

The Heat Capacity of Random Tournaments

A study in nonequilibrium statistical mechanics

by

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Abstract

This thesis characterises the behaviour of the nonequilibrium heat capacity $C_N(\beta)$ on random tournament graphs T_N without an energy landscape for large N . The heat capacity $C_N(\beta)$ is completely determined by the geometry of the underlying network in this case and derived from the quasipotential. The quasipotential is obtained by solving a discrete Poisson equation involving the random generator L_N of a non-reversible Markov chain over the tournament. Numerical results suggest a self-averaging property, leading to our main conjecture

$$N \cdot C_N(\beta) = 4C_2(\beta) (1 + N^{-1/2} \cdot \xi_C),$$

where $C_2(\beta)$ is the heat capacity of a fundamental two-level system and the random variable $\xi_C \rightarrow \mathcal{N}(0, \tau_C)$ in distribution.

To prove the conjecture, we provide three main contributions. We first derive a coarse-grained generator $\overline{G}_N$ in score-space that represents the system in an idealised situation. In this situation, the the score bins B_s that count the number of vertices with a certain score attain their expected value. We solve its corresponding Poisson problem, and prove that it captures the ensemble-averaged $4C_2(\beta)$ limit. We then shift our focus to the statistics of the score bins, which are globally dependent random variables. Using a change of measure, we establish a concentration result and a local Central Limit Theorem for the sizes of the score bins N_s . Finally, we introduce an auxiliary generator G_N that bridges the vertex and score spaces, linking $\overline{G}_N$ and L_N . We demonstrate that G_N satisfies the conjectured Gaussian scaling by deriving the pseudoinverse $\overline{G}_N^\dagger$ and solving the Poisson problem for G_N perturbatively. We conclude by providing a viable route to proving that $C_N(\beta)$ is close to the heat capacity of G_N .

Preface

I first came to Leuven about a year ago to discuss possible thesis topics with Christian Maes. At that point, Julia Komjáthy had contacted him about me possibly working with his group almost another year before that. When I walked into Christian's office that February in 2025, I came fresh off eight months of internships. I had been eager to find a project before starting them, because I knew it would be quite challenging to find a suitable topic for a joint mathematics and physics thesis. Especially because I was intent on finding a topic in physics involving random graphs.

Once we started talking, I was happy to find no lack of ideas. We discussed almost three full projects that morning. I reported back to Julia and my other supervisor in Delft, Marianne Bauer, to form a plan of attack. We decided to focus on the first of these problems and start on one of the others once it had been solved. After all, others had already started a heuristic solution to a part of the problem and the overall goal was well defined. All I had to do was to find an explanation for the self-averaging property of the heat capacity...

What seemed to be a simple problem at face value, turned out to be deceptively difficult. The other problems have remained untouched in my notebook. In my trials trying to solve the Poisson equation for the random tournament graph, I explored many different topics and fields. Some useful, others not (yet) so useful. The most telling of those chapters was the stretch from September to November last year, when I learnt and tried to apply Stein's method to the score bins in Chapter 5. After trying nearly every approach in the book, I gave up, only to stumble across the most *deus ex machina* of all results in the literature making Stein's method no longer needed for this problem. My love for the method has not faded, though.

A year of hard work has led to this thesis. A document that is several orders of magnitude longer and more complex than I could have ever imagined. I am very proud of what I have written down and I am incredibly excited for you to read it. But before we start, I have to thank several people.

To Julia, I owe the beginning of this journey. She first taught me formal probability in the third year of my bachelor's and got me into random graphs the year after. She has been an invaluable help throughout the project, and I want to thank her for guidance. Christian hosted me with open arms in Leuven and often provided perspective changing feedback. It has been a pleasure learning all about nonequilibrium from him and I hope to return to the topic someday. I want to express my gratitude to Marianne for helping me frame the raw ideas and results into something insightful. Without her, the thesis would have been a theoretical exercise only.

Finally, I want to thank my flatmates Noud and Johnny for the unforgettable years in Delft. I want to thank my sister for her endless stream of good advice. I want to thank my parents for quite literally everything. And I want to thank my girlfriend Kaatje for the beautiful memories of the past year.

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Contents

1	Introduction	5
1.1	Tournament graphs	6
1.2	Random walks on graphs	7
1.3	Perturbation theory	9
1.4	Methodological outline and main contributions	10
1.5	Structure of the thesis and reader's guide	11
	Frequently used notation	13
2	Nonequilibrium thermodynamics	14
2.1	What is equilibrium?	14
2.2	Markov processes and equilibrium	15
2.3	Equilibrium response: fluctuation-dissipation	17
2.4	Stationary nonequilibrium and local detailed balance	19
2.5	Nonequilibrium response: dynamical ensembles	22
2.6	Excess heat and nonequilibrium heat capacity	25
2.7	Nonequilibrium heat capacities: examples	26
3	Heat capacity of tournaments	33
3.1	Tournament graphs	33
3.2	Tournament dynamics	38
3.3	Numerical simulations of the tournament heat capacity	39
4	Annealed solutions on random tournaments	48
4.1	Annealed transition rates	49
4.2	Solving the annealed problem	53
5	Statistical properties of random tournaments	57
5.1	The $\mathcal{D}$ -model	58
5.2	Score bins CLT	61
5.3	Improved concentration	70
6	A perturbative approach to random Poisson equations	78
6.1	Annealed pseudoinverse	81
6.2	Heat capacity of G_N	86
6.3	Heat capacity of L_N : proof strategy	96
7	Conclusion	98
A	Useful Theorems	101
B	Calculating tournament heat capacities	105
C	Technical Lemmas: Chapter 1	107
D	Technical Lemmas: Chapter 2	113
E	Technical Lemmas: Chapter 3	115

1. Introduction

The challenge central to nonequilibrium statistical mechanics is to build predictive microscopic models for systems that are driven. In equilibrium systems, statics are sufficient to this end. A static probability distribution over a system’s microstates contains all the information needed to derive its macroscopic thermodynamic properties. This connection is what makes equilibrium statistical mechanics such a useful tool. However, from microscopic molecular motors to large-scale atmospheric heat fluxes, nature is full of systems operating far from equilibrium [1]. These are constantly driven by external forces or thermal gradients, generating persistent, directional currents. A static snapshot of the state space no longer captures the relevant physics in this situation. Instead, the focus shifts from the states themselves to the specific trajectories the system traverses over time [2, 3]. Because the paths are determined by which transitions are physically allowed and where kinetic bottlenecks occur, understanding nonequilibrium behavior becomes linked to the geometry and topology of the underlying state space [4].

This shift in perspective from static states to dynamical trajectories forms the foundation of stochastic thermodynamics [5]. Over the past two decades, this framework has achieved remarkable success at the microscopic scale, yielding exact thermodynamic relations for fluctuating systems such as colloidal particles in optical traps or individual molecular motors [6, 7]. However, while it excels at describing single-trajectory energetics, it has historically struggled to replicate the macroscopic predictive power of classical equilibrium mechanics. Bridging the gap between microscopic path fluctuations and macroscopic bulk responses, such as the heat capacity of a complex system, remains a challenge [8]. The foundational work of Maes, Netočný, and collaborators provides a crucial theoretical bridge to tackle these macroscopic questions. By constructing a dynamic response theory over path-space, they successfully formalised the concept of nonequilibrium calorimetry [9, 10]. Their framework relies on the assumption of local detailed balance [11]: while the global system is actively driven by external forces or temperature gradients, the local Markovian transitions reversibly exchange heat with well-defined equilibrium heat baths.

When a nonequilibrium system possesses a finite number of discrete states, such as a complex chemical reaction network, its state space is naturally modeled as a graph [12]. The vertices of this graph represent specific microscopic configurations and the directed edges denote the physically allowed transitions. If the Markov process is not in physical equilibrium (time-irreversible), properties like the stationary distribution become globally dependent on the shape of the network instead of a local function of an energy landscape [4]. The dependence between movement and graph structure fundamentally modifies how the system absorbs and dissipates heat [13]. Consequently, the nonequilibrium heat capacity of a general driven process on a graph takes the form [14]

$$C(\beta) = -\beta^2 \frac{\partial \langle E \rangle_\rho}{\partial \beta} + \beta^2 \left\langle \frac{\partial V}{\partial \beta} \right\rangle_\rho, \quad (1.1)$$

where $\langle \cdot \rangle_\rho$ denotes the inner product with respect to the stationary measure of the process. The first term is equivalent to the classical equilibrium response of the static energy landscape. The second term, however, is an entirely novel kinetic correction. It represents the temperature dependence of a quasipotential, V , which quantifies the expected excess heat exchanged as the process moves along the structural bottlenecks and asymmetric pathways of the underlying network.

Existing literature on nonequilibrium heat capacity has largely been confined to two regimes. On the one hand, small networks with a handful of states can be solved exactly using algebraic tools like the matrix-forest theorem [15]. On the other hand, exact solutions have been derived for large systems with high symmetry, such as the driven Curie-Weiss model, where uniform mean-field interactions naturally decouple the dynamics [16, 17]. However, a gap remains for large, dynamically frustrated networks. For these systems, direct computations are difficult because of the state space’s size, yet

standard mean-field approximations fail because they inherently wash out the crucial kinetic effects of the underlying disorder [18].

This thesis takes a step in this direction by investigating the nonequilibrium heat capacity of tournaments: fully connected, randomly directed graphs. We omit any underlying energy landscape to isolate the thermal response driven purely by the network’s structural disorder. That is, the first term in (1.1) is zero. We demonstrate that due to quenched frustration, a sampling of a random tournament has a nonzero nonequilibrium heat capacity with asymptotic behaviour

$$N \cdot C_N(\beta) = 4C_2(\beta) (1 + N^{-1/2} \cdot \xi_C),$$

where $C_2(\beta)$ is the heat capacity of a fundamental two-level system and the random variable $\xi_C \rightarrow \mathcal{N}(0, \tau_C)$ in distribution. The bulk macroscopic response of this dense network collapses into the scaled dynamics of a single jump, with variance driven entirely by quenched structural fluctuations. The result suggests that for anomalous, geometry-driven kinetic effects to survive the thermodynamic limit, the underlying network cannot be well-connected.

The field of nonequilibrium thermodynamics provides the physical motivation for this thesis and working with the resulting models requires stepping into several distinct mathematical disciplines. Deriving the exact macroscopic response of our driven system relies on combining tools from the study of directed graphs, random walks on networks, and perturbation theory. In the next sections, we will briefly review these topics.

1.1 Tournament graphs

The specific geometry we focus on throughout the thesis is the tournament graph: a complete graph with randomly directed edges. Harary and Moser standardised the term tournament in a 1966 overview article, noting that the graph encodes the structure of a round-robin tournament [19]. Under this analogy, every vertex in the graph has a score, which is typically set to the number of out-degrees (wins) or the difference between its out-degree and in-degree (net wins). While the structural study of these graphs matured in the 1960s, the origins trace back to the 1930s after Rédei proved that every tournament contains a directed path that visits each vertex once, a Hamiltonian path [20]. In 1943, Szele expanded this result by establishing an upper and lower bound for the number of these paths in a tournament [21]. He proved these bounds by taking the expectation over random tournaments, which is considered to be the first application of the probabilistic method in combinatorics [22].

The next big results were proved by Landau in 1953. He was interested in studying the pecking order of chickens, for which a tournament graph serves as a minimal model [23]. Using the model, he was able to prove that a perfectly linear hierarchies where an absolute transitive ranking naturally emerges from individual matchups are exceedingly rare in nature. Instead, dominance relations are usually full of cycles.

Over the course of his investigation, he proved two important structural properties. First, he established a lower bound for the cumulative scores within a tournament. He proved that the sum of the out-degrees of the k lowest-ranking vertices cannot be less than $\binom{k}{2}$ [24]. Second, he sought a way to define a most dominant individual when the tournament contains cycles and lacks a transitive ordering. He proved that the vertex with the maximum score can reach any other vertex in no more than two steps. In other words, even if this player did not defeat a particular opponent directly, they defeated at least one person who did. This vertex was later dubbed ‘king’ by Maurer in 1980 [25].

As combinatorics boomed throughout the 1960s, the structural theory of tournaments rapidly expanded. For example, Camion improved Rédei’s theorem by showing that every tournament contains a Hamiltonian cycle if and only if it is strongly connected [26]. Following this, Erdős applied probabilistic methods to show the existence of tournaments in which every subset of k vertices is dominated by at least one vertex [27]. In collaboration with Moser, Erdős also established the existence of tournaments on N vertices containing no transitive subtournament larger than $2\lfloor \log_2 N \rfloor + 1$ [28]. The study of cycles was improved when Moon demonstrated that every strongly connected tournament is

vertex-pancyclic, meaning every vertex belongs to cycles of lengths from three up to N [29]. Alspach improved this result by showing that every directed arc is pancyclic on tournaments where all vertices have the same score [30]. The results of the 1960s were consolidated into a cohesive theory in 1968 after the publication of *Topics on Tournaments* by Moon [31].

The mid-century era proved the existence of several extremal properties of random tournaments, mostly using the probabilistic method. However, an estimate for the number of tournaments displaying a certain property requires another method. These types of questions can be handled by asymptotic enumeration, which studies the limiting behaviour of graphs as the number of vertices grows to infinity. For very big systems, the discrete combinatorial problems can be reframed as continuous, allowing for application of tools from complex analysis [32]. One of the main techniques involves using generating functions to represent the graph structure in combination with the saddle-point method to estimate the resulting complex integrals [33, 34]. Using this method, McKay and Wang derived a precise asymptotic formula for the number of tournaments with a given score sequence [35]. This result implies that typical score sequences behave as independent binomial variables conditioned on the fixed total score, which is the $\mathcal{D}$ -model change of measure we will use in this thesis.

Beyond the structural properties of the graphs themselves, the original applied interest came from the social sciences. Just as Landau used tournaments to model biological pecking order, political scientists use tournaments to analyse dominance relations in voting theory [36]. When voters rank a pool of candidates or policies, their collective preferences can be aggregated from every possible head-to-head matchup. However, this often leads to Condorcet’s paradox [37]. This is a scenario where the majority prefers option A over B , B over C , and yet C over A . This means that the voting outcomes form a cycle on the graph, so that the tournament is not transitive. In this situation, no clear winner can be determined by direct matchups alone. This limitation was expanded by Arrow in 1951 [38]. Arrow’s Impossibility Theorem proved that no rank-order voting system can convert individual preferences into a consistent and fair community-wide ranking with a clear winner without violating some axioms of rational choice ¹.

To resolve the ambiguities of cyclic dominance, voting theorists define specific sets that identify the most viable winners based on the network’s structure. These sets are called tournament solutions [36]. For example, the collection of all vertices with Landau’s king property mentioned earlier is such a solution and is called the Landau set [39]. The Smith set is the smallest possible group of candidates who beat everyone outside the group [40]. Even if they cycle among themselves, they structurally dominate the rest of the field. The Copeland set simply takes the candidates with the most wins [41]. The Banks set is constructed by taking all maximum length transitive subtournaments and collecting their respective winners [42]. Using graph theory, each of these sets can be shown to satisfy different axioms terms of rational choice. Choosing one of these options boils down to balancing fairness with computational complexity and how much the winner has to be narrowed down. Because so many ranking problems can be modeled as a series of head-to-head matchups, tournament solutions have found use outside of social choice theory as well. The applications include game theory [43, 44], information retrieval [45] and multi-criteria decision analysis [46, 47].

1.2 Random walks on graphs

The second topic that the thesis touches on is random walks on graphs. Physically, the goal is to analyse the specific trajectories a system takes as it explores the state space. Because we assume that the likelihood of transitioning from state A to state B is governed by a physical transition rate that depends only on the current configuration A , the system is modelled to be memoryless [5, 48]. This Markov property is the defining feature of Markov chains, which have been studied thoroughly. Mapping the physics to a Markov chain allows us to apply a big set of established theoretical tools

¹Specifically, Arrow demonstrated that no system can simultaneously satisfy unanimity (the community follows a universal preference), non-dictatorship (no single voter prevails) and the independence of irrelevant alternatives (the choice between A and B is unaffected by C) while maintaining a clear winner.

to analyse a system's dynamics [49, 50, 51]. Beyond simply determining the stationary distribution, which gives the asymptotic fraction of time spent in each state, many timescales can be quantified. We can calculate the hitting time required to reach a specific target configuration, the cover time needed to explore the entire state space, and the mixing time it takes for the process to fully relax from an initial condition into its steady state. The availability of all these properties make applications of random walks on networks ubiquitous as a model [52].

A famous example of the value of Markov chains is Glauber dynamics used to study the Ising model [53]. The Ising model is an essential model for ferromagnetism, where a collection of N binary spins are arranged on a regular lattice [54]. The energy landscape of the system is shaped by local interactions between the spins. Spins favour alignment with both their nearest neighbours and any applied external magnetic field. Glauber dynamics introduces time into this system by constructing a random walk across the high-dimensional graph of all possible spin configurations. At each step, a random single spin is uniformly chosen, the energy difference ΔE after flipping is calculated and the spin is then flipped with a probability related to the inverse temperature β , $p_{\text{flip}} = (1 + e^{\beta\Delta E})^{-1}$.

Using the mathematics of Markov chains, it is then shown that the stationary distribution of this specific random walk reproduces the system's thermodynamic distribution. More importantly, the Markov framework allows us to calculate how long it takes the system to settle into this state and to observe how this relaxation slows down near phase transitions [55]. This is not possible with a static analytic solution.

Random walks also provide the microscopic basis for modelling macroscopic network diffusion. In this perspective, the flow of heat across a network is mathematically equivalent to the evolving probability distribution of a single random walker [56]. This connection is famously illustrated by taking the continuum limit of an unbiased random walk on $\mathbb{Z}^d$, where the walker's probability density converges to the solution of the heat equation in $\mathbb{R}^d$ [57]. A practical usage of the diffusive behaviour can be found in the Walktrap algorithm used to detect communities in networks [58]. The intuition behind the algorithm is that diffusive processes tend to stay within densely connected subgraphs before escaping. To measure the trapping effect, the algorithm defines a distance between two vertices by comparing their t -step transition probabilities to all other nodes in the network. If two vertices are in the same community, these probabilities will be very similar and the so-called Walktrap distance is small. The properties of Markov chains are then used to show that the Walktrap distance is a valid metric and to provide an estimate for the optimal t [58]. This can not be too big or the walker has left the community, nor can it be too small to avoid too little exploration.

1.2.1 Nonreversible random walks

All of the above examples are random walks on undirected connected networks. As a result, they are inherently reversible. This is because the symmetry of the edges ensures that any movement between nodes is naturally offset by an equal flow in the opposite direction once the walk reaches its steady state. This condition is formally known as detailed balance [49]. Although random walks on directed graphs can satisfy detailed balance, they are generally irreversible processes. This is why we turn to random walks on directed graphs to model nonequilibrium systems [4]. These processes still satisfy the Markov property, so that properties like the stationary distribution and the mixing time can still be calculated. Only specific properties downstream of detailed balance are lost [59].

The most widely known example of a nonreversible random walk on a directed graph is the original version of Google's PageRank algorithm, which computes the relative centrality of websites by treating the entire internet as a directed graph where websites are vertices and hyperlinks are edges [60]. The Markov process models a web surfer who clicks on links at random. These links are directed, because page A linking to B does not imply page B linking back to A . Therefore, the walk is irreversible. To prevent the surfer getting stuck in a dead-end or a cluster that only points towards itself, the algorithm includes a teleportation factor. At any step, the surfer has a 15% chance to jump to any random webpage [61]. Because this ensures all parts of the web can be reached, the Markov chain has a unique stationary distribution. The ranking within the distribution is then exactly the ranking of

the pages after a search.

A second prominent example of a nonreversible process is the nonbacktracking random walk, abbreviated to NBW. This is a walk on an undirected graph forbidden from moving over the edge it has just traversed [62]. The motivation behind this restriction lies in its application to sparse graphs. In a dense graph, a random walker has many neighbours to choose from and naturally diffuses outward. However, backtracking makes a walker spend a lot of time undoing its previous step on a tree-like graph. By removing this behaviour, a sparse graph can be explored more efficiently [63].

At first, the previous step memory restriction appears to destroy the Markov property. However, it can be restored by shifting the perspective from vertices to edges [64]. We can construct a new directed graph by considering each edge in the original graph as a vertex. A transition from (u, v) to (v, w) is then drawn if $v \neq w$. The NBW then becomes a standard Markov chain along this expanded graph, which can be handled using standard methods. It can be shown, for example, that the NBW indeed mixes faster on well-connected sparse graphs [63]. Because the NBW performs better on sparse graphs, they have been widely used to improve methods used for network analysis on sparse graphs. For instance, they can be used to detect communities in graphs where the Walktrap algorithm would fail [65]. Similarly, adapting centrality measures like PageRank to use the NBW has been shown to improve rankings in sparse networks [66].

1.3 Perturbation theory

Perturbation theory broadly refers to a collection of methods used to find an approximate solution to a problem by starting from the exact solution of a related simpler problem [67]. The history of these methods is tied to celestial mechanics. Under Newtonian gravity, the idealised two-body problem is exactly solvable. However, the true solar system contains many planets, all exerting gravitational pulls on each other. Because the general three-body problem lacks a closed-form solution, nineteenth-century astronomers developed a workaround by treating the pull of a second planet as a small perturbation to the exact two-body orbit [68, 69]. By calculating the leading-order effects of this correction, they could predict planetary orbits with high accuracy. This was how Neptune was discovered by Le Verrier and Adams, for example [70]. They noticed that Uranus's orbit was slightly off from its expected path, leading them to calculate the exact location of a new planet.

The core approach to perturbation theory has since spread far beyond classical mechanics. It now appears in fluid dynamics [71], nuclear reactor physics [72], and essentially any field where equations are too complex to solve outright [73]. Most notably, it serves as a cornerstone of quantum mechanics. It provides the standard machinery for refining the energy levels of exact solutions like the hydrogen atom, the harmonic oscillator or the particle in a box by adding small corrections for relativistic effects, field couplings, and other physical refinements [74]. More broadly, perturbative techniques are used to calculate transition rates between states [75], construct Feynman diagrams [76] and obtain effective models at low energy [77]. This wide application has led to the development of many specialised approaches, such as Löwdin partitioning [78], Dyson series expansions [79], Schrieffer-Wolff transformations [80] and Bloch expansions [81].

In a general mathematical setting, suppose we wish to solve an equation involving some complex operator L . If L is too difficult to invert directly, we can split it into an easily invertible operator L_0 and a smaller complicated perturbation ϵL_1 , such that $L = L_0 + \epsilon L_1$. The underlying assumption of perturbation theory is that the full solution x to the problem $Lx = y$ can be expressed as

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots, \quad (1.2)$$

where the zeroth-order term is simply the solution to the unperturbed equation $L_0 x_0 = y$, while the subsequent terms $x_1, x_2, \dots$ act as corrections. By substituting this series back into the original equation and matching the coefficients for each power of ϵ , the problem turns into a sequence of simpler equations that can be solved iteratively [67].

1.4 Methodological outline and main contributions

The main objective of this thesis is to motivate and explain the nonequilibrium heat capacity of a random tournament graph without an energy landscape. In particular, the property that for large N , it behaves as

$$N \cdot C_N(\beta) = 4C_2(\beta) (1 + N^{-1/2} \cdot \xi_C),$$

where $C_2(\beta)$ is the heat capacity of a fundamental two-level system and the random variable $\xi_C \rightarrow \mathcal{N}(0, \tau_C)$ in distribution.

In the absence of an energy landscape, $E_i = 0$ for all vertices i . We thus obtain the heat capacity $C_N(\beta)$ from

$$C_N(\beta) = \beta^2 \left\langle \frac{\partial V_N}{\partial \beta} \right\rangle_{\rho_N}, \quad (1.3)$$

following (1.1) in the introduction. As established in the introduction, V_N quantifies the expected excess heat exchanged as the process moves along the underlying network. It is therefore a property of the random process on the graph. The dynamics of this process are captured by the generator L_N , which is formally constructed in Section 3.2. Because it contains the transition rates between the vertices on the random tournament, it is a random object in our case. While we defer the justification of this claim to Section 2.6, the fundamental equation to solve for the quasipotential is

$$L_N V_N = -Q_N, \quad (1.4)$$

where Q_N represents the expected instantaneous power dissipated at a vertex [14]. The goal of this thesis is to provide a perturbative framework for solving this random generator equation. We build this framework step-by-step, starting with developing numerical intuition and followed by three main theoretical contributions.

Numerical evidence and conjecture Before developing any proofs, we numerically simulate the heat capacity. In the first test, we simulate for an ensemble of random tournament graph realisations at different system sizes. This shows three important properties. First, the ensemble average $C_{N,\text{avg}}(\beta)$ of the heat capacity is stable across system sizes. That is, the average of the heat capacity is the same for tournaments with different N . Second, the fluctuations around the average are multiplicative with respect to the average, diminish as the system size N increases and are approximately Gaussian. This informs that $N \cdot C_N(\beta) = C_{N,\text{avg}}(\beta) (1 + N^{-1/2} \cdot \xi_C)$. Finally, the ensemble average $C_{N,\text{avg}}$ is identified as two-level heat capacity $4C_2(\beta)$ (the exact expression is given in Example 2.4).

In the second test, we average all properties of the vertices over the sets with vertices of the same score (out-degree minus in-degree in our case). We name these sets score bins. The averaging over score bins does not affect the heat capacity. This implies that tournaments with the same score sequence will have the same heat capacity. All of the randomness in the large N limit is captured by the statistics of the score bins themselves. This suggests that the limiting heat capacity $4C_2(\beta)$ corresponds to an annealed situation in which all score bins reach their expected values.

Contribution 1: the annealed baseline solution The first contribution we make in this thesis is the construction of an annealed generator $\overline{G}_N$ that captures the dynamics of the idealised situation in which the score bins reach their expected values. This generator is a coarse graining and is defined on the space of scores instead of the microscopic vertex space. We define the corresponding score-space quasipotential $\overline{W}_N$ and prove that the generator equation

$$\overline{G}_N \overline{W}_N = \overline{Q}_N \quad (1.5)$$

is exactly solvable. From this we derive that the corresponding heat capacity is given by

$$\overline{C}_N(\beta) = \frac{4C_2(\beta)}{N}. \quad (1.6)$$

As a next step, we show that the random transition rates in L_N concentrate around the transition rates in $\overline{G}_N$, showing that they share limiting dynamics. Because the equation for $\overline{G}_N$ is exactly solvable, it will serve as the basis for the perturbative scheme.

Contribution 2: concentration results for random tournaments From the simulations, we postulate that the randomness in the heat capacity is primarily caused by fluctuations in the sizes of the score bins. The sum of all scores adds to zero, because each edge is shared by two vertices. Therefore, these score bins are a collection of globally dependent random variables.

Our second contribution is that we prove two useful properties about the score bins by using the change of measure of the $\mathcal{D}$ -model to handle the dependencies [35]. First, we derive a uniform concentration result that shows the size of the score bins concentrate around their expected value. Second, we derive a local Central Limit Theorem, which shows that the pointwise probability that a score bin attains a certain size converges to a Gaussian density, providing sharp estimates for the fluctuations around the mean. Third, we show that under certain assumptions, linear combinations of score bins converge in distribution to a Normal random variable.

From our methods, we also obtain a better concentration result for the rates of L_N around $\overline{G}_N$. We prove that the rates exhibit a CLT property. This may be of independent interest, especially relating to functionals of dependent random variables.

Contribution 3: perturbative route Our final contribution combines the annealed solution with the statistical properties of the tournament to construct a perturbative scheme to solve the generator equation involving the random matrix L_N . The first result will help the perturbative approach is a derivation of the exact entries of the pseudoinverse of $\overline{G}_N$. This will be used to calculate the higher-order corrections in the perturbation series.

We then move from $\overline{G}_N$ to L_N in two steps. The generator $\overline{G}_N$ represents the situation in which the score bins reach their expected values and all scores can be reached, so that everything is deterministic. The first step is thus to restrict $\overline{G}_N$ to the observed scores and add random fluctuations of the score bin sizes. This leads to the generator G_N . We solve the random generator equation for G_N by decomposing $G_N = \overline{G}_N + \Delta$ and using the exact solution for $\overline{W}_N$ to obtain the random W_N . We show that the resulting heat capacity of G_N yields the conjectured form $N \cdot \tilde{C}_N(\beta) = 4C_N(1 + N^{-1/2}\xi)$.

The generator G_N is both defined on the score space and the microscopic vertex space. This allows us to take the second step and directly compare G_N and L_N . While a full proof of the fact that the heat capacities $C_N(\beta)$ and $\tilde{C}_N(\beta)$ agree is left as a formal conjecture for future work, we provide a viable analytical route towards the proof.

1.5 Structure of the thesis and reader's guide

The thesis is divided into five chapters. The first chapter is dedicated to providing the reader with context on the nonequilibrium framework. The remaining four chapters build towards the perturbative framework outlined in the methodology. Here we give a detailed overview.

Reader's guide To accommodate both mathematics and physics readers, the chapters have been structured in a specific way. Each begins with an introduction presenting the main takeaways and is followed by subsections that are dedicated to substantiating each of the claims made in the introduction. We therefore recommend that readers who are interested in the overarching narrative to first go through all of the introductory sections and finish with the conclusion before reading the proofs.

Physics readers who are mainly interested in the methods developed to derive the average behaviour of the system should focus on Chapters 3 and 4, starting with Chapter 2 if they are unfamiliar with nonequilibrium response theory. In Chapter 3, we recommend prioritising the simulations in Section 3.3 and skimming Sections 3.1 and 3.2 for notation and definitions. In Chapter 4, we recommend the

derivations in Section 4.2 leading to the solution of the annealed system. They can then finish with the introduction to Chapter 6, which provides a review of the calculations in the perturbation step.

The mathematics reader who primarily wants to focus on the probabilistic approach to the random Poisson equation should start with the introduction to Chapter 2 for physical context. They can then move to Chapter 3.1 and the introduction of Chapter 4 to review the derivation of the annealed baseline. For the perturbative step, they should read the introduction of Chapter 6 up to Theorem 6.2, which explains why the score bin sizes are key to the perturbative step. After that, they can read Chapter 5 for a detailed treatment of the random tournament graph and conclude with the proofs of Chapter 6.

Chapter 2: Nonequilibrium Thermodynamics (Physics) The opening chapter gives a review of nonequilibrium thermodynamics. We start off with a characterisation of equilibrium and we explain its equivalence to time-reversibility in continuous-time Markov chains. After this, the effects of breaking time-reversibility are explored to illustrate where nonequilibrium theory diverges. We then formally introduce the concept of local detailed balance for nonequilibrium systems, and show that it allows for the construction of a dynamical ensemble over paths. This allows us to formalise the concepts of frenesy, background dissipation and excess heat, ultimately deriving the nonequilibrium heat capacity. We subsequently present three illustrative examples that show nonequilibrium heat capacity in action.

Chapter 3: Heat Capacity of Tournaments (Maths + Physics) Chapter 3 introduces the tournament T_N and starts with proving some essential properties of this directed graph. We then define a continuous-time Markov process on the graph and prove that detailed balance is strictly broken for any $N \geq 3$. The chapter concludes with numerical simulations motivating the emergence of the $4C_2(\beta)/N$ limit that will guide the proofs in later sections.

Chapter 4: Annealed Solutions on Random Tournaments (Maths + Physics) In Chapter 4 we construct the annealed version of the Markov process on the tournament T_N . The annealed process restores detailed balance, which enables us to derive an exact analytical expression of its heat capacity. We show that this heat capacity exactly matches the empirical $4C_2(\beta)/N$ of the tournament model. Furthermore, we show that the transition rates of the microscopic model concentrate around the rates of the annealed process.

Chapter 5: Statistical Properties of Random Tournaments (Maths) Chapter 5 is dedicated to statistical properties of the random tournament itself. In particular, we focus on the score bins that count the number of vertices with a given score, defined as the difference between out-degree and in-degree. We prove a uniform concentration bound for this statistic and establish a local Central Limit Theorem. This allows us to substantially tighten the concentration results from Chapter 4 by proving a Central Limit Theorem for microscopic coarse-grained transition rates.

Chapter 6: A Perturbative Approach to Random Poisson Equations (Maths + Physics) Chapter 6 constructs a full perturbation theory for the Poisson equations. To do so, we first derive the exact entries of the pseudoinverse for the annealed generator in order, which allows us to explicitly calculate corrections. By treating the quenched randomness of the score bin sizes and restriction to the scores present in a realised tournament as a perturbation around the annealed model, we derive a convergent series expansion for the system's heat capacity. We use the expansion to extract the random leading-order correction.

Chapter 7: Conclusion The final chapter concludes the thesis by first summarising the central thread explaining the self-averaging behaviour of the heat capacity. We discuss the main results of the thesis and explain how they fit together. Following this, we outline several potential further research directions.

Frequently used notation

General notation

$\mathbb{P}(\cdot), \mathbb{E}[\cdot], \mathbb{V}(\cdot), \text{Cov}(\cdot, \cdot)$	Probability, expectation, variance, and covariance.
$\mathcal{N}(\mu, \tau^2)$	Normal distribution with mean μ and variance τ^2 .
$\text{Bin}(N, p)$	Binomial distribution with N trials and probability p .
$\mathbb{1}_{\{\cdot\}}$	The indicator function of an event.
$f(N) = O(g(N))$	Big-O: there exist an $M > 0$ and N_0 s.t. $f(N) \leq Mg(N)$ for $N > N_0$.
$f(N) = \Theta(g(N))$	Same order: $f(N) = O(g(N))$ and $g(N) = O(f(N))$.
$f(N) = o(g(N))$	Little-o: $f(N)/g(N) \rightarrow 0$ as $N \rightarrow \infty$.
$\langle f, g \rangle$	Inner product: $\sum_{x \in \mathcal{X}} f(x)g(x)$.
$\langle f \rangle_\rho$	Expectation with respect to a stationary measure ρ : $\sum_{x \in \mathcal{X}} \rho(x)f(x)$.

Tournament graph

K_N	The complete graph on N vertices.
$\mathcal{V}, E$	The vertex set $\{1, \dots, N\}$ and the edge set.
T_N	Tournament graph on N vertices: a directed complete graph.
σ_{ij}, σ	Edge direction (i, j) and tournament $\sigma = (\sigma_{ij})_{i,j \leq N}$ (3.3).
$s_i, \mathbf{s}$	Score of vertex i (3.4) and score vector $\mathbf{s} = (s_1, \dots, s_N)^T$.
Σ_{ij}, Σ	Random edge direction (i, j) and tournament $\Sigma = (\Sigma_{ij})_{i,j \leq N}$ (3.6).
$S_i, \mathbf{S}$	Random score induced by Σ and vector $\mathbf{S}$.
$\mathcal{R}$	Range of possible scores $\{-(N-1), -(N-3), \dots, N-1\}$ (3.5).
B_s	The score bin containing all vertices with score s (3.11).
N_s	The size of the score bin B_s .
$\mathcal{T}_{\mathbf{S}}$	Set of unique scores realised in a random tournament (3.10).

Processes on the tournament

β, ω	The inverse temperature and bias parameter.
$\mathcal{P}(i)$	The expected instantaneous power dissipated at state i (3.40).
$\mathcal{Q}(i)$	The excess dissipation at state i (2.70).
$C_N(\beta)$	Nonequilibrium heat capacity of the N -vertex random tournament.
L_N	Generator of the Markov chain on the tournament graph (3.36).
$k(i, j)$	The transition rate from state i to state j (3.35).
$\lambda_N(i, s')$	Coarse-grained transition rate from state i to bin $B_{s'}$ (4.17).
$\Lambda_N(i, s')$	The random transition rate from state i to score bin $B_{s'}$ induced by Σ .
ρ_N	The stationary distribution of L_N .
V_N	Quasipotential obtained by solving the Poisson equation for L_N .
$\overline{G}_N, G_N$	The annealed (4.44) and auxiliary (6.79) generators.
$\overline{\pi}_N, \pi_N$	Stationary distributions of $\overline{G}_N$ and G_N , respectively.
$\overline{W}_N, W_N$	The quasipotentials for $\overline{G}_N$ and G_N .
$\overline{C}_N(\beta), \tilde{C}_N(\beta)$	The heat capacities for $\overline{G}_N$ and G_N .

2. Nonequilibrium thermodynamics

In this section, we provide a motivation for the theoretical framework underpinning the heat capacity for systems driven out of equilibrium.

Theorem 2.1. *For a continuous-time Markov process on state space $\mathcal{X}$ with rates $k(x, y)$, the expected instantaneous power is given by*

$$\mathcal{P}(x) = \sum_{y \neq x} k(x, y) \frac{1}{\beta} \log \frac{k(x, y)}{k(y, x)}. \quad (2.1)$$

The quasipotential, representing the expected total excess heat along relaxation paths, is the unique zero-mean solution to the Poisson equation

$$LV = -Q \quad \text{with} \quad Q = \mathcal{P} - \langle \mathcal{P} \rangle_\rho \quad (2.2)$$

where L is the infinitesimal generator of the process and $\mathcal{P}$ the expected instantaneous power (2.64). Consequently, the nonequilibrium heat capacity is given by

$$C(\beta) = -\beta^2 \frac{\partial \langle E \rangle_\rho}{\partial \beta} + \beta^2 \left\langle \frac{\partial V}{\partial \beta} \right\rangle_\rho. \quad (2.3)$$

We begin in Section 2.1 by arguing that equilibrium statistical mechanics relies on an explicit connection between the energy landscape of a system and a static probability distribution over its microscopic configurations. To that end, we review the Gibbs variational principle as a large deviation property relating the large deviation rate and free energy. Section 2.2 then introduces continuous-time Markov chains as a model for the evolution of a system over its microstates. We explicitly define time-reversibility as the condition under which a CTMC describes a thermodynamic equilibrium. In that case, the stationary distribution is directly related to the energy landscape of the system.

In Section 2.3, we rely on the equilibrium connection to derive the fluctuation-dissipation relation. This gives us the standard heat capacity and highlights what is lost when detailed balance is violated. An example in Section 2.4 shows that in the absence of detailed balance, there is no connection between the stationary distribution and the energy landscape. To recover a thermodynamic description, we introduce the assumption of local detailed balance. This ensures entropy production remains well-defined along individual trajectories and allows us to shift our analysis from states to paths.

Section 2.5 follows this chain of thought by constructing the dynamical ensemble. We show that by changing from measure on states to a measure on paths, we recover a consistent response theory. Finally, in Section 2.6, we apply the dynamic response theory to derive the heat capacity formula presented in Theorem 2.1. Several examples are shown in Section 2.7.

2.1 What is equilibrium?

Consider a system on a finite state space $\mathcal{X}$. In the canonical ensemble, a system of fixed volume is in thermal equilibrium with a reservoir at inverse temperature β and number of particles N . The probability of finding the system in a certain configuration $x \in \mathcal{X}$ is then given by the Gibbs distribution

$$\rho(x) = \frac{1}{Z(\beta)} e^{-\beta E(x)}, \quad (2.4)$$

where $E : \mathcal{X} \rightarrow \mathbb{R}$ is the energy function and $Z(\beta) = \sum_{x \in \mathcal{X}} e^{-\beta E(x)}$ the partition function [82]. This result follows from the Gibbs variational principle.

Remark 2.2 (The Gibbs variational principle [83]). The variational principle sets out to find the equilibrium distribution ρ by minimising the free energy functional

$$\mathcal{F}[\mu] = \sum_{x \in \mathcal{X}} \mu(x) \left(E(x) + \frac{1}{\beta} \log \mu(x) \right). \quad (2.5)$$

over probability measures $\mu : \mathcal{X} \rightarrow [0, 1]$. In doing so, it states two things at once. The equilibrium distribution is the unique measure that minimises $\mathcal{F}$ and the minimum value attained at $\mathcal{F}[\rho]$ is the free energy of the system. $\circ$

The reason for minimising the free energy is grounded in the theory of large deviations [84, 85]. Ellis [85] gives an intuitive motivation for this by considering N independent coin tosses $(X_i)_{i \leq N} \sim \text{Ber}(1/2)$. If we set $S_N = \sum_{i=1}^N X_i$ so that the sample mean is S_N/N , the Law of Large Numbers implies that $S_N/N \rightarrow \mathbb{E}[X_i]$. However, a stronger result comes from Cramér's Theorem, which says that for large N

$$\mathbb{P}(|S_N/N - z| < \varepsilon) \asymp \exp(-NI(z)). \quad (2.6)$$

The rate function $I(z)$ for $X_i \sim \text{Ber}(1/2)$ is given by

$$I(z) = z \log 2z + (1 - z) \log 2(1 - z), \quad (2.7)$$

which is minimised at $z = 1/2$. For big systems, it is thus exponentially unlikely to encounter a sequence of coin tosses for which the sample mean does not minimise the rate function.

The Gibbs variational principle is an exact analogue of this situation. However, instead of minimising a rate function $I(z)$ over values, it minimises a rate functional $I[\mu]$ over measures. The parallel to Cramér's Theorem in that case is Sanov's Theorem [86].

The special property of equilibrium is that it is the unique situation in which the rate function to be minimised corresponds to the thermodynamic free energy [84]. In equilibrium statistical mechanics, the thermodynamic properties of a system are thus completely captured by a static probability distribution over its microstates. The specific trajectory the system takes to explore the space of microstates is insignificant.

2.2 Markov processes and equilibrium

The previous section motivates us to say that statics suffice in equilibrium. To expand on this claim and move towards nonequilibrium, we add dynamics on the state space. The type of evolution over microstates we will use throughout the rest of the thesis is a continuous time Markov chain. These processes have a countable set $\mathcal{X}$ as state space and infinitesimal time evolution.

Definition 2.3 (Continuous-time Markov chains (CTMC) [50]). *A time-homogeneous continuous-time Markov chain is a stochastic process $(X_t)_{t \geq 0}$ taking values in a countable state space $\mathcal{X}$ and satisfying the Markov property*

$$\mathbb{P}(X_t = y \mid \sigma(X_u : u \leq s)) = \mathbb{P}(X_t = y \mid X_s = x), \quad s \leq t. \quad (2.8)$$

Once the present state is known, the future is independent of the past. The evolution of a time-homogeneous CTMC is described by a family of operators $(P_t)_{t \geq 0}$ acting on bounded functions $f : \mathcal{X} \rightarrow \mathbb{R}$ via

$$P_t f(x) = \mathbb{E}_x[f(X_t)], \quad (2.9)$$

which play the role of transition matrices in continuous time. They satisfy the semigroup property

$$P_0 = I, \quad P_{s+t} = P_s P_t. \quad (2.10)$$

The infinitesimal generator L is defined by the derivative at the origin,

$$(Lf)(x) = \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t}. \quad (2.11)$$

For a jump process with rates $k(x, y)$ on a finite state space $\mathcal{X}$ it acts on functions $f : \mathcal{X} \rightarrow \mathbb{R}$ as

$$Lf(x) = \sum_{y \in \mathcal{X}} k(x, y)(f(y) - f(x)). \quad (2.12)$$

It can thus be represented as a matrix L_{xy} where the off-diagonal entries are the jump rates $k(x, y)$ and the diagonal entries are given by the negative escape rate $r(x)$ from state x

$$L_{xy} = k(x, y), \quad L_{xx} = -r(x), \quad r(x) := \sum_{y \neq x} k(x, y), \quad (2.13)$$

The waiting time before the next jump from x is exponential with rate $r(x)$, and the destination is y with probability $k(x, y)/r(x)$. The paths of a CTMC are therefore piecewise constant and right-continuous (càdlàg). A probability measure ρ on S is stationary if

$$\rho P_t = \rho \quad \text{for all } t \geq 0, \quad \text{equivalently } \rho L = 0. \quad (2.14)$$

When the chain is irreducible and positive recurrent, such a ρ exists, is unique, and satisfies $\rho(x) > 0$ for all x .

Note that assuming that the dynamics satisfy the Markov property is only possible if the time scales are well separated in the physical system that is modelled. In other words, the memory of the environment must be much shorter than the time between transitions. An example where this is the case would be a massive particle moving randomly in a fluid, where the fluid molecules around the particle relax long before the massive particle moves again [87].

Although any recurring irreducible CTMC reaches a stationary distribution ρ , stationarity does not ensure thermodynamic equilibrium. Thermodynamic equilibrium requires the stronger condition of time-reversibility. In that case, the stationary measure ρ satisfies the detailed balance condition

$$\rho(x)k(x, y) = \rho(y)k(y, x) \quad \forall x, y \in \mathcal{X}, \quad (2.15)$$

which couples the dynamics to the static probability mass.

We can physically parameterise the transition rates of a detailed balance process in Arrhenius form as

$$k(x, y) = \Gamma_\beta(x, y) \exp\left(-\frac{\beta}{2}[E(y) - E(x)]\right) \quad (2.16)$$

where $\Gamma_\beta(x, y)$ is any symmetric function possibly depending on β . From (2.15) it then follows that any reversible CTMC is completely determined by the energy landscape $E(x)$, since the stationary measure is directly given by the Gibbs distribution

$$\rho(x) \propto e^{-\beta E(x)}. \quad (2.17)$$

A rate functional $I[\mu]$ for the large deviation of the empirically observed measure from the stationary measure ρ can also be constructed in the case of a CTMC using Donsker-Varadhan theory [86]. The proof that a time-reversible CTMC is in equilibrium then shows that $I[\mu]$ simplifies to the free energy if and only if detailed balance (2.15) holds [88]. When detailed balance is broken, the stationary measure ρ still exists, but is no longer related to an energy landscape. As a result, the rate functional $I[\mu]$ decouples from the thermodynamic potential.

2.3 Equilibrium response: fluctuation-dissipation

Because of the added dynamics, we can derive the thermal response of a CTMC in equilibrium. The heat capacity C is the response of the system to a change in temperature while holding external parameters constant.

To that end, consider a time dependent protocol $\beta(\varepsilon t)$ for some $\varepsilon > 0$, so that the inverse temperature is varied. This results in time dependent transition rates

$$k(t; x, y) = \exp\left(\frac{\beta(t)}{2} [E(x) - E(y)]\right). \quad (2.18)$$

We define quasistatic as the limit $\varepsilon \rightarrow 0$. This allows us to assume that at any time t , the chain has reached stationarity long before the next change. The distribution over the states is then given by

$$\rho(t; x) = e^{-\beta(t)E(x)} / Z(t). \quad (2.19)$$

The First Law of Thermodynamics identifies the heat exchange δQ as the change in internal energy minus the work done on the system

$$\delta Q = d\langle E \rangle - \sum_{x \in \mathcal{X}} \rho(x) dE(x) \quad (2.20)$$

see [2]. In our case, the energy landscape is fixed, so there is only a change in internal energy due to the temperature difference. Therefore, we have the infinitesimal relation

$$\delta Q = d\langle E \rangle \quad (2.21)$$

This then allows us to derive a heat capacity for systems in detailed balance. The heat capacity is defined as

$$C(\beta) = \frac{\delta Q}{dT}, \quad (2.22)$$

Here, we can use (2.21) to see that

$$C(\beta) = -\beta^2 \frac{\partial \langle E \rangle}{\partial \beta}. \quad (2.23)$$

Now, the equilibrium assumption is used. Namely, for a reversible process, the average energy is related to the partition function through $\langle E \rangle = -\partial \log Z / \partial \beta$. This relation then yields

$$\begin{aligned} \frac{\partial \langle E \rangle}{\partial \beta} &= \frac{1}{Z^2} \left[\sum_x E(x) e^{-\beta E(x)} \right]^2 - \frac{1}{Z} \sum_x E(x)^2 e^{-\beta E(x)} \\ &= \langle E \rangle^2 - \langle E^2 \rangle, \end{aligned} \quad (2.24)$$

which means that for reversible Markov chains we end up with the fluctuation-dissipation relation

$$C(\beta) = \beta^2 (\langle E^2 \rangle - \langle E \rangle^2). \quad (2.25)$$

It tells us that the heat capacity for a detailed balance Markov process is directly proportional to the variance in energy with respect to the stationary distribution. Because detailed balance ensures that the probability of any fluctuation from equilibrium is equal to a relaxation back into it, the system does not care in which direction it was kicked. Exactly this is why the system's response is governed entirely by the static weights of the microstates as given by the energy landscape.

Example 2.4 (Two-level system). We define a general two-level system between two states 1, 2, coupled to a heat bath at inverse temperature β , by setting

$$k_{\pm}(\beta) = e^{\pm \beta \omega / 2}, \quad k(1, 2) = k_+(\beta) \quad \text{and} \quad k(2, 1) = k_-(\beta), \quad (2.26)$$



Figure 2.1: A two-level system. The black arrow shows the preferred direction and the coloured arrows denote the transition rates of the temperature dependent process.

where $\omega > 0$ denotes the energy gap. As such, $E(1) = -\omega/2$ and $E(2) = \omega/2$. The setup is shown in Figure 2.1. Checking the detailed balance condition gives that

$$\frac{\rho(2)}{\rho(1)} = e^{\beta\omega}, \quad (2.27)$$

which means that the system has stationary distribution

$$\rho(1) = \frac{e^{-\beta\omega/2}}{e^{\beta\omega/2} + e^{-\beta\omega/2}} \quad \text{and} \quad \rho(2) = \frac{e^{\beta\omega/2}}{e^{\beta\omega/2} + e^{-\beta\omega/2}}, \quad (2.28)$$

which matches the Gibbs distribution (2.17). We can subsequently calculate the expected values

$$\langle E \rangle = -\frac{\omega}{2} \tanh \frac{\beta\omega}{2} \quad \text{and} \quad \langle E^2 \rangle = \frac{\omega^2}{4}. \quad (2.29)$$

The heat capacity (2.25) is then given by

$$C_2(\beta) = \frac{(\beta\omega)^2}{4} \left[1 - \tanh^2 \frac{\beta\omega}{2} \right] = \frac{(\beta\omega)^2}{4} \text{sech}^2 \frac{\beta\omega}{2} \quad (2.30)$$

for the two-level system. The result is plotted in Figure 2.2b. In the resulting plot, we observe a distinct peak known as the Schottky anomaly [5]. It shows the location of the maximum energy fluctuations, marking the temperature where the stationary distribution is most sensitive to temperature changes in the environment. $\triangle$

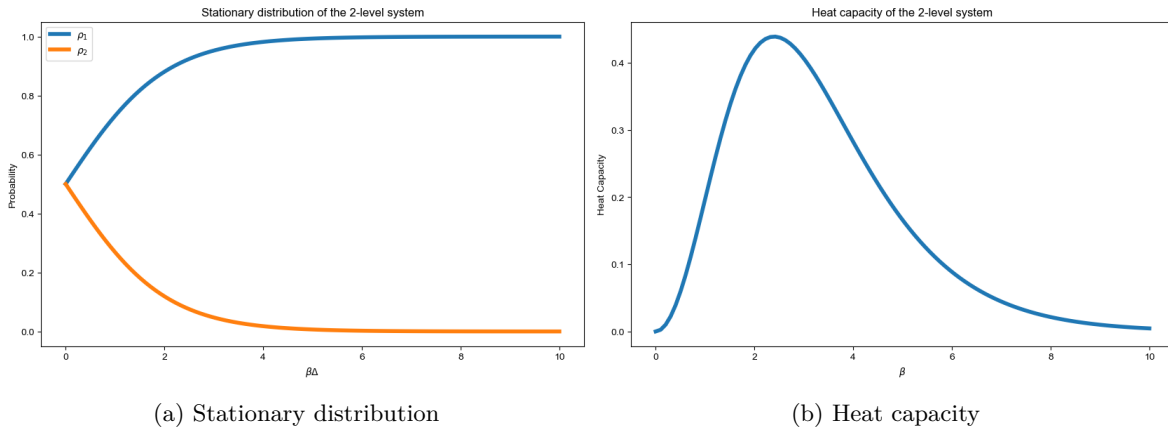


Figure 2.2: (b) The heat capacity of the two-level system (2.30) for $\beta \in [0, 10]$ and $\omega = 1$ showing a Schottky peak.

