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To István Juhász on his 80th birthday

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ABSTRACT

We investigate closed copies of $\mathbb{N}$ in powers of $\mathbb{R}$ with respect to C^*- and C-embedding. We show that $\mathbb{R}^{\omega_1}$ contains closed copies of $\mathbb{N}$ that are not C^*-embedded.

Introduction

In [5] we presented examples of realcompact spaces with closed subsets that are C^*-embedded but not C-embedded.

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One of these spaces, call it X, even contains a closed copy of $\mathbb{N}$ (the discrete space of natural numbers) that is C^*-embedded but not C-embedded. It is well known that the diagonal map from X into $\mathbb{R}^{C(X)}$ embeds X as a closed C-embedded subspace. The closed copy of $\mathbb{N}$ in X then becomes a closed copy of $\mathbb{N}$ in $\mathbb{R}^{C(X)}$ that is C^*-embedded but not C-embedded.

An intermediate realcompact space, Y say, contains a closed copy of $\mathbb{N}$ that is not C^*-embedded and, as above, this yields a closed copy of $\mathbb{N}$ in $\mathbb{R}^{C(Y)}$ that is not C^*-embedded.

For both spaces the cardinality of the set of continuous functions is equal to $\mathfrak{c}$, which yields the interesting result that one can find closed copies of $\mathbb{N}$ in $\mathbb{R}^{\mathfrak{c}}$, that are not C^*-embedded, and that are C^*-embedded but not C-embedded.

In the first version of [5] we posed two questions suggested by these results. We repeat them here.

Question 1. What is the minimum cardinal κ such that $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded?

Question 2. What is the minimum cardinal κ such that $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is not C^*-embedded?

Given that $\mathbb{R}^{\omega_0}$ is metrizable we know that in both cases we have $\aleph_0 < \kappa \leq \mathfrak{c}$.

After we posted the first version of the present paper on `arxiv.org` Roman Pol kindly drew our attention to three papers, [7], [12], and [10], containing results that address the two questions above.

These are:

(1) The main result, Theorem 10, of [7] implies that there are many closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that are not C^*-embedded.

(2) The paper [12] contains another example, Example 1.1, of a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that is not C^*-embedded, and an example of a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\mathfrak{c}}$ that is C^*-embedded but not C-embedded.

(3) In [10] one finds a result, Theorem 3.1, that implies that under the assumption of the inequality $\mathfrak{r} > \aleph_1$ every C^*-embedded subset of $\mathbb{R}^{\omega_1}$ is C-embedded.
The cardinal $\mathfrak{r}$ is the *reaping number*: the minimum cardinality of a family of subsets $\mathcal{R}$ of $\mathbb{N}$ that behaves like an ultrafilter but for the finite intersection property: for every subset X of $\mathbb{N}$ there is a member R of $\mathcal{R}$ such that $R \subseteq^* X$ or $R \cap X =^* \emptyset$; see [3].

Thus, Question 2 was answered before we posed it and the answer to Question 1 depends on one's assumptions: the Continuum Hypothesis implies the minimum is $\aleph_1$, and it is also consistent that it is larger than $\aleph_1$.

The result from [10] can be viewed as a local version of the main result of [2]: in a model obtained by adding supercompact many Random reals to a model of CH every C^*-embedded subspace of every space of character less than $\mathfrak{c}$ is C-embedded. Indeed, one can create a model of $\mathfrak{r} > \aleph_1$ by adding $\aleph_2$ or more Random reals to a model of CH.

In retrospect our paper [5] should have contained references to [7,10,12] and we regret not finding these references ourselves. Nevertheless the methods and results of [5] and the present paper are sufficiently different from the earlier ones that we feel they merit publication.

In Sections 2 and 3 we give new examples and obtain topological and combinatorial translations of the statement "$\mathbb{R}^{\omega_1}$ contains a closed copy of $\mathbb{N}$ that is not C^*-embedded" that suggest further interesting questions.

In Section 2 we present three constructions of closed copies of $\mathbb{N}$ that are not C^*-embedded in $\mathbb{R}^{\omega_1}$: one directly from an Aronszajn tree, one directly from an Aronszajn continuum, and one as the path space of an Aronszajn tree. We decided to give all three examples because they show how versatile these objects are.

In Section 3 we give the translations mentioned above and give a fourth example that is of a somewhat different nature.

Section 4 deals with a class of topological spaces that feature in the translations, and in Section 5 we present models where CH fails but where the answer to Question 1 is still $\aleph_1$.

1. Preliminaries

By now the reader may have guessed that by "a closed copy of $\mathbb{N}$" in some space X we mean a closed subspace of X that is homeomorphic to the discrete space $\mathbb{N}$, in other words: a countably infinite closed and discrete subspace.

In general we say that a subspace Y of a space X is C-embedded if every continuous function $f : Y \to \mathbb{R}$ has a continuous extension to all of X. If this holds for all *bounded* continuous functions then we say that Y is C^*-embedded in X.

The way we shall show that a closed copy of $\mathbb{N}$ is not C^*-embedded in X is by exhibiting disjoint subsets A and B of $\mathbb{N}$ that are not *completely separated*, which means that whenever $g : X \to \mathbb{R}$ is bounded and continuous the closures of $g[A]$ and $g[B]$ intersect. This then implies that the characteristic function of A has no continuous extension to X.

As mentioned in the introduction we shall use Aronszajn trees and continua in some of our constructions; Todorčević's article [13] contains all the information that we need.

As is common we use starred versions of the inclusion and equality signs to indicate 'mod finite'. So $A \subseteq^* B$ means that $A \setminus B$ is finite, $A \subset^* B$ means that $A \setminus B$ is finite but $B \setminus A$ is not, and $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$.

We use the well-known fact that if $\langle A_n : n \in \omega\rangle$ is a sequence of infinite subsets of $\mathbb{N}$ such that $A_{n+1} \subseteq^* A_n$ for all n then there is an infinite subset A of $\mathbb{N}$ such that $A \subseteq^* A_n$ for all n.

