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Unbounded Domain Modeling in Axisymmetric Rotor Wake Analysis

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Abstract. Mitigating the effects of wind turbine wakes is a central part of wind farm design. This paper proposes a spectral solver for the axisymmetrical Ainslie wake model based on modified Laguerre basis functions over the semi-infinite radial domain and a marching scheme in the downstream direction. This orthogonal basis promises fast convergence and a low number of DOFs for discretizing the continuity and axial momentum equation, promising a computationally efficient method. The numerical implementation of the model could not be finished in time; preliminary results are presented but still show non-negligible conservation errors. The focus of this work lies, therefore, on a detailed derivation of the method and a discussion of the sources of numerical errors in the nonlinear terms, and due to the truncation of the spectral basis, a comparison of the solver outputs to other methods remains part of the ongoing work.

1. Introduction

Industrial wake models require speed and accuracy to compute AEP losses and induced loads for the wake effects. Typically, more refined wake models come with a steep increase in computational costs [3]. This paper investigates a numerical basis that could reduce the system's computational cost while increasing the order of the method. To that end, a spectral method based on modified Laguerre functions is proposed. The motivation behind these particular functions comes from an inherent fit to the boundary conditions, an explicit expression for the triple integral, and the performance [6] on an unbound radial domain.

Consequently, the spectral method has a lower degree of freedom and spectral convergence, and the modified Laguerre functions are infinitely differentiable. Employing these modified Laguerre functions, a two-equation wake model with a fixed viscosity has been reduced to a simple fixed-point iteration, which may be straightforwardly solved.

This explorative study into the potential of Laguerre functions may motivate the use of this basis for similar problems. If the model obtains an increase in computational speed, the Laguerre basis could be a logical pick for wake modeling and similarly shaped problems. In particular, axisymmetric problems can be described well with this paper's formulation.

The fields of interest for this implementation are wake and wind modeling in aerodynamic models for wind turbines. This research intends to prove the concept of implementing a Laguerre basis for axisymmetric flow modeling. Although Ainslie's model does not necessitate such a high-order basis, the result contains a sophisticated method which shows similarly defined problems with higher requirements for the numerical basis.

In Section 2, the details of the Modified Laguerre basis are laid out, which are then manipulated to arrive at a concise system solution for the governing equations. Section 3 presents the verification of the model by modeling the diffusion term and the full system separately. Section 4 shows the model's results.

2. Methodology

The equations described by Ainslie J F[4] provide the physical basis for the wake model. The functions used as the basis are a combination of a scaled Laguerre polynomial of the order k and a scaled Gaussian weight

$$\lambda_k(r) = \lambda_k\left(\frac{r^2}{r_0^2}\right) = L_k\left(\frac{r^2}{r_0^2}\right)\mathrm{e}^{-\frac{r^2}{2r_0^2}} \tag{1}$$

Asymptotically for $r \to \infty$, the functions $\lambda_k(r)$ converge to zero. This property may be exploited to develop a velocity description that approaches the free-stream velocity U_∞, hence naturally enforcing a Dirichlet boundary condition of the axial velocity component at $r \to \infty$. The resulting ansatz is

$$U(x,r) = U_\infty - \sum_{k=0}^{K} \underbrace{L_k\left(\frac{r^2}{r_0^2}\right)\mathrm{e}^{-\frac{r^2}{2r_0^2}}}_{\lambda_k(r)} \left[u_{k,r}\frac{x - x_l}{x_r - x_l} + u_{k,l}\frac{x_r - x}{x_r - x_l}\right] \tag{2}$$

where $u_{k,r}$ is the velocity coefficient for the k-th Laguerre function for the downstream component in the marching scheme (the "right" side, index r), and $u_{k,l}$, represents the upstream component (the "left" side, index l).

