AUTOMATIC GUIDANCE OF SHIPS AS A CONTROL PROBLEM

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AUTOMATIC GUIDANCE OF SHIPS AS A CONTROL PROBLEM

(AUTOMATISCHE GELEIDING VAN SCHEPEN ALS REGELPROBLEEM)

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS DR. IR. C. J. D. M. VERHAGEN, HOOGLERAAR IN DE AFDELING DER TECHNISCHE NATUURKUNDE, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP DONDERDAG 25 JUNI 1970 TE 16.00 UUR

DOOR

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geboren te Den Helder

UITGEVERIJ WALTMAN – DELFT

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOREN
PROF. IR. R. G. BOITEN EN PROF. IR. J. GERRITSMA.

Aan Thea

Aan Han en Karel

Dit proefschrift is tot stand gekomen met medewerking van de Koninklijke Marine.

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Chapter 1

INTRODUCTION

1.1 Marine automation and automatic navigation: survey

In shipping, both in the merchant service and in the navy, automation is substantially gaining ground. There is a complex of interrelated factors which may be expected to lead to a continuation of this development, such as:

- The growing necessity to reduce crews, primarily because of the rapid rise of wages and other constituents of the cost of manpower, but sometimes for sheer shortage of qualified personnel also.
- The military requirement that certain ships are able to pass through areas of radioactive contamination, which implies that it must be possible to leave part of the ship unmanned for some time.
- As a result of the social evolution: the fact that personnel are less willing to do strenuous or tedious work, and demand, in general, more favourable working and living conditions than before.
- The growing necessity to operate ships „optimally" from the point of view of economy or military effectiveness, for instance by minimizing travelling time and fuel consumption.
- The increasing traffic densities on some waterways, and also the increased size of some types of ships, making certain situations difficult to handle by a human operator if unassisted by automatic equipment.
- The advances in technology: automation equipment is becoming less expensive, more reliable, smaller and lighter. Besides, computer software is being improved continuously, and much progress is being made in theoretical systems engineering.

The objects of marine automation can be divided into several catechories, such as:

- The propulsion plant, along with related or connected systems, like the electric power plant.
- The navigation and steering.
- The stabilization.
- For naval ships: the weapons systems and combat direction systems.

The topic of this thesis can be classified as automatic navigation.

The obvious example of automation in navigation is the utilization of the course-keeping auto-pilot. For many years this instrument has practically been standard equipment on sea-going vessels. The introduction of this type of auto-pilot, however, is no more than a first step in automatic navigation. Many far more sophisticated

systems are not only conceivable but will be, or even have been put into service. Possible projects of advanced automation, within the category of navigation, are:

- *Evasion:* the problem of manoeuvring one or more ships such, that the probability of a collision is minimized.
- *Station-keeping:* the problem of keeping the ship's position relative to another ship constant, in spite of the movements of the other ship.
  Station-keeping is necessary e.g. when fuelling at sea, and when sailing in a convoy.
- *Track-keeping:* the problem of guiding the ship along a pre-specified path.

It will be clear, that these examples by no means form an exhaustive list.

The topic of this thesis belongs to the class of track-keeping problems. For this reason, that class of problems will be elaborated on in the next section.

1.2 Track-keeping

Track-keeping systems operating independently of vision would make possible an extremely efficient use of the fairways with a high degree of safety. Hence it is not surprising, that the need for them has been felt for many years. Even more attractive is a flexible track-keeping system, in the sense, that the tracks are not fixed but can be chosen. Besides that a more sophisticated traffic control would become possible, a system like that could be useful in case of operations like mine-hunting, where areas have to be searched accurately. Track-keeping is not conceivable but as a feedback process, i.e. the ship has to be steered through knowledge of the ship's position relative to the desired track, and this information must be obtained in one way or another. This constitutes the principal reason why installation of track-keeping systems is not yet widespread: accurate feedback of the ship's position necessitates the use of complicated and expensive equipment.

The oldest type of a track-keeping system is the leader-cable system [1]: the first patent (British) was granted in 1892. A leader-cable is an isolated electric cable lying on the bottom of the fairway and following the desired track. The cable carries an alternating current, and produces a magnetic field which is sensed by a pair of coils on board the ship. The difference between the voltages induced in these coils shows whether the cable lies to starboard or to port, while at the same time an indication of the distance from the cable to the ship is obtained. Though most leader-cable systems are non-automatic in that the ship is steered by a human helmsman, some automatic systems are known too, where the positional information is coupled directly to the steering machine, so that the ship is steered without human intervention. One example is described in [2], another is the Datawell system used in the Shipbuilding Laboratory of the Technological University at Delft and in the Netherlands Ship Model Basin at Wageningen.

Though the idea of leader-cables has not yet been fully abandoned, nor has it ever found general application either. Reasons for this are:

- The advent of radar, radio-navigation etc.
- The difficult and expensive maintenance of leader-cables.
- The poor flexibility of the system.

Other automatic and non-automatic track-keeping systems exist or are being considered where the ship's position is obtained by means of radar [3], radio-navigation or inertial navigation.

1.3 The problem considered in this thesis

The main problem of this thesis is the designing, from the point of view of control theory, of a ship guidance controller for track-keeping. We restrict ourselves to the case where the r.p.m. of the ship's propeller is fixed and where consequently the forward speed is approximately constant too. One is confined to this case owing to the limitations imposed by current naval architecture: the knowledge about a ship's behaviour in other cases is still incomplete. For the same reason, some more restrictions have to be made, such as: the water must be sufficiently deep, there must be no other ships in the near vicinity, etc. These are stated in sec. 2.2.

It is assumed, that sensors observing the ship's heading and position are available. These sensors in themselves are practically left out of consideration, though the fact that their observations are contaminated with random measurement errors is taken into account. Also allowed for are the characteristics of the disturbances, i.e. the deterministic and stochastic forces and moments exerted on the ship as a result of waves, wind and current.

Short outlines of the several chapters are given in the summary.

References Chapter 1

1. J. Th. Verstelle, Private communication, 1967.
2. O. A. Kolody and G. A. Praver, Marine navigation traffic control system. Electrical Engineering **80**-11, 1961.
3. J. Goclowski and A. Gelb, Dynamics of an automatic ship steering system. IEEE Trans. on Automatic Control **11**-3, 1966; also 1966 Joint Automatic Control Conference.

Chapter 2

A MATHEMATICAL MODEL OF THE STEERING DYNAMICS OF A SHIP

2.1 Some conventions and definitions

A ship has three principal axes which are called the longitudinal axis, the lateral axis and the vertical axis. They intersect perpendicularly in the ship's center of gravity.

An orthogonal system of coordinates x_b, y_b, z_b is introduced. It is fixed with respect to the ship: the x_b-axis coincides with the longitudinal axis, the y_b-axis with the lateral axis and the z_b-axis with the vertical axis. The positive directions are forward, to port and upward, respectively; see fig. 2.1-1.

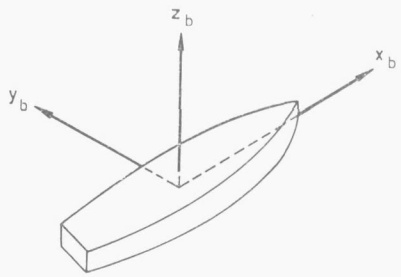

Fig. 2.1-1 The system of coordinates x_b, y_b, z_b

The ship will be considered as a rigid body. Six different motions then remain possible, which go by the names of *surge*, *sway*, *heave*, *roll*, *pitch* and *yaw*. *Surge* is the horizontal longitudinal oscillatory motion of the center of gravity of the ship, *sway* is the horizontal transverse oscillatory motion of the ship's center of gravity, and *heave* is the vertical oscillatory motion of the ship's center of gravity. *Roll*, *pitch* and *yaw* are the rotations about the x_b-, y_b- and z_b-axis, respectively.

For the remainder of this thesis, we shall restrict ourselves to the case where the heave, roll and pitch of the ship are sufficiently small for the influence of these motions on the other three possible motions of the ship to be negligible. As we are not interested in the heave, pitch and roll motions for their own sake, we may assume that the ship's center of gravity is constrained to a horizontal plane, to be referred to as the *plane of motion*, and that the longitudinal and lateral axes remain in this plane at all times.

Another orthogonal right-handed system of coordinates, x_0, y_0, z_0 is introduced. This is a system of coordinates which is fixed with respect to the earth: the x_0, y_0-plane coincides with the dead calm water-surface, and the z_0-axis points upward.

The direction, relative to the x_0-axis, of the ship's forward longitudinal axis, with

the positive sense being taken counterclockwise* is called the *heading* ψ_0. The time derivative of the heading is the *angular rate of yaw* r_0:

$$r_0 \stackrel{\text{def}}{=} \frac{\mathrm{d}\psi_0}{\mathrm{d}t} \tag{2.1-1}$$

The velocity of the ship's center of gravity has as its components u_0 and v_0, with

$$\begin{aligned} u_0 &\stackrel{\text{def}}{=} \frac{\mathrm{d}x_0}{\mathrm{d}t} \\ v_0 &\stackrel{\text{def}}{=} \frac{\mathrm{d}y_0}{\mathrm{d}t} \end{aligned} \tag{2.1-2}$$

The velocity vector can also be decomposed into components in the x_b- and y_b-directions. The component in the x_b-direction is the *forward speed* u, and the other one is the *velocity of sway* v, with

$$\begin{aligned} u &= u_0 \cos\psi_0 + v_0 \sin\psi_0 \\ v &= -u_0 \sin\psi_0 + v_0 \cos\psi_0 \end{aligned} \tag{2.1-3}$$

whence

$$\begin{aligned} u_0 &= u \cos\psi_0 - v \sin\psi_0 \\ v_0 &= u \sin\psi_0 + v \cos\psi_0 \end{aligned} \tag{2.1-4}$$

Components in the y_b-direction of forces exerted on the ship are denoted Y. Components about the z_b-axis of moments exerted on the ship are denoted N; the positive direction is counterclockwise.

The *rudder angle*, which is taken to be positive if the rudder is to port, is denoted δ. The time derivative of this is the *rudder angular velocity* ϱ:

$$\varrho \stackrel{\text{def}}{=} \frac{\mathrm{d}\delta}{\mathrm{d}t} \tag{2.1-5}$$

Fig. 2.1-2 shows the situation in order to clarify the definitions of several quantities given in this section.

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* In this thesis, the plane of motion and the x_0, y_0-plane are always viewed from above.

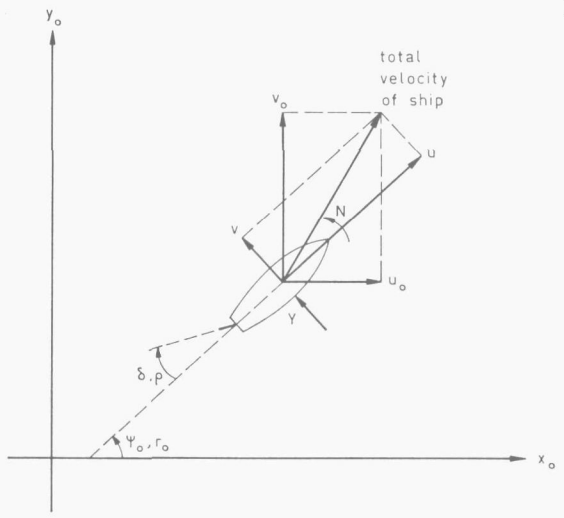

Fig. 2.1-2 Definitions of x_0, y_0, ψ_0, r_0, u_0, v_0, u, v, Y and N

2.2 Basic differential equation for the steering dynamics of a ship

From elementary mechanics, we know the relations

$$M \frac{\mathrm{d}v}{\mathrm{d}t} = Y_{\text{tot}} - Mur_0$$

$$I_{zz} \frac{\mathrm{d}r_0}{\mathrm{d}t} = N_{\text{tot}} \tag{2.2-1}$$

where

M = the ship's mass,
Y_{tot} = the sum of the Y-components of all forces exerted on the ship,
I_{zz} = the ship's moment of inertia about the vertical axis, and
N_{tot} = the sum of the N-components of all moments exerted on the ship.

We shall restrict ourselves to the case where the propeller is kept at a constant r.p.m., while the forward speed relative to the water is approximately constant too. Provided that certain conditions are satisfied, Y_{tot} and N_{tot} can be written as

$$Y_{\text{tot}} = Y_{\dot{v}} \frac{\mathrm{d}v}{\mathrm{d}t} + Y_v v + Y_{\dot{r}} \frac{\mathrm{d}r_0}{\mathrm{d}t} + Y_r r_0 + Y_\delta \delta + Y_d \tag{2.2-2a}$$

$$N_{\text{tot}} = N_{\dot{v}} \frac{\mathrm{d}v}{\mathrm{d}t} + N_v v + N_{\dot{r}} \frac{\mathrm{d}r_0}{\mathrm{d}t} + N_r r_0 + N_\delta \delta + N_d \tag{2.2-2b}$$

where Y_d and N_d serve to account for the disturbances, i.e. the influences of waves, wind and current, whereas $Y_{\dot{v}}$, Y_v, $Y_{\dot{r}}$, Y_r, Y_δ, $N_{\dot{v}}$, N_v, $N_{\dot{r}}$, N_r and N_δ are constants.

The main conditions for the validity of (2.2-2) are:
- δ, ϱ, v and r_0 remain sufficiently small.
- The influence of the heave, roll and pitch motions is negligible.
- The water is deep relative to the ship's draught.
- There are neither other ships nor banks, shores, dams or other similar objects in the near vicinity of the ship.
- The forward speed is moderate.
- The trim is moderate.

Substitution of (2.2-2) in (2.2-1) and rearrangement of terms yields [1]

$$\begin{bmatrix} M-Y_{\dot{v}} & -Y_{\dot{r}} \\ -N_{\dot{v}} & I_{zz}-N_{\dot{r}} \end{bmatrix} \begin{bmatrix} \dfrac{\mathrm{d}v}{\mathrm{d}t} \\ \dfrac{\mathrm{d}r_0}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} Y_v & Y_r-Mu \\ N_v & N_r \end{bmatrix} \begin{bmatrix} v \\ r_0 \end{bmatrix} + \begin{bmatrix} Y_\delta \\ N_\delta \end{bmatrix} \delta + \begin{bmatrix} Y_d \\ N_d \end{bmatrix} \tag{2.2-3}$$

From this follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} v \\ r_0 \end{bmatrix} = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} v \\ r_0 \end{bmatrix} + \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} \delta + \begin{bmatrix} D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix} \begin{bmatrix} Y_d \\ N_d \end{bmatrix} \tag{2.2-4}$$

with

$$\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} M-Y_{\dot{v}} & -Y_{\dot{r}} \\ -N_{\dot{v}} & I_{zz}-N_{\dot{r}} \end{bmatrix}^{-1} \begin{bmatrix} Y_v & Y_r-Mu \\ N_v & N_r \end{bmatrix} \tag{2.2-5}$$

$$\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} M-Y_{\dot{v}} & -Y_{\dot{r}} \\ -N_{\dot{v}} & I_{zz}-N_{\dot{r}} \end{bmatrix}^{-1} \begin{bmatrix} Y_\delta \\ N_\delta \end{bmatrix} \tag{2.2-6}$$

$$\begin{bmatrix} D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix} = \begin{bmatrix} M-Y_{\dot{v}} & -Y_{\dot{r}} \\ -N_{\dot{v}} & I_{zz}-N_{\dot{r}} \end{bmatrix}^{-1} \tag{2.2-7}$$

Combining (2.1-5), (2.2-4), (2.1-1) and (2.1-4), we obtain the following set of equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \delta \\ v \\ r_0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ v \\ r_0 \\ \psi_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varrho + \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Y_d \\ N_d \end{bmatrix} \tag{2.2-8a}$$

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = u \cos \psi_0 - v \sin \psi_0 \tag{2.2-8b}$$

$$\frac{\mathrm{d}y_0}{\mathrm{d}t} = u \sin \psi_0 + v \cos \psi_0 \tag{2.2-8c}$$

In the special case where the desired path of the ship is a straight line, it is advantageous to choose the x_0-axis coincident with it. If ψ_0 is small at all times, we are allowed to write

$$\begin{aligned} \cos \psi_0 &\approx 1 \\ \sin \psi_0 &\approx \psi_0 \end{aligned} \tag{2.2-9}$$

and put (2.2-8) into the following linear form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \delta \\ v \\ r \\ \psi_0 \\ y_0 \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} \delta \\ v \\ r \\ \psi_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\varrho + \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_d \\ N_d \end{bmatrix} \tag{2.2-10a}$$

$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = u - v\psi_0 \approx u \tag{2.2-10b}$$

with

$$\boldsymbol{A} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & u & 0 \end{bmatrix} \tag{2.2-11}$$

2.3 Modification of the mathematical model for guidance along a curved track

Let the desired path of the ship be the curve C_{AB}, lying in the plane of motion. Let A and B be the starting point and the end point of C_{AB}, respectively: the positive direction on C_{AB} is from A to B. We assume that C_{AB} has no angles.

Again, we introduce a new system of coordinates. These coordinates, labelled x, y, are defined in the plane of motion, and only in the neighbourhood of C_{AB}. The origin is chosen in a point O located on C_{AB}. If the ship's center of gravity is at a point P, and if P′ is the projection of P on C_{AB}, then x is defined as OP′, measured along C_{AB} and positive if P′ is between O and B, while y is defined as PP′, positive if P is on the left-hand side of C_{AB}. These definitions are illustrated in fig. 2.3-1.

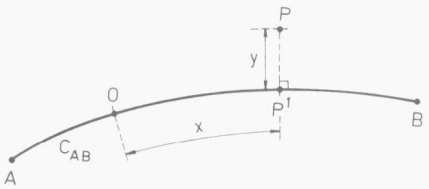

Fig. 2.3-1 Definitions of x and y

The tangent of C_{AB} at P′ is labelled $L_{P'}$, and the positive direction on it is defined in correspondence with the positive direction on C_{AB}.

The angle of $L_{P'}$ relative to the x_0-axis is denoted ψ_C, and the heading of the ship relative to $L_{P'}$ is denoted ψ, where the positive senses of these angles are counter-clockwise. The relation between ψ_C, ψ and ψ_0 is

$$\psi_0 = \psi + \psi_C \tag{2.3-1}$$

The definitions of $L_{P'}$, ψ_C and ψ are illustrated in fig. 2.3-2.

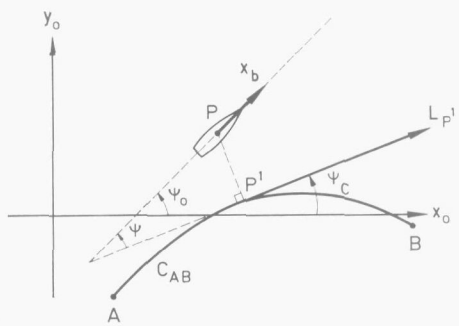

Fig. 2.3-2 Definitions of $L_{P'}$, ψ_C and ψ

The quantity ψ_C is primarily a function of the ship's position, particularly of its coordinate x. If the ship is in motion, however, ψ_C can also be looked upon as a function of time, and consequently an "angular velocity" and an "angular acceleration" can be defined:

$$r_C \stackrel{\text{def}}{=} \frac{\mathrm{d}\psi_C}{\mathrm{d}t} \tag{2.3-2}$$

$$a_C \stackrel{\text{def}}{=} \frac{\mathrm{d}r_C}{\mathrm{d}t} \tag{2.3-3}$$

We observe that

$$\frac{\mathrm{d}\psi_C}{\mathrm{d}t} = \frac{\mathrm{d}\psi_C}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} \tag{2.3-4}$$

and that $\mathrm{d}\psi_C/\mathrm{d}x$ is the curvature of the desired track C_{AB} at the location P′.

We also introduce the angular velocity of the ship relative to the (moving) tangent

$$r \stackrel{\text{def}}{=} \frac{\mathrm{d}\psi}{\mathrm{d}t} \tag{2.3-5}$$

The relation between r_C, r and r_0 is

$$r_0 = r + r_C \tag{2.3-6}$$

Utilizing the newly introduced quantities, (2.2-8a) can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \delta \\ v \\ r+r_C \\ \psi+\psi_C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \delta \\ v \\ r+r_C \\ \psi+\psi_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\varrho + \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_d \\ N_d \end{bmatrix} \tag{2.3.7}$$

Furthermore, it is easy to see, that

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= u\cos\psi - v\sin\psi \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= u\sin\psi + v\cos\psi \end{aligned} \tag{2.3-8}$$

If the ship is guided along the desired track with a reasonable accuracy, ψ will be small at all times, and the following approximations will be admissible:

$$\begin{aligned} \cos\psi &\approx 1 \\ \sin\psi &\approx \psi \end{aligned} \tag{2.3-9}$$

Upon substituting this in (2.3-8) and combining the result with (2.3-7), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \delta \\ v \\ r \\ \psi \\ y \end{bmatrix} = A\begin{bmatrix} \delta \\ v \\ r \\ \psi \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\varrho + \begin{bmatrix} 0 & 0 \\ 0 & A_{23} \\ -1 & A_{33} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_C \\ r_C \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Y_d \\ N_d \end{bmatrix} \tag{2.3-10a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u - v\psi \approx u \tag{2.3-10b}$$

By examination of (2.3-10a) we find, that the influence of a_C and r_C can be accounted for by a fictitious force Y_{curv} and a fictitious moment N_{curv}, satisfying

$$\begin{bmatrix} D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix}\begin{bmatrix} Y_{\mathrm{curv}} \\ N_{\mathrm{curv}} \end{bmatrix} = \begin{bmatrix} 0 & A_{23} \\ -1 & A_{33} \end{bmatrix}\begin{bmatrix} a_C \\ r_C \end{bmatrix} \tag{2.3-11}$$

From (2.2-5) and (2.2-7), the solution of (2.3-11) is found to be

$$\begin{bmatrix} Y_{\text{curv}} \\ N_{\text{curv}} \end{bmatrix} = \begin{bmatrix} Y_{\dot{r}} & Y_r - Mu \\ N_{\dot{r}} - I_{zz} & N_r \end{bmatrix} \begin{bmatrix} a_C \\ r_C \end{bmatrix} \tag{2.3-12}$$

2.4 The sampled-data form of the mathematical model

Let δ, v, r, ψ, ϱ, a_C, r_C, Y_d and N_d be sampled synchronously, with a constant sampling period T_s. Suppose that the sampled values of ϱ, a_C, r_C, Y_d and N_d "enter" by way of a zero order hold-circuit. In other words, suppose that these quantities are constant between any two successive sampling instants.

Instead of (2.3-10), we now have

$$\begin{bmatrix} \delta(k+1) \\ v(k+1) \\ r(k+1) \\ \psi(k+1) \\ y(k+1) \end{bmatrix} = \boldsymbol{\Phi} \begin{bmatrix} \delta(k) \\ v(k) \\ r(k) \\ \psi(k) \\ y(k) \end{bmatrix} + \boldsymbol{\Delta}\varrho(k) + \boldsymbol{\Gamma} \begin{bmatrix} a_C(k) \\ r_C(k) \end{bmatrix} + \boldsymbol{\Theta} \begin{bmatrix} Y_d(k) \\ N_d(k) \end{bmatrix} \tag{2.4-1a}$$

$$x(k+1) = x(k) + u(k) \tag{2.4-1b}$$

where

$$\boldsymbol{\Phi} \overset{\text{def}}{=} \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \Phi_{45} \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} \end{bmatrix} = \exp(\boldsymbol{A}T_s) \tag{2.4-2}$$

$$\boldsymbol{\Delta} \overset{\text{def}}{=} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \end{bmatrix} = \left\{ \int_0^{T_s} \exp(\boldsymbol{A}t)\mathrm{d}t \right\} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.4-3}$$

$$\boldsymbol{\Gamma} \overset{\text{def}}{=} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \\ \Gamma_{31} & \Gamma_{32} \\ \Gamma_{41} & \Gamma_{42} \\ \Gamma_{51} & \Gamma_{52} \end{bmatrix} = \left\{ \int_0^{T_s} \exp(\boldsymbol{A}t)\mathrm{d}t \right\} \begin{bmatrix} 0 & 0 \\ 0 & A_{23} \\ -1 & A_{33} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.4-4}$$

$$\boldsymbol{\Theta} \overset{\text{def}}{=} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \\ \Theta_{31} & \Theta_{32} \\ \Theta_{41} & \Theta_{42} \\ \Theta_{51} & \Theta_{52} \end{bmatrix} = \left\{ \int_0^{T_s} \exp(\boldsymbol{A}t)\mathrm{d}t \right\} \begin{bmatrix} 0 & 0 \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.4-5}$$

In reality, ϱ, a_C, r_C, Y_d and N_d may vary considerably over a sampling interval. Provided that the sampling period is small in comparison with the ship's dominant time constant, (2.4-1) then remains applicable as an approximation, if $\varrho(k)$ is interpreted as the average of $\varrho(t)$ over the time interval between the sampling instants k and $k+1$, and if $a_C(k)$, $r_C(k)$, $Y_d(k)$ and $N_d(k)$ are interpreted similarly.

