Statistics of Temperature Fluctuations in an Electron System out of Equilibrium

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We study the statistics of the fluctuating electron temperature in a metallic island coupled to reservoirs via resistive contacts and driven out of equilibrium by either a temperature or voltage difference between the reservoirs. The fluctuations of temperature are well defined provided that the energy relaxation rate inside the island exceeds the rate of energy exchange with the reservoirs. We quantify these fluctuations in the regime beyond the Gaussian approximation and elucidate their dependence on the nature of the electronic contacts.

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The temperature of a given system is well defined in the case where the system is coupled to and in equilibrium with a reservoir at that temperature. Out of equilibrium, the temperature is determined by a balance of the different heat currents from or to the system [1]. However, this applies only to the average temperature: the heat currents fluctuate, giving rise to temperature fluctuations. Although the equilibrium fluctuations have been discussed in textbooks [2], their existence was still debated around the turn of the 1990s [3].

In this Letter we generalize the concept of temperature fluctuations to the nonequilibrium case by quantifying their statistics in an exemplary system: a metal island coupled to two reservoirs (see Fig. 1). The island can be biased either by a voltage or temperature difference between the reservoirs. In this case, the temperature of the electrons is not necessarily well defined. The electron-electron scattering inside the island may, however, provide an efficient relaxation mechanism to drive the energy distribution of the electrons towards a Fermi distibution with a well defined, but fluctuating, temperature [1,4]. Here we assume this quasiequilibrium limit where the time scale τ_{e-e} of internal relaxation is much smaller than the scale τ_E related to the energy exchange with the reservoirs.

In equilibrium, the only relevant parameters characterizing the temperature fluctuation statistics are the average temperature T_a, fixed by the reservoirs, and the heat capacity $C = \pi^2 k_B^2 T_a/(3\delta_I)$ of the system. The latter is inversely proportional to the effective level spacing δ_I on the island. In terms of these quantities, the probability of the electrons being at temperature T_e reads [2,5]

$$P_{\rm eq}(T_e) \propto \exp\left[-\frac{C(T_e - T_a)^2}{k_B T_a^2}\right] = \exp\left[-\frac{\pi^2 k_B (T_e - T_a)^2}{3T_a\delta_I}\right], \quad (1)$$

corresponding to the Boltzmann distribution of the total energy of the island. The probability has a Gaussian form even for large deviations from T_a, apart from the fact that it naturally vanishes for $T_e < 0$. From this distribution we can, for example, infer the variance, $\langle(\Delta T_I)^2\rangle = k_B T^2/C$. As we show below, the scale of the probability log, $\ln P \sim T_a/\delta_I$, is the same for the nonequilibrium case, while its dependence on (T_e/T_a) is essentially different.

To generalize the concept of temperature fluctuations to the nonequilibrium case, we examine the probability that the temperature of the island measured within a time interval $\tau_0 \dots \tau_0 + \Delta\tau$ and averaged over the interval equals T_e:

$$P(T_e) = \left\langle \delta_I\left[\frac{1}{\Delta\tau}\int_{\tau_0}^{\tau_0+\Delta\tau} T_I(t)dt - T_e\right]\right\rangle = \left\langle \int \frac{dk}{2\pi}\exp\left(ik\left\{\int_{\tau_0}^{\tau_0+\Delta\tau}[T_I(t) - T_e]dt\right\}\right)\right\rangle. \quad (2)$$