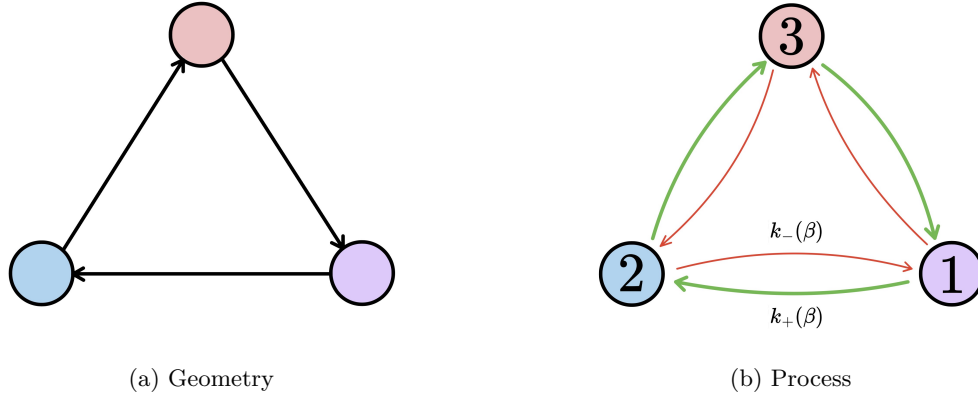


Figure 2.3: A 3-state cycle oriented in the clockwise direction. The black arrows show the preferred direction and the coloured arrows denote the transition rates of the temperature dependent process.

2.4 Stationary nonequilibrium and local detailed balance

The obvious next frontier is to expand to processes that do not satisfy (2.15). Under the assumptions in Definition 2.3, we know that these processes still reach a steady-state. Only in this situation, the stationary measure is not directly useful as there is no straightforward link between the energy landscape probabilities. The stationary measure ρ becomes a function that is globally dependent on all transition rates, involving spanning trees of the graph via the Matrix Tree Theorem [89].

Example 2.5 (3-state cycle). Consider a continuous-time Markov chain on three states arranged in a cycle shown in Figure 2.3 with rates

$$k(1, 2) = k(2, 3) = k(3, 1) = k_+(\beta) \quad \text{and} \quad k(2, 1) = k(3, 2) = k(1, 3) = k_-(\beta), \quad (2.31)$$

where $k_{\pm}(\beta)$ is given by (2.26). This setup creates a preferred clockwise direction when $\omega > 0$. Solving the global balance equations yields the stationary distribution

$$\rho_1 = \rho_2 = \rho_3 = \frac{1}{3} \quad (2.32)$$

However, if one incorrectly assumes detailed balance holds and attempts to assign energies E_1, E_2, E_3 such that $\rho_i \propto e^{-\beta E_i}$, a contradiction arises. The detailed balance condition $\frac{k(i, j)}{k(j, i)} = e^{-\beta(E_j - E_i)}$ would require

$$E_2 - E_1 = -\omega, \quad E_3 - E_2 = -\omega, \quad E_1 - E_3 = -\omega \quad (2.33)$$

Summing these gives $0 = -3\omega$, which forces $\omega = 0$. For $\omega \neq 0$, no consistent energy landscape exists, and the naive Gibbs distribution $\rho_i \propto e^{-\beta E_i}$ fails. Physically, the transition rates purely resemble a driving force. $\triangle$

The previous example demonstrates that we can no longer take ω to be the energy difference between the states if detailed balance is broken. Otherwise, the system would correspond to the famous Penrose staircase in Figure 2.4. However, physical systems resembling the previous example exist in nature in the form of driven systems. For instance, if we couple the system to three independent reservoirs and associate each with one of the specific transitions, the system resembles a microscopic motor. This situation is sketched in Figure 2.5. The first reservoir at inverse temperature β_H is hot and allows the $a \rightarrow b$ transition to absorb heat. The reservoir in between $b \rightarrow c$ allows it to perform work at β_W . The final cold reservoir at β_C is in between $c \rightarrow a$ and allows the system to relax back into the ground

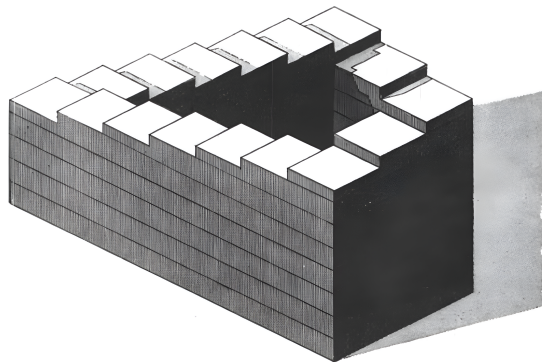


Figure 2.4: The Penrose staircase, an impossible 4-state cycle of ascending steps that can be climbed forever. Adapted from [90].

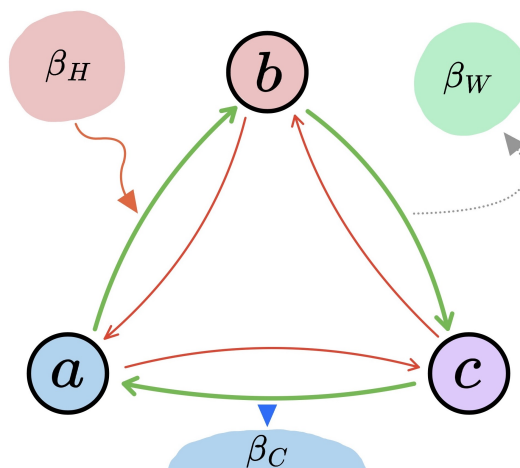


Figure 2.5: A schematic of a molecular motor. Arrows indicate transition rates between states. Heat is absorbed from the hot reservoir ($a \rightarrow b$), converted to work ($b \rightarrow c$), and finally dumped into the cold reservoir ($c \rightarrow a$) as the system returns to the ground state. A clockwise current is maintained by the temperature gradient $\beta_H < \beta_W < \beta_C$.

state while dumping leftover heat. Importantly, detailed balance is broken if the temperatures differ between the reservoirs $\beta_H < \beta_W < \beta_C$. In that case, there is a nonzero current flowing powering the motor in steady-state.

There are several underlying assumptions we need to thermodynamically describe the molecular motor. The most obvious one being that the reservoirs are well-separated. Drawing from the hot reservoir at β_H , should not influence any of the reservoirs. Additionally, the reservoirs should remain in equilibrium at constant temperature, which allows them to be treated using equilibrium thermodynamics. This means that their relaxation time is fast with respect to the interaction. Finally, the local interactions between the system and a reservoir have to be reversible, so that the entropy of the environment is well defined. If these conditions are satisfied, a constant dissipation can be achieved while each reservoir maintains its equilibrium state.

This combination of assumptions is called local detailed balance [91]. It allows us to identify the ratio of transition rates with a change in entropy in the environment. Given the above assumptions, it can be derived that for each single transition $x \rightarrow y$ in a trajectory

$$\log \frac{k(x, y)}{k(y, x)} = \Delta S_{\text{env}}(x \rightarrow y). \quad (2.34)$$

For one of our heat reservoirs in the example, we use the microscopic heat dissipated by the transition $x \rightarrow y$. That is, $\Delta S_{\text{env}} = q_\beta(x, y)/T$.

Example 2.6 (3-state cycle: steady-state dissipation). By (2.34), the heat exchanged with the reservoir during a transition $i \rightarrow j$ is

$$q_\beta(i, j) = \frac{1}{\beta} \log \frac{k(i, j)}{k(j, i)}. \quad (2.35)$$

Using the expressions for the transition rates $k_\pm(\beta)$ in (2.26), we find

$$q_\beta(1, 2) = \omega, \quad q_\beta(2, 3) = \omega, \quad q_\beta(3, 1) = \omega. \quad (2.36)$$

The average heat flux $\langle \dot{Q} \rangle$ is then given by the sum

$$\langle \dot{Q} \rangle = \sum_{i=1}^3 \sum_{j=1}^3 \rho_i k(i, j) q_\beta(i, j). \quad (2.37)$$

In the 3-state cycle, where $\rho_i = 1/3$ for all i , the average heat flux

$$\langle \dot{Q} \rangle = \omega e^{\beta\omega/2} - \omega e^{-\beta\omega/2}, \quad (2.38)$$

which is positive for $\omega > 0$. Even though the state probabilities are constant, the system constantly exchanges heat with the environment. $\triangle$

Local detailed balance is not an inherent property of the system that we can deduce from the graph. Rather, it is the physical requirement for a well-interpretable model that includes nonequilibrium driving induced by coupling to equilibrium reservoirs.

In conclusion, state probabilities and transition rates are no longer derivable from a single global potential for systems in which detailed balance is broken. The challenge to overcome is that this property enabled the derivation of thermodynamical relations from static probabilities. The connection between thermodynamics and static probabilities cannot be repaired, but local detailed balance at least allows us to do pathwise entropy bookkeeping in the equilibrium reservoirs connected to the nonequilibrium system.

This means that to understand the physics of systems violating detailed balance, we stop relying on static states. Instead, we start weighing the probabilities of the paths it took over time.

2.5 Nonequilibrium response: dynamical ensembles

The goal now is to find a way to compare the probabilities of observing the system following a specific path. In other words, we need to define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over all possible paths. This is what we will refer to as the dynamical ensemble.

Although dynamical ensembles can be constructed for other random processes [92], we restrict ourselves to the continuous-time Markov processes in Definition 2.3 that we have been working with so far.

Definition 2.7 (Dynamical ensemble for CTMCs). *Because the paths that $(X_t)_{t \in [0, T]}$ takes are càdlàg, we first set*

$$\Omega = \{\omega : [0, T] \rightarrow \mathcal{X} \mid \omega \text{ càdlàg}\}. \quad (2.39)$$

The corresponding σ -algebra of measurable sets is then the one that is generated by setting $\omega(t) = X_t(\omega)$. That is,

$$\mathcal{F} = \sigma(\{X_t = x\} : t \in [0, T], x \in \mathcal{X}). \quad (2.40)$$

The correct probability distribution on this space is now given through the semigroup property (2.10) for the process $(X_t)_{t \in [0, T]}$.

Take a path ω on $[0, T]$ with exactly N jumps, visiting the states $(x_1, x_2, \dots, x_{N+1})$. For a CTMC with rates $k(x, y)$, the time spent waiting at x before the next jump is exponentially distributed with the escape rate $r(x) = \sum_y k(x, y)$. The joint probability density of waiting a period of τ_i and then transitioning to x_{i+1} is thus given by

$$p_{\tau_i}(x_i, x_{i+1}) = k(x_i, x_{i+1})e^{-r(x_i)\tau_i}. \quad (2.41)$$

There is no jump after reaching the final state x_{N+1} , which has a probability

$$p_{\tau_{N+1}}(x_{N+1}, x_{N+1}) = e^{-r(x_{N+1})\tau_{N+1}}, \quad (2.42)$$

where $\tau_{N+1} = T - \sum_{i=1}^N \tau_i$.

Given an initial distribution μ_0 and a fixed sequence of N jumps $(x_1, x_2, \dots, x_{N+1})$, we can then write down the probability density of a path ω with respect to the Lebesgue measure on the holding times as

$$d\mathbb{P}(\omega) = \mu_0(x_1) \left[\prod_{i=1}^N k(x_i, x_{i+1})e^{-r(x_i)\tau_i} \right] e^{-r(x_{N+1})\tau_{N+1}} d\tau_1 \dots d\tau_N. \quad (2.43)$$

We can simplify the expression further by taking the transition rates into the exponent, which gives

$$d\mathbb{P}(\omega) = \mu_0(x_1) \exp \left(\sum_{i=1}^N \log k(x_i, x_{i+1}) - \sum_{i=1}^{N+1} r(x_i)\tau_i \right) d\tau_1 \dots d\tau_N. \quad (2.44)$$

We then define the action functional of a path $\omega = (x_t, t \in [0, T])$ as

$$\mathcal{A}[\omega] = \sum_{i=1}^{N+1} r(x_i)\tau_i - \sum_{i=1}^N \log k(x_i, x_{i+1}), \quad (2.45)$$

which allows us to write the path density in the compact form

$$d\mathbb{P}(\omega) = \mu_0(x_1)e^{-\mathcal{A}[\omega]} d\tau_1 \dots d\tau_N \quad (2.46)$$

We use the density to reproduce the derivation of a response formula in [93]. To find the expectation of a path-dependent observable $\mathcal{O}(\omega)$, we must integrate over the entire pathspace Ω . We will denote

the sum over the number of jumps N , sum over all possible sequences $\mathbf{x} \in \mathcal{X}^N$ and integral over the holding times by

$$\int_{\Omega} \mathbf{D}[\omega] = \sum_{N=0}^{\infty} \sum_{\mathbf{x} \in \mathcal{X}^N} \int_0^T d\tau_1 \int_0^{T-\tau_1} d\tau_2 \dots \int_0^{T-\sum_{i=1}^N \tau_i} d\tau_N. \quad (2.47)$$

Using (2.46), the expectation of $\mathcal{O}(\omega)$ looks like

$$\langle \mathcal{O}(\omega) \rangle = \int_{\Omega} \mathcal{O}(\omega) \mu_0(x_1) e^{-\mathcal{A}[\omega]} \mathbf{D}[\omega]. \quad (2.48)$$

Taking the derivative with respect to a parameter α for an integrable observable $\mathcal{O}(\omega)$ leads to a general linear response formula

$$\begin{aligned} \frac{d}{d\alpha} \langle \mathcal{O}(\omega) \rangle &= \frac{d}{d\alpha} \int_{\Omega} \mathcal{O}(\omega) \mu_0(x_1) e^{-\mathcal{A}_{\alpha}[\omega]} \mathbf{D}[\omega] \\ &= \int_{\Omega} \mathcal{O}(\omega) \mu_0(x_1) \frac{d}{d\alpha} e^{-\mathcal{A}_{\alpha}[\omega]} \mathbf{D}[\omega] \\ &= - \int_{\Omega} \mathcal{O}(\omega) \frac{d}{d\alpha} \mathcal{A}_{\alpha}[\omega] \mu_0(x_1) e^{-\mathcal{A}_{\alpha}[\omega]} \mathbf{D}[\omega] \\ &= - \left\langle \mathcal{O}(\omega) \frac{d}{d\alpha} \mathcal{A}_{\alpha}[\omega] \right\rangle, \end{aligned} \quad (2.49)$$

where in the second line we assumed that the transition rates are uniformly bounded and continuously differentiable in α in order to apply the dominated convergence theorem.

This is an operational expression, but it pays to decompose $\mathcal{A}(\omega)$ into its time symmetric and antisymmetric part. Let θ be the time-reversal operator, so that $\theta\omega = (x_{T-t}, t \in [0, T])$. Then using (2.45), we obtain

$$\mathcal{A}(\theta\omega) - \mathcal{A}(\omega) = \sum_{i=1}^N \log \frac{k(x_i, x_{i+1})}{k(x_{i+1}, x_i)}. \quad (2.50)$$

Here, we can apply the local detailed balance condition 2.34, to find that

$$\begin{aligned} \mathcal{A}(\theta\omega) - \mathcal{A}(\omega) &= \sum_{i=1}^N \Delta S_{\text{env}}(x_i \rightarrow x_{i+1}) \\ &= \Delta S_{\text{env}}(\omega). \end{aligned} \quad (2.51)$$

With this in mind, we decompose $\mathcal{A}$ as

$$\begin{aligned} \mathcal{A}(\omega) &= \frac{\mathcal{A}(\theta\omega) + \mathcal{A}(\omega)}{2} - \frac{\mathcal{A}(\theta\omega) - \mathcal{A}(\omega)}{2} \\ &= \mathcal{D}(\omega) - \frac{\mathcal{S}(\omega)}{2}, \end{aligned} \quad (2.52)$$

after which (2.49) takes the form

$$\frac{d}{d\alpha} \langle \mathcal{O}(\omega) \rangle = \frac{1}{2} \left\langle \mathcal{O}(\omega) \frac{d}{d\alpha} \mathcal{S}(\omega) \right\rangle - \left\langle \mathcal{O}(\omega) \frac{d}{d\alpha} \mathcal{D}(\omega) \right\rangle. \quad (2.53)$$

The first term is related to the path's entropy by (2.51), while the second term is something new that shows up in nonequilibrium. It is called the frenetic contribution. If we look at its definition, we find that it is directly related to the escape rate of the states along ω and the symmetric part of the transition rates

$$\mathcal{D}(\omega) = \sum_{i=1}^{N+1} r(x_i) \tau_i - \frac{1}{2} \sum_{i=1}^N \log [k(x_i, x_{i+1}) k(x_{i+1}, x_i)], \quad (2.54)$$

which means that it tells us how easy it is for a system to move along this path at all. Large $\mathcal{D}$ corresponds to a path with many quick transitions and vice versa.

In nonequilibrium steady states, the violation of detailed balance brings about probability currents. Suppose that along the path of such a current, the system encounters a configuration with a very low escape rate. The system will spend much of its time trapped there simply because it cannot easily leave, but its arrival rate cannot compensate for this because of the current. The system could thus end up spending a lot of time in a state that is energetically unfavourable. In this case, variations in $\mathcal{D}$ unrelated to an energy landscape can modify the response.

In equilibrium, however, this information plays no role. Because (2.15) is fulfilled, any slow escape rate from a state is exactly compensated for by an equally slow arrival rate. Changing the escape rates therefore does not change the relative occupation probabilities. This leaves the entropic term $\mathcal{S}$ as the only influence.

To make the distinction precise, we can look at the relation between the average changes of the two parts under a small perturbation. Taking $\mathcal{O} = 1$, we see that (2.53) implies

$$\frac{1}{2} \left\langle \frac{d}{d\alpha} \mathcal{S}(\omega) \right\rangle = \left\langle \frac{d}{d\alpha} \mathcal{D}(\omega) \right\rangle \quad (2.55)$$

for any initial condition. If the system is in equilibrium and prepared in the stationary distribution, there is another relation courtesy of time reversal invariance that clarifies the point made before. Taking an observable that depends on the state of the system at final time t after a parameter change, $\mathcal{O}(\omega) = \mathcal{O}(x_t)$, the following holds for a detailed balance process

$$\begin{aligned} \left\langle \mathcal{O}(x_t) \frac{d}{d\alpha} \mathcal{D}(\omega) \right\rangle &= \left\langle \mathcal{O}(x_0) \frac{d}{d\alpha} \mathcal{D}(\omega) \right\rangle \\ &= \frac{1}{2} \left\langle \mathcal{O}(x_0) \frac{d}{d\alpha} \mathcal{S}(\omega) \right\rangle \\ &= -\frac{1}{2} \left\langle \mathcal{O}(x_t) \frac{d}{d\alpha} \mathcal{S}(\omega) \right\rangle. \end{aligned} \quad (2.56)$$

Here, we used time-reversal invariance to move from x_t to the initial state x_0 and then applied (2.55). If we now substitute this into (2.53), we find that in equilibrium

$$\frac{d}{d\alpha} \langle \mathcal{O}(x_t) \rangle = \left\langle \mathcal{O}(x_t) \frac{d}{d\alpha} \mathcal{S}(\omega) \right\rangle. \quad (2.57)$$

We can now use (2.57) to show that the dynamical ensemble is a generalisation of our earlier approach. If we apply (2.57) for the derivative with respect to β on (2.23), we find

$$C(\beta) = -\beta^2 \left\langle E(x_t) \frac{d}{d\beta} \mathcal{S}(\omega) \right\rangle \quad (2.58)$$

Note that by detailed balance for a path with initial condition x_0 and endpoint x_t , (2.51) reduces to

$$\mathcal{S}(\omega) = \Delta S_{\text{env}}(\omega) = \beta(E(x_0) - E(x_t)). \quad (2.59)$$

Applying (2.57), we obtain

$$\begin{aligned} \frac{d}{d\beta} \langle E(x_t) \rangle &= \langle E(x_t) [E(x_0) - E(x_t)] \rangle \\ &= \langle E(x_t) E(x_0) \rangle - \langle E(x_t) E(x_t) \rangle. \end{aligned} \quad (2.60)$$

The last term is simply $\langle E^2 \rangle$ and the second term reduces to $\langle E(x_t) \rangle \langle E(x_0) \rangle$ given the fact that our parameter change is quasistatic. In the quasistatic limit, correlations between times 0 and t vanish,

and the product factorises. This means that (2.60) reduces to

$$\frac{d}{d\beta}\langle E \rangle = \langle E \rangle^2 - \langle E^2 \rangle. \quad (2.61)$$

Which matches (2.24) exactly. The dynamical response framework therefore reproduces equilibrium thermodynamics as a special case.

2.6 Excess heat and nonequilibrium heat capacity

The final conceptual hurdle to defining a heat capacity in nonequilibrium concerns the presence of currents in steady state. In Example 2.6 we showed that nonequilibrium steady states under local detailed balance correspond to situations in which there is constant dissipation into the environment. As a result, the nonequilibrium system exchanges an infinite amount of heat over the infinite duration of a quasistatic parameter change. This background heat must be distinguished from the finite excess heat to extract the physically relevant response to a parameter change [94].

To be more specific, we follow [14] by writing down the expected rate of dissipation in a given state $x \in \mathcal{X}$ for a physical system satisfying the assumptions of local detailed balance. For a general continuous time Markov process on the state space $\mathcal{X}$, the rates can be parameterised by extending (2.16) into the form

$$k(x, y) = \Gamma_\beta(x, y) \exp\left(\frac{\beta}{2} [E(x) - E(y) + F(x, y)]\right), \quad (2.62)$$

where the non-conservative work function $F(x, y)$ drives the system and satisfies $F(x, y) = -F(y, x)$. Local detailed balance (2.34) tells us that the microscopic heat exchanged by the transition $x \rightarrow y$ is given by

$$q_\beta(x, y) = \frac{1}{\beta} \log \frac{k(x, y)}{k(y, x)}. \quad (2.63)$$

which can subsequently be used to calculate the expected instantaneous power given that the system is in state x from

$$\mathcal{P}(x) = \sum_{y \neq x} k(x, y) q_\beta(x, y). \quad (2.64)$$

The background rate of dissipation can then be calculated by taking the average with respect to the stationary distribution

$$\langle \mathcal{P} \rangle_\rho = \sum_{x \in \mathcal{X}} \mathcal{P}(x) \rho(x), \quad (2.65)$$

which in the absence of any parameter change means that the background heat scales as $t \langle \mathcal{P} \rangle_\rho$ in time and thus diverges as $t \rightarrow \infty$. This term must therefore be renormalised in order to possibly identify the transient response caused by relaxation into the new steady-state. This prompts the definition of the excess heat

$$Q^{\text{ex}}(\omega) = \int_0^\infty \left[\mathcal{P}(X_t(\omega)) - \langle \mathcal{P} \rangle_\rho \right] dt, \quad (2.66)$$

which is a path dependent random variable with $\langle Q^{\text{ex}} \rangle = 0$ if the system is stationary.

Because Q^{ex} is in the right form for (2.53), a response coefficient can be derived. In [14] it is shown that the heat capacity associated with this excess heat is given by

$$C(\beta) = \frac{d}{d\beta} \langle Q^{\text{ex}} \rangle = -\beta^2 \frac{\partial \langle E \rangle_\rho}{\partial \beta} + \beta^2 \left\langle \frac{\partial V}{\partial \beta} \right\rangle_\beta. \quad (2.67)$$

If we heat up a nonequilibrium system by $\partial\beta$, the excess heat we need to put into the system with respect to the background dissipation (2.65) is therefore determined by two terms. The first term is

exactly (2.23), but the second term does not have an equilibrium analogue. This term corresponds to the expected total excess heat released or absorbed over all paths starting from a state $x \in \mathcal{X}$

$$V(x) = \mathbb{E} \left[\int_0^\infty \mathcal{P}(X_t) - \langle P \rangle_\rho dt | X_0 = x \right]. \quad (2.68)$$

This shows that V acts similar to a potential, but on the path space. This motivates the coining of the term quasipotential for V . The quasipotential assesses how favourable the surrounding transitions are from state x rather than its position on the energy landscape. It therefore arises as a correction associated with the frenetic $\mathcal{D}$ term in (2.53). In that role, its derivative with respect to β measures a nonzero change in heat caused by shifts in how easily the process can move across certain paths, even when the energy landscape is unchanged.

While (2.68) provides an intuitive interpretation, it is often more convenient to obtain the quasipotential V as the solution of the Poisson equation

$$LV = \langle \mathcal{P} \rangle_\rho - \mathcal{P}. \quad (2.69)$$

This linear system is solved on the space perpendicular to stationary distribution. Here, L is the infinitesimal generator of the process $(X_t)_{t \geq 0}$. This follows from applying Dynkin's formula on (2.68) [95]. For shorthand notation, we will often write

$$\mathcal{Q}(x) := \mathcal{P}(x) - \langle \mathcal{P} \rangle_\rho, \quad (2.70)$$

which we refer to as the excess dissipation.

2.7 Nonequilibrium heat capacities: examples

Building on the previous section, we provide three examples of how the addition of a driving term to the transition rates leads to a nonequilibrium heat capacity. The first example builds on the 3-state cycle of Example 2.5. We introduce a kinetic barrier to the system to artificially clash with the driving force in the clockwise direction. Because the system lacks a conventional energy landscape, the resulting heat capacity is purely a consequence of this nonequilibrium bottleneck.

Example 2.8 (3-state cycle: high driving limit). Consider the 3-state cycle with a finite driving force $\omega > 0$ of Example 2.5. To model the effect of a physical obstruction, we add a kinetic barrier Δ on the $1 - 2$ transition. This scales the transition rate by a factor of $\epsilon = e^{-\beta\Delta}$. The setup is shown in Figure 2.6. The resulting generator in this situation is

$$L = \begin{pmatrix} -(e^{\beta\omega/2}\epsilon + e^{-\beta\omega/2}) & e^{\beta\omega/2}\epsilon & e^{-\beta\omega/2} \\ e^{-\beta\omega/2}\epsilon & -(e^{\beta\omega/2} + e^{-\beta\omega/2}\epsilon) & e^{\beta\omega/2} \\ e^{\beta\omega/2} & e^{-\beta\omega/2} & -(e^{\beta\omega/2} + e^{-\beta\omega/2}) \end{pmatrix}. \quad (2.71)$$

In the high-driving regime ($\beta\omega \gg 1$), the anticlockwise transitions are suppressed as $e^{-\beta\omega/2} \approx 0$. The rate matrix simplifies to

$$L = e^{\beta\omega/2} \begin{pmatrix} -\epsilon & \epsilon & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad (2.72)$$

Solving the global balance equations gives the corresponding stationary distribution

$$\rho_1 = \frac{1}{1+2\epsilon}, \quad \rho_2 = \rho_3 = \frac{\epsilon}{1+2\epsilon}, \quad (2.73)$$

from which we see that the system accumulates in state 1 as the barrier becomes infinitely large ($\epsilon \rightarrow 0$). This is illustrated in Figure 2.7a. Even though the energy landscape is flat, we saw in (2.36) that each clockwise jump dissipates heat ω because of the driving. The instantaneous power $\mathcal{P}(i)$ (3.40) for each state is

$$\mathcal{P}_1 = \frac{\omega e^{\beta\omega/2}}{2}(\epsilon - e^{-\beta\omega}), \quad \mathcal{P}_2 = \frac{\omega e^{\beta\omega/2}}{2}(1 - \epsilon e^{-\beta\omega/2}), \quad \mathcal{P}_3 = \frac{\omega e^{\beta\omega/2}}{2}(1 - e^{-\beta\omega}). \quad (2.74)$$

Neglecting the $e^{-2\beta\omega}$ terms in the high-driving regime, background dissipation (2.65) is

$$\langle \mathcal{P} \rangle_\rho = \frac{\omega}{2} \frac{3e^{\beta\omega/2}\epsilon}{1+2\epsilon}. \quad (2.75)$$

The quasipotential V is now determined by solving the corresponding Poisson equation (2.69). The exact matrix equation takes the form

$$\begin{pmatrix} -\epsilon & \epsilon & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \frac{\omega}{2} \frac{1-\epsilon}{1+2\epsilon} \begin{pmatrix} 2\epsilon \\ -1 \\ -1 \end{pmatrix}, \quad (2.76)$$

subject to the mean-zero condition. Solving this equation results in the following expressions for the quasipotential

$$V_1 = -\frac{\omega}{2} \frac{3\epsilon(1-\epsilon)}{(1+2\epsilon)^2}, \quad V_2 = \frac{\omega}{2} \frac{(1-\epsilon)(2+\epsilon)}{(1+2\epsilon)^2} \quad \text{and} \quad V_3 = \frac{\omega}{2} \frac{(1-\epsilon)^2}{(1+2\epsilon)^2}. \quad (2.77)$$

By substituting these terms into the formula for the nonequilibrium heat capacity (2.67) and writing out $\epsilon = e^{-\beta\Delta}$, we find

$$C(\beta) = \beta^2 \left\langle \frac{\partial V}{\partial \beta} \right\rangle_\rho = \frac{3\beta^2 \omega \Delta e^{-\beta\Delta} (1 - e^{-\beta\Delta})}{2(1 + 2e^{-\beta\Delta})^3} \quad (2.78)$$

for the driven 3-cycle in the high-driving limit. It is shown in Figure 2.7b. The heat capacity $C(\beta)$ now does not reflect a change in the energy landscape, but rather how easily the driven steady-state current overcomes the kinetic barrier as the temperature changes. The solution V to the Poisson equation (2.69) is exactly the quasipotential on the path space that quantifies this effect. $\triangle$

The previous example demonstrates that a nonequilibrium heat capacity can arise from the interaction between a driving force and an explicit kinetic barrier. However, such a barrier is not strictly necessary to create a bottleneck. Importantly, the geometry of the graph itself can force movement against the preferred nonequilibrium current. This is also what happens in the random tournaments that we will study in the following sections. Structural frustration in the graph is enough to produce a nonzero heat capacity.

Example 2.9 (Frustrated 3-state system). Consider the frustrated version of the 3-state system without an energy landscape visualised in Figure 2.8, so that the forward rates are

$$k(1,2) = k(2,3) = k(1,3) = k_+(\beta) \quad (2.79)$$

and the backward transition rates are

$$k(2,1) = k(3,2) = k(3,1) = k_-(\beta), \quad (2.80)$$

with $k_\pm(\beta)$ as given by (2.26). In this situation, the vertex 3 has an in-degree of 2. As a result, the process has a preferred direction towards this vertex. Figure 2.9a, which plots the stationary distribution, shows this funneling effect.

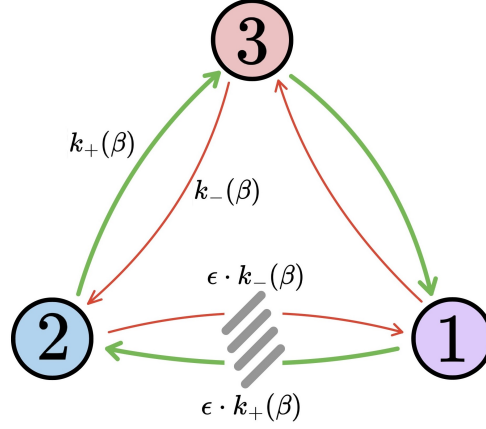


Figure 2.6: The 3-state cycle oriented in the clockwise direction with a kinetic barrier between states 1 and 2 modifying the transition rate with a factor ϵ .

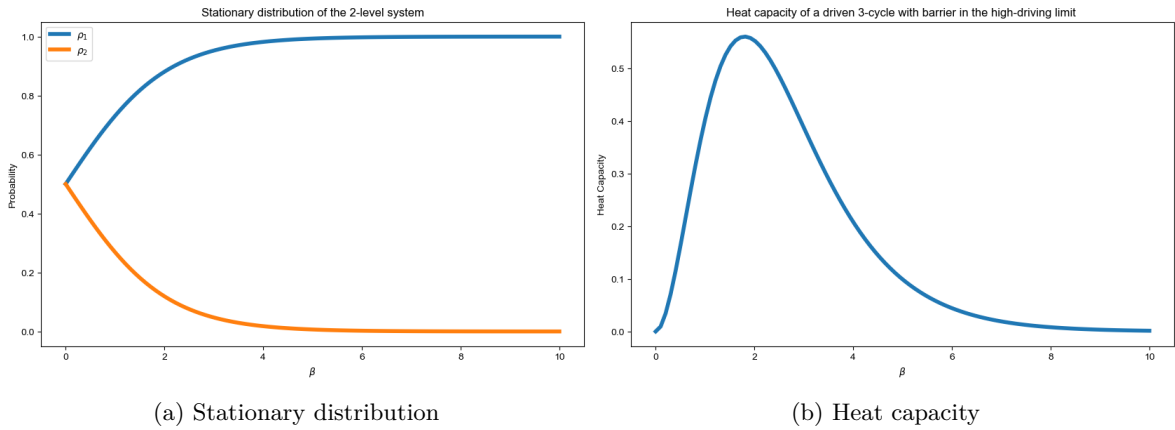


Figure 2.7: The driven 3-state cycle in the high-driving limit evaluated against inverse temperature β . (a) The stationary distribution for $\beta \in [0, 10]$, illustrating the accumulation of the system in state 1 as the transition across the barrier Δ is exponentially suppressed. (b) Nonequilibrium heat capacity $C(\beta)$ of the driven 3-state cycle in the high-driving limit ($\beta\omega \gg 1$), plotted as a function of inverse temperature $\beta \in [0, 10]$. Because there is no energy landscape, the heat capacity instead captures the extent to which the kinetic barrier bottlenecks the driven process as a function of temperature.

At the same time, the rates in this specific setup break detailed balance. The current of the clockwise cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$

$$k(1, 2)k(2, 3)k(3, 1) = e^{\beta\omega/2} \quad (2.81)$$

does not match with the current in the counterclockwise direction

$$k(1, 3)k(3, 2)k(2, 1) = e^{-\beta\omega/2}. \quad (2.82)$$

This imbalance guarantees a nonzero current in the steady state.

The competition between these two properties creates the nonzero heat capacity shown in Figure 2.9b. On the one hand, the structure of the graph naturally funnels the system towards the 3 state. On the other hand, the nonzero clockwise current forces the process to transition out of 3. As the inverse temperature β varies, the escape rate changes, which alters the severity of the bottleneck at 3. The heat capacity captures the excess heat needed to overcome this bottleneck.

In the limit $\beta \rightarrow \infty$, the escape rate vanishes. The system freezes entirely into the 3 state, the cyclical currents disappears and $C(\beta)$ drops to zero. At finite temperatures, however, the frustration alone generates a non-zero heat capacity. $\triangle$

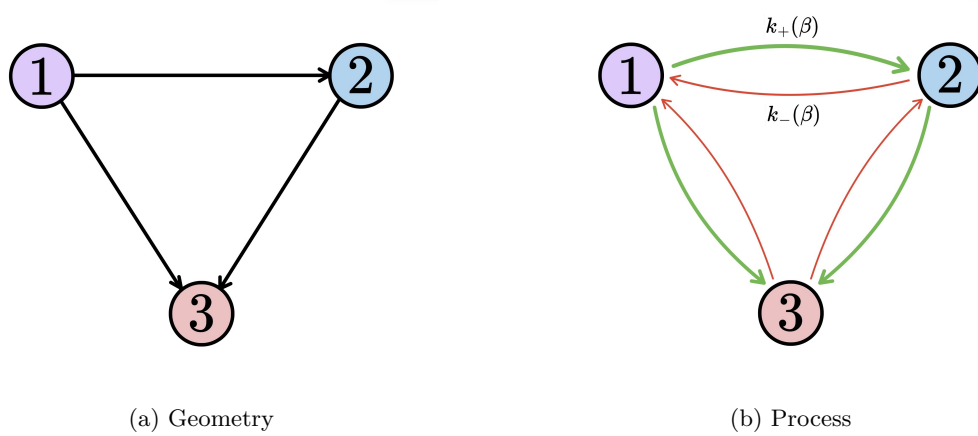


Figure 2.8: A frustrated 3-state system. The black arrows show the preferred direction and the coloured arrows denote the transition rates of the temperature dependent process.

To better understand the geometric bottleneck, the frustrated 3-state system can be mapped onto an equilibrium 3-level system where a single transition is subjected to an external work term. This decomposition allows us to track exactly how the heat capacity evolves as the drive is increased and the system is pushed away from detailed balance. Furthermore, attempting to calculate the heat capacity the 'normal way' using only the first term in (2.67) on the energy landscape shows why the quasipotential is required in nonequilibrium settings.

Example 2.10 (Frustrated 3-state system: driven 3-level). Physically, we can recast the example of the frustrated 3-state system as an equilibrium 3-level system that is driven out of equilibrium by a work term, as formulated in (2.62). We begin with a purely conservative system, illustrated in Figure 2.10a, governed by the energy landscape

$$E_1 = 2\omega, \quad E_2 = \omega, \quad \text{and} \quad E_3 = 0. \quad (2.83)$$

Following the parameterisation for detailed balance processes in (2.16), the forward transition rates are

$$k(1, 2) = k(2, 3) = e^{\beta\omega/2} = k_+(\beta) \quad \text{and} \quad k(1, 3) = e^{\beta\omega} \quad (2.84)$$

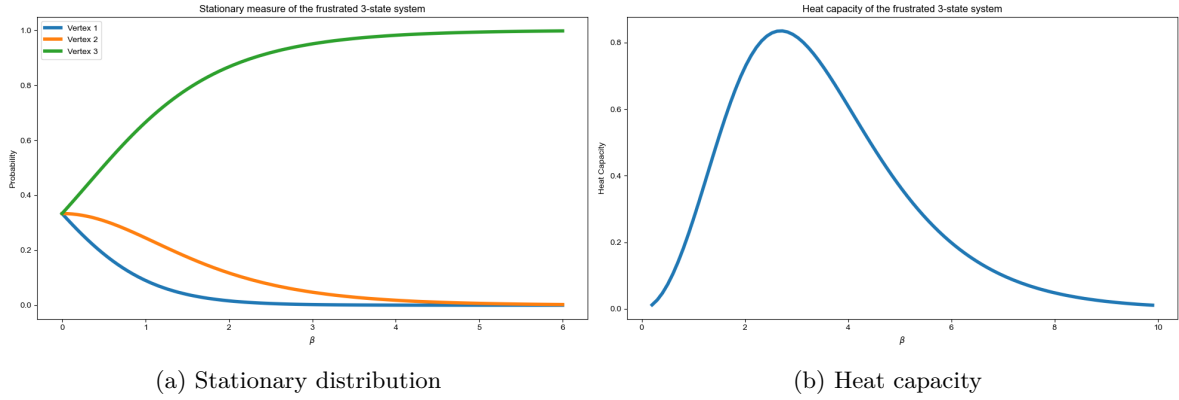


Figure 2.9: The frustrated 3-state system evaluated against inverse temperature β . (a) The stationary distribution for $\beta \in [0, 10]$, showing funneling effect that progressively traps the system in vertex 3 as β increases. (b) The corresponding nonequilibrium heat capacity $C(\beta)$ for $\beta \in [0, 1]$. The heat capacity captures the competition between the trapping at vertex 3 and current out of vertex 3. As $\beta \rightarrow \infty$, the escape rate from vertex 3 vanishes, halting the steady-state current and causing the heat capacity to strictly decay to zero.

and the backward transition rates are

$$k(2, 1) = k(3, 2) = e^{-\beta\omega/2} = k_-(\beta) \quad \text{and} \quad k(3, 1) = e^{-\beta\omega}. \quad (2.85)$$

Because the 3-level system satisfies detailed balance, its stationary distribution is simply the Gibbs distribution (2.17). Its heat capacity can be calculated via the standard fluctuation-dissipation relation in (2.57). The results are shown in Figure 2.11.

We can now directly link this equilibrium setup to the frustrated 3-state system from Example 2.9. Their geometries are identical. There is only a mathematical difference in the rates of the $1 \leftrightarrow 3$ transition. Following the parameterisation in (2.62), we introduce a non-conservative work function to this specific transition

$$F(3, 1; f) = f \implies k(3, 1) = e^{-\beta\omega + \beta f/2} \quad \text{and} \quad k(1, 3) = e^{+\beta\omega - \beta f/2}. \quad (2.86)$$

This generalisation is sketched in Figure 2.10b. Setting the drive to $f = 0$ recovers the equilibrium landscape of (2.84). Setting $f = \omega$ exactly reproduces the frustrated 3-state in Example 2.9.

To see how the driving term changes the heat capacity, we compare the resulting heat capacities in Figure 2.12a. The equilibrium heat capacity peaks earlier and at a lower magnitude. As the contour plot in Figure 2.12b demonstrates, driving the system further out of equilibrium spreads the curve of the heat capacity across β and amplifies its magnitude.

Finally, the decomposition and mapping allows us to investigate how much each term in the nonequilibrium heat capacity (2.67) contributes. The first term represents the naive correction and is equal to (2.23) for equilibrium. It measures how the average internal energy of the stationary state shifts with temperature, but simply reweighted against the changed stationary measure due to the driving. Figure 2.13 shows that this naive term alone fails to capture the true behaviour of the system. It overestimates the magnitude of the peak and misplaces its location.

Physically, this overestimation by the naive term occurs because it is blind to the steady-state current. The naive term evaluates the sensitivity of the internal energy to temperature changes, interpreting the elevated energy of the driven stationary state as a sign of large thermal fluctuations. However, the system is not thermally fluctuating up the energy gradient. The non-conservative drive f is pumping the system from state 3 back into state 1.

The second term explicitly corrects for this. As we saw in Example 2.6, the quasipotential quantifies how the steady-state current interacts with the network as temperature changes. In the barrier model,

the current was choked, and the quasipotential added to the heat capacity to reflect the thermal easing of the bottleneck. Here, it penalises the naive term by accounting for the external work done to drive from 3 to 1. $\triangle$

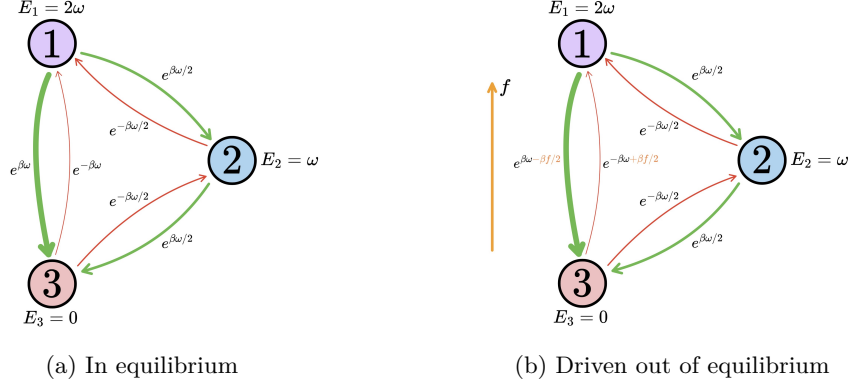


Figure 2.10: Network representations of the 3-level mapping. (a) The purely equilibrium 3-level system ($f = 0$) satisfying detailed balance. The thickness of the arrows illustrates the preferred transition down the energy landscape from $E_1 = 2\omega$ to $E_3 = 0$. (b) The generalised nonequilibrium system. The underlying energy landscape remains identical, but a non-conservative work term f is applied specifically to the $1 \leftrightarrow 3$ transition, modifying the detailed balance rates. Setting $f = \omega$ exactly recovers the fully frustrated tournament graph of Example 2.9.

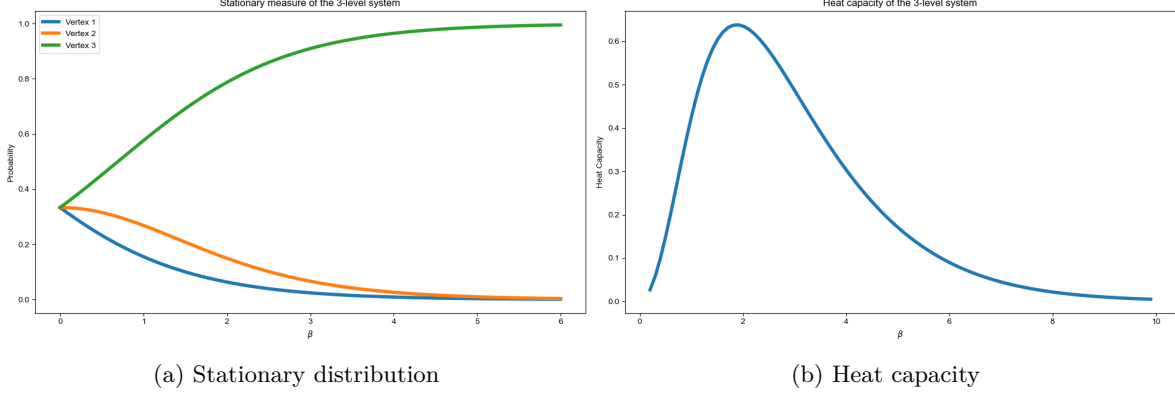


Figure 2.11: The equilibrium 3-level system evaluated against inverse temperature β . (a) The stationary distribution for $\beta \in [0, 10]$, showing the thermal freeze-out that progressively traps the system in the lowest energy state (vertex 3) as β increases. The probability measure of the 2 state decreases more rapidly than in the frustrated setting, as it is not sustained by a steady-state cyclic current. (b) The corresponding equilibrium heat capacity $C(\beta)$ for $\beta \in [0, 10]$ calculated using the equilibrium fluctuation-dissipation relation (2.57).

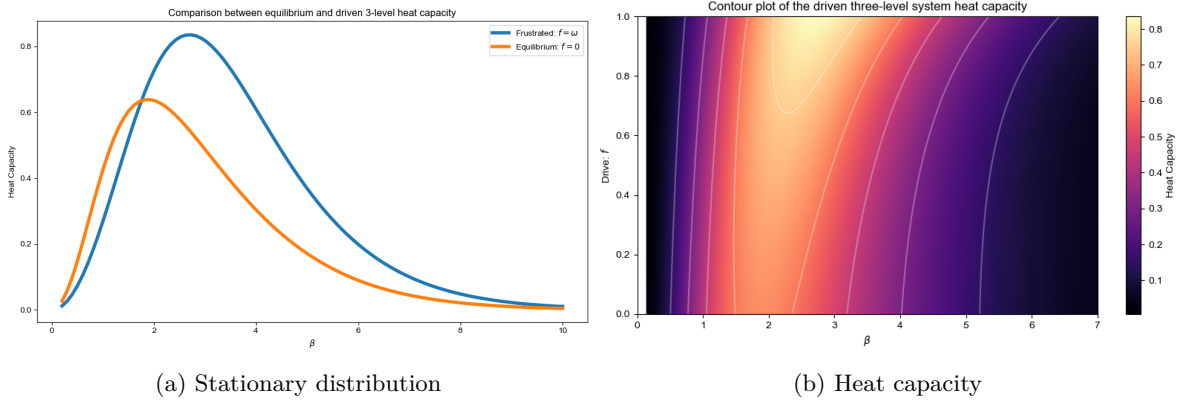


Figure 2.12: Thermodynamic impact of the non-conservative driving force f on the 3-level system. (a) Comparison of the heat capacity profiles as a function of inverse temperature β . The purely equilibrium 3-level system ($f = 0$, orange line) peaks at lower β and with a smaller magnitude compared to the frustrated nonequilibrium system ($f = \omega$, blue line). (b) A contour plot showing the heat capacity across a continuous range of driving strengths f . The colour gradient and contour lines show that the peak shifts towards higher β , increases in magnitude and widens as the system is driven further out of equilibrium.

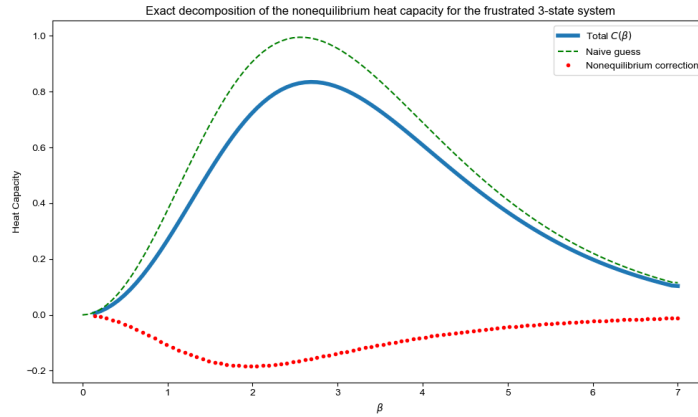


Figure 2.13: Decomposition of the nonequilibrium heat capacity for the frustrated 3-state system ($f = \omega$). The total heat capacity (solid blue line) is split into the naive internal energy variance (dashed green line) and the quasipotential correction (dotted red line). The naive term overestimates the magnitude of the peak and misplaces its location in β because it misinterprets the deterministic steady-state cyclic current as large thermal fluctuations.

3. Heat capacity of tournaments

In this chapter, we introduce the randomly directed complete graph as the tournament graph T_N . Section 3.1 is dedicated to proving some useful properties of the graph that will be used throughout the thesis. In Section 3.2, we define a continuous-time Markov chain on the graph, driven by a bias parameter ω and the inverse temperature β by assigning transition rates

$$k(i, j) = \begin{cases} \frac{1}{N-1} & \text{if } i \rightarrow j, \\ \frac{e^{-\beta\omega}}{N-1} & \text{if } i \leftarrow j. \end{cases} \quad (3.1)$$

The $N-1$ in the denominator, which corresponds to the frequency, is natural as it keeps the total escape rate finite for $N \uparrow \infty$. The section closes with Lemma 3.10, where we show that this process on the random tournament violates detailed balance for all tournaments with number of vertices $N > 2$.

Section 3.3 provides numerical simulations of the quasipotential and stationary distribution of the process on the tournament graph. Figure 3.1 shows the key finding. As the number of vertices N in the graph increases, the heat capacity converges to a deterministic limit while the ensemble average over different graph realisations is stable. Because the limiting corresponds directly to the stable average of the fluctuations, it points toward a self-averaging property. The numerical results lead to the main conjecture of the thesis.

Conjecture 1. *There exists a random variable $\xi_{C,N}$, such that the nonequilibrium heat capacity (2.67) for a realisation of a typical random tournament behaves as*

$$N \cdot C_N(\beta) = 4C_2(\beta) (1 + N^{-1/2} \cdot \xi_C), \quad (3.2)$$

where C_2 is the two-level heat capacity (2.30). Moreover, the random variable $\xi_C \xrightarrow{D} \mathcal{N}(0, \tau_C^2)$ in distribution.

3.1 Tournament graphs

In this subsection, we introduce the random tournament graph and establish its basic properties. We consider the complete graph $K_N = (\mathcal{V}, E)$ on the vertex set $\mathcal{V} = \{1, 2, \dots, N\}$. The set of edges E connects every distinct pair of vertices, giving a total number of edges of $\binom{N}{2}$. We obtain a tournament graph from K_N by assigning an orientation to each edge in E .

Definition 3.1. *A tournament $\sigma = (\sigma_{ij})_{i,j \leq N}$ on K_N is a skew-symmetric labelling $\sigma : \mathcal{V} \times \mathcal{V} \rightarrow \{+1, -1, 0\}$ that satisfies*

$$\sigma_{ij} = -\sigma_{ji} \quad \text{and} \quad \sigma_{ii} = 0 \quad \text{for all } i, j \in \mathcal{V}, \quad (3.3)$$

where $\sigma_{ij} = +1$ indicates an edge $x \rightarrow y$. We denote the resulting tournament graph by $T_N = (\mathcal{V}, E, \sigma)$.

An example of a tournament is shown in Figure 3.2. Instead of using the vertex degree, the natural way to distinguish vertices on T_N is by the difference between the number of incoming and outgoing vertices.

Definition 3.2. *On a tournament graph T_N , the score of a vertex $i \in \mathcal{V}$, is defined as*

$$s_i = \sum_{j \neq i} \sigma_{ij}. \quad (3.4)$$

This is equivalent to $s_i = d_{\text{out}}(i) - d_{\text{in}}(i)$. We denote the set of all possible scores by

$$\mathcal{R} = \{-(N-1), -(N-3), \dots, N-1\}. \quad (3.5)$$

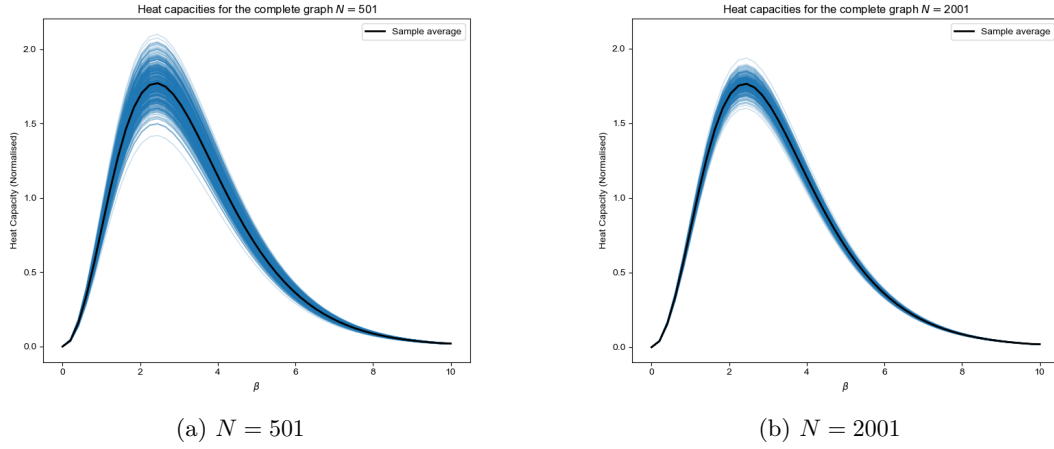


Figure 3.1: Ensemble realisations of the normalised heat capacity $N \cdot C_N(\beta)$ (blue lines) for system sizes $N = 501$ and $N = 2001$ at $\omega = 1$. The ensemble average $C_{N,\text{avg}}(\beta)$ is overlaid as a solid black line. The panels show a reduction in variance across single realisations and stability of the mean as N increases, implying a self-averaging property.