Laguerre polynomials may also be raised to an order α. The generalized form of Laguerre polynomials are defined by the Rodrigues formula [1]

$$L_k^\alpha(r) = \frac{r^{-\alpha}}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}r} - 1\right)^k r^{k+\alpha} \tag{3}$$

In this work, this particular notation is only needed for solving the derivative of the Laguerre basis

$$\begin{aligned}\lambda_k'(r) &= \frac{d}{dr}\left(L_k\left(\frac{r^2}{r_0^2}\right)\mathrm{e}^{-r^2/2r_0^2}\right) = -\frac{r}{r_0^2}\left(L_k\left(\frac{r^2}{r_0^2}\right) + 2L_{k-1}^1\left(\frac{r^2}{r_0^2}\right)\right)\mathrm{e}^{-r^2/2r_0^2}\\ &= -\frac{r}{r_0^2}\left(L_k\left(\frac{r^2}{r_0^2}\right) + 2\sum_{i=0}^{k-1} L_i\left(\frac{r^2}{r_0^2}\right)\right)\mathrm{e}^{-r^2/2r_0^2}\\ \lambda_k'(r) &= -\frac{r}{r_0^2}\left(\lambda_k(r) + 2\sum_{i=0}^{k-1}\lambda_i(r)\right) \end{aligned} \tag{4}$$

Alternatively, the derivative can also be expressed using $L_k = L_k^1 - L_{k-1}^1$[1] as

$$\lambda_k'(r) = -\frac{r}{r_0^2}\left(L_k^1\left(\frac{r^2}{r_0^2}\right) + L_{k-1}^1\left(\frac{r^2}{r_0^2}\right)\right)\mathrm{e}^{-r^2/2r_0^2} \tag{5}$$

Another benefit of the Laguerre functions is their orthogonality relation[1]

$$\int_0^\infty \lambda_k(r)\lambda_m(r)\, r\mathrm{d}r = \frac{r_0^2}{2}\int_0^\infty L_k(\frac{r^2}{r_0^2})L_m(\frac{r^2}{r_0^2})\mathrm{e}^{-s^2}\mathrm{d}r = \frac{r_0^2}{2}\delta_{k,m} \tag{6}$$

After the Galerkin projection, the non-linear terms in the model reduce to triple product integral over the Laguerre basis functions, for which the explicit solution exists[5]

$$\int_0^\infty \lambda_k(r)\lambda_q(r)\lambda_m(r) r\,\mathrm{d}r = \frac{r_0^2}{2}\int_0^\infty L_k(s)L_q(s)L_m(s)\mathrm{e}^{-\frac{3s^2}{2}}\,\mathrm{d}s = \frac{r_0^2}{2}C_{k,q,m} \tag{7}$$

2.1. Marching Scheme

For the downstream discretization, a Finite Element marching scheme is employed. The functions form a piecewise linear basis, of which the first derivatives are constants. For a single element in the marching scheme, the right and left sides are discretized, respectively, by $t(x) = \frac{x-x_l}{x_r-x_l}$ and $(1-t(x))$

$$U_k(x) = [u_{k,r}t(x) + u_{k,l}(1-t(x))] \tag{8}$$

The average of the two linear functions forms the Galerkin projection in the axial dimension. Thus, the projection function becomes: $b_m(x,r) = \lambda_m(r)\int_{x_l}^{x_r}\mathrm{d}x$.

This method combines an infinitely often continuously differentiable basis in the radial dimension with a piecewise linear and only globally continuous but not differentiable basis in the axial direction. This odd choice was made to simplify the discretisation, focus on the spectral method for the radial domain only, and solve a simple problem (Ainslie's wake model) with it. The authors hope this radial basis will be useful for a wider range of axisymmetrical problems.

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2.2. The Natural Basis for the Radial Velocity

Deriving the Unbounded Domain Laguerre model starts with coming to an expression for V(x,r). Firstly, unlike the original publication, the mass equation(9) requires additional arithmetic to arrive at an elegant result.

Although the divergence is defined in a three-dimensional cylindrical space, the basis is azimuthally invariant. As a result, the azimuthally invariant terms in the divergence cancel out, and the azimuthal components are neglected.

$$\nabla \cdot \begin{pmatrix} U(x,r) \\ V(x,r) \\ W(x,r) \end{pmatrix} = \frac{\partial U(x,r)}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV(x,r)) + \underbrace{\frac{1}{r}\frac{\partial W}{\partial \theta}}_{\frac{\partial \mathbf{W}}{\partial \theta}=0} = 0 \tag{9}$$

$$\frac{\partial U(x,r)}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV(x,r)) = 0 \tag{10}$$