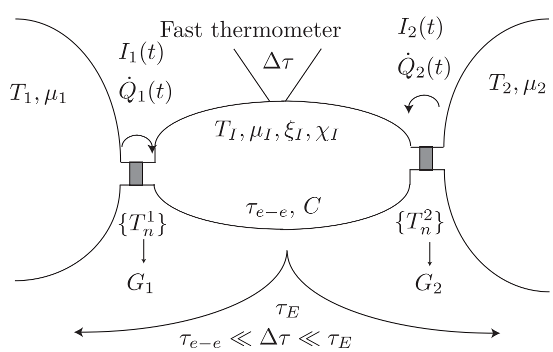

FIG. 1. Setup and limit considered in this Letter: A conducting island is coupled to reservoirs via electrical contacts characterized by the transmission eigenvalues $\{T_n^\alpha\}$, which, for example, yield the conductances G_α of the contacts. The temperature fluctuates on a time scale τ_E characteristic for the energy transport through the junctions. We assume the limit $\tau_{e-e} \ll \tau_E$ where the internal relaxation within the island is much faster than the energy exchange with reservoirs. In this limit both the temperature and its fluctuations are well defined.

The average $\langle\cdot\rangle$ is over the nonequilibrium state of the system. This is evaluated using an extension of the Keldysh technique [6] where the fluctuations of charge and heat are associated with two counting fields, χ and ξ, respectively [7–9]. The technique allows one to evaluate the full statistics of current fluctuations both for charge [7] and heat currents [8] in an arbitrary multiterminal system. In terms of the fluctuating temperature and chemical potential of the island, $T_I(t)$ and $\mu_I(t)$, and the associated counting fields $\xi_I(t)$ and $\chi_I(t)$, the average in Eq. (2) is presented in the form

$$P(T_e) \propto \int \mathcal{D}\xi_I(t)\mathcal{D}T_I(t)\mathcal{D}\chi_I(t)\mathcal{D}\mu_I(t)dk \times \exp\left(-\mathcal{A} + ik\left\{\int_{\tau_0}^{\tau_0+\Delta\tau} dt[T_I(t) - T_e]\right\}\right). \quad (3)$$

Here $\mathcal{A} = \mathcal{A}[\xi_I(t), T_I(t), \chi_I(t), \mu_I(t)]$ is the Keldysh action of the system. The counting fields $\xi_I(t)$ and $\chi_I(t)$ enter as Lagrange multipliers that ensure the conservation of charge and energy [9].

The Keldysh action consists of two types of terms, $\mathcal{A} = \int dt[S_I(t) + S_c(t)]$, with $S_I(t) = Q_I\dot{\chi}_I + E_I\dot{\xi}_I$ describing the storage of charge and heat on the island and S_c describing the contacts to the reservoirs. Here $Q_I = C_c\mu_I$ is the charge on the island, $E_I = C(T_I)T_I/2 + C_c\mu_I^2/2$ gives the total electron energy of the island, and C_c is the electrical capacitance of the island. For the electrical contacts, the action can be expressed in terms of the Keldysh Green's functions as [10] (we set $\hbar = e = k_B = 1$ for intermediate results)

$$S_{c,\mathrm{el}} = \frac{1}{2}\sum_\alpha\sum_{n\in\alpha} \mathrm{Tr}\ln\left[1 + T_n^\alpha \frac{\{\check{G}_\alpha, \check{G}_I\} - 2}{4}\right]. \quad (4)$$

The sums run over the lead and channel indices α and n. All products are convolutions over the inner time variables. The trace is taken over the Keldysh indices, and the action is evaluated with equal outer times. This action is a functional of the Keldysh Green's functions $\check{G}_\alpha$ and $\check{G}_I$ of the reservoirs and the island, respectively. It also depends on the transmission eigenvalues $\{T_n^\alpha\}$, characterizing each contact. The counting fields enter the action by the gauge transformation of Green's function [8]

$$\check{G}(t,t') = e^{-(1/2)[\chi_I + i\xi_I(t)\partial_t]\check{\tau}_3}\check{G}_0(t,t')e^{(1/2)[\chi_I - i\xi_I(t')\partial_{t'}]\check{\tau}_3}, \quad (5)$$

where the Keldysh Green's function reads

$$\check{G}_0(t,t') = \int \frac{d\epsilon}{2\pi} e^{-i\epsilon(t-t')} \begin{pmatrix} 1-2f(\epsilon) & 2f(\epsilon) \\ 2-2f(\epsilon) & -1+2f(\epsilon) \end{pmatrix}. \quad (6)$$

For quasiequilibrium $f(\epsilon) = \{\exp[(\epsilon - \mu)/T] + 1\}^{-1}$ is a Fermi distribution. In what follows, we assume the fields $\xi(t)$, $T(t)$ to vary slowly at the time scale T^{-1}, in which case we can approximate $i\xi(t)\partial_t \mapsto \xi(t)\epsilon$.