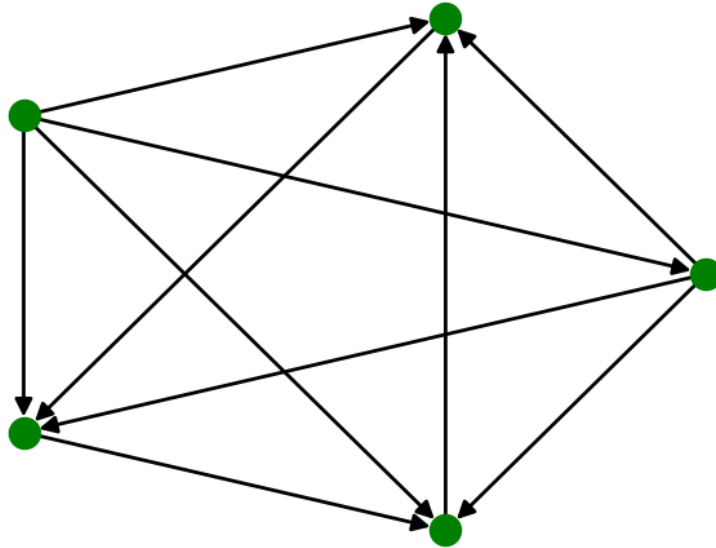


Figure 3.2: A realisation of a tournament graph for $N = 5$.

In order to study the limiting behaviour of a whole class of tournaments, we consider an ensemble where edge orientations are chosen randomly. For every pair $1 \leq i < j \leq N$, we assign the orientation independently with equal probability

$$\mathbb{P}(\Sigma_{ij} = \pm 1) = \frac{1}{2}, \quad (3.6)$$

which induces a probability distribution over the vertex scores. Under this model, the deterministic scores s_i become random variables S_i .

Proposition 3.3. *Consider a random tournament $\Sigma = (\Sigma_{ij})_{i,j \leq N}$. The score S_i of vertex i is the sum of $N - 1$ independent Rademacher variables, that is*

$$S_i \sim 2 \text{Bin}(N - 1, 1/2) - (N - 1). \quad (3.7)$$

The probability mass function is given by

$$p_{N-1}(s) := \mathbb{P}(S_i = s) = \binom{N-1}{\frac{N-1+s}{2}} 2^{-(N-1)} \quad (3.8)$$

for all integers $s \in \mathcal{R}$.

Proof. The score S_i can be written as $S_i = \sum_{j \neq i} \Sigma_{ij}$. For a fixed vertex $i \in \mathcal{V}$, the set of outgoing edges $\{\Sigma_{ij}\}_{i \neq j}$ are mutually independent Rademacher variables. Therefore, $(\Sigma_{ij} + 1)/2$ are independent Bernoulli(1/2). Summing these yields (3.7). $\square$

We denote the random vector of all scores in T_N by

$$\mathbf{S} = (S_1, S_2, \dots, S_N). \quad (3.9)$$

and for the set of unique scores realised in a specific instance of T_N , we write

$$\mathcal{T}_{\mathbf{S}} = \bigcup_{i=1}^N \{S_i\}. \quad (3.10)$$

We can partition the vertex set $\mathcal{V}$ along the scores in $\mathcal{T}_{\mathbf{S}}$ by grouping vertices into score bins.

Definition 3.4. *Given a random tournament Σ , for any $s \in \mathcal{T}_{\mathbf{S}}$, the score bin B_s is the set of vertices with score s*

$$B_s = \{i \in \mathcal{V} : S_i = s\}. \quad (3.11)$$

We denote its size by the shorthand $N_s := |B_s|$.

Because each of the scores in $\mathcal{T}_{\mathbf{S}}$ marginally follows a scaled and centred binomial distribution, linearity of expectation shows that the expected value of the size of a score bin for N vertices is given by $\mathbb{E}[N_s] = N p_{N-1}(s)$. However, because each edge is shared by a pair of vertices, we have a global constraint

$$\sum_{i=1}^N S_i = 0. \quad (3.12)$$

The result is that scores are negatively correlated for every pair i, j , with covariance

$$\text{Cov}(S_i, S_j) = -1. \quad (3.13)$$

However, if we condition on the direction of the edge connecting i and j , the reduced score becomes independent.

Proposition 3.5. *For distinct vertices $i, j \in \mathcal{V}$, the scores S_i and S_j are conditionally independent given the random edge direction Σ_{ij} . Let $S_i^{(j)} := S_i - \Sigma_{ij}$ denote the reduced score of i excluding the edge shared with j . Then*

$$S_i^{(j)} \sim 2 \text{Bin}\left(N-2, \frac{1}{2}\right) - (N-2). \quad (3.14)$$

The conditional probabilities satisfy

$$\mathbb{P}(S_i = s, S_j = s' \mid \Sigma_{ij} = +1) = p_{N-2}(s-1)p_{N-2}(s'+1), \quad (3.15)$$

and

$$\mathbb{P}(S_i = s, S_j = s' \mid \Sigma_{ij} = -1) = p_{N-2}(s+1)p_{N-2}(s'-1). \quad (3.16)$$

Proof. We consider the case $\Sigma_{ij} = +1$. We decompose the joint probability.

$$\begin{aligned} \mathbb{P}(S_i = s, S_j = s' \mid \Sigma_{ij} = +1) &= \mathbb{P}(S_i^{(j)} + 1 = s, S_j^{(i)} - 1 = s') \\ &= \mathbb{P}(S_i^{(j)} = s-1) \mathbb{P}(S_j^{(i)} = s'+1). \end{aligned} \quad (3.17)$$

The factorisation in the second line holds because the reduced scores $S_i^{(j)}$ and $S_j^{(i)}$ sum over disjoint subsets. Since edges are independent, these sums are independent. Substituting the definition of p_{N-2} yields (3.15). The derivation for (3.16) is identical. $\square$

From the opposite perspective, conditioning on the score of a vertex biases the distribution of the edge directions.

Lemma 3.6. *Let i, j and k be distinct vertices in T_N . Given $i \in B_s$ the marginal edge probability is given by*

$$\mathbb{P}(\Sigma_{ij} = \pm 1 \mid i \in B_s) = \frac{1}{2} \left(1 \pm \frac{s}{N-1}\right) := \frac{\alpha_{\pm}}{2}. \quad (3.18)$$

Furthermore, the edges are correlated with covariance

$$\text{Cov}(\mathbb{1}_{\{\Sigma_{ij}=+1\}}, \mathbb{1}_{\{\Sigma_{ik}=+1\}} \mid i \in B_s) = -\frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2} \quad (3.19)$$

Proof. For the marginal edge probability, we write out the definition of the conditional probability and to find

$$\mathbb{P}(\Sigma_{ij} = \pm 1 \mid S_i = s) = \frac{\mathbb{P}(S_i = s, \Sigma_{ij} = \pm 1)}{\mathbb{P}(S_i = s)}. \quad (3.20)$$

We can then use the conditional independence from Proposition 3.5 to get

$$\mathbb{P}(\Sigma_{ij} = \pm 1 \mid S_i = s) = \frac{1}{2} \frac{\mathbb{P}(S_i^{(j)} = s \mp 1)}{\mathbb{P}(S_i = s)}, \quad (3.21)$$

Next, we use that from Lemma C.1 we have that the binomial distribution admits

$$p_{N-2}(s \mp 1) = \frac{N-1 \pm s}{N-1} p_{N-1}(s) \quad (3.22)$$

to find

$$\mathbb{P}(\Sigma_{ij} = \pm 1 \mid S_i = s) = \frac{1}{2} \left(1 \pm \frac{s}{N-1}\right). \quad (3.23)$$

For the covariance, we write out

$$\begin{aligned} \text{Cov}(\mathbb{1}_{\{\Sigma_{ij}=+1\}}, \mathbb{1}_{\{\Sigma_{ik}=+1\}} \mid S_i = s) &= \mathbb{P}(\Sigma_{ij} = +1, \Sigma_{ik} = +1 \mid S_i = s) - \\ &\quad \mathbb{P}(\Sigma_{ik} = +1 \mid S_i = s)^2. \end{aligned} \quad (3.24)$$

The first probability is given by

$$\mathbb{P}(\Sigma_{ij} = +1, \Sigma_{ik} = +1 \mid S_i = s) = \frac{1}{4} \frac{\mathbb{P}(S_i^{(j,k)} = s-2)}{\mathbb{P}(S_i = s)}. \quad (3.25)$$

Using Lemma C.1, we can derive that

$$p_{N-3}(s-2) = \frac{(N-3+s)(N-1+s)}{(N-2)(N-1)} p_{N-1}(s), \quad (3.26)$$

so that (3.25) reduces to

$$\mathbb{P}(\Sigma_{ij} = +1, \Sigma_{ik} = +1 \mid S_i = s) = \frac{1}{4} \frac{(N-3+s)\alpha_+}{N-2}. \quad (3.27)$$

For the second term in (3.24) we use (3.23) to see that the covariance is given by

$$\begin{aligned} \text{Cov}(\mathbb{1}_{\{\Sigma_{ij}=+1\}}, \mathbb{1}_{\{\Sigma_{ik}=+1\}} \mid S_i = s) &= \frac{1}{4} \frac{(N-3+s)\alpha_+}{N-2} - \frac{\alpha_+^2}{4} \\ &= \frac{1}{4} \frac{(N-1+s)(N-1-s)}{(N-2)(N-1)^2} \\ &= \frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2}, \end{aligned} \quad (3.28)$$

which is what we wanted to show. $\square$

The last property we will prove is an estimate on the size of the maximum score in the tournament. Even though the scores are dependent, the following property holds for a general collection of Binomially distributed random variables.

Lemma 3.7. *Let $X_1, \dots, X_N$ be a collection of N random variables $X_i \sim \text{Bin}(N-1, 1/2)$ that are not necessarily independent. Set $Y_i := 2X_i - (N-1)$ and let $Y^* := \max_{1 \leq i \leq N} |Y_i|$. Then the probability*

$$\mathbb{P}(Y^* \geq \sqrt{2(k+1)(N \log N)}) = O(N^{-k}) \quad (3.29)$$

Proof. Fix an index i . Then $X_i = \sum_{j=1}^{N-1} Y_j$, where the Y_j are an independent collection of $\text{Ber}(1/2)$ random variables, so that $0 \leq Y_i \leq 1$ and $\mathbb{E}[X_i] = (N-1)/2$. We can thus apply Hoeffding's inequality A.1 to obtain

$$\mathbb{P}\left(\left|X_i - \frac{N-1}{2}\right| \geq t\right) \leq 2 \exp\left(-\frac{2t^2}{N-1}\right). \quad (3.30)$$

To bound the maximum of the collection $Y_1, \dots, Y_N$, we now use a union bound and use (3.30) to find

$$\mathbb{P}(\exists Y_i : |Y_i| \geq t) \leq N \cdot \mathbb{P}\left(\left|X_i - \frac{N-1}{2}\right| \geq \frac{t}{2}\right) \leq 2N \exp\left(-\frac{t^2}{2(N-1)}\right). \quad (3.31)$$

Taking $t = \sqrt{2(k+1)(N \log N)}$, we get that the maximum Y^* satisfies

$$\mathbb{P}(Y^* \geq \sqrt{2(k+1)(N \log N)}) \leq 2N \exp\left(-(k+1) \log N\right) = 2N^{-k}, \quad (3.32)$$

which is what we set out to prove. $\square$

This lemma motivates the definition of a typical tournament.

Definition 3.8. For any $k > 0$, we define the event of a typical tournament, denoted $\Omega_{\text{typ}}(k)$, as the set of tournament graphs T_N on which the maximum score $S^* := \max_{1 \leq i \leq N} |S_i|$ satisfies $S^* \leq \sqrt{2(k+1)N \log N}$

$$\Omega_{\text{typ}} = \left\{ T_N : S^* \leq \sqrt{2(k+1)N \log N} \right\}. \quad (3.33)$$

From Lemma 3.7, the probability of this event is

$$\mathbb{P}(T_N \in \Omega_{\text{typ}}(k)) = 1 - O(N^{-k}). \quad (3.34)$$

3.2 Tournament dynamics

In this section, we impose a dynamic on the random tournaments that is related to the directed edges. We define a continuous time Markov process $(X_t)_{t \geq 0}$ by assigning a transition rates depending on the edge directions σ_{ij} given by

$$k(i, j) = \begin{cases} \frac{1}{N-1} & \text{if } \sigma_{ij} = +1, \\ \frac{e^{-\beta\omega}}{N-1} & \text{if } \sigma_{ij} = -1 \\ 0 & \text{otherwise.} \end{cases} \quad (3.35)$$

Physically, β represents the inverse temperature and ω the strength of the directional biasing. The rates contain a $1/(N-1)$ prefactor because there are $N-1$ possible transitions for each vertex. Adding this prefactor is a choice of time scale and ensures that the total escape rates do not diverge as $N \uparrow \infty$. If we compare the form of the rates to (2.62), we see that there is no underlying energy landscape. This ensures that we will strictly study the thermal response related to the tournament geometry.

As a final remark, we note that the backward generator of the process as given in (2.12) will be denoted by L_N . Because the backward generator of the process explicitly depends on the underlying tournament σ through the transition rates k_σ it would be more accurate to write L_N^σ . However, we drop the dependence to keep the notation light. For any function $f : \mathcal{V} \rightarrow \mathbb{R}$, we have

$$L_N f(i) = \sum_{j=1}^N k(i, j)(f(j) - f(i)) \quad (3.36)$$

The first property of the process X_t that we prove is that the expected instantaneous power $\mathcal{P}(i)$ (2.64) exerted at a vertex $i \in \mathcal{V}$ depends on the score s_i .

Lemma 3.9. Let $(X_t)_{t \geq 0}$ be the continuous-time Markov process on the tournament graph $T_N = (\mathcal{V}, E, \sigma)$ with rates (3.35), such that

$$k_\sigma(i, j) = \frac{1 + e^{-\beta\omega}}{2(N-1)} + \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij}. \quad (3.37)$$

Under local detailed balance (see Section 2.4), the expected instantaneous power (2.64) in state $i \in \mathcal{V}$ is

$$\mathcal{P}(s_i) = \frac{\omega(1 + e^{-\beta\omega})}{2} \left[\frac{s_i}{N-1} + \tanh \frac{\beta\omega}{2} \right]. \quad (3.38)$$

Proof. Writing (3.35) out in terms of σ gives (3.37). By local detailed balance, the heat exchanged during a jump from x to y is

$$\begin{aligned} q_\beta(i, j) &= \frac{1}{\beta} \log \frac{k(i, j)}{k(j, i)} \\ &= \sigma_{ij} \omega. \end{aligned} \quad (3.39)$$

Then the expected instantaneous power will be

$$\begin{aligned}
\mathcal{P}(i) &= \sum_{j=1}^N k(i, j) \sigma_{ij} \omega \\
&= \omega \sum_{j=1}^N \left[\frac{1 + e^{-\beta\omega}}{2(N-1)} + \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij} \right] \sigma_{ij} \\
&= \frac{\omega}{2} \sum_{j \neq i}^N \left[\frac{1 + e^{-\beta\omega}}{N-1} \sigma_{ij} - (1 - e^{-\beta\omega}) \right] \\
&= \frac{\omega(1 + e^{-\beta\omega})}{2} \left[\frac{s_i}{N-1} + \tanh \frac{\beta\omega}{2} \right], \tag{3.40}
\end{aligned}$$

where we used the definition of score (3.4) to obtain the last line. This proves the claim. $\square$

We also prove that a Markov process with rates (3.37) violates the detailed balance condition on any tournament for which $N \geq 3$ and $\beta\omega > 0$.

Lemma 3.10. *The continuous-time Markov process $(X_t)_{t \geq 0}$ with transition rates given by (3.35) is not reversible for $\beta\omega > 0$ on any tournament graph T_N with $N \geq 3$ vertices.*

Proof. To avoid confusion in notation, we denote the vertices $u, v, w \in \mathcal{V}$. If the process were reversible, there would exist a stationary distribution ρ satisfying detailed balance

$$\rho(u)k(u, v) = \rho(v)k(v, u) \quad \forall u, v \in \mathcal{V}. \tag{3.41}$$

In the case where $N \geq 3$, this would mean that for any triple $u, v, w \in \mathcal{V}$

$$\frac{k(u, v) k(v, w)}{k(v, u) k(w, v)} = \frac{k(u, w)}{k(w, u)}. \tag{3.42}$$

By definition, $\frac{k(u, w)}{k(w, u)} = \exp(\sigma_{uw}\beta\omega)$. However, the left-hand side of the equation is a product of those two possibilities, which means it takes values in $\{e^{-2\beta\omega}, 1, e^{+2\beta\omega}\}$. Therefore, equality can only hold if $\beta\omega = 0$ for $N \geq 3$, completing the proof. $\square$

To summarise, in this section we defined random tournaments, constructed a Markov process on these graphs and showed that this resulted in nonequilibrium dynamics. In the next section, we numerically calculate the nonequilibrium heat capacity (2.67) and show the existence of self-averaging behaviour.

3.3 Numerical simulations of the tournament heat capacity

In this section, we numerically calculate the heat capacity of tournaments. Following Section 2.6, this amounts to solving

$$L_N V_N = -Q \quad \text{and} \quad \rho_N L_N = 0. \tag{3.43}$$

The methods to do so are described in Appendix B.

Figure 3.3 shows a collection of heat capacities for random tournaments of different system sizes. The first thing to note is that the heat capacity itself decreases inversely with the system size N . Therefore, we normalise the heat capacity by taking $N \cdot C_N(\beta)$. After this transformation, the heat capacities remain of similar size as N increases.

Figure 3.4 shows ensemble realisations of the normalised heat capacity as system size increases. For each fixed N , we calculate the heat capacity as a function of β for a number of independently sampled random tournaments. Plotting the ensemble average $C_{N, \text{avg}}(\beta)$ over the individual realisations reveals that two main effects are present in the system. It is clear that the fluctuations in the heat capacity

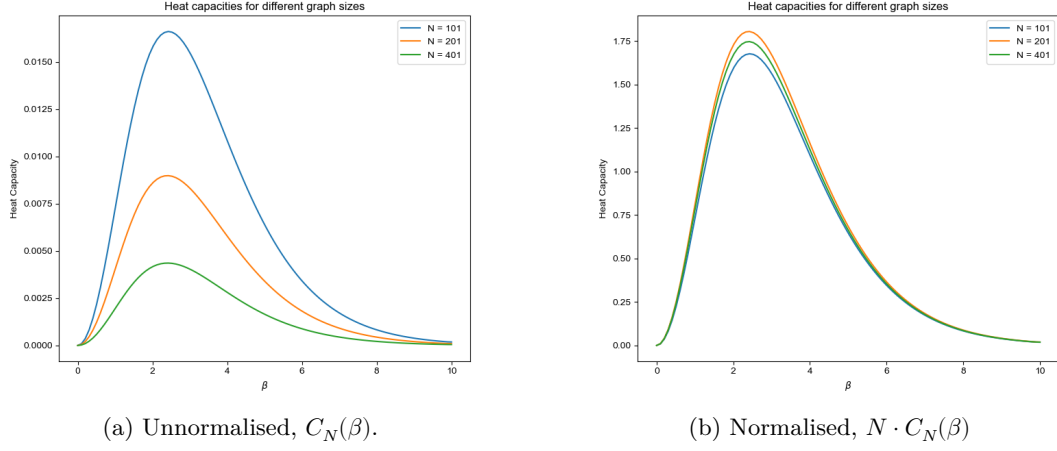


Figure 3.3: Heat capacities as a function of β for three single random tournaments at different system sizes $N \in \{101, 201, 401\}$ and $\omega = 1$. The left panel shows the raw heat capacity $C_N(\beta)$, which decays as $1/N$, while the right panel shows heat capacity multiplied by system size $N \cdot C_N(\beta)$ to show that the heat capacity scales with $O(N^{-1})$

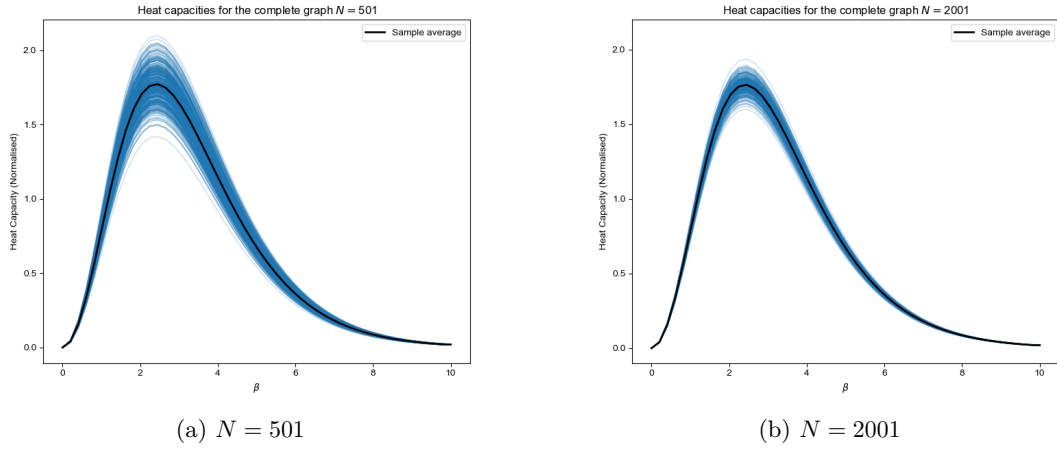


Figure 3.4: Ensemble realisations of the normalised heat capacity $N \cdot C_N(\beta)$ (blue lines) for system sizes $N = 501$ and $N = 2001$ at $\omega = 1$. The ensemble average $C_{N,\text{avg}}(\beta)$ is overlaid as a solid black line. The panels show a reduction in variance across single realisations and stability of the mean as N increases, implying a self-averaging property.

reduce as N increases. At the same time, the shape of the ensemble mean $C_{N,\text{avg}}(\beta)$ remains stable. This implies a self-averaging property.

The profile of the average heat capacity resembles the heat capacity of the two-level system $C_2(\beta)$ introduced in Example 2.4. The clearest similarity is that both the two-level system and the complete graph reach their maximum value at $\beta_{\text{max}} \approx 2.6$. However, while the two-level system peaks at $C_2(\beta_{\text{max}}) \approx 0.43$, the maximum for the random tournament is $C_{N,\text{avg}}(\beta_{\text{max}}) \approx 1.75$. This suggests that the average heat capacity of a tournament is related to the two-level system as $N \cdot C_{N,\text{avg}}(\beta) = 4C_2(\beta)$. The numerical validity of this assumption is shown in Figure 3.5.

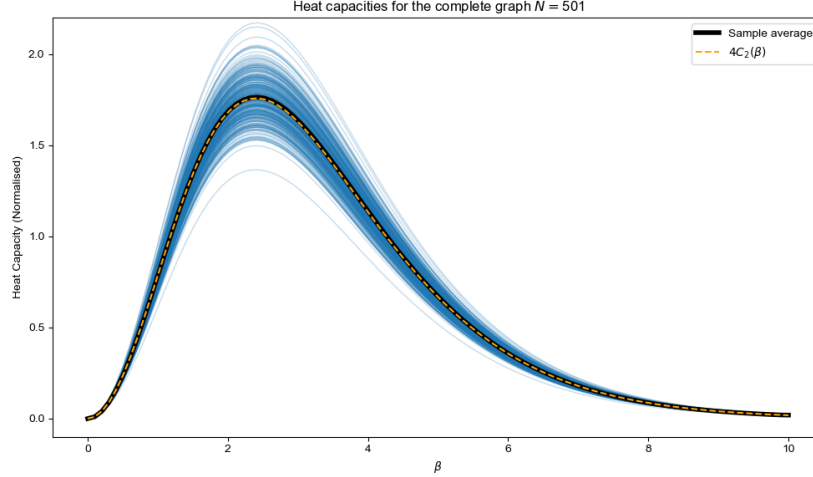


Figure 3.5: Ensemble realisations of the normalised heat capacity $N \cdot C_N(\beta)$ (blue lines) for a system size of $N = 501$ with weight $\omega = 1$. The ensemble average $C_{N,\text{avg}}(\beta)$ (solid black line) closely matches the scaled two-level system heat capacity $4C_2(\beta)$ (dashed orange line). This relatively low- N case shows that the ensemble mean converges to the theoretical limit even in the presence of significant sample-to-sample fluctuations.

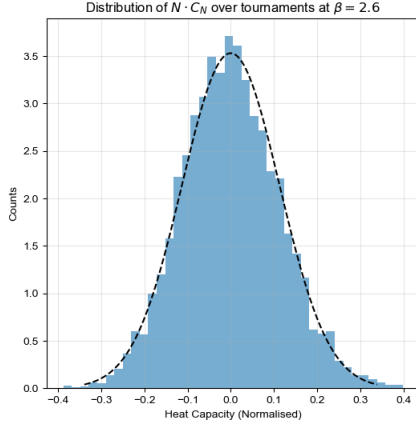
To study the shape of the fluctuations around the average, we examine a slice of the distribution at fixed N and β . We set $\beta = 2.6$ to capture the maximum fluctuations. We calculate the normalised heat capacity for a large ensemble and compare the resulting empirical distribution against a Gaussian distribution in Figure 3.6.

The histogram in combination with the Q-Q plot strongly suggests that the fluctuations converge to a Normal distribution. Moreover, the width of the histograms show that the sample variance scales inversely with N . We conjecture that the Normal distribution has a standard deviation $N^{-1/2}\tau_C$ for β fixed.

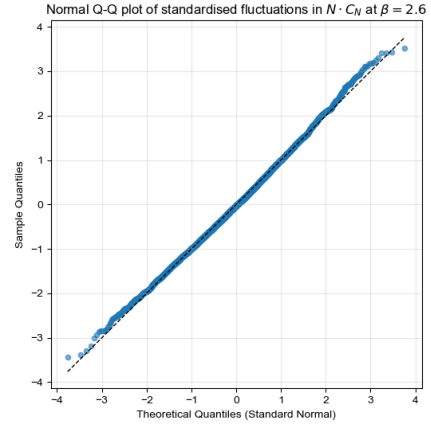
Figure 3.4 suggests that the variance of the fluctuations depends on β . In fact, we see that the variance τ_C^2 is proportional to the size of $C_N(\beta)$. This leads to a model of the form

$$N \cdot C_N(\beta) = 4C_2(\beta)(1 + N^{-1/2}\tau_C\xi_C), \quad (3.44)$$

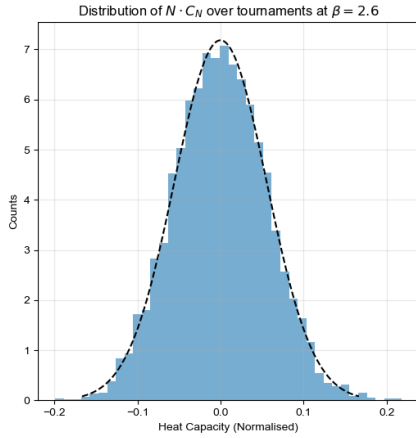
where $\xi_C \rightarrow \mathcal{N}(0, 1)$ in distribution as $N \uparrow \infty$. Because the model implies $\sqrt{\mathbb{V}(NC_N)} = 4C_2(\beta)N^{-1/2}\tau_C$, we can test the assumption by plotting the sample standard deviation against the sample mean. The results are shown in Figure 3.7. The relationship is highly linear up to elliptical hysteresis for large values of $C_N(\beta)$. Because this effect disappears as N increases, it is most likely a finite-size effect. Hence, (3.44) seems like a reasonable model.



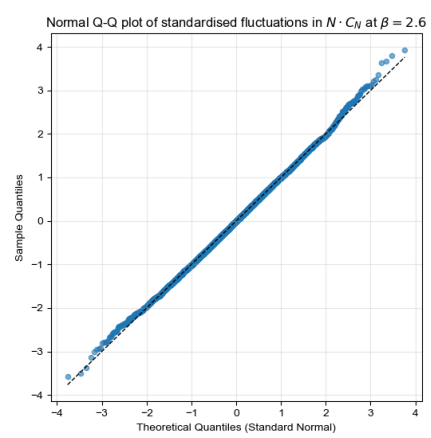
(a) Histogram of 4000 realisations of $C_N(\beta)$ for $\beta = 2.6$ and $N = 501$.



(b) QQ-plot of the histogram for $\beta = 2.6$ and $N = 501$ against Normal distribution.



(c) Histogram of 4000 realisations of $C_N(\beta)$ for $\beta = 2.6$ and $N = 2001$.



(d) QQ-plot of the histogram for $\beta = 2.6$ and $N = 2001$ against Normal distribution.

Figure 3.6: Distribution of the normalised heat capacity fluctuations evaluated at the peak $\beta = 2.6$ and $\omega = 1$, based on 6000 independent realisations. (a, c) Centred histograms for system sizes $N = 501$ and $N = 2001$, respectively. (b, d) The corresponding Q-Q plots against a standard normal distribution. The alignment in the Q-Q plots demonstrates that the fluctuations converge in distribution to a Gaussian as N increases. A comparison between the width of the centred histograms show that the sample variance scales as $1/N$.

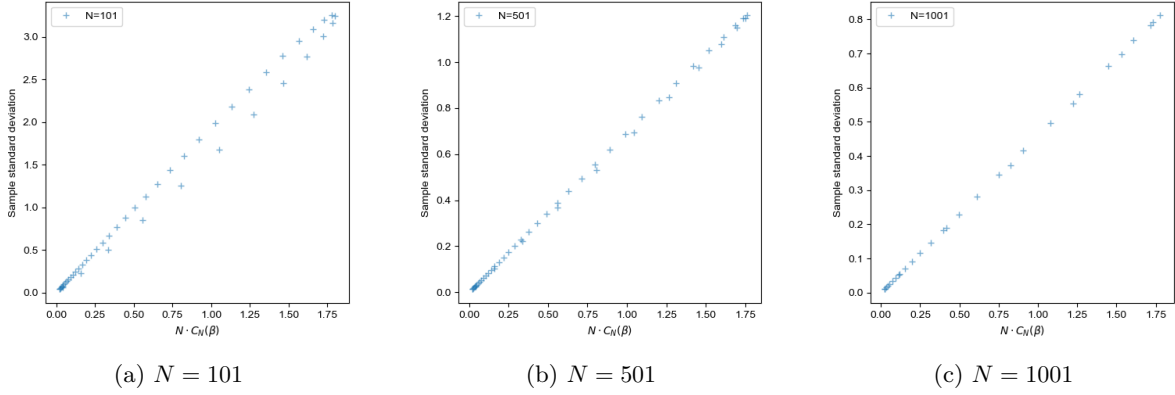


Figure 3.7: Sample standard deviation of the normalised heat capacity plotted against the ensemble mean $N \cdot C_{N,\text{avg}}(\beta)$ for inverse temperatures $\beta \in [0, 10]$ and $\omega = 1$. The relationship is shown for system sizes (a) $N = 101$, (b) $N = 501$ and (c) $N = 1001$. The curves show that the elliptical hysteresis collapses into a straight line as N increases, which supports the multiplicative noise model proposed in (3.44) for large systems.

3.3.1 Microscopic fluctuations

To understand (3.44) the origins of the limiting behaviour of the heat capacity, we inspect the properties of the stationary distribution ρ_N and the quasipotential V_N . Figure 3.8a shows the normalised stationary distribution $N \cdot \rho_N$ as a function of β for a single random tournament realisation. The curves are coloured according to the vertex's score.

At $\beta = 0$ the system is uniformly distributed. As the transitions get more asymmetric, the stationary distribution more strongly correlates with vertex score. A low score corresponds to a high in-degree, which leads to many visits. Singling out all vertices of a particular score s as in Figure 3.8b, we see comparable behaviour to the heat capacity. The stationary distribution fluctuates around a mean with standard deviation dependent on β .

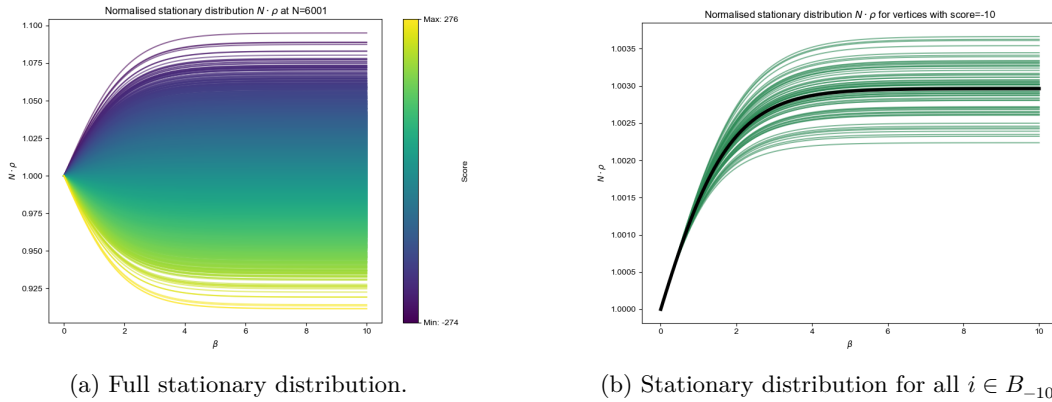
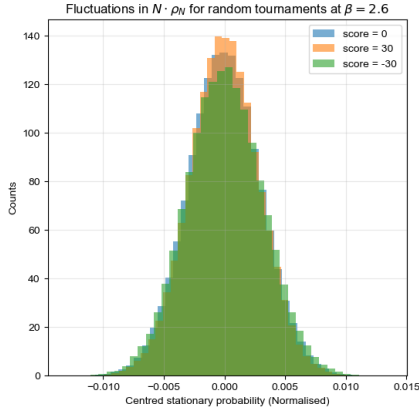


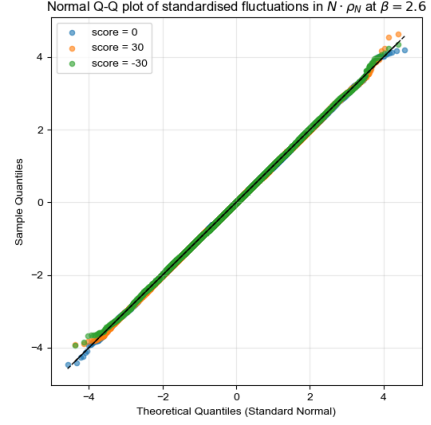
Figure 3.8: (a) Normalised stationary distribution $N \cdot \rho_N(i, \beta)$ of a single tournament realisation for a system size of $N = 6001$ plotted as a function of β individually for all $i \in \mathcal{V}$. Each curve is coloured according to the vertex's respective score S_i (3.2), showing score dependence as β increases. (b) $N \cdot \rho_N(i, \beta)$ isolated for all in the fixed score bin $i \in B_{-10}$ (3.4) ($s = -10$). The plot shows that the individual microscopic trajectories fluctuate around their mean with a β -dependent standard deviation, mirroring the behaviour of the heat capacity.

Fluctuations within a specific score bin can be explained by the local connectivity of the vertices. Consider two vertices $i, j \in \mathcal{V}$ with identical score $S_i = S_j$. If i has incoming edges predominantly from vertices with very negative scores and j from vertices with very positive scores, vertex i will be visited more often.

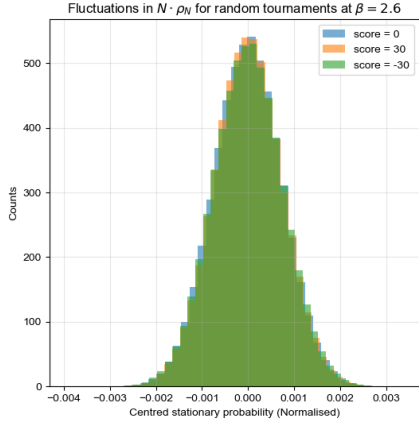
The question is then whether this effect is tournament specific or a general statistical property. To test this, we simulate the stationary distribution for an ensemble of tournaments at fixed β and N to construct histograms. The results for two values of N and three different score classes are presented in Figure 3.9. The histograms and Q-Q plots suggest that the fluctuations converge in distribution to a Gaussian.



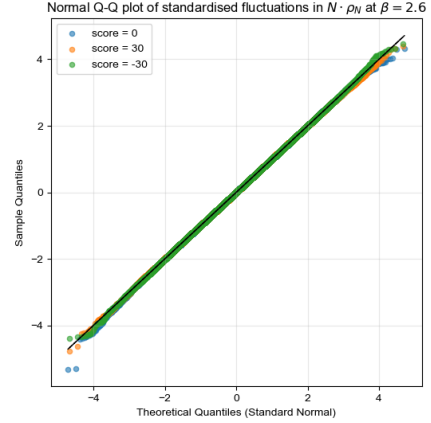
(a) Histogram of the fluctuations for $\beta = 2.6$ and $N = 501$.



(b) QQ-plot of the fluctuations for $\beta = 2.6$ and $N = 501$.



(c) Histogram of the fluctuations for $\beta = 2.6$ and $N = 2001$.



(d) QQ-plot of the histogram for $\beta = 2.6$ and $N = 2001$ against Normal distribution.

Figure 3.9: Histograms of the normalised stationary distribution $N \cdot \rho_N(i, \beta)$ within fixed score bins evaluated at $\beta = 2.6$ and $\omega = 1$ for 4000 iterations. (a,c) Centered histograms for system sizes $N = 501$ and $N = 2001$ across three distinct score bins $s \in \{-30, 0, 30\}$. (b,d) The corresponding Q-Q plots against a standard normal distribution. The increasing alignment in the Q-Q plots demonstrates that intra-bin fluctuations converge in distribution to a Gaussian as N increases.

Turning to the quasipotential V_N , we plot it for all $i \in \mathcal{V}$ as a function of β and coloured according vertex score in Figure 3.10. There is an ordering by score present, approximately related to S_i/N .

Physically, the quasipotential measures the excess heat exchanged with the environment during relaxation. Since the vertex score reflects the balance of incoming and outgoing edges, it shows whether a vertex is more likely to export or absorb heat. Due to the full connectivity of the complete graph, the system relaxes very quickly. We thus expect the expected excess heat to be dominated by the first step, and thus by the vertex score.

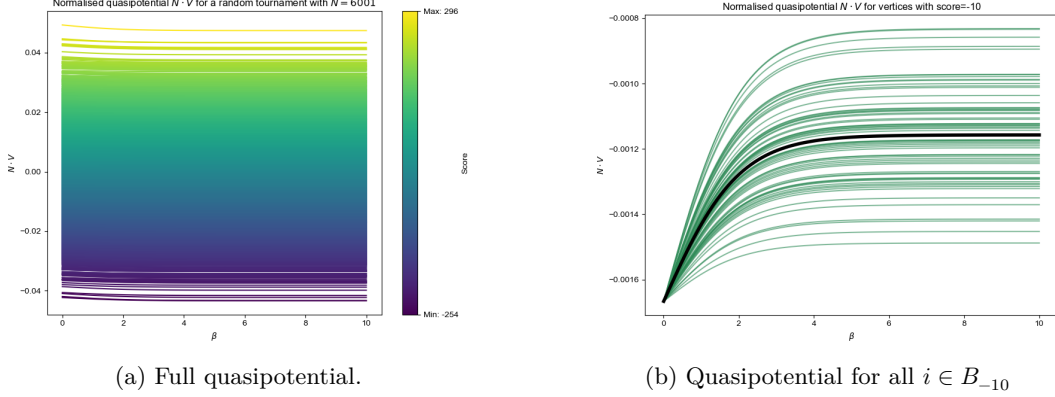


Figure 3.10: (a) Quasipotential $V_N(i, \beta)$ as a function of inverse temperature β for a single tournament realisation of size $N = 6001$. The curves are coloured according to the vertex scores S_i , revealing an ordering proportional to S_i/N . (b) $N \cdot V_N(i, \beta)$ isolated for all vertices $i \in B_{-10}$ (3.4) ($s = -10$). Like the heat capacity and the stationary measure, the individual trajectories fluctuate around a mean with a β -dependent standard deviation.

Centering the quasipotential by subtracting the value at $\beta = 0$ reveals its β dependence more clearly. Figure 3.11a shows that a straightforward score dependence is not immediately obvious from the full ensemble. If we instead divide the figure into extreme and bulk scores, a clearer picture emerges. Figure 3.11b shows that only extreme scores, both strongly positive and negative, can get negative corrections.

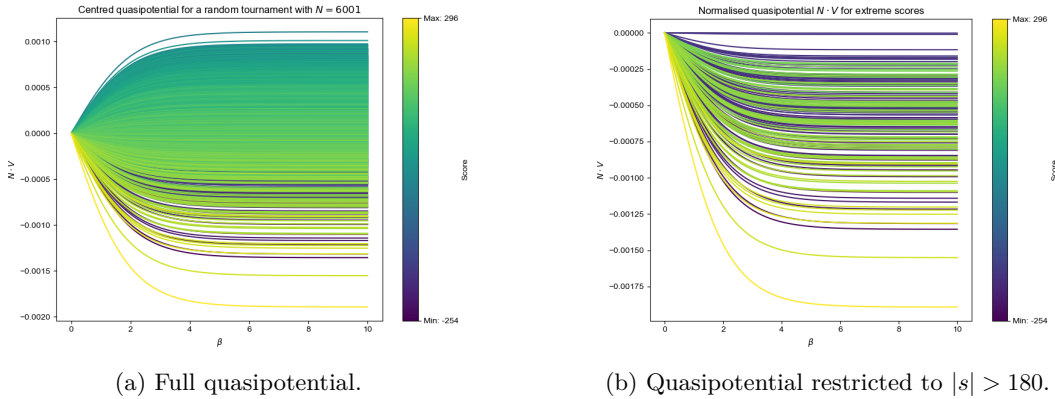
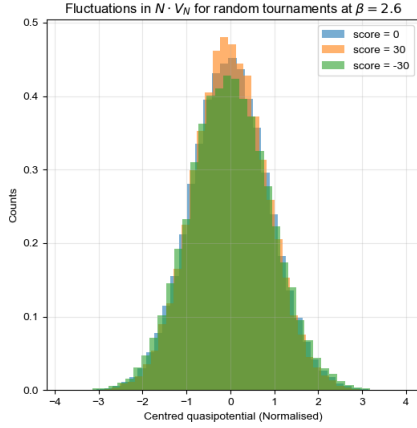


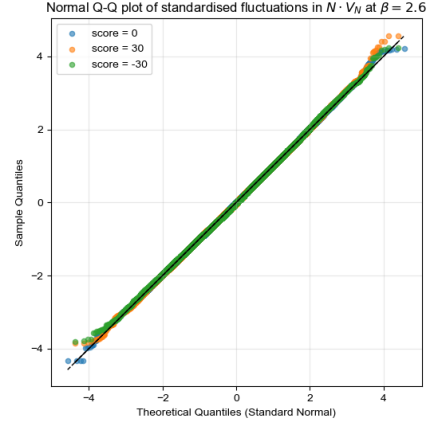
Figure 3.11: Centred quasipotential as a function of inverse temperature β , coloured by vertex score S_i for a system size $N = 6001$ and $\omega = 1$. (a) The leading-order correction $V_N(i, \beta) - S_i/N$, representing the deviation from the uniform $\beta = 0$ state. (b) The centred quasipotential isolated for extreme scores $|s| > 180$. The panels illustrate that while bulk scores experience an increase in quasipotential due to the suppression of unfavorable transitions at higher β , both positive and negative extreme scores show a decrease.

For vertices with very low scores, the picture is clear. A transition from such a vertex predominantly

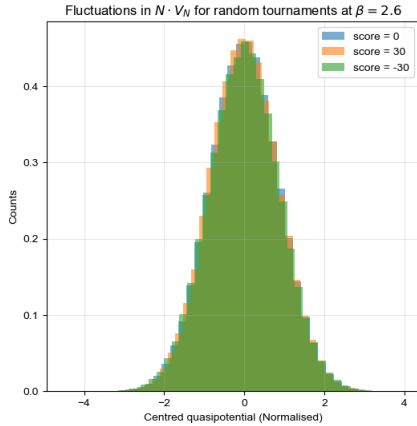
absorbs heat from the environment. Increasing β strengthens the asymmetry of individual transitions and further decreases the quasipotential. For very positive vertices, the story is more complicated. Because vertices with intermediate scores have a good mixture of favourable and unfavourable directions, increasing β suppresses unfavourable transitions, increasing the quasipotential. Vertices with extremely positive scores already possess mostly favourable edges, which means that this blocking effect is negligible. However, since the bulk vertices increasingly take favourable jumps, their quasipotential will increase. The relative excess heat of the positive extremes thus decreases.



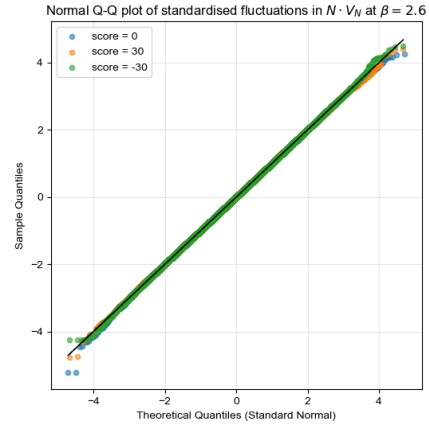
(a) Histogram of the fluctuations for $\beta = 2.6$ and $N = 501$.



(b) QQ-plot of the fluctuations for $\beta = 2.6$ and $N = 501$.



(c) Histogram of the fluctuations for $\beta = 2.6$ and $N = 2001$.



(d) QQ-plot of the histogram for $\beta = 2.6$ and $N = 2001$ against Normal distribution.

Figure 3.12: Empirical distributions of the normalised quasipotential $N \cdot V(i, \beta)$ within fixed score bins, evaluated at $\beta = 2.6$ and $\omega = 1$. (a,c) Centred histograms for system sizes $N = 501$ and $N = 2001$ across three distinct score classes: $s \in \{-20, 0, 20\}$. (b,d) The corresponding Q-Q plots against a standard normal distribution. The increasing alignment in the Q-Q plots demonstrates that intra-bin fluctuations converge in distribution to a Gaussian as N increases.

Fluctuations within score bins, meanwhile, are again related to the microscopic properties of the vertices. Two vertices of the same score in the bulk will both have an increasing quasipotential in β . However, if one vertex disproportionately points toward higher-score vertices, its second jump will also likely dissipate heat to the environment. As a result, the quasipotential of that specific vertex will

have a larger relative increase. The histograms and Q-Q plots in Figure 3.12 show that the resulting fluctuations are Gaussian.

3.3.2 Coarse graining over score bins

Having established the existence of fluctuations within individual score bins due to the microscopic structure of the tournament, we now investigate whether they actually impact the heat capacity. Specifically, we check whether all tournaments with the same realised score sequence $\mathcal{T}_{\mathbf{s}}$ are thermodynamically equivalent. To do so, we introduce an averaging projection P onto the score bins. For any $f : \mathcal{V} \rightarrow \mathbb{R}$ it averages the function over all vertices sharing the same score

$$Pf(i) = \frac{1}{N_{s_i}} \sum_{j \in B_{s_i}} f(j). \quad (3.45)$$

Applying this operator to the stationary distribution ρ_N and quasipotential V_N projects the system onto the set of realised scores $\mathcal{T}_{\mathbf{s}}$. The result of this operation on the heat capacity for a single tournament realisation is shown in Figure 3.13

Figure 3.13 shows that this coarse-graining over the score bins has negligible effect on the heat capacity. This implies that microscopic fluctuations within the score bin are irrelevant to the thermodynamic properties. Any two tournaments with the same score sequence will yield identical heat capacities.

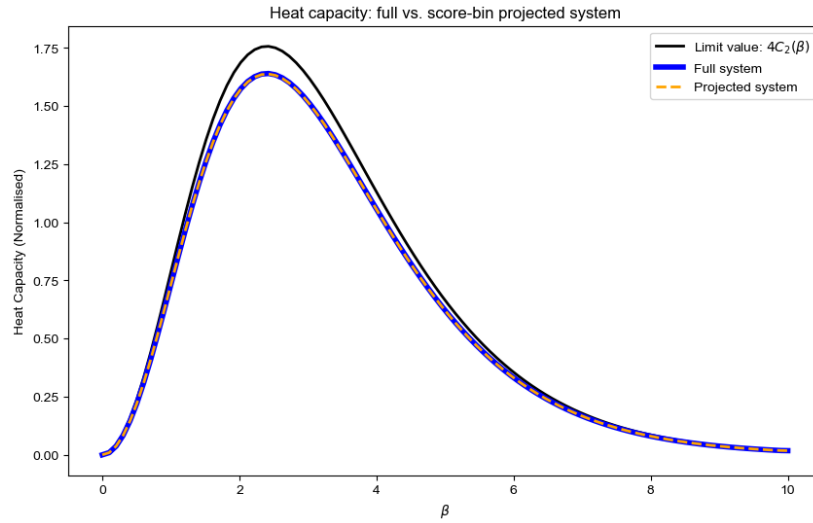


Figure 3.13: Comparison of the heat capacity for the fully microscopic random tournament (solid blue line), the score-bin projected system (dashed orange line) and the limiting heat capacity $4C_2(\beta)$ (solid black line). The heat capacity was calculated for a single tournament with system size of $N = 2001$ with weight $\omega = 1$. The agreement between the microscopic and projected systems, despite their deviation from the limit, demonstrates that the coarse-graining captures all relevant finite-size effects.

All of the randomness in the large N limit is captured by the statistics of the score bins themselves. This suggests that the limiting heat capacity $4C_2(\beta)$ corresponds to an annealed situation in which all score bins reach their expected values. The remaining finite-size effects observed in simulations are then treated as perturbations around this annealed situation.

4. Annealed solutions on random tournaments

In this section, we derive an annealed version of the process with generator $\overline{G}_N$ and obtain an analytic expression for its heat capacity $\overline{C}(\beta)$. Unlike the approach in [16], we do not derive an effective equation by taking $N \uparrow \infty$. That approach is intractable for two reasons. First, we would like to obtain a solution at fixed N to perturb around. Second, the heat capacity vanishes in the limit $N \uparrow \infty$ as illustrated by Figure 3.3.

On the other hand, Figure 3.4 shows that averaging V_N and ρ_N over the score bins B_s into functions of vertex score preserves the heat capacity. Motivated by this observation, we construct a coarse-grained effective process on the score range $\mathcal{R}$ that approximates the dynamics of L_N at N fixed. Theorem 4.1 presents the main result.

This approach is more in line with the one used to find limits for epidemic spreading on a fixed network. In that context, equations for the density of infected nodes are also derived by assuming vertices with similar degree are statistically equivalent. We refer the reader to [96] for details.

Theorem 4.1. *The solution $\overline{W}_N(s)$ to annealed Poisson problem is given by*

$$\overline{W}_N(s) = \omega \frac{s}{N-1} \exp\left(-\frac{s}{N-1} \tanh \frac{\beta\omega}{2}\right) + \frac{3}{N-1} \tanh \frac{\beta\omega}{2}, \quad (4.1)$$

such that

$$\overline{C}_N(\beta) = \frac{(\beta\omega)^2}{N-1} \operatorname{sech}^2 \frac{\beta\omega}{2}. \quad (4.2)$$

Comparing the annealed heat capacity $\overline{C}_N(\beta)$ with $C_2(\beta)$ for the two-level system (2.30), we see that $(N-1) \cdot \overline{C}_N = 4C_2(\beta)$. This is in accordance with the empirical result in Section 3.3. There, Figure 3.5 showed a near exact match between $4C_2(\beta)$ and the normalised average heat capacity for a sample of random tournaments. Of course, for N large there is a negligible difference between N and $N-1$. This indicates that the annealed generator we derive in this section sufficiently captures the ensemble-averaged behaviour. As such, it serves as the starting point of the perturbative approach in Section 6.