The radial velocity component's basis function is yet to be defined. Referring to the mass equation, there is a way of coming to a natural definition for the velocity component. One gets from the definition in equation 10:

$$\frac{1}{r}\frac{\partial}{\partial r}(rV(x,r)) = -\frac{\partial U(x,r)}{\partial x}$$

$$rV(x,r) = -\int_0^r \rho \frac{\partial U(x,\rho)}{\partial x} \mathrm{d}\rho \tag{11}$$

ρ is the alternative notation for the radial dimension but serves to differentiate the integrated domain. By including the radial definition for U, an expression may be derived.

$$rV(x,r) = -\sum_{k=0}^{N_r} \frac{\partial U_k(x)}{\partial x} \int_0^r \rho \lambda_k(\rho) \mathrm{d}\rho \tag{12}$$

By integrating the derivative of a laguerre function and some arithmetic effort, one comes to a natural expression for the axial velocity

$$V(x,r) = \sum_{k=0}^{N_r} \frac{1}{r}(1-\lambda_k(r))[v_{k,r}t(x) + v_{k,l}(1-t(x))] \tag{13}$$

One should note that this expression for V supports the axisymmetric boundary condition and the condition for V going to zero for $r \to \infty$. At r=0, the exponential part of the Laguerre function (λ) tends faster to 1 than $\frac{1}{r}$ tends to ∞, thus, enforcing the axisymmetric boundary condition

$$\lim_{r\to 0} \frac{1}{r}(1-\lambda_k(r)) = 0 \tag{14}$$

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2.3. Mass equation

The equations can be worked out now that both velocity components are expressed. Filling in the terms of equation 10

$$\sum_{k=0}^{N_r} \iiint\limits_{\Omega} b_m(x,r)\frac{\partial U(x,r)}{\partial x} r\mathrm{d}r\mathrm{d}x + \sum_{k=0}^{N_r} \iiint\limits_{\Omega} b_m(x,r)\frac{1}{r}\frac{\partial}{\partial r}(rV(x,r))r\mathrm{d}r\mathrm{d}x = 0 \tag{15}$$

Moving onto the first term

$$\begin{aligned}\iint\limits_{\Omega} b_m(x,r)\frac{\partial U(x,r)}{\partial x} r\,\mathrm{d}r\,\mathrm{d}x &= \iint\limits_{\Omega} \sum_{k=0}^{N_r} \lambda_k(r)\lambda_m(r)\left[\frac{1}{\Delta x}u_{kr} - \frac{1}{\Delta x}u_{kl}\right] r\,\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}\phi \\ &= \frac{r_0^2}{2}\sum_{k=0}^{N_r} \delta_{k,m}\left[u_{kr} - u_{kl}\right]\end{aligned} \tag{16}$$

and the second term

$$\begin{aligned}\iint\limits_{\Omega} b_m(x,r)\frac{1}{r}\frac{\partial (rV)}{\partial r} r\,\mathrm{d}r\,\mathrm{d}x &= -\iint\limits_{\Omega} \sum_{k=0}^{N_r} \left[t(x)v_{kr} + (1-t(x))v_{kl}\right] \lambda_k'(r)\lambda_m(r)\frac{r}{r}\,\mathrm{d}r\,\mathrm{d}x \\ &= -\iint\limits_{\Omega} \sum_{k=0}^{N_r} \left[t(x)v_{kr} + (1-t(x))v_{kl}\right] \underbrace{\left(-\frac{r}{r_0^2}\right)\left(\lambda_k(r) + 2\sum_{i=0}^{k-1}\lambda_i(r)\right)}_{=\lambda_k'(r)} \lambda_m(r)\frac{r}{r}\,\mathrm{d}r\,\mathrm{d}x \\ &= -\sum_{k=0}^{N}\left[\frac{1}{2}v_{kr} + \frac{1}{2}v_{kl}\right]\left(-\frac{1}{r_0^2}\right)\frac{r_0^2}{2}\left(\delta_{k,m} + 2\sum_{i=0}^{k-1}\delta_{i,m}\right) \\ &= \frac{1}{4}\sum_{k=0}^{N}\left[v_{kr} + v_{kl}\right]\left(\delta_{k,m} + 2\sum_{i=0}^{k-1}\delta_{i,m}\right)\end{aligned} \tag{17}$$

with $\int_{x_l}^{x_r} 1 - t(x)\,\mathrm{d}x = \int_{x_l}^{x_r} t(x)\,\mathrm{d}x = \Delta x/(2\Delta x) = 1/2$.