The saddle point of the total action at $\chi = \xi = 0$ yields the balance equations for charge and energy. Assuming that the electrical contacts dominate the energy transport, we get

$$\frac{\partial Q_I}{\partial t} = C_c\partial_t\mu_I = \sum_\alpha \mathrm{Tr}\check{\tau}_3 \sum_n T_n^\alpha \frac{[\check{G}_\alpha, \check{G}_I]}{4 + T_n^\alpha(\{\check{G}_I, \check{G}_\alpha\} - 2)}, \quad (7a)$$

$$\frac{\partial E_I}{\partial t} = C\partial_t T_I = \sum_\alpha \mathrm{Tr}(\epsilon - \mu_I)\check{\tau}_3 \sum_n T_n^\alpha \frac{[\check{G}_\alpha, \check{G}_I]}{4 + T_n^\alpha(\{\check{G}_I, \check{G}_\alpha\} - 2)}. \quad (7b)$$

The right-hand sides are sums of the charge and heat currents, respectively, flowing through the contacts α [11].

The time scale for the charge transport is given by $\tau_c = C_c/G$, with $G = \sum_\alpha\sum_n T_n^\alpha/(2\pi)$. This is typically much smaller than the corresponding time scale for heat transport, $\tau_E = C_h/G_{\mathrm{th}}$, where $G_{\mathrm{th}} = \pi^2 GT/3$. We assume that the measurement takes place between these time scales, $\tau_c \ll \Delta\tau \ll \tau_E$. In this limit the potential and its counting field μ_I and χ_I follow adiabatically the $T_I(t)$ and $\xi_I(t)$, and there is no charge accumulation on the island. As a result, we can neglect the charge capacitance C_c concentrating on the zero-frequency limit of charge transport.

To determine the probability, we evaluate the path integral in Eq. (3) in the saddle-point approximation. There are four saddle-point equations,

$$\partial_{\chi_I} S_c = 0, \qquad \partial_{\mu_I} S_c = 0, \quad (8a)$$

$$\frac{\pi^2}{6}\frac{\dot{\xi}_I}{\delta_I} = -\partial_{T_I^2} S_c - \frac{ikM_b(t;\tau_0,\Delta\tau)}{2T_I}, \quad (8b)$$

$$\frac{\pi^2}{6}\frac{\dot{T}_I^2}{\delta_I} = \partial_{\xi_I} S_c. \quad (8c)$$

Here $M_b(t) = 1$ inside the measurement interval $(\tau_0, \tau_0 + \Delta\tau)$ and zero otherwise. The formulas in Eq. (8a) express the chemical potential and charge counting field in terms of instant values of temperature T_I and energy counting field ξ_I, $\mu_I = \mu_I(\xi_I, T_I)$, $\chi_I = \chi_I(\xi_I, T_I)$. The third and fourth equations give the evolution of these variables. It is crucial for our analysis that these equations are of Hamilton form, ξ_I and T_I^2 being conjugate variables, the total connector action S_c being an integral of motion. Boundary conditions at $t \to \pm\infty$ correspond to most probable configuration $T_e = T_a$. This implies $S_c = 0$ at trajectories of interest.