To get to the heat capacity (4.2), we start in Section 4.1 by constructing the effective generator $\overline{G}_N$. The first step in this construction is to examine the action of L_N on functions $g : \mathcal{R} \rightarrow \mathbb{R}$ of vertex score. We show in Lemma 4.3 that for such functions on

$$L_N g(s_i) = \sum_{j \in B_{s'}} \lambda_N(i, s') [g(s') - g(s_i)]. \quad (4.3)$$

From the microscopic $\lambda_N(i, s')$, we obtain an annealed transition rate $\gamma_N(s, s')$ by randomising the tournament and taking the conditional expectation of the resulting random rate $\Lambda_N(i, s')$, given that $S_i = s$. We then prove that these transition rates concentrate around their conditional mean.

Theorem 4.2. *Let $(\Sigma_{ij})_{ij \leq N}$ be a random tournament. Fix a vertex $i \in B_s$ with score $|s| \leq t\sqrt{N \log N}$. For any target score s' , the coarse grained rates Λ_N satisfy*

$$\mathbb{P}\left(\sqrt{N} |\Lambda_N(i, s') - \mathbb{E}[\Lambda_N(i, s') | i \in B_s]| \geq \varepsilon \mid i \in B_s\right) \leq \varepsilon^{-2} \cdot O(N^{-1/2}). \quad (4.4)$$

for all $s' \in \mathcal{R}$.

Once we have this result, we construct $\gamma_N(s, s')$ from $\mathbb{E}[\lambda_N(i, s') | i \in B_s]$ by neglecting all terms related to extreme scores. We then define the annealed generator $\overline{G}_N$ for all $s, s' \in \mathcal{R}$ in Section 4.2 using $\gamma_N(s, s')$. By defining $\overline{G}_N$ across the entire possible score range $\mathcal{R}$ (3.5), rather than limiting it to the

scores present on a single tournament $\mathcal{T}_{\mathbf{S}}$ (3.10), we ignore any specific tournament realisation. This allows us to work on a deterministic score space. The absence of any microscopic structure in $\overline{G}_N$ is exactly why we call it 'annealed'.

Lemma 4.7 shows that the annealed process generated by $\overline{G}_N$ satisfies detailed balance, which allows us to directly derive its stationary distribution

$$\overline{\pi}_N(s) = \frac{1}{\overline{Z}\sqrt{2\pi(N-1)}} \exp\left(-\frac{(s + 2 \tanh \frac{\beta\omega}{2})^2}{2(N-1)}\right), \quad (4.5)$$

where $\overline{Z}$ is the correction due to discretisation. We finish the section by using this result to calculate the background rate of dissipation $\langle \mathcal{P} \rangle_{\overline{\pi}_N}$ (2.65). We then use the definition of the quasipotential (2.68) to motivate the definition of the annealed Poisson equation

$$\overline{G}_N \overline{W}_N(s) = -\frac{1 + e^{-\beta\omega}}{2} \left[\frac{2\omega}{N-1} \tanh \frac{\beta\omega}{2} + \frac{\omega s}{N-1} \right]. \quad (4.6)$$

We finish the section by solving this equation, which proves Theorem 4.1.

4.1 Annealed transition rates

Figure 3.13 established a crucial invariance. Projecting the quasipotential $V_N(i)$ and stationary measure $\rho_N(i)$ onto the attained scores $\mathcal{T}_{\mathbf{S}}$ leaves the heat capacity unchanged. Because this implies that all relevant information is seemingly captured by functions of score, we start by deriving how L_N acts on such functions.

Lemma 4.3. *Let $T_N = (\mathcal{V}, E, \sigma)$ be a tournament graph and write*

$$\lambda_{N,\pm}(i, s') = \frac{1}{N-1} |\{i \in \mathcal{V} : s_j = s', \sigma_{ij} = \pm 1\}| \quad (4.7)$$

given any $i \in \mathcal{V}$. Then, for any function $g : \mathcal{R} \rightarrow \mathbb{R}$, the generator $L_N g(s_i)$ can be written in the form

$$L_N g(s_i) = \sum_{s'} \lambda_N(i, s') [g(s') - g(s_i)], \quad (4.8)$$

with

$$\lambda_N(i, s') = e^{-\beta\omega} \lambda_{N,+}(i, s') + \lambda_{N,-}(i, s'). \quad (4.9)$$

Proof. Fixing a tournament graph T_N , consider a function $g(s_i)$. Let $s_i = s$. From the definition of the transition rates $k(i, j)$ 3.37, the backward generator L_N (3.36) applied to $g(s_i)$ gives

$$\begin{aligned} L_N g(s_i) &= \sum_{j=1}^N \left[\frac{1 + e^{-\beta\omega}}{2(N-1)} + \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij} \right] (g(s_j) - g(s)) \\ &= \sum_{s' \in \mathcal{T}_{\mathbf{S}}} \sum_{j \in B_s} \left[\frac{1 + e^{-\beta\omega}}{2(N-1)} + \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij} \right] (g(s') - g(s)). \end{aligned} \quad (4.10)$$

Grouping transitions by target score s' , we define

$$\lambda_N(i, s') := \sum_{j \in B_s} \left[\frac{1 + e^{-\beta\omega}}{2(N-1)} + \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij} \right]. \quad (4.11)$$

Splitting into edge-dependent and constant terms

$$\lambda(i, s') = \sum_{j \in B_s} \frac{1 + e^{-\beta\omega}}{2(N-1)} + \sum_{j \in B_s} \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij}. \quad (4.12)$$

Set $\lambda_{N,\pm}(i, s')$ as in (4.7), then

$$\sum_{j \in B_s} \frac{1 + e^{-\beta\omega}}{2(N-1)} = \frac{1 + e^{-\beta\omega}}{2} (\lambda_{N,+}(i, s') + \lambda_{N,-}(i, s')). \quad (4.13)$$

and

$$\sum_{j \in B_s} \frac{1 - e^{-\beta\omega}}{2(N-1)} \sigma_{ij} = \frac{1 - e^{-\beta\omega}}{2} (\lambda_{N,+}(i, s') - \lambda_{N,-}(i, s')), \quad (4.14)$$

Combining terms

$$\lambda_N(i, s') = \lambda_{N,+}(i, s') + e^{-\beta\omega} \lambda_{N,-}(i, s'), \quad (4.15)$$

which is what we wanted to show. $\square$

The definition of $\lambda_{N,\pm}$ in (4.7) indicates that for functions $g(s)$, the deviation from the average behaviour is caused by the unique composition of in- and out-edges of a vertex to each score bin. This matches the behaviour of the stationary measure $\rho(i)$ and the quasipotential $V(i)$ in Figure 3.8a and 3.10. A vertex's ranking within a score bin was decided by the number of out-edges towards extreme score vertices.

We determine the average transition rate of a vertex in B_s towards $B_{s'}$ on tournament graphs by randomising the tournament and taking the expected value of the resulting random $\Lambda_N(i, s')$.

Lemma 4.4. *Let Σ be a random tournament and fix an $i \in B_s$ for $|s| \leq t\sqrt{N \log N}$. The conditional expectation of $\Lambda_N(i, s')$ with respect to $i \in B_s$ given by*

$$\mathbb{E}[\lambda_N(i, s') | i \in B_s] = \frac{1}{2} \frac{(1 + e^{-\beta\omega})}{\sqrt{2\pi(N-1)}} \exp\left(-\frac{(s')^2}{2(N-1)} + \frac{s-s'}{N-1} \tanh\left(\frac{\beta w}{2}\right) - \frac{ss'}{(N-1)^2}\right), \quad (4.16)$$

up to a correction of order $o\left(\frac{1}{\sqrt{N}}\right)$.

Proof. First, we use linearity of expectation to find that

$$\mathbb{E}[\lambda_N(i, s') | i \in B_s] = e^{-\beta\omega} \mathbb{E}[\lambda_{N,+}(i, s') | i \in B_s] + \mathbb{E}[\lambda_{N,-}(i, s') | i \in B_s] \quad (4.17)$$

We can subsequently use Lemma C.2 on both terms to find

$$\mathbb{E}[\lambda_N(i, s') | i \in B_s] = (F_+ + e^{-\beta\omega} F_-) \mathbb{P}(s_j = s'), \quad (4.18)$$

where F_+ and F_- are the corresponding prefactors in (C.5). The prefactors combine into

$$\begin{aligned} F_+ + e^{-\beta\omega} F_- &= \frac{1 + e^{-\beta\omega}}{2} \left(1 - \frac{ss'}{(N-1)^2}\right) + \frac{1 - e^{-\beta\omega}}{2} \frac{s-s'}{N-1} \\ &= \frac{1 + e^{-\beta\omega}}{2} \left(1 - \frac{ss'}{(N-1)^2} + \frac{1 - e^{-\beta\omega}}{1 + e^{-\beta\omega}} \frac{s-s'}{N-1}\right) \\ &= \frac{1 + e^{-\beta\omega}}{2} \left(1 - \frac{ss'}{(N-1)^2} + \frac{s-s'}{N-1} \tanh\left(\frac{\beta w}{2}\right)\right). \end{aligned} \quad (4.19)$$

Pushing the term in brackets into the exponential using a Taylor expansion reduces (4.18) to

$$F_+ + e^{-\beta\omega} F_- = \frac{1 + e^{-\beta\omega}}{2} \left(e^{\frac{s-s'}{N-1} \tanh(\frac{\beta w}{2}) - \frac{ss'}{(N-1)^2}} + O(s^2/N^2) \right). \quad (4.20)$$

It follows that (4.17) can be written as

$$\mathbb{E}[\Lambda_N(i, s') | i \in B_s] = \frac{1 + e^{-\beta\omega}}{2} \left(e^{\frac{s-s'}{N-1} \tanh(\frac{\beta w}{2}) - \frac{ss'}{(N-1)^2}} + O(s^2/N^2) \right) \mathbb{P}(s_j = s') \quad (4.21)$$

Finally, we use the local CLT on $\mathbb{P}(s_j = s')$ to see that

$$\mathbb{E}[\Lambda_N(i, s') \mid i \in B_s] = \frac{1}{2} \frac{(1 + e^{-\beta w})}{\sqrt{\pi(N-1)}} e^{-\frac{s^2}{2(N-1)} + \frac{s-s'}{N-1} \tanh(\frac{\beta w}{2}) - \frac{ss'}{(N-1)^2}} + o\left(\frac{1}{\sqrt{N}}\right), \quad (4.22)$$

which is what we wanted to show. $\square$

The resulting typical behaviour for a vertex $i \in B_s$ is Gaussian to leading order. As a result, the process biases scores towards $s' = 0$. The small correction term accounting for the score of i itself increases the likelihood of transitioning to negative score if i has a positive score, and vice versa.

The next step is to quantify how much $\Lambda_N(i, s)$ fluctuates around its mean behaviour for a vertex $i \in B_s$. To that end, we compute the conditional variance.

Lemma 4.5. *Let Σ be a random tournament and fix an $i \in B_s$ for $|s| \leq t\sqrt{N \log N}$, we have*

$$\mathbb{V}(\Lambda_N(i, s') \mid i \in B_s) = \frac{1}{N} f_+(s, s') + \frac{e^{-2\beta\omega}}{N} f_-(s, s') + O(N^{-2}), \quad (4.23)$$

where

$$f_{\pm}(s, s') = \frac{\alpha_{\pm} p_{N-2}(s' \pm 1)}{2} \left[1 - \frac{\alpha_{\pm} p_{N-2}(s' \pm 1)}{2} \right]. \quad (4.24)$$

and $\alpha_{\pm}$ is given by (3.18).

Proof. By definition

$$\Lambda_N(i, s') = \frac{1}{N-1} \sum_{j \neq i} \mathbb{1}_{\{S_j = s', \Sigma_{ij} = +1\}} + e^{-\beta\omega} \mathbb{1}_{\{S_j = s', \Sigma_{ij} = -1\}}, \quad (4.25)$$

where we will use the shorthand $I(j) = \mathbb{1}_{\{S_j = s', \Sigma_{ij} = +1\}} + e^{-\beta\omega} \mathbb{1}_{\{S_j = s', \Sigma_{ij} = -1\}}$. We start by expanding the variance of the sum

$$\mathbb{V}\left(\sum_{j \neq i} I(j) \mid i \in B_s\right) = \sum_{j \neq i} \mathbb{V}(I(j) \mid i \in B_s) + \sum_{j \neq k} \text{Cov}(I(j), I(k) \mid i \in B_s). \quad (4.26)$$

To find the variance of $I(j)$, note that

$$\begin{aligned} I^2(j) &= \left(\mathbb{1}_{\{S_j = s', \Sigma_{ij} = +1\}} + e^{-\beta\omega} \mathbb{1}_{\{S_j = s', \Sigma_{ij} = -1\}} \right)^2 \\ &= \mathbb{1}_{\{S_j = s', \Sigma_{ij} = +1\}} + e^{-2\beta\omega} \mathbb{1}_{\{S_j = s', \Sigma_{ij} = -1\}}, \end{aligned} \quad (4.27)$$

because the indicators are disjoint. Now, the probability

$$\mathbb{P}(S_j = s', \Sigma_{ij} = \pm 1 \mid i \in B_s) = \mathbb{P}(S^{(i)}(j) = s' \pm 1) \mathbb{P}(\Sigma_{ij} = \pm 1 \mid i \in B_s), \quad (4.28)$$

which by Lemma 3.6 simplifies to

$$\mathbb{P}(S_j = s', \Sigma_{ij}(i, j) = \pm 1 \mid i \in B_s) = \frac{\alpha_{\pm}}{2} p_{N-2}(s' \pm 1). \quad (4.29)$$

Therefore, we find

$$\mathbb{E}[I^2(j) \mid i \in B_s] = \frac{\alpha_+}{2} p_{N-2}(s' + 1) + \frac{\alpha_-}{2} e^{-2\beta\omega} p_{N-2}(s' - 1) \quad (4.30)$$

and

$$\mathbb{E}[I(j) \mid i \in B_s]^2 = \left(\frac{\alpha_+}{2} p_{N-2}(s' + 1) + \frac{\alpha_-}{2} e^{-\beta\omega} p_{N-2}(s' - 1) \right)^2. \quad (4.31)$$

Setting

$$f_{\pm}(s, s') = \frac{\alpha_{\pm}}{2} p_{N-2}(s' \pm 1) \left(1 - \frac{\alpha_{\pm}}{2} p_{N-2}(s' \pm 1) \right) \quad (4.32)$$

we find that first term in (4.26) is given by

$$\mathbb{V}(I(j) \mid i \in B_s) = f_+(s, s') + e^{-2\beta\omega} f_-(s, s') - \frac{\alpha_+ \alpha_-}{2} e^{-\beta\omega} p_{N-2}(s' + 1) p_{N-2}(s' - 1). \quad (4.33)$$

Now, we know that

$$p_{N-2}(s' + 1) p_{N-2}(s' - 1) = O(N^{-1}) \quad (4.34)$$

such that the first term in (4.26) takes the form

$$\sum_{j \neq i} \mathbb{V}(I(j) \mid i \in B_s) = (N-1)(f_+(s, s') + e^{-2\beta\omega} f_-(s, s')) + O(1). \quad (4.35)$$

By Lemma C.5, we get that

$$\sum_{j \neq k} \text{Cov}(I(j), I(k) \mid i \in B_s) = O(1). \quad (4.36)$$

Thus, by (4.25), it follows that

$$\mathbb{V}(\Lambda_N(i, s') \mid i \in B_s) = \frac{1}{N^2} \mathbb{V} \left(\sum_{j \neq i} I(j) \mid i \in B_s \right) = \frac{1}{N} f_+(s, s') + \frac{e^{-2\beta\omega}}{N} f_-(s, s') + O(N^{-2}). \quad (4.37)$$

□

This result enables us to show that $\Lambda_N(i, s')$ concentrates around its conditional expectation. It follows from (4.16) that $\mathbb{E}[\Lambda_N(i, s') \mid i \in B_s] \rightarrow 0$ as $N \uparrow \infty$ due to the prefactor of order $O(N^{-1/2})$. Therefore, we apply Chebyshev's inequality to the scaled difference to prove Theorem 4.2.

Proof of Theorem 4.2. Let $|s| \leq t\sqrt{N \log N}$ and fix an $i \in B_s$. By Chebyshev's inequality, it then follows that

$$\mathbb{P} \left(\sqrt{N} |\Lambda_N(i, s') - \mathbb{E}[\Lambda_N(i, s') \mid i \in B_s]| \geq \varepsilon \mid i \in B_s \right) \leq \frac{\mathbb{V}(\sqrt{N} \Lambda_N(i, s') \mid i \in B_s)}{\varepsilon^2}. \quad (4.38)$$

The variance is given by Lemma 4.5 and for $|s| \leq t\sqrt{N \log N}$ takes the form

$$\mathbb{V}(\Lambda_N(i, s') \mid i \in B_s) = f_+(s, s') + e^{-2\beta\omega} f_-(s, s') + O(N^{-1}). \quad (4.39)$$

By the definition in (4.24), we know that

$$f_{\pm}(s, s') \propto p_{N-2}(s' \pm 1) \quad (4.40)$$

We can use the LCLT (A.5) to estimate the size $p_{N-2}(s') = O(N^{-1/2})$. We conclude that for $|s| \leq t\sqrt{N \log N}$, we have

$$\mathbb{V}(\sqrt{N} \Lambda_N(i, s') \mid i \in B_s) = O(N^{-1/2}) \quad (4.41)$$

and that by (4.38) for all $\varepsilon > 0$

$$\mathbb{P} \left(\sqrt{N} |\Lambda_N(i, s') - \mathbb{E}[\Lambda_N(i, s') \mid i \in B_s]| \geq \varepsilon \mid i \in B_s \right) \leq \varepsilon^{-2} \cdot O(N^{-1/2}), \quad (4.42)$$

which is what we set out to prove. □

4.2 Solving the annealed problem

In this subsection, we work towards proving Theorem 4.1, which provides average value of the heat capacity $\overline{C_N}(\beta)$. The first step is to use the expression derived for $\mathbb{E}[\Lambda_N(i, s') | i \in B_s]$ to define an annealed process that we can use to analytically solve for the heat capacity.

Definition 4.6. *Let*

$$\gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2\sqrt{2\pi(N-1)}} \exp\left(-\frac{(s')^2}{2(N-1)} + \frac{s-s'}{N-1} \tanh \frac{\beta\omega}{2}\right). \quad (4.43)$$

We define the annealed process $(Y_t)_{t \geq 0}$ on the score range $\mathcal{R}$ through the generator $\overline{G_N}$ with entries

$$(\overline{G_N})_{ss'} = \gamma_N(s, s') \quad \text{and} \quad (\overline{G_N})_{ss} = -\sum_{j=1}^N \gamma_N(s, s_j) \quad (4.44)$$

for $s, s' \in \mathcal{R}$.

While any specific tournament graph T_N is restricted to the subset of realised scores $\mathcal{T}_{\mathbf{s}}$, the annealed process is defined on $\mathcal{R}$. This represents an idealised situation in which all potential scores are accessible. The Gaussian factor accounts for the low probability of encountering extreme scores in any random tournament realisation.

Note also that we have dropped the symmetric $\exp(-ss'/N^2)$ term in $\gamma_N(s, s')$, which will simplify the analysis in Section 6.1 by making the expression factorisable in s and s' . Because this term is only significant when the scores are $O(N)$, where the Gaussian is $O(e^{-N})$, this approximation has negligible effect on the dynamics.

The next result shows that $(Y_t)_{t \geq 0}$ satisfies detailed balance and hence reversible. This directly characterises the stationary measure. The proof also shows that dropping the symmetric term from $\gamma_N(s, s')$ has no effect on the stationary measure.

Lemma 4.7. *The chain given by the generator $\overline{G_N}$ is reversible. Furthermore, its stationary distribution*

$$\overline{\pi_N}(s) = \frac{1}{\overline{Z}} \phi(s; -2 \tanh \frac{\beta\omega}{2}, N-1), \quad (4.45)$$

where $\phi(s; \mu, \tau^2)$ denotes the density of a Gaussian random variable with mean μ and variance τ^2 evaluated at s . The normalisation constant is given by

$$\overline{Z} = \sum_{s \in \mathcal{R}} \phi(s; -2 \tanh \frac{\beta\omega}{2}, N-1) = 1 + O(e^{-N/2}). \quad (4.46)$$

For N sufficiently large, $\overline{\pi_N}$ approximates a Normal distribution with $\mu = -2 \tanh \frac{\beta\omega}{2}$ and $\tau^2 = N-1$ with error $O(e^{-N/2})$.

Proof. To show that $\overline{G_N}$ characterises a reversible process, we check the detailed balance condition

$$\pi_N(s) \gamma_N(s, s') = \pi_N(s') \gamma_N(s', s). \quad (4.47)$$

Using the definition of γ_N in (4.43), we find that

$$\frac{\pi_N(s)}{\pi_N(s')} = \exp\left(-\frac{s^2}{2(N-1)} + \frac{(s')^2}{2(N-1)} + 2\frac{s'-s}{N-1} \tanh \frac{\beta\omega}{2}\right). \quad (4.48)$$

Since this must hold for all $s, s' \in \mathcal{R}$ we see that

$$\pi_N(s) \propto \exp\left(-\frac{s^2}{2(N-1)} - \frac{2s}{N-1} \tanh \frac{\beta\omega}{2}\right). \quad (4.49)$$

Then we can complete the square to see that

$$\pi_N(s) \propto \exp \left(-\frac{\left[s + 2 \tanh \frac{\beta\omega}{2} \right]^2}{2(N-1)} \right) \propto \phi(s; -2 \tanh \frac{\beta\omega}{2}, N-1). \quad (4.50)$$

Direct application of Lemma A.7 in the appendix now gives

$$\bar{Z} = \sum_{s \in \mathcal{R}} \phi(s; -2 \tanh \frac{\beta\omega}{2}, N-1) = 1 + O(e^{-N/2}), \quad (4.51)$$

which finishes the proof. $\square$

The transition rates' Gaussian shape directly translates into a Gaussian stationary distribution. The mean is slightly negative and increases in magnitude with β , reflecting the increased the asymmetry of the rates favouring negative score as β increases.

The expression for the stationary distribution $\bar{\pi}_N$ allows us to calculate the background dissipation rate for $(Y_t)_{t \geq 0}$ with respect to the instantaneous power (2.64) a vertex of T_N . We use the fact that $\bar{\pi}_N$ approximates a Normal distribution to prove the following corollary.

Corollary 4.8. *The mean of the instantaneous power (3.40) with respect to the annealed stationary measure is given by*

$$\langle \mathcal{P} \rangle_{\bar{\pi}} = \frac{(1 + e^{-\beta})}{2} \frac{\omega(N-3)}{N-1} \tanh \frac{\beta\omega}{2} \quad (4.52)$$

As a result, the annealed excess dissipation is given by

$$\bar{\mathcal{Q}}(s) := \mathcal{P}(s) - \langle \mathcal{P} \rangle_{\bar{\pi}} = \frac{1 + e^{-\beta\omega}}{2} \left[\frac{\omega s}{N-1} + \frac{2\omega}{N-1} \tanh \frac{\beta\omega}{2} \right], \quad (4.53)$$

for all $s \in \mathcal{R}$.

Proof. From Lemma 4.7 we know that π_N behaves like a Gaussian with $\mu = -2 \tanh \frac{\beta\omega}{2}$ and $\tau^2 = N$. Given the expression for the instantaneous power $\mathcal{P}$ on a tournament graph (3.40), we thus find

$$\begin{aligned} \langle \mathcal{P} \rangle_{\bar{\pi}} &= \sum_{s'} \frac{\omega(1 + e^{-\beta\omega})}{2(N-1)} \left[s' + (N-1) \tanh \frac{\beta\omega}{2} \right] \pi_N(s') \\ &= \frac{\omega(1 + e^{-\beta})}{2(N-1)} \sum_{s'} s' \pi_N(s') + (N-1) \tanh \frac{\beta\omega}{2} \pi_N(s') \\ &= \frac{(1 + e^{-\beta})}{2} \frac{\omega(N-3)}{N-1} \tanh \frac{\beta\omega}{2} + O(e^{-N/2}). \end{aligned} \quad (4.54)$$

Neglecting the exponentially small error term, we find

$$\mathcal{P}(s) - \langle \mathcal{P} \rangle_{\bar{\pi}} = \frac{1 + e^{-\beta\omega}}{2} \left[\frac{\omega s}{N-1} + \frac{2\omega}{N-1} \tanh \frac{\beta\omega}{2} \right], \quad (4.55)$$

which finishes the proof. $\square$

We can substitute the excess dissipation rate $\bar{\mathcal{Q}}(s)$ (4.53) into the definition of the excess heat in (2.66) find that for the process $(Y_t)_{t \geq 0}$ starting from s at t_0 it is given by

$$\bar{W}_N(s) = \int_0^\infty \langle \bar{\mathcal{Q}}(s) |_{s_{t_0} = s} \rangle dt. \quad (4.56)$$

As a result, the quasipotential $\bar{W}_N$ satisfies the Poisson equation as described in (2.69).

Definition 4.9. We call the annealed Poisson problem the equation

$$\overline{G_N W_N}(s) = -\frac{1 + e^{-\beta\omega}}{2} \left[\frac{\omega s}{N-1} + \frac{2\omega}{N-1} \tanh \frac{\beta\omega}{2} \right], \quad (4.57)$$

which is defined for all $s \in \mathcal{R}$ on the zero-mean subspace, that is, $\sum_s \overline{W_N}(s) \pi_N(s) = 0$.

To solve this equation, a first attempt would be to use a polynomial ansatz in powers of s/N . Because annealed transition rates $\gamma_N(s, s')$ (4.43) factorise into a prefactor and a Gaussian density function, $\gamma_N(s, s') \propto \exp\left(\frac{s}{N-1} \tanh \frac{\beta\omega}{2}\right) \cdot \phi(s; \dots)$, this approach fails. The specific form means that the generator $\overline{G_N}$ acts on a generic function $g(s)$ as

$$\overline{G_N W_N}(s) \propto \exp\left(\frac{s}{N-1} \tanh \frac{\beta\omega}{2}\right) (\langle g \rangle - g(s)), \quad (4.58)$$

and a finite polynomial cannot cancel the exponential prefactor.

Since the right-hand side of (4.57) is linear in s , the polynomial ansatz for $\overline{W_N}(s)$ would have to be an infinite series to cancel the prefactor order-by-order. We instead incorporate the inverse of the exponential factor directly into the ansatz to prove Theorem 4.1.

Proof of Theorem 4.1. Let $T = \tanh \beta\omega/2$ for brevity of notation. We start by setting

$$\overline{W_N}(s) = a \frac{s}{N-1} \exp\left(-\frac{sT}{N-1}\right) + b, \quad (4.59)$$

Using Lemma C.6 we find that

$$\begin{aligned} \sum_{s' \in \mathcal{R}} \gamma_N(s, s') \frac{s'}{N-1} \exp\left(-\frac{s'T}{N-1}\right) &= \frac{1 + e^{-\beta\omega}}{2(N-1)} \exp\left(\frac{sT}{N-1}\right) \sum_{s' \in \mathcal{R}} \frac{s' \exp\left(-\frac{(s'+2T)^2}{2(N-1)}\right)}{\sqrt{2\pi(N-1)}} \\ &= \frac{1 + e^{-\beta\omega}}{2(N-1)} \exp\left(\frac{sT}{N-1}\right) (-2T + O(e^{-N/2})), \end{aligned} \quad (4.60)$$

where we used Lemma A.7 to approximate the sum. Neglecting the exponentially small error term, the action of the generator on the full ansatz is then

$$\overline{G_N W_N}(s) = -\frac{1 + e^{-\beta\omega}}{2} \left[\exp\left(\frac{sT}{N-1}\right) \frac{2aT}{N-1} + \frac{as}{N-1} \right]. \quad (4.61)$$

We can now Taylor expand the prefactor $\exp\left(\frac{sT}{N-1}\right)$ and collect the s dependent terms to get

$$\overline{G_N W_N}(s) = -\frac{1 + e^{-\beta\omega}}{2} \left[\frac{2aT}{N-1} + \frac{as}{N-1} \left(1 + O\left(\frac{1}{N}\right)\right) \right]. \quad (4.62)$$

Hence, a comparison to the right-hand side of (4.57) shows that $a = \omega$, up to corrections of at most $O(s/N^2)$. As a result, we have

$$\overline{W_N}(s) = \omega \frac{s}{N-1} \exp\left(-\frac{sT}{N-1}\right) + b \quad (4.63)$$

Next, we subject this expression to the mean-zero condition of the Poisson equation (4.57) to find b . Using Lemma 4.7, we see that

$$\sum_s \overline{W_N}(s) \pi_N(s) = \omega \left\langle \frac{s}{N-1} \exp\left(-\frac{sT}{N-1}\right) \right\rangle_{\pi_N} + b = 0. \quad (4.64)$$

Using Lemma 4.7, we write out the mean

$$\left\langle \frac{s}{N-1} \exp\left(-\frac{sT}{N-1}\right) \right\rangle_{\pi_N} = \sum_{s \in \mathcal{R}} \frac{s}{N-1} \exp\left(-\frac{sT}{N-1}\right) \frac{1}{Z} \phi(s; -2T, N-1). \quad (4.65)$$

As in (4.60), we can absorb the exponential factor into the mean of the Gaussian to obtain

$$\left\langle \frac{s}{N-1} \exp\left(-\frac{sT}{N-1}\right) \right\rangle_{\pi_N} = \sum_{s \in \mathcal{R}} \frac{s}{N-1} \frac{1}{Z} \phi(s; -3T, N-1). \quad (4.66)$$

Therefore, we see that (4.64) reduces to

$$b = \frac{3T}{N-1}. \quad (4.67)$$

We again used Lemma A.7 to approximate the sum. Hence, the final expression for the quasipotential is

$$\overline{W}_N(s) = \omega \frac{s}{N-1} \exp\left(-\frac{s}{N-1} \tanh \frac{\beta\omega}{2}\right) + \frac{3}{N-1} \tanh \frac{\beta\omega}{2}. \quad (4.68)$$

Differentiating with respect to beta gives

$$\beta^2 \frac{\partial \overline{W}_N}{\partial \beta}(s) = -\frac{(\beta\omega)^2}{2} \operatorname{sech}^2 \frac{\beta\omega}{2} \left(\frac{s}{N-1}\right)^2 \exp\left(-\frac{s}{N-1} T\right) + \frac{3(\beta\omega)^2}{2(N-1)} \operatorname{sech}^2 \frac{\beta\omega}{2}. \quad (4.69)$$

After taking the average with respect to π_N and using the property we derive in (4.64), we find that the heat capacity is

$$\begin{aligned} \left\langle \beta^2 \frac{\partial \overline{W}_N}{\partial \beta}(s) \right\rangle &= -\frac{(\beta\omega)^2}{2(N-1)} \operatorname{sech}^2 \frac{\beta\omega}{2} + \frac{3(\beta\omega)^2}{2(N-1)} \operatorname{sech}^2 \frac{\beta\omega}{2} \\ &= \frac{(\beta\omega)^2}{N-1} \operatorname{sech}^2 \frac{\beta\omega}{2}, \end{aligned} \quad (4.70)$$

which completes the proof. $\square$

5. Statistical properties of random tournaments

This section examines the statistical properties of the tournament graph T_N introduced in Definition 3.1. In particular, we prove two useful results regarding the random size of the score bins B_s (3.11) and we give the conditions under which a linear combination converges in distribution to a Normal random variable. The first result we prove bounds the fluctuation of the bin size $N_s = |B_s|$ from its expected value.

Theorem 5.1. *Let $\Sigma = (\Sigma_{ij})_{i,j \leq N}$ be a random tournament. The counts N_s concentrate around their means uniformly for $s \in \mathcal{R}$ with failure probability $O(N^{-k})$ for any arbitrary $k > 0$. Specifically,*

$$\sup_{s \in \mathcal{R}} \left| \frac{N_s - \mathbb{E}[N_s]}{\sqrt{N}} \right| \leq c N^{-1/4} \sqrt{\log N}, \quad (5.1)$$

where c is a constant chosen such that $c > \sqrt{k\sqrt{\pi/2}}$.

The second result provides a local central limit theorem for the size of each score bin, showing that the marginal probability mass function of the bin sizes N_s approximates a Gaussian density. This is shown in Figure 5.1.

Theorem 5.2. *Uniformly for all $s \in \mathcal{T}_{\mathbf{S}}$ and uniformly for all integers t such that $|t - \mathbb{E}[N_s]| = O(N^{-1/4}\sqrt{\log N})$, we have*

$$\mathbb{P}(N_s = t) = \frac{1}{\sqrt{2\pi N p_{N-1}(s)(1 - p_{N-1}(s))}} \exp\left(-\frac{(t - N p_{N-1}(s))^2}{2N p_{N-1}(s)(1 - p_{N-1}(s))}\right) + o(N^{-1/2}). \quad (5.2)$$

Similar results are well known for the Erdős-Rényi graph [97, 98], but have not yet been proved for random tournaments. Because directing the edges in T_N is a procedure similar to removing edges, it might seem that this result could carry over. However, the tournament has a different dependence structure. The vertex degrees in an Erdős-Rényi graph are positively correlated, while vertex scores are negatively correlated.

To handle this dependence structure when analysing functions of multiple score bins, we also prove a more general CLT for linear combinations of the bin counts.

Theorem 5.3. *Let $\{w_s\}_{s \in \mathcal{R}}$ be a sequence of deterministic weights and define the centred sum*

$$Z_N(\mathbf{S}) = \sum_{s \in \mathcal{R}} w_s (N_s - \mathbb{E}[N_s]). \quad (5.3)$$

Denote the weight w_s evaluated at an arbitrary vertex $i \in \mathcal{V}$ by w_{S_i} . Assume the maximum centred weight is bounded

$$M_N = \max_{s \in \mathcal{R}} |w_s - \mathbb{E}[w_{S_i}]| = O(N^t), \quad (5.4)$$

for some fixed $t \geq 0$. If the weights satisfy

$$M_N = o\left(\sqrt{N \mathbb{V}(w_{S_i})}\right) \quad \text{and} \quad \limsup_{N \rightarrow \infty} |\text{Corr}(w_{S_i}, S_i)| < 1, \quad (5.5)$$

then the standardised sum converges in distribution to a standard Normal

$$\frac{Z_N}{\sqrt{\mathbb{V}(Z_N)}} \xrightarrow{D} \mathcal{N}(0, 1). \quad (5.6)$$

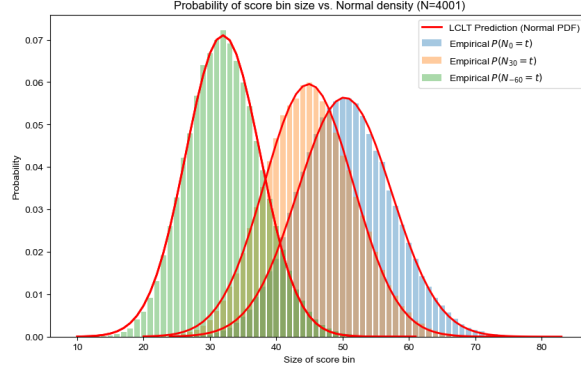


Figure 5.1: Local Central Limit Theorem for the score bin size N_s for $s \in \{-60, 30, 0\}$. The empirical probability mass function (bars) is compared against the theoretical Gaussian density (red line). The data was generated from 10^5 random tournaments at $N = 4001$. The agreement of the discrete mass points with the Gaussian density illustrates the local approximation of the marginal distribution.

We conclude the section by applying the results to improve the convergence of the coarse-grained rates $\Lambda_N(i, s')$ in Theorem 4.2. In particular, we prove that $\Lambda_N(i, s')$ has a CLT property.

Theorem 5.4. *Let Σ be a random tournament and fix an arbitrary vertex $i \in B_s$ for any score $|s| < \frac{1}{2}\sqrt{N \log N}$. Let $\mu_N = \mathbb{E}[\Lambda(i, s') | i \in B_s]$ and $\tau_N^2 = \mathbb{V}(\Lambda_N | i \in B_s)$. We have*

$$\frac{\Lambda_N(i, s') - \mu_N}{\tau_N} \xrightarrow{D} \mathcal{N}(0, 1). \quad (5.7)$$

5.1 The $\mathcal{D}$ -model

Generating a random tournament involves independently sampling edge directions Σ_{ij} , resulting in scores $S_i \sim 2 \cdot \text{Bin}(N-1, 1/2) - (N-1)$ for an $i \in \mathcal{V}$. However, the shared edges impose a global constraint $\sum_{i=1}^N S_i = 0$, creating global dependence. This global coupling hinders standard CLT techniques for dependent variables, such as Stein's method or the Method of Moments, since they typically rely on local independence or sparse dependency graphs [99, 100].

Instead, we apply a change of measure to prove Theorem 5.1 and Theorem 5.2. Rather than working with the tournament scores $\mathbf{S} = (S_1, \dots, S_N)$ directly, we introduce a collection of independent binomial random variables conditioned on the score constraint. We call this the $\mathcal{D}$ -model.

Definition 5.5. *Let $\mathcal{V}$ be a set of N vertices. Let $\tilde{\mathbf{S}} = (\tilde{S}_1, \dots, \tilde{S}_N)$ be a collection of independent random variables such that $\tilde{S}_i = 2 \cdot \text{Bin}(N-1, 1/2) - (N-1)$. The $\mathcal{D}$ -model is the probability measure $\mathbb{P}_{\mathcal{D}}$ resulting from conditioning these variables on the constraint $\tilde{S}_N := \sum_{i=1}^N \tilde{S}_i = 0$.*

McKay and Wang derive the density of the tournament measure with respect to the $\mathcal{D}$ -model in [35].

Theorem 5.6 ([35] Theorem 4.5). *Let Σ be a random tournament. For any score sequence $(s_1, \dots, s_N)$, the density of this sequence is with respect to the probability under the $\mathcal{D}$ -model is given by*

$$\begin{aligned} \mathbb{P}(S_1 = s_1, \dots, S_N = s_N) &= \mathbb{P}_{\mathcal{D}}(\tilde{S}_1 = s_1, \dots, \tilde{S}_N = s_N) \exp \left(-\frac{3}{4} + \frac{1}{N^2} \sum_{j=1}^N s_j^2 - \frac{1}{4N^4} \left(\sum_{j=1}^N s_j^2 \right)^2 \right. \\ &\quad \left. - \frac{1}{6N^6} \left(\sum_{j=1}^N s_j^3 \right)^2 - \frac{1}{2N^7} \left(\sum_{j=1}^N s_j^2 \right)^3 + O \left(\frac{(s^*)^4}{N^3} + N^{-1/4+\epsilon} \right) \right). \end{aligned} \quad (5.8)$$

The following useful transfer theorem is mentioned in [35] as a result of Theorem 5.6, but not proved. We will therefore provide our own proof.

Theorem 5.7 ([35] Theorem 4.6). *Let $g(s_1, \dots, s_N)$ be any function such that $|g(s_1, \dots, s_N)| = O(N^t)$ for some fixed $t \geq 0$. Denote the expectation under the $\mathcal{D}$ -model as $\mathbb{E}_{\mathcal{D}}[\cdot]$. For any $\varepsilon > 0$ and $k > t$, we have*

$$\mathbb{E}[g(S_1, \dots, S_N)] = O(N^{t-k}) + (1 + O(N^{-1/4+\varepsilon})) \mathbb{E}_{\mathcal{D}}[g(\tilde{S}_1, \dots, \tilde{S}_N)]. \quad (5.9)$$

We prove this theorem by splitting into a case where the scores are typical and a case where the scores are atypical. The error term $O(N^{t-k})$ accounts for the event that the scores $\mathbf{S}$ fall outside of $\Omega_{\text{typ}}(k)$. The first step in proving Theorem 5.7 is therefore to provide estimates on the sum of squared and cubed scores that hold on the typical set $\Omega_{\text{typ}}(k)$ (3.33). These will help bound the exponent in (5.8). McKay and Wang [35] mention these in passing, but do not prove them.

Proposition 5.8. *Fix $\varepsilon > 0$. For any $k > 0$ and all N sufficiently large, if a tournament graph $T_N \in \Omega_{\text{typ}}(k)$, then the score sequence satisfies*

$$\sum_{i=1}^N s_i^2 = (1 + O(N^{-1/2+\varepsilon})) N^2 \quad \text{and} \quad \sum_{i=1}^N s_i^3 = O(N^{5/2+\varepsilon}). \quad (5.10)$$

We can substitute for these estimates in (5.8) in order to obtain a direct bound on the density.

Lemma 5.9. *Fix $\varepsilon > 0$. For any score sequence $(s_1, \dots, s_N)$ corresponding to a tournament in $T_N \in \Omega_{\text{typ}}(k)$, the density of this sequence is with respect to the probability under the $\mathcal{D}$ -model is given by*

$$\mathbb{P}(S_1 = s_1, \dots, S_N = s_N) = \mathbb{P}_{\mathcal{D}}(\tilde{S}_1 = s_1, \dots, \tilde{S}_N = s_N) (1 + O(N^{-1/4+\varepsilon})). \quad (5.11)$$

The bound on the density allows us to relate the expectation under the $\mathcal{D}$ -model to a random tournament, so that we may prove Theorem 5.7.

Proofs: Section 5.1

Proof of Theorem 5.7. Let $\Omega_{\text{typ}}(k)$ be the set defined in (3.33). We can rewrite the expectation by splitting the expression into

$$\mathbb{E}[g] = \mathbb{E}[g \cdot \mathbb{1}_{\{\Omega_{\text{typ}}(k)\}}] + \mathbb{E}[g \cdot \mathbb{1}_{\{\Omega_{\text{typ}}^c(k)\}}]. \quad (5.12)$$

By Lemma 3.7, the probability that the tournament T_N is non-typical is $\mathbb{P}(T_N \notin \Omega_{\text{typ}}(k)) = O(N^{-k})$. Combining this with the assumption that $|g(s_1, \dots, s_N)| = O(N^t)$, yields a direct bound on the second term

$$|\mathbb{E}[g \cdot \mathbb{1}_{\Omega_{\text{typ}}^c(k)}]| \leq |g(s_1, \dots, s_N)| \cdot \mathbb{P}(\Omega_{\text{typ}}^c(k)) = O(N^t) \cdot O(N^{-k}) = O(N^{t-k}). \quad (5.13)$$

On the typical set $\Omega_{\text{typ}}(k)$, the estimates of Proposition 5.8 hold. The expectation over the typical set now becomes

$$\mathbb{E}[g \cdot \mathbb{1}_{\Omega_{\text{typ}}(k)}] = \sum_{s \in \Omega_{\text{typ}}(k)} g(s) \mathbb{P}(s) \quad (5.14)$$

$$= \sum_{s \in \Omega_{\text{typ}}(k)} g(s) \mathbb{P}_{\mathcal{D}}(s) (1 + O(N^{-1/4+\varepsilon})) \quad (5.15)$$

$$= (1 + O(N^{-1/4+\varepsilon})) \mathbb{E}_{\mathcal{D}}[g \cdot \mathbb{1}_{\Omega_{\text{typ}}(k)}]. \quad (5.16)$$

We must relate the restricted expectation $\mathbb{E}_{\mathcal{D}}[g \cdot \mathbb{1}_{\Omega_{\text{typ}}(k)}]$ back to the full expectation $\mathbb{E}_{\mathcal{D}}[g]$. We use the fact that the atypical set also has probability $O(N^{-k})$ under the $\mathcal{D}$ -model, so that

$$\mathbb{E}_{\mathcal{D}}[g \cdot \mathbb{1}_{\Omega_{\text{typ}}(k)}] = \mathbb{E}_{\mathcal{D}}[g] - \mathbb{E}_{\mathcal{D}}[g \cdot \mathbb{1}_{\Omega_{\text{typ}}^c(k)}] = \mathbb{E}_{\mathcal{D}}[g] + O(N^{-(k-t)}). \quad (5.17)$$

Combining (5.14) and (5.17) results in

$$\mathbb{E} \left[g \cdot \mathbb{1}_{\Omega_{\text{typ}}(k)} \right] = (1 + O(N^{-1/4+\varepsilon})) \mathbb{E}_{\mathcal{D}}[g] + O(N^{t-k}). \quad (5.18)$$

We find the final bound by substituting (5.13) and (5.18) into (5.12)

$$\mathbb{E}[g] = O(N^{t-k}) + (1 + O(N^{-1/4+\varepsilon})) \mathbb{E}_{\mathcal{D}}[g], \quad (5.19)$$

for any $k, \varepsilon > 0$. $\square$

Proof of Proposition 5.8. We work on the event that $T_N \in \Omega_{\text{typ}}(k)$. By Definition 3.8, this implies that $\max_i |s_i| \leq s^*$, where $s^* = \sqrt{2(k+1)N \log N}$. Let $S_3 = \sum_i s_i^3$. To derive the bound on S_3 , we use the triangle inequality to write

$$\left| \sum_{i=1}^N s_i^3 \right| \leq \sum_{i=1}^N |s_i|^3 \leq N(s^*)^3. \quad (5.20)$$

Substituting the value of s^* , we obtain

$$\sum_{i=1}^N s_i^3 \leq N(2(k+1)N \log N)^{3/2}. \quad (5.21)$$

Because $(\log N)^{3/2}$ is asymptotically smaller than any N^ε , we get

$$\sum_{i=1}^N s_i^3 = O(N^{5/2+\varepsilon}). \quad (5.22)$$

To bound $S_2 = \sum s_i^2$, we use the McDiarmid inequality. Flipping a single edge Σ_{ij} changes the respective scores as $s_i \rightarrow s_i \pm 2$ and $s_j \rightarrow s_j \mp 2$. The change in the sum of squares is

$$c_{ij} = \left| (s_i \pm 2)^2 + (s_j \mp 2)^2 - s_i^2 - s_j^2 \right| \leq 4(|s_i| + |s_j|) + 8. \quad (5.23)$$

Note that on the event $\Omega_{\text{typ}}(k)$, we have that the maximum score $\max_i |s_i| \leq s^*$. This means that for all sufficiently large N , the change in the truncated sum is bounded by

$$c'_{ij} \leq 8s^* + 8 \leq 16\sqrt{2(k+1)N \log N}. \quad (5.24)$$

Summing over all edges, we find

$$\sum_{(i,j) \in E} (c'_{ij})^2 \leq \frac{N^2}{2} \cdot 256 \cdot 2(k+1)N \log N = 256(k+1)N^3 \log N. \quad (5.25)$$

McDiarmid's inequality now yields

$$\mathbb{P}(|S'_2 - \mathbb{E}[S'_2]| > t) \leq 2 \exp \left(-\frac{2t^2}{256(k+1)N^3 \log N} \right). \quad (5.26)$$

Taking $t = N^{3/2+\varepsilon}$, the exponent is of order $N^{2\varepsilon}/\log N$, which tends to infinity faster than $k \log N$ for any $\varepsilon > 0$. Thus,

$$\mathbb{P}(|S'_2 - \mathbb{E}[S'_2]| > N^{3/2+\varepsilon}) = O(N^{-k}). \quad (5.27)$$

Finally, we relate this to the full sum S_2 . We have

$$\mathbb{P}(|S_2 - \mathbb{E}[S_2]| > t) \leq \mathbb{P}(S_2 \neq S'_2) + \mathbb{P}(|S'_2 - \mathbb{E}[S'_2]| + |\mathbb{E}[S'_2] - \mathbb{E}[S_2]| > t). \quad (5.28)$$

The first term is exactly the probability that a tournament is not in $\Omega_{\text{typ}}(k)$, which is $O(N^{-k})$. For the expectation difference, we have

$$|\mathbb{E}[S'_2] - \mathbb{E}[S_2]| \leq \mathbb{E}[|S'_2 - S_2| \mathbb{1}_{\{S_2 \neq S'_2\}}] \leq S_{2,\max} \mathbb{P}(S_2 \neq S'_2), \quad (5.29)$$

where $S_{2,\max} < N^3/4$. Since $\mathbb{P}(S_2 \neq S'_2) = O(N^{-k})$, the difference in expectations is $O(N^{3/4-k})$, which dominated by $N^{3/2} \log N$ for any $k > 0$. Combining the bounds, we thus get

$$\mathbb{P}(|S_2 - \mathbb{E}[S_2]| > N^{3/2+\varepsilon}) \leq O(N^{-(k+3/2)}) + O(N^{-k}) = O(N^{-k}) \quad (5.30)$$

for all $k > 0$, which is what we set out to show. $\square$

5.2 Score bins CLT

In this subsection, we state and prove the variants of Theorem 5.1 and Theorem 5.2 for the $\mathcal{D}$ -model, so that we can transfer them to the random tournament using Theorem 5.7. First, we calculate the density of the scores in the $\mathcal{D}$ -model with respect to the fully independent scores.

Lemma 5.10. *Let $\tilde{N}_s$ be the number of vertices in the $\mathcal{D}$ -model with score s . Then, it follows that*

$$\mathbb{P}_{\mathcal{D}}(\tilde{N}_s = t) = \mathbb{P}(\tilde{N}_s = t) R(t; N, s), \quad (5.31)$$

where

$$R(t; N, s) = \sqrt{\frac{1 - p_{N-1}(s)}{\nu}} \exp\left(-\frac{s^2}{2} \frac{(N p_{N-1}(s) - t)^2}{N(N-1)(1 - p_{N-1}(s))\nu}\right) \quad (5.32)$$

and

$$\nu = \left(1 - \frac{t}{N}\right) \left(1 - \frac{s^2 p_{N-1}(s)}{(N-1)(1 - p_{N-1}(s))}\right) \quad (5.33)$$

Using the standard concentration results for independent random variables to bound $\mathbb{P}(\tilde{N}_s = t)$, the primary task becomes controlling the density $R(t; N, s)$ (5.32). This is exactly what we do to prove concentration of the bin sizes B_s in the $\mathcal{D}$ -model. Because the size of $R(t; N, s)$ is determined by the size of t , we control the deviation of the bin sizes $\tilde{N}_s$ from their expectation.

Lemma 5.11. *For any fixed score such that $|s| < N/2$, $\tilde{N}_s$ concentrates around its unconditional mean. Specifically, for any $c > 0$ and for any s in the range of scores*

$$\mathbb{P}_{\mathcal{D}}\left(\left|\frac{\tilde{N}_s - \mathbb{E}[\tilde{N}_s]}{\sqrt{N}}\right| > cN^{-1/4}\sqrt{\log N}\right) \leq 4N^{-c^2\sqrt{\pi/2}} \quad (5.34)$$

The exponentially decreasing bound allows us to upgrade the result to hold uniformly on all $s \in \mathcal{R}$, proving the $\mathcal{D}$ -model version of Theorem 5.1.

Lemma 5.12. *In the $\mathcal{D}$ -model, the counts $\tilde{N}_s$ concentrate around their means uniformly in s with failure probability $O(N^{-k})$ for any arbitrary $k > 0$. Specifically,*

$$\sup_{s \in \mathcal{R}} \left| \frac{\tilde{N}_s - \mathbb{E}[\tilde{N}_s]}{\sqrt{N}} \right| \leq cN^{-1/4}\sqrt{\log N} \quad (5.35)$$

where c is a constant chosen such that $c > \sqrt{k\sqrt{\pi/2}}$.

Using this result to bound the density $R(t; N, s)$ (5.32), the following local central limit theorem holds with high probability for all score bins B_s .

Corollary 5.13. *Uniformly for all scores such that $|s| = O(\sqrt{N \log N})$ and counts t such that $|t - \mathbb{E}[N_s]| = O(N^{1/4} \sqrt{\log N})$, we have*

$$\mathbb{P}_{\mathcal{D}}(\tilde{N}_s = t) = \frac{1 + o(1)}{\sqrt{2\pi N p_{N-1}(s)(1 - p_{N-1}(s))}} \exp\left(-\frac{(t - N p_{N-1}(s))^2}{2N p_{N-1}(s)(1 - p_{N-1}(s))}\right). \quad (5.36)$$

The final step transfers the results in Lemma 5.12 and Corollary 5.13 to the random tournament by using Theorem 5.7. A direct application of Theorem 5.1 proves the existence of an interval I_N within which no scores are missing with high probability.