2.5 Some comments on the mathematical model

2.5.1 *Directional stability*

In most cases, both eigenvalues of the matrix consisting of A_{22} etc. in (2.2-4) are real and negative. In some cases, however, one of the eigenvalues is zero or positive, which implies that the system described by (2.2-4) is marginally stable or unstable, respectively.

Consider the situation where the rudder of a ship is amidships, where no disturbances are exerted on the ship, and where the initial values of r_0 and v differ slightly from zero. If both eigenvalues are negative, the ship will eventually sail along a straight path, and is said to be *directionally stable*. If an eigenvalue is positive, the ship will follow a path with increasing curvature (due to non-linear effects not accounted for by the mathematical model, the path will approach a limit circle), and the ship is said to be *directionally unstable*. Many big ships, like supertankers and bulkcarriers, are found to be directionally unstable when fully loaded. If an eigenvalue is zero, the ship neither tends to decrease nor to increase its rate of turn, and is said to be *directionally neutral*.

2.5.2 *Improper disturbances*

The only difference between (2.2-10) and (2.3-10) is the occurrence of the terms with a_C and r_C. We have shown, that the effect of a_C and r_C can be interpreted as the effect of a fictitious force Y_{curv} and a fictitious moment N_{curv}. Thus, guidance along a curved track can be regarded as guidance along a straight track, where the influence of the curves can be accounted for by an extra disturbing force and moment. Since this force and this moment are fictitious, we shall call them *improper disturbances*.

If the desired path is given in advance, and if the distance travelled along the desired path at each time is known in advance also, the improper disturbances are deterministic.

2.5.3 *The steering machine*

The rudder of a ship is actuated by a steering machine. The steering machine usually is a feedback control system in itself. A simple mathematical model is given by the following equation:

$$\frac{\mathrm{d}\varrho}{\mathrm{d}t} = \begin{cases} K(\delta_{\text{ref}} - \delta) & \text{if} \quad |\delta_{\text{ref}} - \delta| < \varepsilon_{\max} \\ K\varepsilon_{\max} \operatorname{sign}(\delta_{\text{ref}} - \delta) & \text{if} \quad |\delta_{\text{ref}} - \delta| \geqslant \varepsilon_{\max} \end{cases} \tag{2.5-1}$$

where

δ = the actual rudder angle,
δ_{ref} = the ordered rudder angle, i.e. the input signal of the steering machine, and
ε_{max} and K are positive constants.

Fig. 2.5-1 represents this mathematical model in the form of a block diagram.

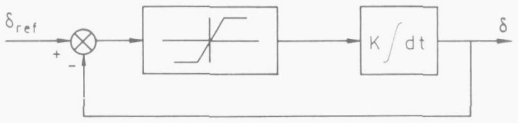

Fig. 2.5-1 Block diagram of a steering machine

If the ship is considered apart from the steering machine, the most natural form of the mathematical model of the ship would be one where the rudder angle is treated as an input variable.

If the mathematical model is to represent the combination of the ship and its steering machine, the ordered rudder angle, which is the command signal to the steering machine, is the most natural input variable.

In the mathematical models of this chapter neither the actual rudder angle nor the ordered angle, but the actual rudder angular velocity, which is the time derivative of the actual rudder angle, is regarded as an input, while the actual rudder angle is treated as a state variable. One could say, that the steering machine is incorporated in the mathematical model, but with the feedback loop opened, the saturation block removed, and K made equal to one. This is an artifice to the purpose of making the system amenable to certain methods of control engineering; a further elucidation on this point will be given in chapter 4.

2.5.4 *Verity of the mathematical model*

The mathematical model developed in this chapter is based on (2.2-2). We already know, that for this pair of equations to be valid, a number of rather severe conditions have to be satisfied. Unfortunately, the knowledge about the ship's behaviour if its forward speed and the r.p.m. of its propeller vary, or in other cases where the aforementioned conditions are not satisfied, is still incomplete.

It is important to note, that the degree of verity of (2.2-2), which is often called the *linearity*, may vary considerably for different ships. As a general rule, reasonable linearity can only be expected for ships showing a good directional stability, i.e. ships for which both eigenvalues of the matrix mentioned before are negative and relatively large. Attempts have been made to make (2.2-2) valid for less linear ships, or, which is equivalent, to relax the conditions, through the addition of non-linear terms [2]. Further improvement still remains desirable, though.

Another noteworthy point is, that the "constants" $Y_{\dot{v}}$ etc. not only depend on the

ship's speed and the r.p.m. of its propeller, but also on the loading of the ship and the specific mass of the water.

2.5.5 *Nomoto's model*

A simpler mathematical model can be obtained if instead of (2.2-4), the following equation is used:

$$\frac{dr_0}{dt} = A'_{33}r_0 + A'_{31}\delta + D'_{31}Y_d + D'_{32}N_d \tag{2.5-2}$$

where A'_{33}, A'_{31}, D'_{31} and D'_{32} are constants, while v is entirely neglected.

This mathematical model, which is generally named after Nomoto [3], is sufficiently accurate in many cases. It has an order which is lower by one compared with the mathematical model given in the preceding sections of this chapter.

References Chapter 2

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2. G. van Leeuwen and C. C. Glansdorp, Experimental determination of linear and non-linear hydrodynamic derivatives of a "Mariner"-type ship model. Delft Technological University, Shipbuilding Laboratory, Report 145, 1966.
3. K. Nomoto, Frequency response research on steering qualities of ships. Techn. Report Osaka University, **8**-294, 1958.

Chapter 3

MATHEMATICAL MODELS OF THE DISTURBANCES ACTING ON A SHIP

3.1 Introduction

The three principal factors causing disturbing forces and moments to be exerted on the ship are waves, wind and current. This chapter is divided into three parts which are devoted to these three factors, respectively.

We shall primarily be concerned with the Y-components of forces and the N-components of moments.

The improper disturbances which were introduced in the previous chapter will not be considered here.

PART I. WAVES

3.2 The regular sea

In fluid mechanics, the following approximate description of waves propagated over the surface of the sea is known [1]. The waves are characterized by a scalar called the *velocity potential* $\varphi = \varphi(x_0, y_0, z_0, t)$, which satisfies the partial differential equation

$$\frac{\partial^2 \varphi}{\partial x_0^2} + \frac{\partial^2 \varphi}{\partial y_0^2} + \frac{\partial^2 \varphi}{\partial z_0^2} = 0 \tag{3.2-1}$$

while one of the boundary conditions is

$$\frac{\partial \varphi}{\partial z_0} + \frac{1}{g}\frac{\partial^2 \varphi}{\partial t^2} = 0 \qquad \text{for} \qquad z_0 = 0 \tag{3.2-2}$$

where g is the acceleration due to gravity. The z_0-coordinate of the surface $\zeta = \zeta(x_0, y_0, t)$ and the pressure $p = p(x_0, y_0, z_0, t)$ then follow from the velocity potential by

$$\zeta = -\frac{1}{g}\frac{\partial \varphi}{\partial t} \qquad \text{for} \qquad z_0 = 0 \tag{3.2-3}$$

$$p = -\varrho_w\left(g z_0 + \frac{\partial \varphi}{\partial t}\right) \tag{3.2-4}$$

where ϱ_w is the specific mass of the liquid.

We shall here consider waves on the surface of a sea whose depth is infinite and whose area is unlimited. A set of possible solutions of (3.2-1) and (3.2-2) then is

$$\varphi = -\hat{\varphi}\ e^{\varkappa z_0} \sin\{\omega_w t + \theta - \varkappa(x_0 \cos\mu + y_0 \sin\mu)\} \tag{3.2-5}$$

where $\hat{\varphi}$, ω_w, μ and θ are arbitrary constants ($\hat{\varphi}$ and ω_w non-negative) with $\hat{\varphi}$ being sufficiently small, and where

$$\varkappa = \frac{\omega_w^2}{g} \tag{3.2-6}$$

From (3.2-5) together with (3.2-3) and (3.2-4), we get

$$\zeta = \hat{\zeta} \cos\{\omega_w t + \theta - \varkappa(x_0 \cos\mu + y_0 \sin\mu)\} \tag{3.2-7}$$

$$p = \varrho_w g(-z_0 + e^{\varkappa z_0}\zeta) \tag{3.2-8}$$

where $\hat{\zeta}$ is a constant satisfying

$$\hat{\zeta} = \frac{\omega_w \hat{\varphi}}{g} \tag{3.2-9}$$

Equation (3.2-7) describes a sea with sinusoidal waves and infinitely long crests. Such a sea is commonly referred to as a *regular sea*. The circular frequency is ω_w, the wave amplitude is $\hat{\zeta}$ (while $2\hat{\zeta}$ is said to be the *wave height*), and the direction of propagation relative to the x_0-axis is μ (with the positive sense taken counterclockwise). The quantity $\varkappa$ is called the *wave number*. Some other important quantities are the *wave velocity* u_w and the *wave length* λ_w: we have

$$u_w = \frac{\omega_w}{\varkappa} = \frac{g}{\omega_w} \tag{3.2-10}$$

$$\lambda_w = \frac{2\pi}{\varkappa} = \frac{2\pi g}{\omega_w^2} \tag{3.2-11}$$

Equation (3.2-10) indicates that the surface of the sea has the property of *dispersion*, i.e. the velocity of wave propagation varies with the wave frequency.

3.3 Characterization of the random sea

In reality, the waves on the surface of the sea are of a stochastic nature. Hence, statistical methods have to be used for their description. Rather than giving an exhaustive

and mathematically rigorous treatment of the subject, we shall outline a somewhat intuitive approach [2], [3].

The actual motion of the sea is approximated by a sum of an infinite number of small sinusoidal components as described in the preceding section. Each one of these components is characterized by a pair of indices (i, j), with

$$i \in \{0, 1, \ldots\} \tag{3.3-1}$$

$$j \in \{-\tfrac{1}{2}n_\mu + 1,\ -\tfrac{1}{2}n_\mu + 2,\ \ldots,\ \tfrac{1}{2}n_\mu\} \tag{3.3-2}$$

where n_μ is a positive and even integer. These indices specify the circular frequency $\omega_w(i)$ and the direction of propagation $\mu(j)$, respectively, as

$$\omega_w(i) = (i - \tfrac{1}{2})\Delta\omega_w \tag{3.3-3}$$

$$\mu(j) = j\Delta\mu \tag{3.3-4}$$

where

$$\Delta\mu = \frac{2\pi}{n_\mu} \tag{3.3-5}$$

and where $\Delta\omega_w$ is fixed and positive.

For the height of the sea surface and the pressure associated with the component identified by the pair of indices (i, j), we can write

$$\zeta_{ij}(x_0, y_0, t) = \hat{\zeta}(i,j)\cos\left[\omega_w(i)t + \theta(i,j) - \varkappa(i)\{x_0\cos\mu(j) + y_0\sin\mu(j)\}\right] \tag{3.3-6}$$

$$p_{ij}(x_0, y_0, z_0, t) = \varrho_w g e^{\varkappa z_0}\zeta_{ij}(x_0, y_0, t) \tag{3.3-7}$$

where, in accordance with (3.2-6),

$$\varkappa(i) \stackrel{\text{def}}{=} \frac{\omega_w^2(i)}{g} \tag{3.3-8}$$

For the total surface height and the total pressure we have

$$\zeta(x_0, y_0, t) = \sum_{i=1}^{\infty} \sum_{j=-\frac{1}{2}n_\mu + 1}^{\frac{1}{2}n_\mu} \zeta_{ij}(x_0, y_0, t) \tag{3.3-9}$$

$$p(x_0, y_0, z_0, t) = -\varrho_w g z_0 + \sum_{i=1}^{\infty} \sum_{j=-\frac{1}{2}n_\mu + 1}^{\frac{1}{2}n_\mu} p_{ij}(x_0, y_0, z_0, t) \tag{3.3-10}$$

In order that the motion of the sea be completely specified, the quantities $\hat{\zeta}_{ij}$ and $\theta(i,j)$ must be given for each pair (i, j).

A statistical characterization is usually given in the following manner. A real non-negative function $\mathscr{P}(\omega_w, \mu)$ is specified, defined for $0 \leqslant \omega_w < \infty$ and $-\pi < \mu \leqslant \pi$, and usually referred to as the *energy spectrum*. For each pair of indices (i, j), we can derive $\hat{\zeta}_{ij}$ from $\mathscr{P}(\omega_w, \mu)$ in the following manner:

$$\hat{\zeta}_{ij} = [\mathscr{P}\{\omega_w(i), \mu(j)\}\,\Delta\omega_w\,\Delta\mu]^{\frac{1}{2}} \tag{3.3-11}$$

Furthermore, $\theta(i, j)$, for each pair (i, j), is considered a random variable restricted to the interval $-\pi < \theta(i,j) \leqslant \pi$ with a probability density which is constant over the entire interval, while there is supposed to be no correlation between any pair of θ's. One could say that the θ's constitute the random nature of the sea.

The approximation of the actual motion of the sea by a sum of sinusoidal components is better as $\Delta\omega_w$ and $\Delta\mu$ are smaller.

Several proposals for $\mathscr{P}(\omega_w, \mu)$ of a wind generated sea have been made, but mostly for the case where the following conditions are satisfied:

- The sea is sufficiently deep.
- The coast is sufficiently far away.
- The wind has been constant for a sufficiently long time over a sufficiently large area („fully developed sea").

Little is known about $\mathscr{P}(\omega_w, \mu)$ in other cases.

The best formula for $\mathscr{P}(\omega_w, \mu)$ under the above conditions is due to Pierson and Moskowitz [4]: in units based on the meter and the second,

$$\mathscr{P}(\omega_w, \mu) = \begin{cases} \dfrac{8.10 \cdot 10^{-3} g^2}{\omega^5} \exp\left(-\dfrac{0.745 g^4}{u_{\text{wind}}^4 \omega^4}\right) \cos^2\mu & \text{if} \quad -\dfrac{\pi}{2} < \mu < \dfrac{\pi}{2} \\ 0 \quad \text{if} \quad -\pi < \mu \leqslant -\dfrac{\pi}{2} \quad \text{or} \quad \dfrac{\pi}{2} \leqslant \mu \leqslant \pi \end{cases} \tag{3.3-12}$$

where u_{wind} is the wind velocity, while the direction *into* which the wind blows is taken $\mu = 0$.

3.4 The forces and moments exerted on a ship in a regular or random sea

We consider the problem of calculating the force Y_{waves} and the moment N_{waves} exerted by the waves on a ship proceeding in a regular sea.

We note that by putting the problem this way, and also by writing (2.2-2), we assume tacitly that the total force and moment acting on the ship are the linear superpositions of (i): the forces and moments acting on the moving ship in undisturbed water, and (ii): the forces and moments acting on the restrained ship as a result of the waves. The force and moment we consider here are in group (ii).

The problem can be solved by means of a simple principle [2], [5], if the following additional assumptions can be made:

- The forces and moments only result from pressure. The effect of the exchange of kinetic energy between the fluid particles and the ship can be ignored.
- The wave field is not disturbed by the presence of the ship.

Under these assumptions, the force Y_{waves} and the moment N_{waves} can be obtained by computing the elementary contributions due to pressure dY_{waves} and dN_{waves} on each hull surface element and by integrating these over the entire immersed hull surface.

It is important to observe that the ship's own forward speed may be in the same order as the wave velocity. As a result of this, the frequency of the forces and moments may differ substantially from the wave frequency as measured by an observer whose position is fixed with respect to the earth.

The method just discussed leads to a Y_{waves} and an N_{waves} which are linear functions of the wave field, i.e. for which the superposition principle holds. This makes it relatively easy to find the statistical characterization (the spectral densities) of the force Y_{waves} and the moment N_{waves} exerted on the ship when proceeding in a random sea whose energy spectrum is given.

New results of theoretical and experimental studies [6] have indicated that the Y_{waves} and N_{waves} (or their spectral densities) calculated under the above simplifying assumptions may show non-negligible differences with the actual values. Unfortunately, a general solution of the problem is not yet available.

3.5 Computation of the force and moment exerted on a block-shaped ship in a regular sea

In this section, an approximate solution of the following problem will be presented. The wetted part of a ship is a rectangular parallelepiped with length L, breadth B and draught D. The ship's center of gravity is exactly amidships. The ship is sailing in a regular sea as described in sec. 3.2. It has a constant forward speed u. Find Y_{waves}, the Y-component of the force, and N_{waves}, the N-component of the moment, exerted on the ship. The problem will be solved under the simplifying assumptions as discussed in the preceding section:

- The forces and moments only result from pressure.
- The wave field is not disturbed by the presence of the ship.

We shall use the symbols and definitions introduced in chapter 2. The y_0-direction will be taken as the direction into which the waves are propagated. The situation is shown in fig. 3.5-1.

Before embarking on the computation, we shall make two further simplifying assumptions:

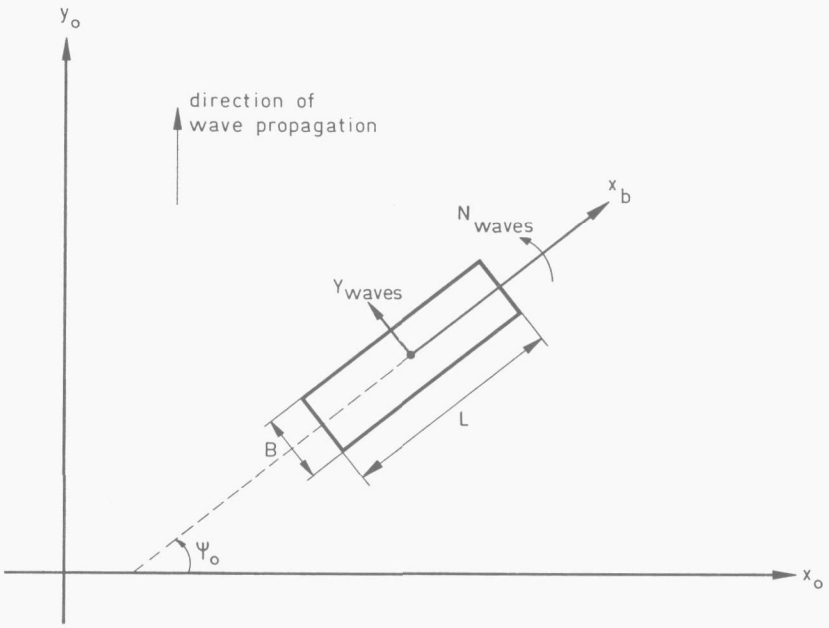

Fig. 3.5-1 Block-shaped ship in a regular sea

- The mathematical model of the regular sea is slightly modified: the influence of the waves is accounted for by assuming a fluctuating pressure distribution below the water-surface, whereas the surface itself is assumed to be undisturbed. Thus we have

$$\zeta(x_0, y_0, t) \equiv 0 \tag{3.5-1}$$

$$p(x_0, y_0, z_0, t) = -\varrho_w g z_0 + \hat{p} e^{\varkappa z_0} \cos(\omega_w t - \varkappa y_0) \tag{3.5-2}$$

where

$$\hat{p} \stackrel{\text{def}}{=} \varrho_w g \hat{\zeta} \tag{3.5-3}$$

with $\hat{\zeta}$ being the (equivalent) wave amplitude.

- The forces on the front and rear surfaces of the ship can be neglected for the calculation of N_{waves} (and Y_{waves}, of course).