We also remind the reader of the notation $\mathbb{N}^*$ for $\beta\mathbb{N} \setminus \mathbb{N}$ and, generally, $A^* = \operatorname{cl} A \cap \mathbb{N}^*$ for subsets of $\mathbb{N}$.

Any potentially unfamiliar topological notions will be defined when needed; definitions not given here can be found in Engelking's book [6].

One piece of possibly non-standard notation: if $t : \alpha \to \omega$ is a sequence of finite ordinals and if $n \in \omega$ then $t * n$ denotes the sequence with domain $\alpha + 1$ that coincides with t on α and takes on value n at α. In one formula: $t * n = t \cup \{\langle\alpha, n\rangle\}$.

We use ${}^{<\omega_1}\omega$ to denote the tree of all sequences of finite ordinals whose domains are countable ordinals.

2. Closed copies of $\mathbb{N}$ that are not C^*-embedded

This section contains further examples that show that the answer to the Question 2 is $\aleph_1$. We give three examples, based on Aronszajn trees and lines, of closed copies of $\mathbb{N}$ that are not C^*-embedded in $\mathbb{R}^{\omega_1}$. This may seem like overdoing things somewhat but we think that this presentation is more informative.

From our first two constructions we extract a few translations of "$\mathbb{R}^{\omega_1}$ contains a closed copy of $\mathbb{N}$ that is not C^*-embedded" that allow us to construct a relatively simple third example and an even simpler fourth one.

2.1. A closed copy of $\mathbb{N}$ that is not C^*-embedded, from an Aronszajn tree

The first construction uses an Aronszajn tree to guide an embedding of $\mathbb{N}$ into $\mathbb{R}^{\omega_1}$.

Using the fact about decreasing sequences of infinite subsets of $\mathbb{N}$ mentioned above we define a family $\{A_t : t \in {}^{<\omega_1}\omega\}$ of infinite subsets of $\mathbb{N}$ such that $A_\emptyset = \mathbb{N}$ and

(1) if $s \subset t$ then $A_t \subset^* A_s$, and
(2) for every t the family $\{A_{t*n} : n \in \omega\}$ is a partition of A_t.

Now take an Aronszajn subtree T of ${}^{<\omega_1}\omega$ as in [11, Theorem II.5.9]: it consists of finite-to-one members of ${}^{<\omega_1}\omega$, and is such that $\{t * n : n \in \omega\} \subseteq T$ whenever $t \in T$.

For every non-zero α in ω_1 we let $\langle t(\alpha, n) : n \in \omega\rangle$ enumerate the αth level T_α of T in a one-to-one fashion. We abbreviate $A_{t(\alpha,n)}$ as $A(\alpha, n)$.

By construction each of the families $\{A(\alpha, n) : n \in \omega\}$ is pairwise almost disjoint. We can assume, after making finite modifications to the $A(\alpha, n)$, that every family $\{A(\alpha, n) : n \in \omega\}$ is in fact a partition of $\mathbb{N}$.

We use the partitions to define a map $k \mapsto x_k$ from $\mathbb{N}$ to $\mathbb{R}^{\omega_1}$.

First we set $x_k(0) = 2^{-k}$ for all k. This ensures that $X = \{x_k : k \in \mathbb{N}\}$ is a relatively discrete subspace of $\mathbb{R}^{\omega_1}$.

Second, for every non-zero α in ω_1 we define

$$x_{2k}(\alpha) = x_{2k+1}(\alpha) = m \text{ iff } k \in A(\alpha, m).$$

This will ensure that X is closed in $\mathbb{R}^{\omega_1}$ and that the sets $\{x_{2k} : k \in \mathbb{N}\}$ and $\{x_{2k+1} : k \in \mathbb{N}\}$ are not completely separated in $\mathbb{R}^{\omega_1}$.

To see that X is closed let $x \in \operatorname{cl} X$ and let u be an ultrafilter such that $x = u\text{-}\lim x_k$. We claim u is in fact a fixed ultrafilter and hence that $x \in X$.

Since u is a filter there is for every β at most one n such that $A(\beta, n) \in u$. Let $B = \{\langle\beta, n\rangle : A(\beta, n) \in u\}$. If u were free then $A(\beta, n) \cap A(\gamma, m)$ would be infinite whenever $\langle\beta, n\rangle, \langle\gamma, m\rangle \in B$. By the construction of the family $\{A_t : t \in {}^{<\omega_1}\omega\}$ this would mean that $\{t(\beta, n) : \langle\beta, n\rangle \in B\}$ is linearly ordered in T, and hence countable.

Take α such that $T_\alpha \cap \{t(\beta, n) : \langle\beta, n\rangle \in B\} = \emptyset$, and let m be the smallest natural number larger than or equal to $x(\alpha)$. Then $U = \mathbb{N} \setminus \bigcup_{i \le m} A(\alpha, i)$ belongs to u, and $x_{2k}(\alpha) = x_{2k+1}(\alpha) \ge x(\alpha) + 1$ for all $k \in U$. This shows that $x(\alpha) \ne u\text{-}\lim x_k(\alpha)$, which contradicts the assumption that $x = u\text{-}\lim x_k$.

To see that $\{x_{2k} : k \in \mathbb{N}\}$ and $\{x_{2k+1} : k \in \mathbb{N}\}$ are not completely separated in $\mathbb{R}^{\omega_1}$ let $g : \mathbb{R}^{\omega_1} \to [0, 1]$ be continuous. It is well-known, see [6, Problem 2.7.12], that there are $\delta < \omega_1$ and a continuous function $h : \mathbb{R}^\delta \to [0, 1]$ such that $g = h \circ \pi_\delta$. Here π_δ is the projection from $\mathbb{R}^{\omega_1}$ onto $\mathbb{R}^\delta$.