Corollary 5.14. *Let Σ be a random tournament on N vertices. For all $\epsilon < 1$, define the interval*

$$I_N = \left[-\sqrt{(1 - \epsilon)N \log N}, \sqrt{(1 - \epsilon)N \log N}\right]. \quad (5.37)$$

Then, all scores in I_N are observed on T_N with failure probability $O(N^{-k})$ for all $k > 0$.

The same machinery of proving a result in the $\mathcal{D}$ -model and subsequently lifting it to the tournament using Theorem 5.7 is used to prove Theorem 5.3. In the $\mathcal{D}$ -model, the scores are initially independent before conditioning on the total score constraint. This means that if we rewrite the sum as a sum over the vertices

$$Z_N(\tilde{\mathbf{S}}) = \sum_{s \in \mathcal{R}} w_s (N_s - \mathbb{E}[N_s]) = \sum_{i=1}^N w_{\tilde{S}_i} - \mathbb{E}[w_{\tilde{S}_i}], \quad (5.38)$$

the $w_{\tilde{S}_i}$ are independent before conditioning. By pairing each term with its score $(w_{\tilde{S}_i}, \tilde{S}_i)$, the bivariate central limit theorem guarantees the sum of these are jointly Gaussian. Provided that the weights satisfy some regularity conditions.

A standard property of multivariate Gaussians is that conditioning a continuous joint Gaussian on a linear constraint yields a new, lower-dimensional Gaussian. However, we are not conditioning the continuous limit. We condition the discrete pre-limit on the exact event $\sum_{i=1}^N S_i = 0$. Because the LCLT tells us that the probability of this event decays as $O(N^{-1/2})$, the standard central limit theorem is not strong enough to rigorously justify the limit of the conditional distribution.

Instead, we rely on a conditional limit theorem by Janson (Theorem A.9) in the appendix, which handles exactly our situation. Verifying the regularity conditions of this framework leads directly to the following result for the $\mathcal{D}$ -model.

Lemma 5.15. *For each N , let $\{w_k\}_{k \in \mathcal{R}}$ be a deterministic weight sequence. Define the centered linear combination by*

$$Z_N(\tilde{\mathbf{S}}) = \sum_{s \in \mathcal{R}} w_s (\tilde{N}_s - \mathbb{E}[\tilde{N}_s]) \quad (5.39)$$

If the weight w_s evaluated at an arbitrary vertex $i \in \mathcal{V}$ satisfies

$$\mathbb{E} \left| w_{S_i} - \mathbb{E}[w_{S_i}] \right|^3 = o\left(N^{1/2} \mathbb{V}(w_{S_i})^{3/2}\right) \quad \text{and} \quad \limsup_{N \rightarrow \infty} |\text{Corr}(w_{S_i}, S_i)| < 1 \quad (5.40)$$

then, conditioned on $\sum_{i=1}^N \tilde{S}_i = 0$, the standardised sum $Z_N/(N^{1/2} \tau_N)$ converges in distribution to $\mathcal{N}(0, 1)$, where $\tau_N^2 = \sigma_{Y_N}^2 - \text{Cov}(\tilde{S}, Y_N)^2 / \text{Var}(\tilde{S})$.

Proofs: Section 5.2

Proof of Theorem 5.1 and Theorem 5.2. In order to prove Theorem 5.1 and Theorem 5.2, we lift the results of Lemma 5.12 and Corollary 5.13 onto the tournament using Theorem 5.7. Because the size

of the score bin N_s , is entirely determined by the score sequence $\mathbf{S}$ of the tournament graph T_N , we define the failure event for the concentration bound as

$$\mathcal{E}(\mathbf{S}) = \bigcup_{s \in \mathcal{R}} \left\{ \left| \frac{N_s - \mathbb{E}[N_s]}{\sqrt{N}} \right| > cN^{-1/4} \sqrt{\log N} \right\}. \quad (5.41)$$

Let $g(\mathbf{S}) = \mathbb{1}_{\{\mathbf{S} \in \mathcal{E}\}}$ be the indicator function of this event. Because $|g| \leq 1$, we can apply Theorem 5.7 with $t = 0$ so that $r = k$. This gives

$$\mathbb{P}(\mathbf{S} \in \mathcal{E}) = \mathbb{E}[g(\mathbf{S})] = O(N^{-k}) + (1 + O(N^{-1/4+\varepsilon})) \mathbb{P}_{\mathcal{D}}(\mathcal{E}). \quad (5.42)$$

We can now apply Lemma 5.12 to bound $\mathbb{P}_{\mathcal{D}}(\mathcal{E})$. Setting $c > \sqrt{k\sqrt{\pi/2}}$ then results in

$$\mathbb{P}(\mathbf{S} \in \mathcal{E}) = O(N^{-k}) + (1 + O(N^{-1/4+\varepsilon})) O(N^{-k}) = O(N^{-k}), \quad (5.43)$$

which proves Theorem 5.1.

To transfer Corollary 5.13 to the tournament, we fix s such that $|s| = O(\sqrt{N \log N})$ and t such that $|t - \mathbb{E}[N_s]| = O(N^{1/4} \sqrt{\log N})$. We define the indicator function $h(N_s) = \mathbb{1}_{\{N_s = t\}}$. Since $|h| \leq 1$, we again apply Theorem 5.7 with $t = 0$ and $r = k$ to obtain

$$\mathbb{P}(N_s = t) = O(N^{-k}) + (1 + O(N^{-1/4+\varepsilon})) \mathbb{P}_{\mathcal{D}}(\tilde{N}_s = t). \quad (5.44)$$

Substituting the local limit approximation from Corollary 5.13 yields

$$\mathbb{P}(N_s = t) = O(N^{-k}) + (1 + O(N^{-1/4+\varepsilon})) \frac{1 + o(1)}{\sqrt{2\pi N p_{N-1}(s)(1 - p_{N-1}(s))}} \exp\left(-\frac{(t - N p_{N-1}(s))^2}{2N p_{N-1}(s)(1 - p_{N-1}(s))}\right). \quad (5.45)$$

Because $p_{N-1}(s) = O(N^{-1/2})$ for $|s| = O(\sqrt{N \log N})$, the explicit density term is of order $O(N^{-1/4})$. We can thus choose k such that the error $O(N^{-k})$ is asymptotically smaller than the density. Consequently, we obtain

$$\mathbb{P}(N_s = t) = \frac{1 + o(1)}{\sqrt{2\pi N p_{N-1}(s)(1 - p_{N-1}(s))}} \exp\left(-\frac{(t - N p_{N-1}(s))^2}{2N p_{N-1}(s)(1 - p_{N-1}(s))}\right), \quad (5.46)$$

which completes proof of Theorem 5.2. $\square$

Proof of Theorem 5.3. To prove the claim, we transfer the result in 5.15 to the tournament using Theorem 5.7. First note that our extra assumption of boundedness on the weights simplifies the third moment condition. Because the weights are bounded, the maximum distance from the mean is given by

$$|Y_N - \mathbb{E}[Y_N]| \leq 2M_N. \quad (5.47)$$

as a result, the absolute third central moment condition is bounded by

$$|Y_N - \mathbb{E}[Y_N]| \leq 2M_N \mathbb{V}(Y_N). \quad (5.48)$$

Substituting this bound into the third moment condition (5.40), we find that

$$M_N = o(N^{1/2} \mathbb{V}(Y_N)). \quad (5.49)$$

Next, we know that under this assumption, the statistic $Z_N(\tilde{\mathbf{S}})$ converges

$$\frac{Z_N(\tilde{\mathbf{S}})}{\sqrt{N} \tau_N} \rightarrow \mathcal{N}(0, 1). \quad (5.50)$$

As a result, we know that its characteristic function also converges. That is,

$$\mathbb{E}_{\mathcal{D}} \left(e^{i\lambda Z_N(\tilde{\mathbf{S}})/\sqrt{N}\tau_N} \right) \rightarrow e^{-\lambda^2/2}. \quad (5.51)$$

Taking $g(s_1, \dots, s_N) = e^{itZ_N(\tilde{\mathbf{S}})/(\sqrt{N}\tau_N)}$, we find that $|g(s_1, \dots, s_N)| = O(1)$. This means that we can set $t = 0$ in Theorem 5.7, so that taking $k > 0$ implies

$$\mathbb{E} \left[e^{i\lambda Z_N(\mathbf{S})/\sqrt{N}\tau_N} \right] = O(N^{-k}) + (1 + O(N^{-1/4+\epsilon})) \mathbb{E}_{\mathcal{D}} \left[e^{i\lambda Z_N(\tilde{\mathbf{S}})/\sqrt{N}\tau_N} \right] \rightarrow e^{-\lambda^2/2}. \quad (5.52)$$

Next, we show that the result holds when standardising with respect to the tournament variance. Because the weights are bounded, we have the trivial bound

$$Z_N(\tilde{\mathbf{S}}) \leq 2NM_N. \quad (5.53)$$

Because we assumed that $M_N = O(N^t)$, the squared statistic is bounded by $O(N^{2(t+1)})$. We again apply Theorem 5.7. This shows that

$$\mathbb{E} [(Z_N(\mathbf{S}))^2] = O(N^{2(t+1)-k}) + (1 + O(N^{-1/4+\epsilon})) \mathbb{E}_{\mathcal{D}} [Z_N(\tilde{\mathbf{S}})]. \quad (5.54)$$

By choosing k sufficiently large such that $N^{2(t+1)-k}/(N^{1/2}\tau_N) \rightarrow 0$, we thus see that

$$\frac{\mathbb{E} [(Z_N(\mathbf{S}))^2]}{\mathbb{E}_{\mathcal{D}} [(Z_N(\tilde{\mathbf{S}}))^2]} \rightarrow 1. \quad (5.55)$$

Therefore, it also follows that

$$\frac{Z_N(\mathbf{S})}{\sqrt{\mathbb{V}(Z_N(\mathbf{S}))}} = \frac{N\tau_N}{\sqrt{\mathbb{V}(Z_N(\mathbf{S}))}} \frac{Z_N(\mathbf{S})}{N\tau_N} \xrightarrow{D} \mathcal{N}(0, 1). \quad (5.56)$$

□

Proof of Lemma 5.10. We use the shorthand $p_{N-1}(s)$ from (3.8) for $\mathbb{P}(S_i = s)$. The first step is to apply Bayes on the conditional probability to rewrite

$$\mathbb{P}(\tilde{N}_s = t | \tilde{S}_N = 0) = \mathbb{P}(\tilde{N}_s = t) \frac{\mathbb{P}(\tilde{S}_N = 0 | \tilde{N}_s = t)}{\mathbb{P}(\tilde{S}_N = 0)}. \quad (5.57)$$

By conditioning on $N_s = t$, we fix N_s of the terms in $\tilde{S}_N$ to s . We can thus remove these variables from S_N , so that

$$(\tilde{S}_N | \tilde{N}_s = t) = ts + \sum_{i \notin \tilde{B}_s} \tilde{S}_i = ts + J. \quad (5.58)$$

With this simplification, we can write

$$\mathbb{P}(\tilde{S}_N = 0 | \tilde{N}_s = t) = \mathbb{P}(J = -ts). \quad (5.59)$$

Note that the remaining scores in T are conditioned on the fact that $\tilde{S}_i \notin \tilde{N}_s$. Therefore, J is a sum of $N - t$ i.i.d. binomials with a hole in the distribution at s . Lemma D.1 in the Appendix gives the mean and variance of these random variables. The expressions in (D.3) show that,

$$\mathbb{E}[J] = -(N - t) \left(\frac{sp_{N-1}(s)}{1 - p_{N-1}(s)} \right) \quad (5.60)$$

and

$$\mathbb{V}(J) = (N - t) \left(\frac{(N - 1)(1 - p_{N-1}(s)) - s^2 p_{N-1}(s)}{(1 - p_{N-1}(s))^2} \right). \quad (5.61)$$

We can now apply a local CLT approximation (A.6) on $\mathbb{P}(T = t)$, so that

$$\mathbb{P}(J = -ts) = \frac{1}{\sqrt{2\pi\mathbb{V}(J)}} \exp\left(-\frac{1}{2} \left[\frac{(ts + \mathbb{E}[J])^2}{\mathbb{V}(J)} \right]\right) + o(N^{-1/2}). \quad (5.62)$$

For the denominator in (5.57) we can apply the same approximation to find

$$\mathbb{P}(\tilde{S}_N = 0) = \frac{1}{\sqrt{2\pi\mathbb{V}(\tilde{S}_N)}} + o(N^{-1/2}). \quad (5.63)$$

The ratio in (5.57) can thus be written as

$$\frac{\mathbb{P}(\tilde{S}_N = 0 | \tilde{N}_s = t)}{\mathbb{P}(\tilde{S}_N = 0)} = \sqrt{\frac{\mathbb{V}(\tilde{S}_N)}{\mathbb{V}(T)}} \exp\left(-\frac{1}{2} \left[\frac{(ts + \mathbb{E}[J])^2}{\mathbb{V}(T)} \right]\right) + o(N^{-1/2}). \quad (5.64)$$

We start by computing the square root term. Substituting (5.61) for $\mathbb{V}(T)$ and using the standard expression $\mathbb{V}(\tilde{S}_N) = N(N-1)$ for a sum of N i.i.d. $\text{Bin}(N-1, 1/2)$ random variables gives

$$\begin{aligned} \frac{\mathbb{V}(T)}{\mathbb{V}(\tilde{S}_N)} &= \frac{N-t}{N} \frac{(N-1)(1-p_{N-1}(s)) - s^2 p_{N-1}(s)}{(N-1)(1-p_{N-1}(s))^2} \\ &= \frac{1}{1-p_{N-1}(s)} \left(1 - \frac{t}{N}\right) \left[1 - \frac{s^2 p_{N-1}(s)}{(N-1)(1-p_{N-1}(s))}\right] \\ &= \frac{\nu}{1-p_{N-1}(s)}. \end{aligned} \quad (5.65)$$

In the last line we used the definition of ν in (5.33). Taking the reciprocal and the square root then yields

$$\sqrt{\frac{\mathbb{V}(\tilde{S}_N)}{\mathbb{V}(T)}} = \sqrt{\frac{1-p_{N-1}(s)}{\nu}}. \quad (5.66)$$

Next, we simplify the argument in the exponential term of (5.64). Writing out the numerator, we get

$$\begin{aligned} ts + \mathbb{E}[J] &= ts - (N-t) \left(\frac{s p_{N-1}(s)}{1-p_{N-1}(s)} \right) \\ &= \frac{s(t - N p_{N-1}(s))}{1-p_{N-1}(s)} \end{aligned} \quad (5.67)$$

Combining (5.61) and (5.67) yields

$$\begin{aligned} \frac{(ts + \mathbb{E}[J])^2}{\mathbb{V}(T)} &= \frac{s^2 (t - N p_{N-1}(s))^2}{(1-p_{N-1}(s))^2} \frac{1}{(N-t)} \frac{(1-p_{N-1}(s))^2}{(N-1)(1-p_{N-1}(s)) - s^2 p_{N-1}(s)} \\ &= \frac{s^2 (N p_{N-1}(s) - t)^2}{(1-p_{N-1}(s))(N-t)(N-1 - s^2 \frac{p_{N-1}(s)}{1-p_{N-1}(s)})}. \end{aligned} \quad (5.68)$$

We can factor out the $N(N-1)$ in the denominator and use (5.33) to write

$$\frac{(ts + \mathbb{E}[J])^2}{\mathbb{V}(T)} = \frac{s^2 (t - N p_{N-1}(s))^2}{N(N-1)(1-p_{N-1}(s))\alpha}. \quad (5.69)$$

Substituting (5.66) and (5.69) into (5.64) gives

$$\frac{\mathbb{P}(\tilde{S}_N = 0 | \tilde{N}_s = t)}{\mathbb{P}(\tilde{S}_N = 0)} = \sqrt{\frac{1-p_{N-1}(s)}{\nu}} \exp\left(-\frac{s^2}{2} \frac{(t - N p_{N-1}(s))^2}{N(N-1)(1-p_{N-1}(s))\nu}\right) \quad (5.70)$$

Thus, we can conclude that (5.57) reduces to

$$\mathbb{P}(\tilde{N}_s = t | \tilde{S}_N = 0) = \mathbb{P}(\tilde{N}_s = t) \sqrt{\frac{1 - p_{N-1}(s)}{\nu}} \exp\left(-\frac{s^2}{2} \frac{(Np_{N-1}(s) - t)^2}{N(N-1)(1 - p_{N-1}(s))\nu}\right), \quad (5.71)$$

which is what we set out to prove. $\square$

Proof of Lemma 5.11. For any event $\mathcal{E}$, the failure probability is given by

$$\mathbb{P}_{\mathcal{D}}(\tilde{N}_s \notin \mathcal{E}) = \sum_{t \notin \mathcal{E}} \mathbb{P}_{\mathcal{D}}(\tilde{N}_s = t). \quad (5.72)$$

We write $\mathbb{P}_{\mathcal{D}}(\tilde{N}_s \notin \mathcal{E})$ in terms of $R(t; N, s)$ in the proof of Lemma 5.10

$$\begin{aligned} \mathbb{P}(\tilde{N}_s \notin \mathcal{E} \mid \tilde{S}_N = 0) &= \sum_{t \in \mathcal{E}} \mathbb{P}(\tilde{N}_s = t \mid \tilde{S}_N = 0) \\ &= \sum_{t \in \mathcal{E}} \mathbb{P}(\tilde{N}_s = t) \frac{\mathbb{P}(\tilde{S}_N = 0 \mid \tilde{N}_s = t)}{\mathbb{P}(\tilde{S}_N = 0)} \\ &= \sum_{t \in \mathcal{E}} \mathbb{P}(\tilde{N}_s = t) R(t; N, s). \end{aligned} \quad (5.73)$$

We split this sum into two parts by setting a cutoff threshold

$$\sum_{t \notin \mathcal{E}} \mathbb{P}(\tilde{N}_s = t) R(t; N, s) = \sum_{t \notin \mathcal{E}, t \leq N/2} \mathbb{P}(\tilde{N}_s = t) R(t; N, s) + \sum_{t \notin \mathcal{E}, t > N/2} \mathbb{P}(\tilde{N}_s = t) R(t; N, s). \quad (5.74)$$

We start by bounding the main term where $t < N/2$. By Lemma 5.10

$$R(t; N, s) = \sqrt{\frac{1 - p_{N-1}(s)}{\alpha}} \exp\left(-\frac{s^2}{2} \frac{(Np_{N-1}(s) - t)^2}{N(N-1)(1 - p_{N-1}(s))\alpha}\right). \quad (5.75)$$

and

$$\alpha = \left(1 - \frac{t}{N}\right) \left(1 - \frac{s^2 p_{N-1}(s)}{(N-1)(1 - p_{N-1}(s))}\right). \quad (5.76)$$

Because we set $t < N/2$, it directly follows that

$$\alpha \geq \frac{1}{2} \left(1 - \frac{s^2 p_{N-1}(s)}{(N-1)(1 - p_{N-1}(s))}\right) \quad (5.77)$$

To bound the remaining term, we apply the local CLT (Theorem A.4) to bound $p_{N-1}(s) \leq CN^{-1/2}e^{-2s^2/N}$. Writing

$$\frac{s^2 p_{N-1}(s)}{N-1} \leq CN^{-1/2} \frac{s^2}{N} e^{-2s^2/N} \quad (5.78)$$

and noting that the function xe^{-2x} is bounded for all $x \geq 0$, we conclude that

$$\alpha \geq \frac{1}{2} (1 - O(N^{-1/2})). \quad (5.79)$$

Therefore, the first term in (5.75) admits the bound

$$\sqrt{\frac{1 - p_{N-1}(s)}{\alpha}} \leq \sqrt{2} (1 - O(N^{-1/2}))^{-1/2} (1 - O(N^{-1/2}))^{1/2} < 2. \quad (5.80)$$

For the exponential term in (5.75), we can also use (5.79) to find that

$$\exp\left(-\frac{s^2}{2} \frac{(t - Np_{N-1}(s))^2}{N(N-1)(1-p_{N-1}(s))\alpha}\right) \leq \exp\left(-\frac{s^2(t - Np_{N-1}(s))^2}{N(N-1)(1-p_{N-1}(s))}\right). \quad (5.81)$$

Because $s^2(Np_{N-1}(s) - t)^2 > 0$ for any s , $p_{N-1}(s)$ and t , and $p_{N-1}(s) = O(N^{-1/2})$, we get

$$\exp\left(-\frac{s^2}{2} \frac{(t - Np_{N-1}(s))^2}{N(N-1)(1-p_{N-1}(s))\alpha}\right) \leq 1, \quad (5.82)$$

The correction term in (5.75) therefore admits the simple bound

$$R(t; N, s) \leq 2 \quad (5.83)$$

The first sum in (5.74) is then bounded by

$$\sum_{t \notin \mathcal{E}, t \leq N/2} \mathbb{P}(\tilde{N}_s = t) R(t) \leq 2\mathbb{P}(\tilde{N}_s \in \mathcal{E}). \quad (5.84)$$

For the second term in (5.74), we use the definition of $R(t)$ and the trivial bound $\mathbb{P}(\tilde{S}_N = \mathbb{E}[S_N] | \tilde{N}_s = t) \leq 1$ to obtain

$$R(t) \leq \frac{1}{\mathbb{P}(\tilde{S}_N = 0)} = O(\sqrt{N}). \quad (5.85)$$

But since $\tilde{N}_s \sim \text{Bin}(N, p_{N-1}(s))$, we also have that

$$\mathbb{P}(\tilde{N}_s = N/2) = \binom{N}{N/2} p_{N-1}(s)^{N/2} (1 - p_{N-1}(s))^{N/2} \leq 2^{N/2} N^{-N/4} = O(N^{-N/4}). \quad (5.86)$$

This means that the sum for all $t > N/2$ has size

$$\sum_{t \notin \mathcal{E}, t > N/2} \mathbb{P}(\tilde{N}_s = t) R(t) \leq O(\sqrt{N}) \mathbb{P}(\tilde{N}_s > N/2) = O(N^{-N/4}). \quad (5.87)$$

Substituting (5.84) and (5.87) into (5.74) results in

$$\mathbb{P}(\tilde{N}_s \notin \mathcal{E} | \tilde{S}_N = 0) \leq 2\mathbb{P}(N_s \notin \mathcal{E}) + O(N^{-N/4}). \quad (5.88)$$

To finish the proof, we fix the event

$$\mathcal{E} = \left\{ t : |t - Np_{N-1}(s)| \leq cN^{1/4} \sqrt{\log N} \right\} \quad (5.89)$$

Because the unconditioned $\tilde{N}_s \sim \text{Bin}(N, p_{N-1}(s))$, we can apply a Bernstein bound A.2 to obtain

$$\mathbb{P}(\tilde{N}_s \notin \mathcal{E}) = \mathbb{P}(|\tilde{N}_s - Np_{N-1}(s)| > \delta_N) \leq 2 \exp\left(-\frac{\delta_N}{\mathbb{V}(\tilde{N}_s) + \frac{2}{3}\delta_N}\right). \quad (5.90)$$

which for $\delta_N = cN^{1/4} \sqrt{\log N}$ and $\mathbb{V}(\tilde{N}_s) = Np_{N-1}(s)(1 - p_{N-1}(s))$ gives

$$\mathbb{P}(\tilde{N}_s \notin \mathcal{E}) \leq 2 \exp\left(-\frac{c^2 N^{1/2} \log N}{Np_{N-1}(s)(1 - p_{N-1}(s)) + N^{1/4} \sqrt{\log N}}\right). \quad (5.91)$$

The worst case occurs when the denominator is maximalised. The probability $p_{N-1}(s)$ is maximalised at $s = 0$, where the local CLT (A.5) ensures $p_{N-1}(s) = (2\pi N)^{-1/2} + o(N^{-1/2})$. Therefore, it follows that

$$\mathbb{P}(\tilde{N}_s \notin \mathcal{E}) \leq 2 \exp\left(-c^2 \sqrt{\frac{\pi}{2}} \log N\right) = 2N^{-c^2 \sqrt{\pi/2}}. \quad (5.92)$$

Substituting this into (5.88), we can conclude that

$$\mathbb{P}(\tilde{N}_s \notin \mathcal{E} \mid \tilde{S}_N = 0) \leq 4N^{-c^2 \sqrt{\pi/2}} + O(N^{-N/4}), \quad (5.93)$$

so that if we choose $c^2 \sqrt{\pi/2} > k$ the result follows. $\square$

Proof of Corollary 5.13. From Lemma 5.10, we know that the probability of the bin sizes under $\mathcal{D}$ -model can be written as

$$\mathbb{P}_{\mathcal{D}}(\tilde{N}_s = t) = \mathbb{P}(\tilde{N}_s = t) R(t; N, s). \quad (5.94)$$

In the independent model, the number of vertices with score $\tilde{N}_s \sim \text{Bin}(N, p_{N-1}(s))$. Therefore, the local CLT A.6 gives

$$\mathbb{P}(\tilde{N}_s = t) = \frac{1}{\sqrt{2\pi N p_{N-1}(s)(1 - p_{N-1}(s))}} \exp\left(-\frac{(t - N p_{N-1}(s))^2}{2N p_{N-1}(s)(1 - p_{N-1}(s))}\right) + o(N^{-1/2}). \quad (5.95)$$

It thus remains to show that the density $R(t; N, s) = 1 + o(1)$ uniformly for the specified s and t . We start by approximating the terms in ν . Applying the local CLT on $p_{N-1}(s)$ gives that $p_{N-1}(s) = O(N^{-1/2})$. Using the assumption that $s^2 = O(N \log N)$, we can also estimate the second factor of (5.33) with

$$\frac{s^2 p_{N-1}(s)}{(N-1)(1 - p_{N-1}(s))} = O\left(\frac{N \log N \cdot N^{-1/2}}{N}\right) = O(N^{-1/2} \log N) = o(1). \quad (5.96)$$

For the first factor, substituting $t = N p_{N-1}(s) + O(N^{1/4} \sqrt{\log N})$ gives

$$1 - \frac{t}{N} = 1 - p_{N-1}(s) + O(N^{-3/4} \sqrt{\log N}). \quad (5.97)$$

This implies that $\nu = (1 - p_{N-1}(s))(1 + o(1))$, and therefore the prefactor in $R(t; N, s)$ satisfies

$$\sqrt{\frac{1 - p_{N-1}(s)}{\nu}} = 1 + o(1). \quad (5.98)$$

Next, we bound the argument of the exponential in (5.32). Using $(N p_{N-1}(s) - t)^2 = O(N^{1/2} \log N)$ and $s^2 = O(N \log N)$, we have

$$\frac{s^2 (N p_{N-1}(s) - t)^2}{N(N-1)(1 - p_{N-1}(s))\nu} = O\left(\frac{N \log N \cdot N^{1/2} \log N}{N^2}\right) = O(N^{-1/2} \log^2 N). \quad (5.99)$$

Now, since $N^{-1/2} \log^2 N$ is asymptotically $o(1)$, the exponential term has size $\exp(o(1)) = 1 + o(1)$. Combining this with (5.98) yields

$$R(t; N, s) = 1 + o(1), \quad (5.100)$$

which concludes the proof. $\square$

Proof of Corollary 5.14. Take an arbitrary $s \in I_N$. We want to bound the probability of a missing score, which corresponds to the event $\tilde{N}_s = 0$. First, we evaluate the expected count $\mathbb{E}[N_s] = N p_{N-1}(s)$ in the independent model. By the De Moivre-Laplace local CLT A.4, we have

$$p_{N-1}(s) = \sqrt{\frac{2}{\pi N}} \exp\left(-\frac{s^2}{2N}\right) (1 + o(1)). \quad (5.101)$$

Since $s \in I_N$, we have $s^2 \leq (1 - \epsilon)N \log N$, which implies

$$\begin{aligned} Np_{N-1}(s) &\geq N\sqrt{\frac{2}{\pi N}} \exp\left(-\frac{1-\epsilon}{2} \log N\right) (1 + o(1)) \\ &= \sqrt{\frac{2}{\pi}} N^{\epsilon/2} (1 + o(1)). \end{aligned} \quad (5.102)$$

Thus, the expected number of vertices with score s grows as $\Theta(N^{\epsilon/2})$. To bound $\mathbb{P}_{\mathcal{D}}(\tilde{N}_s = 0)$, we distinguish two cases based on the magnitude of ϵ . If $\epsilon \leq 1/2$, then $\mathbb{E}[N_s] = O(N^{1/4})$. This means $t = 0$ satisfies the condition $|t - \mathbb{E}[N_s]| = O(N^{1/4}\sqrt{\log N})$, and we can directly apply the bound on $R(t; N, s)$ in (5.100)

$$\mathbb{P}_{\mathcal{D}}(\tilde{N}_s = 0) = (1 + o(1))\mathbb{P}(\tilde{N}_s = 0) = (1 + o(1))(1 - p_s)^N \leq (1 + o(1))e^{-Np_{N-1}(s)} \quad (5.103)$$

By the lower bound on $Np_{N-1}(s)$ in (5.102), this probability decays as $\exp(-\Theta(N^{\epsilon/2}))$, which is asymptotically smaller than N^{-k} for any $k > 0$.

If $\epsilon > 1/2$, $\mathbb{E}[N_s]$ grows strictly faster than $N^{1/4}\sqrt{\log N}$. We can therefore directly use the uniform concentration of Lemma 5.12, which guarantees

$$\mathbb{P}_{\mathcal{D}}(|\tilde{N}_s - \mathbb{E}[N_s]| > cN^{1/4}\sqrt{\log N}) = O(N^{-k}). \quad (5.104)$$

Since $\mathbb{E}[N_s] \gg N^{1/4}\sqrt{\log N}$, the event $\tilde{N}_s = 0$ implies a deviation far exceeding this bound, so $\mathbb{P}_{\mathcal{D}}(\tilde{N}_s = 0) = O(N^{-k})$. Finally, we take a union bound over all valid scores in I_N . The number of such scores is bounded by the length of the interval, $|I_N| \leq 2\sqrt{N \log N}$. Let E be the event that any valid score in I_N is missing. Then,

$$\mathbb{P}_{\mathcal{D}}(\tilde{\mathbf{S}} \in E) \leq \sum_{s \in I_N} \mathbb{P}_{\mathcal{D}}(\tilde{N}_s = 0) \leq 2\sqrt{N \log N} \cdot O(N^{-(k+1)}) = O(N^{-k}). \quad (5.105)$$

Using Theorem 5.7, we can transfer this result onto the tournament

$$\mathbb{P}(\mathbf{S} \in E) = \mathbb{P}_{\mathcal{D}}(\tilde{\mathbf{S}} \in E) + O(N^{-k}) = O(N^{-k}). \quad (5.106)$$

□

Proof of Lemma 5.15. Under the $\mathcal{D}$ -model, the vertex scores are i.i.d. with $\tilde{S}_i \sim 2 \cdot \text{Bin}(N-1, 1/2) - (N-1)$. We rewrite Z_N into a summation over vertices by substituting $\tilde{N}_k = \sum_{i=1}^N \mathbf{1}_{\{\tilde{S}_i=k\}}$. This gives

$$Z_N = \sum_{s \in \mathcal{R}} \sum_{i=1}^N w_s \left(\mathbf{1}_{\{\tilde{S}_i=s\}} - \mathbb{E}[\mathbf{1}_{\{\tilde{S}_i=s\}}] \right) = \sum_{i=1}^N \left(w_{\tilde{S}_i} - \mathbb{E}[w_{\tilde{S}_i}] \right), \quad (5.107)$$

This frames Z_N as a sum of N independent random variables, allowing us to apply Janson's Theorem A.9 in the Appendix. To satisfy the unit span requirement, we rewrite the conditioning $\tilde{S}_N = 0$ in terms of the constraint variables $X_i = \tilde{S}_i/2$

We first verify the moment conditions (ii, iii, iv) for the constraint variables X_i . By definition, they are centred with variance $\tau_{X_i}^2 = (N-1)/4$. This implies that (ii) is immediately satisfied and that $\gamma = 1$ in (iv). The absolute third central moment of a Binomial random variable is $\Theta(N^{3/2})$ by Lemma D.2 in the Appendix. This satisfies the required bound

$$e|X_i|^3 = o(N^{1/2}\tau_{X_N}^3) = o(N^2). \quad (5.108)$$

Because the variance grows linearly with N , we set the parameter $\gamma = 1$ to satisfy the variance growth condition $\sigma_{X_N}^2 = O(N^{2/\gamma-1})$.

Next, we verify the conditions on the characteristic function (v). The standard expression, obtained by writing $S_i/2$ as a sum of independent centred Bernoulli random variables is

$$\phi_{X_i}(t) = (\cos(t/2))^{N-1}. \quad (5.109)$$

Since $\gamma = 1$, we check for $|t| \leq \pi$ if

$$1 - |\phi_{X_i}(t)| \geq c \min\left(|t|, \frac{(N-1)t^2}{4}\right) \quad (5.110)$$

Because $\cos(t/2)$ is symmetric around zero and positive, we only have to check $0 \leq t \leq \pi$. On this interval, a standard inequality holds

$$\cos(t/2) \leq e^{-t^2/8}. \quad (5.111)$$

This can be verified by taking the difference of the functions and checking that the derivative is negative. If we set $v = (N-1)t^2/8$, we see that (6.138) then reduces to showing

$$1 - e^{-v} \geq c \min(t, v), \quad 0 \leq t \leq \pi. \quad (5.112)$$

For $v \leq 1$ the function $1 - e^{-v}$ is concave, which implies that

$$1 - e^{-v} \geq (1 - v) + v(1 - e^{-1}) = (1 - 1/e)v. \quad (5.113)$$

This means that for $v \leq 1$, we have

$$1 - e^{-v} \geq (1 - 1/e)v \geq \frac{1 - 1/e}{2} \min(t, 2v). \quad (5.114)$$

For $v \geq 1$, the function $1 - e^{-v}$ is always bigger than $1 - e^{-1}$. Because $|t| \leq \pi$, we thus have that

$$1 - e^{-v} \geq (1 - 1/e) \geq \frac{1 - 1/e}{\pi} \min(t, 2v). \quad (5.115)$$

The second condition is more strict than the first, which means that

$$1 - |\phi_{X_i}(t)| \geq \frac{1 - 1/e}{\pi} \left(t, \frac{(N-1)t^2}{4}\right), \quad 0 \leq t \leq \pi. \quad (5.116)$$

Therefore, condition (v) is also satisfied.

The conditions on the weights w_{S_i} are satisfied by the assumptions. However, note that we can generally assume that the correlation ρ_N between X_i and w_{S_i} satisfies $\limsup |\rho_N| < 1$. If $|\rho_N| = 1$ this would imply that $w_{S_i} = aX_{N,i} + b$, so that the sum is trivially zero due to the conditioning. Having met all criteria, Theorem A.9 ensures that conditioned on $\sum_{i=1}^N \tilde{S}_i = 0$, the sum $Z_N/(N^{1/2}\tau_N)$ converges in distribution to $\mathcal{N}(0, 1)$. $\square$

5.3 Improved concentration

In this subsection, we prove a CLT for the coarse grained rates acting on functions of score

$$\Lambda_N(i, s') = \underbrace{\frac{|j \in \mathcal{V} : S_j = s', \Sigma_{ij} = +1|}{N}}_{\Lambda_{N,+}(i, s')} + e^{-\beta\omega} \underbrace{\frac{|j \in \mathcal{V} : S_j = s', \Sigma_{ij} = +1|}{N}}_{\Lambda_{N,-}(i, s')} \quad (5.117)$$

as defined in (4.9). Because the value of $\Lambda_N(i, s')$ depends on the specific configuration of the edges of the vertex i , we cannot directly apply the $\mathcal{D}$ -model. While the $\mathcal{D}$ -model handles the global zero-sum constraint for macroscopic vertex scores, it is inherently blind to the specific microscopic edge directions.

To circumvent this, we conceptually construct the graph in two steps. We first isolate a tournament graph T_{N-1} of $N-1$ vertices, on which we can apply the $\mathcal{D}$ -model to handle the score bin statistics. The problem then reduces to wiring vertex i into T_{N-1} .

If we condition on the vertex i belonging to a specific score bin B_s , which we do to derive $\gamma_N(s, s')$, the total number of outgoing edges is strictly fixed. The only remaining randomness is the specific destination of these edges within T_{N-1} and the size of its score bins. Conditioned on the tournament graph T_{N-1} , the independence of the edge orientation Σ_{ij} means that the random variable $\Lambda_{N,\pm}(i, s')$ (4.7) is identical to drawing a random sample without replacement.

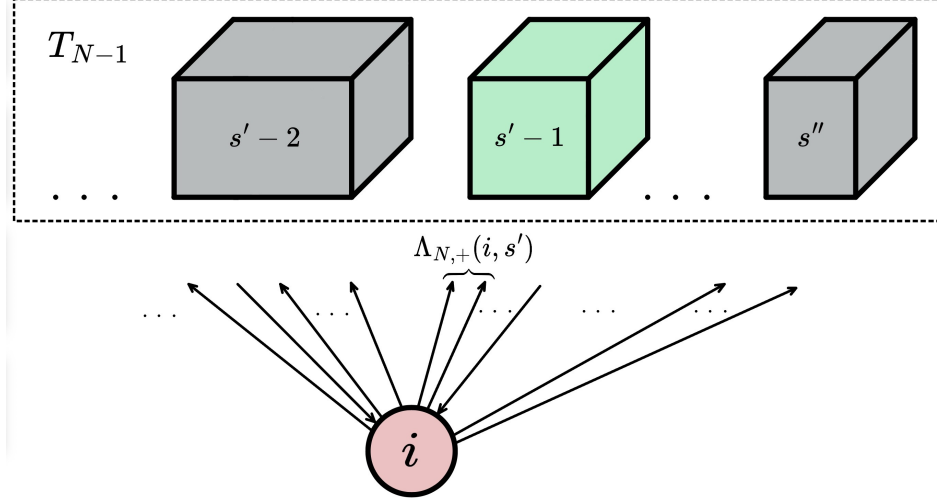


Figure 5.2: The two-step construction of $\Lambda_{N,+}(i, s')$. First, the subtournament T_{N-1} is partitioned by conditioning on the score bins. Second, vertex i is wired into T_{N-1} . Since each edge is placed independently without replacement, the number of edges landing in the $s' - 1$ bin follows a hypergeometric distribution.

Proposition 5.16. *Let T_N be a tournament graph. Fix a vertex $i \in B_s$. Let T_{N-1} be a tournament with vertex set $\mathcal{V}^{(i)} = \mathcal{V} \setminus \{i\}$. Conditioned on the subtournament having exactly $K_+ = |\{j \in \mathcal{V}^{(i)} : s^{(i)}(j) = s' + 1\}|$ vertices with score $s' + 1$, $\Lambda_{N,+}(i, s')$ follows a hypergeometric distribution*

$$(\Lambda_{N,+}(i, s') \mid i \in B_s, K_+) \sim \text{Hyp}\left(N-1, K_+, \frac{N-1+s}{2}\right). \quad (5.118)$$

While $\Lambda_{N,+}(i, s')$ describes the sampling of out-edges, $\Lambda_{N,-}(i, s')$ counts in-edges. To find the total rate $\Lambda_N(i, s')$ in terms of out-edge sampling, we need to rewrite $\Lambda_{N,-}(i, s')$. The intuition for this relies on how connections to vertex i shift the scores of the background graph. Consider the pool of background vertices that currently have a reduced score of $s' - 1$, defined as

$$K_- = |\{j \in \mathcal{V}^{(i)} : s^{(i)}(j) = s' - 1\}| \quad (5.119)$$

Every vertex in this pool must eventually connect to i . If a vertex $j \in K_-$ receives an in-edge from i so that $\Sigma_{ji} = -1$, its final score drops to $s' - 2$. If instead $\Sigma_{ji} = +1$, its final score rises to s' . This means that the entire population K_- is strictly partitioned between these two outcomes. As a result, the number of in-edges that produce a score of s' is simply the size of the total pool minus the out-edges that produced a score of $s' - 2$

$$\Lambda_{N,-}(i, s') = K_- - \Lambda_{N,+}(i, s' - 2) \quad (5.120)$$

This means that we can rewrite the total coarse-grained rate only in terms of $\Lambda_{N,+}$ as

$$\Lambda_N(i, s') = \Lambda_{N,+}(i, s') - e^{-\beta\omega} \Lambda_{N,+}(i, s' - 2) + e^{-\beta\omega} K_- \quad (5.121)$$

By conditioning on the fixed background populations K_+ and K_- , the entire expression reduces to a finite population sampling of out-edges. We can apply the appropriate limit theorem for such problems (Theorem A.10 in the Appendix) to establish the next Lemma.

Lemma 5.17. *Fix a vertex $i \in B_s$. Condition on the counts $K_+ = |\{j \in \mathcal{V}^{(i)} : s^{(i)}(j) = s' + 1\}|$ and $K_- = |\{j \in \mathcal{V}^{(i)} : s^{(i)}(j) = s' - 1\}|$ in the subtournament. Let $\mu_c = \mathbb{E}[\Lambda_N(i, s') | i \in B_s, K_+, K_-]$ and $\tau_c^2 = \mathbb{V}(\Lambda_N(i, s') | i \in B_s, K_+, K_-)$. If $K_+, K_- \rightarrow \infty$ and $N - |s| \rightarrow \infty$ as $N \uparrow \infty$, then*

$$\frac{\Lambda_N(i, s') - \mu_c}{\tau_c} \xrightarrow{D} \mathcal{N}(0, 1). \quad (5.122)$$

The previous Lemma gives us the CLT for $\Lambda_N(i, s')$, given the sizes of K_+ and K_- . To extend this result to the fully random tournament, we relax the conditioning on K_+ and K_- . The main step here is to decompose

$$\frac{\Lambda_N(i, s') - \mu_N}{\tau_N} = \frac{\tau_c}{\tau_N} \frac{\Lambda_N(i, s') - \mu_c}{\tau_c} + \frac{\mu_c - \mu_N}{\tau_N}. \quad (5.123)$$

Proving that $\tau_c/\tau_N \rightarrow c$ and applying Lemma 5.17 then shows that the first term is Normally distributed. The result for the second term follows from the fact that μ_c is a linear combination of K_+ and K_- , which are bin sizes on T_{N-1} , on which Theorem 5.3 of the previous section can be applied.

Proofs: Section 5.3

Proof of Theorem 4.2. The first step of the proof is to decompose $\lambda(x, s')$ into two distinct parts. Because it contains randomness from both the sampling of the directions $(\Sigma_{ij})_{j \neq i}$ for the edges incident to i and the random populations, we write

$$\frac{\Lambda_N(i, s') - \mu_N}{\tau_N} = \frac{\tau_c}{\tau_N} \frac{\Lambda_N(i, s') - \mu_c}{\tau_c} - \frac{\mu_c - \mu_N}{\tau_N}. \quad (5.124)$$

We will work on a typical subtournament $T_{N-1} \in \Omega_{\text{typ}}(k)$. From the proof of Lemma 5.17 we have from (5.176) that

$$\tau_c^2 = \frac{(N-1)^2 - s^2}{4(N-2)} \left[\frac{K_+ + e^{-2\beta\omega} K_-}{N-1} - \left(\frac{K_+ - e^{-\beta\omega} K_-}{N-1} \right)^2 \right]. \quad (5.125)$$

We assumed that $|s| \leq t\sqrt{N \log N}$, which means that

$$\frac{(N-1)^2 - s^2}{4(N-2)} = \frac{N}{4} \left(1 + O\left(\frac{\log N}{N}\right) \right). \quad (5.126)$$

Furthermore, Theorem 5.1 shows that K_+ and K_- concentrate around their expectation on the typical set. In particular

$$K_{\pm} = N p_{N-2}(s' \pm 1) \left(1 + O\left(N^{-1/4} \sqrt{\log N}\right) \right). \quad (5.127)$$

Because $p_{N-2}(s' \pm 1) = O(N^{-1/2})$, it thus follows that

$$\left(\frac{K_+ - e^{-\beta\omega} K_-}{N-1} \right)^2 = O(N^{-1}) \quad (5.128)$$

and

$$\frac{K_+ + e^{-2\beta\omega} K_-}{N-1} = O(N^{-1/2}). \quad (5.129)$$

Substituting (5.128) and (5.129) into (5.125), we observe that the squared mean term is of order $O(N^{-1})$, whereas the second moment term is of order $O(N^{-1/2})$. Thus, the variance is dominated by the second moment

$$\tau_c^2 = \frac{1}{4} (K_+ + e^{-2\beta\omega} K_-) + O(1), \quad (5.130)$$

so that applying (5.127) once more yields

$$\tau_c^2 = \frac{N-1}{4} (p_{N-2}(s' + 1) + e^{-2\beta\omega} p_{N-2}(s' - 1)) + O(N^{1/4} \sqrt{\log N}). \quad (5.131)$$

For τ_N , Lemma 4.5 gives

$$\tau_N^2 = (N-1) (f_+(s, s') + e^{-2\beta\omega} f_-(s, s')) + O(1), \quad (5.132)$$

where

$$f_{\pm}(s, s') = \frac{\alpha_{\pm} p_{N-2}(s' \pm 1)}{2} \left(1 - \frac{\alpha_{\pm} p_{N-2}(s' \pm 1)}{2} \right). \quad (5.133)$$

On a typical subtournament $T_{N-1} \in \Omega_{\text{typ}}(k)$, we then have

$$\alpha_{\pm} = 1 \pm \frac{s}{N-1} = 1 \pm O\left(\sqrt{\frac{\log N}{N}}\right) \quad (5.134)$$

such that

$$f_{\pm}(s, s') = \frac{p_{N-2}(s' \pm 1)}{2} + O\left(\frac{\sqrt{\log N}}{N}\right). \quad (5.135)$$

Consequently, (5.132) reduces to

$$\tau_N^2 = \frac{N-1}{2} (p_{N-2}(s' + 1) + e^{-2\beta\omega} p_{N-2}(s' - 1)) + O(\sqrt{\log N}), \quad (5.136)$$

which means that combining (5.131) and (5.136) yields

$$\frac{\tau_c^2}{\tau_N^2} = \frac{1}{2} (1 + O(N^{-1/4} \sqrt{\log N})). \quad (5.137)$$

As a result, it follows that

$$\frac{\tau_c^2}{\tau_N^2} \longrightarrow \frac{1}{2}. \quad (5.138)$$

Furthermore, the restriction on s satisfies $N - |s| = O(N)$ and working on a typical tournament ensures that $K_{\pm} = O(\sqrt{N})$. This means we can apply Theorem 5.4 to see that, conditioned on $T_{N-1} \in \Omega_{\text{typ}}$, Z_1 satisfies

$$\frac{\tau_c}{\tau_N} \frac{\Lambda_N(i, s') - \mu_N}{\tau_N} \xrightarrow{D} \frac{\sqrt{2}}{2} \cdot \mathcal{N}(0, 1). \quad (5.139)$$

For the second term, we first write out

$$\begin{aligned} \mu_c &= \frac{N-1+s}{2(N-1)} K_+ + e^{-\beta\omega} \frac{N-1-s}{2(N-1)} K_- \\ &= \frac{\alpha_+}{2} K_+ + e^{-\beta\omega} \frac{\alpha_-}{2} K_-. \end{aligned} \quad (5.140)$$

and

$$\mu_N = \mathbb{E}[\mu_c | x \in B_s] = \frac{\alpha_+}{2} \mathbb{E}[K_+] + e^{-\beta\omega} \frac{\alpha_-}{2} \mathbb{E}[K_-]. \quad (5.141)$$

Here, we used that

$$\mathbb{E}[K_{\pm}|i \in B_s] = \mathbb{E}[K_{\pm}], \quad (5.142)$$

which follows from the fact that the $K_{\pm}$ are strictly a function of the random subtournament T_{N-1} . As such, we have

$$\mu_c - \mu_N = \frac{\alpha_+}{2} (K_+ - \mathbb{E}[K_+]) + e^{-\beta\omega} \frac{\alpha_-}{2} (K_- - \mathbb{E}[K_-]). \quad (5.143)$$

To apply Theorem 5.3, we set the weights $w_{s'+1} = \alpha_+/2$, $w_{s'-1} = e^{-\beta\omega} \alpha_-/2$, and $w_k = 0$ otherwise. These weights are uniformly bounded. The restriction $|s| < \frac{1}{2}\sqrt{N \log N}$ guarantees that $N \text{Var}(w_{S_i}) \rightarrow \infty$ so that the variance condition is met. Because the weights are non-zero at only two discrete points, the weight function is strictly non-linear in S_i , trivially satisfying the correlation condition. Therefore, it follows that

$$\frac{\mu_c - \mu_N}{\sqrt{\mathbb{V}(\mu_c)}} \xrightarrow{d} \mathcal{N}(0, 1) \quad (5.144)$$

Now, the variance is given by

$$\mathbb{V}(\mu_c) = \frac{\alpha_+^2}{4} \mathbb{V}(K_+) + e^{-2\beta\omega} \frac{\alpha_-^2}{4} \mathbb{V}(K_-) + e^{-\beta\omega} \frac{\alpha_+ \alpha_-}{2} \text{Cov}(K_+, K_-). \quad (5.145)$$

where $\alpha_{\pm}$ is defined as in (3.18) and

$$\mathbb{V}(K_{\pm}) = (N-1)p_{N-2}(s' \pm 1) + O(1) \quad (5.146)$$

and because a vertex can only have one score, the indicators in K_+ and K_- are disjoint, so that

$$\text{Cov}(K_+, K_-) = -(N-1)p_{N-2}(s' + 1)p_{N-2}(s' - 1) = O(1). \quad (5.147)$$

Substituting (5.146) and (5.147) into (5.145) yields

$$\mathbb{V}(\mu_c) = \frac{N-1}{4} (\alpha_+^2 p_{N-2}(s' + 1) + \alpha_-^2 e^{-2\beta\omega} p_{N-2}(s' - 1)) + O(1). \quad (5.148)$$

Bounding $\alpha_{\pm}$ using (5.134), we know that

$$\alpha_{\pm}^2 p_{N-2}(s' \pm 1) = p_{N-2}(s' \pm 1) \pm \frac{\sqrt{\log N}}{N}. \quad (5.149)$$

Which means that (5.148) takes the form

$$\mathbb{V}(\mu_c) = \frac{N-1}{4} (p_{N-2}(s' + 1) + p_{N-2}e^{-2\beta\omega} p_{N-2}(s' - 1)) + O(\log N). \quad (5.150)$$

Combining this with (5.136) leaves us with

$$\frac{\mathbb{V}(\mu_c)}{\tau_N^2} = \frac{1}{2} \left(1 + O\left(\sqrt{\frac{\log N}{N}}\right) \right), \quad (5.151)$$

so that

$$\frac{\mathbb{V}(\mu_c)}{\tau_N^2} \rightarrow \frac{1}{2}. \quad (5.152)$$

Combining (5.144) and (5.152), we have

$$\frac{\mu_c - \mu_N}{\tau_N} \xrightarrow{D} \frac{\sqrt{2}}{2} \cdot \mathcal{N}(0, 1). \quad (5.153)$$

To finish the proof, we argue that the sum of the two terms converge to a standard normal distribution using characteristic functions. For ease of notation, we set Z_1 and Z_2 as

$$Z_1 := \frac{\tau_c}{\tau_N} \frac{\Lambda_N(i, s') - \mu_c}{\tau_c} \quad \text{and} \quad Z_2 := \frac{\mu_c - \mu_N}{\tau_N}, \quad (5.154)$$

so that we analyse the characteristic function

$$\varphi_{\Lambda_N}(t) = \mathbb{E} \left[e^{it(Z_1+Z_2)} \middle| i \in B_s \right]. \quad (5.155)$$

First, we split the characteristic function into

$$\varphi_{\Lambda_N}(t) = \underbrace{\mathbb{E} \left[e^{it(Z_1+Z_2)} \mathbb{1}_{\{T_{N-1} \in \Omega_{\text{typ}}(k)\}} \middle| i \in B_s \right]}_{E_1} + \underbrace{\mathbb{E} \left[e^{it(Z_1+Z_2)} \mathbb{1}_{\{T_{N-1} \notin \Omega_{\text{typ}}\}} \middle| i \in B_s \right]}_{E_2}. \quad (5.156)$$

Taking the absolute value and using that $|e^{it(Z_1+Z_2)}| \leq 1$ for all $t \in \mathbb{R}$, we find that

$$|\varphi_{\Lambda_N}(t) - E_1| \leq \mathbb{E} \left[|e^{it(Z_1+Z_2)}| \mathbb{1}_{\{T_{N-1} \notin \Omega_{\text{typ}}\}} \middle| i \in B_s \right] = \mathbb{P}(T_{N-1} \notin \Omega_{\text{typ}}), \quad (5.157)$$

so that it suffices to analyse E_1 . Let $\mathcal{F} = \sigma(T_{N-1})$ denote the sigma-algebra generated by the sub-tournament. Note that by (5.139) we have that the characteristic function of Z_1 converges to

$$\mathbb{E} [e^{itZ_1} \mid \mathcal{F}] \longrightarrow e^{-t^2/4}. \quad (5.158)$$

in probability. Similarly, (5.144) tells us that for Z_2

$$\mathbb{E} [e^{itZ_2} \mathbb{1}_{\{T_{N-1} \in \Omega_{\text{typ}}(k)\}} \mid i \in B_s] \longrightarrow e^{-t^2/4}. \quad (5.159)$$

We thus split up E_1 in (5.155) along

$$E_1 = \underbrace{\mathbb{E} [e^{itZ_2} e^{-t^2/4} \mathbb{1}_{\{T_{N-1} \in \Omega_{\text{typ}}(k)\}} \mid i \in B_s]}_{E_3} + \underbrace{\mathbb{E} [e^{itZ_2} \mathbb{1}_{\{T_{N-1} \in \Omega_{\text{typ}}(k)\}} (e^{itZ_1} - e^{-t^2/4}) \mid i \in B_s]}_{E_4}. \quad (5.160)$$

By (5.159), the first term gives

$$E_3 \longrightarrow e^{-t^2/2}. \quad (5.161)$$

For the second term in (5.160), we note that the tower property gives

$$E_4 = \mathbb{E} \left[e^{itZ_2} \mathbb{1}_{\{T_{N-1} \in \Omega_{\text{typ}}(k)\}} \left(\mathbb{E} [e^{itZ_1} \mid \mathcal{F}] - e^{-t^2/4} \right) \middle| i \in B_s \right], \quad (5.162)$$

because $\mathbb{1}_{\{T_{N-1} \in \Omega_{\text{typ}}(k)\}}$ and Z_2 are $\mathcal{F}$ -measurable. Furthermore, we also have

$$|e^{itZ_2}| \leq 1 \quad \text{and} \quad |e^{itZ_1} - e^{-t^2/4}| \leq 2, \quad (5.163)$$

so that the entire term is uniformly bounded by 2. We can then apply the dominated convergence theorem, so that (5.158) gives that

$$E_4 \rightarrow 0. \quad (5.164)$$

Combining this with the result for E_3 in (5.161) shows that (5.160) implies

$$E_1 \longrightarrow e^{-t^2/2}, \quad (5.165)$$

which means that (5.157) implies that

$$\varphi_{\Lambda_N}(t) \longrightarrow e^{-t^2/2} \quad (5.166)$$

as $N \uparrow \infty$. We can therefore conclude that

$$\frac{\Lambda_N(i, s') - \mu_N}{\tau_N} \xrightarrow{D} \mathcal{N}(0, 1), \quad (5.167)$$

which is what we wanted to prove. $\square$

Proof of Proposition 5.16. The random variable $\Lambda_{N,+}(i, s')$ counts the number of edges $i \rightarrow j$, so that $S_j = s'$. Conditioning on $S_i = s$ fixes the number of out-edges to $d_{\text{out}}(i) = (N - 1 + s)/2$. For a random tournament, the edges incident to i are independent of the edges internal to $T_N^{(i)}$. Therefore, conditioned on $S_i = s$, the set of destinations is a random subset of $\mathcal{V}^{(i)}$ of size t . For any such destination j , we have $\Sigma_{ij} = +1$, which implies the score relationship

$$S_j = S_j^{(i)} + \Sigma_{ji} = S_j^{(i)} - 1. \quad (5.168)$$

Consequently, the condition $S_j = s'$ for an out-neighbour is equivalent to $S_j^{(i)} = s' + 1$. In any fixed subtournament $T_N^{(i)}$, the set of possible success vertices is then

$$S = \{y \in \mathcal{V}^{(i)} : S_y^{(i)} = s' + 1\}, \quad (5.169)$$

with size $|S| = K_+$.