The total force $Y^{(s)}_{\text{waves}}$ exerted on the starboard plane of the ship is

$$Y^{(s)}_{\text{waves}} = \int_{x_b=-\frac{1}{2}L}^{\frac{1}{2}L} \int_{z_0=-D}^{0} [-\varrho_w g z_0 +$$

$$+ \hat{p}\, e^{\varkappa z_0} \cos\{\omega_w t - \varkappa(y_0 - \tfrac{1}{2}B\cos\psi_0 + x_b \sin\psi_0)\}]\, \mathrm{d}z_0\, \mathrm{d}x_b \tag{3.5-4}$$

$$= \tfrac{1}{2}\varrho_w g L D^2 + \hat{P} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} \cos\{\omega_w t - \varkappa(y_0 - \tfrac{1}{2}B\cos\psi_0 + x_b \sin\psi_0)\}\, \mathrm{d}x_b \tag{3.5-5}$$

where y_0 refers to the position of the ship's center of gravity, and where

$$\hat{P} \stackrel{\text{def}}{=} \int_{-D}^{0} \hat{p}\, e^{\varkappa z_0}\, \mathrm{d}z_0 \tag{3.5-6}$$

$$= \hat{p}\frac{1-e^{-\varkappa D}}{\varkappa} \tag{3.5-7}$$

The corresponding expression for the total force $Y^{(p)}_{\text{waves}}$ exerted on the port plane of the ship is

$$Y^{(p)}_{\text{waves}} = -\tfrac{1}{2}\varrho_w g L D^2 - \hat{P}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} \cos\{\omega_w t - \varkappa(y_0 + \tfrac{1}{2}B\cos\psi_0 + x_b \sin\psi_0)\}\,\mathrm{d}x_b \tag{3.5-8}$$

The total disturbing force Y_{waves} is obtained by addition of $Y^{(s)}_{\text{waves}}$ and $Y^{(p)}_{\text{waves}}$. The result can be written, after a simple goniometric manipulation,

$$Y_{\text{waves}} = -2\hat{P}\sin(\tfrac{1}{2}\varkappa B\cos\psi_0)\int_{-\frac{1}{2}L}^{\frac{1}{2}L} \sin\{\omega_w t - \varkappa(y_0 + x_b\sin\psi_0)\}\,\mathrm{d}x_b \tag{3.5-9}$$

Upon working out the integral, we find

$$Y_{\text{waves}} = -2\hat{P}L\sin(\tfrac{1}{2}\varkappa B\cos\psi_0)\frac{\sin(\frac{1}{2}\varkappa L\sin\psi_0)}{\frac{1}{2}\varkappa L\sin\psi_0}\sin(\omega_w t - \varkappa y_0) \tag{3.5-10}$$

Likewise, the total moment $N^{(s)}_{\text{waves}}$ about the z_0-axis exerted on the starboard plane of the ship is

$$N^{(s)}_{\text{waves}} = \hat{P}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} x_b\cos\{\omega_w t - \varkappa(y_0 - \tfrac{1}{2}B\cos\psi_0 + x_b\sin\psi_0)\}\,\mathrm{d}x_b \tag{3.5-11}$$

A corresponding expression for $N^{(p)}_{\text{waves}}$, the total moment about the z_0-axis exerted on the port plane of the ship, is easy to write. By adding $N^{(s)}_{\text{waves}}$ and $N^{(p)}_{\text{waves}}$, and performing the same goniometric manipulation as before, we obtain

$$N_{\text{waves}} = -2\hat{P}\sin(\tfrac{1}{2}\varkappa B\cos\psi_0)\int_{-\frac{1}{2}L}^{\frac{1}{2}L} x_b\sin\{\omega_w t - \varkappa(y_0 + x_b\sin\psi_0)\}\,\mathrm{d}x_b \tag{3.5-12}$$

Upon working out the integral, we find

$$= -\hat{P}L^2\sin(\tfrac{1}{2}\varkappa B\cos\psi_0)\frac{\frac{1}{2}\varkappa L\sin\psi_0\cos(\frac{1}{2}\varkappa L\sin\psi_0) - \sin(\frac{1}{2}\varkappa L\sin\psi_0)}{\frac{1}{4}\varkappa^2 L^2\sin^2\psi_0}\cos(\omega_w t - \varkappa y_0) \tag{3.5-13}$$

We shall also derive expressions for the averages of Y_{waves} and N_{waves} over the sampling interval k, characterized by

$$kT_s < t < (k+1)T_s \tag{3.5-14}$$

where k is an integer. The quantities to be calculated are

$$Y_{\text{waves}}(k) \stackrel{\text{def}}{=} \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} Y_{\text{waves}}(t)\,\mathrm{d}t \tag{3.5-15}$$

$$N_{\text{waves}}(k) \stackrel{\text{def}}{=} \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} N_{\text{waves}}(t)\,\mathrm{d}t \tag{3.5-16}$$

We can substitute (3.5-10) and (3.5-13) in these integrals, but we then have to take into account that y_0 is a function of t. As an approximation, we have

$$y_0(t) = y_0(k) + (t - kT_s)\{u \sin \psi_0(k) + v(k) \cos \psi_0(k)\} \tag{3.5-17}$$

where

$$y_0(k) \stackrel{\text{def}}{=} y_0(t) \qquad \text{for} \qquad t = kT_s \tag{3.5-18}$$

$$\psi_0(k) \stackrel{\text{def}}{=} \psi_0(t) \qquad \text{for} \qquad t = kT_s \tag{3.5-19}$$

$$v(k) \stackrel{\text{def}}{=} v(t) \qquad \text{for} \qquad t = kT_s \tag{3.5-20}$$

We substitute (3.5-17) in (3.5-10) and (3.5-13), putting

$$\omega_w t - \varkappa y_0 =$$

$$= \omega_w t - \varkappa y_0(k) - (t - kT_s)\{\varkappa u \sin \psi_0(k) + \varkappa v(k) \cos \psi_0(k)\} \tag{3.5-21}$$

$$= \{\omega_w - \varkappa u \sin \psi_0(k) - \varkappa v(k) \cos \psi_0(k)\}(t - kT_s) + k\omega_w T_s - \kappa y_0(k) \tag{3.5-22}$$

$$= \omega_w^{(\text{rel})}(k) t' + \varphi(k) \tag{3.5-23}$$

where

$$\omega_w^{(\text{rel})}(k) \stackrel{\text{def}}{=} \omega_w - \varkappa u \sin \psi_0(k) - \varkappa v(k) \cos \psi_0(k) \tag{3.5-24}$$

$$t' \stackrel{\text{def}}{=} t - kT_s \tag{3.5-25}$$

$$\varphi(k) \stackrel{\text{def}}{=} k\omega_w T_s - \varkappa y_0(k) \tag{3.5-26}$$

If (3.5-10) is modified in this manner, and then substituted in (3.5-15), the latter becomes

$$Y_{\text{waves}}(k) = -2\hat{P}L\sin\{\tfrac{1}{2}\varkappa B\cos\psi_0(k)\}\,\frac{\sin\{\tfrac{1}{2}\varkappa L\sin\psi_0(k)\}}{\tfrac{1}{2}\varkappa L\sin\psi_0(k)}\cdot$$

$$\cdot\frac{1}{T_s}\int_0^{T_s}\sin\{\omega_w^{(\text{rel})}(k)t'+\varphi(k)\}\,\mathrm{d}t' \tag{3.5-27}$$

$$= -2\hat{P}L\sin\{\tfrac{1}{2}\varkappa B\cos\psi_0(k)\}\,\frac{\sin\{\tfrac{1}{2}\varkappa L\sin\psi_0(k)\}}{\tfrac{1}{2}\varkappa L\sin\psi_0(k)}\cdot$$

$$\cdot\frac{\sin\{\tfrac{1}{2}\omega_w^{(\text{rel})}(k)T_s\}}{\tfrac{1}{2}\omega_w^{(\text{rel})}(k)T_s}\sin\{\varphi(k)+\tfrac{1}{2}\omega_w^{(\text{rel})}(k)T_s\} \tag{3.5-28}$$

Similarly, (3.5-16) leads to

$$N_{\text{waves}}(k) = -\hat{P}L^2\sin\{\tfrac{1}{2}\varkappa B\cos\psi_0(k)\}\cdot$$

$$\cdot\frac{\tfrac{1}{2}\varkappa L\sin\psi_0(k)\cos\{\tfrac{1}{2}\varkappa L\sin\psi_0(k)\}-\sin\{\tfrac{1}{2}\varkappa L\sin\psi_0(k)\}}{\tfrac{1}{4}\varkappa^2L^2\sin^2\psi_0(k)}\cdot$$

$$\cdot\frac{\sin\{\tfrac{1}{2}\omega_w^{(\text{rel})}(k)T_s\}}{\tfrac{1}{2}\omega_w^{(\text{rel})}(k)T_s}\cos\{\varphi(k)+\tfrac{1}{2}\omega_w^{(\text{rel})}(k)T_s\} \tag{3.5-29}$$

PART 2. WIND

3.6 Characterization of wind and its influence on a ship

For our purpose, the wind in a ship's neighbourhood is to be regarded as a superposition of two components. The first component, called the *mean wind*, is a horizontal and homogeneous air-current, constant both with respect to position and time. The second, called *turbulence*, consists of stochastic fluctuations, hence it can only be described by means of statistical methods.

In the literature, some information can be found on forces and moments exerted on a ship in a turbulence-free wind. A short account of this is given in the next section. To the knowledge of the author, information on the influence of turbulence has not yet been published. There are several factors which may explain this deficiency. Firstly, in most cases the influence of turbulence will be small in comparison with the influences of other factors. Secondly, the state of turbulence in the atmosphere is highly variable, depending upon meteorological and oceanographical data such as the mean wind field, static stability and the wave spectrum of the sea. Thirdly, due to the fact that generally wind pressure at a point is proportional to the square of the local wind speed, the influence of turbulence is a non-linear phenomenon which cannot be considered apart from the mean wind and the ship's own forward speed.

3.7 The force and moment exerted on a ship in a constant homogeneous air-current

In [7], an account is given of the measurement, through model experiments, of forces and moments exerted on a ship in a constant homogeneous air-current.

According to that paper, the force Y_{wind} and the moment N_{wind} are expressible, respectively, as

$$Y_{\text{wind}} = C_Y(\mu_a) q A_l \tag{3.7-1}$$

$$N_{\text{wind}} = C_N(\mu_a) q A_l L_{oa}, \tag{3.7-2}$$

with

$$q = \tfrac{1}{2} \varrho_a u_{\text{air}}^2 \tag{3.7-3}$$

where

μ_a = the direction of the air velocity relative to the ship, measured with respect to the negative x_b-axis and taken positive if windward is to port; the definition of μ_a is clarified in fig. 3.7-1,

q = the wind pressure experienced on board the ship,

A_l = the area above the waterline of the ship's projection on the x_b, y_b-plane,

L_{oa} = the ship's over-all length,

ϱ_a = the specific mass of air, and

u_{air} = the air velocity relative to the ship.

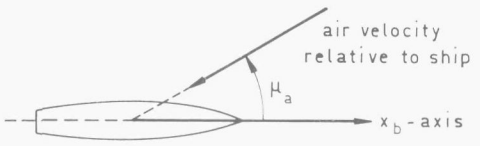

Fig. 3.7-1 Definition of μ_a

In [7], $C_Y(\mu_a)$ and $C_N(\mu_a)$ are given for several types of ships. Summarizing the results, we can say that for most ships being not too asymmetrical with respect to the y_b, z_b-plane, coarse approximations for the two above coefficients are

$$C_Y(\mu_a) = \hat{C}_Y \sin \mu_a \tag{3.7-4}$$

$$C_N(\mu_a) = \hat{C}_N \sin 2\mu_a, \tag{3.7-5}$$

where $\hat{C}_Y$ is a constant in the order of -0.9, while $\hat{C}_N$ is a constant usually between -0.05 and -0.20.

A point of significance is that μ_a, u_{air} and q may be affected to a non-negligible degree by the ship's own forward speed.

3.8 Restriction

In this part of the chapter we shall be concerned with the influence of current, under the restriction that in the neighbourhood of the ship, the current is constant and homogeneous. Thus, we shall neglect the influence of the derivatives with respect to position and time of the current vector. This approximation will be admissible in the majority of cases.

3.9 The force and moment accounting for the influence of current

If there are neither waves nor wind nor current, the quantities Y_d and N_d in (2.2-2) will be zero.

In case the ship is exposed to the influence of current, the hydrodynamic force and moment exerted on the ship will only depend on the motion of the ship relative to the water. From this we deduce that the influence of current can be accounted for in (2.2-2) by leaving Y_d and N_d zero, but interpreting $\mathrm{d}v/\mathrm{d}t$, v, $\mathrm{d}r_0/\mathrm{d}t$ and r_0 as being pertinent to the motion of the ship relative to the water. By the assumption that the current is constant and homogeneous, this leaves the quantities $\mathrm{d}r_0/\mathrm{d}t$ and r_0 unaffected. The quantity v, however, has to be replaced by $v - v_{\text{current}}$, where v_{current} is the y_b-component of the current vector. For $\mathrm{d}v_{\text{current}}/\mathrm{d}t$ we have

$$\frac{\mathrm{d}v_{\text{current}}}{\mathrm{d}t} = -r_0 u_{\text{current}} \approx 0 \tag{3.9-1}$$

where u_{current} is the x_b-component of the current vector.

In view of this, we may rewrite (2.2-2) as

$$Y_{\text{tot}} = Y_{\dot{v}}\frac{\mathrm{d}v}{\mathrm{d}t} + Y_v v + Y_{\dot{r}}\frac{\mathrm{d}r_0}{\mathrm{d}t} + Y_r r_0 + Y_\delta \delta - Y_v v_{\text{current}} + Y_{\text{waves}} + Y_{\text{wind}} \tag{3.9-2a}$$

$$N_{\text{tot}} = N_{\dot{v}}\frac{\mathrm{d}v}{\mathrm{d}t} + N_v v + N_{\dot{r}}\frac{\mathrm{d}r_0}{\mathrm{d}t} + N_r r_0 + N_\delta \delta - N_v v_{\text{current}} + N_{\text{waves}} + N_{\text{wind}} \tag{3.9-2b}$$

We draw the conclusion that the influence of current can be accounted for by a disturbing force Y_{current} and a disturbing moment N_{current}, defined as

$$Y_{\text{current}} \stackrel{\text{def}}{=} -Y_v v_{\text{current}} \tag{3.9-3}$$

$$N_{\text{current}} \stackrel{\text{def}}{=} -N_v v_{\text{current}} \tag{3.9-4}$$

Another point that needs to be considered is that in addition to the velocity of sway and the angular rate of yaw, the ship's forward speed is affected by current as well. We have the relationship

$$u = u_{\text{rel}} + u_{\text{current}} \tag{3.9-5}$$

where

u = the ship's forward speed relative to the earth, and
u_{rel} = the ship's forward speed relative to the water.

Since we have assumed that the propeller r.p.m. is kept constant, u_{rel} will be approximately constant, while u may vary substantially.

This has several consequences. First, A_{32} and A_{33} in (2.2-4) are no longer constants, as can be seen from (2.2-5). We also find, by (2.2-11) and the equations of sec. 2.4, that several elements in Φ, Δ, Γ and Θ of (2.4-1) will vary. Secondly, since the speed along the desired track C_{AB} is not constant, the a priori calculation of the improper disturbances $a_C(t)$ and $r_C(t)$ is more toilsome than in the current-free case.

References Chapter 3

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Chapter 4

A DESIGN PROCEDURE FOR AN AUTOMATIC SHIP GUIDANCE CONTROLLER

4.1 Block diagram of the control system

If an automatic controller guides a ship along a pre-specified track C_{AB}, the ship and the controller will be constituents of a feedback control system as shown in fig. 4.1-1.

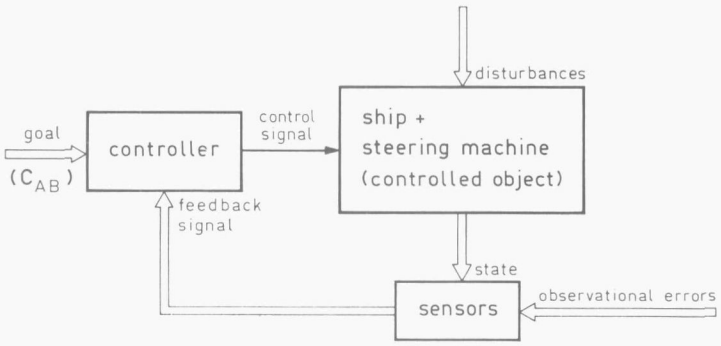


Fig. 4.1-1 Block diagram of the control system

This control system consists of three principal blocks, which are

- The ship together with its steering machine as the *controlled object*, also called *plant*.
- The *controller*.
- The *sensors*.

The *controlled object* is exposed to the influences of two signals: *a control signal* supplied by the controller, and *disturbances*. Disturbances are primarily forces and moments exerted on the ship as a result of waves, wind and current. We have seen, however, that „improper" disturbances also play a role, unless the desired path C_{AB} is a straight line. The *state* of the controlled object is observed by the *sensors*, which send their observations to the controller. This flow of data is the *feedback signal* of the control system. It should be realized that the sensors always will introduce *observational errors*, owing to inevitable imperfections.

Another input signal to the controller is the specification of the *goal* of the control system, the desired track C_{AB}. We assume that this information is entirely fed into the controller in advance.

The controller has to generate a control signal such that the ship keeps to the

desired track in the best possible way, despite the proper and improper disturbances, and despite the fact that the feedback signal is contaminated with observational errors.

In this chapter a design procedure for the controller, i.e. for the control strategy, will be proposed. Since a natural way to implement the controller is utilizing a general-purpose digital computer, the design problem will be treated as a control problem of the sampled-data type: the feedback signal and the control signal are assumed to be sampled synchronously, at a fixed sampling frequency. The theoretical material underlying the designing method to be proposed is presented in chapter 7.

4.2 Further description of the design problem

The design problem of the automatic ship guidance controller, to be dealt with in this chapter, is further described by the following points:

4.2.1 *The timing*

The sampling instant numbered 0 coincides with the moment when the ship's center of gravity passes the starting point A of the pre-specified track C_{AB}.

The last sampling instant considered is numbered n. At that moment, or within the following sampling period, the ship's center of gravity passes the end point B of C_{AB}.

So we have

$$x(0) = x_A \tag{4.2-1}$$

$$x(n) \approx x_B \tag{4.2-2}$$

where x_A and x_B are the x-coordinates of points A and B, respectively.

4.2.2 *The ship's equation*

The ship is sailing at a known constant forward speed u_{rel} relative to the water.

For the quantities δ, v, r, ψ and y, (2.4-1a) is valid for each k. The matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Theta}$ are constant and known.

For the sake of notational convenience, we shall frequently make use of the vector $\boldsymbol{\xi}(k)$, defined by

$$\boldsymbol{\xi}(k) \stackrel{\text{def}}{=} \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \\ \xi_3(k) \\ \xi_4(k) \\ \xi_5(k) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \delta(k) \\ v(k) \\ r(k) \\ \psi(k) \\ y(k) \end{bmatrix} \tag{4.2-3}$$

4.2.3 *The sensors*

At each sampling instant $k \in \{0, 1, \ldots, n-1\}$, sensors observing the quantities $\xi_1(k)$, $\xi_4(k)$ and $\xi_5(k)$ produce the vector of observations

$$\boldsymbol{\eta}(k) \stackrel{\text{def}}{=} \begin{bmatrix} \eta_1(k) \\ \eta_4(k) \\ \eta_5(k) \end{bmatrix} = \begin{bmatrix} \xi_1(k) \\ \xi_4(k) \\ \xi_5(k) \end{bmatrix} + \begin{bmatrix} 0 \\ w_4^{(2)}(k) \\ w_5^{(2)}(k) \end{bmatrix} \tag{4.2-4}$$

The quantities $w_4^{(2)}(k)$ and $w_5^{(2)}(k)$ represent the observational errors. Apparently, $\xi_1(k)$ is observed without error.

4.2.4 *The initial state*

The initial state $\boldsymbol{\xi}(0)$, when considered a priori with respect to $\boldsymbol{\eta}(0)$, is a gaussian random variable, with

$$\mathrm{E}\,\boldsymbol{\xi}(0) = \boldsymbol{\xi}^{(0)} \tag{4.2-5}$$

$$\mathrm{cov}\,\{\boldsymbol{\xi}(0), \boldsymbol{\xi}(0)\} = \boldsymbol{S}^{(0)} \tag{4.2-6}$$

where

$$\boldsymbol{\xi}^{(0)} = \begin{bmatrix} \xi_1^{(0)} \\ \xi_2^{(0)} \\ \xi_3^{(0)} \\ \xi_4^{(0)} \\ \xi_5^{(0)} \end{bmatrix} \tag{4.2-7}$$

and

$$\boldsymbol{S}^{(0)} \stackrel{\text{def}}{=} \begin{bmatrix} S_{11}^{(0)} & S_{12}^{(0)} & S_{13}^{(0)} & S_{14}^{(0)} & S_{15}^{(0)} \\ S_{21}^{(0)} & S_{22}^{(0)} & S_{23}^{(0)} & S_{24}^{(0)} & S_{25}^{(0)} \\ S_{31}^{(0)} & S_{32}^{(0)} & S_{33}^{(0)} & S_{34}^{(0)} & S_{35}^{(0)} \\ S_{41}^{(0)} & S_{42}^{(0)} & S_{43}^{(0)} & S_{44}^{(0)} & S_{45}^{(0)} \\ S_{51}^{(0)} & S_{52}^{(0)} & S_{53}^{(0)} & S_{54}^{(0)} & S_{55}^{(0)} \end{bmatrix} \tag{4.2-8}$$

are given. The latter is positive semi-definite and symmetric.

4.2.5 *The disturbances*

The ship is subject to the influences of waves, wind and current. In addition, if the pre-specified path C_{AB} is curved, there are improper disturbances.

As we know, the proper and improper disturbances are accounted for by the fourth and third term, respectively, on the right-hand side of the equality sign in (2.4-1a). For $Y_d(k)$ and $N_d(k)$, we write

$$\begin{bmatrix} Y_d(\mathrm{k}) \\ N_d(k) \end{bmatrix} = \begin{bmatrix} Y_{\text{waves}}(k) \\ N_{\text{waves}}(k) \end{bmatrix} + \begin{bmatrix} Y_{\text{wind}}(k) \\ N_{\text{wind}}(k) \end{bmatrix} + \begin{bmatrix} Y_{\text{current}}(k) \\ N_{\text{current}}(k) \end{bmatrix} \tag{4.2-9}$$

The forward speed of the ship as well as the current everywhere on and near C_{AB} is supposed to be known. Therefore, $x(k)$, $(k = 1, 2, \ldots, n)$, is considered a sequence of quantities known beforehand.

From this, $a_C(k)$, $r_C(k)$, $v_{\text{current}}(k)$, $Y_{\text{current}}(k)$ and $N_{\text{current}}(k)$, $(k = 0, 1, \ldots, n-1)$, can be computed beforehand. Furthermore, we assume that also $Y_{\text{wind}}(k)$ and $N_{\text{wind}}(k)$, $(k = 0, 1, \ldots, n-1)$, can be computed beforehand, which means that turbulence is neglected.

Thus, the influences of wind and current as well as the improper disturbances are deterministic, and the only stochastic component of the disturbances is the influence of waves. It is assumed that $Y_{\text{waves}}(k)$ and $N_{\text{waves}}(k)$ form a gaussian random process with

$$\mathrm{E}\left\{\begin{bmatrix} Y_{\text{waves}}(k) \\ N_{\text{waves}}(k) \end{bmatrix}\right\} = \mathbf{0} \tag{4.2-10}$$

for each $k \in \{0, 1, \ldots, n-1\}$

The quantities

$$\operatorname{cov}\left\{\begin{bmatrix} Y_{\text{waves}}(k_1) \\ N_{\text{waves}}(k_1) \end{bmatrix}, \begin{bmatrix} Y_{\text{waves}}(k_2) \\ N_{\text{waves}}(k_2) \end{bmatrix}\right\}$$

k_1 and $k_2 \in \{0, 1, \ldots, n-1\}$
are *not* specified. We shall discuss this point in sec. 4.8.

4.2.6 *The observational errors*

The vectors $\boldsymbol{w}^{(2)}(k)$, $(k = 0, 1, \ldots, n-1)$, defined

$$\boldsymbol{w}^{(2)}(k) \stackrel{\text{def}}{=} \begin{bmatrix} w_4^{(2)}(k) \\ w_5^{(2)}(k) \end{bmatrix} \tag{4.2-11}$$

form a gaussian random process, with

$$\mathrm{E}\,\boldsymbol{w}^{(2)}(k) = \mathbf{0} \tag{4.2-12}$$

$$\operatorname{cov}\{\boldsymbol{w}^{(2)}(k_1), \boldsymbol{w}^{(2)}(k_2)\} = \begin{cases} \boldsymbol{R}^{(2)} \stackrel{\text{def}}{=} \begin{bmatrix} R_{44}^{(2)} & 0 \\ 0 & R_{55}^{(2)} \end{bmatrix} & \text{if} \quad k_1 = k_2 \\ \mathbf{0} \quad \text{if} \quad k_1 \neq k_2 \end{cases} \tag{4.2-13}$$

for each k, k_1 and k_2, where $R_{44}^{(2)}$ and $R_{55}^{(2)}$ are positive and known.

The observational errors are supposed to be statistically independent of both the initial state and the disturbances.

4.2.7 *The performance index*

We shall have to choose a certain functional, called the performance index, as the object of optimization: the ship guidance controller will be said to be optimal if it optimizes that functional.

The choice of this criterion must be a compromise. On one hand, the optimal ship guidance controller should be the best possible one from a nautical point of view. On the other hand, the mathematical form of the performance index should be such that the designing of the optimal controller is relatively simple from the mathematical and computational points of view, and leads to a result which can be implemented without excessive complication. A particularly favourable choice of the optimization object is one such that the design problem can be classified as a case of the basic problem of chapter 7.

As is easy to see, at sampling instant k, the greatest horizontal distance $l(k)$ from any point of the ship's lateral plane to the pre-specified track C_{AB}, which may be called the „corridor width", is

$$l(k) = |y(k)| + \tfrac{1}{2}L|\sin\psi(k)| \tag{4.2-14}$$

provided that everywhere on C_{AB} the radius of curvature is large relative to the ship's length L, and that the ships' center of gravity is exactly amidships.

An a posteriori measure of the extent to which the ship has failed to sail along the desired path C_{AB}, is the root-mean-square of the greatest horizontal distance

$$l_{\text{rms}} = \sqrt{\frac{1}{n}\sum_{1}^{n} l^2(k)} \tag{4.2-15}$$

Therefore, a reasonable choice for the object to be minimized by an optimal ship guidance controller for track-keeping seems to be the a priori expectation

$$\mathscr{W}^{(l)} \overset{\text{def}}{=} \mathrm{E}\sum_{1}^{n} l^2(k) \tag{4.2-16}$$

Unfortunately, $\mathscr{W}^{(l)}$ is not attractive from a mathematical point of view. We have

$$l^2(k) = y^2(k) + (\tfrac{1}{2}L)^2\psi^2(k) + L|y(k)\psi(k)| \tag{4.2-17}$$

and owing to the absolute value term in this expression the problem of minimizing $\mathscr{W}^{(l)}$ cannot be solved with the aid of the methods of chapter 7.