Consider $A(\delta, 0)$. By construction we know that for every non-zero $\alpha < \delta$ there is a single n_α such $A(\delta, 0) \subset^* A(\alpha, n_\alpha)$. Let $x \in \mathbb{R}^\delta$ be given by $x(0) = 0$ and $x(\alpha) = n_\alpha$, then the subsequences $\langle\pi_\delta(x_{2k}) : k \in A(\delta, 0)\rangle$ and $\langle\pi_\delta(x_{2k+1}) : k \in A(\delta, 0)\rangle$ of $\langle x_k : k \in \mathbb{N}\rangle$ both converge to x and so $h(x)$ is in the closure of both $\{g(x_{2k}) : k \in \mathbb{N}\}$ and $\{g(x_{2k+1}) : k \in \mathbb{N}\}$.

2.2. Another closed copy of $\mathbb{N}$ that is not C^*-embedded, from an Aronszajn line

Let L be an Aronszajn continuum: a first-countable linearly ordered continuum of weight $\aleph_1$ with the property that the closure of every countable set is second-countable, see [13, Section 5]. We can also assume, without loss of generality, that L has no non-trivial separable intervals.

Let $\langle x_\alpha : \alpha \in \omega_1\rangle$ enumerate a dense subset of L, where we assume that $x_0 = \min L$ and $x_1 = \max L$. Using the first-countability of L we find that $L = \bigcup_{\alpha<\omega_1} \operatorname{cl}\{x_\beta : \beta \le \alpha\}$, that is, L is the union of an increasing sequence of second-countable compact subsets. Upon thinning out the sequence we obtain a strictly increasing sequence $\langle K_\alpha : \alpha \in \omega_1\rangle$ of second-countable compact subsets whose union is equal to L. The assumption on the intervals of L implies that each K_α is nowhere dense.

We claim that every K_α is a G_δ-set of L. By the first-countability of L this is clear if α is finite, so we assume below that α is infinite, and hence that $\min L$ and $\max L$ belong to K_α.

Since K_α is second-countable we can find a countable family $\mathcal{I}$ of open intervals in L such that $\{I \cap K_\alpha : I \in \mathcal{I}\}$ is a base for the topology of K_α.

Every convex component C of $L \setminus K_\alpha$ is of the form (a_C, b_C), with $a_C, b_C \in K_\alpha$. If C and D are two such components then $b_C < a_D$ or $b_D < a_C$. For each C take $I_C \in \mathcal{I}$ such that $b_C \in I_C \cap K_\alpha \subseteq [b_C, \max L]$. Then $a_C \notin I_C$ and so $b_D \notin I_C$ whenever $b_D < a_C$. It follows that $I_C \neq I_D$ whenever $C \neq D$. This shows that there are at most countably many convex components in the complement of K_α.

Enumerate these components as $\langle C_n : n \in \omega\rangle$ and choose for every $n \in \omega$ sequences $\langle a(n,k) : k \in \omega\rangle$ and $\langle b(n,k) : k \in \omega\rangle$ in C_n such that $a(n,k) \downarrow a_{C_n}$ and $b(n,k) \uparrow b_{C_n}$.

Then $C_n = \bigcup_{k\in\omega}[a(n,k), b(n,k)]$ for all n. Define $F_k = \bigcup_{n\le k}[a(n,k), b(n,k)]$ for all k. Then $\langle F_k : k \in \omega\rangle$ is a sequence of closed sets and its union is equal to the complement of K_α.

Since L has weight $\aleph_1$ there is a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ such that $\gamma\mathbb{N} \setminus \mathbb{N}$ is (homeomorphic to) L, see [6, Problem 3.12.18(c)]. Take the quotient of $\gamma\mathbb{N} \times \{0,1\}$ obtained by identifying $\langle x, 0\rangle$ and $\langle x, 1\rangle$ for all $x \in L$.

The result is a new compactification $\delta\mathbb{N}$ of $\mathbb{N}$ with remainder equal to L and in which $\mathbb{N}$ is the union of two subsets A and B such that $L = \operatorname{cl} A \cap \operatorname{cl} B$.

We map $\delta\mathbb{N}$ into $[0,1]^{\omega_1}$ in such a way that the image of $\mathbb{N}$ will be a closed subset of $(0,1)^{\omega_1}$ that is not C^*-embedded.

For every $\alpha \ge 1$ we let $f_\alpha : \delta\mathbb{N} \to [0,1]$ be continuous such that $K_\alpha = f_\alpha^{\leftarrow}(0)$ and $f_\alpha[\mathbb{N}] \subseteq (0,1)$. We let $f_0 : \delta\mathbb{N} \to [0,1]$ be the continuous map determined by $f_0(k) = \frac{1}{2} + 2^{-k-2}$; it maps L to $\{\frac{1}{2}\}$ and $\mathbb{N}$ into $(\frac{1}{2}, 1)$.

The diagonal map F of $\langle f_\alpha : \alpha \in \omega_1\rangle$ maps $\delta\mathbb{N}$ to $[0,1]^{\omega_1}$ and maps $\mathbb{N}$ into $(0,1)^{\omega_1}$.

The first coordinate f_0 ensures that $F[\mathbb{N}]$ is relatively discrete in $(0,1)^{\omega_1}$; it remains to show that it is closed and not C^*-embedded.

To see that $F[\mathbb{N}]$ is closed in $(0,1)^{\omega_1}$ observe that for every $x \in L$ there is an α such that $x \in K_\alpha$; but then $f_\beta(x) = 0$ for $\beta \ge \alpha$. It follows that $F[\mathbb{N}] = F[\delta\mathbb{N}] \cap (0,1)^{\omega_1}$.

To see that $F[\mathbb{N}]$ is not C^*-embedded in $(0,1)^{\omega_1}$ let $g : (0,1)^{\omega_1} \to [0,1]$ be continuous. We show that the closures of $g[F[A]]$ and $g[F[B]]$ intersect.