Thus, $\Lambda_{N,+}(i, s')$ is the count of success vertices in a sample of size $d_{\text{out}}(i)$ drawn without replacement from a population of size $N - 1$ containing K_+ successes. This fits definition of the Hypergeometric distribution. $\square$

Proof of Lemma 5.17. From Proposition 5.16, we know that $\Lambda_{N,+}(i, s')$ is properly described by hypergeometric sampling of out-edges. However, when treating the full $\Lambda_N(i, s')$, we are simultaneously sampling in- and out-edges. It thus helps note that

$$\Lambda_{N,-}(i, s') + \Lambda_{N,+}(i, s' - 2) = K_-, \quad (5.170)$$

which comes from the fact that all vertices in K_- that receive an out-edge end up in $s' - 2$ and vice versa. Therefore, we can write

$$\Lambda_N(i, s') = \Lambda_{N,+}(i, s') - e^{-\beta\omega} \Lambda_{N,+}(i, s' - 2) - e^{-\beta\omega} K_- \quad (5.171)$$

Conditioned on K_+ and K_- , $\Lambda_N(i, s)$ is then sampling problem on the deterministic population $\{c_{N,1}, c_{N,2}, \dots, c_{N,N-1}\}$ with

$$c_{Ni} = \begin{cases} 1 & \text{for } K_+ \text{ items,} \\ -e^{-\beta\omega} & \text{for } K_- \text{ items,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.172)$$

Given $S_i = s$, the number of out edges is fixed to $d_{\text{out}}(i) = (N - 1 + s)/2$, so that

$$p_N = \frac{(N - 1 + s)}{N - 1} \quad \text{and} \quad \overline{c}_N = \frac{K_+ - e^{-\beta\omega} K_-}{N - 1} \quad (5.173)$$

We then take a random sample of k indices $\{i_1, i_2, \dots, i_k\}$, so that

$$\Lambda_N(i, s') = -e^{-\beta\omega} K_- + \sum_{l=1}^k c_{N,i_l} = S_N - e^{-\beta\omega} K_-. \quad (5.174)$$

For the population sample S_N , we have

$$\begin{aligned}
D_N^2 &= \sum_{i=1}^{N-1} (c_{N,i} - \overline{c_N})^2 \\
&= \sum_{i=1}^{N-1} c_{N,i}^2 - 2c_{i,N} \overline{c_N} + \overline{c_N}^2 \\
&= \left(\sum_{i=1}^{N-1} c_{N,i}^2 \right) - (N-1) \overline{c_N}^2 \\
&= K_+ + e^{-2\beta\omega} K_- - \frac{(K_+ - e^{-\beta\omega} K_-)^2}{N-1},
\end{aligned} \tag{5.175}$$

so that the sample population has a variance

$$s_N^2 = p_N(1-p_N) D_N^2 = \frac{(N-1)^2 - s^2}{4(N-2)} \left[\frac{K_+ + e^{-2\beta\omega} K_-}{N-1} - \left(\frac{K_+ - e^{-\beta\omega} K_-}{N-1} \right)^2 \right]. \tag{5.176}$$

Under the assumption that $K_+, K_- \rightarrow \infty$ and $N - |s| \rightarrow \infty$, we thus see that

$$D_N^2 \rightarrow \infty \quad \text{and} \quad s_N^2 \rightarrow \infty. \tag{5.177}$$

It rests for us to verify condition (A.22). That is, for all $\varepsilon > 0$

$$\frac{1}{s_N^2} \sum_{|c_{N,i}| > \varepsilon s_N} c_{N,i}^2 \rightarrow 0 \quad \text{as } N \uparrow \infty. \tag{5.178}$$

But note that in our case the $c_{N,i} \in \{1, -e^{-\beta\omega}, 0\}$ for all N and i . Because the $c_{N,i}$ uniformly bounded while $s_N \rightarrow \infty$, we can fix ε and freely pick an N_0 , such that for all $N \geq N_0$

$$\varepsilon s_N > \max(1, e^{-\beta\omega}). \tag{5.179}$$

This implies that for $N > N_0$

$$\sum_{|c_{N,i}| > \varepsilon s_N} c_{N,i}^2 = 0, \tag{5.180}$$

which means that the condition is trivially satisfied. By Theorem A.10, we thus have

$$\frac{S_N - \overline{c_N}}{s_N} \xrightarrow{D} \mathcal{N}(0, 1). \tag{5.181}$$

Recall that (5.174) shows that Λ_N is S_N shifted by a constant, so that convergence of S_N immediately implies the result for Λ_N . $\square$

6. A perturbative approach to random Poisson equations

In this section, we characterise the fluctuations of the random heat capacity $C_N(\beta)$ around the annealed heat capacity $\overline{C_N}(\beta)$ by constructing a perturbation scheme. The scheme combines the exact solutions for the annealed process in Theorem 4.1 with the statistical properties in Chapter 5. We perturb around $\overline{G_N}$ in three steps.

The initial step requires finding the inverse of $\overline{G_N}$ to transform perturbative equations of the form

$$(\overline{G_N} + \Delta)f = -g \quad \text{into} \quad (I + (\overline{G_N})^{-1}\Delta)f = -(\overline{G_N})^{-1}g, \quad (6.1)$$

which is solvable if the perturbation Δ is small. Because $\overline{G_N}$ is singular due to its stationary measure $\overline{\pi_N}$, we solve for its pseudoinverse $\overline{G_N}^\dagger$, which solves systems on the correct subspace orthogonal to $\overline{\pi_N}$.

Theorem 6.1. *The inverse of $\overline{G_N}$ on the subspace perpendicular to $\overline{\pi_N}$ is given by*

$$(\overline{G_N}^\dagger)_{ss'} = \frac{2}{1 + e^{-\beta\omega}} [\pi_N(s')(v(s) + v(s') - 1 + O(N^{-1})) - v(s)\delta_{ss'}] \quad (6.2)$$

for all $s \in \mathcal{R}$, where $\delta_{ss'}$ is the Kronecker delta and

$$v(s) = e^{-\frac{s}{(N-1)}} \tanh \frac{\beta\omega}{2}. \quad (6.3)$$

The pseudoinverse alone is not enough to establish a direct perturbation relation between the microscopic generator L_N with rates (3.37) and the annealed generator $\overline{G_N}$. The microscopic generator L_N operates on the vertex space $\mathcal{V} \times \mathcal{V}$, while $\overline{G_N}$ operates on the score space $\mathcal{R} \times \mathcal{R}$. To bridge the dimensional gap, we construct an auxiliary generator G_N .

We define G_N on $\mathcal{V} \times \mathcal{V}$, by setting the transition rates

$$(G_N)_{ij} = \frac{\gamma_N(s_i, s_j)}{\mathbb{E}[N_{s_j}]}, \quad (6.4)$$

for $i \neq j$. A crucial property of these rates is that they only depend on vertex score, so that when we consider the total transition rate from a vertex i to a specific score bin $B_{s'}$, we sum over the amount of vertices in that bin. As such, for any $g : \mathcal{T}_{\mathbf{S}} \rightarrow \mathbb{R}$, it follows that for any vertex $i \in B_s$

$$G_N g(s_i) = \sum_{s' \in \mathcal{T}_{\mathbf{S}}} \gamma_N(s, s') \frac{N_{s'}}{\mathbb{E}[N_{s'}]} [g(s') - g(s)] \quad (6.5)$$

The generator G_N is thus also defined on the set of realised scores $\mathcal{T}_{\mathbf{S}} \times \mathcal{T}_{\mathbf{S}}$, where the rates are given by

$$(G_N)_{ss'} = \gamma_N(s, s') \frac{N_{s'}}{\mathbb{E}[N_{s'}]} \quad (6.6)$$

for both $s, s' \in \mathcal{T}_{\mathbf{S}}$ and $(G_N)_{ss'} = 0$ otherwise

With this auxiliary generator, we define the perturbation matrix $\Delta = G_N - \overline{G_N}$. Perturbation theory then yields the following theorem.

Theorem 6.2. *The heat capacity of the process generated by G_N satisfies*

$$\tilde{C}_N(\beta) = \frac{4C_2(\beta)}{N-1} \left(1 + \sum_{s \in \mathcal{T}_{\mathbf{S}}} \mathcal{K}(s)(N_s - \mathbb{E}[N_s]) \right), \quad (6.7)$$

with failure probability $O(N^{-k})$ for any $k > 0$. The weights $\mathcal{K}(s)$ are given by

$$\mathcal{K}(s) = \frac{1}{\omega} \left(\frac{s+2T}{N-1} v(s) \overline{W}_N(s) + \Psi(s) \right), \quad (6.8)$$

the function $\Psi(s)$ by

$$\Psi(s) = v(s) \sum_{s' \in \mathcal{T}_{\mathbf{s}}} \left[(s' + 2T) \overline{\pi}_N(s') + K_v \frac{\overline{\pi}_N(s')}{v(s')} \right] \left(\overline{W}_N(s) - \overline{W}_N(s') \right), \quad (6.9)$$

and the constant K_v by

$$K_v = \sum_{s \in \mathcal{T}_{\mathbf{s}}} (s + 2T) \overline{\pi}_N(s) v(s), \quad (6.10)$$

where $v(s)$ is given by (6.3).

In order to show that this result has exactly the form of Conjecture 1, we first approximate $\Psi(s)$. Before expanding $\Psi(s)$, we must address the domain of the sums. The sums over the empirical score space $\mathcal{T}_{\mathbf{s}}$ are capped at the maximum observed score in the tournament. This is $O(\sqrt{N \log N})$ with high probability, following Definition 3.8. However, because $\overline{\pi}_N(s)$ has a Gaussian tail with variance $O(N)$, the mass excluded in the truncated region decreases exponentially. To study the general behaviour of (6.7), we can safely neglect this penalty.

Remark 6.3. To understand the limiting distribution of the heat capacity of G_N (6.7), we derive the leading order behaviour of the weights $\mathcal{K}(s)$ for typical tournaments where the maximum score is $O(\sqrt{N \log N})$. This boils down to evaluating $\Psi(s)$ in (6.9). To do so, we first Taylor expand the annealed quasipotential $\overline{W}_N$ in (4.1) and $v(s)$ in (6.3) to find

$$\overline{W}_N(s) = \frac{\omega s}{N-1} + O(N^{-1}) \quad \text{and} \quad v(s) = \exp\left(-\frac{sT}{N-1}\right) = 1 - \frac{sT}{N-1} + O(N^{-1}). \quad (6.11)$$

This means that the difference in quasipotentials can be estimated by

$$\overline{W}_N(s) - \overline{W}_N(s') = \frac{\omega}{N-1} (s - s') + O(N^{-1}), \quad (6.12)$$

Substituting this into the full expression for $\Psi(s)$, we obtain

$$\Psi(s) \approx \frac{\omega v(s)}{N} \sum_{s' \in \mathcal{T}_{\mathbf{s}}} \left[(s' + 2T)(s - s') \overline{\pi}_N(s') + K_v \frac{\overline{\pi}_N(s')}{v(s')} (s - s') \right]. \quad (6.13)$$

We can treat the first term as an expectation with respect to the stationary measure $\overline{\pi}_N$, which is approximately $\mathcal{N}(-2T, N-1)$. Recognising that the sum corresponds to the variance, we find

$$\sum_{s' \in \mathcal{T}_{\mathbf{s}}} (s' + 2T)(s - s') \overline{\pi}_N(s') \approx -(N-1). \quad (6.14)$$

For the second sum, we use the Taylor expansion of $v(s)$ in (6.11). We can then calculate the constant

$$K_v \approx \sum_{s \in \mathcal{T}_{\mathbf{s}}} (s + 2T) \left(1 - \frac{sT}{N-1} \right) \overline{\pi}_N(s) \approx -T, \quad (6.15)$$

so that the second sum in (6.13) can be approximated as

$$\sum_{s' \in \mathcal{T}_{\mathbf{s}}} K_v \frac{\overline{\pi}_N(s')}{v(s')} (s - s') \approx \sum_{s' \in \mathcal{T}_{\mathbf{s}}} K_v \left(1 + \frac{s'T}{N-1} \right) (s - s') \overline{\pi}_N(s') \approx -T(s + T). \quad (6.16)$$

Therefore, we find that

$$\Psi(s) \approx v(s) \frac{\omega}{N-1} [-(N-1) - T(s+T)], \quad (6.17)$$

so that after Taylor expanding $v(s)$ again, we can conclude that

$$\Psi(s) \approx \left(1 - \frac{sT}{N-1}\right) \frac{\omega}{N-1} [-(N-1) - T(s+T)] \approx -\omega, \quad (6.18)$$

Substituting this result and the Taylor expanded expression for $\overline{W}_N$ (4.1) into the definition of $\mathcal{K}(s)$, we find

$$\mathcal{K}(s) \approx \frac{s^2 + 2Ts}{N^2} - 1, \quad (6.19)$$

where we used that $N-1 \approx N$ for N large. Next, we apply Theorem 5.3 on these weights. We see that

$$\mathcal{K}(S_i) - \mathbb{E}[\mathcal{K}(S_i)] = \frac{S_i^2 - \mathbb{E}[S_i^2] + 2TS_i}{N^2}, \quad (6.20)$$

so that $M_N = O(N^{-1} \log N)$ for typical tournaments. Because the odd moments of a binomial distribution are zero, we also have that S_i and S_i^2 are uncorrelated, so that

$$\mathbb{V}(\mathcal{K}(S_i)) = \frac{\mathbb{V}(S_i^2) + 4T^2 \mathbb{V}(S_i)}{N^4} = O(N^{-2}). \quad (6.21)$$

Therefore, the condition in (5.5) is satisfied

$$M_N = O(N^{-1} \log N) = o(N^{-1/2}). \quad (6.22)$$

We can thus conclude that the sum in (6.7) converges in distribution to a Normal random variable. To compute the variance of this sum, we have to account for the covariances. For any two vertices i and j , we can extract the shared edge by writing $S_i = \Sigma_{ij} + S_i^{(j)}$ and $S_j = -\Sigma_{ij} + S_j^{(i)}$. Because $S_i^{(j)}$ and $S_j^{(i)}$ are independent symmetric random variables with zero odd moments, all cross-terms in the quadratic covariance vanish, yielding exactly $\text{Cov}(S_i^2, S_j^2) = 0$. By the same argument, the cross-covariances $\text{Cov}(S_i^2, S_j)$ are also zero. The only non-zero contribution arises from the linear covariance $\text{Cov}(S_i, S_j) = -1$. Evaluating the covariance of the centered weights gives

$$\text{Cov}(\mathcal{K}(S_i), \mathcal{K}(S_j)) \approx \frac{1}{N^4} \text{Cov}(S_i^2 + 2TS_i, S_j^2 + 2TS_j) = -\frac{4T^2}{N^4}. \quad (6.23)$$

The total variance of the sum is the sum of the individual variances and the $N(N-1)$ covariance pairs. Because the covariance sum scales as $\mathcal{O}(N^{-2})$, it is strictly sub-leading to the diagonal variance sum

$$\mathbb{V}\left(\sum_{s \in \mathcal{T}(\mathbf{S})} \mathcal{K}(s)(N_s - \mathbb{E}[N_s])\right) \approx \sum_{i=1}^N \mathbb{V}(\mathcal{K}(S_i)) = O(N^{-1}). \quad (6.24)$$

Therefore, we can conclude that the heat capacity of G_N behaves exactly as in Conjecture 1. $\circ$

Simulations show that the auxiliary generator G_N and the tournament generator L_N have nearly identical heat capacities. An example of this is shown in Figure 6.1a. In fact, Figure 6.1b suggests that the absolute error to scales with $O(N^{-1})$. This observation leads to the final statement of the thesis. Due to time constraints, it is stated as a conjecture. However, a viable route towards proving this result is given in Section 6.3.

Conjecture 2. *Let $\Sigma = (\Sigma(i, j))_{i, j \leq N}$ be a random tournament. Let $C_N(\beta)$ be the heat capacity of the process corresponding to L_N (3.36) and let $\tilde{C}_N(\beta)$ be the heat capacity for G_N (6.5). Then, as $N \rightarrow \infty$*

$$N \cdot |C_N(\beta) - \tilde{C}_N(\beta)| = O(N^{-1}), \quad (6.25)$$

with high probability.

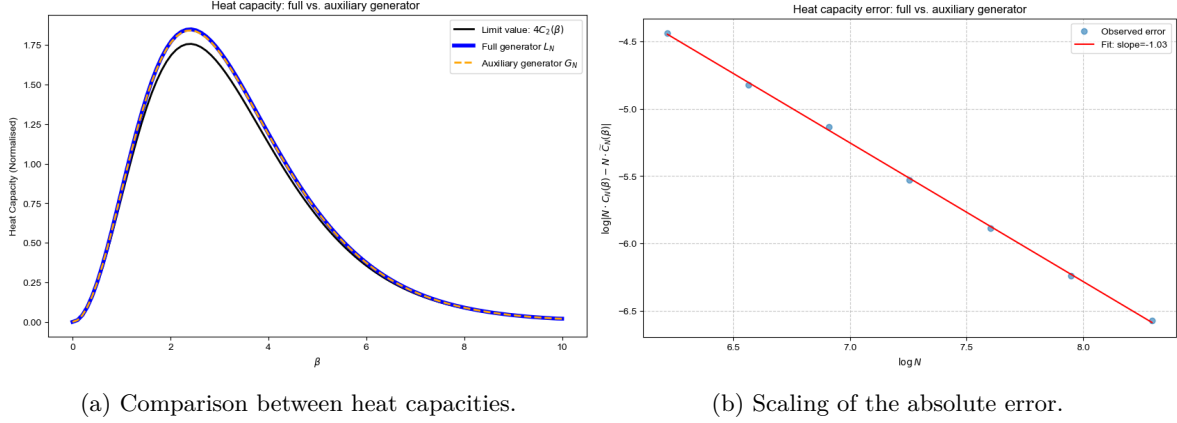


Figure 6.1: Comparison of the normalised heat capacity $N \cdot C_N(\beta)$ for the microscopic random tournament and the process generated by G_N . (a) The heat capacity plotted as a function of β at $\omega = 1$ for a single tournament realisation with $N = 2001$. It shows agreement between G_N (dashed orange) and the microscopic system (solid blue) despite the deviation from the limiting value $4C_2(\beta)$ (solid black). Despite the deviation of the microscopic system from the limit, the heat capacity of G_N is in agreement, which implies that G_N acts similarly to the coarse graining in Section 3.3.2. (b) Log-log plot of the absolute difference $N \cdot |C_N(\beta) - \tilde{C}_N(\beta)|$ as a function of system size $N \in \{501, \dots, 4001\}$ averaged over 100 realisations. The linear fit suggests a scaling of $\mathcal{O}(N^{-1})$.

6.1 Annealed pseudoinverse

In this section, we derive the pseudoinverse of the annealed generator $\overline{G_N}$ to invert the matrix on its range while projecting elements in the kernel to zero. While singular value decomposition offers a general method, the detailed balance property of $\overline{G_N}$ allows for a more direct approach.

Let Π be the diagonal matrix with entries $(\Pi)_{ss} = \overline{\pi_N}(s)$. We define the similarity transformation

$$M_N = \Pi^{1/2} \overline{G_N} \Pi^{-1/2}. \quad (6.26)$$

Detailed balance ensures that M_N is symmetric and therefore normal. The advantage is that the pseudoinverse of a normal matrix can be computed by diagonalizing the matrix, taking the reciprocal of non-zero eigenvalues, and keeping zero eigenvalues at zero. For a proof of this claim, we refer the reader to [101].

The following lemma computes the entries of M_N and shows that it is in fact a rank-one update of a diagonal matrix.

Lemma 6.4. *Let Π be a diagonal matrix with entries $(\Pi)_{ss} = \overline{\pi_N}(s)$. The similarity transformation*

$$M_N = \Pi^{1/2} \overline{G_N} \Pi^{-1/2} \quad (6.27)$$

yields a symmetric matrix M_N acting on $\mathcal{R} \times \mathcal{R}$. The matrix M_N can be decomposed into a diagonal matrix D and a rank-one perturbation $\mathbf{q}\mathbf{q}^T$

$$M_N = D + \mathbf{q}\mathbf{q}^T. \quad (6.28)$$

The entries of the diagonal matrix are the negative row-sums of the annealed rates $\gamma_N(s, s')$ (4.43)

$$D_{ss} = -\frac{1 + e^{-\beta\omega}}{2} \exp\left(\frac{s}{(N-1)} \tanh \frac{\beta\omega}{2}\right) + \mathcal{O}(N^{-1}), \quad (6.29)$$

and the components of $\mathbf{q}$ are given by

$$q_s = \sqrt{\frac{1 + e^{-\beta\omega}}{2\sqrt{2\pi(N-1)}}} \exp\left(-\frac{s^2}{4(N-1)}\right). \quad (6.30)$$

The next step is to characterise the kernel of M_N . The matrices $\overline{G_N}$ and M_N share the same spectrum and kernel dimension because they are similar. The unique stationary measure of $\overline{G_N}$ thus implies that M_N has a unique zero eigenvalue. The eigenvector spanning the kernel of M_N is identified in the next lemma.

Lemma 6.5. *The kernel of M_N is one-dimensional and is spanned by the unit vector $\mathbf{w} = \Pi^{1/2}\mathbf{1}$. It holds that $\mathbf{w} \propto D^{-1}\mathbf{q}$.*

The final important observation is that the inverse of the diagonal D^{-1} acts as a right inverse for M_N restricted to $\mathbf{w}^\perp$. Specifically, for any $\mathbf{v} \perp \mathbf{w}$, the rank-one structure implies

$$M_N D^{-1} \mathbf{v} = (D + \mathbf{q}\mathbf{q}^T) D^{-1} \mathbf{v} = \mathbf{v} + \mathbf{q}\mathbf{q}^T \mathbf{v} = \mathbf{v}, \quad (6.31)$$

where the final equality holds because $\mathbf{v}$ perpendicular to $\mathbf{w} \propto D^{-1}\mathbf{q}$ by Lemma 6.5.

Therefore, constructing the pseudoinverse reduces to projecting the input vector onto the subspace $\mathbf{w}^\perp$ before and after applying the diagonal inverse D^{-1} . This ensures all kernel components are eliminated. The next lemma formalises this claim.

Lemma 6.6. *Fix $\mathbf{w} = D^{-1}\mathbf{q}$. The pseudoinverse of M_N is given by*

$$M_N^\dagger = (I - K) D^{-1} (I - K), \quad (6.32)$$

where K is the orthogonal projection onto the kernel of M_N

$$K = \frac{\mathbf{w}\mathbf{w}^T}{\mathbf{w}^T \mathbf{w}}. \quad (6.33)$$

Using the previous lemma to compute the pseudoinverse $M_N^\dagger$ gives its closed form expression.

Lemma 6.7. *The entries of the pseudoinverse of M_N are given by*

$$(M_N^\dagger)_{ss'} = \frac{\delta_{ss'}}{D_{ss}} - \sqrt{\pi_N(s)\pi_N(s')} \left(\frac{1}{D_{ss}} + \frac{1}{D_{s's'}} - \frac{\nu}{\kappa} \right), \quad (6.34)$$

where $\delta_{ss'}$ denotes the Kronecker delta and

$$\kappa = \sum_{s \in S} \frac{q_s^2}{D_{ss}^2} \quad \text{and} \quad \nu = \sum_{s \in S} \frac{q_s^2}{D_{ss}^3}. \quad (6.35)$$

Using the inverse of the similarity transform we applied in the beginning to obtain $\overline{G_N}^\dagger = \Pi^{-1/2} \overline{G_N}^\dagger \Pi^{1/2}$, we prove Theorem 6.1.

Proofs: section 6.1

Proof of Theorem 6.1. We obtain $\overline{G_N}^\dagger$ by applying the similarity transform to $M_N^\dagger$

$$\overline{G_N}^\dagger = D_{\pi_N}^{-1/2} M_N^\dagger D_{\pi_N}^{1/2} \implies (\overline{G_N}^\dagger)_{ss'} = \sqrt{\frac{\pi_N(s')}{\pi_N(s)}} (M_N^\dagger)_{ss'} \quad (6.36)$$

Substituting for the definition of $(M_N^\dagger)_{ss'}$ in (6.34) we have

$$(\overline{G_N}^\dagger)_{ss'} = \frac{\delta_{ss'}}{D_{ss}} - \pi_N(s') \left(\frac{1}{D_{ss}} + \frac{1}{D_{s's'}} - \frac{\nu}{\kappa} \right). \quad (6.37)$$

From an explicit calculation of κ and ν in the appendix (E.9), it follows that

$$\frac{\nu}{\kappa} = \frac{-4/(1 + e^{-\beta\omega})^2 + O(N^{-1})}{2/(1 + e^{-\beta\omega}) + O(N^{-1})} = -\frac{2}{1 + e^{-\beta\omega}} + O(N^{-1}), \quad (6.38)$$

such that (6.37) reduces to

$$(\overline{G_N}^\dagger)_{ss'} = \frac{\delta_{ss'}}{D_{ss}} - \pi_N(s') \left(\frac{1}{D_{ss}} + \frac{1}{D_{s's'}} + \frac{2}{1 + e^{-\beta\omega}} + O(N^{-1}) \right) \quad (6.39)$$

Here, we used that $\pi_N(s) = O(N^{-1/2})$ from the local CLT (A.6). Now, substituting for D_{ss} from (6.29) into (6.39) yields (6.2). $\square$

Proof of Lemma 6.4. We start by noting that the entries of M_N are given by the similarity transformation

$$(M_N)_{ss'} = \sqrt{\frac{\pi_N(s)}{\pi_N(s')}} \gamma_N(s, s'). \quad (6.40)$$

Since the chain is reversible, it satisfies the detailed balance condition

$$\pi_N(s) \gamma_N(s, s') = \pi_N(s') \gamma_N(s', s), \quad (6.41)$$

so that

$$\frac{\pi_N(s)}{\pi_N(s')} = \frac{\gamma_N(s', s)}{\gamma_N(s, s')}. \quad (6.42)$$

Substituting (6.42) into (6.40) then yields

$$(M_N)_{ss'} = \sqrt{\frac{\gamma_N(s', s)}{\gamma_N(s, s')}} \gamma_N(s, s') = \sqrt{\gamma_N(s, s') \gamma_N(s', s)} \quad (6.43)$$

Now, we substitute for the explicit expression for the rates γ_N in (4.43). Let us use the abbreviation $T = \tanh \beta\omega/2$ for brevity. The rate is then

$$\gamma_N(s, s') = \frac{1}{2} \frac{1 + e^{-\beta\omega}}{\sqrt{2\pi(N-1)}} \exp \left(-\frac{s^2}{2(N-1)} + \frac{s-s'}{N-1} T \right) \quad (6.44)$$

The product under the square root in (6.43) is therefore given by

$$\gamma_N(s, s') \gamma_N(s', s) = \left(\frac{1}{2} \frac{1 + e^{-\beta\omega}}{\sqrt{2\pi(N-1)}} \right)^2 \exp \left(-\frac{s^2}{2(N-1)} - \frac{(s')^2}{2(N-1)} \right). \quad (6.45)$$

Taking the square root, following (6.43), we obtain for $(M_N)_{ss'}$

$$(M_N)_{ss'} = \frac{1}{2} \frac{1 + e^{-\beta\omega}}{\sqrt{2\pi(N-1)}} \exp \left(-\frac{s^2}{4(N-1)} \right) \exp \left(-\frac{(s')^2}{4(N-1)} \right). \quad (6.46)$$

Indeed then, if we take

$$q_s = \sqrt{\frac{1 + e^{-\beta\omega}}{2\sqrt{2\pi(N-1)}}} \exp \left(-\frac{s^2}{4(N-1)} \right), \quad (6.47)$$

it follows that $(M_N)_{ss'} = q_s q_{s'}$ for $s \neq s'$. For the diagonal, the transformation yields the original diagonal of the generator

$$(M_N)_{ss} = \sqrt{\frac{\pi_N(s)}{\pi_N(s)}} (\overline{G_N})_{ss} = (\overline{G_N})_{ss}. \quad (6.48)$$

Because $\overline{G_N}$ is a generator, the diagonal entry of M_N is the negative sum of the off-diagonal rates

$$(M_N)_{ss} = - \sum_{s' \neq s} \gamma_N(s, s'). \quad (6.49)$$

We will show that this equals the diagonal of the proposed decomposition $D + \mathbf{q}\mathbf{q}^T$. From (6.42), we get that

$$q_s q_s = \gamma_N(s, s) \quad (6.50)$$

So that indeed

$$(M_N)_{ss} = \gamma_N(s, s) - \sum_{s' \in \mathcal{R}} \gamma_N(s, s'), \quad (6.51)$$

which is what we set out to prove. By Lemma E.1, the sum over the $\gamma_N(s, s')$ correspond to (6.29). $\square$

Proof of Lemma 6.5. Because $\overline{G_N}$ is the generator of a Markov process, it holds that

$$\overline{G_N} \mathbf{1} = 0. \quad (6.52)$$

Therefore, M_N satisfies

$$M_N(D_{\pi_N}^{1/2} \mathbf{1}) = D_{\pi_N}^{1/2} \overline{G_N} \mathbf{1} = 0 \quad (6.53)$$

such that indeed

$$\mathbf{w} = D_{\pi_N}^{1/2} \mathbf{1} \implies w_s = \sqrt{\pi_N(s)}. \quad (6.54)$$

Because all diagonal entries D_{ss} are negative otherwise, the interlacing property implies that 0 is the unique, biggest eigenvalue. As such, M_N has a nonzero spectral gap for all N . $\square$

Proof of Lemma 6.6. We verify that the proposed $M_N^\dagger$ satisfies the Moore-Penrose conditions for a symmetric matrix. We start by simplifying the product $M_N D^{-1}$. Substituting $M_N = D + \mathbf{q}\mathbf{q}^T$, we get

$$M_N D^{-1} = (D + \mathbf{q}\mathbf{q}^T) D^{-1} = I + \mathbf{q}(\mathbf{q}^T D^{-1}). \quad (6.55)$$

Using the definition $\mathbf{w} = D^{-1} \mathbf{q}$, we have $\mathbf{q} = D \mathbf{w}$ and $\mathbf{q}^T D^{-1} = \mathbf{w}^T$. Thus

$$M_N D^{-1} = I + D \mathbf{w} \mathbf{w}^T. \quad (6.56)$$

Computing the product $M_N M_N^\dagger$, we find

$$\begin{aligned} M_N M_N^\dagger &= M_N (I - P) D^{-1} (I - P) \\ &= (M_N - M_N P) D^{-1} (I - P) \\ &= M_N D^{-1} (I - P), \end{aligned} \quad (6.57)$$

where in the last line we used that $M_N \mathbf{w} = 0$. Substituting our expression for $M_N D^{-1}$ then gives

$$\begin{aligned} M_N M_N^\dagger &= (I + D \mathbf{w} \mathbf{w}^T) (I - P) \\ &= (I - P) + D \mathbf{w} \mathbf{w}^T (I - P) \\ &= I - P + D \mathbf{w} (\mathbf{w}^T - \mathbf{w}^T P). \end{aligned} \quad (6.58)$$

Using the definition of P and the fact that $\mathbf{w}$ is a unit vector, we know that $\mathbf{w}^T P = \mathbf{w}^T$. Therefore, (6.58) reduces to

$$M_N M_N^\dagger = I - P. \quad (6.59)$$

For the symmetry condition, note that

$$(M_N M_N^\dagger)^T = (I - \mathbf{w} \mathbf{w}^T)^T = I - P \quad (6.60)$$

Furthermore, we immediately obtain from the definition that $(M_N^\dagger)^T = M_N^\dagger$. This now gives that

$$M_N M_N^\dagger = M_N^\dagger M_N, \quad (6.61)$$

such that we immediately get that $M_N^\dagger M_N$ is also symmetric. For the second condition, we simply calculate

$$M_N M_N^\dagger M_N = (I - P) M_N = M_N - P M_N = M_N. \quad (6.62)$$

And for the final condition, we write

$$M_N^\dagger M_N M_N^\dagger = M_N^\dagger (I - P) = M_N^\dagger - M_N^\dagger P. \quad (6.63)$$

Now, it holds that

$$M_N^\dagger P = (I - P) D^{-1} (I - P) P = 0 \quad (6.64)$$

and therefore (6.63) reduces to

$$M_N^\dagger M_N M_N^\dagger = M_N^\dagger. \quad (6.65)$$

All conditions are satisfied, which means that $M_N^\dagger$ is indeed the correct pseudoinverse. $\square$

Proof of Lemma 6.7. We work out the brackets in the definition of $M_N^\dagger$ in (6.32) to obtain

$$M_N^\dagger = D^{-1} - D^{-1} P - P D^{-1} + P D^{-1} P. \quad (6.66)$$

From (6.33), we get that

$$P = \frac{\mathbf{w} \mathbf{w}^T}{\mathbf{w}^T \mathbf{w}}. \quad (6.67)$$

We compute the denominator to see

$$\mathbf{w}^T \mathbf{w} = \sum_{s \in S} \frac{q_s^2}{D_{ss}^2} = \kappa, \quad (6.68)$$

so that the entries of P are given by

$$P_{ss'} = \frac{1}{\kappa} \frac{q_s}{D_{ss}} \frac{q_{s'}}{D_{s's'}}. \quad (6.69)$$

We can now treat (6.66) term by term. Because D is a diagonal matrix, we get

$$(D^{-1})_{ss'} = \frac{\delta_{ss'}}{D_{ss}}. \quad (6.70)$$

The second term yields

$$(D^{-1} P)_{ss'} = \sum_k (D^{-1})_{sk} P_{ks'} = P_{ss'} \frac{1}{D_{ss}}. \quad (6.71)$$

Using (6.69) it thus follows that

$$(D^{-1} P)_{ss'} = \frac{1}{\kappa} \frac{q_s q_{s'}}{D_{ss}^2 D_{s's'}}. \quad (6.72)$$

Similarly, the third term becomes

$$(P D^{-1})_{ss'} = \sum_k P_{sk} (D^{-1})_{ks'} = \frac{1}{\kappa} \frac{q_s q_{s'}}{D_{ss} D_{s's'}^2}. \quad (6.73)$$

Finally, we write

$$(PD^{-1}P)_{ss'} = \sum_k P_{sk}(D^{-1}P)_{ks'}. \quad (6.74)$$

Using (6.71) and (6.69) this sum simplifies to

$$\begin{aligned} (PD^{-1}P)_{ss'} &= \sum_k P_{sk}P_{ks'} \frac{1}{D_{kk}} \\ &= \frac{1}{\kappa} \frac{q_s}{D_{ss}} \frac{q'_{s'}}{D_{s's'}} \sum_k \frac{q_k^2}{D_{kk}^3} \\ &= \frac{\nu}{\kappa} \frac{q_s}{D_{ss}} \frac{q'_{s'}}{D_{s's'}}. \end{aligned} \quad (6.75)$$

Combining (6.70), (6.71), (6.73) and (6.75) then gives

$$(M_N^\dagger)_{ss'} = \frac{\delta_{ss'}}{D_{ss}} - \frac{q_s q'_{s'}}{\kappa} \left(\frac{1}{D_{ss}^2 D_{s's'}} + \frac{1}{D_{ss} D_{s's'}^2} - \frac{\nu}{\kappa} \frac{1}{D_{ss} D_{s's'}} \right). \quad (6.76)$$

Now, to finalise the proof, note that

$$\frac{q_s q'_{s'}}{D_{ss} D_{s's'}} = \kappa \sqrt{\pi_N(s) \pi_N(s')}. \quad (6.77)$$

This follows from the property that $\mathbf{w} = D^{-1}\mathbf{q}$ is the zero eigenvector of M_N , whose entries we know by Lemma 6.5. As such, $w_s \propto \sqrt{\pi_N(s)}$. In particular

$$\frac{q_s}{D_{ss}} = b \sqrt{\pi_N(s)} \implies b^2 = \sum_{s \in S} \left(\frac{q_s}{D_{ss}} \right)^2 = \kappa. \quad (6.78)$$

Substituting (6.77) into (6.76) then yields (6.32), which is what we wanted to show. $\square$

6.2 Heat capacity of G_N

We start this section by providing the formal definition of the generator G_N and its associated process.

Definition 6.8. We define the process $(Z_t)_{t \geq 0}$ on the vertex space through the generator G_N with entries

$$(G_N)_{ij} = \frac{\gamma_N(s_i, s_j)}{\mathbb{E}[N_{s_j}]} \quad \text{and} \quad (G_N)_{ii} = - \sum_{j=1}^N (G_N)_{ij} \quad (6.79)$$

for $i, j \in \mathcal{V}$. For any function $g : \mathcal{R} \rightarrow \mathbb{R}$ of score only, the generator G_N is defined over the realised vertex scores $\mathcal{T}_{\mathbf{S}}$ as

$$(G_N)_{ss'} = \gamma_N(s, s') \frac{N_{s'}}{\mathbb{E}[N_{s'}]} \quad \text{and} \quad (G_N)_{ss} = - \sum_{s' \in \mathcal{T}_{\mathbf{S}}} (G_N)_{ss'} \quad (6.80)$$

for all $s, s' \in \mathcal{T}_{\mathbf{S}}$ and 0 elsewhere. We denote its stationary measure by $\pi_N(s)$.

The specific structure of its transition ensures that G_N inherits the detailed balance properties of the annealed generator $\overline{G}_N$. We use this to derive an exact expression for the stationary measure $\pi_N(s)$.

Lemma 6.9. The generator G_N satisfies the detailed balance condition for the stationary distribution

$$\pi_N(s) = \begin{cases} \frac{1}{Z} \overline{\pi}_N(s) \frac{N_s}{\mathbb{E}[N_s]} & \text{if } s \in \mathcal{T}_{\mathbf{S}}, \\ 0 & \text{if } s \notin \mathcal{T}_{\mathbf{S}}, \end{cases} \quad (6.81)$$

where

$$Z = \sum_{s \in \mathcal{T}_{\mathbf{S}}} \overline{\pi}_N(s) \frac{N_s}{\mathbb{E}[N_s]}. \quad (6.82)$$

The lemma shows that the stationary distribution of G_N is essentially Gaussian reweighted with the size of a score bin relative to its expectation. This is illustrated in Figure 6.2.

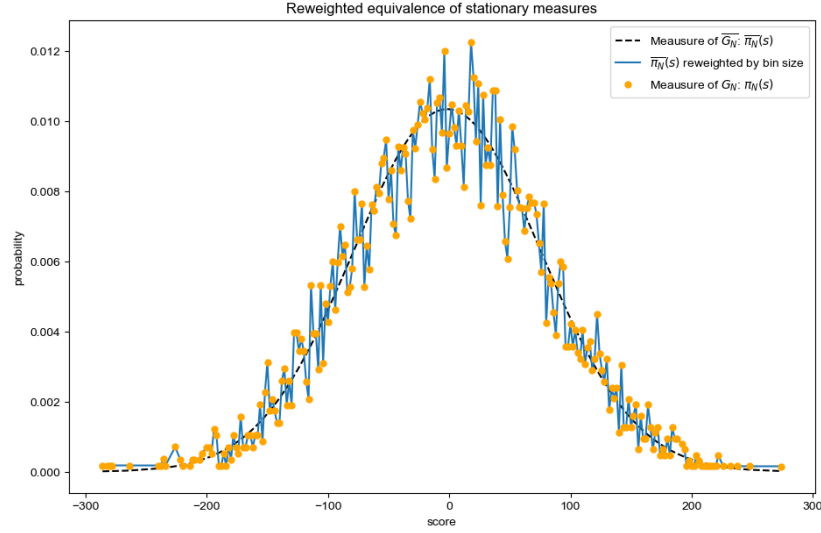


Figure 6.2: Comparison of stationary measures $\overline{\pi}_N$ and π_N associated with $\overline{G}_N$ and G_N , respectively. The unweighted measure $\overline{\pi}_N$ is shown as a dotted line. The solid line represents $\overline{\pi}_N$ reweighted as in (6.81). The scatter points show the true measure of G_N , π_N . The figure shows that (6.81) is the correct stationary measure. Simulation parameters: system size $N = 6001$, inverse temperature $\beta = 2.6$.

The aim is now to solve the Poisson equation

$$G_N W_N = \mathcal{Q}_N. \quad (6.83)$$

By defining $\Delta = G_N - \overline{G}_N$ as the deviation of the auxiliary generator from the annealed generator and $\delta W = W_N - \overline{W}_N$ as the corresponding difference in quasipotential, we can expand the Poisson equation in terms of these perturbations

$$(\overline{G}_N + \Delta)(\overline{W}_N + \delta W) = \mathcal{Q}_N. \quad (6.84)$$

Since the annealed quasipotential $\overline{W}_N$ satisfies the annealed Poisson equation (4.57), the problem reduces to an equation for the correction term δW

$$(\overline{G}_N + \Delta)\delta W = \overline{\mathcal{Q}}_N - \mathcal{Q}_N - \Delta \overline{W}_N. \quad (6.85)$$

Solving this equation amounts to applying the pseudoinverse $\overline{G}_N^\dagger$ (6.2) and inverting $(I + \overline{G}_N^\dagger \Delta)$.

It is convenient to adopt block matrix notation to analyse the structure of the perturbation matrix Δ . We partition the score space $\mathcal{R}$ into the bulk $B = \mathcal{T}_{\mathbf{S}}$, consisting of the scores realised in the tournament and, and the tail of unrealised scores $T = \mathcal{R} \setminus \mathcal{T}_{\mathbf{S}}$.

In this basis, the annealed generator decomposes as

$$\overline{G}_N = \begin{pmatrix} (\overline{G}_N)_{BB} & (\overline{G}_N)_{BT} \\ (\overline{G}_N)_{TB} & (\overline{G}_N)_{TT} \end{pmatrix}, \quad (6.86)$$

while the auxiliary generator G_N is only defined on the realised scores, so that

$$G_N = \begin{pmatrix} (G_N)_{BB} & 0 \\ 0 & 0 \end{pmatrix}. \quad (6.87)$$

The perturbation matrix $\Delta = G_N - \overline{G_N}$ thus takes the form

$$\Delta = \begin{pmatrix} (G_N)_{BB} - (\overline{G_N})_{BB} & -(\overline{G_N})_{BT} \\ -(\overline{G_N})_{TB} & -(\overline{G_N})_{TT} \end{pmatrix} \quad (6.88)$$

The blocks involving the tail region consists entirely of entries removed by the restriction to realised scores. The randomness of the tournament is confined to the bulk-bulk block Δ_{BB} . For off-diagonal entries, we have

$$(G_N)_{ss'} - (\overline{G_N})_{ss'} = \frac{\gamma_N(s, s')}{\mathbb{E}[N_{s'}]} (N_{s'} - \mathbb{E}[N_{s'}]). \quad (6.89)$$

For diagonal entries, the perturbation has to account for the total change across the row

$$(G_N)_{ss} - (\overline{G_N})_{ss} = - \sum_{s' \in \mathcal{T}_S} \frac{\gamma_N(s, s')}{\mathbb{E}[N_{s'}]} (N_{s'} - \mathbb{E}[N_{s'}]) + \sum_{s' \notin \mathcal{T}_S} \gamma_N(s, s'). \quad (6.90)$$

Since the generator G_N is restricted to the realised scores $\mathcal{T}_S$, the quasipotential W_N vanishes on the tail set T . Therefore, the correction on the tail is fixed by $\delta W_T = -(\overline{W_N})_T$. The following lemma derives the effective correction δW_B .

Lemma 6.10. *The perturbative correction on the $\mathcal{T}_S$, denoted by δW_B , is the solution to the equation*

$$(I + H_N)\delta W_B = (\overline{G_N}^\dagger)_{BB}U_B + (\overline{G_N}^\dagger)_{BT}U_T, \quad (6.91)$$

where the matrix H_N is given by

$$H_N = -(\overline{G_N}^\dagger)_{BB}\Delta_{BB} + (\overline{G_N}^\dagger)_{BT}(\overline{G_N})_{TB} \quad (6.92)$$

and the source vector U by

$$U = \langle \mathcal{P} \rangle_{\overline{\pi_N}} - \langle \mathcal{P} \rangle_{\pi_N} - \Delta_{BB}\overline{W_B} - \Delta_{TB}\overline{W_B}. \quad (6.93)$$

Formally solving the equation for δW_B (6.91) requires verifying the invertibility of $(I + H_N)$. The following lemma bounds the norm of H_N , which confirms the existence of the inverse and justifies its expansion as a power series.

Lemma 6.11. *The matrix norm of H_N satisfies the bound*

$$\|H_N\|_\infty \leq O(N^{-1/4} \log N) < 1, \quad (6.94)$$

with failure probability $O(N^{-k})$ for all $k > 0$. Consequently, the operator $I + H_N$ is invertible on $\mathcal{T}_S$ with high probability. The inverse is given by the Neumann series

$$(I + H_N)^{-1} = \sum_{k=0}^{\infty} (-1)^k (H_N)^k. \quad (6.95)$$

The Neumann series representation of $(I + H_N)^{-1}$ allows us to express the full solution for the bulk correction δW_B as an infinite sum.

Lemma 6.12. *Let the initial correction to the quasipotential be given by*

$$\delta W_B^{(0)} = (\overline{G_N}^\dagger)_{BB}U_B + (\overline{G_N}^\dagger)_{BT}U_T. \quad (6.96)$$

Then, (6.91) has the solution

$$\delta W_B = \sum_{k=0}^{\infty} (-1)^k (H_N)^k \delta W_B^{(0)}. \quad (6.97)$$

The convergence of the series expansion means that the zero-order term is the dominant correction. In fact, a direct application the bound in (6.94) shows that

$$\|\delta W_B - \delta W_B^{(0)}\|_\infty = O(N^{-1/4}) \cdot \|\delta W_B^{(0)}\|_\infty. \quad (6.98)$$

The next lemma gives a closed form expression for $\delta W_B^{(0)}$

Lemma 6.13. *The correction term $\delta W^{(0)}$ is given by*

$$\delta W^{(0)}(s) = \frac{2}{1 + e^{-\beta\omega}} \left[(v(s) - 1) \langle U \rangle_{\pi_N} + \mathcal{M} - U_B(s)v(s) \right], \quad (6.99)$$

where the constant $\mathcal{M}$ is given by the weighted mean

$$\mathcal{M} = \sum_{s \in \mathcal{R}} \pi_N(s) v(s) U(s), \quad (6.100)$$

and $v(s)$ is as defined in Theorem 6.1.

So far, we have the stationary measure π_N and the leading-order correction $\delta W_B^{(0)}$ to the annealed quasipotential $\bar{W}_N$. Notably, the expression for π_N is exact and differentiable with respect to β , as the bin sizes N_s are independent of β . The zero-mean property $\langle W_N \rangle_{\pi_N} = 0$ now implies that

$$\sum_{s \in \mathcal{T}_s} \beta^2 \frac{\partial W_N(s)}{\partial \beta} \pi_N(s) = - \sum_{s \in \mathcal{T}_s} \beta^2 W_N(s) \frac{\partial \pi_N(s)}{\partial \beta}, \quad (6.101)$$

which means that we can calculate the heat capacity of G_N .

Lemma 6.14. *The heat capacity of the process generated by G_N is given by*

$$\tilde{C}_N(\beta) = \frac{\beta^2 \omega}{N-1} \text{sech}^2 \frac{\beta \omega}{2} \sum_{s \in \mathcal{T}_s} (s + 2T) W_N(s) \pi_N(s) \frac{N_s}{\mathbb{E}[N_s]} \quad (6.102)$$

The proof of Theorem 6.2 now follows by substituting the approximate quasipotential $W_N \approx \bar{W}_N + \delta W_B^{(0)}$ into the formula for the heat capacity (6.102).