2.4. Diffusion term

The general formulation of the two-dimensional diffusion equation is the starting point for the term.

$$\iint\limits_{\Omega} \nu_T \nabla \cdot \left(\nabla \begin{bmatrix} U(x,r) \\ V(x,r) \end{bmatrix}\right) r\,\mathrm{d}r\,\mathrm{d}x \tag{18}$$

The viscosity is kept to a constant value in this paper and, thus, may be taken out of the integral. After applying the product rule to the filled-in equation 18 and applying the divergence theorem

$$\bar{\nu}_T \iint\limits_{\Omega} b_m(x,r)\nabla\cdot(\nabla U(x,r))\,\mathrm{d}V = \underbrace{\bar{\nu}_T \iint\limits_{\Omega} \nabla\cdot(b_m(x,r)\nabla U(x,r))\,\mathrm{d}V}_{\oiint\limits_{\partial\Omega} b_m(x,r)\bar{\nu}_T(\vec{n}\cdot\nabla U(x,r))\mathrm{d}A} - \bar{\nu}_T \iint\limits_{\Omega} (\nabla b_m(x,r))\cdot(\nabla U(x,r))\,\mathrm{d}V$$

The only nonzero boundaries of the axisymmetric system are the two boundaries $\partial\Omega_1$, $\partial\Omega_2$, for which the integrals become

$$\oiint\limits_{\partial\Omega} b_m(x,r)\bar{\nu}_T(\vec{n}\cdot\nabla U(x,r))\,\mathrm{d}A = \underbrace{-\oiint\limits_{\partial\Omega_1:x=x_l} b_m(x,r)\bar{\nu}_T(\vec{n}\cdot\nabla U(x_l,r))\,\mathrm{d}A \quad \oiint\limits_{\partial\Omega_2:x=x_r} b_m(x,r)\bar{\nu}_T(\vec{n}\cdot\nabla U(x_r,r))\,\mathrm{d}A} \tag{19}$$

$$\oiint\limits_{\partial\Omega} b_m(x,r)\bar{\nu}_T(\vec{n}\cdot\nabla U(x,r))\,\mathrm{d}A = \frac{\bar{\nu}_T}{2\Delta x}\mathbb{I}[\mathbf{u}_r - \mathbf{u}_l] \tag{20}$$

thus, the diffusion term reads

$$\bar{\nu}_T \iint\limits_{\Omega} b_m(x,r)\nabla\cdot(\nabla U(x,r))\,\mathrm{d}V = \frac{\bar{\nu}_T}{2\Delta x}\mathbb{I}[\mathbf{u}_d - \mathbf{u}_u] - \bar{\nu}_T \iint\limits_{\Omega} (\nabla b_m(x,r))\cdot(\nabla U(x,r))\,\mathrm{d}V \tag{21}$$

Singling out the most right-hand term and writing it out becomes

$$-\bar{\nu}_T \iint\limits_{\Omega} \frac{\partial b_m(x,r)}{\partial r}\frac{\partial U(x,r)}{\partial r} r\,\mathrm{d}r\,\mathrm{d}x = -\bar{\nu}_T \iint\limits_{\Omega} \sum_{k,q=0}^{Nr} \lambda'_m(r)\lambda'_k(r)\left(t(x)u_{k,r} + (1-t(x))u_{k,l}\right) r\,\mathrm{d}r\,\mathrm{d}x \tag{22}$$

and solving for the radial dependent term only to make the system parabolic, using equations 6 and 5, gives