As above there is an α such that g factors through the first α coordinates, that is, there is a continuous map $h : (0,1)^\alpha \to [0,1]$ such that $g = h \circ \pi_\alpha$. Take $x \in L \setminus K_\alpha$. Then $x \in \operatorname{cl} A \cap \operatorname{cl} B$, hence $\pi_\alpha(x) \in \operatorname{cl}(\pi_\alpha[A]) \cap \operatorname{cl}(\pi_\alpha[B])$. But because $x \notin K_\beta$ for all $\beta \le \alpha$ we find that $\pi_\alpha(x) \in (0,1)^\alpha$ and hence we conclude that $h(\pi_\alpha(x)) \in \operatorname{cl}(g[F[A]]) \cap \operatorname{cl}(g[F[B]])$.

2.3. A characterization

From the foregoing example we extract a characterization of there being a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that is not C^*-embedded.

Theorem 2.1. *The following three statements are equivalent:*

(1) *There is closed copy of* $\mathbb{N}$ *in* $\mathbb{R}^{\omega_1}$ *that is not* C^**-embedded.*
(2) *There is closed copy of* $\mathbb{N}$ *in* $\mathbb{R}^{\omega_1}$ *that is not* C*-embedded.*
(3) *There is a compact space* X *with a cover consisting of* $\aleph_1$ *many zero-sets that has no countable subcover.*

Proof. That (1) implies (2) is clear.

To prove (2) implies (3) we take a countable closed and discrete subset N of $(0,1)^{\omega_1}$ that is not C-embedded. Let $K = \operatorname{cl} N \setminus N$, where we take the closure in $[0,1]^{\omega_1}$. For every $\alpha \in \omega_1$ and $i \in \{0,1\}$ we let

$A(\alpha,i)=\{x\in K: x_\alpha=i\}$. Then $\{A(\alpha,i):\langle\alpha,i\rangle\in\omega_1\times 2\}$ is a cover of K by $\aleph_1$ many G_δ-sets. We show that there is no $\alpha\in\omega_1$ such that $\{A(\beta,i):\langle\beta,i\rangle\in\alpha\times 2\}$ covers K.

Let $\alpha\in\omega_1$; we can assume that the projection $\pi_\alpha:[0,1]^{\omega_1}\to[0,1]^\alpha$ is one-to-one on N. If $\{A(\beta,i):\langle\beta,i\rangle\in\alpha\times 2\}$ covers K then for every $x\in K$ there is a $\beta\in\alpha$ such that $x_\beta\in\{0,1\}$, and hence $\pi_\alpha(x)\notin(0,1)^\alpha$. We see that $\pi_\alpha[K]$ is disjoint from $(0,1)^\alpha$ and hence that $\pi_\alpha[N]$ is closed in $(0,1)^\alpha$ and hence also C-embedded because $(0,1)^\alpha$ is metrizable. But then N would be C-embedded in $(0,1)^{\omega_1}$.

To prove that (3) implies (1) we proceed as in Section 2.2. Let X be a space as in (3) and let $\{A_\alpha:\alpha\in\omega_1\}$ be the cover by zero-sets without a countable subcover. We may assume that X has weight $\aleph_1$, for example, by choosing a sequence $\langle f_\alpha:\alpha\in\omega_1\rangle$ of continuous functions from X to $[0,1]$ such that $A_\alpha=f_\alpha^{\leftarrow}(0)$ for all α. The image K of X under the diagonal map of the sequence has the same property as X itself, where $B_\alpha=\{x\in K:x_\alpha=0\}$ defines the family of zero-sets.

The construction in Section 2.2 now yields a closed copy of $\mathbb{N}$ in $(0,1)^{\omega_1}$ that is not C^*-embedded. □

Remark 2.2. Of course $2^2=4$ is also an equivalent of statement (1), as both are true, but this theorem should be understood as a translation: to construct the desired embedding it is necessary and sufficient to construct a particular type of compact topological space.

Remark 2.3. It is interesting to see that the formally weaker statement (2) implies statement (1); what is hidden in the proof is that from the copy that is not C-embedded one constructs a copy that is not C^*-embedded by taking its closure in $[0,1]^{\omega_1}$, doubling the resulting compactification, then glueing the remainders onto each other and find a suitable embedding of the resulting space.

2.4. Yet another closed copy of $\mathbb{N}$ that is not C^*-embedded, from an Aronszajn tree

To see an application of Theorem 2.1 we create yet another closed copy of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that is not C^*-embedded, by exhibiting a space that satisfies the properties in (3) in the theorem.

We let T be an Aronszajn tree and we take its *path space* σT, where a path is a linearly ordered subset P that is also an initial segment: if $t\in P$ and $s\le t$ then $s\in P$. We view σT, via characteristic functions, as a subspace of the Cantor cube $\{0,1\}^T$. For more on this construction see [14].

If a point $x\in\{0,1\}^T$ is not in σT then there are s and t in T with $x(s)=x(t)=1$ that are incomparable, or there are s and t with $x(s)=0$, $x(t)=1$ and $s<t$. In either case x has a neighborhood that is disjoint from σT. We see that σT is closed and hence compact. The weight of σT is at most that of $\{0,1\}^T$, that is $\aleph_1$.

For $\alpha\in\omega_1$ we let K_α be the set of paths that are of length at most α. To see that K_α is closed note that $p\in K_\alpha$ iff $p\cap T_\alpha=\emptyset$. That is $K_\alpha=\sigma T\setminus\bigcup_{t\in T_\alpha}O_t$, where $O_t=\{p:t\in p\}$. The sets O_t are clopen, so the union $\bigcup_{t\in T_\alpha}O_t$ is an open F_σ-set.

Because T is uncountable no countable subfamily of $\{K_\alpha:\alpha\in\omega_1\}$ covers σT.

Note that, as every path is countable, the space σT is actually Corson-compact.

3. The connection with Aronszajn trees and lines

Each of the three constructions in the previous section uses an Aronszajn tree or line as input. The following theorem, which adds three more statements to the list in Theorem 2.1, makes precise how these structures enter the constructions.

