}(e|a) < P_i(e) \cdot \mu_i^a(h(e)) \cdot \overline{\mathsf{pre}}(e|a).$$

The inequality above, the definition of μ_i' (see Definition 155) and the monotonicity of μ_i' (see Proposition 156) imply that $\mu_i'([g]) < \mu_i'([h])$, which in turn implies that $\mu_i^{\mathbb{E}}([g]) < \mu_i^{\mathbb{E}}([h])$.

□

5

The updated algebra for the classical case.

In this section, we conclude the proof of Proposition 128 by showing that the pseudo-quotient construction described above, applied to the complex algebras of the intermediate classical models, dualizes the submodel construction in Section 5.2.2.

The definition of the complex algebra of a PES-model (Definition 127) can be equivalently reformulated as follows.

Definition 171 (Complex algebra)**.** For any PES-model $\mathbb{M} = \langle S, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, [\![\cdot]\!] \rangle$, its complex algebra is the tuple

$$\mathbb{M}^+ := \big(\mathcal{P}S, (\Diamond_i)_{i \in \mathsf{Ag}}, (\Box_i)_{i \in \mathsf{Ag}}, (P_i^+)_{i \in \mathsf{Ag}}\big)$$

where

1. for each $i \in \mathsf{Ag}$ and $X \in \mathcal{P}S$,

$$\begin{aligned}\Diamond_i X &= \{s \in S \mid \exists x\, (s \sim_i x \text{ and } x \in X)\}, \\ \Box_i X &= \{s \in S \mid \forall x\, (s \sim_i x \implies x \in X)\},\end{aligned}$$

2. $\mathbb{A} := \langle \mathcal{P}S, (\Diamond_i)_{i \in \mathsf{Ag}}, (\Box_i)_{i \in \mathsf{Ag}} \rangle$ is an epistemic Heyting algebra,
3. for each $i \in \mathsf{Ag}$ and $X \in \mathcal{P}S$,

$$\begin{aligned}P_i^+ &: \mathsf{Min}_i(\mathbb{A}){\downarrow} \to \mathbb{A} \\ &X \mapsto \sum_{x \in X} P_i(x).\end{aligned}$$

Notice that the domain of P_i^+ consists of all the subsets of the equivalence classes of $\sim_i$.

Lemma 172. *For any PES-model* $\mathbb{M}$ *and any event structure* $\mathcal{E}$ *over* $\mathcal{L}$*,*

$$(P_i^+)^{\mathbb{E}_\mathcal{E}} \cong (P_i^\mathcal{E})^+.$$

Proof. Using Definitions 122 and 127, we get that: for any $X \in \mathsf{Min}_i((\mathbb{M}^\mathcal{E})^+){\downarrow}$,

$$(P_i^\mathcal{E})^+(X) = \sum_{(s,e)\in X} \frac{P_i(e)\cdot P_i(s)\cdot \mathsf{pre}(e \mid s)}{\sum_{(s',e')\sim_i(s,e)} P_i(e')\cdot P_i(s')\cdot \mathsf{pre}(e' \mid s')}.$$

5

By using Definitions 127 and 168, we get that: for any $[g] \in \mathsf{Min}_i((\mathbb{M}^+)^\mathbb{E}){\downarrow}$,

$$(P_i^+)^{\mathbb{E}_\mathcal{E}}([g]) = \frac{\sum_{e\in E}\sum_{\phi\in\Phi} P_i(e)\cdot (P_i^+)^{[\![\phi]\!]}(g(e))\cdot \overline{\mathsf{pre}}(e \mid [\![\phi]\!])}{\sum_{e\in E}\sum_{\phi\in\Phi} P_i(e)\cdot (P_i^+)^{[\![\phi]\!]}(f(e))\cdot \overline{\mathsf{pre}}(e \mid [\![\phi]\!])}.$$

Let $X \in \mathsf{Min}_i((\mathbb{M}^\mathcal{E})^+){\downarrow}$. Following the notation introduced in the proof of Proposition 160, let $[g] \in \mathsf{Min}_i((\mathbb{M}^+)^\mathbb{E}){\downarrow}$ be the map such that

$$\begin{aligned} g : E &\to \mathcal{P}(S) \\ e &\mapsto X_e := \{s \in S \mid (s,e) \in X\}. \end{aligned}$$

Notice that X is a subset of one of the i-equivalence classes of $(\mathbb{M}^\mathcal{E})^+$, hence $g = g \wedge \overline{pre}$ and $[g] \leq [f]$ for some $[f] \in \mathsf{Min}_i((\mathbb{M}^+)^\mathbb{E}){\downarrow}$. Let

$$[X]_i := \{(s,e) \mid \exists (s',e') \in X,\ (s,e) \sim_i (s',e')\}.$$

We can easily see that $([X]_i)_e = f(e)$ where f is the canonical representative of $[f]$.

We have:

$$
\begin{aligned}
&(P_i^{\mathcal{E}})^+(X) \\
&= \sum_{(s,e)\in X} \frac{P_i(e)\cdot P_i(s)\cdot \mathsf{pre}(e\mid s)}{\sum_{(s',e')\sim_i(s,e)} P_i(e')\cdot P_i(s')\cdot \mathsf{pre}(e'\mid s')} \\
&= \frac{\sum_{(s,e)\in X} P_i(e)\cdot P_i(s)\cdot \mathsf{pre}(e\mid s)}{\sum_{(s',e')\in[X]_i} P_i(e')\cdot P_i(s')\cdot \mathsf{pre}(e'\mid s')} \\
&\qquad\qquad (X \text{ is a subset of the equivalence classes } [X]_i) \\
&= \frac{\sum_{e\in E} P_i(e)\cdot \sum_{s\in X_e} P_i(s)\cdot \mathsf{pre}(e\mid s)}{\sum_{e'\in E} P_i(e')\cdot \sum_{s'\in f(e')} P_i(s')\cdot \mathsf{pre}(e'\mid s')} \qquad (([X]_i)_e = f(e)) \\
&= \frac{\sum_{e\in E} P_i(e)\cdot \sum_{\phi\in\Phi} \mathsf{pre}(e\mid\phi)\cdot \sum_{s\in g(e)\cap[\![\phi]\!]} P_i(s)}{\sum_{e'\in E} P_i(e')\cdot \sum_{\phi\in\Phi} \mathsf{pre}(e'\mid\phi)\cdot \sum_{s'\in f(e')\cap[\![\phi]\!]} P_i(s')} \\
&\qquad\qquad (\text{In the classical case, } \Phi \text{ gives a partition of } S^{\mathcal{E}}) \\
&= \frac{\sum_{e\in E} P_i(e)\cdot \sum_{\phi\in\Phi} \mathsf{pre}(e\mid\phi)\cdot (P_i^+)(g(e)\cap[\![\phi]\!])}{\sum_{e'\in E} P_i(e')\cdot \sum_{\phi\in\Phi} \mathsf{pre}(e'\mid\phi)\cdot (P_i^+)(f(e)\cap[\![\phi]\!])} \qquad (\text{Definition } 171) \\
&= \frac{\sum_{e\in E} P_i(e)\cdot \sum_{\phi\in\Phi} \mathsf{pre}(e\mid\phi)\cdot (P_i^+)^{[\![\phi]\!]}(g(e))}{\sum_{e'\in E} P_i(e')\cdot \sum_{\phi\in\Phi} \mathsf{pre}(e'\mid\phi)\cdot (P_i^+)^{[\![\phi]\!]}(f(e))} \qquad (\text{Remark } 159 : \mathrm{mb}([\![\phi]\!]) = \varnothing) \\
&= \frac{\sum_{e\in E}\sum_{\phi\in\Phi} P_i(e)\cdot (P_i^+)^{[\![\phi]\!]}(g(e))\cdot \overline{\mathsf{pre}}(e\mid[\![\phi]\!])}{\sum_{e\in E}\sum_{\phi\in\Phi} P_i(e)\cdot (P_i^+)^{[\![\phi]\!]}(f(e))\cdot \overline{\mathsf{pre}}(e\mid[\![\phi]\!])} \\
&= (P_i^+)^{\mathbb{E}_{\mathcal{E}}}([g]).
\end{aligned}
$$

□

5.5 Algebraic semantics of intuitionistic PDEL

In this section, we introduce the Intuitionistic Probabilistic Dynamic Epistemic Logic (IPDEL). We first define the syntax in Section 5.5.4 and the semantics (Definition 181) of IPDEL in Section 5.5.2. Then, in Section 5.5.3, we present the axiomatisation of IPDEL (Table 5.2). Finally, in Section 5.5.4, we prove that the axiomatization is sound (Proposition 184). The completeness of IPDEL is treated in Section 5.6.

5.5.1 Syntax

Definition 173 (IPDEL syntax)**.** The set $\mathcal{L}$ of *IPDEL-formulas* φ and the class of *intuitionistic probabilistic event structures* $\mathbb{E}$ over $\mathcal{L}$ are built by simultaneous recursion as follows:

$$\varphi ::= p \mid \bot \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi\to\varphi \mid \Diamond_i\varphi \mid \Box_i\varphi \mid \langle\mathcal{E},e\rangle\varphi \mid [\mathcal{E},e]\varphi \mid (\sum_{k=1}^{n}\alpha_k\mathring{u}\mu_i(\varphi)) \geq \beta,$$

where $i \in \mathsf{Ag}$, and the event structures $\mathcal{E}$ are such as in Definition 174.
The connectives $\top$, $\neg$, and $\leftrightarrow$ are defined by the usual abbreviations.

Definition 174 (Intuitionistic probabilistic event structure)**.** An *intuitionistic probabilistic event structure over $\mathcal{L}$* is a tuple

$$\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub}),$$

such that

- E is a non-empty finite set;
- each $\sim_i$ is an equivalence relation on E;
- each $P_i : E \to]0,1]$ assigns a probability distribution over each $\sim_i$-equivalence class, i.e.
$$\sum\{P_i(e') \mid e' \sim_i e\} = 1;$$
- Φ is a finite set of formulas in $\mathcal{L}$ such that, for all $\phi_k, \phi_j \in \Phi$, one and only one of the following conditions is true:
  - $\vdash (\phi_j \wedge \phi_k) \to \bot$,
  - $\vdash \phi_k \to \phi_j$,
  - $\vdash \phi_j \to \phi_k$;
- the map $\mathsf{pre} : E \times \Phi \to [0,1]$ assigns a probability distribution $\mathsf{pre}(\bullet|\phi)$ over E for every $\phi \in \Phi$;
- the map $\mathbf{sub} : E \to Sub_{\mathcal{L}}$ assigns a substitution function (see Definition 115) to each event in E;
- for all $\phi_j \in \Phi$ and $e \in E$, if $\mathsf{pre}(e|\phi_j) = 0$ then $\mathsf{pre}(e|\phi_k) = 0$ for all $\phi_k \in \Phi$ such that $\vdash \phi_j \to \phi_k$.

Remark 175. The conditions on Φ match the conditions of $\boldsymbol{\Phi}$ given in Definition 146 (cf. Proposition 178). The requirement in Definition 146 that $\boldsymbol{\Phi}$ is a multiset stems from the fact that the interpretation of distinct formulas ϕ_k, ϕ_j such that $\phi_k \to \phi_j$ might coincide in a model.

Remark 176. The conditions on the preconditions are given using $\vdash$. One should refer to Section 5.5.3 and Table 5.2 for the axiomatisation of IPDEL.

5.5.2 Semantics

In what follows, we define the models, the event structures on the language, the event structures on the model, the updated models and the semantics. Notice that the definition of the event structure on the model relies on the definition of the event structure on the language, and that the definitions of the event structure on the model, the updated models and the semantics are given by simultaneous induction.

Definition 177 (APE-models)**.** *Algebraic probabilistic epistemic models (APE-models)* are tuples $\mathcal{M} = \langle \mathcal{F}, v \rangle$ such that $\mathcal{F} = (\mathbb{A}, (\mu_i)_{i \in \mathsf{Ag}})$ is an APE-structure and v : AtProp $\to \mathbb{A}$.

The update construction of Section 5.4 extends from APE-structures to APE-models. Indeed, for any APE-model $\mathcal{M} = \langle \mathbb{A}, (\mu_i)_{i \in \mathsf{Ag}}, v \rangle$ and any event structure $\mathcal{E}$ (see Definition 174), the event structure $\mathcal{E}$ induces an event structure over the algebra $\mathbb{A}$ (see Definition 146) as follows.

Proposition 178. *For any APE-model* $\mathcal{M} = \langle \mathbb{A}, (\mu_i)_{i \in \mathsf{Ag}}, v \rangle$ *and any event structure*

$$\mathcal{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub}),$$

over $\mathcal{L}$*, the following tuple is an event structure over* $\mathbb{A}$

$$\mathbb{E}_{\mathcal{E}} := (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \mathbf{\Phi}_{\mathcal{M}}, \overline{\mathsf{pre}}_{\mathcal{M}}),$$

where

- $\mathbf{\Phi} := (\Phi_{\mathcal{M}}, \prec_{\mathcal{M}})$ *with* $\Phi_{\mathcal{M}} := \{[\![\phi]\!]_{\mathcal{M}} \mid \phi \in \Phi\}$ *and* $\prec_{\mathcal{M}} := \{([\![\phi_j]\!], [\![\phi_k]\!]) \mid \vdash \phi_j \to \phi_k\}$*, and*
- $\overline{\mathsf{pre}}_{\mathcal{M}}$ *assigns a probability distribution* $\overline{\mathsf{pre}}_{\mathcal{M}}(\bullet | a)$ *over* E *for every* $a \in \Phi_{\mathcal{M}}$*.*

Proof. Trivial. □

Definition 179 (Updated model)**.** The update of the APE-model $\mathcal{M} = \langle \mathcal{F}, v \rangle$ by the intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ is given by the APE-model

$$\mathcal{M}^{\mathcal{E}} := \langle \mathcal{F}^{\mathcal{E}}, v^{\mathcal{E}} \rangle,$$

where

- $\mathcal{F}^{\mathcal{E}} := \mathcal{F}^{\mathbb{E}_{\mathcal{E}}}$ as in Definition 168,
- and the map $v^{\mathcal{E}}$ is defined as follows:

$$v^{\mathcal{E}} : \mathsf{AtProp} \to \mathbb{A}^{\mathbb{E}_{\mathcal{E}}}$$

$$p \mapsto \begin{cases} [v\textstyle\prod(\mathbf{sub}(e)(p))] & \text{if } p \in dom(\mathbf{sub}(e)) \\ [v\textstyle\prod(p)] & \text{otherwise} \end{cases}$$

where

$$v\textstyle\prod(p) : E \to \displaystyle\prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{A} \qquad \text{and} \qquad v\textstyle\prod(\mathbf{sub}(e)(p)) : E \to \displaystyle\prod_{\mathbb{E}_{\mathcal{E}}} \mathbb{A}$$

$$e \mapsto v(p) \qquad\qquad e \mapsto v(\mathbf{sub}(e)(p)).$$

Notation 180. We define the e-th projection π_e for every $e \in E$, the quotient map π and the map ι as follows:

$$\pi_e : \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A} \to \mathbb{A} \quad \text{and} \quad \pi : \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A} \to \mathbb{A}^{\mathbb{E}_\mathcal{E}} \quad \text{and} \quad \iota : \mathbb{A}^{\mathbb{E}_\mathcal{E}} \to \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A}$$

$$g \mapsto g(e) \qquad g \mapsto [g] \qquad [g] \mapsto g \wedge \overline{pre}.$$

As explained in [36, Section 3.2], the map ι is well-defined.

Definition 181 (Semantics)**.** The interpretation of $\mathcal{L}$-formulas on any APE-model $\mathcal{M}$ is defined recursively as follows:

$$\begin{aligned}
[\![\bot]\!]_\mathcal{M} &= \bot^\mathbb{A} & [\![\top]\!]_\mathcal{M} &= \top^\mathbb{A}\\
[\![p]\!]_\mathcal{M} &= v(p) & [\![\varphi \to \psi]\!]_\mathcal{M} &= [\![\varphi]\!]_\mathcal{M} \to^\mathbb{A} [\![\psi]\!]_\mathcal{M}\\
[\![\varphi \wedge \psi]\!]_\mathcal{M} &= [\![\varphi]\!]_\mathcal{M} \wedge^\mathbb{A} [\![\psi]\!]_\mathcal{M} & [\![\varphi \vee \psi]\!]_\mathcal{M} &= [\![\varphi]\!]_\mathcal{M} \vee^\mathbb{A} [\![\psi]\!]_\mathcal{M}\\
[\![\Diamond_i \varphi]\!]_\mathcal{M} &= \Diamond_i [\![\varphi]\!]_\mathcal{M} & [\![\Box_i \varphi]\!]_\mathcal{M} &= \Box_i [\![\varphi]\!]_\mathcal{M}\\
[\![\langle \mathcal{E}, e\rangle \varphi]\!]_\mathcal{M} &= [\![pre(e)]\!]_\mathcal{M} \wedge^\mathbb{A} \pi_e \circ \iota([\![\varphi]\!]_{\mathcal{M}^{\mathbb{E}_\mathcal{E}}}) & [\![[\mathcal{E}, e]\varphi]\!]_\mathcal{M} &= [\![pre(e)]\!]_\mathcal{M} \to^\mathbb{A} \pi_e \circ \iota([\![\varphi]\!]_{\mathcal{M}^{\mathbb{E}_\mathcal{E}}})
\end{aligned}$$

$$\left[\!\!\left[\left(\sum_{k=1}^{n} \alpha_k \mathring{u}\mu_i(\varphi_k)\right) \geq \beta\right]\!\!\right]_\mathcal{M} = \bigvee \left\{ a \in \mathbb{A} \;\middle|\; a \in \mathsf{Min}_i(\mathbb{A}) \text{ and } \left(\sum_{k=1}^{n} \alpha_k \mu_i([\![\varphi_k]\!]_\mathcal{M} \wedge a)\right) \geq \beta \right\}$$

5.5.3 Axiomatisation

IPDEL is intended as the intuitionistic counterpart of classical PDEL. The full axiomatisation of IPDEL is given in Table 5.2 (see page 214).

The main differences between the axiomatisation of IPDEL and the axiomatisation of classical PDEL presented in Table 5.1 are that the axioms for S5 are replaced by the axioms of intuitionistic modal logic MIPC and axiom E (see Definition 133), and that the axioms capturing classical probability theory are replaced by axioms capturing intuitionistic probability theory. In particular, axioms p3 and p4 in Table 5.1 are different from the axioms P3 and P4 in Table 5.2. It is not hard to see that axiom p3 implies P3 and $\mu_i(\varphi) + \mu_i(\neg\varphi) = 1$ in the presence of p1 and p2. Axiom P3 is strictly weaker that p3, since the aforementioned equality is generally false in intuitionistic probabilities. In classical logic axioms p4 and P4 are equivalent. In intuitionistic logic P4 is strictly stronger than p4. Indeed, as Lemma 183 shows, p4 is not strong enough to express the strict monotonicity of i-measures.

Finally, notice that axioms M8 and M9 from Definition 129 are not in Table 5.2. Indeed, they follow from the rest of the axioms and the necessitation rules (see Lemma 182 and also compare with [10]).