$$\int_0^\infty \lambda'_k(r)\lambda'_m(r) r\,\mathrm{d}r = \int_0^\infty \frac{r}{r_0^2}\left(L_k^1\left(\frac{r^2}{r_0^2}\right) + L_{k-1}^1\left(\frac{r^2}{r_0^2}\right)\right)\frac{r}{r_0^2}\left(L_k^1\left(\frac{r^2}{r_0^2}\right) + L_{k-1}^1\left(\frac{r^2}{r_0^2}\right)\right)e^{-r^2/r_0^2}\,r\,\mathrm{d}r$$
$$= \underbrace{\frac{1}{2r_0^2}\sum_{k=0}^{N_r}\left((2k+1)\delta_{k,m} + k\delta_{k-1,m} + m\delta_{k,m-1}\right)}_{\mathbf{D}} \tag{23}$$

$$-\bar{\nu}_T \iint_\Omega \frac{\partial b_m(x,r)}{\partial r}\frac{\partial U(x,r)}{\partial r} r\,\mathrm{d}r\,\mathrm{d}x = -\bar{\nu}_T \frac{1}{2r_0^2}\Delta x \mathbf{D}\left(\frac{1}{2}u_{k,r} + \frac{1}{2}u_{k,l}\right) \tag{24}$$

2.5. Advection terms

Starting from the axial dominant advection term, this term may be straightforwardly solved

$$\iint_\Omega U(x,r)\frac{\partial U(x,r)}{\partial x}\lambda_m(r) r\,\mathrm{d}r\,\mathrm{d}x = \underbrace{\sum_{k,l=0}^{N_r}\int_0^\infty \lambda_k(r)\lambda_l(r)\lambda_m(r) r\,\mathrm{d}r}_{\frac{r_0^2}{2}\mathbf{C}_m}$$
$$\int_{x_l}^{x_r}\left[t(x)u_{k,r} + (1-t(x))u_{k,l}\right]\left[u_{l,r} - u_{l,l}\right]\mathrm{d}x \tag{25}$$

$$\iint_\Omega U(x,r)\frac{\partial U(x,r)}{\partial x}\lambda_m(r) r\,\mathrm{d}r\,\mathrm{d}x = \frac{r_0^2}{4}\vec{u}\left(\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\otimes \mathbf{C}_m\right)\vec{u} \tag{26}$$

$$\iint_\Omega U(x,r)\frac{\partial U(x,r)}{\partial x}\lambda_m(r) r\,\mathrm{d}r\,\mathrm{d}x = \frac{r_0^2}{4}(\vec{u}_r + \vec{u}_l)\mathbf{C}_m(\vec{u}_r - \vec{u}_l) \tag{27}$$

introducing the vectors $\vec{u}_r = (u_{0,r}, \ldots, u_{N_r,r})^T$ and $\vec{u}_l = (u_{0,l}, \ldots, u_{N_r,l})^T$.

The final term to be solved is the advection dependent on the radial velocity

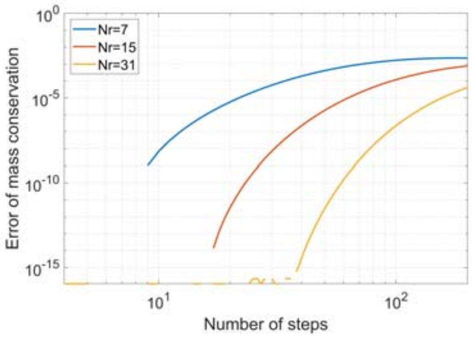
$$\iint_\Omega V(x,r)\frac{\partial U(x,r)}{\partial r}\lambda_m(r) r\,\mathrm{d}r\,\mathrm{d}x = \sum_{k,q=0}^{N_r}\int_0^\infty \lambda_k'(r)(1-\lambda_q(r))\lambda_m(r) r\,\mathrm{d}r$$
$$\cdot\int_{x_l}^{x_r}\left[t(x)u_{k,r} + (1-t(x))u_{k,l}\right]\left[t(x)v_{q,r} + (1-t(x))v_{q,l}\right]\mathrm{d}x \tag{28}$$

$$= \sum_{k,q=0}^{N_r}\int_0^\infty \lambda_m(r)(\lambda_k(r) + 2\sum_{i=0}^{k-1}\lambda_i(r))(1-\lambda_q(r))\,\mathrm{d}r$$
$$\cdot\frac{1}{6}\left[2u_{k,r}v_{qr} + u_{k,l}v_{qr} + u_{k,r}v_{ql} + 2u_{k,l}v_{ql}\right] \tag{29}$$