Theorem 3.1. *The following statements are equivalent.*

(1) *There is closed copy of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that is not C^*-embedded.*

(2) *There is closed copy of* $\mathbb{N}$ *in* $\mathbb{R}^{\omega_1}$ *that is not* C*-embedded.*
(3) *There is a compact space* X *with a cover consisting of* $\aleph_1$ *many zero-sets that has no countable subcover.*
(4) *There is a compact space* X *of weight* $\aleph_1$ *with a cover consisting of* $\aleph_1$ *many zero-sets that has no countable subcover.*
(5) *The space* $\mathbb{N}^*$ *has a cover by* $\aleph_1$ *many zero-sets that has no countable subcover.*
(6) *There is an* $\omega_1 \times \omega$*-matrix* $\langle A(\alpha, n) : \langle \alpha, n\rangle \in \omega_1 \times \omega\rangle$ *of infinite subsets of* $\mathbb{N}$ *such that*
   (a) *for every countable subset* C *of* ω_1 *there is a function* $f : C \to \omega$ *such that* $\{A(\alpha, f(\alpha)) : \alpha \in C\}$ *has the strong finite intersection property, and*
   (b) *there is* no function $f : \omega_1 \to \omega$ *such that* $\{A(\alpha, f(\alpha)) : \alpha \in \omega_1\}$ *has the strong finite intersection property.*

Proof. Theorem 2.1 established the equivalence of (1), (2), and (3). In the proof that (3) implies (1) we proved implicitly that (3) implies (4) and (4) implies (1).

Clearly (5) implies (3).

To prove that (4) implies (5) we take a continuous map f from $\mathbb{N}^*$ onto X and take the preimages of the members of the given cover. This yields the desired cover of $\mathbb{N}^*$.

It remains to show that (5) and (6) are equivalent. This follows from the strong zero-dimensionality of $\mathbb{N}^*$: if Z is a zero-set in $\mathbb{N}^*$ then one can cover $\mathbb{N}^* \setminus Z$ by a countable pairwise disjoint family of clopen sets. This family can be expressed as $\{A_n^* : n \in \omega\}$, where each A_n is an infinite subset of $\mathbb{N}$.

Conversely if $\{A_n : n \in \omega\}$ is a family of infinite subsets of $\mathbb{N}$ then $\mathbb{N}^* \setminus \bigcup_{n\in\omega} A_n^*$ is a zero-set.

Thus a family $\{Z_\alpha : \alpha \in \omega_1\}$ of zero-sets of $\mathbb{N}^*$ can be represented by a matrix $\langle A(\alpha, n) : \langle \alpha, n\rangle \in \omega_1 \times \omega\rangle$ of infinite subsets of $\mathbb{N}$ such that $Z_\alpha = \mathbb{N}^* \setminus \bigcup_{n\in\omega} A(\alpha, n)^*$.

Then condition (a) expresses that no countable subfamily covers $\mathbb{N}^*$, and condition (b) expresses that the family does cover $\mathbb{N}^*$. □

The matrix $\langle A(\alpha, n) : \langle \alpha, n\rangle \in \omega_1 \times \omega\rangle$ of sets from Section 2.1, that resulted from enumerating the levels of the Aronszajn tree as $\langle t(\alpha, n) : n \in \omega\rangle$, satisfies the conditions in item (6) of Theorem 3.1.

It would seem natural to call such a matrix an Aronszajn matrix and a compact space with a cover of cardinality $\aleph_1$ by closed G_δ-sets without a countable subcover an Aronszajn compactum. This usage would conflict with that of Hart and Kunen in [8]; and, more importantly, it would not be quite correct, as we show next.

3.1. A matrix and space that are not derived from an Aronszajn tree

The three examples constructed in Section 2 all involve compact spaces with an *increasing* cover of length ω_1 by closed G_δ-sets. These spaces were constructed explicitly in the second and third example.

In the first example we get a cover of $\mathbb{N}^*$ from the matrix $\langle A(\alpha, n) : \langle \alpha, n\rangle \in \omega_1 \times \omega\rangle$ used in that example. The proof of "(5) implies (6)" in Theorem 3.1 yields the G_δ-sets $Z_\alpha = \mathbb{N}^* \setminus \bigcup_{n\in\omega} A(\alpha, n)^*$ in $\mathbb{N}^*$. The matrix has the additional property that $\{A(\alpha, n)^* : n \in \omega\}$ refines $\{A(\beta, n)^* : n \in \omega\}$ whenever $\beta < \alpha$, so that $\langle Z_\alpha : \alpha \in \omega_1\rangle$ is an increasing sequence. Finally condition (b) of (6) implies that $\mathbb{N}^* = \bigcup_{\alpha\in\omega_1} Z_\alpha$.

Here we construct a compact space of weight $\aleph_1$ with an $\aleph_1$-sized cover by closed G_δ-sets that has no countable subcover, and that is definitely not increasing. The space is a variation of Example 7 in [1].

To begin we take an injective map $f : \omega_1 \to \mathbb{R}$ with the property that for every α the image of the interval $I_\alpha = \big[\omega \cdot \alpha, \omega \cdot (\alpha + 1)\big)$ under f is dense in $\mathbb{R}$. This is easily arranged, for example by taking $\aleph_1$ many cosets of the subgroup of rationals and mapping each interval I_α onto one of these cosets.

We let X be the set of all subsets of ω_1 on which f is increasing; we identify X, via characteristic functions, with a subset of 2^{ω_1} and give it the subspace topology.

The complement of X is open: if $x \notin X$ then there are two ordinals α and β such that $x_\alpha = x_\beta = 1$, $\alpha \in \beta$, and $f(\beta) < f(\alpha)$. Then $\{y : y_\alpha = y_\beta = 1\}$ is an open set disjoint from X. It follows that X is compact.

As subsets of $\mathbb{R}$ that are well-ordered by the normal order are countable the space X is Corson compact.

It remains to exhibit a cover of X by closed G_δ-sets that has no countable subcover.