The zeros of S_c in the ξ_I-T_I plane are concentrated in two branches that cross at the equilibrium point $\xi_I = 0$, $T_e = T_a$ (for illustration, see [12]). The saddle-point solutions $\xi_I(t)$, $T_I(t)$ describing the fluctuations follow these branches (see Fig. 2 for an example). Branch B ($\xi = 0$) corresponds to the usual "classical" relaxation to the equilibrium point from either higher or lower temperatures. Branch A corresponds to "antirelaxation": the trajectories following the curve quickly depart from equilibrium to either higher or lower temperatures. The solution of the saddle-point equations follows A before the measurement and B after it.

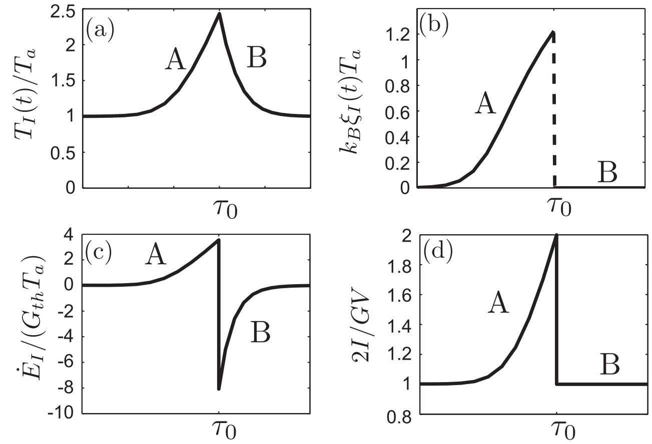

FIG. 2. Time line of a huge fluctuation. The measurement is made at $t \approx \tau_0$ with the result $T_e = 2.5T_a$, and thereby the time lines here are conditioned to give this (very unprobable) outcome at $t = \tau_0$. For $t < \tau_0$, the temperature follows the "antirelaxation" branch A, whereas after the measurement, it relaxes as predicted by a "classical" equation in branch B. (a) and (b) show the time dependence of the fluctuation for $T_I(t)$ and $\xi_I(t)$, respectively. The heat current into the island corresponding to this fluctuation is plotted in (c), and (d) shows the charge current flowing through the island. The statistical fluctuations of these curves are small (with amplitude $\sim\sqrt{T_a\delta_I/k_B}$) on the plot scale $\sim T_a$.

Since $S_c = 0$, the only contribution to path integral (3) comes from the island term $C\dot{\xi}_I T_I^2$, and thus

$$P(T_e) = \exp\left[\frac{\pi^2}{3\delta_I}\int \dot{\xi}T^2 dt\right] = \exp\left[-\frac{2\pi^2}{3\delta_I}\int_{T_a}^{T_e} T\xi_I^S(T)dT\right]. \quad (9)$$

Thus, in order to find $P(T_e)$, we only need a function $\xi_I^S(T_I)$ satisfying $S_c[\xi_I^S(T_I), T_I] = 0$ at branch A.

The connector action can generally be written in the form

$$S_c = \sum_\alpha \sum_{n\in\alpha} \int \frac{d\epsilon}{2\pi} \ln\{1 + T_n^\alpha[f_I(1-f_\alpha)(e^{-\chi_I - \xi_I\epsilon} - 1) + f_\alpha(1-f_I)(e^{\chi_I+\xi_I\epsilon} - 1)]\}, \quad (10)$$

with $f_{\alpha/I} = \{\exp[(\epsilon - \mu_{\alpha/I})/T_{\alpha/I}] + 1\}^{-1}$.

To prove the validity of the method for the equilibrium case, let us set all the chemical potentials to 0 and all the reservoir temperatures to T_a. This implies $\mu_I = \chi_I = 0$. Using the fact that for a Fermi function $f = -e^{\epsilon/T}(1-f)$, we observe that $S_c = 0$ regardless of contact properties provided $\xi_I = \xi_I^S(T_I) = 1/T_L - 1/T_I$. Substituting this into Eq. (9) reproduces the equilibrium distribution, Eq. (1).