Proofs: section 6.2

Proof of Theorem 6.2. For clearness of notation, we will abbreviate

$$\eta_s = \frac{N_s - \mathbb{E}[N_s]}{\mathbb{E}[N_s]}. \quad (6.103)$$

From Lemma E.3, we get that the expression for the heat capacity in (6.102) can be written in the form

$$\tilde{C}_N(\beta) = \bar{C}(\beta) \left(1 + \underbrace{\sum_{s \in B} \frac{s + 2T}{\omega(N-1)} v(s) \bar{W}_N(s) (N_s - \mathbb{E}[N_s])}_{S_1} + \underbrace{\sum_{s \in B} \frac{s + 2T}{\omega} \pi_N(s) \delta W(s)}_{S_2} \right) \quad (6.104)$$

The first sum (6.104) is already written in terms of bin fluctuations $N_s - \mathbb{E}[N_s]$. It thus remains for us to rewrite

$$S_2 = \sum_{s \in B} \frac{s + 2T}{\omega} \pi_N(s) \delta W(s). \quad (6.105)$$

We start by replacing δW with the dominant correction $\delta W^{(0)}$. Lemma E.4 bounding the size of U in combination with (6.96) shows that this introduces a negligible error. From the expression in Lemma 6.13, we then have that

$$\delta W^{(0)}(s) = \frac{2}{1 + e^{-\beta\omega}} \left[(v(s) - 1) \langle U \rangle_{\pi_N} + \mathcal{M} - v(s) U_B(s) \right]. \quad (6.106)$$

Neglecting the $O(N^{-(1-\epsilon)/2})$ in the unrealised tail $s \in T$, the term $(s + 2T)\pi_N(s)$ is centered on the bulk B . Consequently, the sum over the constant $\mathcal{M}$ and the $\langle U \rangle_{\pi_N}$ term vanishes. This leaves

$$S_2 = \frac{2}{\omega(1 + e^{-\beta\omega})} \sum_{s \in B} (s + 2T)\pi_N(s) v(s) (\langle U \rangle_{\pi_N} - U_B(s)). \quad (6.107)$$

Applying the definition of $U_B(s)$ from (6.93), the constant power differences $\langle \mathcal{P} \rangle_{\pi_N} - \langle \mathcal{P} \rangle_{\pi_N}$ cancel exactly, yielding

$$\langle U \rangle_{\pi_N} - U_B(s) = (\Delta_{BB} \bar{W}_B)(s) - \langle \Delta \bar{W}_B \rangle_{\pi_N} \quad (6.108)$$

Substituting this into (6.107), we find

$$S_2 = \frac{2}{\omega(1 + e^{-\beta\omega})} \left[\underbrace{\sum_{s \in B} (s + 2T)\pi_N(s) v(s) (\Delta_{BB} \bar{W}_B)}_F - K_v \langle \Delta \bar{W}_B \rangle_{\pi_N} \right], \quad (6.109)$$

where we defined the constant K_v as

$$K_v = \sum_{s \in B} (s + 2T)\pi_N(s) v(s). \quad (6.110)$$

Next, we write the action of the perturbation Δ in terms of the random variables η_s . Note that for off-diagonal entries $\Delta_{ss'} = (\bar{G}_N)_{ss'} \eta_{s'}$, so that the perturbation acts as

$$\begin{aligned} (\Delta_{BB} \bar{W}_B)(s) &= \sum_{s' \in B} (\bar{G}_N)_{ss'} \eta_{s'} (\bar{W}_N(s') - \bar{W}_N(s)) + \sum_{s' \in T} (\bar{G}_N)_{ss'} \bar{W}_N(s) \\ &= \sum_{s' \in B} (\bar{G}_N)_{ss'} \eta_{s'} (\bar{W}_N(s') - \bar{W}_N(s)) + O(N^{(1-\epsilon)/2}). \end{aligned} \quad (6.111)$$

For the averaged term, we apply detailed balance $\pi_N(s) (\bar{G}_N)_{ss'} = \pi_N(s') (\bar{G}_N)_{s's}$ to swap the order of summation

$$\begin{aligned} \langle \Delta \bar{W}_B \rangle_{\pi_N} &= \sum_{s \in B} \pi_N(s) \sum_{s' \in B} (\bar{G}_N)_{ss'} \eta_{s'} (\bar{W}_N(s') - \bar{W}_N(s)) \\ &= \sum_{s, s' \in B} \pi_N(s') (\bar{G}_N)_{s's} \eta_{s'} (\bar{W}_N(s') - \bar{W}_N(s)). \end{aligned} \quad (6.112)$$

Relabeling the indices $s \leftrightarrow s'$, we obtain

$$\langle \Delta \bar{W}_B \rangle_{\pi_N} = - \sum_{s \in B} \pi_N(s) \eta_s \sum_{s' \in B} (\bar{G}_N)_{ss'} (\bar{W}_N(s') - \bar{W}_N(s)). \quad (6.113)$$

Using the definition of η_s (6.103), we can rewrite everything in terms of the bin fluctuations $(N_s - \mathbb{E}[N_s])$. Swapping the summation indices s and s' in the first part F of S_2 in (6.109), we use (6.111) to obtain

$$\begin{aligned} F &= \sum_{s \in B} (s + 2T)\pi_N(s) v(s) \sum_{s' \in B} \frac{(\bar{G}_N)_{ss'}}{\mathbb{E}[N_{s'}]} (\bar{W}_N(s') - \bar{W}_N(s)) (N_s - \mathbb{E}[N_s]) \\ &= \sum_{s \in B} (N_s - \mathbb{E}[N_s]) \sum_{s' \in B} (s' + 2T)\pi_N(s') v(s') \frac{(\bar{G}_N)_{s's}}{\mathbb{E}[N_s]} (\bar{W}_N(s) - \bar{W}_N(s')). \end{aligned} \quad (6.114)$$

Similarly, substituting η_s (6.103) into our result for the averaged term $\langle \Delta \overline{W}_B \rangle_{\overline{\pi}_N}$ in (6.113) yields

$$\langle \Delta \overline{W}_B \rangle_{\overline{\pi}_N} = - \sum_{s \in B} \frac{\overline{\pi}_N(s)}{\mathbb{E}[N_s]} (N_s - \mathbb{E}[N_s]) \sum_{s' \in B} (\overline{G}_N)_{ss'} (\overline{W}_N(s') - \overline{W}_N(s)). \quad (6.115)$$

Substituting these expressions back into (6.109), the full second sum S_2 can be written as a single linear combination of the bin count fluctuations

$$S_2 = \frac{1}{\omega} \sum_{s \in B} \Psi(s) (N_s - \mathbb{E}[N_s]), \quad (6.116)$$

where the weight function $\mathcal{K}(s)$ contains the deterministic perturbative corrections, given by

$$\Psi(s) = \frac{2}{1 + e^{-\beta\omega}} \sum_{s' \in B} \left((s' + 2T) \overline{\pi}_N(s') v(s') \frac{(\overline{G}_N)_{s's}}{\mathbb{E}[N_s]} + K_v \frac{\overline{\pi}_N(s)}{\mathbb{E}[N_s]} (\overline{G}_N)_{ss'} \right) (\overline{W}_N(s) - \overline{W}_N(s')). \quad (6.117)$$

We can simplify this sum by noting $(\overline{G}_N)_{ss'} = \mathbb{E}[N_{s'}] v(s)/v(s')$, so that

$$v(s') \frac{(\overline{G}_N)_{s's}}{\mathbb{E}[N_s]} = \frac{1 + e^{-\beta\omega}}{2} v(s) \quad \text{and} \quad \frac{\overline{\pi}_N(s)}{\mathbb{E}[N_s]} (\overline{G}_N)_{ss'} = \frac{v(s')}{v(s)} \overline{\pi}_N(s') \quad (6.118)$$

This means that (6.117) reduces to

$$\Psi(s) = v(s) \sum_{s' \in B} \left[(s' + 2T) \overline{\pi}_N(s') + K_v \frac{\overline{\pi}_N(s')}{v(s')} \right] (\overline{W}_N(s) - \overline{W}_N(s')) \quad (6.119)$$

Substituting S_2 into (6.104) and grouping it with the direct contribution from the first sum, we arrive at the final linearised form for the heat capacity

$$\tilde{C}_N(\beta) = \overline{C}(\beta) \left[1 + \frac{1}{\omega} \sum_{s \in B} \left(\frac{s + 2T}{N - 1} v(s) \overline{W}_N(s) + \Psi(s) \right) (N_s - \mathbb{E}[N_s]) \right]. \quad (6.120)$$

Finally, setting

$$\mathcal{K}(s) = \frac{1}{\omega} \left(\frac{s + 2T}{N - 1} v(s) \overline{W}_N(s) + \Psi(s) \right), \quad (6.121)$$

then gives the result. $\square$

Proof of Lemma 6.9. We aim to show that for any $s, s' \in \mathcal{T}_{\mathbf{S}}$

$$\pi_N(s) (G_N)_{ss'} = \pi_N(s') (G_N)_{s's}. \quad (6.122)$$

Using the definition of π_N in (6.81) and $(G_N)_{ss'}$ in (6.80), we get

$$\begin{aligned} \pi_N(s) (G_N)_{ss'} &= \left(\frac{1}{Z} \overline{\pi}_N(s) \frac{N_s}{\mathbb{E}[N_s]} \right) \left(\gamma_N(s, s') \frac{N_{s'}}{\mathbb{E}[N_{s'}]} \right) \\ &= \frac{1}{Z} \frac{N_s}{\mathbb{E}[N_s]} \frac{N_{s'}}{\mathbb{E}[N_{s'}]} [\overline{\pi}_N(s) \gamma_N(s, s')]. \end{aligned} \quad (6.123)$$

The same substitution for the right-hand side gives

$$\pi_N(s') (G_N)_{s's} = \frac{1}{Z} \frac{N_s}{\mathbb{E}[N_s]} \frac{N_{s'}}{\mathbb{E}[N_{s'}]} [\overline{\pi}_N(s') \gamma_N(s', s)], \quad (6.124)$$

so that both terms carry the same random prefactor. Since we set $s, s' \in \mathcal{T}_{\mathbf{S}}$, we have $N_s > 0$ and $N_{s'} > 0$, which means that (6.122) reduces to checking

$$\overline{\pi}_N(s') \gamma_N(s', s) = \overline{\pi}_N(s) \gamma_N(s, s'). \quad (6.125)$$

By Lemma 4.7, the annealed generator satisfies detailed balance. Thus, we know that this equality holds. For the case where $s \notin \mathcal{T}_{\mathbf{S}}$, then $(G_N)_{ks} = 0$ and $(G_N)_{ks} = 0$ for all $k \in S$. This also implies that $\pi_N(s) = 0$, so that detailed balance is trivially satisfied. $\square$

Proof of Lemma 6.10. Let $R = -(\mathcal{Q} - \overline{\mathcal{Q}}) - \Delta \overline{W}_N$. We begin with the full equation in (6.85)

$$\overline{G}_N \delta W = R \quad (6.126)$$

and our ansatz for W

$$W = \begin{pmatrix} W_B \\ W_T \end{pmatrix} = \begin{pmatrix} \overline{W}_B + \delta W_B \\ 0 \end{pmatrix}. \quad (6.127)$$

First note that (6.126) is trivially satisfied for any choice of δW_T in the tail, which means that we are allowed to set $W_T = 0$. We can then simplify

$$\Delta W = \begin{pmatrix} \Delta_{BB} & \Delta_{BT} \\ \Delta_{TB} & \Delta_{TT} \end{pmatrix} \begin{pmatrix} W_B \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta_{BB} W_B \\ \Delta_{TB} W_B \end{pmatrix}. \quad (6.128)$$

This means that the right hand side R of (6.126) is given in the bulk by

$$R_B = -(\mathcal{Q}_B - \overline{\mathcal{Q}}_B) - \Delta_{BB}(\overline{W}_B + \delta W_B), \quad (6.129)$$

while in the tail it takes the form

$$R_T = -(\mathcal{Q}_T - \overline{\mathcal{Q}}_T) - \Delta_{TB}(\overline{W}_B + \delta W_B) = \overline{\mathcal{Q}}_T - \Delta_{TB}(\overline{W}_B + \delta W_B). \quad (6.130)$$

Using the fact that in the tail $\Delta_{TB} = -(\overline{G}_N)_{TB}$, we get

$$R_T = \overline{\mathcal{Q}}_T + (\overline{G}_N)_{TB}(\overline{W}_B + \delta W_B) \quad (6.131)$$

Applying the pseudoinverse on the right-hand side of (6.126) then gives

$$\delta W_B = (\overline{G}_N^\dagger)_{BB} R_B + (\overline{G}_N^\dagger)_{BT} R_T. \quad (6.132)$$

Working out the first term using (6.129), we find

$$\begin{aligned} (\overline{G}_N^\dagger)_{BB} R_B &= -(\overline{G}_N^\dagger)_{BB} [(\mathcal{Q}_B - \overline{\mathcal{Q}}_B) + \Delta_{BB}(\overline{W}_B + \delta W_B)] \\ &= -(\overline{G}_N^\dagger)_{BB} [(\mathcal{Q}_B - \overline{\mathcal{Q}}_B) + \Delta_{BB} \overline{W}_B] - (\overline{G}_N^\dagger)_{BB} \Delta_{BB} \delta W_B \\ &= (\overline{G}_N^\dagger)_{BB} U_B - (\overline{G}_N^\dagger)_{BB} \Delta_{BB} \delta W_B, \end{aligned} \quad (6.133)$$

where we used the definition of U (6.93) to obtain the final equality. Similarly, using (6.131) we find

$$(\overline{G}_N^\dagger)_{BT} R_T = (\overline{G}_N^\dagger)_{BT} U_T + (\overline{G}_N^\dagger)_{BT} (\overline{G}_N)_{TB} \delta W_B. \quad (6.134)$$

Substituting (6.133) and (6.2) into (6.132), we conclude

$$\delta W_B = \left[-(\overline{G}_N^\dagger)_{BB} \Delta_{BB} + (\overline{G}_N^\dagger)_{BT} (\overline{G}_N)_{TB} \right] \delta W_B + (\overline{G}_N^\dagger)_{BB} U_B + (\overline{G}_N^\dagger)_{BT} U_T, \quad (6.135)$$

where we recognise the first term as our H_N in (6.92). $\square$

Proof of Lemma 6.11. From the definition of H_N in (6.92) it follows that

$$\|H_N\|_\infty \leq \|(\overline{G}_N^\dagger)_{BB} \Delta_{BB}\|_\infty + \|(\overline{G}_N^\dagger)_{BT} (\overline{G}_N)_{TB}\|_\infty. \quad (6.136)$$

Starting with the second term, we first write out the definition of the infinity norm

$$\|(\overline{G}_N^\dagger)_{BT} (\overline{G}_N)_{TB}\|_\infty = \sup_{s \in B} \sum_{s' \in B} \left| \sum_{t \in T} (\overline{G}_N^\dagger)_{st} (\overline{G}_N)_{ts'} \right|. \quad (6.137)$$

Next, we take the absolute value inside the second sum using the triangle inequality and switch the order of the sums

$$\|(\overline{G_N}^\dagger)_{BT}(\overline{G_N})_{TB}\|_\infty \leq \sup_{s \in B} \sum_{t \in T} |(\overline{G_N}^\dagger)_{st}| \sum_{s' \in B} |(\overline{G_N})_{ts'}|. \quad (6.138)$$

Using the fact that $\overline{G_N}$ is a generator, we know that $(\overline{G_N})_{ts'} > 0$ for $t \in T$ and $s' \in B$, so that

$$\sum_{s' \in B} |(\overline{G_N})_{ts'}| \leq \sum_{s' \neq t} (\overline{G_N})_{ts'} = -(\overline{G_N})_{tt} = O(1) \quad (6.139)$$

Furthermore, Theorem 6.1 tells us that

$$|(\overline{G_N}^\dagger)_{st}| = \left| \frac{2}{1 + e^{\beta\omega}} \overline{\pi_N}(t) [v(s) + v(t) - 1 + O(N^{-1})] \right| \leq \frac{6}{1 + e^{\beta\omega}} \overline{\pi_N}(t) \quad (6.140)$$

for all $s, t \in S$. Hence, (6.138) is upper bounded by

$$\|(\overline{G_N}^\dagger)_{BT}(\overline{G_N})_{TB}\|_\infty \leq \sum_{t \in T} -(\overline{G_N})_{tt} \frac{6}{1 + e^{\beta\omega}} \overline{\pi_N}(t) = O(1) \cdot \sum_{t \in T} \overline{\pi_N}(t). \quad (6.141)$$

Because $T = S \setminus \mathcal{T}_S$ is a random set, we upper bound

$$\sum_{t \in T} \overline{\pi_N}(t) \leq \sum_{t \notin I_N} \overline{\pi_N}(t). \quad (6.142)$$

and note that by 5.14 this sum is now contiguous with failure probability $O(N^{-k})$. Because the Gaussian is strictly decreasing on the tail regions, we can apply a standard Riemann sum bound to find

$$\sum_{t \notin I_N} \overline{\pi_N}(t) \leq \overline{\pi_N}(\sqrt{(1-\epsilon)N \log N}) + 4 \int_{\sqrt{(1-\epsilon)N \log N}}^\infty \frac{1}{\sqrt{2\pi(N-1)}} e^{-\frac{t^2}{2(N-1)}} dt \quad (6.143)$$

Using Mill's inequality A.11 we thus find

$$\sum_{t \notin I_N} \overline{\pi_N}(t) \leq \frac{1}{\sqrt{\log N}} N^{-(1-\epsilon)/2} = O(N^{-(1-\epsilon)/2}). \quad (6.144)$$

Substituting the result into (6.141), we conclude

$$\|(\overline{G_N}^\dagger)_{BT}(\overline{G_N})_{TB}\|_\infty = O(N^{-(1-\epsilon)/2}) \quad (6.145)$$

with failure probability $O(N^{-k})$ for all $k > 0$.

For the first term in (6.136), the definition of the infinity norm in combination with the triangle inequality gives

$$\begin{aligned} \|(\overline{G_N}^\dagger)_{BB} \Delta_{BB}\|_\infty &= \sup_{s \in B} \sum_{s' \in B} \left| \sum_{r \in B} (\overline{G_N}^\dagger)_{sr} \Delta_{rs'} \right| \\ &= \sup_{s \in B} \sum_{s' \in B} \left(\left| (\overline{G_N}^\dagger)_{ss'} \Delta_{s's'} \right| + \sum_{r \neq s'} \left| (\overline{G_N}^\dagger)_{sr} \Delta_{rs'} \right| \right). \end{aligned} \quad (6.146)$$

Evaluating $\Delta_{rs'}$ for $r, s' \in B = \mathcal{T}_S$ and $r \neq s'$ gives

$$\begin{aligned} \Delta_{rs'} &= \gamma_N(r, s') \left(\frac{N_{s'}}{\mathbb{E}[N_{s'}]} - 1 \right) \\ &= \frac{\gamma_N(r, s')}{\mathbb{E}[N_{s'}]} (N_{s'} - \mathbb{E}[N_{s'}]) \end{aligned} \quad (6.147)$$

for the off-diagonal terms of Δ_{BB} . We can now use Lemma C.6 for γ_N in the numerator and the LCLT for $\mathbb{E}[N_{s'}]$ in the denominator to obtain

$$\begin{aligned}\Delta_{rs'} &= \frac{1 + e^{-\beta\omega}}{2} \frac{\phi(s'; -\tanh(\frac{\beta\omega}{2}), N-1)}{v(r)N\phi(s'; 0, N-1)} (N_{s'} - \mathbb{E}[N_{s'}]) \\ &= \frac{1 + e^{-\beta\omega}}{2\sqrt{N-1}} e^{(r-s') \tanh \frac{\beta\omega}{2}/N} \left(\frac{N_{s'} - \mathbb{E}[N_{s'}]}{\sqrt{N}} \right),\end{aligned}\tag{6.148}$$

where we used that $v(r) = e^{-r \tanh \frac{\beta\omega}{2}/(N-1)}$ for the final equality. Using Theorem 5.1 and that $|e^{(r-s') \tanh \frac{\beta\omega}{2}}| \leq 1$ for all $r, s' \in B$, we thus find that

$$|\Delta_{rs'}| \leq \frac{1 + e^{-\beta\omega}}{2\sqrt{N}} \left| \frac{N_{s'} - \mathbb{E}[N_{s'}]}{\sqrt{N}} \right| = O(N^{-3/4} \sqrt{\log N})\tag{6.149}$$

for all $r, s' \in \mathcal{T}_{\mathbf{S}}$ and $r \neq s'$. Therefore, we get using the fact that $\mathcal{T}_{\mathbf{S}}$ has $O(\sqrt{N \log N})$ elements that the diagonal term

$$|\Delta_{rr}| \leq \sum_{s' \in B} |\Delta_{rs'}| = O(N^{-1/4} \log N),\tag{6.150}$$

with failure probability $O(N^{-k})$. The first sum in (6.146) thus has size

$$\sum_{s' \in B} \left| (\overline{G_N}^\dagger)_{ss'} \Delta_{s's'} \right| \leq \sup_{k \in B} |\Delta_{kk}| \sum_{s' \in B} (\overline{G_N}^\dagger)_{ss'} = O(N^{-1/4} \log N),\tag{6.151}$$

where we used Theorem 6.1 to bound the sum over $(\overline{G_N}^\dagger)_{ss'}$ as $O(1)$. Similarly, using Theorem 6.1 and (6.151), we can estimate the second sum

$$\begin{aligned}\sum_{s' \in B} \sum_{r \neq s'} \left| (\overline{G_N}^\dagger)_{sr} \Delta_{rs'} \right| &\leq \sup_{r \neq s'} |\Delta_{rs'}| \sum_{s' \in B} \sum_{r \neq s'} (\overline{G_N}^\dagger)_{sr} \\ &= \sup_{r \neq s'} |\Delta_{rs'}| |\mathcal{T}_{\mathbf{S}}| \sum_{r \neq s'} (\overline{G_N}^\dagger)_{sr}\end{aligned}\tag{6.152}$$

Because $|\mathcal{T}_{\mathbf{S}}| = O(\sqrt{N \log N})$, we have

$$\sum_{s' \in B} \sum_{r \neq s'} \left| (\overline{G_N}^\dagger)_{sr} \Delta_{rs'} \right| = O(N^{-1/4} \log N).\tag{6.153}$$

Therefore, (6.146) admits the bound

$$\|(\overline{G_N}^\dagger)_{BB} \Delta_{BB}\|_\infty = O(N^{-1/4} \log N).\tag{6.154}$$

Now, substituting our bounds of (6.145) and (6.154) into (6.136), we conclude

$$\|H_N\|_\infty = O(N^{-1/4} \log N),\tag{6.155}$$

with failure probability $O(N^{-k})$. Because the operator norm now satisfies $\|H_N\|_\infty < 1$ with high probability, the geometric series $\sum_{k=0}^\infty (-1)^k (H_N)^k$ converges absolutely in the operator norm, which by definition coincides with the inverse $(I + H_N)^{-1}$. $\square$

Proof of Lemma 6.12. We start from (6.91)

$$(I + H_N) \delta W_B = (\overline{G_N}^\dagger)_{BB} U_B + (\overline{G_N}^\dagger)_{BT} U_T.\tag{6.156}$$

Taking the inverse and using the definition of $\delta W_B^{(0)}$, we get

$$\delta W_B = (I + H_N)^{-1} \delta W_B^{(0)}. \quad (6.157)$$

We now use Lemma 6.11 to find

$$\delta W_B = \sum_{k=0}^{\infty} (-1)^k (H_N)^k \delta W_B^{(0)}, \quad (6.158)$$

which is what we wanted to show. $\square$

Proof of Lemma 6.13. Following Lemma 6.12 initial correction is given by

$$\delta W_B^{(0)} = (\overline{G_N}^\dagger)_{BB} U_B + (\overline{G_N}^\dagger)_{BT} U_T. \quad (6.159)$$

Evaluating this expression at a point $s \in \mathcal{T}_S$ gives

$$\delta W_B^{(0)}(s) = \sum_{s' \in B} (\overline{G_N}^\dagger)_{ss'} U_B(s') + \sum_{s' \in T} (\overline{G_N}^\dagger)_{ss'} U_T(s'). \quad (6.160)$$

The first term contains a diagonal, so Theorem 6.1 shows that

$$\sum_{s' \in B} (\overline{G_N}^\dagger)_{ss'} U_B(s') = \frac{2}{1 + e^{-\beta\omega}} \left[-v(s) U_B(s) + \sum_{s' \in B} \overline{\pi_N}(s') (v(s) + v(s') - 1) U_B(s') \right]. \quad (6.161)$$

The second term solely contains off-diagonal terms, which means

$$\sum_{s' \in T} (\overline{G_N}^\dagger)_{ss'} U_T(s') = \frac{2}{1 + e^{-\beta\omega}} \left[\sum_{s' \in T} \overline{\pi_N}(s') (v(s) + v(s') - 1) U_T(s') \right]. \quad (6.162)$$

Recognising that the two remaining sums combine into a full sum over S , substituting (6.161) and (6.162) into (6.160) results in

$$\delta W_B^{(0)}(s) = \frac{2}{1 + e^{-\beta\omega}} \left[-v(s) U_B(s) + \sum_{s' \in S} \overline{\pi_N}(s') (v(s) + v(s') - 1) U(s') \right]. \quad (6.163)$$

Expanding the sum leaves us with two averages to calculate

$$\sum_{s' \in S} \overline{\pi_N}(s') (v(s) + v(s') - 1) U(s') = (v(s) - 1) \langle U \rangle_{\overline{\pi_N}} + \mathcal{M}. \quad (6.164)$$

Substituting (6.164) into (6.163) then gives the result. $\square$

Proof of Lemma 6.14. By definition, the heat capacity $\tilde{C}_N(\beta)$ is given by

$$\tilde{C}_N(\beta) = \beta^2 \left\langle \frac{\partial W_N}{\partial \beta} \right\rangle_{\pi_N}. \quad (6.165)$$

Using the fact that $\langle W_N \rangle_{\pi_N} = 0$, we find that differentiating with respect to β gives

$$\sum_{s \in \mathcal{T}_S} \frac{\partial W_N}{\partial \beta}(s) \pi_N(s) + \sum_{s \in \mathcal{T}_S} W_N(s) \frac{\partial \pi_N}{\partial \beta}(s) = 0. \quad (6.166)$$

This means that we are also free to use

$$\tilde{C}_N(\beta) = -\beta^2 \left\langle W_N \frac{\partial \log \pi_N}{\partial \beta} \right\rangle_{\pi_N}, \quad (6.167)$$

which is helpful, since Lemma 6.9 gives π_N exactly. Substituting for its definition in (6.81), we find

$$\frac{\partial \log \pi_N}{\partial \beta}(s) = \frac{\partial \log \overline{\pi_N}}{\partial \beta}(s) - \frac{\partial \log Z}{\partial \beta}. \quad (6.168)$$

Because the second term is a constant with respect to s and W_N is zero-mean with respect to π_N , we get

$$\tilde{C}_N(\beta) = -\beta^2 \left\langle W_N \frac{\partial \log \overline{\pi_N}}{\partial \beta} \right\rangle_{\pi_N}. \quad (6.169)$$

From (4.45) it follows that

$$\frac{\partial \log \overline{\pi_N}}{\partial \beta} = -\frac{\partial}{\partial \beta} \frac{(s+2T)^2}{2(N-1)} = \frac{-(s+2T)2T'}{N-1}, \quad (6.170)$$

where T' denotes the derivative of $T = \tanh \frac{\beta\omega}{2}$ with respect to β , which is given by

$$T' = \frac{\partial}{\partial \beta} \tanh \frac{\beta\omega}{2} = \frac{\omega}{2} \operatorname{sech}^2 \frac{\beta\omega}{2}. \quad (6.171)$$

Substituting the result into (6.169) yields

$$\tilde{C}_N(\beta) = \frac{\beta^2 \omega}{N-1} \operatorname{sech}^2 \frac{\beta\omega}{2} \sum_{s \in \mathcal{J}_S} (s+2T) W_N(s) \overline{\pi_N}(s) \frac{N_s}{\mathbb{E}[N_s]}, \quad (6.172)$$

which is what we wanted to show $\square$

6.3 Heat capacity of L_N : proof strategy

The route towards proving Conjecture 2 relies on a separating two scales. First, we use a Mori-Zwanzig-like projection to separate the macroscopic score-level dynamics from the microscopic fluctuations within the score bins. This is informed by the observations in Figure 3.13. The projection reframes the problem into showing that the macroscopic component dominates. Second, we treat the projected macroscopic system as the perturbation of the auxiliary generator G_N and note that the deviations in the perturbation matrix are centred random variables.

6.3.1 First step: projection and decoupling

Recall the projection operator (3.45) from Section 3.3.2

$$Pf(i) = \frac{1}{N_{s_i}} \sum_{j \in B_{s_i}} f(j), \quad (6.173)$$

which averages a function over all vertices sharing the same score. We define the complementary operator $Q = I - P$, which captures fluctuations within the score bins. This allows us to decompose the quasipotential and the derivative of the stationary measure into macroscopic and microscopic parts

$$V_N = V_P + V_Q \quad \text{and} \quad \partial_\beta \rho_N = \partial_\beta \rho_P + \partial_\beta \rho_Q. \quad (6.174)$$

Because P and Q are orthogonal projections, the cross-terms in the heat capacity $C_N(\beta) = \beta^2 \langle V_N, \partial_\beta \rho_N \rangle$ cancel. Defining the projected heat capacity as $C_P(\beta) = \beta^2 \langle V_P, \partial_\beta \rho_P \rangle$, the difference between the full and projected heat capacity only contains terms from the Q -space.

$$C_N(\beta) - C_P(\beta) = -\beta^2 \langle V_Q, \partial_\beta \rho_Q \rangle. \quad (6.175)$$

To bound the difference, we thus project the Poisson equation and the stationarity equation onto the Q -space. Given that the operator $L_{QQ} = QL_NQ$ is invertible, we can express the microscopic variables V_Q and $\partial_\beta \rho_Q$ entirely in terms of the macroscopic variables and cross space transitions such as L_{PQ} . Applying these substitutions and taking norms via Hölder's inequality gives the bound

$$|C_N(\beta) - C_P(\beta)| \leq \beta^2 \left(\|L_{PQ}(L_{QQ}^{-1})^2 L_{QP}\|_\infty + \|(\partial_\beta L_{PQ})(L_{QQ}^{-1})^2 L_{QP}\|_\infty \right) \|V_P\|_\infty \|\partial_\beta \rho_P\|_1. \quad (6.176)$$

Anticipating that the projected variables concentrate around their G_N counterparts, so that $V_P \approx W_N$ and $\partial_\beta \rho_P \approx \partial_\beta \pi_N$, the norms will be $\|V_P\|_\infty = O(N^{-1/2})$ and $\|\partial_\beta \rho_P\|_1 = O(N^{-1/2})$. The proof of the first step then reduces to showing that the norms in the brackets are of order $O(N^{-1})$.

6.3.2 Second step: perturbative connection to G_N

To show that V_P and $\partial_\beta \rho_P$ indeed concentrate and that the projected heat capacity is close to $\tilde{C}_N(\beta)$, we regard the projected generator L_{PP} as a perturbation to the auxiliary generator G_N

$$L_{PP} = G_N + E_N. \quad (6.177)$$

By writing $V_P = W_N + \delta V$ and $\partial_\beta \rho_P = \partial_\beta \pi_N + \delta(\partial_\beta \rho)$, the equations for the corrections can be derived. Substituting the perturbed generator into the P projected Poisson equation and neglecting higher-order interactions with the Q -space, we can apply the pseudoinverse $G_N^\dagger$ to find for the quasipotential

$$(I + G_N^\dagger E_N) \delta V = -G_N^\dagger E_N W_N \quad (6.178)$$

and for the stationary measure

$$\delta(\partial_\beta \rho)(I + E_N G_N^\dagger) = -\left[(\partial_\beta \pi_N) E_N + \pi_N(\partial_\beta E_N) + \delta\rho(\partial_\beta G_N) \right] G_N^\dagger. \quad (6.179)$$

The leading order terms can be calculated using a series expansion as in Lemma 6.11 after controlling the norm of E_N . The perturbation matrix E_N itself captures the empirical deviations from the expected transition rates

$$(E_N)_{ss'} = \left(\frac{1}{N_s} \sum_{i \in B_s} \lambda_N(i, s') \right) - \gamma_N(s, s') \frac{N_{s'}}{\mathbb{E}[N_{s'}]}. \quad (6.180)$$

The empirical average over λ_N can be written out as

$$\left(\frac{1}{N_s} \sum_{i \in B_s} \lambda_N(i, s') \right) = \frac{N_{s'}}{N} e^{-\beta\omega} + \frac{e(B_s, B_{s'})}{N_s N} (1 - e^{-\beta\omega}), \quad (6.181)$$

where $e(B_s, B_{s'})$ denotes the number of edges directed from B_s to $B_{s'}$. Conditioned on the score vector of the tournament $\mathbf{S}$, the out-degree of every vertex is fixed. As a result, the total number of out-edges from score bin B_s is also fixed. If we regard the in-edges of possible half-edges to match the out-edges with, this turns into finite population sampling problem. A short calculation also shows that

$$\mathbb{E} \left[\frac{N_{s'}}{N} e^{-\beta\omega} + \frac{e(B_s, B_{s'})}{N_s N} (1 - e^{-\beta\omega}) \mid \mathbf{S} \right] = \gamma_N(s, s') \frac{N_{s'}}{\mathbb{E}[N_{s'}]}. \quad (6.182)$$

Any sum over the matrix E_N is thus a sum over centred random variables from a finite population sampling. This transform the task of bounding the perturbation expansions into a concentration of measure problem. Proving the scale of these centred fluctuations scale appropriately then completes the proof.

7. Conclusion

Summary

In this thesis, we have investigated the nonequilibrium heat capacity $C_N(\beta)$ of randomly directed tournament graphs, defined in Section 3.1. We defined an irreversible continuous-time Markov chain on this graph in Section 3.2 with transition rates $k(i, j) = 1/(N-1)$ for $i \rightarrow j$ and $k(i, j) = e^{-\beta\omega}/(N-1)$ for $i \leftarrow j$, creating a system with a nonzero nonequilibrium heat capacity while lacking a traditional energy landscape.

Our primary objective was to motivate and explain the scaling behaviour of the heat capacity for large system sizes N . Numerical simulations of the quasipotential and stationary distribution in Section 3.3 revealed a self-averaging property: while single realisations of a tournament fluctuate, the ensemble average $C_{N,\text{avg}}(\beta)$ remains stable and matches the heat capacity of a two-level system, $4C_2(\beta)$. These observations led to the formulation of our main conjecture

$$N \cdot C_N(\beta) = 4C_2(\beta) (1 + N^{-1/2} \cdot \xi_C)$$

where ξ_C converges in distribution to a Gaussian random variable $\mathcal{N}(0, \tau_C^2)$. Furthermore, we conjectured that the fluctuations ξ_C were primarily caused by the random score bin sizes.

Because the system lacks an energy landscape, the heat capacity is derived from the quasipotential V_N , which is a structural property of the graph. Section 2.6 explored how V_N quantifies the expected excess heat exchanged as the process moves along the network. To find the heat capacity, we must solve the equations

$$L_N V_N = -Q \quad \text{and} \quad \rho_N L_N = 0,$$

where L_N is the random generator that contains the transition rates and Q_N the expected excess power (2.70) dissipated at a particular vertex. To perturbatively solve the random generator equations and provide a basis for the conjecture, we gave three main contributions.

The annealed baseline We constructed an annealed generator $\overline{G_N}$ in Chapter 4 that operates on the entire score range $\mathcal{R}$ (3.5) rather than individual vertices. By solving the associated Poisson problem, we proved that this idealised coarse-grained system yields a heat capacity of exactly $4C_2(\beta)/(N-1)$. This further solidified the intuition that the average behaviour of the tournament is captured by a process where score bins reach their expected sizes.

Statistical properties of tournaments Using the $\mathcal{D}$ -model to handle the negative correlations between vertex scores, we derived a uniform concentration result and a local Central Limit Theorem for bin sizes in Chapter 5. We further established that linear combinations of these bins converge to a Normal random variable, providing the necessary tools to analyse functionals of the random graph. We were also able to prove that the random coarse grained rates $\Lambda_N(i, s')$ (4.17) in L_N have a CLT property around the rates $\gamma_N(s, s')$ of the annealed generator $\overline{G_N}$.

Perturbative route We combined results of Chapters 4 and 5 into a perturbative scheme in Chapter 6 to solve the random generator equation $L_N V_N = -Q_N$. By finding the pseudoinverse of the annealed generator $\overline{G_N}$ and introducing an auxiliary generator G_N , we showed that adding random bin-size fluctuations to the annealed baseline exactly reproduces the conjectured Gaussian fluctuations.

While we have provided strong numerical evidence and a viable analytical route showing that the microscopic generator L_N and the auxiliary generator G_N share the same heat capacity, a formal proof of their equivalence remains a conjecture for future work.

Further research

The framework developed in this thesis opens several options for future work, ranging from direct extensions to broader physical generalisations.

Proving Conjecture 2 and other graphs (Mathematics) The most immediate next step is to complete the proof of Conjecture 2 outlined in Section 6.3, formally proving that the heat capacity $C_N(\beta)$ converges to the auxiliary $\tilde{C}_N(\beta)$. Moving beyond complete graphs, the perturbative scheme could be extended to directed random networks, such as the oriented Erdős-Rényi graph $G(N, p)$. We expect the self-averaging property to survive for these graphs in the connected regime where $p > (\log N)/N$. Based on our conditional expectation approach, the heat capacity should pick up a $1/p$ scaling factor.

Conjecture 3. *The heat capacity of the Erdős-Rényi graph $G(N, p)$ in the regime where $p > (\log N)/N$ converges to the limit*

$$\lim_{N \rightarrow \infty} N \cdot C_N(\beta) = \frac{4}{p} C_2(\beta). \quad (7.1)$$

The broader goal would be to prove that the annealed limit holds for any graph sequence $\mathbb{G}(N, W)$ converging to a (scaled) graphon W , which generalises the result to an entire class of quasirandom and inhomogeneous random networks [102, 103]. Because the $\mathcal{D}$ -model change of measure is unique to tournament graphs, proving this requires a new statistical approach. Because the scores will no longer be globally coupled, which was the main bottleneck on the tournament, Stein’s method might prove to be useful.

On the other hand, the above conjecture implies that a non-vanishing macroscopic heat capacity requires looking at the subcritical regime of the Erdős-Rényi graph where $p < 1/N$. In this regime, the concentration of the score bins fails, the annealed approximation $\overline{G}_N$ breaks down, and the system enters a percolation-like state. This model would therefore be a first step toward developing a method that survives the sparse limit.

Finally, expander graphs could provide a theoretical middle ground. These graphs are strictly sparse while remaining strongly connected [104]. We conjecture that despite their sparsity, expander graphs will still exhibit the self-averaging annealing property due to their connectivity.

Atypical networks and large deviations (Mathematics) The results in this thesis rely on the typicality of the random tournament, where the score sequence tightly concentrates around a Gaussian profile. It is natural to ask how the heat capacity behaves on atypical graphs, such as perfectly transitive or perfectly regular tournaments. Formulating a Large Deviation Principle for the nonequilibrium heat capacity would allow us to condition the ensemble on observing a macroscopic deviation from the expected $4C_2(\beta)$. Rather than giving the typical heat capacity, it would reveal how or in what way the tournament has to be untypical to drive an extreme thermal response. This requires deriving a rate function for the empirical score distribution to quantify the probability of these rare graph configurations [86].

Dynamic networks and timescale separation (Physics) In many physical systems, such as active matter, the underlying graph changes alongside the state of the system [105]. This can be modelled by introducing a symmetric edge-flipping rate γ , creating a competition between the relaxation time of the random walk and the dynamic timescale of the network. In the limit $\gamma \rightarrow 0$, the system changes slow enough for the process over the vertices to reach the stationary state before the geometry changes. The solution in between each edge flip then corresponds to the solution for the quenched process L_N studied in the thesis. Conversely, the limit $\gamma \rightarrow \infty$ describes the situation in which the geometry has been completely randomised before each new step. We expect that this will recover the annealed generator $\overline{G}_N$.

Investigating the intermediate regime where $\gamma = O(N^{-1})$, comparable to the jump rates of the process, could reveal whether the heat capacity exhibits any fluctuations or resonances when the network changes at the same speed as the particle explores it. In particular, this approach could possibly add a phase transition to the system. Analytically, this approach requires defining a joint Markov process on the product space of the vertices and the graph configurations with combined generator $L_N + \gamma L_{\text{graph}}$. Similar work in the field of contact processes could be a helpful starting point [106, 107].

Exclusion process (Physics) Another interesting extension to the tournament model would be to move from a single particle to an exclusion process with several particles. This is a process where particles move according to the underlying Markov chain, but with the added constraint that no two particles can occupy the same site at the same time [108]. On a network, these can be used to model the transport of active particles such as kinesin over a network of protein filaments, for example [109]. The tournament graph could serve as a test-case here as well. The first step would then be to numerically test to see how the heat capacity reacts to an increasing number of particles and in which regimes the self-averaging property survives. The major mathematical challenges would be to handle the size of the state space exploding from N to $\binom{N}{k}$ and the non-linear nature of the exclusion rule, which breaks the exact coarse-graining we used to derive $\overline{G_N}$.

A. Useful Theorems

Concentration inequalities

Throughout the thesis, we want to bound sums of random variables. The following concentration inequalities are used to that end. The first inequality only assumes independence and boundedness of the individual variables.

Theorem A.1 (Hoeffding's Inequality, Theorem 2.2.6 in [110]). *Let $X_1, \dots, X_N$ be independent random variables such that $X_i \in [a_i, b_i]$ for every i . Then, for any $t > 0$, we have*

$$\mathbb{P} \left(\sum_{i=1}^N (X_i - \mathbb{E}[X_i]) \geq t \right) \leq \exp \left(-\frac{2t^2}{\sum_{i=1}^N (b_i - a_i)^2} \right) \quad (\text{A.1})$$

The second inequality provides a tighter bound by using additional information about the variance of the variables. This is particularly useful for variables with parameters that vary in N , so that the variance vanishes as $N \rightarrow \infty$. In that situation, the Hoeffding inequality would stay stuck treating the variable as if it still had the potential for maximum fluctuation.

Theorem A.2 (Bernstein Inequality, Theorem 2.9.5 in [110]). *Let $X_1, \dots, X_N$ be independent, mean-zero random variables satisfying $|X_i| \leq K$ for all i . Then for every $t \geq 0$, we have*

$$\mathbb{P} \left(\left| \sum_{i=1}^N X_i \right| \geq t \right) \leq 2 \exp \left(-\frac{t^2/2}{\sigma^2 + Kt/3} \right) \quad (\text{A.2})$$

where $\sigma^2 = \sum_{i=1}^N \mathbb{E}[X_i^2]$.

The final inequality is a generalisation of Hoeffding and useful for bounding general functions of independent random variables. The only condition is that the function is bounded when changing a single coordinate. That is, for all i and all possible coordinates

$$|f(x_1, \dots, x_i, \dots, x_N) - f(x_1, \dots, x'_i, \dots, x_N)| \leq c_i. \quad (\text{A.3})$$

If this condition holds, the following inequality is valid.

Theorem A.3 (McDiarmids Inequality, Theorem 5.7.1 in [110]). *Let $X = (X_1, \dots, X_N)$ be a random vector with independent entries. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function. Assume that the value of $f(x)$ can change by at most $c_i > 0$ under an arbitrary change of a single coordinate of $x \in \mathbb{R}^n$. Then for any $t > 0$, we have*

$$\mathbb{P}(f(X) - \mathbb{E}[f(X)] \geq t) \leq \exp \left(-\frac{2t^2}{\sum_{i=1}^N c_i^2} \right) \quad (\text{A.4})$$

Local limit theorems

The following two theorems allow us to approximate sums of centred independent random variables with the density of a Normal random variable. This estimation is extremely useful for dealing with sums or multiplications of point probabilities at large N . The first LCLT deals with sums of independent Rademacher random variables, such as our scores S_i . Because these variables have zero skewness, they allow for a multiplicative bound up to moderate deviations.

Theorem A.4 (De Moivre-Laplace Local CLT, Theorem 2 in [111]). *Let $X_1, X_2, \dots, X_N$ be i.i.d. Rademacher random variables such that $\mathbb{P}(X_i = +1) = \mathbb{P}(X_i = -1) = 1/2$. Let $S_N = \sum_{i=1}^N X_i$. For $N \geq 3$ and $|k| \leq N^{2/3}$ one has*

$$1 - 3N^{-1/3} \leq \frac{\mathbb{P}(S_N = k)}{\frac{1}{\sqrt{2\pi N}} e^{-k^2/(2N)}} \leq 1 + 6N^{-1/3} \quad (\text{A.5})$$

The Local CLT holds for any collection of i.i.d. random lattice variables with finite variance. However, generality is exchanged for a weaker condition on the tails. The additive $o(N^{-1/2})$ error term becomes uninformative in the tails where the true probabilities drop below this scale.

Theorem A.5 (Local CLT, Theorem 3.5.3 in [112]). *Let $X_1, \dots, X_N$ be i.i.d., mean-zero random variable satisfying $\mathbb{E}[X_i^2] = \sigma^2 \in (0, \infty)$ with common lattice distribution of span h . If $S_N = \sum_{i=1}^N X_i$ and $\mathbb{P}(X_i \in b + h\mathbb{Z}) = 1$ then $\mathbb{P}(S_N \in b + h\mathbb{Z})$. As $N \rightarrow \infty$*

$$\sup_{x \in \mathcal{L}_N} \left| \frac{N^{1/2}}{h} \mathbb{P}(S_N/\sqrt{N} = x) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| \rightarrow 0, \quad (\text{A.6})$$

where $\mathcal{L}_N = \{(nb + hz)/\sqrt{N} : z \in \mathbb{Z}\}$.

Approximating sums with integrals

Because we are working with Markov chains on a large finite state space, all of the relevant moments against the stationary measure or applications of the backward generator are sums. The following theorem allows us to approximate these sums with an integral and gives the error in terms of the Fourier transform of the summand.

Theorem A.6 (Theorem 3.1.17 in [113]). *Let $\hat{f}$ denote the Fourier transform of a function f . Suppose that $f, \hat{f} \in L^1(\mathbb{R}^n)$ satisfy*

$$|f(x)| + |\hat{f}(x)| \leq C(1 + |x|)^{-n-\delta} \quad (\text{A.7})$$

for some $C, \delta > 0$. Then f and $\hat{f}$ are both continuous, and for all $x \in \mathbb{R}^n$ we have

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x} = \sum_{m \in \mathbb{Z}^n} f(x + m). \quad (\text{A.8})$$

In particular $\sum_{m \in \mathbb{Z}^n} \hat{f}(m) = \sum_{m \in \mathbb{Z}^n} f(m)$, so that

$$\sum_{m \in \mathbb{Z}^n} f(m) - \int_{-\infty}^{\infty} f(x) dx = \sum_{m \neq 0} \hat{f}(m). \quad (\text{A.9})$$

Because the Fourier transform of a Gaussian is again a Gaussian, the next Lemma follows from a direct application of the previous lemma.

Lemma A.7. *Let $\phi(x) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{x^2}{2N}}$ be the normal probability density function with mean 0 and variance $N > 0$. For any integers $a, b > 0$, the sum of $f(n)$ over the discrete interval $[-a, b]$ is given by*

$$\sum_{m=-a}^b \phi(m) = 1 + O(e^{-2\pi^2 N}) - O\left(\frac{\sqrt{N}}{a} e^{-\frac{a^2}{2N}}\right) - O\left(\frac{\sqrt{N}}{b} e^{-\frac{b^2}{2N}}\right). \quad (\text{A.10})$$

Before proving this Lemma, we first state a useful result that bounds the tails of a Gaussian integral.

Lemma A.8 (Mill's inequality, Proposition 2.1.2 in [110]). *Let $Z \sim \mathcal{N}(0, 1)$ be a standard normal random variable. Then for all $z > 0$, we have*

$$\frac{z}{z^2 + 1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq \mathbb{P}(Z \geq z) \leq \frac{1}{z} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}. \quad (\text{A.11})$$

We now proceed with proving Lemma A.7.

Proof of Lemma A.7. The first step is to split the sum into three parts

$$\sum_{m=-a}^b \phi(m) = \sum_{m=-\infty}^{\infty} \phi(m) - \sum_{m=-\infty}^{-a-1} \phi(m) - \sum_{m=b+1}^{\infty} \phi(m). \quad (\text{A.12})$$

We use (A.9) on the first term to find

$$\sum_{m=-\infty}^{\infty} \phi(m) = \sum_{k=-\infty}^{\infty} e^{-2\pi^2 N k^2} = 1 + 2 \sum_{k=1}^{\infty} e^{-2\pi^2 N k^2} = 1 + O(e^{-2\pi^2 N}). \quad (\text{A.13})$$

For the second term in (A.12) we use that $\phi(x)$ is a decreasing function for any $x > 0$, so that it is bounded by the integral

$$\sum_{n=b+1}^{\infty} \phi(n) \leq \int_b^{\infty} \frac{1}{\sqrt{2\pi N}} e^{-\frac{x^2}{2N}} dx. \quad (\text{A.14})$$

We can now use Mill's inequality (A.11) to find

$$\sum_{m=-\infty}^{\infty} \phi(m) \leq \frac{\sqrt{N}}{b\sqrt{2\pi}} e^{-\frac{b^2}{2N}} = O\left(\frac{\sqrt{N}}{b} e^{-\frac{b^2}{2N}}\right), \quad (\text{A.15})$$

which by symmetry of $\phi(x)$ means that the third term in (A.12) is bounded by

$$\sum_{n=-\infty}^{-a-1} \phi(n) \leq \frac{\sqrt{N}}{a\sqrt{2\pi}} e^{-\frac{a^2}{2N}} = O\left(\frac{\sqrt{N}}{a} e^{-\frac{a^2}{2N}}\right). \quad (\text{A.16})$$

Therefore, substituting (A.15) and (A.16) into (A.12) then yields

$$\sum_{m=-a}^b \phi(m) = 1 + O(e^{-2\pi^2 N}) - O\left(\frac{\sqrt{N}}{a} e^{-\frac{a^2}{2N}}\right) - O\left(\frac{\sqrt{N}}{b} e^{-\frac{b^2}{2N}}\right), \quad (\text{A.17})$$

which finishes the proof. $\square$

CLTs in specific settings

In Section 5, we are interested in proving that certain random variables on the random tournament graph have a CLT property. The first theorem we cite gives the conditions under which the conditional distribution of a sum of random variables converges to a normal distribution. The conditioning is on the fixed sum of another sequence of random variables. The result ensures that Gaussianity is preserved under conditioning, even as the probability of the conditioning event tends to zero. This is exactly the case in the $\mathcal{D}$ -model (Definition 5.5).

Theorem A.9 (Theorem 2.1 in [114]). *(a) Suppose that, for each n , (X_n, Y_n) is a pair of random variables such that X_n is integer valued, and that N_n and m_n are integers. Suppose further that for some γ and c (independent of n), with $0 < \gamma \leq 2$ and $c > 0$, the following hold, where all limits are taken as $n \rightarrow \infty$:*

- (i) $\mathbb{E}[X_n] = m_n/N_n$;
- (ii) $0 < \sigma_{X_n}^2 < \infty$;
- (iii) $\mathbb{E}|X_n - EX_n|^3 = o(N_n^{1/2}\sigma_{X_n}^3)$;
- (iv) $\sigma_{X_n}^2 = O(N_n^{2/\gamma-1})$;
- (v) $\varphi_{X_n}(s) := Ee^{isX_n}$ satisfies $1 - |\varphi_{X_n}(s)| \geq c \min\{|s|^\gamma, s^2\sigma_{X_n}^2\}$ for $|s| \leq \pi$;
- (vi) $0 < \sigma_{Y_n}^2 < \infty$;
- (vii) $\mathbb{E}|Y_n - EY_n|^3 = o(N_n^{1/2}\sigma_{Y_n}^3)$;
- (viii) The correlation $\rho_n := \text{cov}(X_n, Y_n)/(\sigma_{X_n}\sigma_{Y_n})$ satisfies $\limsup_{n \rightarrow \infty} |\rho_n| < 1$.