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Table 5.2: Axioms of IPDEL

Axioms of IPL	
H1.	$A \to (B \to A)$
H2.	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
H3.	$A \to (B \to A \wedge B)$
H4.	$(A \to C) \to ((B \to C) \to (A \vee B \to C))$
H5.	$A \wedge B \to A$
H6.	$A \wedge B \to B$
H7.	$A \to A \vee B$
H8.	$B \to A \vee B$
H9.	$\bot \to A$
Axioms for static modalities	
M1.	$p \to \Diamond_i p$
M2.	$\Box_i p \to p$
M3.	$\Diamond_i (p \vee q) \to \Diamond_i p \vee \Diamond_i q$
M4.	$\Box_i (p \to q) \to (\Box_i p \to \Box_i q)$
M5.	$\Diamond_i p \to \Box_i \Diamond_i p$
M6.	$\Diamond_i \Box_i p \to \Box_i p$
M7.	$\Box_i (p \to q) \to (\Diamond_i p \to \Diamond_i q)$
E.	$\Diamond_i p \vee \neg \Diamond_i p$
Axioms for inequalities	
N0.	$t \geq t$
N1.	$(t \geq \beta) \leftrightarrow (t + 0 \cdot \mu_i(\varphi) \geq \beta)$
N2.	$(\sum_{k=1}^{n} \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \to (\sum_{k=1}^{n} \alpha_{\sigma(k)} \cdot \mu_i(\varphi_{\sigma(k)}) \geq \beta)$ for any permutation σ over $\{1, ..., n\}$
N3.	$(\sum_{k=1}^{n} \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \wedge (\sum_{k=1}^{n} \alpha'_k \cdot \mu_i(\varphi_k) \geq \beta') \to$ $(\sum_{k=1}^{n} (\alpha_k + \alpha'_k) \cdot \mu_i(\varphi_k) \geq (\beta + \beta'))$
N4.	$((t \geq \beta) \wedge (d \geq 0)) \to (d \cdot t \geq d \cdot \beta)$
N5.	$(t \geq \beta) \vee (\beta \geq t)$
N6.	$(t \geq \beta) \to (t > \gamma)$ for all $\gamma < \beta$
Axioms for Intuitionistic Probabilities	
P1.	$\mu_i(\bot) = 0$
P2.	$\mu_i(\top) = 1$
P3.	$\mu_i(\varphi) + \mu_i(\psi) = \mu_i(\varphi \vee \psi) + \mu_i(\varphi \wedge \psi)$
P4.	$((\Box_i(\varphi \to \psi)) \wedge (\mu_i(\varphi) = \mu_i(\psi))) \leftrightarrow \Box_i(\psi \leftrightarrow \varphi)$
P5.	$(\sum_{k=1}^{n} \alpha_k \cdot \mu_i(\varphi_k) \geq \beta) \to \Box_i (\sum_{k=1}^{n} \alpha_k \cdot \mu_i(\varphi_k) \geq \beta)$

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Table 5.3: Axioms of IPDEL

Reduction Axioms	
I1.	$[\mathcal{E}, e]\, p \leftrightarrow pre(e) \rightarrow \mathbf{sub}(e, p)$
I2.	$\langle \mathcal{E}, e\rangle p \leftrightarrow pre(e) \wedge \mathbf{sub}(e, p)$
I3.	$[\mathcal{E}, e]\, \top \leftrightarrow \top$
I4.	$\langle \mathcal{E}, e\rangle \top \leftrightarrow pre(e)$
I5.	$[\mathcal{E}, e]\, \bot \leftrightarrow \neg pre(e)$
I6.	$\langle \mathcal{E}, e\rangle \bot \leftrightarrow \bot$
I7.	$[\mathcal{E}, e]\, (\psi_1 \wedge \psi_2) \leftrightarrow [\mathcal{E}, e]\, \psi_1 \wedge [\mathcal{E}, e]\, \psi_2$
I8.	$\langle \mathcal{E}, e\rangle(\psi_1 \wedge \psi_2) \leftrightarrow \langle \mathcal{E}, e\rangle\psi_1 \wedge \langle \mathcal{E}, e\rangle\psi_2$
I9.	$[\mathcal{E}, e]\, (\psi_1 \vee \psi_2) \leftrightarrow pre(e) \rightarrow \langle \mathcal{E}, e\rangle\psi_1 \vee \langle \mathcal{E}, e\rangle\psi_2$
I10.	$\langle \mathcal{E}, e\rangle(\psi_1 \vee \psi_2) \leftrightarrow \langle \mathcal{E}, e\rangle\psi_1 \vee \langle \mathcal{E}, e\rangle\psi_2$
I11.	$[\mathcal{E}, e]\, (\psi_1 \rightarrow \psi_2) \leftrightarrow \langle \mathcal{E}, e\rangle\psi_1 \rightarrow \langle \mathcal{E}, e\rangle\psi_2$
I12.	$\langle \mathcal{E}, e\rangle(\psi_1 \rightarrow \psi_2) \leftrightarrow pre(e) \wedge (\langle \mathcal{E}, e\rangle\psi_1 \rightarrow \langle \mathcal{E}, e\rangle\psi_2)$
I13.	$[\mathcal{E}, e]\, \Diamond_i\psi \leftrightarrow pre(e) \rightarrow \bigvee_{e' \sim_i e} \Diamond_i(\langle \mathcal{E}, e'\rangle\psi)$
I14.	$\langle \mathcal{E}, e\rangle\Diamond_i\psi \leftrightarrow pre(e) \wedge \bigvee_{e' \sim_i e} \Diamond_i(\langle \mathcal{E}, e'\rangle\psi)$
I15.	$[\mathcal{E}, e]\, \Box_i\psi \leftrightarrow pre(e) \rightarrow \bigwedge_{e' \sim_i e} \Box_i([\mathcal{E}, e']\psi)$
I16.	$\langle \mathcal{E}, e\rangle\Box_i\psi \leftrightarrow pre(e) \wedge \bigwedge_{e' \sim_i e} \Box_i([\mathcal{E}, e']\psi)$
I17.	$[\mathcal{E}, e]\, (\alpha\mu_i(\psi) \geq \beta) \leftrightarrow pre(e) \rightarrow (C + D \geq 0)$
I18.	$\langle \mathcal{E}, e\rangle(\alpha\mu_i(\psi) \geq \beta) \leftrightarrow pre(e) \wedge (C' + D \geq 0)$
	where
	$C := \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}([\mathcal{E}, e']\, \psi)$
	$C' := \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\langle \mathcal{E}, e'\rangle\psi)$
	$D := \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\top)$,
	with
	$\mu_i^{\phi}(\psi) := \mu_i(\psi \wedge \phi) - \sum_{\sigma \in \mathrm{mb}(\phi)} \mu_i(\psi \wedge \sigma)$
	and
	$\mathrm{mb}(\phi) := \max_{\rightarrow} \Phi \cap \downarrow \phi$.
Inference Rules	
MP.	if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$
Nec$_i$	if $\vdash A$, then $\vdash \Box_i A$
Nec$_\alpha$	if $\vdash A$, then $\vdash [\mathcal{E}, e]A$
Sub$_\mu$	if $\vdash A \rightarrow B$, then $\vdash \mu_i(A) \leq \mu_i(B)$
SubEq	if $\vdash A \leftrightarrow B$, then $\vdash \phi \leftrightarrow \phi[A/B]$

Lemma 182. *Axioms M8 and M9 from Definition 129 are derivable from rules and axioms in Table 5.2.*

Proof. Axiom M9 (i.e. $\top \leq \Box_i\top$) is a direct consequence of the necessitation rule. Axiom M8 (i.e. $\Diamond_i\bot \leq \bot$) can be derived as follows: by instantiating axiom M6 with $\bot$, one gets $\Diamond_i\Box_i\bot \to \Box_i\bot$; by instantiating axiom M2 with $\bot$, one gets $\Box_i\bot \to \bot$; since, in addition, $\bot \to \Box_i\bot$ (axiom H9), one gets that $\Box_i\bot \leftrightarrow \bot$; by substitution of logical equivalence (rule SubEq) in $\Diamond_i\Box_i\bot \to \Box_i\bot$, one gets $\Diamond_i\bot \to \bot$ as required. □

Lemma 183. *Axiom P4 in Table 5.2 implies axiom p4 in Table 5.1. In classical logic the two formulas are equivalent in the context of the rest of the axioms. Finally, there exists an ApPE-structure that validates axiom p4 but doesn't validate axiom P4.*

Proof. Recall that

$$\begin{array}{ll} \text{(P4)} & ((\Box_i(\phi \to \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi))]) \leftrightarrow \Box_i(\psi \leftrightarrow \phi), \\ \text{(p4)} & \Box_i\varphi \leftrightarrow (\mu_i(\varphi) = 1). \end{array}$$

That P4 implies p4 follows immediately by replacing ψ with $\top$. Now, let us prove that p4 implies P4 in classical logic. We first show that p4 implies $\Box_i(\psi \leftrightarrow \phi) \to ((\Box_i(\phi \to \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi))])$ as follows.

$$\begin{array}{lll} & \Box_i(\psi \leftrightarrow \phi) & \\ \Leftrightarrow & \mu_i(\psi \leftrightarrow \phi) = 1 & \text{(Axiom p4)} \\ \Leftrightarrow & \mu_i((\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi)) = 1 & \text{(classical logic equivalence)} \end{array}$$

Notice that

$$(\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi) \quad \to \quad (\neg\psi \vee \phi) \tag{5.5.1}$$
$$(\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi) \quad \to \quad (\neg\phi \vee \psi) \tag{5.5.2}$$

Hence, using the rule Sub_μ : if $\vdash A \to B$, then $\vdash \mu_i(A) \leq \mu_i(B)$, the equality $\mu_i((\neg\psi \vee \phi) \wedge (\neg\phi \vee \psi)) = 1$ and the equations (5.5.1) and (5.5.2), one can prove that

$$(\mu_i(\neg\psi \vee \phi) = 1) \wedge (\mu_i(\neg\phi \vee \psi) = 1)$$

Using p4, we can derive that $\Box_i(\phi \to \phi)$. It remains to derive that $\mu_i(\psi) = \mu_i(\phi)$ as follows.

$$
\begin{aligned}
& (\mu_i(\neg\psi \vee \phi) = 1) \wedge (\mu_i(\neg\phi \vee \psi) = 1) \\
\Rightarrow\ & (\mu_i(\neg(\neg\psi \vee \phi)) = 0) \wedge (\mu_i(\neg\phi \vee \psi) = 1) \\
& \qquad (\mu_i(\varphi) = 1 - \mu_i(\neg\varphi) \text{ in PDEL, see Table 5.1}) \\
\Rightarrow\ & (\mu_i(\psi \wedge \neg\phi) = 0) \wedge (\mu_i(\neg\phi \vee \psi) = 1) \qquad \text{(De Morgan laws)} \\
\Rightarrow\ & (\mu_i(\psi \wedge \neg\phi) = 0) \wedge (\mu_i(\neg\phi) + \mu_i(\psi) - \mu_i(\psi \wedge \neg\phi) = 1) \\
& \qquad (\mu_i(\varphi) + \mu_i(\psi) = \mu_i(\varphi \vee \psi) + \mu_i(\varphi \wedge \psi) \text{ in PDEL, see Table 5.1}) \\
\Rightarrow\ & \mu_i(\neg\phi) + \mu_i(\psi) = 1 \\
\Rightarrow\ & \mu_i(\neg\phi) + \mu_i(\psi) = \mu_i(\phi) + \mu_i(\neg\phi) \\
& \qquad (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL, by axioms p2 and p3}) \\
\Rightarrow\ & \mu_i(\psi) = \mu_i(\phi).
\end{aligned}
$$

Now, we show that p4 implies $((\Box_i(\phi \to \psi)) \wedge (\mu_i(\phi) = \mu_i(\psi))]) \to \Box_i(\psi \leftrightarrow \phi)$ as follows.

$$
\begin{aligned}
& \Box_i(\phi \to \psi) \wedge (\mu_i(\phi) = \mu_i(\psi)) \\
\Rightarrow\ & (\mu_i(\neg\phi \vee \psi) = 1) \wedge (\mu_i(\phi) = \mu_i(\psi)) \qquad \text{(Axiom p4)} \\
\Rightarrow\ & (\mu_i(\neg\phi) + \mu_i(\psi) - \mu_i(\neg\phi \wedge \psi) = 1) \wedge (\mu_i(\phi) = \mu_i(\psi)) \\
& \qquad (\mu_i(\varphi) + \mu_i(\psi) = \mu_i(\varphi \vee \psi) + \mu_i(\varphi \wedge \psi) \text{ in PDEL}) \\
\Rightarrow\ & (\mu_i(\neg\phi) + \mu_i(\psi) - \mu_i(\neg\phi \wedge \psi) = 1) \wedge (\mu_i(\neg\phi) = \mu_i(\neg\psi)) \\
& \qquad (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\
\Rightarrow\ & (\mu_i(\neg\psi) + \mu_i(\psi) - \mu_i(\neg\phi \wedge \psi) = 1) \\
\Rightarrow\ & (1 - \mu_i(\neg\phi \wedge \psi) = 1) \qquad (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\
\Rightarrow\ & (\mu_i(\neg\phi \wedge \psi) = 0) \qquad (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\
\Rightarrow\ & (\mu_i(\phi \vee \neg\psi) = 1) \qquad (\mu_i(\phi) + \mu_i(\neg\phi) = 1 \text{ in PDEL}) \\
\Rightarrow\ & (\mu_i(\psi \to \phi) = 1) \\
\Rightarrow\ & \Box_i(\psi \to \phi) \qquad \text{(Axiom p4)}
\end{aligned}
$$

This concludes the proof that in classical logic p4 and P4 are equivalent. Finally, consider the Heyting algebra $\mathbb{H}$ in Figure 5.1 with

$$
\Diamond x := \begin{cases} \top & \text{if } x \neq \bot, \\ \bot & \text{if } x = \bot \end{cases} \qquad \Box x := \begin{cases} \bot & \text{if } x \neq \top, \\ \top & \text{if } x = \top \end{cases}
$$

and $\mu(\bot) = 0$, $\mu(a) = 0.5$, $\mu(b) = 0.5$ and $\mu(\top) = 1$.

It is easy to see that the Heyting algebra in Figure 5.1 satisfies all axioms of IPDEL except for $P4$ and it satisfies p4. It falsifies P4 because $(\Box(a \to b)) \wedge (\mu(a) = \mu(b)) = \top$, while $\Box(a \leftrightarrow b) = \bot$. □

5.5.4 Soundness

Proposition 184 (Soundness)**.** *The axiomatization for IPDEL given in Table 5.2 is sound w.r.t. APE-models.*

Figure 5.1: Heyting algebra $\mathbb{H}$

Proof. By definition, the underlying structure of an APE-structures is an epistemic Heyting algebra. Hence, it satisfies the axioms of intuitionistic propositional logic and the axioms M1 – M7 and E for static modalities.

Axioms for inequalities. As discussed in Remark 136, it is the case that $\bigvee \mathsf{Min}_i(\mathbb{A}) = \top$ for every epistemic Heyting algebra $\mathbb{A}$. This implies that axioms N0 and N5 are satisfied in every APE-model. Axioms N1, N2, N3, N4 and N6 are also satisfied because if the valuation of their antecedent is above any i-minimal element a then so will be the valuation of their succedent.

Axioms for probabilities. The fact that axioms P1-P3 are satisfied in every APE-model is shown similarly as axiom N0. Since $\lozenge_i\mathbb{A}$ is a subalgebra of $\mathbb{A}$ for every epistemic Heyting algebra $\mathbb{A}$, it is the case that $[\![\varphi]\!]_{\mathcal{M}} \in \lozenge_i\mathbb{A}$ for every i-probability formula φ and every APE-model based on $\mathbb{A}$. Hence, Lemma 135 implies the satisfiability of P5.

Finally, let us show that P4 is satisfied in every APE-model based on $\mathbb{A}$. For the right to left direction, as discussed in Remark 136, every element of $\lozenge_i\mathbb{A}$ can be written as a union of i-minimal elements and therefore $[\![\Box_i(\varphi \leftrightarrow \psi)]\!] = \bigvee\{a \in \mathsf{Min}_i(\mathbb{A}) \mid a \wedge [\![\varphi]\!] = a \wedge [\![\psi]\!]\}$. This of course implies that $\bigvee\{a \in \mathsf{Min}_i(\mathbb{A}) \mid a \wedge [\![\varphi]\!] = a \wedge [\![\psi]\!]\} \leq [\![\mu_i(\varphi) = \mu_i(\psi)]\!]$. As for the left to right direction, we have that $[\![\Box_i(\varphi \to \psi) \wedge (\mu(\varphi) = \mu(\psi))]\!] = \bigvee\{a \in \mathsf{Min}_i(\mathbb{A}) \mid a \wedge [\![\varphi]\!] \leq a \wedge [\![\psi]\!] \text{ and } \mu_i(a \wedge [\![\varphi]\!]) = \mu_i(a \wedge [\![\psi]\!])\}$. By the strict monotonicity of the i-measure μ_i, we have

$$[\![\Box_i(\varphi \to \psi) \wedge (\mu(\varphi) = \mu(\psi))]\!] \leq \bigvee\{a \in \mathsf{Min}_i(\mathbb{A}) \mid a \wedge [\![\varphi]\!] = a \wedge [\![\psi]\!]\} = [\![\Box_i(\varphi \leftrightarrow \psi)]\!]$$

as required.

Reduction axioms. See Section 5.10 (page 246). □

5.6 Completeness

In the present section, we prove the weak completeness of IPDEL w.r.t. APE-models. Recall that a calculus is *weakly complete* w.r.t. a semantics if it provides a proof for every validity, namely, for any formula ϕ, if $\models \phi$ then $\vdash \phi$. Similarly to akin logical systems (cf. [9, 34, 36, 41] [2, 6, 13]), the proof relies on a reduction procedure of IPDEL-formulas to formulas of the static fragment of IPDEL (referred to in what follows as IPEL), which preserves provable equivalence. This reduction procedure is effected using the interaction axioms and the rule of substitution of equivalent formulas. We omit the details since this procedure is standard (see for instance [8, 9, 43] for details). In the reminder of the present section, we prove the weak completeness of IPEL w.r.t. APE-models, i.e., we show that every APE-validity in the language of IPEL is a theorem of IPEL. By contraposition, this is equivalent to proving that for any IPEL-formula φ which is not an IPEL-theorem, there exists an APE-model $\mathcal{M}$ that does not satisfy φ in the sense that $[\![\varphi]\!]_{\mathcal{M}} \neq \top$.

The proof will proceed as follows. In Section 5.6.1, we extract a finite sublattice of the Lindenbaum-Tarski algebra of the logic that contains φ and we prove that it is an Epistemic Heyting Algebra satisfying certain properties akin to those described in [22]. Then, in Section 5.6.2, following ideas from [21] adapted to the algebraic setting, we define appropriate i-measures over the finite Epistemic Heyting Algebra to turn it into an APE-model that does not satisfy φ.

5.6.1 The epistemic Heyting algebra $\mathbb{A}^{\star}_{\varphi}$

In this subsection, we construct the finite epistemic Heyting algebra on which the counter-model for φ is based. The construction consists of a number of steps, starting with the Lindenbaum-Tarski algebra of $\mathcal{L}$ and restricting it accordingly.

Henceforth, we let

$$\mathbb{A} = (A, \top_{\mathbb{A}}, \bot_{\mathbb{A}}, \vee_{\mathbb{A}}, \wedge_{\mathbb{A}}, \rightarrow_{\mathbb{A}}, (\Diamond_i)_{i\in \mathsf{Ag}}, (\Box_i)_{i\in \mathsf{Ag}}) \tag{5.6.1}$$

denote the Lindenbaum-Tarski algebra of IPEL. We will use $\neg_{\mathbb{A}}(\bullet)$ as shorthand for $\bullet \rightarrow_{\mathbb{A}} \bot_{\mathbb{A}}$. For any agent i, we define:

$$\Diamond_i\mathbb{A} := \{\Diamond_i a \in \mathbb{A} \mid a \in \mathbb{A}\}.$$

For any formula $\sigma \in \mathcal{L}_{IPEL}$, we let $\sigma^{\mathbb{A}} \in \mathbb{A}$ denote the equivalence class of σ modulo provable equivalence in IPEL. Let

$$\mathbb{B} := (B, \top_{\mathbb{B}}, \bot_{\mathbb{B}}, \vee_{\mathbb{B}}, \wedge_{\mathbb{B}}, \neg_{\mathbb{B}})$$

be the Boolean Extension of the Heyting algebra reduct of $\mathbb{A}$ (see [37, Section 13, page 450]).[6] To enhance readability, we identify $\mathbb{A}$ with its image through the embedding $A \hookrightarrow B$. Recall that $\mathbb{A}$ is a sublattice of $\mathbb{B}$. Henceforth, we will use $\vee$ and $\wedge$ and $\top$ and $\bot$ ambiguously to denote the operations on both algebras. Since $\Diamond_i\mathbb{A}$ is a Boolean algebra (see Lemma 135) and, in every Boolean algebra, negation is unique, we have that $\neg_{\mathbb{A}} a = \neg_{\mathbb{B}} a$ for every $a \in \Diamond_i\mathbb{A}$ and for every agent $i \in \mathsf{Ag}$.

[6] The Boolean extension of $\mathbb{A}$ can be identified with the algebra of clopens of the Esakia space dual to $\mathbb{A}$. Notice that this is exactly the same construction semantically underlying the Gödel-Tarski translation (cf. [17, Section 3] for an expanded discussion).

Let φ be an IPEL-formula that is not a theorem. Let

$$S_\varphi := \{\sigma^{\mathbb{A}} \mid \sigma \text{ is a subformula of } \varphi\},$$

let Ag_φ be the set of agents that appear in φ and let $S_\varphi^{\Diamond} \supseteq S_\varphi$ be

$$S_\varphi^{\Diamond} := S_\varphi \cup \{(\Diamond_i \sigma)^{\mathbb{A}}, (\neg \Diamond_i \sigma)^{\mathbb{A}} \mid \sigma \in S_\varphi \text{ and } i \in \mathsf{Ag}_\varphi\}.$$

Notice that the sets S_φ and $S_\varphi^{\Diamond}$ are finite. Now let $\mathbb{B}_\varphi \subseteq \mathbb{B}$ be the Boolean subalgebra of $\mathbb{B}$ generated by $S_\varphi^{\Diamond}$. Since $S_\varphi^{\Diamond}$ is finite, so will be the domain of $\mathbb{B}_\varphi$ (which we denote with B_φ). It follows that $\mathbb{B}_\varphi$ is generated by its atoms. In view of what will follow, let us endow $\mathbb{B}_\varphi$ with a measure $\mu_{\mathbb{B}}$ as follows: Let n be the number of atoms of $\mathbb{B}_\varphi$. For every $a \in \mathbb{B}_\varphi$ that is above exactly m atoms, let

$$\mu_{\mathbb{B}}(a) = \frac{m}{n}. \tag{5.6.2}$$

Now, let

$$\mathbb{A}_\varphi := (A_\varphi, \top, \bot, \wedge, \vee)$$

with $A_\varphi := A \cap B_\varphi$. Notice that, since both $\mathbb{A}$ and $\mathbb{B}_\varphi$ are distributive lattices, so is $\mathbb{A}_\varphi$. For every agent $i \in \mathsf{Ag}_\varphi$, we define

$$A_\varphi^{\Diamond_i} := \{a \in \mathbb{A}_\varphi \mid \text{ there exists } \sigma \in \mathcal{L} \text{ such that } \Diamond_i \sigma \in a\} = A_\varphi \cap \Diamond_i \mathbb{A}.$$

Notice that, if $a \in A_\varphi^{\Diamond_i}$, then $\neg_{\mathbb{A}} a \in A_\varphi^{\Diamond_i}$ as well (since $\neg_{\mathbb{B}} a \in B_\varphi$ and $\neg_{\mathbb{B}} a = \neg_{\mathbb{A}} a$). Hence for every agent $i \in \mathsf{Ag}_\varphi$, $\left(A_\varphi^{\Diamond_i}, \top, \bot, \wedge, \vee, \neg_{\mathbb{A}}\right)$ is a Boolean subalgebra of $\mathbb{A}_\varphi$. We are now ready to endow $\mathbb{A}_\varphi$ with an epistemic Heyting algebra structure.