To this end we let $G_\alpha = \{x \in X : (\forall \beta \in I_\alpha)(x_\beta = 0)\}$. This is a closed G_δ-set; it is the intersection of countably many basic clopen sets: $G_\alpha = \bigcap_{\beta \in I_\alpha} \{x : x_\beta = 0\}$.

To see that $\{G_\alpha : \alpha \in \omega_1\}$ is a cover of X, let $x \in X$. Then, because $S = \{\beta : x_\beta = 1\}$ is countable, there is an α such that $S \subset \alpha$; then $S \cap I_\alpha = \emptyset$ and so $x \in G_\alpha$.

To see that no countable subfamily covers X we let $\delta \in \omega_1$. We take a subset A of $\mathbb{R}$ that is ordered in order-type $\delta + 2$ by the normal order of $\mathbb{R}$ and we list A as $\langle a_\alpha : \alpha < \delta + 2\rangle$ in increasing order. Next we take a sequence $\langle \gamma_\alpha : \alpha < \delta + 2\rangle$ of ordinals such that $\gamma_\alpha \in I_\alpha$ and $a_\alpha < f(\gamma_\alpha) < a_{\alpha+1}$ for all α. Then the set $\{\gamma_\alpha : \alpha < \delta + 2\}$ determines a point in X that is not in $\bigcup_{\alpha \in \delta} G_\alpha$.

The same argument enables one to show that the sets G_α are quite independent: given two disjoint countable sets of ordinals A and B one can find points in $\bigcap_{\alpha \in A} G_\alpha \setminus \bigcup_{\beta \in B} G_\beta$.

Via a map from $\mathbb{N}^*$ onto X we can then create a matrix that is quite different from the ones derived from Aronszajn trees.

4. Pseudo-Aronszajn compacta

Let us, for the nonce, call a compact space a *pseudo-Aronszajn compactum* if it has a cover of cardinality $\aleph_1$ by closed G_δ-sets that has no countable subcover. We let $\mathcal{A}$ denote the class of these compacta.

It is readily seen that $\mathcal{A}$ is closed under taking (compact) preimages: simply pull back the cover.

We have established that every Aronszajn continuum is in $\mathcal{A}$, and hence that a Souslin continuum is a ccc compactum in $\mathcal{A}$.

The ordinal space $\omega_1 + 1$ does not belong to $\mathcal{A}$ as every G_δ-set that contains the point ω_1 is co-countable.

Somewhat surprisingly, uncountable compact metrizable spaces may or may not all be pseudo-Aronszajn compacta. They all are under CH and they all are not under $\mathsf{MA} + \neg\mathsf{CH}$.

Proposition 4.1 *(CH). If X is compact and admits a continuous map $f : X \to \mathbb{R}$ such that $f[X]$ is uncountable, then $X \in \mathcal{A}$.*

Proof. The image $f[X]$ is in $\mathcal{A}$, as witnessed by the family of singleton subsets. □

Proposition 4.2 *($\mathsf{MA} + \neg\mathsf{CH}$). If X is compact, uncountable and hereditarily Lindelöf, then $X \notin \mathcal{A}$.*

Proof. Let $\mathcal{Z}$ be a witness of the fact that the uncountable compact hereditarily Lindelöf space X is in $\mathcal{A}$. We will derive a contradiction.

Let $X_0 = X$ and $U_0 = \bigcup_{Z \in \mathcal{Z}} \operatorname{int}_{X_0} Z$. There is a countable subfamily $\mathcal{Z}_0$ of $\mathcal{Z}$ such that $U_0 = \bigcup_{Z \in \mathcal{Z}_0} \operatorname{int}_{X_0} Z$.

Assume that for some $\alpha < \omega_1$, we defined closed sets X_β, open sets U_β, and subfamilies $\mathcal{Z}_\beta$ of $\mathcal{Z}$, for all $\beta < \alpha$.

Let $V = \bigcup_{\beta < \alpha} U_\beta$, $X_\alpha = (\bigcap_{\beta < \alpha} X_\beta) \setminus V$, and $\mathcal{S} = \bigcup_{\beta < \alpha} \mathcal{Z}_\beta$.

Inside X_α let $W = \bigcup_{Z \in \mathcal{Z}} \operatorname{int}_{X_\alpha}(Z \cap X_\alpha)$. Then $U_\alpha = V \cup W$ is open in X, and there is a countable subcollection $\mathcal{T}$ of $\mathcal{Z}$ such that $W = \bigcup_{Z \in \mathcal{T}} \operatorname{int}_{X_\alpha}(Z \cap X_\alpha)$. We let $\mathcal{Z}_\alpha = \mathcal{S} \cup \mathcal{T}$.

There is a first $\alpha \in \omega_1$ such that $U_\alpha = U_{\alpha+1}$. If $Y = X \setminus U_\alpha$ is countable, then we are clearly done. If Y is uncountable, then for every $Z \in \mathcal{Z}$, the intersection $Z \cap Y$ is nowhere dense in Y. But this contradicts $\mathsf{MA} + \neg\mathsf{CH}$, for Y is an uncountable compact ccc space with a cover by fewer than $\mathfrak{c}$ many nowhere dense sets. □

One may wonder whether $\mathsf{MA} + \neg\mathsf{CH}$ prevents more compact spaces from being pseudo-Aronszajn. We have seen that a Souslin line is a pseudo-Aronszajn compactum and we also know that $\mathsf{MA} + \neg\mathsf{CH}$ implies there are no Souslin lines. Thus we may conjecture that it implies that there are no pseudo-Aronszajn compacta that are ccc.

However, as there are pseudo-Aronszajn compacta of weight $\aleph_1$ one can construct a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ with a pseudo-Aronszajn remainder. That compactification is itself also pseudo-Aronszajn: simply add the isolated points to the cover of the remainder. Thus we see that $\mathcal{A}$ contains separable spaces.