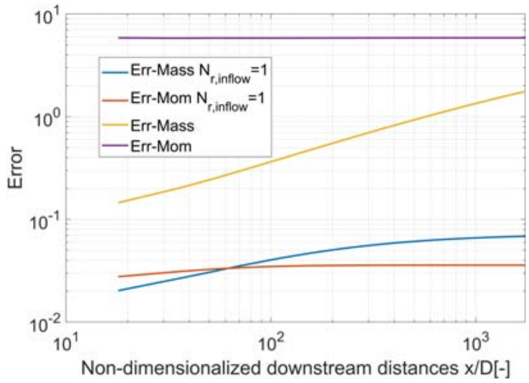
$$\iint_\Omega V(x,r)\frac{\partial U(x,r)}{\partial r}\lambda_m(r) r\,\mathrm{d}r\,\mathrm{d}x = \frac{1}{6}\underbrace{\sum_{k,q=0}^{N_r}\delta_{k,m} - C_{k,q,m} + 2\sum_{i=0}^{k-1}\delta_{i,m} - C_{i,q,m}}_{\mathbf{B}_m} V_q\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}U_k \tag{30}$$

in matrix notation,

$$\iint_\Omega V(x,r)\frac{\partial U(x,r)}{\partial r}\lambda_m(r) r\,\mathrm{d}r\,\mathrm{d}x = \frac{1}{6}\begin{bmatrix}\vec{v}_r & \vec{v}_l\end{bmatrix}\left(\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\otimes\mathbf{B}_m\right)\begin{bmatrix}\vec{u}_r\\ \vec{u}_l\end{bmatrix} \tag{31}$$



(a) Mass conservation for the heat equation solved for different numbers of radial modes (Nr) used to model the domain. The number of inflow modes remains constant.



(b) Mass and momentum conservation for the full system described in the methodology. $Nr = 1$ corresponds to a boundary input of one radial mode, whereas the regular inflow represents half the domain modes. The domain is modeled with 64 modes.

Figure 1: Errors in the conservation quantities. Subfigure (a) focuses solely on the diffusion term, whereas (b) uses the full equations.

The non-linearity of the terms necessitates the inclusion of a solver. In this research, the Newton-Raphson method is chosen. The fixed point form of the equations

$$\frac{r_0^2}{4}(\vec{u}_r + \vec{u}_l)\mathbf{C}_m(\vec{u}_r - \vec{u}_l) - \frac{1}{6}\begin{bmatrix}\vec{v}_r & \vec{v}_l\end{bmatrix}\left(\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \otimes \mathbf{B}_m\right)\begin{bmatrix}\vec{u}_r\\ \vec{u}_l\end{bmatrix} - \frac{1}{4r_0^2}(\mathbf{D})(\vec{u}_r - \vec{u}_l) = 0 \quad (32)$$

With the fixed point form of the equations in place, the results may be computed straightaway. Beforehand, the model should be verified.

3. Verification

Although the resulting model from the derivations above comes about quite neatly, the results show poor, conservative properties and instability for higher numbers of modes. The system has been extensively tested and debugged to determine the issue. The continuity of mass and momentum for the separate diffusion terms (heat equation) and the full system are used as verification.

Figure 1a shows the conservation of mass for the heat equation. The error remains at machine precision up to a certain point; the limit corresponds to the point where the diffusion can no longer expand, and the error rises due to truncation.

The tri-diagonal structure of the diffusion operator lets one additional coefficient be occupied with a finite value for each diffusion step. Starting with only the order-zero coefficient $\neq 0$, a model truncated at N has all entries in the coefficient vector $\neq 0$ after

$N - 1$ steps. After the N-th step, the truncated system can no longer represent the solution. Figure 1a shows the error in the pure diffusion equation (preservation of the integral under the function). The integral is preserved until the highest-order coefficient is occupied; beyond that point, the integral is not preserved, and the error increases.

This has consequences for the solver. Due to the dense matrices $\mathbf{B}_m$ and $\mathbf{C}_m$, the Jacobian of the Newton-Raphson iteration is dense, resulting in all vector entries $\neq 0$ after one single solver step already. Despite the values in the higher-order coefficients being small, the truncation of the diffusion term leads to a non-recoverable loss of mass and momentum after the second downstream step. This problem is enhanced if more coefficients are $\neq 0$ at the inflow, as shown in Figure 1b.