Definition 185. Let

$$\mathbb{A}_\varphi^\star := (\mathbb{A}_\varphi, \to^\star, (\Diamond_i^\star)_{i \in \mathsf{Ag}}, (\Box_i^\star)_{i \in \mathsf{Ag}})$$

where, for all $a, b \in \mathbb{A}_\varphi$,

$$a \to^\star b := \bigvee \{c \in \mathbb{A}_\varphi \mid c \leq a \to_{\mathbb{A}} b\} = \bigvee \{c \in \mathbb{A}_\varphi \mid c \wedge a \leq b\},$$

$$\Diamond_i^\star a := \bigwedge \{b \in A_\varphi^{\Diamond_i} \mid a \leq b\} \qquad \Box_i^\star a := \bigvee \{b \in A_\varphi^{\Diamond_i} \mid b \leq a\} \qquad \text{for } i \in \mathsf{Ag}_\varphi$$

$$\Diamond_i^\star a := \begin{cases} \top & \text{if } a \neq \bot, \\ \bot & \text{if } a = \bot \end{cases} \qquad \Box_i^\star a := \begin{cases} \bot & \text{if } a \neq \top, \\ \top & \text{if } a = \top \end{cases} \qquad \text{for } i \notin \mathsf{Ag}_\varphi$$

The operations above are well-defined since $\mathbb{A}_\varphi$ is a finite distributive lattice and hence all the joins and meets exist.

Lemma 186. *For every $i \in \mathsf{Ag}_\varphi$, the algebra $\mathbb{A}_\varphi^\star$ satisfies the following properties:*

1. $\Diamond_i^\star \mathbb{A}_\varphi^\star = \{\Diamond_i^\star a \mid a \in \mathbb{A}_\varphi^\star\} \subseteq A_\varphi^{\Diamond_i}$;

2. $\lozenge_i^\star \mathbb{A}_\varphi^\star = \Box_i^\star \mathbb{A}_\varphi^\star$;

3. *for all* $a \in A_\varphi^{\lozenge_i}$, *it holds that* $\lozenge_i^\star a = a$ *and* $\Box_i^\star a = a$;

4. *for all* $a, b \in \mathbb{A}_\varphi^\star$, *if* $a \to_\mathbb{A} b \in \mathbb{A}_\varphi^\star$, *then* $a \to^\star b = a \to_\mathbb{A} b$;

5. *for all* $a \in \mathbb{A}_\varphi^\star$, *if* $\lozenge_i a \in \mathbb{A}_\varphi^\star$ *(resp.* $\Box_i a \in \mathbb{A}_\varphi^\star$*), then* $\lozenge_i^\star a = \lozenge_i a$ *(resp.* $\Box_i^\star a = \Box_i a$*);*

6. *for all formulas* $\psi, \varphi \in \mathcal{L}$, *if* $(\lozenge_i \psi)^\mathbb{A} \in S_\varphi^\lozenge$ *(resp.* $(\Box_i \psi)^\mathbb{A} \in S_\varphi^\lozenge$ *or* $(\psi \to \chi)^\mathbb{A} \in S_\varphi^\lozenge$*), then* $\lozenge_i^\star \psi^\mathbb{A} = \lozenge_i \psi^\mathbb{A}$ *(resp.* $\Box_i^\star \psi^\mathbb{A} = \Box_i \psi^\mathbb{A}$ *or* $\psi^\mathbb{A} \to^\star \chi^\mathbb{A} = \psi^\mathbb{A} \to_\mathbb{A} \chi^\mathbb{A}$*).*

7. $\lozenge_i^\star \mathbb{A}_\varphi^\star = \{\lozenge_i^\star a \mid a \in \mathbb{A}_\varphi^\star\} = A_\varphi^{\lozenge_i}$;

Proof. The first five items follow immediately from the definition of $\lozenge_i^\star$ and $\Box_i^\star$. Item 6 is an application of items 4 and 5. Item 7 follows from items 1 and 3. □

Lemma 187. *The algebra* $\mathbb{A}_\varphi^\star$ *is an epistemic Heyting algebra.*

Proof. As mentioned early on, $\mathbb{A}_\varphi$ is a distributive lattice. Moreover, by definition, $\to^\star$ is the right residual of $\wedge$ in $\mathbb{A}_\varphi$. This shows that $\mathbb{A}_\varphi^\star$ is a Heyting algebra. To prove that $\mathbb{A}_\varphi^\star$ is an epistemic Heyting algebra, it remains to show that $\mathbb{A}_\varphi^\star$ satisfies the following axioms (c.f. Definitions 129 and 133):

$$a \leq \lozenge_i a \tag{M1}$$
$$\Box_i a \leq a \tag{M2}$$
$$\lozenge_i(a \vee b) \leq \lozenge_i a \vee \lozenge_i b \tag{M3}$$
$$\Box_i(a \to b) \leq \Box_i a \to \Box_i b \tag{M4}$$
$$\lozenge_i a \leq \Box_i \lozenge_i a \tag{M5}$$
$$\lozenge_i \Box_i a \leq \Box_i a \tag{M6}$$
$$\Box_i(a \to b) \leq \lozenge_i a \to \lozenge_i b \tag{M7}$$
$$\lozenge_i \bot \leq \bot \tag{M8}$$
$$\top \leq \Box_i \top \tag{M9}$$
$$\lozenge_i a \vee \neg \lozenge_i a = \top. \tag{E}$$

Let $i \in \mathsf{Ag}_\varphi$. By definition, it immediately follows that $\lozenge_i^\star$ and $\Box_i^\star$ verify axioms M1 and M2. Axiom M3 holds because $\lozenge_i^\star a \vee \lozenge_i^\star b \in \lozenge_i \mathbb{A}_\varphi$ and $a \vee b \leq \lozenge_i^\star a \vee \lozenge_i^\star b$ (and similarly for axiom M4).

As for axioms M5 and M6, since $\lozenge_i^\star a, \Box_i^\star a \in \lozenge_i \mathbb{A}_\varphi$, by item 3 of Lemma 186, we obtain that $\lozenge_i^\star \Box_i^\star a = \Box_i^\star a$ and $\lozenge_i^\star a = \Box_i^\star \lozenge_i^\star a$, which imply the axioms.

In the context of axioms M1 through M6, axiom M7 is equivalent to $\lozenge_i(\lozenge_i p \to \lozenge_i q) \to (\lozenge_i p \to \lozenge_i q)$ (see [10, Lemma 2]), so let us show that $\mathbb{A}_\varphi^\star$ satisfies $\lozenge_i(\lozenge_i p \to \lozenge_i q) \to (\lozenge_i p \to \lozenge_i q)$. Observe that for all $a, b \in \mathbb{A}_\varphi^\star$, since $\lozenge_i^\star a, \lozenge_i^\star b \in A_\varphi^{\lozenge_i}$ and $A_\varphi^{\lozenge_i}$ is a Boolean algebra (and hence contains $\neg_\mathbb{A} \lozenge_i^\star a$), we have that

$$\lozenge_i^\star a \to_\mathbb{A} \lozenge_i^\star b = \neg_\mathbb{A} \lozenge_i^\star a \vee \lozenge_i^\star b \quad \in \ A_\varphi^{\lozenge_i}$$

which implies by item 4 of Lemma 186 that

$$\Diamond_i^\star a \to^\star \Diamond_i^\star b = \Diamond_i^\star a \to_{\mathbb{A}} \Diamond_i^\star b. \tag{5.6.3}$$

Now, by item 3 of Lemma 186, we have that

$$\Diamond_i^\star(\Diamond_i^\star a \to_{\mathbb{A}} \Diamond_i^\star b) = \Diamond_i^\star a \to_{\mathbb{A}} \Diamond_i^\star b$$

which by the equation 5.6.3 is equivalent to

$$\Diamond_i^\star(\Diamond_i^\star a \to^\star \Diamond_i^\star b) = \Diamond_i^\star a \to^\star \Diamond_i^\star b,$$

that is, $\mathbb{A}_\varphi^\star$ satisfies $\Diamond_i(\Diamond_i p \to \Diamond_i q) \to (\Diamond_i p \to \Diamond_i q)$.

Axioms M8 and M9 follows from the fact that $\top, \bot \in A_\varphi \cap \Diamond_i \mathbb{A}$ and item 3 of Lemma 186.

5

Finally, axiom E follows immediately from item 4 of Lemma 186 and from the fact that $A_\varphi^{\Diamond_i}$ is a Boolean algebra. Hence if $a \in A_\varphi^{\Diamond_i}$ then $(a \to_{\mathbb{A}} \bot_{\mathbb{A}}) \in A_\varphi^{\Diamond_i}$. □

5.6.2 Measures on $\mathbb{A}_\varphi^\star$

In this section, for each agent $i \in \mathsf{Ag}_\varphi$, we will define an i-measure on the algebra $\mathbb{A}_\varphi^\star$ and a valuation on $\mathbb{A}_\varphi^\star$, so as to define an APE-model $\mathcal{M}_\varphi$ such that $[\![\sigma]\!]_{\mathcal{M}_\varphi} = \sigma^{\mathbb{A}}$ for every subformula σ of φ. Before defining the measures, we will state some auxiliary results.

Lemma 188. *The system IPEL proves all classical truths about linear inequalities.*

Proof. See [21] for an explanation of why axioms N0 to N6 are enough. Notice that, even though the result is proven for classical logic, it still holds for IPEL. Indeed, the fragment of the logic involving inequalities is classical because of the axiom N5: $(\tau \geq \beta) \vee (\neg\tau \geq \beta)$. □

Lemma 189. *The formulas*

$$\left(\Diamond_i\psi \wedge \left(\sum_m \alpha_m \mu_i(\phi_m) \geq \beta\right)\right) \to \left(\sum_m \alpha_m \mu_i(\phi_m \wedge \Diamond_i\psi) \geq \beta\right) \tag{5.6.4}$$

and

$$\left(\Diamond_i\psi \wedge \left(\sum_m \alpha_m \mu_i(\phi_m) < \beta\right)\right) \to \left(\sum_m \alpha_m \mu_i(\phi_m \wedge \Diamond_i\psi) < \beta\right) \tag{5.6.5}$$

are provable in IPEL.

Proof. We only prove (5.6.4), the proof of (5.6.5) being almost verbatim. Early on we observed (see Lemma 183) that axiom P4 implies the validity of $\Box_i\varphi \leftrightarrow (\mu_i(\varphi) = 1)$. This and axiom M5 (i.e. $\Diamond_i\psi \leftrightarrow \Box_i\Diamond_i\psi$) imply

$$\vdash_{\mathsf{IPEL}} \Diamond_i\psi \leftrightarrow (\mu_i(\Diamond_i\psi) = 1). \tag{5.6.6}$$

Since $\vdash_{\mathsf{IPEL}} \Diamond_i\psi \to (\Diamond_i\psi \vee \phi_m)$ for every $\phi_m \in \mathcal{L}$, by rule Sub_μ we obtain

$$\vdash_{\mathsf{IPEL}} \mu_i(\Diamond_i\psi) \leq \mu_i(\Diamond_i\psi \vee \phi_m). \tag{5.6.7}$$

From (5.6.7) and Lemma 188, we deduce that

$$\vdash_{\mathsf{IPEL}} \mu_i(\Diamond_i\psi) = 1 \to \mu_i(\phi_m \vee \Diamond_i\psi) = 1. \tag{5.6.8}$$

Lemma 188 and axiom P3 (i.e. $\mu_i(\phi_m) = \mu_i(\phi_m \vee \Diamond_i\psi) + \mu_i(\phi_m \wedge \Diamond_i\psi) - \mu_i(\Diamond_i\psi)$) entail

$$\vdash_{\mathsf{IPEL}} \left(\sum_m \alpha_m\mu_i(\phi_m) \geq \beta\right) \leftrightarrow \left(\sum_m \alpha_m\Big(\mu_i(\phi_m \wedge \Diamond_i\psi) + \mu_i(\phi_m \vee \Diamond_i\psi) - \mu_i(\Diamond_i\psi)\Big) \geq \beta\right). \tag{5.6.9}$$

Combining (5.6.6), (5.6.8) and (5.6.9), we obtain

$$\vdash_{\mathsf{IPEL}} \Big(\Diamond_i\psi \wedge A\Big) \to \Big(\big(\mu_i(\Diamond_i\psi) = 1\big) \wedge \bigwedge_m \big(\mu_i(\phi_m \vee \Diamond_i\psi) = 1\big) \wedge B\Big). \tag{5.6.10}$$

with

$$A := \sum_m \alpha_m\mu_i(\phi_m) \geq \beta,$$

and

$$B := \sum_m \alpha_m(\mu_i(\phi_m \wedge \Diamond_i\psi) + \mu_i(\phi_m \vee \Diamond_i\psi) - \mu_i(\Diamond_i\psi)) \geq \beta.$$

Again, by using Lemma 188, we obtain that

$$\vdash_{\mathsf{IPEL}} \left((\mu_i(\Diamond_i\psi) = 1) \wedge \bigwedge_m(\mu_i(\phi_m \vee \Diamond_i\psi) = 1) \wedge B\right) \to D \tag{5.6.11}$$

with

$$D := \sum_m \alpha_m\mu_i(\phi_m \wedge \Diamond_i\psi) \geq \beta.$$

Putting (5.6.10) and (5.6.11) together, we finally get:

$$\vdash_{\mathsf{IPEL}} \left(\Diamond_i\psi \wedge \sum_m \alpha_m\mu_i(\phi_m) \geq \beta\right) \to \left(\sum_m \alpha_m\mu_i(\phi_m \wedge \Diamond_i\psi) \geq \beta\right)$$

as desired. □

Observe that for any agent $i \in \mathsf{Ag}_\varphi$, since $\mathbb{A}^\star_\varphi$ is finite and $\Diamond^\star_i\mathbb{A}^\star_\varphi = A^{\Diamond_i}_\varphi$ is a Boolean algebra, it is the case that the i-minimal elements are the atoms of this Boolean algebra and every element of $\mathbb{A}_\varphi^{\Diamond_i}$ can be written as the union of some of these i-minimal elements. Let us call a_k^i for $k \in n_i$ the i-minimal elements of $\mathbb{A}_\varphi^\star$. Now, for each i-probability formula σ with $\sigma^{\mathbb{A}} \in S_\varphi^\Diamond$, we have that $\sigma^{\mathbb{A}} \in A_\varphi^{\Diamond_i}$. Hence, we have that $(\neg\sigma)^{\mathbb{A}} \in A_\varphi^{\Diamond_i}$. This implies that there exists a function $f_\sigma : n_i \to \{0, 1\}$ such that

$$\sigma^{\mathbb{A}} = \bigvee_{f_\sigma(k)=1} a_k^i \quad \text{and} \quad (\neg\sigma)^{\mathbb{A}} = \bigvee_{f_\sigma(k)=0} a_k^i.$$

It should be stressed that since $\vee$ and $\wedge$ in $\mathbb{A}_\varphi^\star$ are inherited by $\mathbb{A}$, these equalities hold in $\mathbb{A}$ as well.

Now, let us fix $i \in \mathsf{Ag}_\varphi$. For every $k \in n_i$, we define a system of equations $E_{a_k^i}$, with variables x_b for every $b \leq a_k^i$ as follows[7]:

$$E_{a_k^i} := \left(\begin{array}{ll} \sum \alpha_m \cdot x^i_{\psi_m^{\mathbb{A}} \wedge a_k^i} \geq \beta, & \text{for all } \sigma := (\sum \alpha_m \cdot \mu_i(\psi_m) \geq \beta) \\ & \ \text{with } \sigma^{\mathbb{A}} \in S_\varphi^\Diamond \text{ and } f_\sigma(k) = 1 \\ \sum \alpha_m \cdot x^i_{\psi_m^{\mathbb{A}} \wedge a_k^i} < \beta, & \text{for all } \sigma := (\sum \alpha_m \cdot \mu_i(\psi_m) \geq \beta) \\ & \ \text{with } \sigma^{\mathbb{A}} \in S_\varphi^\Diamond \text{ and } f_\sigma(k) = 0 \\ x_b^i \geq 0 \text{ and } x_b \leq 1, & \text{for all } b \in \mathbb{A}_\varphi^\star \text{ with } b \leq a_k^i \\ x_b^i + x_c^i = x_{b\wedge c}^i + x_{b \vee c}^i, & \text{for all } b, c \in \mathbb{A}_\varphi^\star \text{ with } b, c \leq a_k^i \\ x_b^i \leq x_c^i, & \text{for all } b, c \in \mathbb{A}_\varphi^\star \text{ with } b \leq c \leq a_k^i \\ x_\perp^i = 0 & \\ x_{a_k^i}^i = 1 & \end{array} \right)$$

For a solution s of the above system, we denote with $(x_b^i)^s$ the solution according to s of x_b^i.

Notice that the system is designed in such a way that particular solutions (cf. Lemma 192) provide an i-measure on $\mathbb{A}_\varphi^\star$ such that will guarantee that the valuation of an i-probability formula σ will be $\sigma^{\mathbb{A}}$. Indeed the first two type of inequalities in the system will guarantee that exactly the i-minimal elements of $\mathbb{A}_\varphi^\star$ below $\sigma^{\mathbb{A}}$ will constitute $[\![\sigma]\!]$ (see Definition 181). The rest of the inequalities will guarantee that the solution satisfies the basic properties of i-measures.

Observe that, for every $b \leq a_k^i$, there exists a formula τ_b such that $b = \tau_b^{\mathbb{A}}$ and if $b \leq c$ then $\vdash_{\mathsf{IPEL}} \tau_b \to \tau_c$. Let $E^\tau_{a_k^i}$ be the system of equations where each x_b^i is replaced by $\mu_i(\tau_b)$. Since a_k^i is i-minimal, we can assume without loss of generality that $\tau_{a_k^i}$ is of the form $\Diamond_i \tau'$. Furthermore, let $PS_i \subseteq S_\varphi^\Diamond$ be the set of i-probability formulas that are subformulas of φ. For every $\sigma^{\mathbb{A}} \in PS_i$ such that $\sigma := (\sum \alpha_m \cdot \mu_i(\psi_m) \geq \beta)$, let $\sigma[a_k^i]$ be the formula $\sum \alpha_m \cdot \mu_i(\psi_m \wedge \tau_{a_k^i}) \geq \beta$.

Lemma 190. *For every $k \in n_i$, the system $E_{a_k^i}$ has a solution.*

Proof. Notice that all but the first two types of inequalities in $E^\tau_{a_k^i}$ are provable in IPEL as they are immediate consequences of axioms P1, P2, P3 and the rule Sub_μ. Heading

[7]The sums in system of equations $E_{a_k^i}$ range over m.

towards a contradiction, let us first assume that $E_{a^i_k}$ does not have a solution at all. This is a truth about linear inequalities of rational numbers, hence, by Lemma 188, it is provable in IPDEL. As mentioned above, since some inequalities are provable this is tantamount to saying that

$$\vdash_{\mathsf{IPEL}} \neg((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma[a^i_k]) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a^i_k])). \tag{5.6.12}$$

Notice that, by Lemma 189, we have: for every $\sigma^{\mathbb{A}} \in PS_i$,

$$\vdash_{\mathsf{IPEL}} \left(\sigma \wedge \tau_{a^i_k}\right) \to \sigma[a^i_k]$$

and

$$\vdash_{\mathsf{IPEL}} \left(\neg\sigma \wedge \tau_{a^i_k}\right) \to \neg\sigma[a^i_k].$$

Therefore,

$$\vdash_{\mathsf{IPEL}} (((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma)) \wedge \tau_{a^i_k}) \to ((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma[a^i_k]) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a^i_k])). \tag{5.6.13}$$

Since one direction of contraposition is provable in intiontionistic logic we obtain that:

$$\vdash_{\mathsf{IPEL}} (\neg((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma[a^i_k]) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a^i_k]))) \to \neg(((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma)) \wedge \tau_{a^i_k}). \tag{5.6.14}$$

(5.6.12) and (5.6.14) imply that

$$\vdash_{\mathsf{IPEL}} \neg(((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma)) \wedge \tau_{a^i_k}). \tag{5.6.15}$$

In addition, $\mathbb{A}^\star_\varphi$ inherits the order from $\mathbb{A}$ and by construction $a^i_k \leq \sigma^{\mathbb{A}}$ when $f_\sigma(k) = 1$ and $a^i_k \leq (\neg\sigma)^{\mathbb{A}}$ when $f_\sigma(k) = 0$. Hence, we have that, for all $\sigma \in PS_i$, if $f_\sigma(k) = 1$ then $\vdash_{\mathsf{IPEL}} \tau_{a^i_k} \to \sigma$ and if $f_\sigma(k) = 0$ then $\vdash_{\mathsf{IPEL}} \tau_{a^i_k} \to \neg\sigma$. Therefore, we have

$$\vdash_{\mathsf{IPEL}} \tau_{a^i_k} \to ((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma)).$$

Hence,

$$\vdash_{\mathsf{IPEL}} \neg\tau_{a^i_k} \leftrightarrow \neg(((\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=1}} \sigma) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}}\in PS_i \\ f_\sigma(k)=0}} \neg\sigma)) \wedge \tau_{a^i_k})$$

and by (5.6.15)

$$\vdash_{\mathsf{IPEL}} \neg\tau_{a^i_k}.$$

We have reached a contradiction because a^i_k is an element of $\mathbb{A}$ different from $\bot$ and hence each formula corresponding to it is consistent. Therefore $E_{a^i_k}$ has a solution. □

Lemma 191. *For every $k \in n_i$ and every $b < c \leq a^i_k$, the system $E_{a^i_k}$ has a solution $s_{b,c}$ such that $(x^i_b)^{s_{b,c}} < (x^i_c)^{s_{b,c}}$.*

Proof.