We can strengthen the ccc assumption by making it hereditary; it is well known that having the hereditary ccc is equivalent to every relatively discrete subspace being countable, see [6, Problem 2.7.9(b)] for example. Thus, the hereditary ccc is also a weakening of the hereditary Lindelöf property and a positive answer to the following question would yield a strengthening of Proposition 4.2.

Question 3. Does $\mathsf{MA} + \neg\mathsf{CH}$ imply that uncountable compact hereditarily ccc spaces are not pseudo-Aronszajn?

We remark in passing that it is also unknown whether compact hereditarily ccc spaces are continuous images of $\mathbb{N}^*$, see [9, Question 44].

5. $\neg\mathsf{CH}$ and a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded

In section 2 we used an Aronszajn tree to guide an embedding of $\mathbb{N}$ into $\mathbb{R}^{\omega_1}$ so as to obtain a closed copy of $\mathbb{N}$ that is not C^*-embedded. In this section we use an Aronszajn tree again, this time to create closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that are C^*-embedded but not C-embedded, in models where CH fails. Thus we see that it is consistent with $\neg\mathsf{CH}$ that the answer to Question 1 be $\aleph_1$.

The embedding will be much like the one from an arbitrary Aronszajn tree but with a few changes. We shall show that the following assumption suffices to create a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that is C^*-embedded but not C-embedded.

Assumption. There are an Aronszajn tree S and a family $\{A_s : s \in S\}$ of infinite subsets of $\mathbb{N}$ such that $A_\emptyset = \mathbb{N}$ and

- if $s < t$ then $A_t \subset^* A_s$, and
- if $Y \subseteq \mathbb{N}$ then there is an ordinal α in ω_1 such that for every $s \in S_\alpha$ either $A_s \subseteq^* Y$ or $A_s \cap Y =^* \emptyset$.

Here S_α denotes the αth level of S. We also assume that every level S_α, except S_0, is infinite and that every node in S has infinitely many direct successors.

In addition we make finite modifications to each A_s so that $\{A_s : s \in S_\alpha\}$ is a partition of $\mathbb{N}$.

5.1. The construction

We shall embed $\mathbb{N}$ into the following product:

$$\Pi = C \times \prod_{1 \le \alpha < \omega_1} S_\alpha$$

where C is the subspace $\{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ of $\mathbb{R}$ and each other factor S_α has the discrete topology. This product is homeomorphic to the product $C \times \mathbb{N}^{\omega_1}$, which in turn can be embedded as a C-embedded subspace into $\mathbb{R}^{\omega_1}$.

Now we are ready to define the embedding.

To begin we set $x_k(0) = 2^{-k}$ for all k; this ensures that the image will be relatively discrete.

If $\alpha \in [1, \omega_1)$ then we set $x_k(\alpha) = s$ iff $k \in A_s$ (and $s \in S_\alpha$ of course).

This defines our copy $N = \{x_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ in Π.

N is closed in Π

Let $v \in \Pi$. Then $\langle v_\alpha : 1 \leq \alpha < \omega_1 \rangle$ is a sequence in S with $v_\alpha \in S_\alpha$ for all α.

As S is an Aronszajn tree there are α and β with $\alpha < \beta$ and such that v_α and v_β are incomparable. Let w be the predecessor of v_β in S_α. Then $A_w \cap A_{v_\alpha} = \emptyset$ and so, because $A_{v_\beta} \subset^* A_w$ the intersection $A_{v_\beta} \cap A_{v_\alpha}$ is finite.

Let U be the basic neighborhood $\{x \in \Pi : x_\alpha = v_\alpha \text{ and } x_\beta = v_\beta\}$ of v. Then $x_k \in U$ iff $k \in A_{v_\beta} \cap A_{v_\alpha}$, hence $U \cap N$ is finite.

We see that N is a locally finite and relatively discrete subset of Π, hence N is closed and discrete.

N is C^*-embedded in Π

Let $Y \subseteq \mathbb{N}$; we show that the sets $\{x_k : k \in Y\}$ and $\{x_k : k \notin Y\}$ are completely separated in Π.

Let α be such that $A_s \subseteq^* Y$ or $A_s \subseteq^* \mathbb{N} \setminus Y$ for all $s \in S_\alpha$ and divide S_α into two sets: $I = \{s \in S_\alpha : A_s \subseteq^* Y\}$ and $J = \{s \in S_\alpha : A_s \cap Y =^* \emptyset\}$.

In this way we create four subsets of $\mathbb{N}$:

(1) $Y_1 = \bigcup\{A_s \cap Y : s \in I\}$,
(2) $Y_2 = \bigcup\{A_s \cap Y : s \in J\}$,
(3) $Z_1 = \bigcup\{A_s \setminus Y : s \in J\}$, and
(4) $Z_2 = \bigcup\{A_s \setminus Y : s \in I\}$.

To begin we observe that $Y_2 \cup Z_2$ intersects every A_s in a finite set. Because $\{A_s : s \in S_\alpha\}$ is a partition of $\mathbb{N}$ this implies, as in the proof that N is closed, that $D = \{x_k \restriction (\alpha + 1) : k \in Y_2 \cup Z_2\}$ is a closed and discrete subset of the subproduct $\Pi_\alpha = C \times \prod_{1 \leq \beta \leq \alpha} S_\beta$. This product is separable and metrizable, hence D is C-embedded in this subproduct, this implies that in particular, $\{x_k \restriction (\alpha + 1) : k \in Y_2\}$ and $\{x_k \restriction (\alpha + 1) : k \in Z_2\}$ are completely separated in Π_α.

Furthermore, because N is relatively discrete in the subproduct the set D is disjoint from the closure of $\{x_k \restriction (\alpha + 1) : k \in Y_1 \cup Z_1\}$.

Finally the αth coordinates of the x_k ensure that $\{x_k(\alpha) : k \in Y_1\}$ and $\{x_k(\alpha) : k \in Z_1\}$ are disjoint. And because S_α has the discrete topology this shows that $\{x_k \restriction (\alpha+1) : k \in Y_1\}$ and $\{x_k \restriction (\alpha+1) : k \in Z_1\}$ are completely separated in Π_α.