For this reason we shall use as the performance index the risk $\mathscr{W}$ defined

$$\mathscr{W} \overset{\text{def}}{=} \mathrm{E}\sum_{1}^{n} \mathscr{L}(k) \tag{4.2-18}$$

based on the following definition of the loss functions:

$$\mathscr{L}(k) \stackrel{\text{def}}{=} y^2(k)+(\tfrac{1}{2}L)^2\psi^2(k) \tag{4.2-19}$$

($k = 1, 2, \ldots, n$)

The loss functions $\mathscr{L}(k)$ thus defined show a reasonable similarity to $l^2(k)$ in that a control strategy minimizing $\mathscr{W}$ may be expected to be not too far from optimal for $\mathrm{E}\sum_1^n l^2(k)$. The important advantage of $\mathscr{L}(k)$ is that it makes the designing problem of the controller amenable to the methods presented in chapter 7.

We shall also be interested in

$$\max_{k \in \{1, 2, \ldots, n\}} \mathrm{E}\,\mathscr{L}(k)$$

because obviously it will be undesirable that this quantity is extremely large compared with the mean, with respect to k, of $\mathrm{E}\,\mathscr{L}(k)$.

4.2.8 *Constraints on the rudder movements*

The rudder movements are subject to two sorts of constraints.

In the first place, there is a maximal rudder angle and there is a maximal rudder angular velocity. The following conditions, which may be called „hard limitations", have to be satisfied:

$$-\delta_{\max} \leqslant \delta(k) \leqslant \delta_{\max} \tag{4.2-20}$$

$$-\varrho_{\max} \leqslant \varrho(k) \leqslant \varrho_{\max} \tag{4.2-21}$$

for each k, where $\delta_{\max}$ and $\varrho_{\max}$ are positive and known.

Secondly, several economic and other factors, such as fuel consumption, loss of speed, wear of the steering gear, and comfort of persons aboard, make it desirable that the root-mean-square values of δ and ϱ be kept substantially below $\delta_{\max}$ and $\varrho_{\max}$, respectively. These requirements, which might be called „soft limitations", can be cast in the mould of the following pair of conditions:

$$\mathrm{E}\sum_1^n \delta^2(k) \leqslant F_\delta n\delta_{\max}^2 \tag{4.2-22}$$

$$\mathrm{E}\sum_0^{n-1} \varrho^2(k) \leqslant F_\varrho n\varrho_{\max}^2 \tag{4.2-23}$$

where F_δ and F_ϱ are constants between 0 and 1 to be pre-specified.

4.2.9 *The mathematical form of the controller*

A controller, i.e. a control strategy, of the mathematical form

$$\varrho(k) = \varrho_k\{\boldsymbol{\eta}(0), \boldsymbol{\eta}(1), \ldots, \boldsymbol{\eta}(k), \varrho(0), \varrho(1), \ldots, \varrho(k-1)\} \tag{4.2-24}$$

$(k = 0, 1, \ldots, n-1)$,
has to be found which optimizes the performance index $\mathscr{W}$ under the constraints on the rudder movements as discussed in point 4.2.8 of this section.

Instead of the optimal controller, sub-optimal controllers which are easier to design and to implement will get the greatest attention in this chapter.

We note that $\varrho(k)$ is allowed to depend on $\boldsymbol{\eta}(k)$. Like we remarked before, this implies that we put no limitations on the speed of computation of the controller.

4.3 Stochastic treatment of the mean disturbances

We introduce the symbol $\boldsymbol{d}(k)$ for the mean, i.e. the ensemble average, of the sum of the disturbance terms occurring in (2.4-1a):

$$\boldsymbol{d}(k) \stackrel{\text{def}}{=} \begin{bmatrix} d_1(k) \\ d_2(k) \\ d_3(k) \\ d_4(k) \\ d_5(k) \end{bmatrix} \stackrel{\text{def}}{=} \mathrm{E}\left\{\boldsymbol{\Gamma}\begin{bmatrix} a_C(k) \\ r_C(k) \end{bmatrix} + \boldsymbol{\Theta}\begin{bmatrix} Y_d(k) \\ N_d(k) \end{bmatrix}\right\} \tag{4.3-1}$$

for each k. We also introduce

$$\boldsymbol{s}(k) \stackrel{\text{def}}{=} \begin{bmatrix} s_1(k) \\ s_2(k) \\ s_3(k) \\ s_4(k) \\ s_5(k) \end{bmatrix} \stackrel{\text{def}}{=} \boldsymbol{\Gamma}\begin{bmatrix} a_C(k) \\ r_C(k) \end{bmatrix} + \boldsymbol{\Theta}\begin{bmatrix} Y_d(k) \\ N_d(k) \end{bmatrix} - \boldsymbol{d}(k) \tag{4.3-2}$$

In point (4.2.5) of the preceding section, we assumed that the influence of waves is stochastic, with zero mean, and that the influences of wind and current as well as the improper disturbances are deterministic. Hence we can say that $\boldsymbol{d}(k)$ and $\boldsymbol{s}(k)$ are the deterministic and stochastic part, respectively, of the proper and improper disturbances; we further define

$$\boldsymbol{D}(k) \stackrel{\text{def}}{=} \begin{bmatrix} Y_{\text{det}}(k) \\ N_{\text{det}}(k) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} Y_{wind}(k) + Y_{current}(k) + Y_{curv}(k) \\ N_{\text{wind}}(k) + N_{\text{current}}(k) + N_{\text{curv}}(k) \end{bmatrix} \tag{4.3-3}$$

where $Y_{\text{curv}}(k)$ and $N_{\text{curv}}(k)$ account for the influences of $a_C(k)$ and $r_C(k)$, as discussed in sec. 2.3.

With respect to the deterministic disturbances, we shall sometimes take the "stochastic" point of view, discussed in sec. 7.8. This means that it is assumed, that these deterministic disturbances can be fed into the controller before a run, and that during each run the controller generates a signal based on this advance information, whereas *we* do *not* wish to consider the particular values of the deterministic disturbances. For us, the sequence $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$, is to be viewed as a random process.

More specifically, we let this sequence be generated by a system characterized by

$$\boldsymbol{D}(k+1) = \boldsymbol{\Upsilon}\boldsymbol{D}(k) + \boldsymbol{w}^{(3)}(k) \tag{4.3-4}$$

for each $k \in \{\ldots, -1, 0, 1, \ldots\}$
where

$$\boldsymbol{\Upsilon} \stackrel{\text{def}}{=} \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{bmatrix} \tag{4.3-5}$$

has two eigenvalues with an absolute value less than unity. The sequence

$$\boldsymbol{w}^{(3)}(k) \stackrel{\text{def}}{=} \begin{bmatrix} w_1^{(3)}(k) \\ w_2^{(3)}(k) \end{bmatrix}, \quad (k = \ldots, -1, 0, 1, \ldots) \tag{4.3-6}$$

is a gaussian random process, with

$$\mathrm{E}\, \boldsymbol{w}^{(3)}(k) = \boldsymbol{0} \tag{4.3-7}$$

$$\mathrm{cov}\{\boldsymbol{w}^{(3)}(k_1),\ \boldsymbol{w}^{(3)}(k_2)\} = \begin{cases} \boldsymbol{R}^{(3)} \stackrel{\text{def}}{=} \begin{bmatrix} R_{11}^{(3)} & R_{12}^{(3)} \\ R_{21}^{(3)} & R_{22}^{(3)} \end{bmatrix} & \text{if} \quad k_1 = k_2 \\ \boldsymbol{0} \quad \text{if} \quad k_1 \neq k_2 \end{cases} \tag{4.3-8}$$

for each k, k_1 and k_2, where $\boldsymbol{R}^{(3)}$ is positive semi-definite, symmetric and known.

By virtue of this characterization of the deterministic disturbances, we have

$$\mathrm{E}\, \boldsymbol{D}(k) = \boldsymbol{0} \tag{4.3-9}$$

$$\mathrm{cov}\{\boldsymbol{D}(k), \boldsymbol{D}(k)\} = \boldsymbol{R}^{(3)} + \boldsymbol{\Upsilon}\boldsymbol{R}^{(3)}\boldsymbol{\Upsilon}^{\mathrm{T}} + \boldsymbol{\Upsilon}^2\boldsymbol{R}^{(3)}(\boldsymbol{\Upsilon}^{\mathrm{T}})^2 + \ldots \tag{4.3-10}$$

for each k. So the random process $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$, is stationary.

For $\boldsymbol{d}(k)$, we have

$$\boldsymbol{d}(k) = \boldsymbol{\Theta} \begin{bmatrix} Y_{\mathrm{det}}(k) \\ N_{\mathrm{det}}(k) \end{bmatrix} \tag{4.3-11}$$

for each k.

Whenever the deterministic disturbances are treated stochastically, the expectation of the initial state $\xi^{(0)}$ will be treated stochastically too, which means that the controller is supposed to generate a control signal based on exact knowledge of $\xi^{(0)}$, whereas for us $\xi^{(0)}$ is to be regarded as a random variable. The probability density of $\xi^{(0)}$ is then assumed to be gaussian, with

$$\mathrm{E}\,\xi^{(0)} = \mathbf{0} \tag{4.3-12}$$

We shall subsequently use the symbol E_d to designate expectations with respect to $\xi^{(0)}$ and $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$, treated stochastically.

4.4 Formulation and solution of a modified version of the control problem

We return to the problem statement given in sec. 4.2. We now make the following two modifications:

1. The covariance of $Y_{\text{waves}}(k)$ and $N_{\text{waves}}(k)$ is specified as follows:

$$\operatorname{cov}\left\{\begin{bmatrix} Y_{\text{waves}}(k_1) \\ N_{\text{waves}}(k_1) \end{bmatrix}, \begin{bmatrix} Y_{\text{waves}}(k_2) \\ N_{\text{waves}}(k_2) \end{bmatrix}\right\} = \begin{cases} \boldsymbol{R}^{(1)} \stackrel{\text{def}}{=} \begin{bmatrix} R_{11}^{(1)} & R_{12}^{(1)} \\ R_{21}^{(1)} & R_{22}^{(1)} \end{bmatrix} & \text{if} \quad k_1 = k_2 \\ \mathbf{0} \quad \text{if} \quad k_1 \neq k_2 \end{cases} \tag{4.4-1}$$

   for each k_1 and k_2, where $\boldsymbol{R}^{(1)}$ is a known positive semi-definite and symmetric matrix.
2. We define

$$\mathscr{L}^{(\lambda)}(k) \stackrel{\text{def}}{=} \mathscr{L}(k) + \lambda_\delta \delta^2(k) + \lambda_\varrho \varrho^2(k-1) \tag{4.4-2}$$

   for each k, where λ_δ and λ_ϱ are given positive constants. We also define

$$\mathscr{W}^{(\lambda)} \stackrel{\text{def}}{=} \mathrm{E} \sum_1^n \mathscr{L}^{(\lambda)}(k) \tag{4.4-3}$$

   The problem is to design the controller so as to minimize $\mathscr{W}^{(\lambda)}$.

The design problem thus specified is a case of the basic problem of chapter 7.

As we know, the optimal controller will consist of two distinct parts, being the estimator and the controller proper.

The estimator can be designed by means of the results of sec. 7.2. We notice that according to (4.2-4), ξ_1 is observed without error. Hence we have the trivial relation

$$\xi_1(k|k) = \eta_1(k) \tag{4.4-4}$$

for each k, where

$$\boldsymbol{K}(k) \stackrel{\text{def}}{=} \begin{bmatrix} K_{24}(k) & K_{25}(k) \\ K_{34}(k) & K_{35}(k) \\ K_{44}(k) & K_{45}(k) \\ K_{54}(k) & K_{55}(k) \end{bmatrix} \tag{4.4-10}$$

$(k = 0, 1, \ldots, n-1)$
are matrices for which we are able to compute the optimal values.

The controller proper can be designed by straightforward application of the results of sec. 7.3. It has the form

$$\varrho(k) = -\boldsymbol{C}_{\xi}(k)\boldsymbol{\xi}(k|k) - c_d(k) \tag{4.4-11}$$

for each k, with $c_d(k)$ being expressible as

$$c_d(k) = \sum_{j=0}^{n-k-1} \boldsymbol{C}_d(k, k+j)\, \boldsymbol{d}(k+j) \tag{4.4-12}$$

which we prefer to rewrite as

$$c_d(k) = \sum_{j=0}^{n-k-1} \boldsymbol{C}_D(k, k+j)\, \boldsymbol{D}(k+j) \tag{4.4-13}$$

$\boldsymbol{d}(k+j)$ and $\boldsymbol{D}(k+j)$ are the quantities defined by (4.3-1) and (4.3-3), respectively; in consequence of (4.3-11),

$$\boldsymbol{C}_D(k, k+j) = \boldsymbol{C}_d(k, k+j)\, \boldsymbol{\Theta} \tag{4.4-14}$$

for each k and j. The quantities

$$\boldsymbol{C}_{\xi}(k) \stackrel{\text{def}}{=} [C_{\xi_1}(k)\ C_{\xi_2}(k)\ C_{\xi_3}(k)\ C_{\xi_4}(k)\ C_{\xi_5}(k)] \tag{4.4-15}$$

and

$$\boldsymbol{C}_D(k, k+j) \stackrel{\text{def}}{=} [C_{D_1}(k, k+j)\ C_{D_2}(k, k+j)] \tag{4.4-16}$$

$(k = n-1, n-2, \ldots, 0;\ j = n-k-1, n-k-2, \ldots, 0)$
are matrices for which we are able to find the optimal values.

The design problem dealt with in this section will be referred to subsequently as *problem 4.4*. The optimal controller forming the solution will be called *controller 4.4*. The closed-loop system consisting of a ship for which all assumptions of problem 4.4 are valid, controlled by controller 4.4, will be called *system 4.4*.

Likewise, we write

$$\mathrm{E}\delta^2(k) = \mathrm{E}\delta'^2(k)+\delta''^2(k) \tag{4.5-10}$$

$$\mathrm{E}\varrho^2(k) = \mathrm{E}\varrho'^2(k)+\varrho''^2(k) \tag{4.5-11}$$

with

$$\delta'(k) \stackrel{\text{def}}{=} \delta(k)-\mathrm{E}\delta(k) \tag{4.5-12}$$

$$\delta''(k) \stackrel{\text{def}}{=} \mathrm{E}\delta(k) \tag{4.5-13}$$

$$\varrho'(k) \stackrel{\text{def}}{=} \varrho(k)-\mathrm{E}\varrho(k) \tag{4.5-14}$$

$$\varrho''(k) \stackrel{\text{def}}{=} \mathrm{E}\varrho(k) \tag{4.5-15}$$

for each k.

Let us now suppose that $n \to \infty$. If we next let $k \to \infty$, keeping $n-k = \infty$, then $\mathrm{E}\mathscr{L}'(k)$, $\mathrm{E}\delta'^2(k)$ and $\mathrm{E}\varrho'^2(k)$ approach finite limits, to be denoted

$$\mathrm{E}\mathscr{L}'_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}\mathscr{L}'(k) \tag{4.5-16}$$

$$\mathrm{E}\delta'^2_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}\delta'^2(k) \tag{4.5-17}$$

$$\mathrm{E}\varrho'^2_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}\varrho'^2(k) \tag{4.5-18}$$

Furthermore, taking the stochastic point of view with respect to the deterministic disturbances, we see that $\mathrm{E}_d\mathscr{L}''(k)$, $\mathrm{E}_d\delta''^2(k)$ and $\mathrm{E}_d\varrho''^2(k)$ approach finite limits, to be denoted

$$\mathrm{E}_d\mathscr{L}''_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}_d\mathscr{L}''(k) \tag{4.5-19}$$

$$\mathrm{E}_d\delta''^2_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}_d\delta''^2(k) \tag{4.5-20}$$

$$\mathrm{E}_d\varrho''^2_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}_d\varrho''^2(k) \tag{4.5-21}$$

Hence we can say that both with respect to the stochastic inputs and with respect to the deterministic disturbances, treated stochastically, system 4.4.TI has quasi-steady states. We further introduce

$$\mathrm{E}_d\mathrm{E}\mathscr{L}_{qs} \stackrel{\text{def}}{=} \mathrm{E}\mathscr{L}'_{qs}+\mathrm{E}_d\mathscr{L}''_{qs} \tag{4.5-22}$$

$$\mathrm{E}_d\mathrm{E}\delta^2_{qs} \stackrel{\text{def}}{=} \mathrm{E}\delta'^2_{qs} +\mathrm{E}_d\delta''^2_{qs} \tag{4.5-23}$$

$$\mathrm{E}_d\mathrm{E}\varrho^2_{qs} \stackrel{\text{def}}{=} \mathrm{E}\varrho'^2_{qs} +\mathrm{E}_d\varrho''^2_{qs} \tag{4.5-24}$$

We next focus our attention on the final period.

With respect to the stochastic input signals, system 4.4.TI is causal. In other words, it is not affected by future inputs. From this follows that if k is large, it will make no difference for the system whether k is close to n or not. The final period is merely a continuation of the intermediate period, the system will remain in the quasi-steady state, and hence $E\mathscr{L}'(k)$, $E\delta'^2(k)$ and $E\varrho'^2(k)$ remain equal to $E\mathscr{L}'_{qs}$, $E\delta'^2_{qs}$ and $E\varrho'^2_{qs}$, respectively.

As for the deterministic disturbances, treated stochastically, system 4.4.TI cannot be said to be causal: future deterministic disturbances affect the system because of the anticipative nature of the control signal. Consequently, the system will not remain in the quasi-steady state.

Let us suppose for a moment that the controller were made to anticipate fictitious deterministic disturbances at $k = n, n+1, \ldots$, and that these disturbances, when treated stochastically, would have the same statistical characteristics as the deterministic disturbances at $k = 0, 1, \ldots, n-1$. In that case, the final period would be a continuation of the intermediate periode again, the system would remain in the quasi-steady state and hence $E_d\mathscr{L}''(k)$, $E_d\delta''^2(k)$ and $E_d\varrho''^2(k)$ in the final period would remain equal to $E_d\mathscr{L}''_{qs}$, $E_d\delta''^2_{qs}$ and $E_d\varrho''^2_{qs}$, respectively.

The actual situation differs from the one just pictured in that the controller does not have to anticipate fictitious deterministic disturbances occurring at $k = n$ and later. It will be intuitively clear, and the proof is straightforward, that this will improve the performance of the controller in the final period. So, $E_d\mathscr{L}''(k)$, $E_d\delta''^2(k)$ and $E_d\varrho''^2(k)$ will be smaller than $E_d\mathscr{L}''_{qs}$, $E_d\delta''^2_{qs}$ and $E_d\varrho''^2_{qs}$, respectively, for each k in the final period.

Finally, we take the initial period under consideration.

In this period, $E\mathscr{L}'(k)$, $E\delta'^2(k)$ and $E\varrho'^2(k)$ are under the influence of $\boldsymbol{S}^{(0)}$. It is important to note that there exist values of $\boldsymbol{S}^{(0)}$ such that the three quantities are all

Table 4.6-1 Summary of the results of sec. 4.6

	Initial period	Intermediate period	Final period
$E\mathscr{L}'(k)$ $E\delta'^2(k)$ $E\varrho'^2(k)$	Dependent on $\boldsymbol{S}^{(0)}$. If $\boldsymbol{S}^{(0)}$ "small enough", then smaller than the quasi-steady state values.	Equal to the quasi-steady state values.	Equal tot the quasi-steady state values.
$E_d\mathscr{L}''(k)$ $E_d\delta''^2(k)$ $E_d\varrho''^2(k)$	Dependent on the relation between $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$ and $\xi^{(0)}$; if (4.5-25), then smaller than the quasi-steady state values.	Equal to the quasi-steady state values.	Smaller than the quasi-steady state values.

under the constraints (4.2-22) and (4.2-23), repeated below:

$$\mathrm{E} \sum_{1}^{n} \delta^2(k) \leqslant F_\delta n \delta_{\max}^2 \tag{4.7-2}$$

$$\mathrm{E} \sum_{0}^{n-1} \varrho^2(k) \leqslant F_\varrho n \varrho_{\max}^2 \tag{4.7-3}$$

It is known [1] that the solution to this problem is found among the set of those controllers which minimize the quantity $\mathscr{W}^{(\lambda)}$ defined by (4.4-3) and (4.4-2), repeated here as

$$\mathscr{W}^{(\lambda)} = \mathrm{E} \sum_{1}^{n} \{y^2(k) + (\tfrac{1}{2}L)^2 \psi^2(k) + \lambda_\delta \delta^2(k) + \lambda_\varrho \varrho^2(k-1)\} \tag{4.7-4}$$

where the Lagrange multipliers λ_δ and λ_ϱ are non-negative. All controllers belonging to that set are of the type 4.4.

Since the controller as well as the quantities $\mathscr{W}^{(\lambda)}$, $\mathrm{E} \sum_1^n \delta^2(k)$ and $\mathrm{E} \sum_0^{n-1} \varrho^2(k)$ can be regarded as images of the $(\lambda_\delta, \lambda_\varrho)$-space, the design problem is a minimization problem, under constraints, in this space. For the solution of such a problem several methods can be found in the literature [1, 2, 3].

In the design problem just discussed, the covariance matrix of the stochastic disturbances $\boldsymbol{R}^{(1)}$ and the sequence of deterministic disturbances $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$, were considered as specified quantities. These quantities, however, not only depend on the specifications of the track C_{AB}, but also on the statistical characteristics of the waves, on the wind and on the current. This means that the controller is not only designed for one particular pre-specified track C_{AB}, but even for only one particular run of the ship, i.e. for each time the ship is to be guided from A to B.

The point that the statistical characteristics of the disturbances are different for each run will be returned on in the next section. We shall here put it aside temporarily, by means of the assumption that the value of the matrix $\boldsymbol{R}^{(1)}$ characterizing these disturbances is the same for each run.

As regards the deterministic disturbances, the design procedure above can be modified such that, in a sense, the resulting controller is optimal not for a particular sequence $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$, but for a set of such sequences. Thus, we not only eliminate the necessity to re-design the controller for each run along a particular pre-specified track C_{AB}, but the controller is also, to a certain extent, made independent of the particular length and shape of that curve. It should be well understood, however, that despite of this generalization, the controller still requires an exact specification of $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$. Only the law according to which the controller signal is generated, in particular the coefficients $\boldsymbol{C}_\xi(k)$ and $\boldsymbol{C}_D(k, k+j)$, $(k = 0, 1, \ldots, n-1;\ j = 0, 1, \ldots, n-k-1)$, in (4.4-11) and (4.4-13) are made the same for each run.

$$\max_{k \in \{1, 2, \ldots, n\}} \mathrm{E}_d \mathrm{E}\mathscr{L}(k) = \mathrm{E}_d \mathrm{E}\mathscr{L}_{qs} \tag{4.7-10}$$

$$\max_{k \in \{1, 2, \ldots, n\}} \mathrm{E}_d \mathrm{E}\delta^2(k) = \mathrm{E}_d \mathrm{E}\delta_{qs}^2 \tag{4.7-11}$$

$$\max_{k \in \{0, 1, \ldots, n-1\}} \mathrm{E}_d \mathrm{E}\varrho^2(k) = \mathrm{E}_d \mathrm{E}\varrho_{qs}^2 \tag{4.7-12}$$

provided that $\boldsymbol{\xi}^{(0)}$ is a suitable function of $\boldsymbol{D}(k)$, $(k = 0, 1, \ldots, n-1)$ and that $\boldsymbol{S}^{(0)}$ is "small enough". The quantity

$$\max_{k \in \{1, 2, \ldots, n\}} \mathrm{E}_d \mathrm{E}\mathscr{L}(k)$$

is indicative of

$$\max_{k \in \{1, 2, \ldots, n\}} \mathrm{E}\mathscr{L}(k)$$

which quantity came up for discussion in point 7 of sec. 4.2. Likewise,

$$\max_{k \in \{1, 2, \ldots, n\}} \mathrm{E}_d \mathrm{E}\delta^2(k) \quad \text{and} \quad \max_{k \in \{0, 1, \ldots, n-1\}} \mathrm{E}_d \mathrm{E}\varrho^2(k)$$

can be used as a measure of the probability that the "hard" and "soft" limitations on the rudder movements, discussed in point 8 of sec. 4.2, are violated.