Heading towards a contradiction, let $b < c \leq a^i_k$ such that in every solution s of $E_{a^i_k}$, we have $(x^i_b)^s = (x^i_c)^s$. This is a fact of inequalities of real numbers and therefore, by Lemma 188, it is provable in IPEL. Since all but the first two types of inequalities in $E_{a^i_k}$ are provable in IPEL, we have that

$$\vdash_{\text{IPEL}} ((\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a^i_k]) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a^i_k])) \to \mu_i(\tau_b) = \mu_i(\tau_c).$$

Since $\vdash_{\text{IPEL}} \tau_b \to \tau_c$, necessitation implies $\vdash_{\text{IPEL}} \Box_i(\tau_b \to \tau_c)$. Using axiom P4

$$\Big(\big(\Box_i(\phi \to \psi)\big) \wedge \big(\mu_i(\phi) = \mu_i(\psi)\big)\Big) \leftrightarrow \Box_i(\psi \leftrightarrow \phi),$$

we obtain that

$$\vdash_{\text{IPEL}} ((\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a^i_k]) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a^i_k])) \to \Box_i(\tau_c \to \tau_b). \tag{5.6.16}$$

Recall that[8]

$$\vdash_{\text{IPEL}} \tau_{a^i_k} \to ((\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma)). \tag{5.6.17}$$

Using Lemma 189 and (5.6.17) (cf. (5.6.13)), we get that

$$\vdash_{\text{IPEL}} \tau_{a^i_k} \to ((\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=1}} \sigma[a^i_k]) \wedge (\bigwedge_{\substack{\sigma^{\mathbb{A}} \in PS_i \\ f_\sigma(k)=0}} \neg\sigma[a^i_k])). \tag{5.6.18}$$

From (5.6.16) and (5.6.18), we deduce that

$$\vdash_{\text{IPEL}} \tau_{a^i_k} \to \Box_i(\tau_c \to \tau_b).$$

By axiom M2 ($\Box_i p \to p$), we have

$$\vdash_{\text{IPEL}} \tau_{a^i_k} \to (\tau_c \to \tau_b),$$

which is equivalent to

$$\vdash_{\text{IPEL}} (\tau_{a^i_k} \wedge \tau_c) \to \tau_b.$$

Since $\vdash_{\text{IPEL}} \tau_c \to \tau_{a^i_k}$, the equation above implies that

$$\vdash_{\text{IPEL}} \tau_c \to \tau_b, .$$

This last equation is a contradiction since in $\mathbb{A}$, the Lindenbaum-Tarski algebra of IPEL, we have that $c \not\leq b$. Therefore, for every such pair $b < c \leq a^i_k$, there exists a solution $s_{b,c}$ of $E_{a^i_k}$ such that $(x^i_b)^{s_{b,c}} < (x^i_c)^{s_{b,c}}$. □

[8] see proof of Lemma 190.

Lemma 192. *For every $k \in n_i$, the system $E_{a^i_k}$ has a solution s, such that $(x^i_b)^s < (x^i_c)^s$ for all $b, c \leq a^i_k$ with $b < c$.*

Proof.

By Lemma 191, for every pair $b < c \leq a^i_k$ there exists a solution $s_{b,c}$ of $E_{a^i_k}$ such that $(x^i_b)^{s_{b,c}} < (x^i_c)^{s_{b,c}}$. Notice that the solution space of $E_{a^i_k}$ is a convex subspace of $\mathbb{R}^l$, for some natural number l. Indeed, it is immediate that the solutions of each linear inequality define a convex space and the intersection of convex spaces is a convex space (cf. [35, Chapter 12]). Let n be the number of aforementioned solutions. Then it is the case that $\sum_{b<c\leq a^i_k} \frac{1}{n} s_{b,c}$ is also a solution of $E_{a^i_k}$ (see e.g. [35, Chapter 12, Theorem 1.2]). Let us call this solution s and show that if $d < e$ then $(x^i_d)^s < (x^i_e)^s$.

Let $d < e$. Notice that, for every $s_{b,c}$, it is the case that $(x^i_d)^{s_{b,c}} \leq (x^i_e)^{s_{b,c}}$ by the restraints of the system $E_{a^i_k}$. Moreover, we have $(x^i_d)^{s_{d,e}} < (x^i_e)^{s_{d,e}}$. Hence,

$$(x^i_d)^s = \sum_{b<c} \frac{1}{n} (x^i_d)^{s_{b,c}} < \sum_{b<c} \frac{1}{n} (x^i_e)^{s_{b,c}} = (x^i_e)^s.$$

Therefore, we have that, for every pair $d < e \leq a^i_k$, we have $(x^i_d)^s < (x^i_e)^s$ s as required. □

For every agent $i \in \mathsf{Ag}_\varphi$ and every system $E_{a^i_k}$, pick a solution s satisfying the conditions of Lemma 192 and define $\mu_i(b) = (x^i_b)^s$, for every $b \in \mathsf{Min}(\mathbb{A}^\star_\varphi){\downarrow}$. For agents $j \notin \mathsf{Ag}_\varphi$, let $\mu_j(b) = \mu_{\mathbb{B}}(b)$ (see (5.6.2)). Now we define an APE-model

$$\mathcal{M}_\varphi = \langle \mathbb{A}^\star_\varphi, (\mu_i)_{i\in\mathsf{Ag}}, v\rangle \tag{5.6.19}$$

such that, for every $p \in AtProp \cap S^\Diamond_\varphi$, it holds that $v(p) = p^\mathbb{A}$.

Lemma 193. *The model $\mathcal{M}_\varphi$ is an APE-model.*

Proof. For any $i \in \mathsf{Ag}_\varphi$ the restrictions imposed by the systems of inequalities and the conditions of Lemma 192 immediately yield that μ_i is an i-measure. For $j \notin \mathsf{Ag}_\varphi$ the only j-minimal element is $\top$. Furthermore, $\mu_{\mathbb{B}}$ is satisfies the restrictions of j-measures by definition. Hence each μ_i is an i-measure, and by Lemma 187 and Definition 141 we have that $\mathcal{M}_\varphi$ is an APE-model. □

Lemma 194 (Truth Lemma). *For every $\psi \in \mathcal{L}$ such that $\psi^\mathbb{A} \in S^\Diamond_\varphi$, it is the case that*

$$[\![\psi]\!]_{\mathcal{M}_\varphi} = \psi^\mathbb{A}.$$

Proof. By definition, $S^\Diamond_\varphi$ is closed under subformulas. The proof proceeds by induction on the complexity of ψ. For the atomic variables, this follows immediately from the definition of v. For formulas of the form $\psi \wedge \tau$ and $\psi \vee \tau$ this follows from the fact that $\mathbb{A}^\star_\varphi$ inherits $\vee$ and $\wedge$ from $\mathbb{A}$. For formulas of the form $\psi \to \tau$, $\Diamond_i\psi$ and $\Box_i\psi$ it follows from item 6 of Lemma 186. Finally, for probability formulas of the form $\sigma := \sum \alpha_m \mu_i(\psi_m) \geq \beta$, notice that, by the choice of μ_i as particular solutions of the systems $E_{a^i_k}$, exactly the i-minimal elements $a^i_k \leq \sigma^\mathbb{A}$ are such that $\sum \alpha_m \mu_i([\![\psi_m]\!]_{\mathcal{M}_\varphi} \wedge a^i_k) \leq \beta$. Hence, $[\![\sigma]\!]_{\mathcal{M}_\varphi} = \sigma^\mathbb{A}$ by definition (cf. Definition 181). This concludes the proof. □

Proposition 195 (Completeness)**.** *The axiomatisation for IPDEL given in Table 5.2 is weakly complete w.r.t. APE-models.*

Proof. As discussed in the beginning of this section, the problem is reduced to proving the weak completeness of IPEL. Let φ be an IPEL formula that is not a theorem. This means that $\varphi^{\mathbb{A}} \neq \top^{\mathbb{A}}$, where $\mathbb{A}$ is the Lindembaum-Tarski algebra of IPEL (see (5.6.1)). By Lemma 193, the model $\mathcal{M}_\varphi$ based on the algebra $\mathbb{A}^\star_\varphi$ defined in (5.6.19) is an APE-model. By Lemma 194, $[\![\varphi]\!]_{\mathcal{M}_\varphi} = \varphi^{\mathbb{A}}$. Since $\top^{\mathbb{A}^\star_\varphi} = \top^{\mathbb{A}}$, this shows that $[\![\varphi]\!]_{\mathcal{M}_\varphi} \neq \top^{\mathbb{A}^\star_\varphi}$, which means that $\mathcal{M}_\varphi$ does not satisfies φ as required. □

5.7 Relational semantics

In this section we introduce the finite relational semantics of IPDEL, as the dual structures of epistemic Heyting algebras within the duality between monadic Heyting algebras and MIPC-frames (cf. [11, 34]). Specifically we specialize this duality[9] by identifying the condition corresponding to axiom E. Moreover, we present a dual correspondence between the probability distributions on intuitionistic Kripke frames and measures on epistemic Heyting algebras. This correspondence appears in [23] in the context of finite GBL-algebras. Furthermore, we generalize the model-theoretic constructions presented in Section 5.2.2 for the Boolean setting and show that they dually correspond to the constructions presented in Section 5.4. Finally, notice that these results readily imply the completeness and the finite model property of IPDEL with respect to this class of relational structures via the algebraic completeness presented in Section 5.6.

Structure of this section. In Section 5.7.1 we introduce the *epistemic intuitionistic Kripke frames* as the class of relational structures dually corresponding to epistemic Heyting algebras. In Section 5.7.2 we introduce the probability distributions associated with any agent i and prove that each dually corresponds to an i-measure. In Section 5.7.3 we introduce the construction of intermediate epistemic intuitionistic Kripke frames and prove that it dually corresponds to the construction of intermediate epistemic Heyting algebras presented in Section 5.4.4. Finally in Section 5.7.4 we define the dual construction to the pseudo-quotient defined in 5.4.5.

5.7.1 Epistemic HAs and epistemic intuitionistic Kripke frames

We first recall the definition on the objects of the duality between finite monadic Heyting algebras and MIPC-frames[10]. We then identify the MIPC-frames corresponding to epistemic intuitionistic Kripke frames and show that their dual algebras exactly correspond to epistemic Heyting algebras.

Definition 196 (Finite MIPC-frames)**.** A *finite MIPC-frame* is a tuple

$$\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$$

[9] Because we consider only finite algebras and finite relational structures we can dispense with the topology.

[10] A complete exposition can be found in [11].

such that $(S, \leq)$ is a finite poset and each R_i is an equivalence relation on S such that

$$(R_i \circ \geq) \subseteq (\geq \circ R_i) \qquad R_i = (\geq \circ R_i) \cap (R_i \circ \leq).$$

Notation 197. For any poset $(S, \leq)$ and any set $X \subseteq S$, we define the downset and the upset generated by X as

$$X\downarrow = \{w \in S \mid \exists v \in X, w \leq v\} \qquad \text{and} \qquad X\uparrow = \{w \in S \mid \exists v \in X w \geq v\}$$

respectively. We let $\mathcal{P}^{\downarrow}(S) = \{X\downarrow \mid X \subseteq S\}$ be the set of all downsets of S.

Definition 198 (Complex algebra of a finite MIPC-frame)**.** For any finite MIPC-frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$, let its *complex algebra* be:

$$\mathbb{F}^+ = (\mathcal{P}^{\downarrow}(S), \wedge, \vee, \rightarrow, (\Diamond_i)_{i \in \mathsf{Ag}}, (\Box_i)_{i \in \mathsf{Ag}}, \bot)$$

where

$$X \wedge Y := X \cap Y; \tag{5.7.1}$$
$$X \vee Y := X \cup Y; \tag{5.7.2}$$
$$X \rightarrow Y := S \setminus ((X \cap (S \setminus Y)) \uparrow); \tag{5.7.3}$$
$$\Diamond_i X := R_i^{-1}[X]; \tag{5.7.4}$$
$$\Box_i X := S \setminus (\geq \circ R_i)^{-1}[S \setminus X]. \tag{5.7.5}$$
$$\bot := \emptyset; \tag{5.7.6}$$

We also use the standard notation

$$\top := S; \tag{5.7.7}$$
$$\neg X := X \rightarrow \bot = S \setminus X\uparrow . \tag{5.7.8}$$

Definition 199 (MIPC frame associated to a finite monadic Heyting algebra)**.** For any finite monadic Heyting algebra[11] $\mathbb{A} = (\mathbb{L}, (\Diamond_i)_{i \in \mathsf{Ag}}, (\Box_i)_{i \in \mathsf{Ag}})$, let its *associated frame* be:

$$\mathbb{A}_+ = \langle \mathcal{J}(\mathbb{A}), \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$$

where

- $\mathcal{J}(\mathbb{A})$ is the set of join-irreducible elements of $\mathbb{A}$;
- $\leq \subseteq \mathcal{J}(\mathbb{A}) \times \mathcal{J}(\mathbb{A})$ is the order inherited from $\mathbb{A}$, i.e. $j \leq j'$ iff $j \leq_{\mathbb{A}} j'$ for all $j, j' \in \mathcal{J}(\mathbb{A})$;
- $R_i \subseteq \mathcal{J}(\mathbb{A}) \times \mathcal{J}(\mathbb{A})$ is defined as follows: $j R_i j'$ if and only if $\Diamond_i j = \Diamond_i j'$ for all $j, j' \in \mathcal{J}(\mathbb{A})$ and every $i \in \mathsf{Ag}$.

The following lemma is stated in [34, Fact 20,Proposition 21] and [11]:

[11]see Definition 129, page 184.

Lemma 200. *If $\mathbb{F}$ is a finite MIPC-frame, then $\mathbb{F}^+$ is a finite monadic Heyting algebra. If $\mathbb{A}$ is a finite monadic Heyting algebra then $\mathbb{A}_+$ is a finite MIPC-frame. Furthermore $(\mathbb{F}^+)_+ \cong \mathbb{F}$ and $(\mathbb{A}_+)^+ \cong \mathbb{A}$.*

Definition 201 (Epistemic intuitionistic Kripke frame)**.** An *epistemic intuitionistic Kripke frame* is a finite $MIPC$-frame $\mathbb{F} = \langle S, \leq, (R_i)_{i\in \mathsf{Ag}}\rangle$ such that, for every $i \in$ Ag, the equivalence relation R_i is upwards and downwards closed w.r.t. the order relation $\leq$.

The following lemma characterises the dual spaces of epistemic Heyting algebras[12]:

Lemma 202. *If $\mathbb{A}$ is an epistemic Heyting algebra, then $\mathbb{A}_+$ is an epistemic intuitionistic Kripke frame. If $\mathbb{F}$ is an epistemic intuitionistic Kripke frame, then $\mathbb{F}^+$ is an epistemic Heyting algebra.*

Proof. Since, by definition, all epistemic Heyting algebras are finite monadic Heyting algebras, it follows from Lemma 200 that their dual spaces are finite MIPC-frames.

Let $\mathbb{A} = (\mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$ and $\mathbb{A}_+ = \langle S, \leq, (R_i)_{i\in\mathsf{Ag}}\rangle$. By Lemma 200, it is enough to show that the equivalence relations R_i are upwards and downwards closed. Since R_i is symmetric it is enough to show that R_i is upwards closed.

Assume for contradiction that the equivalence relation R_i is not upwards closed for some $i \in$ Ag. Hence, there is at least one equivalence class defined by the relation R_i that is not upwards closed. Since the empty set is upwards and downwards closed, this equivalence class is non-empty. Let $w \in S$ be an element of that class, let $v \in S$ be such that $v \geq w$ and $v \notin R_i[w]$, and let a be the element of the dual algebra corresponding to the downset generated by w. Then $\Diamond_i a = R_i^{-1}[w{\downarrow}]$.

First, let us show that $v \notin R_i^{-1}[w{\downarrow}]$. Heading towards a contradiction let us assume that $v \in R_i^{-1}[w{\downarrow}]$. This means that there exists $z \in S$ such that $z \leq w$ and $(v, z) \in R_i$, therefore $(v, w) \in (R_i \circ \leq)$. Furthermore, we have that $(v, w) \in (\geq \circ R_i)$, because $(w, w) \in R_i$ and $v \geq w$. Since $R_i = (\geq \circ R_i) \cap (R_i \circ \leq)$, we deduce that $(v, w) \in R_i$, which is a contradiction. This proves that $v \notin R_i^{-1}[w{\downarrow}]$.

From (5.7.8), we have that $\neg\Diamond_i a = S \setminus ((R^{-1}[w{\downarrow}]){\uparrow})$. By assumption, $w \leq v$, hence $v \in (R^{-1}[w{\downarrow}]){\uparrow}$ and $v \notin \neg\Diamond_i a$. Hence $v \notin \Diamond_i a \vee \neg\Diamond_i a$, and therefore axiom E does not hold, contradicting the assumption that $\mathbb{A}$ is an epistemic Heyting algebra. Hence, R_i is upwards closed.

As to the second part of the statement, let $\mathbb{F} = \langle S, \leq, (R_i)_{i\in\mathsf{Ag}}\rangle$ and $\mathbb{F}^+ = (\mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$. By Lemma 200, it remains to prove that $\mathbb{F}^+$ satisfies axiom E (i.e. $\Diamond_i a \vee \neg\Diamond_i a = \top$) for every $i \in$ Ag. Since R_i is upwards closed for every $i \in$ Ag, it follows that $(R_i^{-1}[X{\downarrow}]){\uparrow} = R_i^{-1}[X{\downarrow}]$. Therefore $R_i^{-1}[X{\downarrow}] \cup (S \setminus ((R_i^{-1}[X{\downarrow}]){\uparrow}) = S$, i.e. axiom E holds in $\mathbb{F}^+$, as required. □

Definition 203 (Epistemic intuitionistic Kripke model)**.** An *epistemic intuitionistic Kripke model* is a tuple $\mathbb{M} = \langle \mathbb{F}, V\rangle$ such that $\mathbb{F}$ is an epistemic intuitionistic Kripke frame and V : AtProp $\to \mathcal{P}^{\downarrow}(S)$.

Corollary 204. *For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i\in\mathsf{Ag}}\rangle$, the i-minimal elements of $\mathbb{F}^+$ are exactly the equivalence cells of R_i.*

[12]see Definition 133, page 186.

Proof. Recall (cf. Definition 131) that an element $a \in \mathbb{F}^+$ is i-minimal if

1. $a \neq \bot$,
2. $\Diamond_i a = a$ and
3. if $b \in \mathbb{F}^+$, $b < a$ and $\Diamond_i b = b$, then $b = \bot$.

Let $X \subseteq S$ be an R_i-equivalence cell of $\mathbb{F}$. Hence, X is a non-empty set, which proves item (1). Moreover, by definition of $\Diamond$ (see (5.7.4)), we have $\Diamond_i X := R_i^{-1}[X] = X$, which proves item (2). Finally, if $\varnothing \neq Y \subseteq X$ then $\Diamond_i Y = R_i^{-1}[Y] = X$, which proves item (3).

Let $a \in \mathbb{F}^+ = \mathcal{P}^{\downarrow}(S)$ be an i-minimal element. To prove that a is an equivalence cell of R_i, we need to show that $a = R_i^{-1}[w]$ for some $w \in S$. By item 1, $a \neq \varnothing$; hence, there exists $w \in a$. Recall that $\Diamond_i X := R_i^{-1}[X]$ (see (5.7.4)). By item 2, $a = \Diamond_i a = R_i^{-1}[a]$; hence, a is the union of equivalence cells. By item 3, the only equivalence cell or union of equivalence cells smaller than a is the empty set; hence, a contains exactly one equivalence cell. ☐

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Corollary 205. *For every epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$, and every join-prime element j of $\mathbb{F}^+$, there exists some i-minimal element a such that $j \leq a$.*

Proof. If j is a join-prime element of $\mathbb{F}^+$, then $j = w{\downarrow}$ for some $w \in S$. Let $a = R_i^{-1}[w]$, which is an i-minimal element by Corollary 204. Since the equivalence relation R_i is upwards and downwards closed for every $i \in$ Ag, we have $w{\downarrow} \subseteq R_i^{-1}[w]$, as required. ☐

5.7.2 Epistemic intuitionistic Kripke frames and probabilities

In this section we define i-probability distributions. Applying ideas of [23] to the setting of epistemic Heyting algebras, we define a correspondence between maps from epistemic intuitionistic Kripke frames to non-negative reals and premeasures on epistemic Heyting algebras (see Definition 139).

Definition 206 (i-probability distribution)**.** Let $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$ be an epistemic intuitionistic Kripke frame. An *i-probability distribution* over S is a map $P_i : S \to \,]0, 1]$ such that $\sum_{w \in X} P_i(w) = 1$ for each equivalence cell X of R_i.

Lemma 207. *For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$, any map $f : S \to \mathbb{R}^+$ defines the i-premeasure f^+ on $\mathbb{F}^+$ as follows:*

$$\begin{aligned} f^+ \ : \ \mathsf{Min}_i(\mathbb{A}){\downarrow} &\to \mathbb{R}^+ \\ a &\mapsto \sum_{x \in a} P_i(x). \end{aligned} \tag{5.7.9}$$

Moreover, if f is an i-probability distribution, then the map f^+ is an i-measure (see Definition 139) on $\mathbb{F}^+$.

Proof. This result directly follows from the definition of f^+ and Corollary 204. ☐

Definition 208. For any finite monadic Heyting algebra $\mathbb{A} = (\mathbb{L}, (\Diamond_i)_{i\in\mathsf{Ag}}, (\Box_i)_{i\in\mathsf{Ag}})$ and any i-premeasure μ_i on $\mathbb{A}$, let

$$\begin{aligned}(\mu_i)_+ \ &: \ \mathcal{J}(\mathbb{A}) \to \mathbb{R}^+ \\ &b \mapsto (\mu_i)_+(b) = \left(\mu_i(b) - \mu_i\left(\bigvee_{c<b} c\right)\right)\end{aligned} \tag{5.7.10}$$

It follows from the monotonicity of μ_i that $(\mu_i)_+$ is well-defined.