The conservation quantities for the model with a single inflow mode show a high initial error, as demonstrated in figure 1b. The regular inflow number used for modeling the wake corresponds to half the number of modes applied in the domain, which in this example of 64 modes means 32 inflow coefficients. A single inflow coefficient is used as the inflow condition to assess the effect on the error.

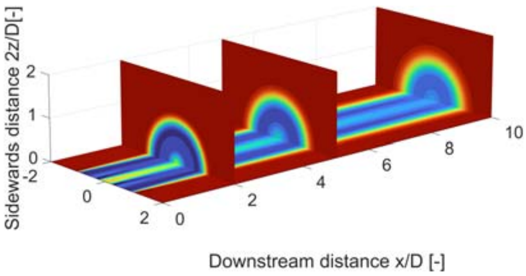
Figure 1b demonstrates the improved performance of the model for a lower number of inflow coefficients. Besides a significant decrease in starting error, the slope for the mass conservation error is smaller. Nonetheless, both inflow conditions show non-negligible starting errors, suggesting another significant error source.

The heat equation is included to assess the performance of the diffusion term. Not only does the fixed viscosity affect the scaling of the diffusion term, but so does the radial scaling (r_0). From equation 32, the direction for the relation becomes obvious. The first advection term increases while the diffusion decreases for each increment in r_0. Changing the current implementation of the radial scaling to a coefficient that is not fixed at the start of the domain but a function of downstream position and wind speed may yield improvements for a similar implementation.

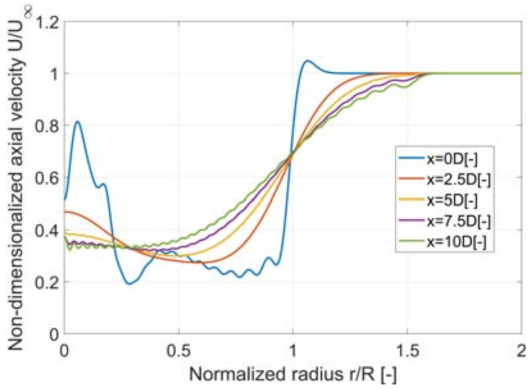
4. Results

This paper has focused on the mathematical system underlying unbounded domain modeling. Based on the wind turbine input of [7], the flow field in subfigure 2a with corresponding cross-sections in subfigure 2b is obtained. The results have been computed for a constant eddy viscosity. The model, as established, shows the typical wake profile of a wind turbine, including its expansion and diffusion.
Interestingly, similarly as for an under-expanded domain in finite element methods, without enough radial expansion available, the truncation error hinders wake expansion. However, focussing on figure 2b, it becomes obvious that numerical instability is starting to develop. Often, the method does not converge for higher diffusive flows, even when the step size is similarly decreased.

Comparing performance with other numerical implementations of the Laguerre functions is impractical with the model in its current state. The comparison of performance should be done at a later time.



(a) The flowfield corresponds to 64 radial modes and a prescribed viscosity of 80.



(b) Cross-sections velocity profiles at different downstream distances.

Figure 2: Velocity in downstream direction demonstrated for the full flow field and cross sections. The eddy viscosity has been kept constant for the domain.

5. Conclusion

This paper shows an implementation of the modified Laguerre functions for modeling wind turbine wakes. Successfully, the DOF may be kept low and still provide a smooth result. Laguerre functions are a logical basis for wind turbine wake modeling or similarly defined problems.

Although the method is unbounded, it encounters truncation errors when the solution expands. Additionally, there is an unexplained high starting error even for an inflow of a vector containing a single coefficient. These errors are non-negligible and require further work for the method to be used for any real-life application. Any comparison with other Ainslie models is counterproductive for now.

In the current approach, the dependency of the solution on the scaling constant demonstrates a critical flaw. Coupling the viscosity with the scaling variable could potentially enhance the method. Ideally, the radial scaling expands with the wake along the domain, and with the viscosity responding accordingly, the diffusion could still be modeled satisfactorily.

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