Comparing a controller 4.4 with the corresponding controller 4.4.TI, we find that for the former the maxima of (4.7-10), (4.7-11) and (4.7-12) will not be smaller than for the latter, because of the identity of the two controllers in the intermediate period.

Lastly, it will be clear that a time-invariant controller, being so much easier to implement, is preferable to a time-varying one.

Summarizing, we propose the following procedure for solving the design problem as described in sec. 4.2, in the case of white stochastic disturbances, with the covariance matrix of the plant disturbances, $\boldsymbol{R}^{(1)}$, being supposed to be the same for each run. The controller is selected from the set of controllers 4.4.TI which is the image of the set of all pairs of non-negative real values of λ_δ and λ_ϱ. To that end, from the latter set the pair of values $(\lambda_\delta, \lambda_\varrho)$ is selected so as to minimize $\mathrm{E}_d \mathrm{E}\mathscr{L}_{qs}$ under the constraints

$$\mathrm{E}_d \mathrm{E}\delta_{qs}^2 \leqslant F_\delta \delta_{\max}^2 \tag{4.7-13}$$

$$\mathrm{E}_d \mathrm{E}\varrho_{qs}^2 \leqslant F_\varrho \varrho_{\max}^2 \tag{4.7-14}$$

where, as the E_d indicates, the deterministic disturbances are treated stochastically. The controller corresponding to that pair $(\lambda_\delta, \lambda_\varrho)$ is proposed as the solution to the control problem considered here.

$$\mathrm{E}_d\mathrm{E}\delta_{qs}^2 \leqslant F_\delta \delta_{\max}^2 \tag{4.8-3}$$

$$\mathrm{E}_d\mathrm{E}\varrho_{qs}^2 \leqslant F_\varrho \varrho_{\max}^2 \tag{4.8-4}$$

again. The significance of the symbols $\mathrm{E}_d\mathrm{E}\mathscr{L}_{qs}$, $\mathrm{E}_d\mathrm{E}\delta_{qs}^2$ and $\mathrm{E}_d\mathrm{E}\varrho_{qs}^2$ is modified as will be explained below.

In the preceding section we had

$$\mathrm{E}_d\mathrm{E}\mathscr{L}_{qs} = \mathrm{E}\mathscr{L}'_{qs} + \mathrm{E}_d\mathscr{L}''_{qs} \tag{4.8-5}$$

The term $\mathrm{E}\mathscr{L}'_{qs}$ represents the response of the system to two sets of inputs: the first consists of the forces Y_{waves} and the moments N_{waves}, the second of the observational errors $w_4^{(2)}$ and $w_5^{(2)}$. Since the control system is supposed to be linear, while $w_4^{(2)}$ and $w_5^{(2)}$ are supposed to be statistically independent of Y_{waves} and N_{waves}, $\mathrm{E}\mathscr{L}'_{qs}$ can be split up as

$$\mathrm{E}\mathscr{L}'_{qs} = \mathrm{E}_{\text{waves}}\mathscr{L}'_{qs} + \mathrm{E}_{w^{(2)}}\mathscr{L}'_{qs} \tag{4.8-6}$$

where $\mathrm{E}_{\text{waves}}\mathscr{L}'_{qs}$ represents the influence of Y_{waves} and N_{waves} while $\mathrm{E}_{w^{(2)}}\mathscr{L}'_{qs}$ represents the influence of $w_4^{(2)}$ and $w_5^{(2)}$. Likewise, we can write

$$\mathrm{E}_d\mathrm{E}\delta_{qs}^2 = \mathrm{E}_{\text{waves}}\delta_{qs}'^2 + \mathrm{E}_{w^{(2)}}\delta_{qs}'^2 + \mathrm{E}_d\delta_{qs}''^2 \tag{4.8-7}$$

$$\mathrm{E}_d\mathrm{E}\varrho_{qs}^2 = \mathrm{E}_{\text{waves}}\varrho_{qs}'^2 + \mathrm{E}_{w^{(2)}}\varrho_{qs}'^2 + \mathrm{E}_d\varrho_{qs}''^2 \tag{4.8-8}$$

For the case at hand, the terms $\mathrm{E}_{w^{(2)}}\mathscr{L}'_{qs}$, $\mathrm{E}_{w^{(2)}}\delta_{qs}'^2$, $\mathrm{E}_{w^{(2)}}\varrho_{qs}'^2$, $\mathrm{E}_d\mathscr{L}''_{qs}$, $\mathrm{E}_d\delta_{qs}''^2$ and $\mathrm{E}_d\varrho_{qs}''^2$ are found in the same manner as in the preceding section, but owing to the variable statistical characteristics of Y_{waves} and N_{waves}, the quantities $\mathrm{E}_{\text{waves}}\mathscr{L}'_{qs}$, $\mathrm{E}_{\text{waves}}\delta_{qs}'^2$ and $\mathrm{E}_{\text{waves}}\varrho_{qs}'^2$ are now undefined.

The last mentioned three quantities are therefore re-defined on the basis of a *norm-experiment*. This term is used to signify that a special sequence of forces $Y(k)$ and moments $N(k)$, $(k = -n'', -n''+1, \ldots, 0, 1, \ldots, n'-1)$, is chosen, representative of the sequences occurring in reality, and that the response of the control system to this sequence is computed. The three quantities are then defined

$$\mathrm{E}_{\text{waves}}\mathscr{L}'_{qs} \overset{\text{def}}{=} \frac{1}{n'} \sum_{1}^{n'} \mathscr{L}(k) \tag{4.8-9}$$

$$\mathrm{E}_{\text{waves}}\delta_{qs}'^2 \overset{\text{def}}{=} \frac{1}{n'} \sum_{1}^{n'} \delta^2(k) \tag{4.8-10}$$

$$\mathrm{E}_{\text{waves}}\varrho_{qs}'^2 \overset{\text{def}}{=} \frac{1}{n'} \sum_{0}^{n'-1} \varrho^2(k) \tag{4.8-11}$$

4.9.3 *The determination of* $\boldsymbol{D}(k)$ *and* $c_d(k)$, $(k = 0, 1, \ldots, n-1)$

In secs. 4.2 and 4.4 we stated that $\boldsymbol{D}(k)$ and $c_d(k)$, $(k = 0, 1, \ldots, n-1)$, are to be considered as known in advance.

At each sampling instant k, the quantity $\boldsymbol{D}(k)$ is a function of $x(k)$, the ship's position along C_{AB}. Therefore, $\boldsymbol{D}(k)$ and $c_d(k)$, $(k = 0, 1, \ldots, n-1)$, can only be computed in advance if $x(k)$, $(k = 0, 1, \ldots, n-1)$, can be computed in advance. But since the ship's forward speed must not be expected to be exactly equal to its nominal value, the actual values of $x(k)$ may differ from the nominal values; the differences in $x(k)$ may be large even though the variations in the ship's forward speed are small.

The difficulty we encounter here can be avoided by storing $\boldsymbol{D}(k)$ and $c_d(k)$, $(k = 0, 1, \ldots, n-1)$, as functions of x rather than k, which means that at each sampling instant k the ship's position $x(k)$ has to be observed before $\boldsymbol{D}(k)$ and $c_d(k)$ can be looked up in the computer memory.

At each sampling instant k, the observed value $\eta_x(k)$ of $x(k)$ will differ from the actual value of $x(k)$, because of observational errors. However, since the positioning system used will be rather accurate, while the curvature of C_{AB} will be small, the observational error in $\eta_x(k)$ will normally have a negligible effect on $\boldsymbol{d}(k)$ and $c_d(k)$. Thus, $x(k)$ can be taken to be equal to $\eta_x(k)$ for each k. If the effect of the observational error in $\eta_x(k)$ is non-negligible, the estimation technique described in sec 7.2 can be used to reduce this effect.

4.9.4 *Inertial navigation*

We assumed that the measurement errors, in addition to being white, have a covariance matrix which is constant with respect to k.

In case the ship's position is measured by means of an inertial navigation system, the observational errors of this system with regard to the ship's position do not comply with those assumptions. A simple mathematical model of the inertial navigation system consists of two pairs of cascaded integrators, one pair for the x_0-direction and one pair for the y_0-direction. The inputs of these integrator pairs are the actual accelerations in the x_0- and y_0-direction, respectively, together with noise possibly to be considered as white noise with a covariance matrix which is constant with respect to k.

The problem in the case of positioning by means of an inertial navigation system can be made amenable to the theory of chapter 7 by including the navigation system in the plant, which means that the state variables pertinent to this system are considered as plant state variables and that the noise input to the navigation system is treated like the plant disturbances.

4.9.5 *Estimation of disturbances*

Our treatment of the problem results in a procedure where the deterministic parts of the disturbances along with the control terms $c_d(k)$, $(k = 0, 1, \ldots, n-1)$, are com-

Chapter 5

RESULTS OF NUMERICAL COMPUTATIONS

5.1 Introduction; characteristics of the ship

This chapter is a report of an investigation consisting of computations on a digital computer. For a particular ship a guidance controller was designed, by means of the procedure proposed in the preceding chapter, and the performance of the resulting controller was evaluated.

The system of units used throughout this chapter is the system based on:

- the kilogram (kg) as the unit of mass,
- the meter (m) as the unit of length, and
- the second (sec) as the unit of time.

Angles are expressed in radians (rad).

The data of the particular ship for which the design problem was worked out were borrowed from Van Leeuwen [1], who carried out experiments on a particular model in a towing tank and determined the force and moment coefficients $M-Y_{\dot{v}}$, $Y_{\dot{r}}$, $N_{\dot{v}}$, $I_{zz}-N_{\dot{r}}$, Y_v, Y_r-Mu, N_v, N_r and Y_δ, introduced in sec. 2.2.

The model used was one of the Todd Sixty Series with block-coefficient 0.70 (a freighter model) and made of polyester. The length between perpendiculars was 2.258 m, while the speed was 0.928 m sec^{-1}.

Since the model tested by Van Leeuwen represents a fictitious ship, we could make an arbitrary choice for the length of the full-size ship. We chose

$$L = 160.0 \text{ m}$$

where L is the length between perpendiculars of the full size ship. According to a well-known rule, the *Froude number* of the full-size ship must be equal to the one of the model. The Froude number F_n is defined

$$F_n \stackrel{\text{def}}{=} \frac{u}{\sqrt{gL}} \tag{5.1-1}$$

where

u = the speed,
L = the length between perpendiculars,
g = the acceleration due to gravity.

Substituting $u = 0.928$ m sec^{-1}, $g = 9.80$ m sec^{-2} and $L = 2.258$ m, we find

$$F_n = 0.197$$

The values of the force and moment coefficients $M - Y_{\dot{v}}$ etc. are given in [1] in the non-dimensional form, designated by primes, which is generally used in naval architecture. These non-dimensional values, the formulae expressing the actual coefficients in the non-dimensional ones, and the values of the actual coefficients* are gathered in table 5.1-2. In the last mentioned formulae we have substituted $L = 160.0$ m, $u = 7.81$ m sec^{-1} and $\varrho_w = 1029$ kg m^{-3}; ϱ_w is the specific mass of the water.

5.2 The choice of the sampling interval and the computation of the ship's sampled-data coefficient values

With the aid of (2.2-5), (2.2-6) and (2.2-7), the values of the coefficients A_{22} etc. in (2.2-4) were calculated. The results are set down in table 5.2-1. The eigenvalues of the matrix

$$\begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}$$

are also given in that table, along with the corresponding time constants, which are the negatives of the reciprocals of the eigenvalues.

Before the element values of the matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Theta}$ could be figured out, the duration of the control system's sampling interval T_s had to be decided on first. In view of the well-known rule that the duration of the sampling interval should be (approximately) one tenth of the dominant time constant of the controlled object, we set

$$T_s = 5.65 \text{ sec}$$

The element values of the matrices $\boldsymbol{\Phi}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Theta}$ as they then follow from (2.4-2), (2.4-3), (2.4-4) and (2.4-5), respectively, are listed in table 5.2-2.

Table 5.2-1 Values of the coefficients in equation (2.2-4)

$A_{22} = -0.04368$ sec^{-1}	$A_{23} = -2.231$ m sec^{-1}
$A_{32} = -0.001340$ m^{-1} sec^{-1}	$A_{33} = -0.1328$ sec^{-1}
eigenvalues: -0.01769 and -0.1587 sec^{-1}	
time constants: 56.52 and 6.300 sec	
$A_{21} = -0.04110$ m sec^{-2}	
$A_{31} = 0.002189$ sec^{-2}	
$D_{21} = 0.2087 \times 10^{-7}$ kg^{-1}	$D_{22} = -0.4240 \times 10^{-10}$ kg^{-1} m^{-1}
$D_{31} = -0.5218 \times 10^{-10}$ kg^{-1} m^{-1}	$D_{32} = 0.1556 \times 10^{-10}$ kg^{-1} m^{-2}

* These values are given to four decimal places, regardless of accuracy considerations. This remark also applies to some of the subsequent tables and lists.

$$\sqrt{E_d E\delta_{qs}^2} \leqslant 0.1309 \text{ rad } (= 7°30')$$

$$\sqrt{E_d E\varrho_{qs}^2} \leqslant 0.01309 \text{ rad sec}^{-1} \ (= 0°45'0'' \text{ sec}^{-1})$$

which means that the probability of δ being outside the linear region is in the order of 0.8%.

We shall now deal with the procedure used to obtain $E_d E\mathscr{L}_{qs}$, $E_d E\delta_{qs}^2$ and $E_d E\varrho_{qs}^2$, given λ_δ, λ_ϱ, $R_{11}^{(1)}$, $R_{12}^{(1)}$ and $R_{22}^{(1)}$. The three quantities are composed of the contributions of the waves, the observational errors $w_4^{(2)}$ and $w_5^{(2)}$, the mean wind, the current and the curvature. These influences will now be discussed separately.

5.3.1 *The waves*

In sec. 3.3 we learned that a random sea can be thought of as a superposition of small "regular" components. In sec. 3.5 we derived formulae approximating the forces and moments exerted on a block-shaped ship proceeding in a regular sea. Thus, using the superposition principle, it is possible to find a statistical characterization of the forces and moments exerted on a block-shaped ship proceeding in a random sea whose energy spectrum $\mathscr{P}(\omega_w, \mu)$ is known. For these computations, the block shape will be a reasonable approximation for the actual shape of the ship.

For each run of the ship along the pre-specified path C_{AB}, however, $\mathscr{P}(\omega_w, \mu)$ may be different. Since we do not wish to re-design the ship guidance controller for each particular run, the influence of the waves on $E_d E\mathscr{L}_{qs}$, $E_d E\delta_{qs}^2$ and $E_d E\varrho_{qs}^2$ was determined by means of a *norm-procedure*: the ship was exposed to a sequence of forces and moments which is supposed to have about the same effect as the waves will have in an average case.

The choice of the norm-procedure was based on three considerations:

1. All directions of propagation, relative to the ship's path, of the principal regular components of the random sea are supposed to be equally probable.
2. It is a well-known fact that (at the North Sea) the circular frequency of the dominant regular component of the random sea is often in the order of 1.0 rad sec^{-1}.
3. For the computation of $E_d E\mathscr{L}_{qs}$, $E_d E\delta_{qs}^2$ and $E_d E\varrho_{qs}^2$, time averages and ensemble averages are interchangeable.

The norm-procedure consisted of letting the controller guide the block-shaped ship along a sufficiently long circular path in a regular sea and finding the averages of $y^2(k)+(\frac{1}{2}L)^2\psi^2(k)$, $\delta^2(k)$ and $\varrho^2(k)$, as far as these are due to the waves, over half a circle. These results were taken as the contribution of the waves to $E_d E\mathscr{L}_{qs}$, $E_d E\delta_{qs}^2$ and $E_d E\varrho_{qs}^2$, respectively.

Some numerical particulars: the regular sea had a circular wave frequency

$$\omega_w = 1.000 \text{ rad sec}^{-1}$$

N_{wind} was built: $Y_{\text{wind}}(k)$ and $N_{\text{wind}}(k)$, $(k = 0, 1, \ldots)$, are generated by a system characterized by

$$\begin{bmatrix} Y_{\text{wind}}(k+1) \\ N_{\text{wind}}(k+1) \end{bmatrix} = \Upsilon_{\text{wind}} \begin{bmatrix} Y_{\text{wind}}(k) \\ N_{\text{wind}}(k) \end{bmatrix} + \begin{bmatrix} w_1^{(\text{wind})}(k) \\ w_2^{(\text{wind})}(k) \end{bmatrix} \tag{5.3-1}$$

for each $k \in \{\ldots, -1, 0, 1, \ldots\}$

where Υ_{wind} is a non-negative constant less than unity, and where

$$\boldsymbol{w}^{(\text{wind})}(k) \stackrel{\text{def}}{=} \begin{bmatrix} w_1^{(\text{wind})}(k) \\ w_2^{(\text{wind})}(k) \end{bmatrix}, (k = \ldots, -1, 0, 1, \ldots) \tag{5.3-2}$$

is a vector-valued random process for which

$$\mathrm{E}\boldsymbol{w}^{(\text{wind})}(k) = \mathbf{0} \tag{5.3-3}$$

$$\operatorname{cov}\{\boldsymbol{w}^{(\text{wind})}(k_1), \boldsymbol{w}^{(\text{wind})}(k_2)\} = \begin{cases} \boldsymbol{R}^{(\text{wind})} & \text{if } k_1 = k_2 \\ \mathbf{0} & \text{if } k_1 \neq k_2 \end{cases} \tag{5.3-4}$$

for each k, k_1 and $k_2 \in \{\ldots, -1, 0, 1, \ldots\}$.

Furthermore,

$$\boldsymbol{R}^{(\text{wind})} \stackrel{\text{def}}{=} \begin{bmatrix} R_{11}^{(\text{wind})} & 0 \\ 0 & R_{22}^{(\text{wind})} \end{bmatrix} \tag{5.3-5}$$

where $R_{11}^{(\text{wind})}$ and $R_{22}^{(\text{wind})}$ are non-negative.

From this characterization of Y_{wind} and N_{wind} follows

$$\sqrt{\mathrm{E}Y_{\text{wind}}^2(k)} = \sqrt{\frac{R_{11}^{(\text{wind})}}{1 - \Upsilon_{\text{wind}}^2}} \tag{5.3-6}$$

$$\sqrt{\mathrm{E}N_{\text{wind}}^2(k)} = \sqrt{\frac{R_{22}^{(\text{wind})}}{1 - \Upsilon_{\text{wind}}^2}} \tag{5.3-7}$$

for each k.

Straightforward application of the material of sec. 7.9 gives us a procedure for finding the contributions to $\mathrm{E}_d\mathrm{E}\mathscr{L}_{qs}$, $\mathrm{E}_d\mathrm{E}\delta_{qs}^2$ and $\mathrm{E}_d\mathrm{E}\varrho_{qs}^2$ due to the mean wind.

For Υ_{wind} the following numerical value was chosen

$$\Upsilon_{\text{wind}} = 0.98$$

For this transverse current, the following model was built: $v_{\text{current}}(k)$, $(k = 0, 1, \ldots)$, is a random process generated by a system characterized by

$$v_{\text{current}}(k+1) = \Upsilon_{\text{current}} v_{\text{current}}(k) + w_{\text{current}}(k) \tag{5.3-9}$$

for each $k \in \{\ldots, -1, 0, 1, \ldots\}$

where $\Upsilon_{\text{current}}$ is a non-negatieve constant smaller than unity, and where $w_{\text{current}}(k)$, $(k = \ldots, -1, 0, 1, \ldots)$, is a gaussian random process for which

$$\mathrm{E} w_{\text{current}}(k) = 0 \tag{5.3-10}$$

$$\operatorname{cov}\{w_{\text{current}}(k_1), w_{\text{current}}(k_2)\} = \begin{cases} R_{\text{current}} & \text{if} \quad k_1 = k_2 \\ 0 & \text{if} \quad k_1 \neq k_2 \end{cases} \tag{5.3-11}$$

for each k, k_1 and $k_2 \in \{\ldots, -1, 0, 1, \ldots\}$

where R_{current} is another non-negative constant.

From this characterization of v_{current}, it follows that

$$\sqrt{\mathrm{E} v^2_{\text{current}}(k)} = \sqrt{\frac{R_{\text{current}}}{1 - \Upsilon^2_{\text{current}}}} \tag{5.3-12}$$

for each k.

Again, a procedure for finding the contributions of the current to $\mathrm{E}_d \mathrm{E} \mathscr{L}_{qs}$, $\mathrm{E}_d \mathrm{E} \delta^2_{qs}$ and $\mathrm{E}_d \mathrm{E} \varrho^2_{qs}$ follows by straightforward application of the material of sec. 7.9.

$\Upsilon_{\text{current}}$ was chosen equal to Υ_{wind}, hence

$$\Upsilon_{\text{current}} = 0.98$$

R_{current} was chosen such that

$$\sqrt{\mathrm{E} v^2_{\text{current}}(k)} = 0.717 \text{ m sec}^{-1}$$

for each k.