Lemma 209. *Let $\mathbb{A}$ be an epistemic Heyting algebra equipped with an i-premeasure μ_i. Let the map $\eta : \mathbb{A} \to (\mathbb{A}_+)^+$ be the natural isomorphism. Then, $((\mu_i)_+)^+(\eta(a)) = \mu_i(a)$ for every $a \in \mathbb{A}$.*

Proof. Notice that, by definition,

$$\begin{aligned}((\mu_i)_+)^+ \circ \eta \ &: \ \mathsf{Min}_i(\mathbb{A})\!\downarrow \to \mathbb{R}^+ \\ &b \mapsto \sum_{x\in b}(\mu_i)_+(x) = \sum_{x\in b}\left(\mu_i(x) - \mu_i\left(\bigvee_{c<x} c\right)\right)\end{aligned}$$

Since $\mathbb{A}$ is a finite poset, we can define the *height* of its elements. The only element of height 0 is $\bot$ (i.e. $\mathtt{h}(\bot) = 0$). For $a \neq \bot$, we define $\mathtt{h}(a) := \max\{\mathtt{h}(b) \mid b < a\} + 1$. The proof will proceed by induction on the height of the elements of $\mathbb{A}$ below the i-minimal elements.

As to the base case, it is immediate to see that $((\mu_i)_+)^+(\eta(\bot)) = ((\mu_i)_+)^+(\varnothing) = \mu_i(\bot) = 0$.

As to the induction step, assume that $\mu_i(a) = ((\mu_i)_+)^+(\eta(a))$ for all $a \in \mathsf{Min}_i(\mathbb{A})$ such that $\mathtt{h}(a) \leq n$. Now let b be such that $\mathtt{h}(b) = n+1$. If b is a join prime element of $\mathbb{A}$, then $\eta(b) = b\!\downarrow$ and by definition $\left(\bigvee_{c<b} c\right) < b$. This implies that $\mathtt{h}\left(\bigvee_{c<b} c\right) < \mathtt{h}(b)$. Hence, by induction hypothesis,

$$\mu_i\left(\bigvee_{c<b} c\right) = ((\mu_i)_+)^+\left(\eta\left(\bigvee_{c<b} c\right)\right) = ((\mu_i)_+)^+\left(b\!\downarrow \setminus \{b\}\right).$$

Therefore,

$$\begin{aligned}((\mu_i)_+)^+(b\!\downarrow) &= \sum_{x\in b\downarrow}((\mu_i)_+)(x) \\ &= ((\mu_i)_+)(b) + \sum_{x\in b\downarrow\setminus\{b\}}((\mu_i)_+)(x) \\ &= ((\mu_i)_+)(b) + ((\mu_i)_+)^+(b\downarrow\setminus\{b\}) \\ &= \mu_i(b) - \mu_i\left(\bigvee_{c<b} c\right) + ((\mu_i)_+)^+\left(\eta\left(\bigvee_{c<b} c\right)\right) \\ &= \mu_i(b). \qquad \text{(by induction hypothesis)}\end{aligned}$$

If b is not a join prime element then it can be written as the union of elements strictly below it. Since both μ_i and $((\mu_i)_+)^+$ satisfy condition 3 of Definition 139 and have the same values on elements of height strictly smaller than $n+1$, it follows that $\mu_i(b) = ((\mu_i)_+)^+(\eta(b))$. □

Corollary 210. *Let $\mathbb{A}$ be an epistemic Heyting algebra equiped with an i-measure $\mu_i : \mathsf{Min}_i(\mathbb{A}){\downarrow} \to \mathbb{R}^+$. Then the map*

$$(\mu_i)_+ \ : \ \mathcal{J}(\mathbb{A}) \to \,]0,1] \qquad (5.7.11)$$
$$a \mapsto \mu_i(a) - \mu_i\left(\bigvee_{b<a} b\right)$$

is an i-probability distribution over $\mathbb{A}_+$.

Proof. First we need to show that the map above is well-defined. Indeed, $(\mu_i)_+(b)$ is strictly positive for any $b \in \mathcal{J}(\mathbb{A})$, because μ_i is strictly monotone (see Definition 139 item 6); $(\mu_i)_+(b) \leq 1$, because there exists an i-minimal element a such that $b \leq a$ (see Corollary 205) and because $\mu_i(a) = 1$ (see Definition 139 item 5). Lemma 209 implies that $1 = \mu_i(a) = ((\mu_i)_+)^+(a) = \sum_{x \in X}(\mu_i)_+(x)$ for every i-minimal element a, which shows that $(\mu_i)_+$ is an i-probability distribution over $\mathbb{A}_+$, as required. □

Lemma 211. *Let $\mathbb{F}$ be an epistemic intuitionistic Kripke frame equipped with a probability distribution P_i. Let the map $\varepsilon : \mathbb{F} \to (\mathbb{F}^+)_+$ be the natural isomorphism. Then $((P_i)^+)_+(\varepsilon(w)) = P_i(w)$ for every $w \in \mathbb{F}$.*

Proof. For every join prime element $w{\downarrow}$ of $\mathbb{F}^+$, we have that $v \in w{\downarrow}$ if and only if $v \leq w$. Thus we obtain:

$$((P_i)^+)_+(\varepsilon(w)) = (P_i)^+(w{\downarrow}) - (P_i)^+\left(\bigvee_{b<w\downarrow} b\right) = \sum_{v \leq w} P_i(v) - \sum_{v<w} P_i(v) = P_i(w).$$

□

5.7.3 Dualizing the product updates of APE-structures

In this section we introduce the generalization of the construction of the intermediate structure presented in Section 5.2.2, and show that it dualizes the intermediate construction on algebras presented in Section 5.4.4.

Definition 212 (Intermediate intuitionistic structure)**.** For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle S, \leq, (R_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$ and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$ (see Definition 174), let the intermediate intuitionistic structure of $\mathbb{M}$ and $\mathcal{E}$ be the tuple:

$$\coprod_{\mathcal{E}}\mathbb{M} := \langle \coprod_{|E|} S, \leq^{\coprod}, (R_i^{\coprod})_{i\in\mathsf{Ag}}, [\![\cdot]\!]_{\coprod}\rangle$$

where

- $\coprod_{|E|} S \cong S \times E$ is the $|E|$-fold coproduct of S,
- the order relation $\leq^{\coprod}$ on $\coprod_{|E|} S$ is defined as follows:

$$(s, e) \leq_i^{\coprod} (s', e') \quad \text{iff} \quad s \leq_i s' \text{ and } e = e',$$

- each binary relation $R_i^{\coprod}$ on $\coprod_{|E|} S$ is defined as follows:

$$(s, e) R_i^{\coprod} (s', e') \quad \text{iff} \quad s R_i s' \text{ and } e \sim_i e',$$

- and the valuation $[\![\cdot]\!]_{\coprod} : \mathsf{AtProp} \to \mathcal{P}S$ is defined by

$$[\![p]\!]_{\coprod} := \{(s, e) \mid s \in [\![p]\!]_{\mathbb{M}}\} = [\![p]\!]_{\mathbb{M}} \times E$$

  for every $p \in \mathsf{AtProp}$.

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For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$, let

$$\coprod_{\mathcal{E}} \mathbb{F} := \langle \coprod_{|E|} S, \leq^{\coprod}, (R_i^{\coprod})_{i \in \mathsf{Ag}} \rangle.$$

Lemma 213. *Let $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$ be an epistemic intuitionistic Kripke model. Then $(\coprod_{\mathcal{E}} \mathbb{F}, [\![\cdot]\!]_{\coprod})$ is also an epistemic intuitionistic Kripke model. Moreover, $(\coprod_{\mathcal{E}} \mathbb{F})^+ = \prod_{\mathbb{E}_{\mathcal{E}}} (\mathbb{F}^+)$.*

Proof. Given [34, Fact 23], Lemma 200 and Lemma 202, it remains to show that each $R_i^{\coprod}$ is upwards closed. This follows from each R_i being upwards closed and the definition of $\leq^{\coprod}$. □

Definition 214. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$, any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$, any i-probability distribution P_i on $\mathbb{F}$ (see Definition 206), and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$, let us define the function $P_i^{\coprod} : S \times E \to \mathbb{R}^+$ by recursion on the order $\leq^{\coprod}$ as follows:

$$P_i^{\coprod}(w, e) = \left(\sum_{\varphi \in \Phi} P_i(e) \cdot P_i^{\varphi}(w) \cdot \mathsf{pre}(e \mid \varphi) \right) - \sum_{v < w} P_i^{\coprod}(v, e) \tag{5.7.12}$$

where

$$P_i^{\varphi}(w) = \sum_{v \leq w} \left\{ P_i(v) \;\middle|\; \mathbb{M}, v \models \varphi \text{ and } \mathbb{M}, v \not\models \psi \text{ for all } \psi \in \mathrm{mb}([\![\varphi]\!]) \right\}. \tag{5.7.13}$$

Recall that $\mathrm{mb}(a)$ denotes the multiset of the $\prec$-maximal elements of $\mathbf{\Phi}$ $\prec$-below a (see Definition 152).

Lemma 215. *For every $\mathbb{M}$, P_i and $\mathcal{E}$ as in Definition 214 and for every $w \in S$,*

$$P_i^{\varphi}(w) = ((P_i)^+)^{[\![\varphi]\!]}(w{\downarrow}). \tag{5.7.14}$$

Proof.

$$\begin{aligned}
P_i^{\varphi}(w) &= \sum_{v \leq w} \left\{ P_i(v) \;\middle|\; \mathbb{M}, v \models \varphi \text{ and } \mathbb{M}, v \not\models \psi \text{ for all } \psi \in \mathrm{mb}([\![\varphi]\!]) \right\} \\
&= \sum_{v \leq w} \left\{ P_i(v) \;\middle|\; M, v \models \varphi \right\} - \sum_{v \leq w} \left\{ P_i(v) \;\middle|\; M, v \models \bigvee_{[\![\psi]\!] \in \mathrm{mb}([\![\varphi]\!])} \psi \right\} \\
&= (P_i)^+(w{\downarrow} \wedge [\![\varphi]\!]) - (P_i)^+(w{\downarrow} \wedge \bigvee_{[\![\psi]\!] \in \mathrm{mb}([\![\varphi]\!])} [\![\psi]\!]) \\
& \qquad \text{(see Lemma 207 and equation (5.7.9))} \\
&= ((P_i)^+)^{[\![\varphi]\!]}(w{\downarrow}). \qquad \text{(see Definition 152 and equation (5.4.4))}
\end{aligned}$$

□

Lemma 216. *For every $\mathbb{M}$, P_i and $\mathcal{E}$ as in Definition 214,*

$$(P_i^{\coprod})^+ = ((P_i)^+)'.$$

Proof. Recall that

$$\begin{aligned}
((P_i)^+)' : \mathsf{Min}_i(\prod_{\mathbb{E}} \mathbb{A}){\downarrow} &\to \mathbb{R}^+ \qquad \text{(see Definition 155)} \\
f &\mapsto \sum_{e \in E} \sum_{a \in \Phi} P_i(e) \cdot \mu_i^a(f(e)) \cdot \overline{\mathsf{pre}}(e \mid a).
\end{aligned}$$

By Lemma 209 and Lemma 211, it is enough to show that $P_i^{\coprod} = (((P_i)^+)')_+$. We show this by induction on the order $\leq^{\coprod}$.

Notice that the element $(w, e){\downarrow}$ corresponds to the map $g_{(w,e)} : E \to S$ such that $g_{(w,e)}(e) = w{\downarrow}$ and $g_{(w,e)}(e') = \varnothing$ for every $e' \neq e$. Hence, we have:

$$\begin{aligned}
((P_i)^+)'(g_{(w,e)}) &= \sum_{\varphi \in \Phi} P_i(e) \cdot (P_i^+)^{[\![\varphi]\!]}(w{\downarrow}) \cdot \overline{\mathsf{pre}}(e \mid [\![\varphi]\!]) \\
&= \sum_{\varphi \in \Phi} P_i(e) \cdot P_i^{\varphi}(w) \cdot \mathsf{pre}(e \mid \varphi) \qquad \text{(Lemma 215 and (5.4.8))}
\end{aligned}$$

Notice that

$$(((P_i)^+)')_+((w, e)) = ((P_i)^+)'(g_{(w,e)}) - ((P_i)^+)'(f) \qquad \text{(see Definition 208)}$$

with $f(e) = w{\downarrow} \setminus \{w\}$ and $f(e') = \varnothing$ for $e' \neq e$.

By the induction hypothesis, we have

$$((P_i)^+)'(f) = (P_i^{\coprod})^+(f) = \sum_{v<w} P_i^{\coprod}(v,e).$$

Hence, we get

$$\begin{aligned}(((P_i)^+)')_+((w,e)) &= ((P_i)^+)'(g_{(w,e)}) - \sum_{v<w} P_i^{\coprod}(v,e)\\ &= \sum_{\varphi\in\Phi} P_i(e)\cdot P_i^{\varphi}(w)\cdot \mathsf{pre}(e\mid\varphi) - \sum_{v<w} P_i^{\coprod}(v,e)\\ &= P_i^{\coprod}((w,e)). \qquad \text{(see Definition 214)}\end{aligned}$$

□

5.7.4 Dualizing the updated APE structures

In the present section we introduce the generalization of the construction of the update model presented in Section 5.2.2 and show that it dualizes the construction of the updated APE structure presented in Section 5.4.5.

Definition 217. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i\in\mathsf{Ag}}\rangle$, any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!]\rangle$ and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$, let

$$\begin{aligned} pre : E &\to \mathcal{L}\\ e &\mapsto \bigvee \{\phi\in\Phi \mid \mathsf{pre}(e\mid\phi)\neq 0\}\,. \end{aligned}$$

Definition 218 (Updated intuitionistic structure)**.** For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle S, \leq, (R_i)_{i\in\mathsf{Ag}}, [\![\cdot]\!]\rangle$ and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i\in\mathsf{Ag}}, (P_i)_{i\in\mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$ (see Definition 206), let the updated intuitionistic structure of $\mathbb{M}$ and $\mathcal{E}$ be the tuple:

$$\mathbb{M}^{\mathcal{E}} := \langle S^{\mathcal{E}}, \leq^{\mathcal{E}}, (R_i^{\mathcal{E}})_{i\in\mathsf{Ag}}, [\![\cdot]\!]^{\mathcal{E}}\rangle$$

where

- $S^{\mathcal{E}} = \{(w,e)\in \coprod_{|E|} S \mid \mathbb{M}, w \models pre(e)\}$,
- $\leq^{\mathcal{E}} = \leq^{\coprod} \cap (S^{\mathcal{E}}\times S^{\mathcal{E}})$,
- $R_i^{\mathcal{E}} = R_i^{\coprod} \cap (S^{\mathcal{E}}\times S^{\mathcal{E}})$ for each $i\in\mathsf{Ag}$,
- $[\![\cdot]\!]_{\mathcal{E}}$: AtProp $\to \mathcal{P}S$ is defined by

  $$[\![p]\!]^{\mathcal{E}} := \{(w,e)\in S^{\mathcal{E}} \mid \mathbb{M}, w\models \mathbf{sub}(e)\}$$

  for every $p\in$ AtProp.

For any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$, let

$$\mathbb{F}^{\mathcal{E}} := \langle S^{\mathcal{E}}, \leq^{\mathcal{E}}, (R_i^{\mathcal{E}})_{i \in \mathsf{Ag}} \rangle.$$

Lemma 219. *If $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$ is an epistemic intuitionistic Kripke model, then so is $\mathbb{M}_{\mathcal{E}}$. Moreover, $(F^{\mathcal{E}})^+ = (F^+)^{\mathbb{E}^{\mathcal{E}}}$.*

Proof. It follows from [34, Definition 22,Fact 23] and Lemma 213. □

Definition 220. For any epistemic intuitionistic Kripke frame $\mathbb{F} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}} \rangle$, any epistemic intuitionistic Kripke model $\mathbb{M} = \langle \mathbb{F}, [\![\cdot]\!] \rangle$, any i-probability distribution P_i on $\mathbb{F}$ (see Definition 206), and any intuitionistic probabilistic event structure $\mathcal{E} = (E, (\sim_i)_{i \in \mathsf{Ag}}, (P_i)_{i \in \mathsf{Ag}}, \Phi, \mathsf{pre}, \mathbf{sub})$ over $\mathcal{L}$, the updated i-probability distribution $P_i^{\mathcal{E}} : S^{\mathcal{E}} \to]0, 1]$ is defined as follows:

$$P_i^{\mathcal{E}}(w, e) = \frac{P_i^{\coprod}(w, e)}{\sum\{P_i^{\coprod}(w', e') \mid (w', e') R_i^{\mathcal{E}} (w, e)\}} \tag{5.7.15}$$

where $P_i^{\coprod}$ is as per Definition 214.

Lemma 221. *For every $\mathbb{M}$, P_i and $\mathcal{E}$ as in Definition 220,*

$$(P_i^{\mathcal{E}})^+ = ((P_i)^+)^{\mathbb{E}_{\mathcal{E}}}.$$

Proof. By Corollary 204 and Lemma 213 the i-minimal elements of $(\mathbb{M}^{\mathcal{E}})^+$ are the equivalence cells of R_i. Now let $g \in (\mathbb{M}^{\mathcal{E}})^+$, and f the i-minimal element above g and let $(w, e) \in g$. By Lemma 216 $\sum\{P_i^{\coprod}(w', e') \mid (w', e') R_i^{\mathcal{E}} (w, e)\} = (P_i^{\coprod})^+(f)$ and $\sum_{(w', e') \in g} P_i^{\coprod}(w', e') = (P_i^{\coprod})^+(g)$. Therefore:

$$((P_i)^+)^{\mathbb{E}^{\mathcal{E}}}(g) = \frac{(P_i^{\coprod})^+(g)}{(P_i^{\coprod})^+(f)} \tag{5.7.16}$$

$$= \frac{\sum_{(w', e') \in g} P_i^{\coprod}(w', e')}{\sum\{P_i^{\coprod}(w', e') \mid (w', e') R_i^{\mathcal{E}} (w, e)\}} \tag{5.7.17}$$

$$= \sum_{(w', e') \in g} \frac{P_i^{\coprod}(w', e')}{\sum\{P_i^{\coprod}(w', e') \mid (w', e') R_i^{\mathcal{E}} (w, e)\}} \tag{5.7.18}$$

$$= \sum_{(w', e') \in g} P_i^{\mathcal{E}}(w, e) \tag{5.7.19}$$

$$= (P_i^{\mathcal{E}})^+(g). \tag{5.7.20}$$

□

5

5.7.5 Relational semantics for IPDEL

Definition 222. An IPDEL-model is a structure $\mathbb{N} = \langle \mathbb{M}, (P_i)_{i \in \mathsf{Ag}} \rangle$ such that $\mathbb{M} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}}, [\![\cdot]\!] \rangle$ is an epistemic intuitionistc Kripke model, and P_i is a probability distribution over S for every $i \in$ Ag. For every IPDEL-model $\mathbb{N}$ and every event structure $\mathcal{E}$, we let $\mathbb{N}^{\mathcal{E}} = \langle \mathbb{M}^{\mathcal{E}}, (P_i^{\mathcal{E}})_{i \in \mathsf{Ag}} \rangle$ (cf. Definitions 218 and 220).

It can be verified straightforwardly that for every IPDEL-model $\mathbb{N}$ and every event structure $\mathcal{E}$, the structure $\mathbb{N}^{\mathcal{E}}$ is an IPDEL-model.

Definition 223 (Semantics of IPDEL). For every IPDEL-model $\mathbb{N} = \langle \mathbb{M}, (P_i)_{i \in \mathsf{Ag}} \rangle$ where $\mathbb{M} = \langle S, \leq, (R_i)_{i \in \mathsf{Ag}}, [\![\cdot]\!] \rangle$ the IPDEL-formulas are interpreted on $\mathbb{N}$ as follows:

$$\begin{array}{rll}
\mathbb{N}, s \models \bot & & \text{never} \\
\mathbb{N}, s \models p & \text{iff} & s \in [\![p]\!] \\
\mathbb{N}, s \models \phi \wedge \psi & \text{iff} & \mathbb{N}, s \models \phi \quad \text{and} \quad \mathbb{N}, s \models \psi \\
\mathbb{N}, s \models \phi \vee \psi & \text{iff} & \mathbb{N}, s \models \phi \quad \text{or} \quad \mathbb{N}, s \models \psi \\
\mathbb{N}, s \models \phi \to \psi & \text{iff} & \mathbb{N}, s' \models \phi \quad \text{implies} \quad \mathbb{N}, s' \models \psi \text{ for every } s' \leq s \\
\mathbb{N}, s \models \Diamond_i \phi & \text{iff} & \text{there exists } s' R_i s \text{ such that } \mathbb{N}, s' \models \phi \\
\mathbb{N}, s \models \Box_i \phi & \text{iff} & \mathbb{N}, s' \models \phi \quad \text{for all } s' (\geq \circ R_i) s \\
\mathbb{N}, s \models \langle \mathcal{E}, e \rangle \phi & \text{iff} & \mathbb{N}, s \models pre(e) \quad \text{and} \quad \mathbb{N}^{\mathcal{E}}, (s, e) \models \phi \\
\mathbb{N}, s \models [\mathcal{E}, e] \phi & \text{iff} & \mathbb{N}, s \models pre(e) \quad \text{implies} \quad \mathbb{N}^{\mathcal{E}}, (s, e) \models \phi \\
\mathbb{N}, s \models \left(\sum_{k=1}^{n} \alpha_k \mathring{u} \mu_i(\varphi) \right) \geq \beta & \text{iff} & \sum_{k=1}^{n} \alpha_k \mathring{u} (P_i)^{+} ([\![\varphi]\!] \cap R_i[s]) \geq \beta.
\end{array}$$

Recalling that in epistemic intuitionistic Kripke frames, and hence on IPDEL-models, the relations R_i are both upwards and downwards closed, this implies that the seventh clause in the definition above can be simplified as follows:

$$\mathbb{N}, s \models \Box_i \phi \quad \text{iff} \quad \mathbb{N}, s' \models \phi \quad \text{for all } s' R_i s.$$

5.8 Case study: Decision-making under uncertainty

In the present section we illustrate the relational semantic update process described in Section 5.7 by means of a case study that involves the assessment of the likelihood of a socially constructed event (a bankruptcy), taking place at some point in the future.