Out of equilibrium, further analytical progress can be made in the case where the connectors are ballistic, $T_n \equiv 1$. Such a situation can be realized in a chaotic cavity connected to terminals via open quantum point contacts. The connector action reads [9],

$$S_c = \sum_\alpha \frac{G_\alpha}{2}\left[\frac{2\mu_\alpha\chi_I + T_\alpha\chi_I^2 + [\pi^2T_\alpha^2/3 + \mu_\alpha^2]\xi_I}{1 - T_\alpha\xi_I} - \frac{2\mu_I\chi_I - T_I\chi_I^2 + [\pi^2T_I^2/3 + \mu_I^2]\xi_I}{1 + T_I\xi_I}\right]. \quad (11)$$

Let us first assume two reservoirs with $T_1 = T_2 \equiv T_L$. In this case the general saddle-point solution for the potential follows from Kirchoff law: $\mu_I = (g\mu_1 + \mu_2)/(1+g)$ with $g \equiv G_L/G_R$. For the charge counting field we get $\chi_I = -\mu_I\xi$. The most probable temperature T_a is given by $T_a^2 = T_L^2 + 3g(\mu_1 - \mu_2)^2/[\pi^2(1+g)^2]$, and function $\xi_I^S(T_I)$ is expressed as

$$\xi_I^S = \frac{T_I^2 - T_a^2}{T_I(T_LT_I + T_a^2)}. \quad (12)$$

Substituting this into Eq. (9) yields for the full probability

$$-\ln P_{\text{ball}} = \frac{\pi^2 k_B}{3\delta_I T_L^3}\left\{T_L(T_e - T_a)[(T_e + T_a)T_L - 2T_a^2] + 2T_a^2(T_a^2 - T_L^2)\ln\left(\frac{T_a^2 + T_eT_L}{T_a^2 + T_aT_L}\right)\right\}. \quad (13)$$

In the strong nonequilibrium limit $V \equiv (\mu_1 - \mu_2) \gg T_L$, i.e., $T_a \gg T_L$, this reduces to

$$P_{\text{ball}} \propto \exp\left\{-\frac{2\pi^2 k_B}{3\delta_I}\frac{(T_e + 2T_a)(T_e - T_a)^2}{3T_a^2}\right\}. \quad (14)$$

The logarithm of this probability is plotted as the lowermost line in Fig. 3.

If the island is biased by temperature difference, $T_1 \equiv T_L \gg T_2$, $V = 0$, the probability obeys the same Eq. (13) with $T_a^2 = gT_1^2/(1+g)$.

For general contacts, the connector action and its saddle-point trajectories have to be calculated numerically. For tunnel contacts, the full probability distribution is plotted in two regimes in Fig. 3. The distribution takes values between the ballistic and equilibrium cases. Let us understand this by concentrating on the Gaussian regime and inspecting the variance of the temperature fluctuations for various contacts. This variance is related to the zero-frequency heat current noise $S_{\dot{Q}}$ via

$$2G_{\text{th}}C\langle\delta T^2\rangle = S_{\dot{Q}} = \partial_\xi^2 S_c|_{\xi\to 0}. \quad (15)$$

In equilibrium, $S_{\dot{Q}}^{(\text{eq})} = 2G_{\text{th}}T^2$ by virtue of the fluctuation-dissipation theorem. For an island with equal ballistic contacts driven far from equilibrium, $V \gg T_L$, $S_{\dot{Q}}^{\text{bal}} = \sqrt{3}GV^3/(8\pi) = G_{\text{th}}(T_a)T_a^2$, i.e., only half of $S_{\dot{Q}}^{(\text{eq})}$. The reduction manifests the vanishing temperature of the reservoirs. Most generally, for contacts of any nature, the heat current noise reads

$$S_{\dot{Q}}/S_{\dot{Q}}^{(\text{eq})} = \frac{1}{2} + a_Q\sum_\alpha F_\alpha, \quad (16)$$