5.3.5 *The curvature*

For the curvature r_C of the ship's pre-specified path C_{AB}, a model was built similar to the one for the current: $r_C(k)$, $(k = 0, 1, \ldots)$, is a random process generated by a system characterized by

$$r_C(k+1) = \Upsilon_{\text{curv}} r_C(k) + w_{\text{curv}}(k) \tag{5.3-13}$$

for each $k \in \{\ldots, -1, 0, 1, \ldots\}$

5.4 Some figures characterizing the best controller found

The best controller that was found is characterized by the following figures:

$$\lambda_\delta = 400\ \mathrm{m}^2$$
$$\lambda_\varrho = 0.290 \times 10^8\ \mathrm{m}^2\,\mathrm{sec}^2$$
$$R_{11}^{(1)} = 0.100 \times 10^{14}\ \mathrm{kg}^2\,\mathrm{m}^2\,\mathrm{sec}^{-4}$$
$$R_{12}^{(1)} = 0\ \mathrm{kg}^2\,\mathrm{m}^3\,\mathrm{sec}^{-4}$$
$$R_{22}^{(1)} = 0.215 \times 10^{14}\ \mathrm{kg}^2\,\mathrm{m}^4\,\mathrm{sec}^{-4}$$

It is interesting to note that since the controller is designed so as to optimize the expectation of

$$y^2(k) + (\tfrac{1}{2}L)^2\psi^2(k) + \lambda_\delta\delta^2(k) + \lambda_\varrho\varrho^2(k-1)$$

Table 5.4-1 Values of the coefficients characterizing the best controller found

$\hat{C}_{\xi_1} = 0.06823\ \mathrm{sec}^{-1}$ $\quad$ $\hat{C}_{\xi_2} = -0.01409\ \mathrm{m}^{-1}$
$\hat{C}_{\xi_3} = 0.8549$ $\quad$ $\hat{C}_{\xi_4} = 0.09686\ \mathrm{sec}^{-1}$
$\hat{C}_{\xi_5} = 0.1498 \times 10^{-3}\ \mathrm{m}^{-1}\mathrm{sec}^{-1}$

j	$\hat{C}_{D_1}(j)$ in $\mathrm{sec\,kg^{-1}\,m^{-1}}$	$\hat{C}_{D_2}(j)$ in $\mathrm{sec\,kg^{-1}m^{-2}}$	j	$\hat{C}_{D_1}(j)$ in $\mathrm{sec\,kg^{-1}m^{-1}}$	$\hat{C}_{D_2}(j)$ in $\mathrm{sec\,kg^{-1}m^{-2}}$
0	-0.1777×10^{-8}	0.7466×10^{-10}	19	0.1881×10^{-9}	-0.4785×10^{-11}
1	-0.1513×10^{-8}	0.6699×10^{-10}	20	0.1626×10^{-9} →	-0.4870×10^{-11}
2	-0.1263×10^{-8}	0.5957×10^{-10}	21	0.1351×10^{-9}	-0.4741×10^{-11}
3	-0.1031×10^{-8}	0.5239×10^{-10}	22	0.1069×10^{-9}	-0.4442×10^{-11}
4	-0.8160×10^{-9}	0.4549×10^{-10}	23	0.7921×10^{-10}	-0.4061×10^{-11}
5	-0.6208×10^{-9}	0.3893×10^{-10}	24	0.5286×10^{-10}	-0.3499×10^{-11}
6	-0.4460×10^{-9}	0.3275×10^{-10}	25	0.2860×10^{-10}	-0.2928×10^{-11}
7	-0.2923×10^{-9}	0.2699×10^{-10}	26	0.6990×10^{-11}	-0.2334×10^{-11}
8	-0.1598×10^{-9}	0.2169×10^{-10}	27	-0.1162×10^{-10}	-0.1742×10^{-11}
9	-0.4837×10^{-10}	0.1687×10^{-10}	28	-0.2704×10^{-10}	-0.1174×10^{-11}
10	0.4270×10^{-10}	0.1256×10^{-10}	29	-0.3920×10^{-10}	-0.6473×10^{-12}
11	0.1144×10^{-9}	0.8760×10^{-11}	30	-0.4818×10^{-10}	-0.1750×10^{-12}
12	0.1681×10^{-9}	0.5480×10^{-11}	31	-0.5414×10^{-10}	0.2342×10^{-12}
13	0.2054×10^{-9}	0.2707×10^{-11}	32	-0.5733×10^{-10}	0.5751×10^{-12}
14	0.2280×10^{-9}	0.4227×10^{-12}	33 →	-0.5805×10^{-10}	0.8458×10^{-12}
15	→ 0.2378×10^{-9}	-0.1399×10^{-11}	34	-0.5666×10^{-10}	0.1047×10^{-11}
16	0.2368×10^{-9}	-0.2790×10^{-11}	35	-0.5351×10^{-10}	0.1183×10^{-11}
17	0.2270×10^{-9}	-0.3790×10^{-11}	36	-0.4898×10^{-10}	0.1258×10^{-11}
18	0.2101×10^{-9}	-0.4440×10^{-11}	37	-0.4343×10^{-10} →	0.1278×10^{-11}

$\hat{K}_{24} = -7.412\ \mathrm{m\,sec}^{-1}$ $\quad$ $\hat{K}_{25} = 0.001890\ \mathrm{sec}^{-1}$
$\hat{K}_{34} = 0.06721\ \mathrm{sec}^{-1}$ $\quad$ $\hat{K}_{35} = -0.1502 \times 10^{-4}\ \mathrm{m}^{-1}\mathrm{sec}^{-1}$
$\hat{K}_{44} = 0.6825$ $\quad$ $\hat{K}_{45} = -0.6377 \times 10^{-4}\ \mathrm{m}^{-1}$
$\hat{K}_{54} = -63.77\ \mathrm{m}$ $\quad$ $\hat{K}_{55} = 0.06182$

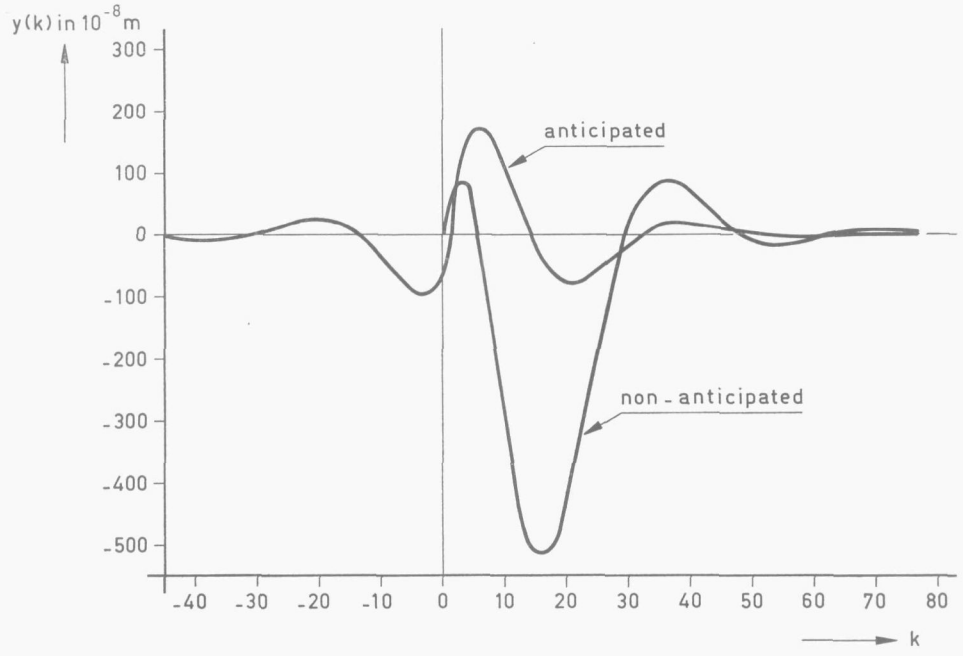


Fig. 5.4-2 Response to an anticipated and to a non-anticipated disturbing unit force

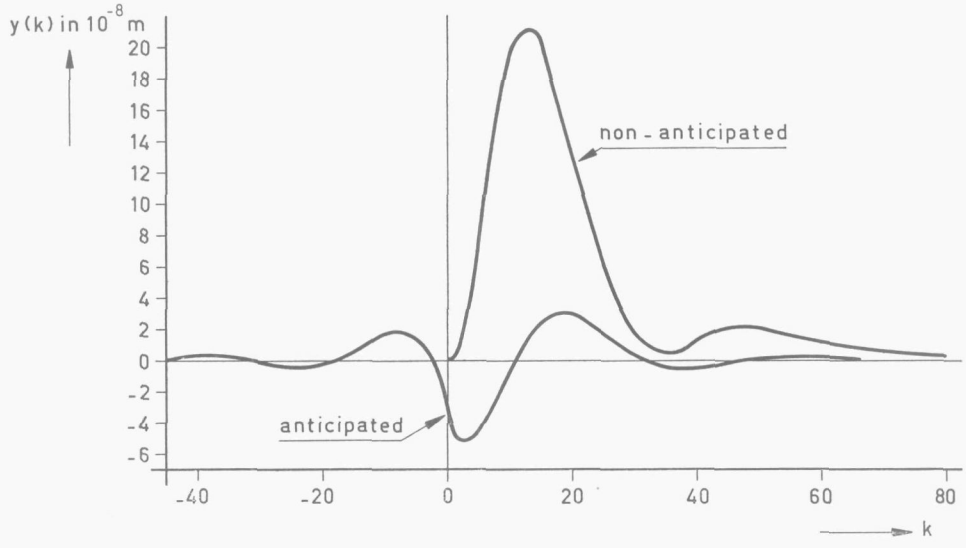


Fig. 5.4-3 Response to an anticipated and to a non-anticipated disturbing unit moment

Table 5.5-1 Variations of $\sqrt{E_d E \mathscr{L}_{qs}}$ due to parameter variations of $\pm$ 10%

normal value $\sqrt{E_d E \mathscr{L}_{qs}} = 13.54$ m

	$(\sqrt{E_d E \mathscr{L}_{qs}})_-$	$(\sqrt{E_d E \mathscr{L}_{qs}})_+$
Φ_{21}	13.57	13.56
Φ_{22}	17.85	→24.71
Φ_{23}	13.59	13.54
Φ_{31}	13.98	13.81
Φ_{32}	17.44	16.81
Φ_{33}	→13.26	14.06
Φ_{41}	13.64	13.59
Φ_{42}	14.62	14.32
Φ_{43}	13.17	14.11
Φ_{51}	13.54	13.54
Φ_{52}	14.12	14.93
Φ_{53}	13.54	13.55
Φ_{54}	14.97	15.03
Δ_2	13.53	13.56
Δ_3	13.54	13.55
Δ_4	13.55	13.54
Δ_5	13.54	13.54
Θ_{21}	15.33	14.59
Θ_{22}	13.55	13.64
Θ_{31}	13.88	13.73
Θ_{32}	→19.18	20.57
Θ_{41}	13.62	13.54
Θ_{42}	14.65	15.43
Θ_{51}	13.51	13.61
Θ_{52}	13.53 m	13.56 m

Table 5.5-3 Variations of $\sqrt{E_d E \varrho_{qs}^2}$ due to parameter variations of $\pm$ 10%

normal value $\sqrt{E_d E \varrho_{qs}^2} = 0.008211$ rad sec^{-1}

	$(\sqrt{E_d E \varrho_{qs}^2})_-$	$(\sqrt{E_d E \varrho_{qs}^2})_+$
Φ_{21}	0.008213	0.008213
Φ_{22}	0.008309	0.008422
Φ_{23}	0.008162	0.008268
Φ_{31}	0.008182	0.008258
Φ_{32}	0.008328	0.008131
Φ_{33}	0.008196	0.008297
Φ_{41}	0.008180	0.008243
Φ_{42}	0.008278	0.008155
Φ_{43}	0.008023	0.008372
Φ_{51}	0.008209	0.008213
Φ_{52}	0.008207	0.008219
Φ_{53}	0.008216	0.008206
Φ_{54}	0.008102	0.008312
Δ_2	0.008220	0.008202
Δ_3	0.008211	0.008212
Δ_4	0.008209	0.008213
Δ_5	0.008211	0.008211
Θ_{21}	$\rightarrow$0.008537	$\rightarrow$0.007935
Θ_{22}	0.008220	0.008213
Θ_{31}	0.008247	0.008180
Θ_{32}	$\rightarrow$0.007888	$\rightarrow$0.008693
Θ_{41}	0.008221	0.008201
Θ_{42}	0.008148	0.008337
Θ_{51}	0.008209	0.008213
Θ_{52}	0.008210 rad sec^{-1}	0.008212 rad sec^{-1}

while in reality

$$\mu_a = -\frac{\pi}{2} \quad \text{but} \quad u_{\text{air}} = 11 \text{ m sec}^{-1}$$

Referring to (3.7-1) and (3.7-2), where we put

$$A_l = 1600 \text{ m}^2$$
$$\varrho_a = 1.293 \text{ kg m}^{-3}$$
$$C_Y(\mu_a) = 1.00$$
$$C_N(\mu_a) = 0$$

we get for the non-anticipated part of the steady wind force and moment

$$Y_d = \tfrac{1}{2} \times 1.293 \times (11^2 - 10^2) \times 1600 = 0.2172 \times 10^5 \text{ kg m sec}^{-2}$$
$$N_d = 0 \text{ kg m}^2\text{sec}^{-2}$$

Substituting this in (5.6-1) and using the values of table 5.6-1, we find

$$\delta = 0.0044 \text{ rad}$$
$$v = 0.0063 \text{ m sec}^{-1}$$
$$\psi = -0.80 \times 10^{-3} \text{ rad}$$
$$y = -1.34 \text{ m}$$

2. The curvature of the path has a steady non-anticipated component such that

$$r_C = 10^{-3} \text{ rad sec}^{-1}$$

which means that the path is a circle with a radius of 7812 m. According to (2.3-12) and table 5.1-2, the non-anticipated curvature can be accounted for by a force and a moment

$$Y_d = (Y_r - Mu)r_C = -0.1251 \times 10^6 \text{ kg m sec}^{-2}$$
$$N_d = N_r r_C = -0.8952 \times 10^7 \text{ kg m}^2\text{sec}^{-2}$$

Substituting this in (5.6-1) and using table 5.6-1, we find

$$\delta = 0.019 \text{ rad}$$
$$v = -0.69 \text{ m sec}^{-1}$$
$$\psi = 0.0088 \text{ rad}$$
$$y = -27.8 \text{ m}$$

$$\sqrt{E_d E \mathscr{L}_{qs}} = 4400 \text{ m}$$

$$\sqrt{E_d E \delta_{qs}^2} = 0.13 \text{ rad}$$

$$\sqrt{E_d E \varrho_{qs}^2} = 0.010 \text{ rad sec}^{-1}$$

Since this controller will not be able to retain the state variables of the controlled object within the region where the linear mathematical model is valid, those figures have little significance. They do demonstrate, however, that the performance of the controller proposed is not matched by those of controllers being substantially simpler to design and implement.

Reference Chapter 5

1. G. VAN LEEUWEN, The lateral damping and added mass of a horizontally oscillating shipmodel. Netherlands' Research Centre T.N.O. for Shipbuilding and Navigation, Report no. 65 S, 1964. Also: Technological University Delft, Shipbuilding Laboratory, Report no. 23, 1964.

6.2.1 *The positioning system*

There will be a sub-system for the purpose of observing the ship's position. Such a positioning system may be based on several principles:

- Measuring, by means of radar, of the ship's bearing and distance relative to a point the position of which is known precisely.
- Inertial navigation. A remark on this was made in sub-sec. 4.9.4.
- The leader-cable, as mentioned in sec. 1.2.
- Radio positioning. On board the ship, the radio signals received from two transmitters at different, precisely known locations are compared. Thus the difference between the distances from the two respective transmitters to the ship is determined, and hence the ship is known to be situated on a particular hyperbola. By means of comparison of the radio signals received on board the ship from a second pair of transmitters, also at precisely known locations, another hyperbola is found on which the ship is known to be situated. The ship's position is the point of intersection of the two hyperbolas.
  Alternative possibilities of applying the same principle exist, where the ship has to transmit signals, while these signals are received by stations at fixed locations. These alternatives are less attractive, for reasons that will become clear below.

We shall further confine ourselves to radio positioning systems. Several systems of this type are commercially available, their main applications being in such fields as surveying, hydrography and mine-hunting. We shall briefly discuss some points relevant to the suitability of a particular system for the application in question.

Accuracy. The accuracy of the system is dependent on several factors. Most of these, like component tolerances and the choice of the frequency used, are in the hands of the designer of the system. The accuracy is also dependent, however, on the properties of the atmosphere, in particular on the departures of the velocity of propagation from the nominal value.

As regards the frequency, the accuracy of the system can be higher as the frequency is higher. On the other hand, if the frequency is chosen too high, the system will turn unreliable as a result of the fact that the signals are reflected on neighbouring ships and other obstacles, while in addition the geographical range of the system will become too small.

The accuracy of a radio positioning system is not the same throughout the area covered by the system. The highest accuracy is found at the center of that area, because the spacing and the angles of intersection of the hyperbolas mentioned above are the most favourable.

Manufacturers of current accurate radio positioning systems claim accuracies mainly in the order of 10 meters.

6.2.3 *The computer*

As we remarked before, the controller, i.e. the sub-system calculating the control signal at each sampling instant, is likely to be a digital general-purpose computer. In subsequent sections we shall see that this computer will have to serve several secondary purposes as well.

The computer can be either ship-borne or shore-based. In case the automatic ship guidance system is installed for the purpose of guiding more than one ship (either simultaneously or one at a time) on one particular fairway, i.e. if the system pertains to the fairway rather than to the ship, then it stands to reason to choose the shore as the location of the computer. In the opposite case, if the automatic ship guidance system pertains to the ship rather than to the fairway, the ship is the more natural location of the computer.

If the computer is based on the shore, radio channels are needed for the continuous exchange of information between the computer and equipment on board the ship. This has an important bearing on the cost of the system. In addition, it makes the system in this form unacceptable from a military point of view unless special measures are taken so as to prevent enemy detection of the ships, deception and jamming.

Since at each sampling instant rather elaborate computations have to be carried out, the computer has to have a high computing speed and a memory with a short access time. As the amount of information to be stored will be rather large, while not all of this information needs to be stored in an equally fast memory, the computer will have a second memory with a larger capacity and longer access time, which may consist, for instance, of one or more magnetic disks.

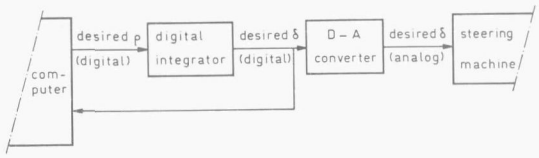
Furthermore, the computer should be equipped with a rather comprehensive peripheral equipment. A typical configuration may comprise, besides the central processor:

- A sufficient number of adequate digital and analog *input* and *output lines*, by way of which feedback and control signals are transferred from the sensors to the computer and from the computer to the controlled object, respectively.
- One or more *operator consoles*, enabling the operator or the operators to converse with the computer.

If the computer is ship-borne, there will be one operator console, with keyboard and printer or comparable equipment. If the computer is shore-based, then in addition to the operator console at the computer station, at least a simple operator console must be present on board each individual ship to be guided by the system.

- A *card reader*, to be used for reading-in programs and such information as data on the pre-specified track C_{AB} and coefficients characterizing the controller for a particular ship. The card reader will be installed at the same location as the computer.

Fig. 6.2-1 A possible system for the linkage of the steering machine to the computer



serve as a link between the computer and the steering machine consists of a digital integrator and a digital-to-analog converter, as shown in fig. 6.2-1.

This sub-system is able to convert a digital signal representing the desired rudder angular velocity, generated by the computer, into an analog signal representing the desired rudder angle, suitable as an input signal to the steering machine.

6.2.5 *Sensors for wind and current*

The directions and velocities of the mean wind and the current have to be measured and fed into the computer.

Instruments measuring the direction and the velocity of the wind (the vane and the anemometer, respectively) are standard equipment in meteorology, while current meters are the same in hydrography.

Since data on the wind and the current have to be put into the computer only infrequently, the instruments do not necessarily have to be linked directly to input channels of the computer, but their data can be put in manually, for instance by way of a keyboard. It will be necessary, though, that the current meters have radio transmitters attached to them, in order that the latest information on the current be available at the computer station at all times.

6.2.6 *Radio equipment*

We have seen that in the automatic ship guidance system several sub-systems will have to exchange information with the computer at a high rate, that several of these sub-systems are on shipboard, and that if the computer is shore-based, the information exchange has to be accomplished via radio channels.

Re-examining the descriptions of the sub-systems of the guidance system given earlier in this section, we find that in case the computer is located ashore, radio transmission of the following signals will be necessary.

1. The ship's observed position $\{\eta_{x_h}(k), \eta_{y_h}(k)\}$ or $\{\eta_{x_0}(k), \eta_{y_0}(k)\}$ – from the output of the radio positioning system to the computer.
2. The ship's observed heading $\eta_{\psi_0}(k)$ – from the compass to the computer.
3. Supposed that the steering machine is linked to the computer by means of the sub-system of fig. 6.2-1, and that this sub-system is installed on board the ship: the desired rudder angular velocity $\varrho(k)$ and the desired rudder angle $\delta(k)$ – from the computer to the linkage system and from the linkage system to the computer, respectively.

4. The signals making up the two-way communication between the operator console and the computer.
5. The signals controlling the display unit installed on board the ship – from the computer to the display unit.
6. The observed direction and velocity of the current – from one or more current meters to a display unit at the computer station.
7. According to circumstances, the observed direction and velocity of the wind – from the vane and the anemometer, respectively, to a display unit at the computer station.

Additional channels may be necessary for such purposes as synchronization of data transfer, safety checks (for instance, besides the desired rudder angle, the actual rudder angle may be fed back too), and voice.

By way of a summary of this section, fig. 6.2-2 gives a block diagram showing the main sub-systems, their locations and their interconnection in a typical configuration.

6.3 Real-time actions

The computations to be carried out and the further actions to be taken by the system at each sampling instant $k \in \{0, 1, \ldots, n-1\}$ are the following:

1. A set of observations of the ship's position is transferred from the positioning system to the computer.
   The observed position is given in hyperbolic coordinates which we shall denote (x_h, y_h), or, if the positioning system includes a coordinate-conversion computer, in rectangular coordinates (x_0, y_0). In order to distinguish the ship's observed position from its actual position, we shall denote the former as $\{\eta_{x_h}(k), \eta_{y_h}(k)\}$ or $\{\eta_{x_0}(k), \eta_{y_0}(k)\}$.
2. An observation of the ship's heading $\psi_0(k)$ relative to the system of coordinates (x_0, y_0) is transferred from the compass to the computer. The observed value, as distinguished from the actual value, will be denoted $\eta_{\psi_0}(k)$.
3. The desired rudder angle (the output of the digital integrator in fig. 6.2-1), and possibly the actual rudder angle, are transferred to the computer.
4. The computer converts $\{\eta_{x_h}(k), \eta_{y_h}(k)\}$ or $\{\eta_{x_0}(k), \eta_{y_0}(k)\}$ to $\{\eta_x(k), \eta_y(k)\}$, where the third pair of symbols denotes the ship's observed position in the system of coordinates (x, y).
   This coordinate conversion can only be carried out if a sufficient amount of information on the desired track C_{AB} has been stored in the computer memory beforehand. The methods of storing C_{AB} and performing the coordinate conversion, and the further computations and look-up operations will be returned on later in this section.
5. From $x(k)$, which is simply taken to be equal to $\eta_x(k)$, (see sub-sec. 4.9.3), the

$$R_C(k') = \frac{1}{r_C'(k')} \tag{6.3-1}$$

for each k'.

$\psi_C(k')$ is the angle, relative to the x_0-axis and with the positive sense taken counterclockwise, of the oriented tangent on $O(k')$ at $P_C(k')$.

$a_C'(k')$ is the spiral coefficient, defined

$$a_C'(k') \stackrel{\text{def}}{=} \frac{r_c'(k'+1) - r_c'(k')}{X_s} \tag{6.3-2}$$

for $k' \in \{0, 1, \ldots, n'-1\}$

Let us assume now that at sampling instant $k \in \{0, 1, \ldots, n-1\}$ the pair of coordinates $\{x_0(k), y_0(k)\}$ of a point $P(k)$ is given, and that the corresponding pair of coordinates $\{x(k), y(k)\}$ together with the quantity $\psi_C(k)$ as defined in sec. 2.3 must be determined.

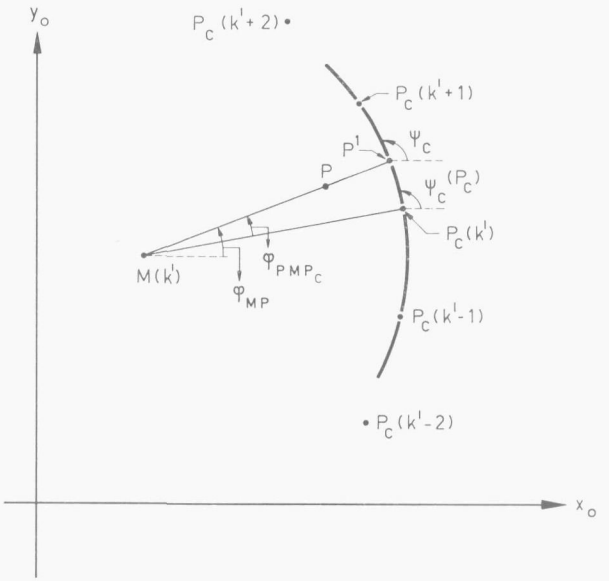
The algorithm proposed comprises the following steps:

1. Among the points $P_C(k')$, $(k' = 0, 1, \ldots, n')$, the one which is nearest to $P(k)$ is found. The particular value of k' associated with that point can be denoted $k'(k)$. $k'(k)$ can be found by trial and error. Evidently, at time $k+1$, when $k'(k)$ is known, the most likely value of $k'(k+1)$, that should be tried first, is $k'(k)+1$; at time 0, the value of k that should be tried first is 0.
2. The quantities $x_0^{(M)}\{k'(k)\}$, $y_0^{(M)}\{k'(k)\}$, $\psi_C\{k'(k)\}$ and $r_C'\{k'(k)\}$ are determined. As concerns this step, there are two main possibilities. First, the quantities just mentioned can be calculated in real time. A second possibility is, that these quantities are obtained through advance calculations, and stored in the computer's fast memory. In the first case, this step will take an amount of computer time during the real-time operation. In the second case, the computer time needed is smaller, but considerably more memory space has to be occupied. An intermediate solution is, for instance, to store only
   $x_0^{(M)}\{k'(k)\}$ and $y_0^{(M)}\{k'(k)\}$.
   For the sake of notational convenience, we shall subsequently abbreviate
   $x_0^{(M)}\{k'(k)\}$ as $x_0^{(M)}(k)$, etc.
3. As can be seen from fig. 6.3-1,

$$\psi_C(k) = \varphi_{MP}(k) + \frac{\pi}{2} \operatorname{sign} R_C(k) \tag{6.3-3}$$

   where $\varphi_{MP}(k)$ is used to denote the angle of line $M(k)P(k)$ relative to the x_0-axis, with the positive sense taken counterclockwise. This leads to the relations given below, from which $\psi_C(k)$ can be computed:

Fig. 6.3-1 Illustration of the algorithm proposed



If the coordinates (x, y) have to be determined of a point P(k) whose position is given in the hyperbolic coordinates (x_h, y_h) instead of in the coordinates (x_0, y_0), then the calculation can be achieved in two steps: first (x_h, y_h) can be converted into (x_0, y_0), and next (x_0, y_0) can be converted into (x, y). The derivation of formulae expressing x_0 and y_0 as functions of x_h and y_h can be performed through classical analytical geometry. Instead of such formulae, methods from numerical analysis can be used for the conversion of (x_h, y_h) into (x_0, y_0).