We conclude that $\{x_k \restriction (\alpha + 1) : k \in Y\}$ and $\{x_k \restriction (\alpha + 1) : k \notin Y\}$ are completely separated in Π_α.

N is not C-embedded in Π

We show that the function $f : N \to \mathbb{R}$ that maps x_k to k has no continuous extension to Π.

Assume $g : \Pi \to \mathbb{R}$ is continuous and such that $g(x_k) = k$ for all k. As before we can factor g through a partial product: there are a δ and a continuous function $h : C \times \prod_{1 \leq \alpha < \delta} S_\alpha$ such that $g = h \circ \pi_\delta$.

Let $s \in S_\delta$ and let s_α denote its predecessor in S_α, for $\alpha \in [1, \delta)$. Take such an α, then by construction $A_s \subseteq^* A_{s_\alpha}$ and so $x_k(\alpha) = s_\alpha$ for all but finitely many $k \in A_s$.

Because A_s is infinite this implies that the point v, with $v(0) = 0$ and $v(\alpha) = s_\alpha$ for $\alpha \in [1, \delta)$, is an accumulation point of $\{\pi_\delta(x_k) : k \in A_s\}$ and hence that $h(v) > k$ for all k, a contradiction.

5.2. A model

To finish we show that our assumption is actually consistent with the negation of CH. Chapters VII and VIII of [11] provide all the forcing background that we need.

We let S be an Aronszajn tree as constructed in [11, Theorem II.5.9]. This tree is a subtree of the subtree T of ${}^{<\omega_1}\omega$ that consists of all finite-to-one sequences of natural numbers and it has the property that for every $s \in S$ the set of direct successors is $\{s * n : n \in \omega\}$. This tree has the advantage that if a partial order preserves ω_1 then it will not add an ω_1-branch to it, as such a branch would give a finite-to-one map from ω_1 to ω.

Next we work Exercise VIII(A10) in [11], that is, we perform an ω_1 long finite support iteration of σ-centered partial orders to create an ultrafilter on $\mathbb{N}$ of character $\aleph_1$.

More explicitly: we form a sequence $\langle M_\alpha : \alpha \le \omega_1\rangle$ of models, together with sequences $\langle u_\alpha : \alpha \in \omega_1\rangle$ and $\langle U_\alpha : \alpha \in \omega_1\rangle$. Together these satisfy

(1) u_α is an ultrafilter on $\mathbb{N}$ in M_α,
(2) $M_{\alpha+1}$ is obtained by forcing over M_α with the partial order $\mathbb{E}(u_\alpha)$ described below, which produces a subset U_α of $\mathbb{N}$ such that $U_\alpha \subseteq^* X$ for all $X \in u_\alpha$, and
(3) $u_{\alpha+1}$ extends $u_\alpha \cup \{U_\alpha\}$.

For a free ultrafilter u on $\mathbb{N}$ we define the partial order

$$\mathbb{E}(u) = \{\langle s, U\rangle : s \in [\mathbb{N}]^{<\omega}, U \in u\}$$

ordered by $\langle s, U\rangle \le \langle t, V\rangle$ iff

- $t \subseteq s$,
- $U \subseteq V$, and
- $s \setminus t \subseteq V$.

If G is a generic filter on $\mathbb{E}(u)$ then $E = \bigcup\{s : (\exists U \in u)(\langle s, U\rangle \in G)\}$ is an infinite subset of ω such that $E \subseteq^* U$ for all $U \in u$.

The assumption

The iteration yields a ccc partial order with a dense subset of cardinality $\mathfrak{c}$. Therefore it preserves all cardinal arithmetic from the ground model M_0. Thus M_{ω_1} can be made to satisfy any consistent cardinal arithmetic, in particular $2^{\aleph_0}$ can be anything it ought to be.

We define a family $\{A_s : s \in S\}$ of infinite subsets as in our assumption. We start by setting $A_\emptyset = \mathbb{N}$.

For the successor steps we fix a definable bijection $f : \mathbb{N}^2 \to \mathbb{N}$, say $f(m, n) = \frac{1}{2}(m+n)(m+n+1)+m$ (compare [4, p. 257]).

Going from α to $\alpha+1$ we assume that $\{A_s : s \in S_\alpha\}$ is in M_α and build $\{A_t : t \in S_{\alpha+1}\}$ in $M_{\alpha+1}$. We take for every $s \in S_\alpha$ the counting function $c_s : \mathbb{N} \to A_s$; these functions belong to M_α. For every $s \in S_\alpha$ and $n \in \mathbb{N}$ we define $A_{s*n} = c_s\big[f[\{n\} \times U_\alpha]\big]$. In words: we use $c_s \circ f$ to create a partition of A_s in M_α and then copy U_α to each element of that partition by maps in M_α.

In this way we ensure that each A_{s*n} has the property that U_α has: for every subset Y of $\mathbb{N}$ that is in M_α we have $A_{s*n} \subseteq^* Y$ or $A_{s*n} \cap Y =^* \emptyset$. The resulting family $\{A_t : t \in S_{\alpha+1}\}$ is defined from U_α and members of M_α, hence it is in $M_{\alpha+1}$.

In case $\alpha \in \omega_1$ is a limit the partial family $\{A_s : s \in \bigcup_{\beta\in\alpha} S_\beta\}$ belongs to M_α. So in M_α we can find a family $\{A_t : t \in S_\alpha\}$ of infinite subsets of $\mathbb{N}$ such that $A_t \subseteq^* A_s$ whenever $s < t$.

To see that the resulting family has the second property in our assumption we let Y, in M_{ω_1}, be a subset of $\mathbb{N}$. By well-known properties of finite-support iterations of ccc partial orders there is an $\alpha \in \omega_1$ such that $Y \in M_\alpha$. But then for all $s \in S_{\alpha+1}$ we have $A_s \subseteq Y$ or $A_s \cap Y =^* \emptyset$.

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