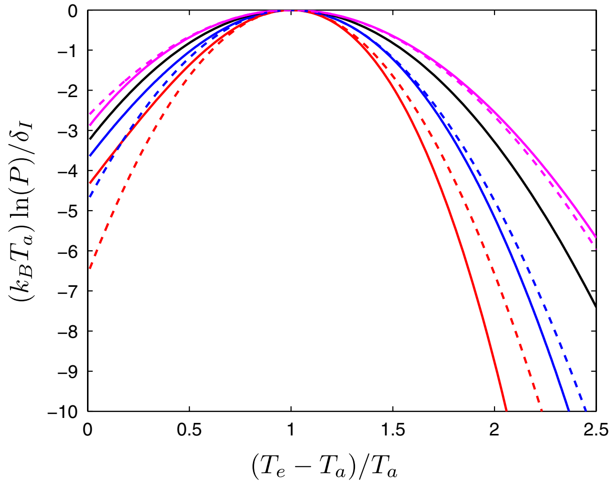

FIG. 3 (color online). Logarithm of temperature fluctuation statistics probability $P(T_e)$ in a few example cases. Solid lines from top to bottom: temperature bias with symmetric tunneling contacts, $T_a = T_1/\sqrt{2}$, $T_2 = 0$ (magenta); Gaussian equilibrium fluctuations (black), nonequilibrium fluctuations with $T_a = \sqrt{3}|eV|/(2\pi k_B)$, $T_1 = T_2 = 0$ for symmetric tunneling and ballistic contacts (blue and red lines, respectively). The dashed lines are Gaussian fits to small fluctuations $(T_e - T_a) \ll T_a$, described by the heat current noise S_Q at $T_e \approx T_a$.

where $F_\alpha = \sum_n T_n^\alpha(1 - T_n^\alpha)/\sum_n T_n^\alpha$ is the Fano factor for a contact α, $a_Q \approx 0.112$ being a numerical factor. For two tunnel contacts we hence obtain $S_{\dot{Q}}^{\rm tun} \approx 0.723 S_{\dot{Q}}^{\rm (eq)}$, a value between the ballistic and equilibrium values. For contacts of any type, the variation of temperature fluctuations is between the ballistic and tunneling values.

For rare fluctuations of temperature, $|T_e - T_a| \simeq T_a$, the probability distribution is essentially non-Gaussian in contrast to the equilibrium case. The skewness of the distribution is negative in the case of voltage driving: low-temperature fluctuations ($T_e < T_a$) are preferred to the high-temperature ones ($T_e > T_a$). In contrast, biasing with a temperature difference (uppermost curve in Fig. 3) favors high-temperature fluctuations.

The non-Gaussian features of the temperature fluctuations can be accessed at best in islands with a large level spacing that is smaller than the average temperature, say, by an order of magnitude. Many-electron quantum dots with spacing up to 0.1 K$/k_B$ seem natural candidates for the measurement of the phenomenon. The most natural way to detect the rare fluctuations is through a threshold detector [13], which produces a response only for temperatures exceeding or going under a certain threshold value. Besides the direct measurement of temperature, one can use the correlation of fluctuations. For example, Fig. 2(d) shows that the fluctuation of the temperature also causes a fluctuation in the charge current. Observing the latter may thus yield information about the former.

It is interesting to note an analogy in our calculation to the problem of tunneling: both the level spacing $\delta_I \sim \hbar$ in S_I and the scattering described in S_c are quantum effects, and thereby the temperature fluctuation probability can be written as $P(T_e) \sim \exp(-S_{\rm classical}/\hbar)$, where $S_{\rm classical}$ can be computed from classical physics. In the case of tunneling, $S_{\rm classical}$ (in imaginary time) is given by classical motion in an inverted potential [14]. This motion, describing the tunneling through a potential barrier, is analogous to our antirelaxation.

To conclude, we have evaluated nonequilibrium temperature fluctuations of an example system beyond the Gaussian regime. The method makes use of saddle-point trajectories and allows us to describe electric contacts of arbitrary transparency.

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