It will be clear that the computational method for the calculation of $x_0(k)$, $y_0(k)$ and $\psi_C(k)$ just outlined is by no means the only one possible. A different method for instance, which sometimes deserves to be considered, is to store tables in the computer's memory such that given $\{x_0(k), y_0(k)\}$ or $\{x_h(k), y_h(k)\}$, the quantities to be determined are found by looking up and interpolation.

Another point we wish to make a remark on is the determination of $\boldsymbol{D}(k)$ and $c_d(k)$. We already pointed out in sub-sec. 4.9.3 that it is preferable to consider $\boldsymbol{D}(k)$ and $c_d(k)$ as functions of x rather than k. Hence, in connection with the method proposed for storing C_{AB}, and in view of the fact that $\boldsymbol{D}(t)$ will vary only slightly during the space of time it takes the ship to travel the distance X_s separating two adjacent points P_C, the natural procedure is that $\boldsymbol{D}$ and c_d are computed beforehand and stored in the computer's fast memory in the form

$$\boldsymbol{D}(k') \stackrel{\text{def}}{=} \boldsymbol{D}\{x^{(Pc)}(k')\} \tag{6.3-15}$$

$$c_d(k') \stackrel{\text{def}}{=} c_d\{x^{(Pc)}(k')\} \tag{6.3-16}$$

On the ground of (4.5-5) and (4.4-13) we have

$$c_d(k) = \hat{C}_D(0)\boldsymbol{D}(k)+\hat{C}_D(1)\boldsymbol{D}(k+1)+\ldots+\hat{C}_D(n-k-1)\boldsymbol{D}(n-1) \qquad (6.4\text{-}1)$$

for each k, whence it is possible to write $c_d(k')$ as a function of $\boldsymbol{D}(k')$, $\boldsymbol{D}(k'+1)$, ..., $\boldsymbol{D}(n'-1)$. This function has the form

$$c_d(k') = \hat{C}_D'(0)\boldsymbol{D}(k')+\hat{C}_D'(1)\boldsymbol{D}(k'+1)+\ldots+\hat{C}_D'(n'-k'-1)\boldsymbol{D}(n'-1) \qquad (6.4\text{-}2)$$

for each k', where $\hat{C}_D'(0)$, $\hat{C}_D'(1)$, ...
are expressible in $\hat{C}_D(0)$, $\hat{C}_D(1)$, ...

8. The desired initial state ξ_0, discussed in the final part of sec. 4.5, is computed and presented to the operator on board the ship.

6.5 Further features of the automatic ship guidance system

Having given a survey of the computations to be carried out and of the other actions to be taken at each sampling instant and beforehand, we shall here mention some of the further features of the automatic ship guidance system that will be necessary, desirable or at least conceivable.

6.5.1 *Automatic emergency actions*

The sequence of computations and other actions in real time, given in sec. 6.3, mainly describes the normal situation. It may be judged necessary, however, that the ship guidance system is designed such that it is able to take special actions automatically, to avert dangers resulting from certain anomalous events.

In the list of real-time actions, we already included a safety check on y, the distance of the ship from the pre-specified track C_{AB}. If y exceeds certain limits, or even better, as soon as this is liable to happen, a jump to an emergency subroutine may be made, causing an alarm signal, or the effectuation of radical interventions like reduction of speed or stopping.

If the system is designed to guide more than one ship at a time, the distance between any two of the ships is another quantity to be checked frequently. Special subroutines may be included in the computer program so as to produce an alarm signal or even effect evasive manoeuvres in case a collision is imminent.

As a third example of circumstances in case of which programming of automatic emergency actions may prove beneficial, we mention malfunction of the radio positioning system, the steering machine or other vital parts of the system.

6.5.2 *Manual override*

At all times, an operator on board the ship must be able to manually override the

Chapter 7

OPTIMAL CONTROL OF STOCHASTIC LINEAR SAMPLED-DATA SYSTEMS WITH QUADRATIC LOSS

7.1 Introduction and problem formulation

This chapter presents the theoretical material underlying the designing method for automatic ship guidance controllers that is proposed in chapter 4. A control problem will be considered that will be referred to subsequently as the *basic problem*. We shall show how the basic problem can be solved, and we shall discuss some properties of the resulting optimal system.

The basic problem is specified by the following points:

1. The plant is a linear system, sampled synchronously, with a fixed sampling period T_s, and characterized by

$$\boldsymbol{\xi}(k+1) = \boldsymbol{\Phi}\boldsymbol{\xi}(k)+\boldsymbol{\Lambda}\boldsymbol{\varrho}(k)+\boldsymbol{d}(k)+\boldsymbol{w}_1(k) \tag{7.1-1}$$

for each $k \in \{0, 1, \ldots, n-1\}$

where

$\boldsymbol{\xi}$ = the $n_\xi \times 1$ plant state vector,
$\boldsymbol{\Phi}$ = the $n_\xi \times n_\xi$ plant transition matrix,
$\boldsymbol{\varrho}$ = the $n_\varrho \times 1$ control vector,
$\boldsymbol{\Lambda}$ = the $n_\xi \times n_\varrho$ control input matrix,
$\boldsymbol{d}$ = the mean of the $n_\xi \times 1$ plant disturbances vector, and
$\boldsymbol{w}_1$ = the zero-mean random part of the plant disturbances vector.

2. At each sampling instant $k \in \{0, 1, \ldots, n-1\}$, a vector of observations is produced as described by

$$\boldsymbol{\eta}(k) = \boldsymbol{M}\boldsymbol{\xi}(k)+\boldsymbol{w}_2(k) \tag{7.1-2}$$

where

$\boldsymbol{\eta}$ = the $n_\eta \times 1$ vector of observations,
$\boldsymbol{M}$ = the $n_\xi \times n_\eta$ observation matrix, and
$\boldsymbol{w}_2$ = the $n_\eta \times 1$ vector of random observational errors.

3. The initial state $\boldsymbol{\xi}(0)$, when considered a priori with respect to $\boldsymbol{\eta}(0)$, is a gaussian random variable, with

$$\mathrm{E}\,\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \tag{7.1-3}$$

$$\mathrm{cov}\,\{\boldsymbol{\xi}(0), \boldsymbol{\xi}(0)\} = \boldsymbol{S}_0 \tag{7.1-4}$$

where

$\boldsymbol{\xi}_0$ = an $n_\xi \times 1$ vector, and
$\boldsymbol{S}_0$ = a positive semi-definite and symmetric $n_\xi \times n_\xi$ matrix.

where

$$\mathscr{S}_{\eta}(k) \stackrel{\text{def}}{=} \{\boldsymbol{\eta}(0), \boldsymbol{\eta}(1), \ldots, \boldsymbol{\eta}(k)\} \tag{7.1-14}$$

$$\mathscr{S}_{\varrho}(k-1) \stackrel{\text{def}}{=} \{\boldsymbol{\varrho}(0), \boldsymbol{\varrho}(1), \ldots, \boldsymbol{\varrho}(k-1)\} \tag{7.1-15}$$

A separate function $\boldsymbol{\varrho}_k(\cdot)$ is called a *control policy*, while a sequence of functions $\boldsymbol{\varrho}_0(\cdot), \boldsymbol{\varrho}_1(\cdot), \ldots, \boldsymbol{\varrho}_{n-1}(\cdot)$ is called a *control strategy*.

7.2 The estimation problem

The procedure for finding the optimal control policies to be discussed in the next section is based on the premise that at each sampling instant $k \in \{0, 1, \ldots, n\}$, the conditional probability density on the state, $\mathrm{p}\{\boldsymbol{\xi}(k)|\mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k-1)\}$, is known. Therefore, we shall now give some results [1, 2] that can be made use of in order to determine this probability density for the case under consideration.

1. If $\mathrm{p}\{\boldsymbol{\xi}(k)|\mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k-1)\}$ is gaussian, then $\mathrm{p}\{\boldsymbol{\xi}(k+1)|\mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k)\}$ is gaussian. If we denote

$$\boldsymbol{\xi}(k|k) \stackrel{\text{def}}{=} \mathrm{E}\{\boldsymbol{\xi}(k)|\mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k-1)\} \tag{7.2-1}$$

$$\boldsymbol{P}(k) \stackrel{\text{def}}{=} \operatorname{cov}\left\{\boldsymbol{\xi}(k), \boldsymbol{\xi}(k)\right\} \Big| \mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k-1) \tag{7.2-2}$$

then we have

$$\boldsymbol{\xi}(k+1|k) \stackrel{\text{def}}{=} \mathrm{E}\{\boldsymbol{\xi}(k+1)|\mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k)\} \tag{7.2-3}$$

$$= \boldsymbol{\Phi}\boldsymbol{\xi}(k|k) + \boldsymbol{\Delta}\boldsymbol{\varrho}(k) + \boldsymbol{d}(k) \tag{7.2-4}$$

$$\boldsymbol{S}(k+1) \stackrel{\text{def}}{=} \operatorname{cov}\left\{\boldsymbol{\xi}(k+1), \boldsymbol{\xi}(k+1)\right\} \Big| \mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k) \tag{7.2-5}$$

$$= \boldsymbol{\Phi}\boldsymbol{P}(k)\boldsymbol{\Phi}^{\mathrm{T}} + \boldsymbol{R}_1 \tag{7.2-6}$$

for each $k \in \{0, 1, \ldots, n-1\}$

2. If $\mathrm{p}\{\boldsymbol{\xi}(k+1)|\mathscr{S}_{\eta}(k), \mathscr{S}_{\varrho}(k)\}$ is gaussian, then $\mathrm{p}\{\boldsymbol{\xi}(k+1)|\mathscr{S}_{\eta}(k+1), \mathscr{S}_{\varrho}(k)\}$ is gaussian, with

$$\boldsymbol{\xi}(k+1|k+1) = \{\boldsymbol{I} - \boldsymbol{K}(k+1)\boldsymbol{M}\}\boldsymbol{\xi}(k+1|k) + \boldsymbol{K}(k+1)\boldsymbol{\eta}(k+1) \tag{7.2-7}$$

$\sum_{k+1}^{n} \mathscr{L}(i)$ is a random variable depending on the value of $\varrho(k)$ and on the sequence of functions $\varrho_i\{\mathscr{S}_\eta(i), \mathscr{S}_\varrho(i-1)\}$, $(i = k+1, k+2, \ldots, n-1)$. The control strategy must be such that

$$\mathrm{E}\left\{\sum_{k+1}^{n} \mathscr{L}(i)\right\}$$
$$\left|\mathscr{S}_\eta(k), \mathscr{S}_\varrho(k-1)\right.$$

is minimized.

We denote

$$\mathscr{V}(k) \stackrel{\text{def}}{=} \min_{\varrho(k), \varrho_{k+1}(\cdot), \varrho_{k+2}(\cdot), \ldots, \varrho_{n-1}(\cdot)} \left[\mathrm{E}\left\{\sum_{k+1}^{n} \mathscr{L}(i)\right\}\right] \tag{7.3-1}$$
$$\left|\mathscr{S}_\eta(k), \mathscr{S}_\varrho(k-1)\right.$$

In words: $\mathscr{V}(k)$ is the expectation of the sum of all future loss terms, provided that at the present sampling instant as well as at all future sampling instants optimal control will be applied. We also introduce, as a "boundary condition",

$$\mathscr{V}(n) = 0 \tag{7.3-2}$$

We shall prove that the following two statements are true:

1. $\mathscr{V}(k)$ can be written as

$$\mathscr{V}(k) = \mathrm{E}\{\boldsymbol{\xi}^{\mathrm{T}}(k)\boldsymbol{A}_2(k)\boldsymbol{\xi}(k) + 2\boldsymbol{A}_1(k)\boldsymbol{\xi}(k) + A_0(k)\} \tag{7.3-3}$$
$$\left|\mathscr{S}_\eta(k), \mathscr{S}_\varrho(k-1)\right.$$

   where $\boldsymbol{A}_2(k)$ and $\boldsymbol{A}_1(k)$ are $n_\xi \times n_\xi$ and $1 \times n_\xi$ constant matrices, respectively, with $\boldsymbol{A}_2(k)$ being symmetric, and where $A_0(k)$ is a constant scalar.

2. The expression between $\{\}$ in (7.3-3) is positive semi-definite.

Referring to (7.3-2), we observe that $\mathscr{V}(n)$, though degenerate, complies with these two statements. Therefore, we only have to prove that statements 1 and 2 are true, given that statements 3 and 4 are true:

3. $\mathscr{V}(k+1)$ can be written as

$$\mathscr{V}(k+1) = \mathrm{E}\{\boldsymbol{\xi}^{\mathrm{T}}(k+1)\boldsymbol{A}_2(k+1)\boldsymbol{\xi}(k+1) + 2\boldsymbol{A}_1(k+1)\boldsymbol{\xi}(k+1) + A_0(k+1)\} \tag{7.3-4}$$
$$\left|\mathscr{S}_\eta(k+1), \mathscr{S}_\varrho(k)\right.$$

   where $\boldsymbol{A}_2(k+1)$ and $\boldsymbol{A}_1(k+1)$ are $n_\xi \times n_\xi$ and $1 \times n_\xi$ constant matrices, respectively, with $\boldsymbol{A}_2(k+1)$ being symmetric, and where $A_0(k+1)$ is a constant scalar.

4. The expression between $\{\}$ in (7.3-4) is positive semi-definite.

In sec. 7.4 it will also be proved that the expression following the second E-operator in (7.3-8) is positive semi-definite.

Using the results of the preceding section along with property 2, we rewrite the first E-term in (7.3-8)

$$\begin{aligned}
&\mathrm{E}\Big|_{\mathscr{S}_\eta(k),\,\mathscr{S}_\varrho(k)} [\{\boldsymbol{\varrho}(k)+\boldsymbol{C}_\xi(k)\boldsymbol{\xi}(k)+\boldsymbol{c}_d(k)\}^{\mathrm{T}}[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+\boldsymbol{Q}_2]\cdot \\
&\quad\cdot\{\boldsymbol{\varrho}(k)+\boldsymbol{C}_\xi(k)\boldsymbol{\xi}(k)+\boldsymbol{c}_d(k)\}] = \\
&\quad= \{\boldsymbol{\varrho}(k)+\boldsymbol{C}_\xi(k)\boldsymbol{\xi}(k|k)+\boldsymbol{c}_d(k)\}^{\mathrm{T}}[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+\boldsymbol{Q}_2]\cdot \\
&\quad\cdot\{\boldsymbol{\varrho}(k)+\boldsymbol{C}_\xi(k)\boldsymbol{\xi}(k|k)+\boldsymbol{c}_d(k)\}+ \\
&\quad+\operatorname{trace}[\boldsymbol{P}(k)\boldsymbol{C}_\xi^{\mathrm{T}}(k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+\boldsymbol{Q}_2]\boldsymbol{C}_\xi(k)] \qquad (7.3\text{-}11)
\end{aligned}$$

From this it becomes evident that the control vector minimizing (7.3-11) is

$$\boldsymbol{\varrho}(k) = -\boldsymbol{C}_\xi(k)\boldsymbol{\xi}(k|k)-\boldsymbol{c}_d(k) \qquad (7.3\text{-}12)$$

while the minimum of the expression in (7.3-11) is the trace term. As a result of this we find that in (7.3-8), $\mathscr{V}(k)$ takes the form of (7.3-3), if we introduce

$$\boldsymbol{A}_2(k) = \boldsymbol{\Phi}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Phi}-\boldsymbol{C}_\xi^{\mathrm{T}}(k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+\boldsymbol{Q}_2]\boldsymbol{C}_\xi(k) \qquad (7.3\text{-}13)$$

$$\boldsymbol{A}_1(k)=[\boldsymbol{d}^{\mathrm{T}}(k)\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}+\boldsymbol{A}_1(k+1)]\boldsymbol{\Phi}-\boldsymbol{c}_d^{\mathrm{T}}(k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+\boldsymbol{Q}_2]\boldsymbol{C}_\xi(k) \qquad (7.3\text{-}14)$$

$$\begin{aligned}
A_0(k) = {}& \boldsymbol{d}^{\mathrm{T}}(k)\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{d}(k)+2\boldsymbol{A}_1(k+1)\boldsymbol{d}(k)+ \\
&+\operatorname{trace}[\boldsymbol{R}_1\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}]+\operatorname{trace}[\boldsymbol{P}(k)\boldsymbol{C}_\xi^{\mathrm{T}}(k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+ \\
&+\boldsymbol{Q}_2]\boldsymbol{C}_\xi(k)]-\boldsymbol{c}_d^{\mathrm{T}}(k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1+\boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta}+\boldsymbol{Q}_2]\boldsymbol{c}_d(k)+A_0(k+1) \qquad (7.3\text{-}15)
\end{aligned}$$

With these definitions, the expression $\boldsymbol{\xi}^{\mathrm{T}}(k)\boldsymbol{A}_2(k)\boldsymbol{\xi}(k)+2\boldsymbol{A}_1(k)\boldsymbol{\xi}(k)+A_0(k)$ consists of the expression following the second E-operator in (7.3-8), added to the trace term of (7.3-11). The former will be shown to be positive semi-definite in sec. 7.4, while the latter is easily shown to be non-negative. Hence the above quadratic form is positive semi-definite.

The set of recursive equations consisting of (7.3-9), (7.3-13), (7.3-10) and (7.3-14) makes it possible to compute $\boldsymbol{C}_\xi(k)$, $\boldsymbol{A}_2(k)$, $\boldsymbol{c}_d(k)$ and $\boldsymbol{A}_1(k)$, ($k = n-1, n-2, \ldots, 0$), in succession, starting from

$$\boldsymbol{A}_2(n) = \boldsymbol{0} \qquad (7.3\text{-}16a)$$

$$\boldsymbol{A}_1(n) = \boldsymbol{0} \qquad (7.3\text{-}16b)$$

7.4 The step from (7.3-7) to (7.3-8)

The purpose of this section is to justify the step from (7.3-7) to (7.3-8).

We shall first prove the following theorem [4]:

Theorem 7.4-1. Let

$$F(\varrho) = \varrho^{\mathrm{T}} Q_{22} \varrho + 2 Q_{02} \varrho + Q_{00} \tag{7.4-1}$$

be a positive semi-definite form in the $n_\varrho \times 1$ vector ϱ, where Q_{22} is a symmetric $n_\varrho \times n_\varrho$ matrix, Q_{02} is a $1 \times n_\varrho$ matrix, and Q_{00} is a scalar. Let

$$\varrho = \varrho_1 + \varrho_2 \tag{7.4-2}$$

where

$$\varrho_1 \in \text{range } (Q_{22})$$

$$\varrho_2 \in \text{nullspace } (Q_{22}) \tag{7.4-3}$$

Then

(i) $$Q_{02} \varrho_2 = 0 \tag{7.4-4}$$

(ii) $F(\varrho)$ can be written

$$F(\varrho) = (\varrho + c)^{\mathrm{T}} Q_{22} (\varrho + c) + Q_{00} - c^{\mathrm{T}} Q_{22} c \tag{7.4-5}$$

where

$$c \stackrel{\text{def}}{=} Q_{22}^{\dagger} Q_{02}^{\mathrm{T}} \tag{7.4-6}$$

(iii) $$Q_{00} - c^{\mathrm{T}} Q_{22} c \geqslant 0 \tag{7.4-7}$$

Proof of (i). From (7.4-1) and (7.4-2) follows

$$F(\varrho) = \varrho_1^{\mathrm{T}} Q_{22} \varrho_1 + 2 Q_{02} (\varrho_1 + \varrho_2) + Q_{00} \tag{7.4-8}$$

$$= F(\varrho_1) + 2 Q_{02} \varrho_2 \tag{7.4-9}$$

If (i) were not true, then a ϱ could be chosen such that $F(\varrho) < 0$. Hence (i) must be true.

Proof of (ii). We have

$$(\varrho + c)^{\mathrm{T}} Q_{22} (\varrho + c) + Q_{00} - \varrho^{\mathrm{T}} Q_{22} \varrho - (\varrho^{\mathrm{T}} Q_{22} \varrho + 2 Q_{02} \varrho + Q_{00}) =$$

$$= \varrho^{\mathrm{T}} Q_{22} c + c^{\mathrm{T}} Q_{22} \varrho - 2 Q_{02} \varrho \tag{7.4-10}$$

The proof of this theorem is similar to the proof of theorem 7.4-1, so there is no need for giving it here.

We remark that the definitions of $\boldsymbol{C}$ and $\boldsymbol{c}$ as given by (7.4-17) and (7.4-18), respectively, can be extended in the manner of (7.4-14).

Form (7.3-8) follows from (7.3-7) by straightforward application of theorem 7.4-2. From point (ii) of this theorem, it is evident that the form behind the second E-operator in (7.3-8) is positive semi-definite in $\xi(k)$.

7.5 The anticipating control term as a linear combination of mean disturbances

Examining (7.3-9), (7.3-10), (7.3-13) and (7.3-14), we find that for each $k \in \{0, 1, \ldots, n-1\}$ the "anticipating control term" $\boldsymbol{c}_d(k)$ and the quantity $\boldsymbol{A}_1(k)$ can be expressed as linear combinations of the mean disturbances at the same sampling instant and at all future sampling instants:

$$\boldsymbol{c}_d(k) = \boldsymbol{C}_d(k,k)\boldsymbol{d}(k) + \boldsymbol{C}_d(k,k+1)\boldsymbol{d}(k+1) + \ldots + \boldsymbol{C}_d(k,n-1)\boldsymbol{d}(n-1) \tag{7.5-1}$$

$$\boldsymbol{A}_1(k) = \boldsymbol{d}^{\mathrm{T}}(k)\boldsymbol{A}_d(k,k) + \boldsymbol{d}^{\mathrm{T}}(k+1)\boldsymbol{A}_d(k,k+1) + \ldots + \boldsymbol{d}^{\mathrm{T}}(n-1)\boldsymbol{A}_d(k,n-1) \tag{7.5-2}$$

where the $\boldsymbol{C}_d$ are $n_\varrho \times n_\xi$ matrices and the $\boldsymbol{A}_d$ are $n_\xi \times n_\xi$ matrices. These matrices can be computed by means of the following set of equations

$$\boldsymbol{C}_d(k,k) = [\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1 + \boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta} + \boldsymbol{Q}_2]^{\dagger}\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1 + \boldsymbol{A}_2(k+1)\} \tag{7.5-3}$$

$$\boldsymbol{A}_d(k,k) = \{\boldsymbol{Q}_1 + \boldsymbol{A}_2(k+1)\}\boldsymbol{\Phi} - \boldsymbol{C}_d^{\mathrm{T}}(k,k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1 + \boldsymbol{A}_2(k+1)\}\boldsymbol{\Delta} + \boldsymbol{Q}_2]\boldsymbol{C}_\xi(k) \tag{7.5-4}$$

$$\boldsymbol{C}_d(i,k) = [\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1 + \boldsymbol{A}_2(i+1)\}\boldsymbol{\Delta} + \boldsymbol{Q}_2]^{\dagger}\boldsymbol{\Delta}^{\mathrm{T}}\boldsymbol{A}_d^{\mathrm{T}}(i+1,k) \tag{7.5-5}$$

$$\boldsymbol{A}_d(i,k) = \boldsymbol{A}_d(i+1,k)\boldsymbol{\Phi} - \boldsymbol{C}_d^{\mathrm{T}}(i,k)[\boldsymbol{\Delta}^{\mathrm{T}}\{\boldsymbol{Q}_1 + \boldsymbol{A}_2(i+1)\}\boldsymbol{\Delta} + \boldsymbol{Q}_2]\boldsymbol{C}_\xi(i) \tag{7.5-6}$$

$(k = n-1, n-2, \ldots, 0;\ i = k-1, k-2, \ldots, 0)$

7.6 Comments on the formulation and on the solution of the basic problem

7.6.1 *Certainty equivalence*

At each sampling instant $k \in \{0, 1, \ldots, n-1\}$, the problem of finding $\varrho(k)$, given $\mathscr{S}_\eta(k)$ and $\mathscr{S}_\varrho(k-1)$, is broken down into two partial problems:

- Find $\mathrm{p}\{\xi(k) | \mathscr{S}_\eta(k), \mathscr{S}_\varrho(k-1)\}$.
- Find $\varrho(k)$, given the above probability density.