The focal feature of the case study is that this assessment depends to a greater extent on the actions, beliefs and expectations of the agents than on factual information.

In what follows, we first present the case study informally, and then we introduce a simplified formalization of the problem using probabilistic epistemic intuitionistic Kripke models and probabilistic intuitionistic epistemic event structures.

5.8.1 Informal presentation

Around 1950, there was a small businessman w in Amsterdam whose main business was to sell the products of foreign textile manufacturers to Dutch clothing firms. Like most small businessmen in Amsterdam at the time, he banked with the Amsterdamsche Bank (which later became the present ABN AMRO).

One day, w received an invitation to lunch with one of the directors of that bank. This invitation puzzled him a great deal, because he did not know this director personally, and a small businessman like him usually only dealt with bank employees at much lower levels. However, he accepted the invitation and showed up for the lunch at the top floor of the bank's headquarters, in the city centre.

During the copious lunch, the bank director talked about all kinds of general subjects and asked w's opinion about the economic climate in Amsterdam. Rather than being flattered, w found it hard to imagine he was invited to provide opinions about matters the bank knew better than he. When the dessert was served, the banker mentioned aside some other matter the name of a certain Amsterdam firm f, which was an important client of w. This firm, the bank director said, was doing very well under the present solid leadership.

The small businessman realised that this must have been the point of the whole lunch. And if this large bank went to so much effort to increase the confidence of one small businessman in this firm, it must have been very important to the bank that w believed that f was doing well.

The small businessman said he wanted to wash his hands, although coffee still needed to be served, but instead of walking to the bathroom he ran down the stairs and on the street to find a telephone booth and call to the office to stop all deliveries to f and also claim back any supplies that had already been delivered.

Two weeks later, f went bankrupt and it turned out that the bank not only was its major creditor but also had preferential right to sell off any stocks in the possession of f to pay back the debt to the bank before other creditors would be satisfied.

5.8.2 Analysis of the situation

Let $\mathtt{B}_f$ be the following proposition:

'Firm f will bankrupt within a month.'

Notice that, while being two-valued, intuitionistic logic allows for $\mathtt{B}_f$ to be either true, or false, or undecided in a model, and the availability of the third option seems to adequately reflect this real-life situation. Indeed, there is a strict judicial procedure which establishes the truth of $\mathtt{B}_f$, and when this procedure is not (yet) in place it seems reasonable to not assign it a truth value.

Accordingly, the sum of the probability attributed to $\mathtt{B}_f$ by w and the probability attributed to $\neg\mathtt{B}_f$ by w does not need to be 1.

For simplicity we regard everything which happened from the invitation to the banker's utterance about firm f as one single event. We also propose that the uncertainty of w concerns how to interpret this event, and very much simplifying this story, the two mutually inconsistent interpretations of this event are

e_1:*'The banker is trying to manipulate my opinions.'*

e_2:*'The banker only wants to exchange information.'*

The uncertainty of w about how to interpret the event is encoded in the shape of the event structure, which consists of two states, corresponding to e_1 and e_2 above respectively, to each of which w assigns his (subjective) probability.

For the sake of illustrating how the substitution map works and to simplify the subsequent treatment we also include the following atomic proposition $\mathtt{M}$ in our language, the intended meaning of which is:

'The banker is manipulative.'

5.8.3 Formalization: initial model and event structure

Let the set of atomic propositions be AtProp $:= \{\mathtt{B}_f, \mathtt{M}\}$ as discussed above.

Initial model. In the formalization discussed below, we only consider the viewpoint of agent w; hence, in the model and the event structure we specify only the subjective probabilities of agent w. The initial model is

$$\mathbb{M} := \langle S, \leq, \sim_w, P_w, [\![\cdot]\!] \rangle$$

with:

- $S := \{s_0, s_1, s_2\}$;
- $\leq := \{(s, s) \mid s \in S\} \cup \{(s_0, s_1), (s_0, s_2)\}$;
- $\sim_w := S \times S$;
- $P_w : S \to]0, 1]$ with

$$P_w(s_0) := 0.1, \qquad P_w(s_1) := 0.1 \qquad P_w(s_2) := 0.8$$

- $[\![\cdot]\!]$: AtProp $\to \mathcal{P}S$ is such that $[\![\mathtt{B}_f]\!] := \{s_1\}$ and $[\![\mathtt{M}]\!] := \bot$.

This model represents a situation in which w has no additional information about the financial health of firm f. Hence we assume that the probability assigned by w to each state of the model reflects the average risk of bankruptcy of firms in that industry during that period. For w to be willing to do business with f it is not just enough that f does not have a higher probability of bankruptcy than the average firm, but also the probability of being in an uncertain state should be low. The model $\mathbb{M}$ is drawn in Figure 5.2.

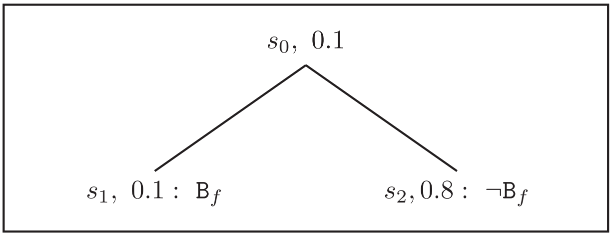

Figure 5.2: Initial model $\mathbb{M}$

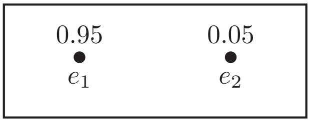

Figure 5.3: Event structure $\mathbb{E}$

Event structure. We consider the following pointed event structure:

$$(\mathbb{E}, e_1) := (E, \sim_w, P_w, \Phi, \mathsf{pre}, \mathbf{sub})$$

where

- $E := \{e_1, e_2\}$;
- $\sim_w := E \times E$,
- $P_w(e_1) = 0.95$ and $P_w(e_2) = 0.05$;
- $\Phi = \{\top, \mathtt{B}_f, \neg\mathtt{B}_f\}$,
- pre : $E \times \Phi \to [0, 1]$ is given in Figure 5.4.
- the definition of the map $\mathbf{sub} : E \times \{\mathtt{M}\} \to \mathcal{L}$ is given in Figure 5.5,

where e_1 and e_2 correspond to the two interpretations of the event discussed in the previous section. The event structure $\mathbb{E}$ is partially represented in Figure 5.3.

By stipulating that $P_w(e_1) = 0.95$ and $P_w(e_2) = 0.05$, we indicate that w believes that it is far more likely that the banker is trying to manipulate his opinion on f.

	e_1	e_2
$\top$	0.8	0.2
$\mathtt{B}_f$	0.99	0.01
$\neg\mathtt{B}_f$	0.05	0.95

Figure 5.4: The map pre

	e_1	e_2
$\mathtt{M}$	$\top$	$\bot$

Figure 5.5: The map **sub**

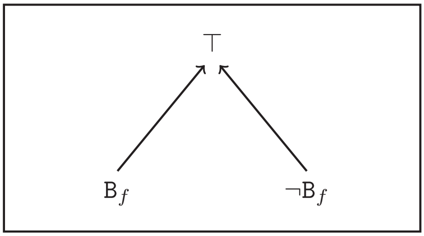

Figure 5.6: The partial order given by $(\Phi, \to)$

The map pre provides the objective probability pre$(e \mid \phi)$ of each event $e \in E$ happening when one assumes that the formula $\phi \in \Phi$ holds. Each line of Figure 5.4 gives the probability distribution pre$(\bullet \mid \phi) : E \times [0,1]$ for each $\phi \in \Phi$. The values in Figure 5.4 are based on the following assumptions:

- If we consider the row where $\phi = \top$, which corresponds to the state in which the bankruptcy of f is undetermined, it is reasonable to assume that the probability of e_1, namely the banker trying to manipulate w's opinion on f, is significantly higher than that of e_2.
- If we consider the row where $\phi = \mathtt{B}_f$, which corresponds to the state in which f is going to be bankrupt within a month, it is reasonable to regard e_1 as almost certain.
- If we consider the row where $\phi = \neg\mathtt{B}_f$, which corresponds to the state in which f is financially healthy then it is reasonable to assign a very low probability to the event in which the banker wants to manipulate w's opinion about f, since the banker has nothing to gain from it.

Remark 224. The poset Φ ordered by logical implication is a tree and is drawn in Figure 5.6.

5.8.4 Updated model

In this section, we show how the initial model described in the section above is updated with the event structure. The updated model

$$\mathbb{M}^{(\mathbb{E},e_1)} := \left\langle S', \leq', \sim'_w, P'_w, [\![\cdot]\!]' \right\rangle$$

is defined as follows:

- $S' := S \times E$;
- $(s,e) \leq' (s',e')$ iff $s \leq s'$ and $e = e'$ for all $(s,e),(s',e') \in S'$;
- $(s,e) \sim'_w (s',e')$ iff $s \sim_w s'$ and $e \sim_w e'$ for all $(s,e),(s',e') \in S'$;
- the map P'_w is shown in Figure 5.7, where the actual values are rounded off.

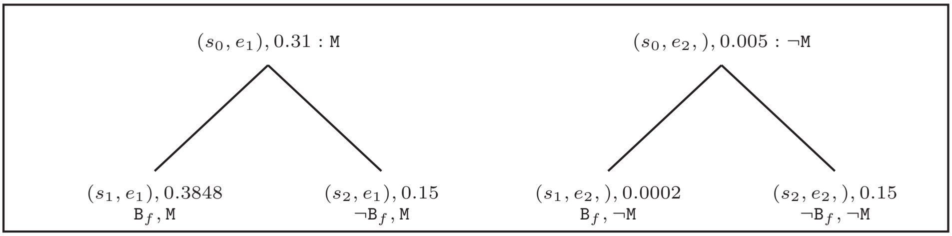

Figure 5.7: Updated model $\mathbb{M}^{\mathbb{E}}$

- the map $[\![\cdot]\!]'$: AtProp $\to \mathcal{P}S'$ is defined as follows:

$$\begin{aligned}
[\![\mathtt{B}_f]\!]' &:= [\![\mathtt{B}_f]\!] \times E;\\
[\![\mathtt{M}]\!]' &:= ([\![\mathbf{sub}(e_1, \mathtt{M})]\!] \times \{e_1\}) \cup ([\![\mathbf{sub}(e_2, \mathtt{M})]\!] \times \{e_2\})\\
&= \{(s_0, e_1), (s_1, e_1), (s_2, e_1)\}.
\end{aligned}$$

The updated model $\mathbb{M}^{(\mathbb{E}, \mathsf{e}_1)}$ is drawn in Figure 5.7.

As expected, the fact that w assigns a much greater probability to e_1 than e_2 implies that the probabilistic weight of the model above is concentrated among the three leftmost states. Of these three states, the weight is shared almost equally between the two in which $\mathtt{B}_f$ is either true or undecided, which reverses the subjective probability assigned in the initial model. This reversal captures w's decision to abruptly stop all deliveries to f.

5.8.5 Syntactic inference of a property of the afternoon event

In the present section we will use the Hilbert style presentation of IPDEL to derive the following formula. The formula (5.8.1) gives the threshold of reasonable optimism which enables w to revise his subjective probability about $\mathtt{B}_f$ after the afternoon event (E, e_1) takes place. Specifically, the probability w assigns to $\mathtt{B}_f$ should not be less than 19.8 times that he assigns to $\neg\mathtt{B}_f$ in order for the event (E, e_1) as specified in the sections above to be enough for w to revert his judgment about $\mathtt{B}_f$.

Proposition 225.

$$(19.8\mu_w(\mathtt{B}_f) > \mu_w(\neg\mathtt{B}_f)) \leftrightarrow [E, e_1](\mu_w(\mathtt{M} \wedge \mathtt{B}_f) > \mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f)), \tag{5.8.1}$$

where $\alpha\mu_i(\varphi) > \beta\mu_i(\psi)$ *is shorthand for* $(\beta\mu_i(\psi) \geq \alpha\mu_i(\varphi)) \to \bot$.

Proof. In order to show the equivalence (5.8.1), we will use the IPDEL axioms to equivalently rewrite its right-hand side into its left-hand side.

	$[E, e_1](\mu_w(\mathtt{M} \wedge \mathtt{B}_f) > \mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f))$	
iff	$[E, e_1]\,((\mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \geq \mu_w(\mathtt{M} \wedge \mathtt{B}_f)) \to \bot)$	(notation for $>$)
iff	$\langle E, e_1\rangle(\mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \geq \mu_w(\mathtt{M} \wedge \mathtt{B}_f)) \to \langle E, e_1\rangle\bot$	(I11 in Table 5.3)
iff	$\langle E, e_1\rangle(\mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \geq \mu_w(\mathtt{M} \wedge \mathtt{B}_f)) \to \bot$	(I6 in Table 5.3)

In what follows we focus on equivalently rewriting the antecedent of the implication above.

$$\begin{array}{lll} & \langle E, e_1\rangle(\mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \geq \mu_w(\mathtt{M} \wedge \mathtt{B}_f)) & \\ \text{iff} & \sum_{\substack{e' \in E \\ \phi \in \Phi}} P_w(e') \cdot \mathsf{pre}(e' \mid \phi) \cdot \mu_w^{\phi}(\langle E, e'\rangle(\neg\mathtt{M} \wedge \neg\mathtt{B}_f)) & \\ & \geq \sum_{\substack{e' \in E \\ \phi \in \Phi}} P_w(e') \cdot \mathsf{pre}(e' \mid \phi) \cdot \mu_w^{\phi}(\langle E, e'\rangle(\mathtt{M} \wedge \mathtt{B}_f)) & \text{(I18 in Table 5.3)} \\ \text{iff} & P_w(e_2) \cdot \mathsf{pre}(e_2 \mid \neg\mathtt{B}_f) \cdot \mu_w(\neg\mathtt{B}_f) \geq P_w(e_1) \cdot \mathsf{pre}(e_1 \mid \mathtt{B}_f) \cdot \mu_w(\mathtt{B}_f) & (*) \\ \text{iff} & 0.05 \cdot 0.95 \cdot \mu_w(\neg\mathtt{B}_f) \geq 0.95 \cdot 0.99 \cdot \mu_w(\mathtt{B}_f) & \text{(definition of } (E, e_1)) \\ \text{iff} & 0.05 \cdot \mu_w(\neg\mathtt{B}_f) \geq 0.99 \cdot \mu_w(\mathtt{B}_f) & \text{(by Lemma 188)} \\ \text{iff} & \mu_w(\neg\mathtt{B}_f) \geq 19.8\mu_w(\mathtt{B}_f). & \text{(by Lemma 188)} \end{array}$$

Hence

$$\begin{array}{ll} & \langle E, e_1\rangle(\mu_w(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \geq \mu_w(\mathtt{M} \wedge \mathtt{B}_f)) \to \bot \\ \text{iff} & (\mu_w(\neg\mathtt{B}_f) \geq 19.8\mu_w(\mathtt{B}_f)) \to \bot \\ \text{iff} & 19.8\mu_w(\mathtt{B}_f) > \mu_w(\neg\mathtt{B}_f), \end{array}$$

as required. □

The equivalence marked by $(*)$ is justified by the following lemma:

Lemma 226. *The following propositions are provable in IPDEL.*

1. $\langle E, e_1\rangle(\mathtt{M} \wedge \mathtt{B}_f) \leftrightarrow \mathtt{B}_f$ *and* $\langle E, e_1\rangle(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \leftrightarrow \bot$;
2. $\langle E, e_2\rangle(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) \leftrightarrow \neg\mathtt{B}_f$ *and* $\langle E, e_2\rangle(\mathtt{M} \wedge \mathtt{B}_f) \leftrightarrow \bot$;
3. $\mu_w^{\top}(\mathtt{B}_f) = 0$ *and* $\mu_w^{\top}(\neg\mathtt{B}_f) = 0$;
4. $\mu_w^{\mathtt{B}_f}(\neg\mathtt{B}_f) = 0$ *and* $\mu_w^{\neg\mathtt{B}_f}(\mathtt{B}_f) = 0$;
5. $\mu_w^{\mathtt{B}_f}(\mathtt{B}_f) = \mu_w(\mathtt{B}_f)$ *and* $\mu_w^{\neg\mathtt{B}_f}(\neg\mathtt{B}_f) = \mu_w(\neg\mathtt{B}_f)$.

Proof. 1.

$$\begin{array}{lll} & \langle E, e_1\rangle(\mathtt{M} \wedge \mathtt{B}_f) & \\ \text{iff} & \langle E, e_1\rangle\mathtt{M} \wedge \langle E, e_1\rangle\mathtt{B}_f & \text{(I8 in Table 5.3)} \\ \text{iff} & pre(e_1) \wedge \mathbf{sub}(e_1, \mathtt{M}) \wedge pre(e_1) \wedge \mathbf{sub}(e_1, \mathtt{B}_f) & \text{(I2 in Table 5.3)} \\ \text{iff} & \mathbf{sub}(e_1, \mathtt{M}) \wedge \mathbf{sub}(e_1, \mathtt{B}_f) & (pre(e_1) \text{ is } \top \vee \mathtt{B}_f \vee \neg\mathtt{B}_f) \\ \text{iff} & \top \wedge \mathtt{B}_f & \text{(definition of } \mathbf{sub}) \\ \text{iff} & \mathtt{B}_f & \end{array}$$

and

$$\begin{array}{lll} & \langle E, e_1\rangle(\neg\mathtt{M} \wedge \neg\mathtt{B}_f) & \\ \text{iff} & \langle E, e_1\rangle\neg\mathtt{M} \wedge \langle E, e_1\rangle\neg\mathtt{B}_f & \text{(I8 Table 5.3)} \\ \text{iff} & (pre(e_1) \wedge (\neg\langle E, e_1\rangle\mathtt{M})) \wedge (pre(e_1) \wedge (\neg\langle E, e_1\rangle\mathtt{B}_f)) & \text{(I12, I6 Table 5.3)} \\ \text{iff} & pre(e_1) \wedge \neg(pre(e_1) \wedge \mathbf{sub}(e_1, \mathtt{M})) \wedge pre(e_1) \wedge \neg(pre(e_1) \wedge \mathbf{sub}(e_1, \mathtt{B}_f)) & \text{(I2 Table 5.3)} \\ \text{iff} & \neg\mathbf{sub}(e_1, \mathtt{M}) \wedge \neg\mathbf{sub}(e_1, \mathtt{B}_f) & (pre(e_1) \text{ is } \top) \\ \text{iff} & \neg\top \wedge \neg\mathtt{B}_f & \text{(def. of } \mathbf{sub}) \\ \text{iff} & \bot & (\neg\top \leftrightarrow \bot) \end{array}$$

2. The proof is similar to that of item 1.

3. Notice that $\mu_w^{\top}(\mathtt{B}_f)$ is shorthand for $\mu_w(\top \wedge \mathtt{B}_f) - (\mu_w(\mathtt{B}_f \wedge \mathtt{B}_f) + \mu_w(\neg\mathtt{B}_f \wedge \mathtt{B}_f))$ (cf.). Therefore:

$$\begin{array}{ll} & \mu_w^{\top}(\mathtt{B}_f) = 0 \\ \text{iff} & \mu_w(\top \wedge \mathtt{B}_f) - (\mu_w(\mathtt{B}_f \wedge \mathtt{B}_f) + \mu_w(\neg\mathtt{B}_f \wedge \mathtt{B}_f)) = 0 \\ \text{iff} & \mu_w(\top \wedge \mathtt{B}_f) - \mu_w(\mathtt{B}_f \wedge \mathtt{B}_f) = 0 \qquad \text{(P1 Table 5.2, Lemma 188)} \\ \text{iff} & \mu_w(\mathtt{B}_f) - \mu_w(\mathtt{B}_f) = 0 \end{array}$$

the last equality follows by N0 in Table 5.2. The proof of the second inequality is similar.

4. Notice that $\mu_w^{\mathtt{B}_f}(\neg\mathtt{B}_f)$ is shorthand for $\mu_w(\mathtt{B}_f \wedge \neg\mathtt{B}_f)$ and $\mu_w^{\neg\mathtt{B}_f}(\mathtt{B}_f)$ is shorthand for $\mu_w(\neg\mathtt{B}_f \wedge \mathtt{B}_f)$ so the equality follows from Axiom P1 in Table 5.2.

5. Notice that $\mu_w^{\mathtt{B}_f}(\mathtt{B}_f)$ is shorthand for $\mu_w(\mathtt{B}_f \wedge \mathtt{B}_f)$ and $\mu_w^{\neg\mathtt{B}_f}(\neg\mathtt{B}_f)$ is shorthand for $\mu_w(\neg\mathtt{B}_f \wedge \neg\mathtt{B}_f)$. Hence the required equality is straightforwardly true. □

Because, as discussed in Section 5.7, IPDEL is sound and complete with respect to the class of relational models, Proposition 225 implies that every IPDEL model $\mathcal{M}$ which supports the left-hand side of the equivalence (5.8.1) will be updated by the event (E, e_1) to a model that satisfies $\mu_w(\mathtt{M}\wedge\mathtt{B}_f) > \mu_w(\neg\mathtt{M}\wedge\neg\mathtt{B}_f)$. Hence in each such model agent w will update his subjective probabilities concerning $\mathtt{B}_f$ analogously to the model in the example above.