Let (X_{ni}, Y_{ni}) , $i = 1, 2, \dots$, be i.i.d. copies of (X_n, Y_n) and let $S_{nN} := \sum_1^N X_{ni}$, $T_{nN} := \sum_1^N Y_{ni}$, and $\tau_n^2 := \sigma_{Y_n}^2(1 - \rho_n^2) = \sigma_{Y_n}^2 - \frac{\text{cov}(X_n, Y_n)^2}{\sigma_{X_n}^2}$. Then, as $n \rightarrow \infty$, the conditional distribution of $(T_{nN} - N_n \mathbb{E}[Y_n])/(N_n^{1/2}\tau_n)$ given $S_{nN} = m_n$ converges to a standard normal distribution. In other words, if U_n is a random variable whose distribution equals the conditional distribution of T_{nN} given $S_{nN} = m_n$, then

$$\frac{U_n - N_n \mathbb{E}[Y_n]}{N_n^{1/2}\tau_n} \rightarrow N(0, 1). \quad (\text{A.18})$$

Moreover, $EU_n = N_n \mathbb{E}[Y_n] + o(N_n^{1/2}\tau_n)$ and $\text{var } U_n \sim N_n \tau_n^2$, and thus also

$$\frac{U_n - \mathbb{E}[U_n]}{(\text{var } U_n)^{1/2}} \rightarrow N(0, 1). \quad (\text{A.19})$$

(b) Assume furthermore that for each even integer $r \geq 4$:

- (ix) $\mathbb{E}(X_n - \mathbb{E}[X_n])^r = O(N_n^{r/2-1}\sigma_{X_n}^r)$;
- (x) $\mathbb{E}(Y_n - \mathbb{E}[Y_n])^r = O(N_n^{r/2-1}\sigma_{Y_n}^r)$.

Then the limits (A.19) and (A.18) hold with convergence of all moments too.

The second theorem is a Finite Population Central Limit Theorem, which provides the conditions for the asymptotic normality of a sum of elements sampled without replacement from a fixed population.

Theorem A.10 (Finite Population CLT, Theorem 3.1 in [115]). *For each N let $c_{N1}, \dots, c_{NN} \in \mathbb{R}$ be a deterministic finite population and let $n_N \in \{1, \dots, N-1\}$ be a sample size. Consider all subsets $\{i_1, \dots, i_{n_N}\} \subset \{1, \dots, N\}$ of size n_N drawn without replacement. Define*

$$p_N = \frac{n_N}{N}, \quad \bar{c}_N = \sum_{i=1}^N \frac{c_{Ni}}{N} \quad \text{and} \quad L_N = \sum_{k=1}^{n_N} c_{Ni_k} \quad (\text{A.20})$$

and put

$$D_N^2 = \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2, \quad s_N^2 = p_N(1 - p_N) D_N^2. \quad (\text{A.21})$$

Assume that for every $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \frac{1}{D_N^2} \sum_{|c_{Ni}| > \varepsilon s_N} c_{Ni}^2 = 0. \quad (\text{A.22})$$

Then as $N \rightarrow \infty$

$$\frac{L_N - \bar{c}_N}{s_N} \xrightarrow{D} \xi, \quad (\text{A.23})$$

where $\xi \sim \mathcal{N}(0, 1)$.

B. Calculating tournament heat capacities

We represent each random tournament by its signed adjacency matrix $A = -A^T$, where

$$A_{ii} = 0 \quad \text{and} \quad A_{ij} = \Sigma_{ij} \quad \forall i \neq j, \quad (\text{B.1})$$

The antisymmetry of A represents the directionality in the graph and sampling a tournament of size N therefore amounts to drawing $N(N-1)/2$ random signs ± 1 . The result is a dense matrix, which unfortunately limits the system sizes that can be treated due to memory issues.

The generator of the process is given by the matrix

$$(L_N)_{ii} = -\sum_{j=1}^N k(i, j) \quad \text{and} \quad (L_N)_{ij} = k(i, j) \quad \forall i \neq j \quad (\text{B.2})$$

with $k(i, j)$ as given in (3.35). Because A contains all Σ_{ij} , we see from the form of $k(i, j)$ in Lemma 3.9 that it can be used to obtain L_N . In particular,

$$L_N = \frac{1 + e^{-\beta\omega}}{2N}(J - NI) + \frac{1 - e^{-\beta\omega}}{2N}A, \quad (\text{B.3})$$

where I the identity matrix and J the matrix of all ones. We can then use the generator to compute the stationary distribution, which satisfies

$$\rho_N^T L_N = 0. \quad (\text{B.4})$$

Directly solving the system is possible, but scales poorly with N . Instead we opt for a power method.

Because the generator is not a direct transition matrix as in discrete time, we cannot simply iterate the generator to obtain the stationary distribution. However, the continuous-time process contains an embedded discrete-time process. We apply the power method to this process to find the corresponding continuous time stationary measure.

Remark B.1 (Power method for CTMCs). Given an irreducible and recurrent CTMC with generator L and stationary measure ρ , the embedded process that is obtained by conditioning on the random transition times has transition probabilities

$$P_{ij} = -\frac{L_{ij}}{L_{ii}} \quad \text{and} \quad P_{ii} = 0 \quad (\text{B.5})$$

The unique stationary distribution ν of P now satisfies $\nu^T P = \nu^T$, which can be computed using standard power iteration.

To relate ν back to ρ , note that (B.4) can be written as

$$\rho_i L_{ii} = \sum_{j \neq i} \rho_j L_{ji}. \quad (\text{B.6})$$

Then substituting for

$$\rho_i \propto \frac{\nu_i}{L_{ii}} \quad (\text{B.7})$$

simply returns $\nu^T P = \nu^T$. Therefore, once we know ν , we find the stationary measure ρ from

$$\rho_i = \frac{\nu_i / L_{ii}}{\sum_{j \neq i} \nu_j / L_{jj}}. \quad (\text{B.8})$$

◦

Once we have found the stationary distribution, we solve the Poisson equation

$$L_N V_N = \langle \mathcal{P} \rangle - \mathcal{P}, \quad (\text{B.9})$$

to obtain the quasipotential. From (B.4) it is clear that the generator L_N has a zero eigenvalue due to the stationary distribution. The Poisson equation must therefore be solved on the subspace orthogonal to ρ_N where $\langle V_N \rangle = 0$. We can either invert L_N on the subspace by projecting out the zero mode. However, this forces us to calculate the eigenspace of L_N . We instead fix one reference value $V_N(1) = 0$ and solve the resulting non-singular system. This solution is then centered by setting $V_N \leftarrow V_N - \langle V_N \rangle$. This is motivated by the fact that $\langle V_N \rangle$ is a constant, such that $L_N \langle V_N \rangle = 0$.

The last piece we need to evaluate the heat capacity is the derivative $\frac{d}{d\beta} V_N$. We could implicitly differentiate (B.9) with respect to β to derive a separate Poisson equation for $\frac{d}{d\beta} V_N$ that is in principle exactly solvable. However, as this would require a second linear system to be solved for each β , we approximate the derivative using a centered finite-difference scheme. This is accurate to second order in step size and computationally more efficient.

Remark B.2 (Calculating the heat capacity of CTMCs on directed graphs). The following steps provide a complete algorithm for calculating the heat capacity of random graphs:

1. Draw a random antisymmetric adjacency matrix $A = -A^T$ with entries $A_{ij} = \Sigma_{ij}$.
2. Calculate the generator matrix of the continuous time Markov process on the graph

$$L_{ii} = -\sum_{j=1}^N k(i, j) \quad \text{and} \quad L_{ij} = k(i, j) \quad \forall i \neq j \quad (\text{B.10})$$

3. Compute the stationary distribution ρ using the power method applied to the embedded chain $P_{ij} = L_{ij}/L_{ii}$.
4. Evaluate the instantaneous power $\mathcal{P}(i)$ for all $i \in \mathcal{V}$ and calculate its mean with respect to the stationary measure ρ .
5. Solve the Poisson equation $LV = \langle \mathcal{P} \rangle - \mathcal{P}$ on the subspace $\langle V \rangle = 0$. An efficient way is by fixing $V(1) = 0$ and recentering.
6. Calculate $\frac{d}{d\beta} V$ using any preferred method, such as a finite difference scheme.
7. Take the stationary average of $\frac{d}{d\beta} V$ to find

$$C(\beta) = \left\langle \beta^2 \frac{d}{d\beta} V \right\rangle. \quad (\text{B.11})$$

◦

C. Technical Lemmas: Chapter 1

When calculating probabilities of the form $\mathbb{P}(S_i = s \mid \Sigma_{ij} = +1)$ in the first chapter, we are moving between Binomial distributions at different N . The following lemma gives two basic properties of the binomial coefficient that are useful to relate these distributions.

Lemma C.1. *The binomial coefficient is given by*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (\text{C.1})$$

for any $k, n \in \mathbb{N}$ such that $0 \leq k \leq n$. It holds that

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} = \frac{n}{n-k} \binom{n-1}{k} \quad (\text{C.2})$$

Proof. Using the definition, we see that

$$\binom{n}{k} = \frac{n}{k} \frac{(n-1)!}{(k-1)!(n-k)!} \quad (\text{C.3})$$

and

$$\binom{n}{k} = \frac{n}{n-k} \frac{(n-1)!}{k!(n-1-k)!}. \quad (\text{C.4})$$

□

A direct application of this lemma yields a compact expression for the conditional expectation of $\Lambda_N(i, s')$.

Lemma C.2. *For a vertex $i \in B_s$ it holds that*

$$\mathbb{E}[\lambda_{N,\pm}(i, s') \mid i \in B_s] = \frac{1}{2} \left(1 \pm \frac{s-s'}{N-1} - \frac{ss'}{(N-1)^2} \right) \mathbb{P}(S_v = s). \quad (\text{C.5})$$

Proof. Using the definition in Lemma 4.3 we can write

$$\lambda_{N,\pm}(i, s') = \frac{1}{N-1} \sum_{j=1}^N \mathbb{1}_{\{S_v = s', \Sigma_{uv} = \pm 1\}}. \quad (\text{C.6})$$

For brevity of notation, we set $D_{s',+} := \{i, j \in \mathcal{V} : S_v = s', \sigma(i, j) = +1\}$. By linearity of expectation, it follows that

$$\begin{aligned} \mathbb{E}[\lambda_{N,\pm}(i, s') \mid i \in B_s] &= \frac{1}{N-1} \mathbb{E} \left[\sum_{j=1}^N \mathbb{1}_{D_{s',+}} \mid i \in B_s \right] \\ &= \mathbb{P}(D_{s',+} \mid i \in B_s). \end{aligned} \quad (\text{C.7})$$

We write out the conditional probability using the definition to see

$$\begin{aligned} \mathbb{P}(D_{s',\pm} \mid i \in B_s) &= \frac{\mathbb{P}(S_u = s, S_v = s', \Sigma_{uv} = \pm 1)}{\mathbb{P}(S_u = s)} \\ &= \frac{1}{2} \frac{\mathbb{P}(S_u = s, S_w = s' \mid \Sigma_{uv} = \pm 1)}{\mathbb{P}(S_u = s)}, \end{aligned} \quad (\text{C.8})$$

where we used $\mathbb{P}(\Sigma_i j = \pm 1) = \frac{1}{2}$. We fix $\Sigma_{uv} = 1$ and prove the claim for $\lambda_{N,+}$. The proof for the opposite case is identical. Using Propositions 3.3 and 3.5 on the probabilities, we find

$$\mathbb{P}(D_{s',+} \mid i \in B_s) = \left(\frac{N-1}{\frac{N-1+s}{2}} \right)^{-1} \left(\frac{N-2}{\frac{N-1+s}{2} - 1} \right) \left(\frac{N-2}{\frac{N-1+s'}{2}} \right) 2^{-N+2}. \quad (\text{C.9})$$

Using the first property of the binomial coefficient in Lemma C.1, we can rewrite

$$\left(\frac{N-2}{\frac{N-1+s}{2} - 1} \right) = \frac{1}{2} \frac{N-1+s}{N-1} \left(\frac{N-1}{\frac{N-1+s}{2}} \right) \quad (\text{C.10})$$

such that

$$\begin{aligned} \mathbb{P}(D_{s',+} \mid i \in B_s) &= \frac{1}{2} \frac{N-1+s}{N-1} \left(\frac{N-2}{\frac{N-1+s'}{2}} \right) 2^{-N+2} \\ &= \frac{N-1+s}{N-1} \left(\frac{N-2}{\frac{N-1+s'}{2}} \right) 2^{-N+1}. \end{aligned} \quad (\text{C.11})$$

Then we use the second property in Lemma C.1 to rewrite

$$\begin{aligned} \left(\frac{N-2}{\frac{N-1+s'}{2}} \right) 2^{-N+1} &= \frac{1}{2} \frac{N-1-s'}{N-1} \left(\frac{N-1}{\frac{N-1+s'}{2}} \right) 2^{-N+1} \\ &= \frac{1}{2} \frac{N-1-s'}{N-1} \mathbb{P}(S_v = s'), \end{aligned} \quad (\text{C.12})$$

where we used the definition of the binomial distribution in combination with Proposition 3.3 to obtain the second line. Substituting (C.12) into (C.11) gives

$$\mathbb{P}(D_{s',+} \mid i \in B_s) = \frac{1}{2} \frac{N-1+s}{N-1} \frac{N-1-s'}{N-1} \mathbb{P}(S_v = s'). \quad (\text{C.13})$$

Working out the fractions then gives (C.5) in the case where $\Sigma_{uv} = +1$. Note that setting $\Sigma_{uv} = -1$ only flips the sign

$$\mathbb{P}(D_{s',-} \mid i \in B_s) = \frac{1}{2} \frac{N-1-s}{N-1} \frac{N-1+s'}{N-1} \mathbb{P}(S_v = s'). \quad (\text{C.14})$$

It is therefore indeed the case that

$$\mathbb{P}(D_{s',\pm} \mid i \in B_s) = \frac{1}{2} \left(1 \pm \frac{s-s'}{N-1} - \frac{ss'}{(N-1)^2} \right) \mathbb{P}(S_v = s'), \quad (\text{C.15})$$

which proves the claim. $\square$

For the proof of Theorem 4.2, we apply Chebyshev's inequality on $\mathbb{V}(\Lambda_N(i, s') \mid i \in B_s)$ in order to prove convergence in probability. Calculating the variance of this term requires us to estimate the size of the covariance term between $\Lambda_N(i, s')$ and $\Lambda_N(j, s')$. The following Lemmas are used in order to do so.

Lemma C.3. *Let u, v, w be distinct vertices in a random tournament T_N . Define the difference function $D(k) = p_{N-3}(k+1) - p_{N-3}(k-1)$. The covariance of the reduced score indicators is given by*

$$\text{Cov} \left(\mathbb{1}_{\{S_v^{(u)}=m\}}, \mathbb{1}_{\{S_w^{(u)}=n\}} \right) = -\frac{1}{4} D(m) D(n). \quad (\text{C.16})$$

Proof. We calculate the covariance of the indicators $\mathbb{1}_{\{S_v^{(u)}=m\}}$ and $\mathbb{1}_{\{S_w^{(u)}=n\}}$. By definition

$$\begin{aligned} \text{Cov} \left(\mathbb{1}_{\{S_v^{(u)}=m\}}, \mathbb{1}_{\{S_w^{(u)}=n\}} \right) &= \mathbb{P} \left(S_v^{(u)} = m, S_w^{(u)} = n \right) \\ &\quad - \mathbb{P} \left(S_v^{(u)} = m \right) \mathbb{P} \left(S_w^{(u)} = n \right). \end{aligned} \quad (\text{C.17})$$

The reduced scores share exactly one random edge direction $\sigma(v, w)$. Using the standard conditional independence argument, the joint probability reduces to

$$\begin{aligned} \mathbb{P} \left(S_v^{(u)} = m, S_w^{(u)} = n \right) &= \frac{1}{2} p_{N-3} (m-1) p_{N-3} (n+1) \\ &\quad + \frac{1}{2} p_{N-3} (m+1) p_{N-3} (n-1). \end{aligned} \quad (\text{C.18})$$

Similarly, the product of the marginals decomposes into

$$\begin{aligned} \mathbb{P} \left(S_v^{(u)} = m \right) \mathbb{P} \left(S_w^{(u)} = n \right) &= \frac{1}{4} (p_{N-3} (m+1) + p_{N-3} (m-1)) \\ &\quad (p_{N-3} (n+1) + p_{N-3} (n-1)). \end{aligned} \quad (\text{C.19})$$

Subtracting the product of marginals from the joint probability simplifies to

$$\begin{aligned} \text{Cov} \left(\mathbb{1}_{\{S_v^{(u)}=m\}}, \mathbb{1}_{\{S_w^{(u)}=n\}} \right) &= -\frac{1}{4} (p_{N-3} (m+1) - p_{N-3} (m-1)) \\ &\quad (p_{N-3} (n+1) - p_{N-3} (n-1)). \end{aligned} \quad (\text{C.20})$$

Setting $D(k) = p_{N-3} (k+1) - p_{N-3} (k-1)$, then reduces the expression to

$$\text{Cov} \left(\mathbb{1}_{\{S_v^{(u)}=m\}}, \mathbb{1}_{\{S_w^{(u)}=n\}} \right) = -\frac{1}{4} D(m) D(n). \quad (\text{C.21})$$

□

Lemma C.4. Let u, v, w be distinct vertices in T_N . Define the auxiliary random variables

$$Q(\Sigma_{uv}) = k(u, v) D(s + \Sigma_{uv}) \quad \text{and} \quad R(\Sigma_{uv}) = k(u, v) p_{N-2}(s + \Sigma_{uv}) \quad (\text{C.22})$$

The conditional covariance of the weighted indicators is

$$\text{Cov} (I(u, v), I(u, w) \mid S_u = s) = -\frac{1}{16} \left(\mathbb{E} [Q \mid S_u = s] \right)^2 - \frac{\alpha_+ \alpha_-}{4(N-2)} (\Delta Q)^2 + \frac{\alpha_+ \alpha_-}{N-2} (\Delta R)^2 \quad (\text{C.23})$$

where

$$\mathbb{E} [Q \mid S_u = s] = \alpha_+ D(s+1) + \alpha_- e^{-\beta\omega} D(s+1) \quad (\text{C.24})$$

and

$$\Delta Q = \alpha_+ D(s+1) + \alpha_- e^{-\beta\omega} D(s+1), \quad \Delta R = p_{N-2}(s'+1) - e^{-\beta\omega} p_{N-2}(s'-1). \quad (\text{C.25})$$

Proof. For brevity of notation we will denote

$$\mathbb{E} [\cdot \mid S_u = s] = \mathbb{E}^{S_u} [\cdot] \quad \text{and} \quad \text{Cov} (\cdot, \cdot \mid S_u = s) = \text{Cov}^{S_u} (\cdot, \cdot) \quad (\text{C.26})$$

throughout this proof. To calculate the covariance between $I(u, v)$ and $I(u, w)$, we use the decomposition

$$\begin{aligned} \text{Cov}^{S_u} (I(u, v), I(u, w)) &= \mathbb{E}^{S_u} \left[\text{Cov}^{S_u, \mathbf{e}} (I(u, v), I(u, w)) \right] + \\ &\quad \text{Cov}^{S_u} (\mathbb{E}^{S_u, \mathbf{e}} [I(u, v)], \mathbb{E}^{S_u, \mathbf{e}} [I(u, w)]), \end{aligned} \quad (\text{C.27})$$

where we set $\mathbf{e} = \{\Sigma_{uv}, \Sigma_{uw}\}$ for the relevant edges. For the first term, note that given $\mathbf{e}$, we have

$$I(u, v) \mid \Sigma_{uv} = k(u, v) \mathbb{1}_{\{S_v^{(u)} = s' + \Sigma_{uv}\}}. \quad (\text{C.28})$$

This means that the conditional covariance takes the form

$$\text{Cov}^{S_u, \mathbf{e}}(I(u, v), I(u, w)) = k(u, v)k(u, w) \text{Cov}\left(\mathbb{1}_{\{S_v^{(u)} = s' + \Sigma_{uv}\}}, \mathbb{1}_{\{S_w^{(u)} = s' + \Sigma_{uw}\}}\right). \quad (\text{C.29})$$

Applying Lemma C.3 then yields

$$\text{Cov}\left(\mathbb{1}_{\{S_v^{(u)} = s' + \Sigma_{uv}\}}, \mathbb{1}_{\{S_w^{(u)} = s' + \Sigma_{uw}\}}\right) = -\frac{1}{4}D(s' + \Sigma_{uv})D(s' + \Sigma_{uw}). \quad (\text{C.30})$$

Setting

$$\begin{aligned} Q(\Sigma_{uv}) &= k(u, v)D(s' + \Sigma_{uv}) \\ &= e^{-\beta\omega}D(s' - 1) + \left(D(s' + 1) - e^{-\beta\omega}D(s' - 1)\right)\mathbb{1}_{\{\Sigma_{uv}=+1\}} \end{aligned} \quad (\text{C.31})$$

we can simplify (C.29) to

$$\text{Cov}^{S_u, \mathbf{e}}(I(u, v), I(u, w)) = -\frac{1}{4}Q(\Sigma_{uv})Q(\Sigma_{uw}). \quad (\text{C.32})$$

Therefore, the first term in (C.27) is given by

$$\mathbb{E}^{S_u} [\text{Cov}^{S_u, \mathbf{e}}(I(u, v), I(u, w))] = -\frac{1}{4}\mathbb{E}^{S_u} [Q(\Sigma_{uv})Q(\Sigma_{uw})]. \quad (\text{C.33})$$

To calculate this expectation, we use that Q is a linear function of $\mathbb{1}_{\{\Sigma_{uv}=+1\}}$ by (C.31), such that

$$\text{Cov}^{S_u}(Q(\Sigma_{uv}), Q(\Sigma_{uw})) = \frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2} \left(D(s' + 1) - e^{-\beta\omega}D(s' - 1)\right)^2. \quad (\text{C.34})$$

Here, we used Lemma 3.6 for $\text{Cov}^{S_u}(\mathbb{1}_{\{\Sigma_{uv}=+1\}}, \mathbb{1}_{\{\Sigma_{uw}=+1\}}) = (\alpha_+ \alpha_-)/(N-2)$. Using the same lemma, we also find that

$$\mathbb{E}^{S_u} [Q(\Sigma_{uv})] = \alpha_+ D(s + 1) + \alpha_- e^{-\beta\omega} D(s + 1). \quad (\text{C.35})$$

Therefore, we can conclude

$$\begin{aligned} \mathbb{E}^{S_u} [\text{Cov}^{S_u, \mathbf{e}}(I(u, v), I(u, w))] &= -\frac{1}{4} \left[\left(\alpha_+ D(s + 1) + \alpha_- e^{-\beta\omega} D(s + 1) \right)^2 + \right. \\ &\quad \left. \frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2} \left(D(s' + 1) - e^{-\beta\omega} D(s' - 1) \right)^2 \right]. \end{aligned} \quad (\text{C.36})$$

For the second term in (C.27), we use that

$$\begin{aligned} \mathbb{E}^{S_u, \mathbf{e}} [I(u, v)] &= k(u, v) \mathbb{P}(S_v^{(u)} = s' + \Sigma_{uv}) \\ &= e^{-\beta\omega} p_{N-2}(s' - 1) + \left(p_{N-2}(s' + 1) - e^{-\beta\omega} p_{N-2}(s' - 1) \right) \mathbb{1}_{\{\Sigma_{uv}=+1\}}. \end{aligned} \quad (\text{C.37})$$

We can apply Lemma 3.6 once more to find

$$\text{Cov}^{S_u}(\mathbb{E}^{S_u, \mathbf{e}} [I(u, v)], \mathbb{E}^{S_u, \mathbf{e}} [I(u, v)]) = \frac{\alpha_+ \alpha_-}{N-2} \left(p_{N-2}(s' + 1) - e^{-\beta\omega} p_{N-2}(s' - 1) \right)^2. \quad (\text{C.38})$$

Combining (C.36) and (C.38) then results in the final expression

$$\begin{aligned} \text{Cov}^{S_u}(I(u, v), I(u, w)) = & -\frac{1}{16} \left(\alpha_+ D(s+1) + \alpha_- e^{-\beta\omega} D(s+1) \right)^2 - \\ & \frac{1}{16} \frac{\alpha_+ \alpha_-}{N-2} \left(D(s'+1) - e^{-\beta\omega} D(s'-1) \right)^2 + \\ & \frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2} \left(p_{N-2}(s'+1) - e^{-\beta\omega} p_{N-2}(s'-1) \right)^2. \end{aligned} \quad (\text{C.39})$$

□

Lemma C.5. *Let u, v, w be distinct vertices on T_N . The covariance for the weighted indicators $I(u, v)$ conditioned on $S_u = s$ has a size of order*

$$\text{Cov}(I(u, v), I(u, w) \mid S_u = s) = O(N^{-2}). \quad (\text{C.40})$$

Proof. From (C.39), we know that

$$\begin{aligned} \text{Cov}(I(u, v), I(u, w) \mid S_u = s) = & -\frac{1}{16} \left(\alpha_+ D(s+1) + \alpha_- e^{-\beta\omega} D(s+1) \right)^2 - \\ & \frac{1}{16} \frac{\alpha_+ \alpha_-}{N-2} \left(D(s'+1) - e^{-\beta\omega} D(s'-1) \right)^2 + \\ & \frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2} \left(p_{N-2}(s'+1) - e^{-\beta\omega} p_{N-2}(s'-1) \right)^2. \end{aligned} \quad (\text{C.41})$$

By the local CLT (A.5), we know that the $p_{N-2}(s' \pm 1) = O(N^{-1/2})$. Therefore, the third term has the straightforward bound

$$\frac{1}{4} \frac{\alpha_+ \alpha_-}{N-2} \left(p_{N-2}(s'+1) - e^{-\beta\omega} p_{N-2}(s'-1) \right)^2 = O(N^{-2}). \quad (\text{C.42})$$

In order to bound the other two terms in (C.41), note that

$$D(k) = p_{N-3}(k+1) - p_{N-3}(k-1). \quad (\text{C.43})$$

By Lemma C.1, we know that

$$\frac{p_{N-3}(k+1)}{p_{N-3}(k-1)} = \frac{N-3-k}{N-1+k}, \quad (\text{C.44})$$

so that we have

$$D(k) = p_{N-3}(k-1) \left(\frac{N-3-k}{N-1+k} - 1 \right) = p_{N-3}(k-1) \left(\frac{-2-k}{N-1+k} \right). \quad (\text{C.45})$$

Applying the local CLT on $p_{N-3}(k-1)$, we see that for $k = O(\sqrt{N})$, we get

$$D(k) = O(N^{-1}). \quad (\text{C.46})$$

This means that the first term in (C.41) is bounded by

$$\frac{1}{16} \left(\alpha_+ D(s+1) + \alpha_- e^{-\beta\omega} D(s+1) \right)^2 = O(N^{-2}) \quad (\text{C.47})$$

and the second term

$$\frac{1}{16} \frac{\alpha_+ \alpha_-}{N-2} \left(D(s' + 1) - e^{-\beta\omega} D(s' - 1) \right)^2 = O(N^{-3}). \quad (\text{C.48})$$

Therefore, it follows that

$$\text{Cov}(I(u, v), I(u, w) \mid S_u = s) = O(N^{-2}) \quad (\text{C.49})$$

□

The final technical lemma of this chapter is a rewrite of the annealed transition rate $\gamma_N(s, s')$ into a useful form involving the Gaussian distribution function.

Lemma C.6. *For large N , the function $\gamma_N(s, s')$ is proportional to a Gaussian probability density function in s' . Specifically, it can be written as*

$$\gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2} \frac{\phi(s'; -\tanh(\frac{\beta\omega}{2}), N-1)}{v(s)} \quad (\text{C.50})$$

up to an error of order $O(N^{-3/2})$. Here, $\phi(s; \mu, \tau^2)$ denotes the density of a Gaussian random variable with mean μ and variance σ^2 evaluated at s , and we have

$$v(s) = e^{-\frac{s}{(N-1)} \tanh \frac{\beta\omega}{2}}, \quad (\text{C.51})$$

as in Theorem 6.1.

Proof. Starting from

$$\gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2\sqrt{2\pi(N-1)}} \exp \left(\frac{(s')^2}{2(N-1)} + \frac{s-s'}{N-1} \tanh \frac{\beta\omega}{2} \right), \quad (\text{C.52})$$

we can complete the square in the exponent with respect to s' yielding

$$\gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2\sqrt{2\pi N}} \exp \left(\frac{(s' + \tanh \frac{\beta\omega}{2})^2}{2N} + \frac{s}{N-1} \tanh \frac{\beta\omega}{2} - \frac{\tanh^2 \frac{\beta\omega}{2}}{2N} \right), \quad (\text{C.53})$$

Then, using the definition of $v(s)$ in (C.51) and Taylor expanding the square, we find that

$$\gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2} \frac{1}{v(s)} \left[\frac{1}{\sqrt{2\pi N}} \exp \left(\frac{(s' + \tanh \frac{\beta\omega}{2})^2}{2N} \right) \right] + O(N^{-3/2}), \quad (\text{C.54})$$

where we recognise $\phi(s'; -\tanh(\frac{\beta\omega}{2}), N)$ in the brackets. □

D. Technical Lemmas: Chapter 2

In section 5.2, we want to calculate the probability of a score bin attaining a certain size t in the $\mathcal{D}$ -model. In the $\mathcal{D}$ -model the scores are independent binomials, conditioned on their fixed score. Rewriting this probability into the form

$$\mathbb{P}(\tilde{N}_s = t \mid \tilde{S}_N = 0) = \frac{\mathbb{P}(\tilde{S}_N = 0 \mid \tilde{N}_s = t)}{\tilde{N}_s = t}, \quad (\text{D.1})$$

we see that we see that calculating the probability of the bin size t reduces to a sum of restricted random variables. Specifically, the event $\{\tilde{S}_N = 0\}$ given $\{\tilde{N}_s = t\}$ implies that the remaining $N - t$ vertices are excluded from attaining the score s . The following Lemma calculates the mean and variance of these variables.

Lemma D.1. *Let $X \sim \text{Bin}(N-1, 1/2)$ and set X' to $(X \mid X \neq (N-1+s)/2)$. Denote by $p_{N-1}(s) = \mathbb{P}(X = (N-1+s)/2)$. Then the m -th moment of X' is given by*

$$\mathbb{E}[(X')^m] = \frac{\mathbb{E}[X^m]}{1 - p_{N-1}(s)} - \left(\frac{N-1+s}{2}\right)^m \frac{p_{N-1}(s)}{1 - p_{N-1}(s)}. \quad (\text{D.2})$$

Consequently, the mean and variance of X' are given by

$$\mathbb{E}[X'] = \frac{N-1}{2} - \frac{s}{2} \frac{p_{N-1}(s)}{1 - p_{N-1}(s)} \quad \text{and} \quad \mathbb{V}(X') = \frac{(N-1)(1 - p_{N-1}(s)) - sp_{N-1}(s)}{4(1 - p_{N-1}(s))^2} \quad (\text{D.3})$$

Proof. Using the Law of Total Probability, we see that

$$\begin{aligned} \mathbb{E}[X^m] &= \mathbb{E}[X^m \mid X = x] \mathbb{P}(X = x) + \mathbb{E}[X^m \mid X \neq x] \mathbb{P}(X \neq x) \\ &= s^m (1 - \mathbb{P}(X \neq x)) + \mathbb{E}[X^m \mid X \neq x] \mathbb{P}(X \neq x). \end{aligned} \quad (\text{D.4})$$

Setting $X' = (X \mid X \neq x)$ and subsequently solving for $\mathbb{E}[X']$ then gives

$$\mathbb{E}[(X')^m] = \frac{\mathbb{E}[X^m] - x^m \mathbb{P}(X = x)}{1 - \mathbb{P}(X = x)}, \quad (\text{D.5})$$

which is what we wanted to show. The mean of X' follows directly by substituting for $m = 1$. To obtain the variance, we set $x = (N-1+s)/2$ and directly compute

$$\begin{aligned} \mathbb{V}(X') &= \mathbb{E}[(X')^2] - \mathbb{E}[X']^2 \\ &= \frac{\mathbb{E}[X^2] - x^2 p_{N-1}(s)}{1 - p_{N-1}(s)} - \left(\frac{\mathbb{E}[X] - x p_{N-1}(s)}{1 - p_{N-1}(s)}\right)^2 \\ &= \frac{\mathbb{V}(X) - (x^2 + \mathbb{E}[X^2])p_{N-1}(s) + 2x\mathbb{E}[X]p_{N-1}(s)}{(1 - p_{N-1}(s))^2}. \end{aligned} \quad (\text{D.6})$$

Note that $(N-1+s)/2 = \mathbb{E}[X] + s/2$. Substituting this for x gives

$$\mathbb{V}(X') = \frac{\mathbb{V}(X)(1 - p_{N-1}(s)) - (s^2/4)p_{N-1}(s)}{(1 - p_{N-1}(s))^2}, \quad (\text{D.7})$$

which after substituting for $\mathbb{V}(X)$ agrees with (D.3). $\square$

In the proof of Lemma 5.15, where we show the CLT for the score bins in the $\mathcal{D}$ -model, we use Janson's Theorem A.9. Condition (iii) is related to the third central moment of the conditioning variable. In our case, this is the score $\tilde{S}_i$. The following Lemma gives the size of this term.

Lemma D.2. *Let $X \sim \text{Bin}(N, 1/2)$. Then the third central moment has size*

$$\mathbb{E} \left| X - \frac{N}{2} \right|^3 = \Theta(N^{3/2}). \quad (\text{D.8})$$

Proof. The first step is to write X as a sum of i.i.d. $Y_i \sim \text{Ber}(1/2)$, so that

$$X = \sum_{i=1}^N Y_i. \quad (\text{D.9})$$

Then, by the standard Central Limit Theorem for sums of independent random variables, we have that

$$\frac{X - N/2}{\sqrt{N}} \xrightarrow{d} \mathcal{N}(0, 1/4). \quad (\text{D.10})$$

Because the Y_i are bounded, we can use the dominated convergence theorem to take the limit inside the expectation in order to find

$$\lim_{N \rightarrow \infty} \mathbb{E} \left| \frac{X - N/2}{\sqrt{N}} \right|^3 = \frac{1}{8} \mathbb{E}|Z|^3, \quad (\text{D.11})$$

, where $Z \sim \mathcal{N}(0, 1)$. Now, the third moment of a standard Normal is $\mathbb{E}|Z| = 2/\sqrt{\pi}$. Therefore, we can conclude that

$$\lim_{N \rightarrow \infty} N^{-3/2} \mathbb{E} |X - N/2|^3 = \frac{1}{4\sqrt{\pi}}, \quad (\text{D.12})$$

which finishes the proof. $\square$

E. Technical Lemmas: Chapter 3

The first step in calculating the pseudoinverse of $\overline{G_N}$ is to apply a similarity transform with respect to the stationary measure $\overline{\pi_N}$. Lemma 6.4 then shows that the resulting matrix, M_N , is the sum of a diagonal matrix D and a rank-one update $\mathbf{q}\mathbf{q}^T$. The entries of the diagonal matrix correspond to a sum over the annealed transition rates $\gamma_N(s, s')$. The following lemma calculates this sum.

Lemma E.1. *The sum over the annealed transition rate is given by*

$$\sum_{s' \in \mathcal{R}} \gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2} \exp\left(\frac{s}{(N-1)} \tanh \frac{\beta\omega}{2}\right) + O(N^{-1}), \quad (\text{E.1})$$

for every $s \in S$.

Proof. As in the proof of Theorem 4.1, we first complete the square in the exponent of $\gamma_N(s, s')$ to obtain

$$\begin{aligned} \gamma_N(s, s') &= \frac{1 + e^{-\beta\omega}}{2\sqrt{2\pi(N-1)}} \exp\left(-\frac{(s')^2}{2(N-1)} + \frac{s-s'}{N-1}T\right) \\ &= E(s) \frac{1}{\sqrt{2\pi(N-1)}} \exp\left(-\frac{1}{2} \frac{(s' + T)^2}{(N-1)}\right), \end{aligned} \quad (\text{E.2})$$

where

$$E(s) = \frac{1 + e^{-\beta\omega}}{2} \exp\left(\frac{T^2 + 2sT}{2(N-1)}\right). \quad (\text{E.3})$$

Since the remaining summand is a Gaussian density, we use Lemma A.7 to approximate this sum by an integral. This means that

$$\sum_{s' \in \mathcal{R}} \gamma_N(s, s') = \int_{-\infty}^{\infty} \gamma_N(s, s') ds' + O(e^{-N/2}). \quad (\text{E.4})$$

This integral is given by

$$\frac{1}{2} \int_{-\infty}^{\infty} \gamma_N(s, s') ds' = \frac{E(s)}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(N-1)}} \exp\left(-\frac{1}{2} \frac{(s' + T)^2}{(N-1)}\right) ds'. \quad (\text{E.5})$$

This is an integral against the Gaussian density, so we find

$$\int_{-\infty}^{\infty} \gamma_N(s, s') ds' = \frac{E(s)}{2} = \frac{1 + e^{-\beta\omega}}{2} \exp\left(\frac{T^2 + 2sT}{2(N-1)}\right). \quad (\text{E.6})$$

And so (E.4) reduces to

$$\sum_{s' \in \mathcal{R}} \gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2} \exp\left(\frac{sT}{(N-1)}\right) \exp\left(\frac{T^2}{2(N-1)}\right) + O(e^{-N/2}), \quad (\text{E.7})$$

such that a Taylor expansion of the second exponential gives

$$\sum_{s' \in \mathcal{R}} \gamma_N(s, s') = \frac{1 + e^{-\beta\omega}}{2} \exp\left(\frac{s}{(N-1)} \tanh \frac{\beta\omega}{2}\right) + O(N^{-1}), \quad (\text{E.8})$$

which finishes the proof. $\square$

Later in Section 6.1, we derive the expression for the pseudoinverse $M_N^\dagger$. This expression includes the constants κ and ν (6.35). The following lemma calculates these constants.

Lemma E.2. *The coefficients κ and ν are given by*

$$\kappa = \frac{2}{1 + e^{-\beta\omega}} + O(N^{-1}) \quad \text{and} \quad \nu = -\frac{4}{(1 + e^{-\beta\omega})^2} + O(N^{-1}). \quad (\text{E.9})$$

Proof. Let $T = \tanh \beta\omega/2$ and $K = (1 + e^{-\beta\omega})/2$. From the definitions of q_s in (6.30), D_{ss} in (6.29) and κ in (6.35), we get

$$\begin{aligned} \kappa &= \sum_{s \in S} \frac{q_s^2}{D_{ss}^2} \\ &= \sum_{s \in S} \frac{2}{K} \frac{1}{\sqrt{2\pi(N-1)}} \exp\left(-\frac{s^2 + 4sT}{2(N-1)}\right) (1 + O(N^{-1})). \end{aligned} \quad (\text{E.10})$$

By Lemma A.7, we may switch the sum to an integral with exponentially small penalty. Therefore,

$$\kappa = \frac{1}{K} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(N-1)}} \exp\left(-\frac{s^2 + 4sT}{2(N-1)}\right) ds + O(N^{-1}). \quad (\text{E.11})$$

Now, we can complete the square in the integral to find

$$\int_{-\infty}^{\infty} \exp\left(-\frac{s^2 + 4sT}{2(N-1)}\right) ds = e^{2T^2/(N-1)} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2(N-1)}\right) du, \quad (\text{E.12})$$

such that after taking the Gaussian integral, (E.11) reduces to

$$\kappa = \frac{1}{K} e^{2T^2/(N-1)} + O(N^{-1}). \quad (\text{E.13})$$

Substituting for the definition of K and Taylor expanding the exponential, then yields the final expression

$$\kappa = \frac{2}{1 + e^{-\beta\omega}} + O(N^{-1}). \quad (\text{E.14})$$

Similarly, using the definition of ν in (6.35), we get

$$\begin{aligned} \nu &= \sum_{s \in S} \frac{q_s^2}{D_{ss}^3} \\ &= -\sum_{s \in S} \frac{4}{K^2} \frac{1}{\sqrt{2\pi(N-1)}} \exp\left(-\frac{s^2 + 6sT}{2(N-1)}\right) (1 + O(N^{-1})). \end{aligned} \quad (\text{E.15})$$

Analogous to the derivation of κ , we can replace the sum by an integral to find

$$\nu = -\frac{2}{K^2} e^{3T^2/(N-1)} + O(N^{-1}), \quad (\text{E.16})$$

so that substituting for K in combination with a Taylor expansion gives

$$\nu = -\frac{4}{(1 + e^{-\beta\omega})^2} + O(N^{-1}). \quad (\text{E.17})$$

□

In Section 6.2, we derive a closed form solution for the heat capacity of the process generated by G_N . The following technical Lemma does some preliminary work in deriving an expression into which the perturbative solution δW can be substituted in order to prove Theorem 6.2.

Lemma E.3. *The heat capacity of the process generated by G_N is given by*

$$\tilde{C}_N(\beta) = \overline{C}_N(\beta) \left(1 + \sum_{s \in \mathcal{J}_S} \frac{s - \mu}{\omega N} v(s) \overline{W}_N(s) (N_s - \mathbb{E}[N_s]) + \sum_{s \in \mathcal{J}_S} \frac{s - \mu}{\omega} \overline{\pi}_N(s) \delta W \right) \quad (\text{E.18})$$

Proof. For clarity of notation, we set

$$h(s) = (s - \mu) \frac{\beta^2 \omega}{N} \text{sech}^2 \frac{\beta \omega}{2} \quad (\text{E.19})$$

Then, from Lemma 6.14, we have

$$\tilde{C}_N(\beta) = \sum_{s \in \mathcal{J}_S} h(s) W_N(s) \overline{\pi}_N(s) \frac{N_s}{\mathbb{E}[N_s]}, \quad (\text{E.20})$$

we use $W_N = \overline{W}_N + \delta W$ to split the sum

$$\tilde{C}_N(\beta) = \underbrace{\sum_{s \in \mathcal{J}_S} h(s) \overline{\pi}_N(s) \frac{N_s}{\mathbb{E}[N_s]} \overline{W}_N(s)}_{S_1} + \underbrace{\sum_{s \in \mathcal{J}_S} h(s) \overline{\pi}_N(s) \frac{N_s}{\mathbb{E}[N_s]} \delta W(s)}_{S_2}. \quad (\text{E.21})$$

From the definition of the annealed heat capacity, we know that

$$\overline{C}_N(\beta) = -\beta^2 \left\langle \overline{W}_N \frac{\partial \log \overline{\pi}_N}{\partial \beta} \right\rangle_{\overline{\pi}_N} = \sum_{s \in \mathcal{J}_S} h(s) \overline{\pi}_N(s) \overline{W}_N(s). \quad (\text{E.22})$$

Subtracting this from (E.21), we get

$$S_1 - \overline{C}_N(\beta) = \sum_{s \in \mathcal{J}_S} h(s) \overline{\pi}_N(s) \left(\frac{N_s}{\mathbb{E}[N_s]} - 1 \right) \overline{W}_N(s). \quad (\text{E.23})$$

From (4.45) and the fact that $\mathbb{E}[N_s] = N p_{N-1}(s)$, we can derive that

$$\overline{\pi}_N(s) \left(\frac{N_s}{\mathbb{E}[N_s]} - 1 \right) = v(s) \frac{N_s - \mathbb{E}[N_s]}{N}. \quad (\text{E.24})$$

Therefore, it follows that

$$S_1 = \overline{C}_N(\beta) + \sum_{s \in \mathcal{J}_S} \frac{h(s) v(s)}{N} \overline{\pi}_N(s) (N_s - \mathbb{E}[N_s]) \overline{W}_N(s). \quad (\text{E.25})$$

For S_2 , we split

$$S_2 = \sum_{s \in \mathcal{J}_S} h(s) \overline{\pi}_N(s) \delta W(s) + \sum_{s \in \mathcal{J}_S} h(s) \overline{\pi}_N(s) \left(\frac{N_s}{\mathbb{E}[N_s]} - 1 \right) \delta W(s). \quad (\text{E.26})$$

and note that the second term is a higher order term that we can neglect. Finally, we note that

$$\frac{h(s)}{\overline{C}(\beta)} = \frac{s - \mu}{\omega}. \quad (\text{E.27})$$

Using (E.25) and (E.26), we can then conclude that (E.21) reduces to

$$\tilde{C}_N(\beta) = \overline{C}_N(\beta) \left(1 + \sum_{s \in \mathcal{J}_S} \frac{s - \mu}{\omega N} v(s) \overline{W}_N(s) (N_s - \mathbb{E}[N_s]) + \sum_{s \in \mathcal{J}_S} \frac{s - \mu}{\omega} \overline{\pi}_N(s) \delta W \right). \quad (\text{E.28})$$

□

The second technical lemma gives the magnitude of the source term U , and shows that it is strictly smaller than the quasipotential $\overline{W}_N$. This can be used, for example, to assess the size of $\delta W_B^{(0)}$.

Lemma E.4. *The source term U has magnitudes*

$$\|U_B\|_\infty = \|(\overline{W}_N)_B\|_\infty \cdot O(N^{-1/4} \log N) \quad \text{and} \quad \|U_T\|_\infty = \|(\overline{W}_N)_T\|_\infty \cdot O(1) \quad (\text{E.29})$$

with failure probability $O(N^{-k})$ for all $k > 0$.

Proof. We start with U_B . In the bulk, the source term is given by

$$U_B = \langle \overline{\mathcal{P}} \rangle_{\overline{\pi}} - \langle \mathcal{P} \rangle_{\pi} - \Delta_{BB}(\overline{W}_N)_B \quad (\text{E.30})$$

Using the triangle inequality, we get

$$\|U_B\|_\infty = |\langle \overline{\mathcal{P}} \rangle_{\overline{\pi}} - \langle \mathcal{P} \rangle_{\pi}| + \|\Delta_{BB}(\overline{W}_N)_B\|_\infty. \quad (\text{E.31})$$

By Lemma 6.10 the first term admits the bound

$$|\langle \overline{\mathcal{P}} \rangle_{\overline{\pi}} - \langle \mathcal{P} \rangle_{\pi}| \leq \|\overline{W}_N\|_\infty O(N^{-1/4} \sqrt{\log N}). \quad (\text{E.32})$$

Evaluating the second term in (E.30) gives for $s \in \mathcal{T}_{\mathbf{S}}$

$$\Delta_{BB}(\overline{W}_N)_B = \sum_{s' \in \mathcal{T}_{\mathbf{S}}} (\overline{G}_N)_{ss'} \left(\frac{N_{s'}}{\mathbb{E}[N_{s'}]} - 1 \right) \overline{W}_N(s'). \quad (\text{E.33})$$

In order to bound this sum, we note that

$$(\overline{G}_N)_{ss'} \left(\frac{N_{s'}}{\mathbb{E}[N_{s'}]} - 1 \right) = \frac{\gamma_N(s, s')}{\mathbb{E}[N_{s'}]} (N_{s'} - \mathbb{E}[N_{s'}]). \quad (\text{E.34})$$

Now, using Lemma C.6 for γ_N and the LCLT for $\mathbb{E}[N_{s'}]$ we find that

$$\begin{aligned} \left\| \frac{\gamma_N(s, s')}{\mathbb{E}[N_{s'}]} \right\|_\infty &= \left\| \frac{1 + e^{-\beta\omega}}{2} e^{\frac{\phi(s'; -\tanh(\frac{\beta\omega}{2}), N)}{v(s)N\phi(s'; 0, N)}} \right\|_\infty \\ &\leq \frac{1 + e^{-\beta\omega}}{2N}. \end{aligned} \quad (\text{E.35})$$

Substituting this result into (E.34) yields

$$\left\| (\overline{G}_N)_{ss'} \left(\frac{N_{s'}}{\mathbb{E}[N_{s'}]} - 1 \right) \right\|_\infty \leq \frac{1 + e^{-\beta\omega}}{2\sqrt{N}} \left\| \frac{N_{s'} - \mathbb{E}[N_{s'}]}{\sqrt{N}} \right\|_\infty, \quad (\text{E.36})$$

so that by Theorem 5.1 it follows that

$$\left\| (\overline{G}_N)_{ss'} \left(\frac{N_{s'}}{\mathbb{E}[N_{s'}]} - 1 \right) \right\|_\infty = O(N^{-3/4} \sqrt{\log N}). \quad (\text{E.37})$$

Taking the norm of (E.33), we see that

$$\|\Delta_{BB}(\overline{W}_N)_B\|_\infty \leq \|(\overline{W}_N)_B\|_\infty \sum_{s' \in \mathcal{T}_{\mathbf{S}}} \left\| (\overline{G}_N)_{ss'} \left(\frac{N_{s'}}{\mathbb{E}[N_{s'}]} - 1 \right) \right\|_\infty. \quad (\text{E.38})$$

Because $\mathcal{T}_{\mathbf{S}}$ has $O(\sqrt{N \log N})$ elements, we have

$$\|\Delta_{BB}(\overline{W}_N)_B\|_\infty = \|(\overline{W}_N)_B\|_\infty \cdot O(N^{-1/4} \sqrt{\log N}). \quad (\text{E.39})$$

Substituting this result and (E.32) into (E.33), we conclude

$$\|U_B\|_\infty = O(N^{-3/4} \log N). \quad (\text{E.40})$$

For the tail estimate, we have

$$U_T = \bar{Q}_T + (\bar{G}_N)_{TB}(\bar{W}_N)_B. \quad (\text{E.41})$$

Using the fact that $\bar{W}_N$ solves the poisson equation

$$\bar{G}_N \bar{W}_N = -\bar{Q}, \quad (\text{E.42})$$

we get

$$(\bar{G}_N)_{TB}(\bar{W}_N)_B + (\bar{G}_N)_{TT}(\bar{W}_N)_T = -\bar{Q}_T. \quad (\text{E.43})$$

This means that we can rewrite (E.41) into

$$U_T = -(\bar{G}_N)_{TT}(\bar{W}_N)_T. \quad (\text{E.44})$$

Because the entries of $\bar{G}_N$ are bounded for all $s, s' \in S$, we conclude

$$\|U_T\|_\infty = \|(\bar{G}_N)_{TT}(\bar{W}_N)_T\|_\infty = \|\bar{W}_N\|_\infty \cdot O(1). \quad (\text{E.45})$$

□

In Section 6.3, we claim that the generator L_N projected onto the subspace of microscopic fluctuations is invertible. The following lemma proves this.

Lemma E.5. *Let P be the projection operator defined in (3.45) and $Q = (I - P)$. We denote $L_{QQ} = QL_NQ$. The matrix L_{QQ} is invertible.*

Proof. We start by proving that

$$QL_NQ = D + A \quad (\text{E.46})$$

where D is diagonal and A skew-symmetric. In order to do so, we show that the symmetric part of L acts diagonally on an arbitrary $g \in \text{Im}(Q)$. To that end, we calculate

$$gL_Ng = \sum_x (L_N)_{xx} v(x)^2 + \sum_{x \neq y} \frac{1}{2} [(L_N)_{xy} + (L_N)_{yx}] v(x)v(y). \quad (\text{E.47})$$

Because any two vertices share one edge, we get a constant

$$\chi_N = \frac{1}{2} [(L_N)_{xy} + (L_N)_{yx}] = \frac{1 + e^{-\beta\omega}}{2N}. \quad (\text{E.48})$$

so that

$$gL_Ng = \sum_{x \in \mathcal{I}_S} (L_N)_{xx} v(x)^2 + \chi_N \sum_{x \neq y} v(x)v(y). \quad (\text{E.49})$$

Since $g \in \text{Im}(Q)$, it sums to zero on every block, which means that

$$\chi_N \left(\sum_{x \in \mathcal{I}_S} g(x) \right)^2 = \chi_N \sum_{x \in \mathcal{I}_S} g(x)^2 + \chi_N \sum_{x \neq y} g(x)g(y) = 0. \quad (\text{E.50})$$

Substituting this into (E.49) yields

$$gL_Ng = \sum_{x \in \mathcal{I}_S} ((L_N)_{xx} - \chi_N) v(x)^2. \quad (\text{E.51})$$

Now, by definition, the diagonal of L_N is related to the score of the vertex by

$$(L_N)_{xx} = - \left[\frac{N-1+s}{2N} + \frac{N-1-s}{2N} e^{-\beta\omega} \right]. \quad (\text{E.52})$$

That means that

$$(L_N)_{xx} - \chi_N = -\frac{1}{2} (1 + e^{-\beta\omega}) - \frac{s}{2N} (1 - e^{-\beta\omega}) - \frac{1 + e^{-\beta\omega}}{2N} < 0 \quad (\text{E.53})$$

for any N and $s \in \mathcal{T}_{\mathbf{S}}$. We conclude that QL_NQ is negative definite on $\text{Im}(Q)$ and thus invertible. $\square$

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