5.9 Conclusion

Present contributions. In this chapter, we have introduced the logic IPDEL, the intuitionistic counterpart of classical PDEL, as an instance of a general methodology, based on the mathematical construction of updates on algebras, which makes it possible to define non-classical counterparts of DEL-type logics on different propositional bases. This methodology makes it possible to also obtain the update construction on relational and topological models via appropriate (extended) dualities, and hence define relational semantics for the defined logics. In this way we have shown that IPDEL, which is sound by construction with respect to the class of algebraic probabilistic epistemic models (cf. Definition 177), is also complete with respect to APE-models and hence also with respect to their dual relational structures. Since these structures are finite by definition, this result immediately implies that IPDEL has the finite model property. The logic IPDEL is intended as a tool to analyze decision-making under uncertainty in situations in which truth is socially constructed and hence decisions are taken in contexts in which the truth value of certain states of affair might be undetermined. To show IPDEL at work, we partially formalize one such situation.

Generalizing APE-structures APE-structures are based on epistemic Heyting algebras (cf. Definition 133), the definition of which requires the image of each diamond operator to have a Boolean algebra structure. Thus, epistemic Heyting algebras are a proper subclass of monadic Heyting algebras. This additional condition guarantees that the i-minimal elements induce a partition on the dual structure of each epistemic Heyting algebra, and hence that axioms such as $\mu(\top) = 1$ or $\mu(\varphi) \geq \alpha \vee \mu(\varphi) < \alpha$ are valid. One natural question that presents itself is whether this condition can be dropped and hence base APE-structures on general monadic Heyting algebras. Addressing this question requires solving issues of technical and conceptual nature. On the technical side, the additional requirement plays a role in the completeness theorem, and specifically makes sure that in the finite lattice that we extract from the Lindenbaum-Tarski algebra, a sublattice can be defined out of the image of each diamond (cf. Lemma 187). This issue would partially be addressed by relaxing the condition that APE-structures be finite (see paragraph below). On the conceptual side, we would need to restructure the definition of probabilistic measure. The axiom $\mu(\varphi) \geq \alpha \vee \mu(\varphi) < \alpha$ is tightly linked to the metatheory of the real numbers and in particular to the validity of trichotomy. Hence in the context of a different metatheory in which trichotomy does not hold such as the constructive metatheory of real numbers, it seems reasonable that this axiom might be dropped. However the condition $\mu(\top) = 1$ expresses the link between probability and the underlying logic. For this reason this axiom should arguably be kept.

Finite to infinite models Another natural question is whether we can drop the condition that APE-structures be finite. A first step would be to investigate the case of APE-structures based on perfect Heyting algebras, i.e. those Heyting algebras which are isomorphic to algebras of upsets or downsets of given posets. Does every probability measure on such a Heyting algebra correspond to a discrete probability distribution on the corresponding dual poset? More generally, possibly infinite APE-structures would dually correspond to relational Esakia spaces endowed with probability distributions. Are there purely algebraic conditions on probability measures guaranteeing that the corresponding probability distribution be discrete?

Proof theory for probabilistic logics. As mentioned in the introduction, the present chapter pertains to a line of research aimed at studying the phenomenon of dynamic (probabilistic epistemic) updates in contexts at odds with classical truth. The language and semantics of the formal settings previously studied (i.e. those of the nonclassical versions of PAL and EAK) have served as a basis for a research program in structural proof theory aimed at developing a uniform methodology for endowing dynamic logics with so-called *analytic calculi* (see [14, 30]). This research program has successfully addressed PAL and DEL [24, 26, 27, 29], and PDL [25], and has been further generalized into the proof-theoretic framework of *multi-type calculi* [24]. This methodology has been successfully deployed to introduce analytic calculi for logics particularly impervious to the standard treatment [12, 28, 31, 32], and is now ready to be applied to the issue of endowing PDEL and its non-classical versions with analytic calculi.

5.10 Soundness of the reduction axioms

In this section we aim at proving the soundness of the reduction axioms as stated in Lemma 184.

5.10.1 Preliminary results

Throughout this section, we let $\mathbb{A}$ denote the complex algebra of a model $\mathcal{M}$ and $\mathcal{E}$ denote an event structure. Recall the definition of the event structure $\mathbb{E}_{\mathcal{E}}$ (cf. Proposition 178). Then we define a map $F : \mathcal{L} \to \prod_{\mathbb{E}_\mathcal{E}} \mathbb{A}$ that associates an element in $\prod_{\mathbb{E}_\mathcal{E}} \mathbb{A}$ to each formula. We want F (Definition 228) to be the map such that

$$[\![\psi]\!]_{\mathcal{M}^{\mathbb{E}_\mathcal{E}}} = [F(\psi)].$$

$[\![\psi]\!]_{\mathcal{M}^{\mathbb{E}_\mathcal{E}}}$ is the evaluation of the formula ψ in the updated algebra $\mathbb{A}^{\mathbb{E}_\mathcal{E}}$ corresponding to the updated model $\mathcal{M}^{\mathbb{E}_\mathcal{E}}$. Hence, $F(\psi)$ is a representative of the equivalence class $[\![\psi]\!]_{\mathcal{M}^{\mathbb{E}_\mathcal{E}}}$ in the product algebra $\mathbb{A}^{\Pi}$.

Since $F(\psi) \in \mathbb{A}^{\Pi}$, $F(\psi)$ is a tuple of elements of the algebra $\mathbb{A}$. To help us in the computation we define the map $f : \mathcal{L} \times E \to \mathcal{L}$ (see definition 227) such that $F(\psi)(e) = [\![f(\psi, e)]\!]_{\mathcal{M}}$. This means that $f(\psi, e)$ is a formula such that its evaluation $[\![f(\psi, e)]\!]_{\mathcal{M}}$ in the algebra $\mathbb{A}$ is equal to the e^{th} coordinate of the tuple $F(\psi)$.

We first prove that the maps F and f have the desired properties in Lemma 229. Then we prove the key lemma 230 that we will use to prove the reduction axioms (see Section 5.10.2).

Definition 227. For every $\psi \in \mathcal{L}$ and $e \in E$ let us define by recursion the formula $f(\psi, e)$:

$$\begin{aligned}
f(p, e) &= \mathbf{sub}(e, p)\\
f(\bot, e) &= \bot\\
f(\top, e) &= \top\\
f(\psi_1 \wedge \psi_2, e) &= f(\psi_1, e) \wedge f(\psi_2, e)\\
f(\psi_1 \vee \psi_2, e) &= f(\psi_1, e) \vee f(\psi_2, e)\\
f(\psi_1 \to \psi_2, e) &= f(\psi_1, e) \to f(\psi_2, e)\\
f(\Diamond_i \psi, e) &= \bigvee_{e' \sim_i e} \Diamond_i (f(\psi, e') \wedge pre(e'))\\
f(\Box_i \psi, e) &= \bigwedge_{e' \sim_i e} \Box_i (pre(e') \to f(\psi, e'))\\
f(\langle \mathcal{E}', e' \rangle \psi, e) &= f(pre(e') \wedge f(\psi, e'), e)\\
f([\mathcal{E}', e']\, \psi, e) &= f(pre(e') \to f(\psi, e'), e)\\
f(\alpha \mu_i(\psi) \geq \beta, e) &= \alpha \sum_{\substack{e' \sim_i e\\ \phi \in \Phi}} \mu_i^{\phi}(f(\psi, e')) \cdot P_i(e') \cdot \mathsf{pre}(e' \mid \phi) +\\
&\quad \sum_{\substack{e' \sim_i e\\ \phi \in \Phi}} -\beta \mu_i^{\phi}(\top) P_i(e') \mathsf{pre}(e' \mid \phi) \geq 0
\end{aligned}$$

Definition 228. Let us define the map $F_{\mathbb{E}_\mathcal{E}} : \mathcal{L} \to \mathbb{A}^{\Pi}$ such that for every $e \in E$, the e^{th} coordinate of $F_{\mathbb{E}_\mathcal{E}}(\psi)$ is equal to $[\![f(\psi, e)]\!]_{\mathcal{M}}$.

For the sake readability, we will omit the subscript when it causes no confusion.

Lemma 229. *For $\mathcal{M}$ and $\mathcal{E}$ as above,*

$$[\![\psi]\!]_{\mathcal{M}^{\mathbb{E}_\mathcal{E}}} = [F(\psi)]$$

where $F(\psi)(e) = [\![f(\psi, e)]\!]_{\mathcal{M}}$.

Proof. By induction on ψ. Trivially true in the base cases and if the main connective belongs to $\{\wedge, \vee, \rightarrow\}$. If $\psi = \Diamond_i \psi'$, then

$$
\begin{aligned}
[\![\Diamond_i \psi']\!]_{\mathcal{M}^{\mathbb{E}\varepsilon}} &= \Diamond^{\mathbb{E}\varepsilon}[\![\psi']\!]_{\mathcal{M}^{\mathbb{E}\varepsilon}} \\
&= \Diamond_i^{\mathbb{E}\varepsilon}[F(\psi')] && \text{(induction hypothesis)} \\
&= [\Diamond_i^{\prod}(F(\psi') \wedge \overline{pre}_{\mathcal{M}})]
\end{aligned}
$$

and

$$
\begin{aligned}
\Diamond_i^{\prod}(F(\psi') \wedge \overline{pre}_{\mathcal{M}})(e) &= \bigvee_{e' \sim_i e} \{\Diamond_i(F(\psi')(e') \wedge \overline{pre}(e'))\} && \text{(definition)} \\
&= \bigvee_{e' \sim_i e} \{\Diamond_i([\![f(\psi', e')]\!]_{\mathcal{M}} \wedge \overline{pre}(e'))\} \\
&= [\![\bigvee_{e' \sim_i e} \Diamond_i(f(\psi', e') \wedge pre(e'))]\!]_{\mathcal{M}} \\
&= [\![f(\Diamond_i \psi', e)]\!]_{\mathcal{M}} && \text{(Definition 227)} \\
&= F(\Diamond_i \psi')(e)
\end{aligned}
$$

If $\psi = \Box_i \psi'$, then

$$
\begin{aligned}
[\![\Box_i \psi']\!]_{\mathcal{M}^{\mathbb{E}\varepsilon}} &= \Box_i^{\mathbb{E}\varepsilon}[\![\psi']\!]_{\mathcal{M}^{\mathbb{E}\varepsilon}} \\
&= \Box_i^{\mathbb{E}\varepsilon}[F(\psi')] && \text{(induction hypothesis)} \\
&= [\Box_i^{\prod}(\overline{pre}_{\mathcal{M}} \rightarrow F(\psi'))]
\end{aligned}
$$

and

$$
\begin{aligned}
\Box_i^{\prod}(\overline{pre}_{\mathcal{M}} \rightarrow F(\psi'))(e) &= \bigwedge_{e' \sim_i e} \{\Box_i(\overline{pre}(e') \rightarrow F(\psi')(e'))\} && \text{(definition)} \\
&= \bigwedge_{e' \sim_i e} \{\Box_i(\overline{pre}(e') \rightarrow [\![f(\psi', e')]\!]_{\mathcal{M}})\} \\
&= [\![\bigwedge_{e' \sim_i e} \Box_i(pre(e') \rightarrow f(\psi', e'))]\!]_{\mathcal{M}} \\
&= [\![f(\Box_i \psi', e)]\!]_{\mathcal{M}} && \text{(Definition 227)} \\
&= F(\Box_i \psi')(e)
\end{aligned}
$$

If $\psi = \alpha\mu_i(\psi') \geq \beta$, then

$$
\begin{aligned}
&[\![\alpha\mu_i(\psi') \geq \beta]\!]_{\mathcal{M}^{\mathbb{E}\mathcal{E}}} \\
&= \bigvee\{[f_{e,a}] \mid \alpha\mu_i^{\mathbb{E}\mathcal{E}}([\![\psi']\!]_{\mathcal{M}^{\mathbb{E}\mathcal{E}}} \wedge [f_{e,a}]) \geq \beta\} \\
&= \bigvee\{[f_{e,a}] \mid \alpha\mu_i^{\mathbb{E}\mathcal{E}}([F(\psi')] \wedge [f_{e,a}]) \geq \beta\} && \text{(induction hypothesis)} \\
&= \bigvee\{[f_{e,a}] \mid \alpha\frac{\mu_i'(F(\psi') \wedge f_{e,a})}{\mu_i'(f_{e,a})} \geq \beta\} && \text{(definition)} \\
&= \left[\bigvee\{f_{e,a} \mid \alpha\frac{\mu_i'(F(\psi') \wedge f_{e,a})}{\mu_i'(f_{e,a})} \geq \beta\}\right] \\
&= \left[\bigvee\{f_{e,a} \mid \alpha\mu_i'(F(\psi') \wedge f_{e,a}) - \beta\mu_i'(f_{e,a}) \geq 0\}\right] \\
&= [\bigvee\{f_{e,a} \mid \alpha\sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^{\phi}(F(\psi')(e') \wedge a) \cdot P_i(e') \cdot \mathsf{pre}(e' \mid \phi) + \\
&\sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta\mu_i^{\phi}(a) \cdot P_i(e') \cdot \mathsf{pre}(e' \mid \phi) \geq 0\}]
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigvee\{f_{e,a} \mid \alpha\sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^{\phi}(F(\psi')(e') \wedge a)P_i(e')\mathsf{pre}(e' \mid \phi) + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta\mu_i^{\phi}(a)P_i(e')\mathsf{pre}(e' \mid \phi) \geq 0\}(d) \\
&= \bigvee\{f_{e,a} \mid \alpha\sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} \mu_i^{\phi}([\![f(\psi', e')]\!]_{\mathcal{M}} \wedge a)P_i(e')\mathsf{pre}(e' \mid \phi) + \sum_{\substack{e' \sim_i e \\ \phi \in \Phi}} -\beta\mu_i^{\phi}(a)P_i(e')\mathsf{pre}(e' \mid \phi) \geq 0\}(d) \\
&= \bigvee\{a \mid \alpha\sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} \mu_i^{\phi}([\![f(\psi', e')]\!]_{\mathcal{M}} \wedge a)P_i(e')\mathsf{pre}(e' \mid \phi) + \sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} -\beta\mu_i^{\phi}(a)P_i(e')\mathsf{pre}(e' \mid \phi) \geq 0\} \\
&= [\![\alpha\sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} \mu_i^{\phi}(f(\psi', e'))P_i(e')\mathsf{pre}(e' \mid \phi) + \sum_{\substack{e' \sim_i d \\ \phi \in \Phi}} -\beta\mu_i^{\phi}(\top)P_i(e')\mathsf{pre}(e' \mid \phi) \geq 0]\!]_{\mathcal{M}} \\
&= [\![f(\alpha\mu_i(\psi') \geq \beta, d)]\!]_{\mathcal{M}} && \text{(Definition 227)} \\
&= F(\alpha\mu_i(\psi') \geq \beta)(d).
\end{aligned}
$$

If $\psi = \langle\mathcal{E}', e'\rangle\psi'$ and $\mathcal{N} = \mathcal{M}^{\mathbb{E}\mathcal{E}}$ then

$$
\begin{aligned}
[\![\langle\mathcal{E}', e'\rangle\psi']\!]_{\mathcal{N}} &= [\![pre(e')]\!]_{\mathcal{N}} \wedge \pi_{e'} \circ i'([\![\psi']\!]_{\mathcal{N}^{\mathbb{E}\mathcal{E}'}}) \\
&= [\![pre(e')]\!]_{\mathcal{N}} \wedge \pi_{e'} \circ i'([F(\psi')]) && \text{(induction hypothesis)} \\
&= [\![pre(e')]\!]_{\mathcal{N}} \wedge \pi_{e'}(F(\psi') \wedge \overline{pre}) \\
&= [\![pre(e')]\!]_{\mathcal{N}} \wedge F(\psi')(e') \wedge [\![pre(e')]\!]_{\mathcal{N}} \\
&= [\![pre(e')]\!]_{\mathcal{N}} \wedge [\![f(\psi', e')]\!]_{\mathcal{N}} \wedge [\![pre(e')]\!]_{\mathcal{N}} \\
&= [\![pre(e')]\!]_{\mathcal{N}} \wedge [\![f(\psi', e')]\!]_{\mathcal{N}} \\
&= [\![pre(e') \wedge f(\psi', e')]\!]_{\mathcal{N}} \\
&= [F(pre(e') \wedge f(\psi', e'))] && \text{(induction hypothesis)}
\end{aligned}
$$

5

and

$$\begin{aligned}F(pre(e') \wedge f(\psi', e'))(e) &= [\![f(pre(e') \wedge f(\psi', e'), e)]\!]_{\mathcal{M}} \\ &= [\![f(\langle \mathbb{E}', e' \rangle \psi', e)]\!]_{\mathcal{M}} && \text{(Definition 227)} \\ &= F(\langle \mathbb{E}', e' \rangle \psi')(e).\end{aligned}$$

Finally if $\psi = [\mathcal{E}', e']\,\psi'$ and $\mathcal{N} = \mathcal{M}^{\mathbb{E}_{\mathcal{E}}}$ then

$$\begin{aligned}[\![[\mathcal{E}', e']\,\psi']\!]_{\mathcal{N}} &= [\![pre(e')]\!]_{\mathcal{N}} \to \pi_{e'} \circ i'([\![\psi']\!]_{\mathcal{N}^{\mathbb{E}_{\mathcal{E}'}}}) \\ &= [\![pre(e')]\!]_{\mathcal{N}} \to \pi_{e'} \circ i'([F(\psi')]) && \text{(induction hypothesis)} \\ &= [\![pre(e')]\!]_{\mathcal{N}} \to \pi_{e'}(F(\psi') \wedge \overline{pre}) \\ &= [\![pre(e')]\!]_{\mathcal{N}} \to F(\psi')(e') \wedge [\![pre(e')]\!]_{\mathcal{N}} \\ &= [\![pre(e')]\!]_{\mathcal{N}} \to [\![f(\psi', e')]\!]_{\mathcal{N}} \wedge [\![pre(e')]\!]_{\mathcal{N}} \\ &= [\![pre(e')]\!]_{\mathcal{N}} \to [\![f(\psi', e')]\!]_{\mathcal{N}} && (a \to (a \wedge b) = a \to b) \\ &= [\![pre(e') \to f(\psi', e')]\!]_{\mathcal{N}} \\ &= [F(pre(e') \to f(\psi', e'))] && \text{(induction hypothesis)}\end{aligned}$$

and

$$\begin{aligned}F(pre(e') \to f(\psi', e'))(e) &= [\![f(pre(e') \to f(\psi', e'), e)]\!]_{\mathcal{M}} \\ &= [\![f([\mathbb{E}', e']\,\psi', e)]\!]_{\mathcal{M}} && \text{(Definition 227)} \\ &= F([\mathbb{E}', e']\,\psi')(e).\end{aligned}$$

□

Lemma 230. *For every $\mathcal{M}$, $\mathcal{E}$, e and ψ,*

$$[\![\langle \mathcal{E}, e \rangle \psi]\!]_{\mathcal{M}} = \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi, e)]\!]_{\mathcal{M}} \quad \textit{and} \quad [\![[\mathcal{E}, e]\,\psi]\!]_{\mathcal{M}} = \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi, e)]\!]_{\mathcal{M}}.$$

Proof. We have

$$\begin{aligned}[\![\langle \mathcal{E}, e \rangle \psi]\!]_{\mathcal{M}} &= [\![pre(e)]\!]_{\mathcal{M}} \wedge \pi_e \circ i'([\![\psi]\!]_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}}) \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge \pi_e \circ i'([F(\psi)]) && \text{(Lemma 229)} \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge \pi_e(F(\psi) \wedge \overline{pre}_{\mathcal{M}}) \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge F(\psi)(e) \wedge \overline{pre}_{\mathcal{M}}(e) \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi, e)]\!]_{\mathcal{M}}\end{aligned}$$

and

$$\begin{aligned}[\![[\mathcal{E}, e]\,\psi]\!]_{\mathcal{M}} &= [\![pre(e)]\!]_{\mathcal{M}} \to \pi_e \circ i'([\![\psi]\!]_{\mathcal{M}^{\mathbb{E}_{\mathcal{E}}}}) \\ &= \overline{pre}_{\mathcal{M}}(e) \to \pi_e \circ i'([F(\psi)]) && \text{(Lemma 229)} \\ &= \overline{pre}_{\mathcal{M}}(e) \to \pi_e(F(\psi) \wedge \overline{pre}_{\mathcal{M}}) \\ &= \overline{pre}_{\mathcal{M}}(e) \to F(\psi)(e) \wedge \overline{pre}_{\mathcal{M}}(e) \\ &= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi, e)]\!]_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e) \\ &= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi, e)]\!]_{\mathcal{M}}. && (a \to (a \wedge b) = a \to b)\end{aligned}$$

□

5.10.2 Proof of soundness

Axiom I1. $[\mathcal{E}, e]\, p \quad \leftrightarrow \quad pre(e) \rightarrow \mathbf{sub}(e, p)$.

$$\begin{aligned} [\![[\mathcal{E}, e]\, p]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow [\![f(p, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)} \\ &= \overline{pre}_{\mathcal{M}}(e) \rightarrow [\![\mathbf{sub}(e, p)]\!]_{\mathcal{M}}. \end{aligned}$$

Axiom I2. $\langle \mathcal{E}, e\rangle p \quad \leftrightarrow \quad pre(e) \wedge \mathbf{sub}(e, p)$.

$$\begin{aligned} [\![\langle \mathcal{E}, e\rangle p]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(p, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)} \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![\mathbf{sub}(e, p)]\!]_{\mathcal{M}}. \end{aligned}$$

Axiom I3. $[\mathcal{E}, e]\, \top \quad \leftrightarrow \quad \top$.

$$\begin{aligned} [\![[\mathcal{E}, e]\, \top]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow [\![f(\top, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)} \\ &= \overline{pre}_{\mathcal{M}}(e) \rightarrow [\![\top]\!]_{\mathcal{M}} \\ &= [\![\top]\!]_{\mathcal{M}}. \end{aligned}$$

Axiom I4. $\langle \mathcal{E}, e\rangle \top \quad \leftrightarrow \quad pre(e)$.

$$\begin{aligned} [\![\langle \mathcal{E}, e\rangle \top]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\top, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)} \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![\top]\!]_{\mathcal{M}} \\ &= \overline{pre}_{\mathcal{M}}(e). \end{aligned}$$

Axiom I5. $[\mathcal{E}, e]\, \bot \quad \leftrightarrow \quad \neg pre(e)$.

$$\begin{aligned} [\![[\mathcal{E}, e]\, \bot]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \rightarrow [\![f(\bot, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)} \\ &= \overline{pre}_{\mathcal{M}}(e) \rightarrow [\![\bot]\!]_{\mathcal{M}} \\ &= [\![\neg pre(e)]\!]_{\mathcal{M}}. \end{aligned}$$

Axiom I6. $\langle \mathcal{E}, e\rangle \bot \quad \leftrightarrow \quad \bot$.

$$\begin{aligned} [\![\langle \mathcal{E}, e\rangle \bot]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\bot, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)} \\ &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![\bot]\!]_{\mathcal{M}} \\ &= \bot. \end{aligned}$$

Axiom I7. $[\mathcal{E}, e]\,(\psi_1 \wedge \psi_2) \quad \leftrightarrow \quad [\mathcal{E}, e]\,\psi_1 \wedge [\mathcal{E}, e]\,\psi_2.$

$$
\begin{aligned}
[\![[\mathcal{E}, e]\,(\psi_1 \wedge \psi_2)]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1 \wedge \psi_2, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1, e) \wedge f(\psi_2, e)]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1, e)]\!]_{\mathcal{M}} \wedge [\![f(\psi_2, e)]\!]_{\mathcal{M}}\\
&= (\overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1, e)]\!]_{\mathcal{M}}) \wedge (\overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
& && (a \to b \wedge c = (a \to b) \wedge (a \to c))\\
&= [\![[\mathcal{E}, e]\,\psi_1]\!]_{\mathcal{M}} \wedge [\![[\mathcal{E}, e]\,\psi_2]\!]_{\mathcal{M}} && \text{(Lemma 230)}
\end{aligned}
$$

Axiom I8. $\langle \mathcal{E}, e\rangle(\psi_1 \wedge \psi_2) \quad \leftrightarrow \quad \langle \mathcal{E}, e\rangle\psi_1 \wedge \langle \mathcal{E}, e\rangle\psi_2.$

$$
\begin{aligned}
[\![\langle \mathcal{E}, e\rangle(\psi_1 \wedge \psi_2)]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1 \wedge \psi_2, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e) \wedge f(\psi_2, e)]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_2, e)]\!]_{\mathcal{M}}\\
&= [\![\langle \mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \wedge [\![\langle \mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}} && \text{(Lemma 230)}
\end{aligned}
$$

Axiom I9. $[\mathcal{E}, e]\,(\psi_1 \vee \psi_2) \quad \leftrightarrow \quad pre(e) \to \langle \mathcal{E}, e\rangle\psi_1 \vee \langle \mathcal{E}, e\rangle\psi_2.$

$$
\begin{aligned}
[\![[\mathcal{E}, e]\,(\psi_1 \vee \psi_2)]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1 \vee \psi_2, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1, e) \vee f(\psi_2, e)]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to \overline{pre}_{\mathcal{M}}(e) \wedge ([\![f(\psi_1, e)]\!]_{\mathcal{M}} \vee [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
& && (a \to b = a \to a \wedge b)\\
&= \overline{pre}_{\mathcal{M}}(e) \to (\overline{pre}_{\mathcal{M}} \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}}) \vee (\overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
& && \text{(distributivity)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![\langle \mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \vee [\![\langle \mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}} && \text{(Lemma 230)}
\end{aligned}
$$

Axiom I10. $\langle \mathcal{E}, e\rangle(\psi_1 \vee \psi_2) \quad \leftrightarrow \quad \langle \mathcal{E}, e\rangle\psi_1 \vee \langle \mathcal{E}, e\rangle\psi_2.$

$$
\begin{aligned}
[\![\langle \mathcal{E}, e\rangle(\psi_1 \vee \psi_2)]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1 \vee \psi_2, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e) \vee f(\psi_2, e)]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= (\overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}}) \vee (\overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
& && \text{(distributivity)}\\
&= [\![\langle \mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \vee [\![\langle \mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}} && \text{(Lemma 230)}
\end{aligned}
$$

Axiom I11. $[\mathcal{E}, e]\,(\psi_1 \to \psi_2) \;\;\leftrightarrow\;\; \langle\mathcal{E}, e\rangle\psi_1 \to \langle\mathcal{E}, e\rangle\psi_2.$

$$
\begin{aligned}
[\![[\mathcal{E}, e]\,(\psi_1 \to \psi_2)]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1 \to \psi_2, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\psi_1, e) \to f(\psi_2, e)]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to ([\![f(\psi_1, e)]\!]_{\mathcal{M}} \to [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}} \to [\![f(\psi_2, e)]\!]_{\mathcal{M}}\\
& && (a \to (b \to c) = a \wedge b \to c)\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}} \to \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}} \wedge [\![f(\psi_2, e)]\!]_{\mathcal{M}}\\
& && (b \to c = b \to b \wedge c))\\
&= [\![\langle\mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \to [\![\langle\mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \wedge [\![\langle\mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= [\![\langle\mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \to [\![\langle\mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}}. && (b \to c = b \to b \wedge c))
\end{aligned}
$$

Axiom I12. $\langle\mathcal{E}, e\rangle(\psi_1 \to \psi_2) \;\;\leftrightarrow\;\; pre(e) \wedge (\langle\mathcal{E}, e\rangle\psi_1 \to \langle\mathcal{E}, e\rangle\psi_2).$

$$
\begin{aligned}
[\![\langle\mathcal{E}, e\rangle(\psi_1 \to \psi_2)]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1 \to \psi_2, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e) \to f(\psi_2, e)]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge ([\![f(\psi_1, e)]\!]_{\mathcal{M}} \to [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge (\overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}} \to [\![f(\psi_2, e)]\!]_{\mathcal{M}})\\
& && (a \wedge (b \to c) = a \wedge (a \wedge b \to c))\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge (\overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}} \to\\
&\quad [\![f(\psi_2, e)]\!]_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e) \wedge [\![f(\psi_1, e)]\!]_{\mathcal{M}}) && (b \to c = b \to b \wedge c))\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge ([\![\langle\mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \to [\![\langle\mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \wedge [\![\langle\mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}})\\
& && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \wedge ([\![\langle\mathcal{E}, e\rangle\psi_1]\!]_{\mathcal{M}} \to [\![\langle\mathcal{E}, e\rangle\psi_2]\!]_{\mathcal{M}}).\\
& && (b \to c = b \to b \wedge c))
\end{aligned}
$$

Axiom I13 $[\mathcal{E}, e]\,\Diamond_i\psi \;\;\leftrightarrow\;\; pre(e) \to \bigvee_{e' \sim_i e} \Diamond_i(\langle\mathcal{E}, e'\rangle\psi).$

$$
\begin{aligned}
[\![[\mathcal{E}, e]\,\Diamond_i\psi]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e) \to [\![f(\Diamond_i\psi, e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![\bigvee_{e' \sim_i e} \Diamond_i(f(\psi, e') \wedge pre(e'))]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to \bigvee_{e' \sim_i e} \Diamond_i([\![f(\psi, e')]\!]_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e'))\\
&= \overline{pre}_{\mathcal{M}}(e) \to \bigvee_{e' \sim_i e} \Diamond_i([\![\langle\mathcal{E}, e'\rangle\psi]\!]_{\mathcal{M}}) && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e) \to [\![\bigvee_{e' \sim_i e} \Diamond_i(\langle\mathcal{E}, e'\rangle\psi)]\!]_{\mathcal{M}}.
\end{aligned}
$$

Axiom I14. $\langle\mathcal{E},e\rangle\Diamond_i\psi \quad\leftrightarrow\quad pre(e)\wedge\bigvee_{e'\sim_i e}\Diamond_i(\langle\mathcal{E},e'\rangle\psi)$.

$$\begin{aligned}
[\![\langle\mathcal{E},e\rangle\Diamond_i\psi]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e)\wedge[\![f(\Diamond_i\psi,e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge[\![\bigvee_{e'\sim_i e}\Diamond_i(f(\psi,e')\wedge pre(e'))]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge\bigvee_{e'\sim_i e}\Diamond_i([\![f(\psi,e')]\!]_{\mathcal{M}}\wedge\overline{pre}_{\mathcal{M}}(e'))\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge\bigvee_{e'\sim_i e}\Diamond_i([\![\langle\mathcal{E},e'\rangle\psi]\!]_{\mathcal{M}}) && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge[\![\bigvee_{e'\sim_i e}\Diamond_i(\langle\mathcal{E},e'\rangle\psi)]\!]_{\mathcal{M}}.
\end{aligned}$$

Axiom I15. $[\mathcal{E},e]\,\Box_i\psi \quad\leftrightarrow\quad pre(e)\rightarrow\bigwedge_{e'\sim_i e}\Box_i([\mathcal{E},e']\psi)$.

$$\begin{aligned}
[\![[\mathcal{E},e]\,\Box_i\psi]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e)\rightarrow[\![f(\Box_i\psi,e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e)\rightarrow[\![\bigwedge_{e'\sim_i e}\Box_i(pre(e')\rightarrow f(\psi,e'))]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e)\rightarrow\bigwedge_{e'\sim_i e}\Box_i(\overline{pre}_{\mathcal{M}}(e')\rightarrow[\![f(\psi,e')]\!]_{\mathcal{M}})\\
&= \overline{pre}_{\mathcal{M}}(e)\rightarrow\bigwedge_{e'\sim_i e}\Box_i([\![[\mathcal{E},e']\,\psi]\!]_{\mathcal{M}}) && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e)\rightarrow[\![\bigwedge_{e'\sim_i e}\Box_i([\mathcal{E},e']\,\psi)]\!]_{\mathcal{M}}.
\end{aligned}$$

Axiom I16. $\langle\mathcal{E},e\rangle\Box_i\psi \quad\leftrightarrow\quad pre(e)\wedge\bigwedge_{e'\sim_i e}\Box_i([\mathcal{E},e']\psi)$.

$$\begin{aligned}
[\![\langle\mathcal{E},e\rangle\Box_i\psi]\!]_{\mathcal{M}} &= \overline{pre}_{\mathcal{M}}(e)\wedge[\![f(\Box_i\psi,e)]\!]_{\mathcal{M}} && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge[\![\bigwedge_{e'\sim_i e}\Box_i(pre(e')\rightarrow f(\psi,e'))]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge\bigwedge_{e'\sim_i e}\Box_i(\overline{pre}_{\mathcal{M}}(e')\rightarrow[\![f(\psi,e')]\!]_{\mathcal{M}})\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge\bigwedge_{e'\sim_i e}\Box_i([\![[\mathcal{E},e']\,\psi]\!]_{\mathcal{M}}) && \text{(Lemma 230)}\\
&= \overline{pre}_{\mathcal{M}}(e)\wedge[\![\bigwedge_{e'\sim_i e}\Box_i([\mathcal{E},e']\,\psi)]\!]_{\mathcal{M}}.
\end{aligned}$$

Axiom I17. $[\mathcal{E}, e]\,(\alpha\mu_i(\psi) \geq \beta) \;\;\leftrightarrow\;\; pre(e) \rightarrow \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}([\mathcal{E}, e']\,\psi) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\top) \geq 0$

$$\llbracket [\mathcal{E}, e]\,(\alpha\mu_i(\psi) \geq \beta) \rrbracket_{\mathcal{M}} = \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket f(\alpha\mu_i(\psi) \geq \beta, e) \rrbracket_{\mathcal{M}}$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \alpha \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \mu_i^{\phi}(f(\psi, e'))P_i(e')\mathsf{pre}(e' \mid \phi) - \beta \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \mu_i^{\phi}(\top)P_i(e')\mathsf{pre}(e' \mid \phi) \geq 0 \rrbracket_{\mathcal{M}} \qquad \text{(Definition 227)}$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee \{a \mid \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\llbracket f(\psi, e') \rrbracket_{\mathcal{M}} \wedge a) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(a) \geq 0\}$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee \{a \mid \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\llbracket f(\psi, e') \rrbracket_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e') \wedge a) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(a) \geq 0\}$$

$$(\; \llbracket \varphi \rrbracket_{\mathcal{M}} \leq \overline{pre}_{\mathcal{M}}(e') \text{ if } \mathsf{pre}(e' \mid \varphi) \neq 0\text{, cf. Proposition 153})$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee \{a \mid \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}((\overline{pre}_{\mathcal{M}}(e') \rightarrow \llbracket f(\psi, e') \rrbracket_{\mathcal{M}}) \wedge \overline{pre}_{\mathcal{M}}(e') \wedge a) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(a) \geq 0\}$$

$$(\; a \wedge (a \rightarrow b) = a \wedge b)$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee \{a \mid \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\llbracket \left[\mathcal{E}, e'\right]\,\psi \rrbracket_{\mathcal{M}} \wedge \overline{pre}_{\mathcal{M}}(e') \wedge a) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(a) \geq 0\} \qquad \text{(Lemma 230)}$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \bigvee \{a \mid \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\llbracket \left[\mathcal{E}, e'\right]\,\psi \rrbracket_{\mathcal{M}} \wedge a) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(a) \geq 0\} \quad (\; \llbracket \varphi \rrbracket_{\mathcal{M}} \leq \overline{pre}_{\mathcal{M}}(e') \text{ if } \mathsf{pre}(e' \mid \varphi) \neq 0)$$

$$= \overline{pre}_{\mathcal{M}}(e) \rightarrow \llbracket \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} \alpha P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\left[\mathcal{E}, e'\right]\,\psi) + \sum_{\substack{e' \sim_i e\\ \phi\in\Phi}} -\beta P_i(e')\mathsf{pre}(e' \mid \phi)\mu_i^{\phi}(\top) \geq 0 \rrbracket_{\mathcal{M}}$$

Axiom I18. $\langle\mathcal{E},e\rangle(\alpha\mu_i(\psi)\geq\beta) \;\leftrightarrow\; pre(e)\wedge\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\alpha P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(\langle\mathcal{E},e'\rangle\psi)+\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}-\beta P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(\top)\geq 0$

$$
\begin{aligned}
&[\![\langle\mathcal{E},e\rangle(\alpha\mu_i(\psi)\geq\beta)]\!]_{\mathcal{M}}\\
&=\overline{pre}_{\mathcal{M}}(e)\wedge[\![f(\alpha\mu_i(\psi)\geq\beta,e)]\!]_{\mathcal{M}}\\
&=\overline{pre}_{\mathcal{M}}(e)\wedge[\![\alpha\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\mu_i^{\phi}(f(\psi,e'))P_i(e')\mathsf{pre}(e'\mid\phi)-\beta\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\mu_i^{\phi}(\top)P_i(e')\mathsf{pre}(e'\mid\phi)\geq 0]\!]_{\mathcal{M}} && \text{(Definition 227)}\\
&=\overline{pre}_{\mathcal{M}}(e)\wedge\bigvee\{a\mid\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\alpha P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}([\![f(\psi,e')]\!]_{\mathcal{M}}\wedge a)+\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}-\beta P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(a)\geq 0\}\\
&=\overline{pre}_{\mathcal{M}}(e)\wedge\bigvee\{a\mid\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\alpha P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}([\![f(\psi,e')]\!]_{\mathcal{M}}\wedge\overline{pre}_{\mathcal{M}}(e')\wedge a)+\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}-\beta P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(a)\geq 0\}\\
& && \text{($[\![\varphi]\!]_{\mathcal{M}}\leq\overline{pre}_{\mathcal{M}}(e')$ if $\mathsf{pre}(e'\mid\varphi)\neq 0$, cf. Proposition 153)}\\
&=\overline{pre}_{\mathcal{M}}(e)\wedge\bigvee\{a\mid\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\alpha P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}([\![\langle\mathcal{E},e'\rangle\psi]\!]_{\mathcal{M}}\wedge a)+\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}-\beta P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(a)\geq 0\} && \text{(Lemma 230)}\\
&=\overline{pre}_{\mathcal{M}}(e)\wedge[\![\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}\alpha P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(\langle\mathcal{E},e'\rangle\psi)+\sum_{\substack{e'\sim_i e\\ \phi\in\Phi}}-\beta P_i(e')\mathsf{pre}(e'\mid\phi)\mu_i^{\phi}(\top)\geq 0]\!]_{\mathcal{M}}
\end{aligned}
$$

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Chapter 6

Conclusion

This is supposed to be the end of a journey, but it rather feels like its beginning. In what follows, I will list the largest among the many questions and directions which are opened up by the results collected in this thesis. Each of these questions is broad enough to give rise to yet another PhD thesis.

1. Having established systematic connections between correspondence theory and the problem of analyticity in structural proof theory (cf. Chapter 2), and having introduced a multi-type environment where the modal perspective on first-order logic of [13] can be developed in a principled way (cf. Chapter 4), we are now in a position to exploit these techniques to address the long-standing open problem of characterizing which rules are admissible in a given propositional or predicate logic. Specifically, in the literature, both proof-theoretic [6] and algebraic/duality-theoretic methods [4] have been used to solve this problem for specific logics. The techniques we have developed can serve to throw light on the connection between these methods and to systematically explore their scope.

2. As discussed in Chapter 3, The logic LRC provides a formal environment where to formalize e.g. Ricardo's theory of comparative advantage regarding the division of labour in organizations. Another example of economic theory which can be formally addressed is the *resource-based view* of the firm (RBV) [15]. It posits that the resources of a firm are key to the success of that firm in market-competition. The RBV starts by observing that resources are not evenly distributed across firms, and their transfer from firm to firm typically has a cost. It suggests that possessing valuable and rare resources provides the basis for competitive advantage, and if these resources are also inimitable and lack substitutes, the competitive advantage will be sustainable over time [2]. The RBV has been hugely influential across all areas of management; however, research has highlighted the existence of logical circularities among basic notions such as value, resources, and competitive advantage [7, 9, 10]. For instance, defining the value of a resource as the extent to which this resource contributes to the competitive advantage of the firm is problematic, since competitive advantage is explained in terms of possessing valuable

resources. Researchers have indicated that formal methods are needed to clarify and resolve these issues [9], and that, besides achieving internal coherence, the RBV should address a wider range of issues to evolve into a fully-fledged theory of market-competition, able to explain the core connections between the resources that the firm uses to produce goods/services, and the competitive environment in which the firm operates to satisfy its customers' demands. Specific questions this theory should address are: how the value of resources explicitly relates to the value of the goods/services the firm produces; how the value of resources is established in the resource-market, in relation to their contribution to the satisfaction of the customers' demands; how the competition between resource-providers within the firm shapes and is shaped by the competitive processes in the resource-market and in the market of goods/services.

The logic of resources and capabilities (LRC) is eminently suited to provide a platform where these issues can be formally addressed, since not only does it focus on core notions of RBV, but it can be modularly expanded so as to account for other key notions such as group capabilities and strategies. Notice that the modularity requirement that is guaranteed by the strong mathematical backbone of LRC is not just a theoretically desirable requirement, but translates into a *necessary* requirement for a formal system to have any hope of tackling an economic theory of such complexity.

3. In Chapter 5, we have introduced the logic IPDEL in which probabilistic and epistemic reasoning are integrated in an intuitionistic background. This logic has been introduced by means of a Hilbert-style axiomatization but lacks a sequent calculus. More in general, the proof-theoretic development of probabilistic logics has hardly begun. Hence, a natural direction to go is to develop proof calculi for IPDEL and other probabilistic logics. This direction would significantly advance logic as a field, both from a theoretical perspective, since we would achieve a stronger integration between the qualitative and quantitative sides of formal reasoning, and from the viewpoint of applications, since probabilistic logics are key to the formalization of core foundational aspects in social science, such as decision-theoretic problems, uncertainty, beliefs and information aggregation, and have been invaluable for the formalization of Bayesian epistemology [16], decision theory [3], knowledge and belief representation and update [1, 5], Dempster-Shafer theory [11, 12], quantum theory [8, 14]. The multi-type methodology has reached an adequate stage of consolidation to tackle this challenge.

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Curriculum Vitæ

Apostolos TZIMOULIS

02-01-1988	Born in Thessaloniki, Greece.

Education

2002–2005	Unified Lyceum Thessaloniki, Greece
2005–2010	Degree of the Department of Mathematics Aristotle University of Thessaloniki, Thessaloniki, Greece
2011–2013	MSc in Logic, with distinction (cum laude) Universiteit van Amsterdam, Amsterdam, The Netherlands
2014–2018	Ph.D., Logic Applied Logic Group, Technische Universiteit Delft, Delft, The Netherlands

List of Publications

6. **Logic of Resources and Capabilities**, with Marta Bílková, Giuseppe Greco, Alessandra Palmigiano, and Nachoem M. Wijnberg, *Review of Symbolic Logic*, in press, doi: 10.1017/S175502031700034X, (2018).

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