Due to the fact that we are dealing with a linear system while the plant disturbances

"cost" associated with them. This modification has also been studied by Gittelman [12].

7.7 Computation of the a priori expected loss terms

We consider the control system consisting of:

- A plant for which points (1), (2), (3), (4), (5) and (6) of sec. 7.1 are valid.
- A controller operating according to (7.2-10), (7.2-4), (7.2-7) and (7.3-12), where $\boldsymbol{K}(k)$, $\boldsymbol{C}_{\xi}(k)$ and $\boldsymbol{c}_{d}(k)$, $(k = 0, 1, \ldots, n-1)$, are given but not necessarily optimal.

We shall devote our efforts to the computation of the a priori expectations of the separate loss terms $\mathrm{E}\,\mathscr{L}(k)$, $(k = 1, 2, \ldots, n)$.

Each one of these expected loss terms can be split up as

$$\mathrm{E}\,\mathscr{L}(k) = \mathrm{E}\{\mathscr{L}'(k)\} + \mathscr{L}''(k) \tag{7.7-1}$$

where

$$\mathscr{L}'(k) \stackrel{\text{def}}{=} \{\xi(k) - \mathrm{E}\xi(k)\}^{\mathrm{T}} \boldsymbol{Q}_1 \{\xi(k) - \mathrm{E}\xi(k)\} + \\ + \{\varrho(k-1) - \mathrm{E}\varrho(k-1)\}^{\mathrm{T}} \boldsymbol{Q}_2 \{\varrho(k-1) - \mathrm{E}\varrho(k-1)\} \tag{7.7-2}$$

$$\mathscr{L}''(k) \stackrel{\text{def}}{=} \{\mathrm{E}\xi(k)\}^{\mathrm{T}} \boldsymbol{Q}_1 \mathrm{E}\xi(k) + \{\mathrm{E}\varrho(k-1)\}^{\mathrm{T}} \boldsymbol{Q}_2 \mathrm{E}\varrho(k-1) \tag{7.7-3}$$

We begin by writing (7.1-1), (7.1-2), (7.2-10), (7.2-4), (7.2-7) and (7.3-12) jointly in the compact form

$$\xi_a(k+1) = \boldsymbol{\Phi}_a(k)\xi_a(k) + \boldsymbol{d}_a(k) + \boldsymbol{W}_1(k)\boldsymbol{w}_1(k) + \boldsymbol{W}_2(k+1)\boldsymbol{w}_2(k+1) \tag{7.7-4}$$

for each $k \in \{0, 1, \ldots, n-1\}$

where *

$$\xi_a(k) \stackrel{\text{def}}{=} \begin{bmatrix} \xi(k) \\ \cdots\cdots \\ \tilde{\xi}(k) \end{bmatrix} \tag{7.7-5}$$

$$\boldsymbol{\Phi}_a(k) = \left[\begin{array}{c|c} \boldsymbol{\Phi} - \boldsymbol{\Delta C}_{\xi}(k) & -\boldsymbol{\Delta C}_{\xi}(k) \\ \hline \boldsymbol{0} & \{\boldsymbol{I} - \boldsymbol{K}(k+1)\boldsymbol{M}\}\boldsymbol{\Phi} \end{array}\right] \tag{7.7-6}$$

$$\boldsymbol{d}_a(k) \stackrel{\text{def}}{=} \left[\begin{array}{c} \boldsymbol{d}(k) - \boldsymbol{\Delta c}_{d}(k) \\ \hline \boldsymbol{0} \end{array}\right] \tag{7.7-7}$$

---

* The occurrence in (7.7-4) of $\boldsymbol{w}_2(n)$, $\hat{\xi}(n|n)$ and $\boldsymbol{K}(n)$, which strictly speaking are non-existent, does not lead to difficulties.

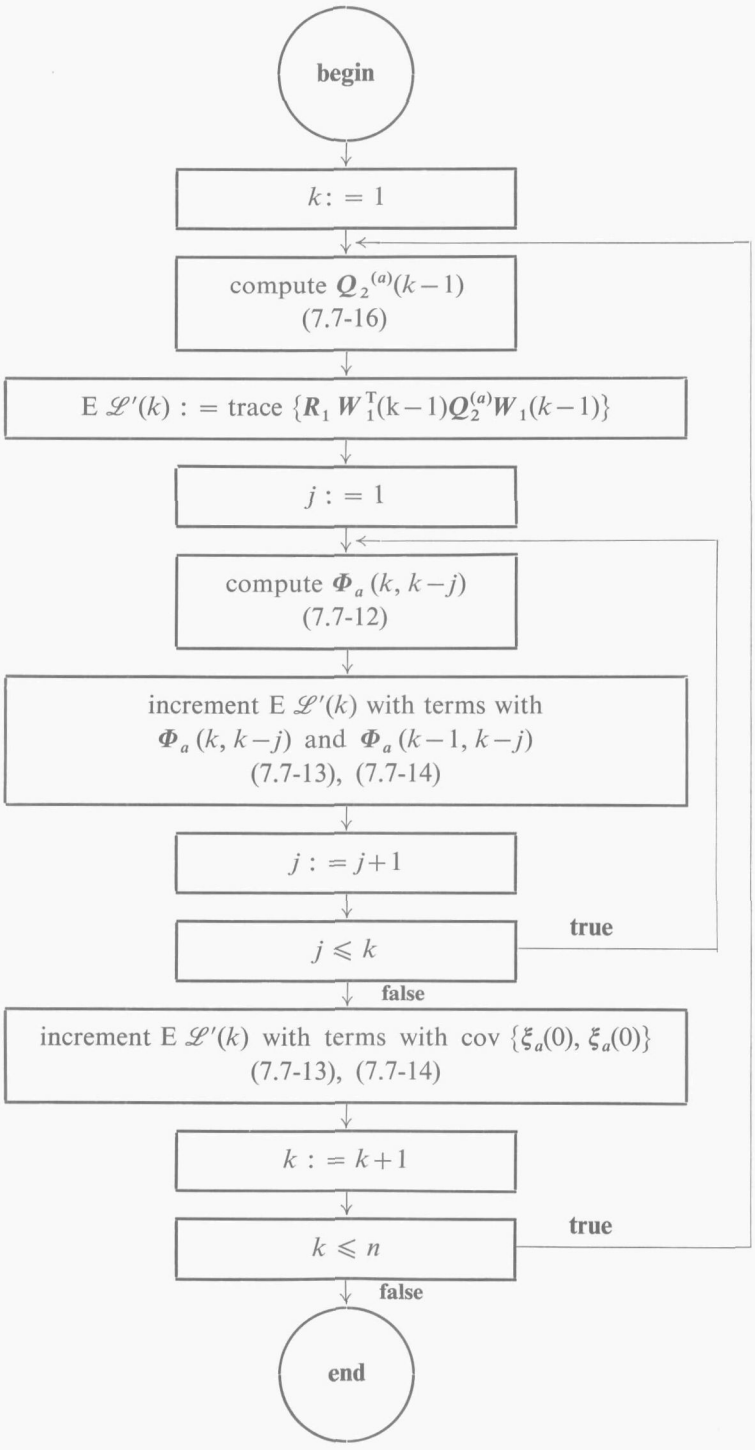


Fig. 7.7-1 Flow diagram for the computation of E $\mathscr{L}'(k)$, $(k = 1, 2, \ldots, n)$

of parameters $\boldsymbol{K}(k)$, $\boldsymbol{C}_{\xi}(k)$ and $\boldsymbol{C}_{d}(k, i)$, $(k = 0, 1, \ldots, n-1;\ i = k, k+1, \ldots, n-1)$, such that a control signal is obtained which, in general, is different for different sequences of mean disturbances.

In view of this, we wish to evaluate the performance of a controller characterized in this fashion, without having to specify one sequence $\boldsymbol{d}(k)$, $(k = 0, 1, \ldots, n-1)$. In order to attain this goal, we shall treat the sequence of mean disturbances "stochastically". This term is used to indicate that we still assume that the sequence of mean disturbances is known beforehand to the *designer* of the control system, that the controller generates a control signal based on the a priori knowledge of this sequence, according to (7.5-1), but that we do *not* identify ourselves with the designer: for us, the sequence $\boldsymbol{d}(k)$, $(k = 0,1, \ldots, n-1)$, is to be viewed as a random process.

More specifically, we shall regard the sequence of mean disturbances as the output of a linear sampled-data system characterized by

$$\boldsymbol{d}(k+1) = \boldsymbol{\Upsilon}\boldsymbol{d}(k) + \boldsymbol{w}_3(k) \tag{7.8-1}$$

for each $k \in \{0, 1, \ldots, n-2\}$
where $\boldsymbol{\Upsilon}$ is an $n_{\xi} \times n_{\xi}$ matrix, while $\boldsymbol{w}_3(k)$, $(k = 0, 1, \ldots, n-2)$, is a gaussian random process of $n_{\xi} \times 1$ vectors, satisfying

$$\mathrm{E}\boldsymbol{w}_3(k) = \boldsymbol{0} \tag{7.8-2}$$

$$\mathrm{cov}\ \{\boldsymbol{w}_3(k_1), \boldsymbol{w}_3(k_2)\} = \begin{cases} \boldsymbol{R}_3 & \text{if} \quad k_1 = k_2 \\ \boldsymbol{0} & \text{if} \quad k_1 \neq k_2 \end{cases} \tag{7.8-3}$$

for each k, k_1 and k_2, with $\boldsymbol{R}_3$ being a positive semi-definite and symmetric $n_{\xi} \times n_{\xi}$ matrix. When considered a priori, the initial state $\boldsymbol{d}(0)$ of the system described by (7.8-1) is a gaussian random variable, for which

$$\mathrm{E}\ \boldsymbol{d}(0) = \boldsymbol{0} \tag{7.8-4}$$

$$\mathrm{cov}\ \{\boldsymbol{d}(0), \boldsymbol{d}(0)\} = \boldsymbol{R}_{d(0)} \tag{7.8-5}$$

with $\boldsymbol{R}_{d(0)}$ being a positive semi-definite and symmetric $n_{\xi} \times n_{\xi}$ matrix. The quantities $\boldsymbol{\Upsilon}$, $\boldsymbol{R}_3$ and $\boldsymbol{R}_{d(0)}$ are supposed to be given in advance. Furthermore, $\boldsymbol{d}(0)$ is statistically independent of $\boldsymbol{w}_3(k)$, $(k = 0, 1, \ldots, n-2)$, while both $\boldsymbol{d}(0)$ and $\boldsymbol{w}_3(k)$, $(k = 0, 1, \ldots, n-2)$, are statistically independent of the stochastic quantities introduced previously in this chapter.

Examining (7.7-4), we find that $\boldsymbol{d}$ and $\boldsymbol{c}_d$ affect only $\boldsymbol{\xi}$, and not $\tilde{\boldsymbol{\xi}}$. Consequently, if we study the influence of $\boldsymbol{d}$ and $\boldsymbol{c}_d$ on the system, then we only have to work with the "upper half" of (7.7-4), keeping $\boldsymbol{w}_1(k)$, $\boldsymbol{w}_2(k)$ and $\boldsymbol{\xi}(k)$, for each k, and $\boldsymbol{\xi}(0)$ zero. Thus the equation is simplified to

The equations (7.8-10) can be written in the recursive form

$$\begin{aligned}
&\boldsymbol{G}(k,n-1) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,n-1)-\boldsymbol{\Delta C}_d(k-1,n-1)\\
&\boldsymbol{G}(k,n-2) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,n-2)-\boldsymbol{\Delta C}_d(k-1,n-2)\\
&\cdots\cdots\cdots\cdots\\
&\boldsymbol{G}(k,k) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,k)-\boldsymbol{\Delta C}_d(k-1,k)\\
&\boldsymbol{G}(k,k-1) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,k-1)+\boldsymbol{I}-\boldsymbol{\Delta C}_d(k-1,k-1)\\
&\boldsymbol{G}(k,k-2) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,k-2)\\
&\cdots\cdots\cdots\cdots\\
&\boldsymbol{G}(k,1) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,1)\\
&\boldsymbol{G}(k,0) = \boldsymbol{\Phi}'(k-1)\boldsymbol{G}(k-1,0)
\end{aligned} \tag{7.8-12}$$

By (7.8-1), we can replace (7.8-9) by

$$\boldsymbol{\xi}(k) = \boldsymbol{H}(k,-1)\boldsymbol{d}(0)+\boldsymbol{H}(k,0)\boldsymbol{w}_3(0)+\boldsymbol{H}(k,1)\boldsymbol{w}_3(1)+\ldots+\boldsymbol{H}(k,n-2)\boldsymbol{w}_3(n-2) \tag{7.8-13}$$

where

$$\begin{aligned}
&\boldsymbol{H}(k,n-2) = \boldsymbol{G}(k,n-1)\\
&\boldsymbol{H}(k,n-3) = \boldsymbol{G}(k,n-1)\boldsymbol{\Upsilon}+\boldsymbol{G}(k,n-2)\\
&\cdots\cdots\cdots\cdots\\
&\boldsymbol{H}(k,0) = \boldsymbol{G}(k,n-1)\boldsymbol{\Upsilon}^{n-2}+\boldsymbol{G}(k,n-2)\boldsymbol{\Upsilon}^{n-3}+\ldots+\boldsymbol{G}(k,2)\boldsymbol{\Upsilon}+\boldsymbol{G}(k,1)\\
&\boldsymbol{H}(k,-1) = \boldsymbol{G}(k,n-1)\boldsymbol{\Upsilon}^{n-1}+\boldsymbol{G}(k,n-2)\boldsymbol{\Upsilon}^{n-2}+\ldots+\boldsymbol{G}(k,1)\boldsymbol{\Upsilon}+\boldsymbol{G}(k,0)
\end{aligned} \tag{7.8-14}$$

This set of equations can be put into the more convenient recursive form

$$\begin{aligned}
&\boldsymbol{H}(k,n-2) = \boldsymbol{G}(k,\mathrm{n}-1)\\
&\boldsymbol{H}(k,n-3) = \boldsymbol{H}(k,\mathrm{n}-2)\boldsymbol{\Upsilon}+\boldsymbol{G}(k,n-2)\\
&\cdots\cdots\cdots\cdots\\
&\boldsymbol{H}(k,0) = \boldsymbol{H}(k,1)\boldsymbol{\Upsilon}+\boldsymbol{G}(k,1)\\
&\boldsymbol{H}(k,-1) = \boldsymbol{H}(k,0)\boldsymbol{\Upsilon}+\boldsymbol{G}(k,0)
\end{aligned} \tag{7.8-15}$$

We wish to derive an equation for $\boldsymbol{\varrho}(k)$, similar to (7.8-13). When $\tilde{\boldsymbol{\xi}}(k) = \boldsymbol{0}$ for each k, (7.3−12) reads

$$\boldsymbol{\varrho}(k) = -\boldsymbol{C}_{\xi}(k)\boldsymbol{\xi}(k)-\boldsymbol{c}_d(k) \tag{7.8-16}$$

$$\mathrm{E}\,\mathscr{L}'_{qs} \overset{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}\,\mathscr{L}'(k) \tag{7.9-4}$$

$$\mathrm{E}_d\mathscr{L}''_{qs} \overset{\text{def}}{=} \lim_{k\to\infty} \mathrm{E}_d\mathscr{L}''(k) \tag{7.9-5}$$

provided that these limits exist.

To begin with, we remark that in view of (7.9-1) and (7.9-3), $\boldsymbol{\Phi}_a(k)$, $\boldsymbol{W}_1(k)$ and $\boldsymbol{W}_2(k+1)$ in (7.7-4) become time-independent.

If k approaches infinity, the following equation replaces (7.7-11):

$$\begin{aligned}&\xi_a(k)-\mathrm{E}\xi_a(k) = \boldsymbol{W}_1\boldsymbol{w}_1(k-1)+\boldsymbol{W}_2\boldsymbol{w}_2(k)+\\&\boldsymbol{\Phi}_a\{\boldsymbol{W}_1\boldsymbol{w}_1(k-2)+\boldsymbol{W}_2\boldsymbol{w}_2(k-1)\}+\boldsymbol{\Phi}_a{}^2\{\boldsymbol{W}_1\boldsymbol{w}_1(k-3)+\boldsymbol{W}_2\boldsymbol{w}_2(k-2)\}+\ldots\end{aligned} \tag{7.9-6}$$

In correspondence to (7.7-13) and (7.7-14), this leads to

$$\begin{aligned}&\mathrm{E}[\{\xi(k)-\mathrm{E}\xi(k)\}^{\mathrm{T}}\boldsymbol{Q}_1\{\xi(k)-\mathrm{E}\xi(k)\}] =\\&= \operatorname{trace}\left[\boldsymbol{R}_1\sum_0^\infty\{(\boldsymbol{\Phi}_a^j\boldsymbol{W}_1)^{\mathrm{T}}\boldsymbol{Q}_1^{(a)}\boldsymbol{\Phi}_a^j\boldsymbol{W}_1\}+\boldsymbol{R}_2\sum_1^\infty\{(\boldsymbol{\Phi}_a^j\boldsymbol{W}_2)^{\mathrm{T}}\boldsymbol{Q}_2^{(a)}\boldsymbol{\Phi}_a^j\boldsymbol{W}_2\}\right]\end{aligned} \tag{7.9-7}$$

and

$$\begin{aligned}&\mathrm{E}[\{\varrho(k-1)-\mathrm{E}\varrho(k-1)\}^{\mathrm{T}}\boldsymbol{Q}_2\{\varrho(k-1)-\mathrm{E}\varrho(k-1)\}] =\\&= \operatorname{trace}\left[\boldsymbol{R}_1\sum_0^\infty\{(\boldsymbol{\Phi}_a^j\boldsymbol{W}_1)^{\mathrm{T}}\boldsymbol{Q}_2^{(a)}\boldsymbol{\Phi}_a^j\boldsymbol{W}_1\}+\boldsymbol{R}_2\sum_0^\infty\{(\boldsymbol{\Phi}_a^j\boldsymbol{W}_2)^{\mathrm{T}}\boldsymbol{Q}_2^{(a)}\boldsymbol{\Phi}_a^j\boldsymbol{W}_2\}\right]\end{aligned} \tag{7.9-8}$$

respectively, where $\boldsymbol{Q}_2{}^{(a)}$ is now time-independent. Addition of (7.9-7) and (7.9-8) gives us $\mathrm{E}\mathscr{L}'_{qs}$.

We next consider the computation of $\mathrm{E}_d\mathscr{L}''_{qs}$. If we let both k and $n-k$ approach infinity, we may write instead of (7.8-9).

$$\xi(k) = \sum_{j=-\infty}^{\infty}\{\boldsymbol{G}(j)\boldsymbol{d}(k+j)\} \tag{7.9-9}$$

where $\boldsymbol{G}(j)$, $(j = \ldots, -1, 0, 1, \ldots)$, is a sequence of matrices following from the set of recursive equations

$$\begin{aligned}&\ldots\ldots\ldots\\&\boldsymbol{G}(1) = \boldsymbol{\Phi}'\boldsymbol{G}(2)-\boldsymbol{\Delta}\boldsymbol{C}_d(2)\\&\boldsymbol{G}(0) = \boldsymbol{\Phi}'\boldsymbol{G}(1)-\boldsymbol{\Delta}\boldsymbol{C}_d(1)\\&\boldsymbol{G}(-1) = \boldsymbol{\Phi}'\boldsymbol{G}(0)+\boldsymbol{I}-\boldsymbol{\Delta}\boldsymbol{C}_d(0)\\&\boldsymbol{G}(-2) = \boldsymbol{\Phi}'\boldsymbol{G}(-1)\\&\boldsymbol{G}(-3) = \boldsymbol{\Phi}'\boldsymbol{G}(-2)\\&\ldots\ldots\ldots\end{aligned} \tag{7.9-10}$$

7.10 Convergency questions

In sec. 7.9 the existence of $E\mathscr{L}'_{qs}$ and $E_d\mathscr{L}''_{qs}$, as defined by (7.9-4) and (7.9-5), respectively, was simply assumed.

In this section we shall deal with some questions concerning the convergency with respect to k of $E\mathscr{L}'(k)$, $E_d\mathscr{L}''(k)$, (k = 1, 2, ...), and some other quantities.

7.10.1 *Convergency of* $E\mathscr{L}'(k)$ *and* $E_d\mathscr{L}''(k)$, $(k = 1, 2, \ldots)$

Evidently, sufficient conditions for the existence of $E\mathscr{L}'_{qs}$ and $E_d\mathscr{L}''_{qs}$ are the existence of

$$\Xi'_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \Xi'(k) \tag{7.10-1}$$

and

$$\Xi''_{qs} \stackrel{\text{def}}{=} \lim_{k\to\infty} \Xi''(k) \tag{7.10-2}$$

respectively, where

$$\Xi'(k) \stackrel{\text{def}}{=} E[\{\xi_a(k)-E\xi_a(k)\}\{\xi_a(k)-E\xi_a(k)\}^T] \tag{7.10-3}$$

$$\Xi''(k) \stackrel{\text{def}}{=} E_d\{E\xi(k)\cdot E\xi^T(k)\} \tag{7.10-4}$$

for each $k \in \{1, 2, \ldots\}$

7.10.2 *Convergency of* $\Xi'(k)$, $(k = 1, 2, \ldots)$

Once again, we consider the system described by (7.7-4) in the time-invariant version

$$\xi_a(k+1) = \boldsymbol{\Phi}_a\xi_a(k)+\boldsymbol{d}_a(k)+\boldsymbol{W}_1\boldsymbol{w}_1(k)+\boldsymbol{W}_2\boldsymbol{w}_2(k+1) \tag{7.10-5}$$

for each $k \in \{0, 1, \ldots\}$
where $\boldsymbol{d}_a(k)$, $E\{\boldsymbol{w}_1(k)\boldsymbol{w}_1{}^T(k)\}$ and $E\{\boldsymbol{w}_2(k+1)\boldsymbol{w}_2{}^T(k+1)\}$, $(k = 0, 1, \ldots)$, are uniformly bounded.

As we know [10], a sufficient condition for the convergency of $\Xi'(k)$, $(k = 1, 2, \ldots)$, is that the system described by the homogeneous equation

$$\xi_a(k+1) = \boldsymbol{\Phi}_a\xi(k) \tag{7.10-6}$$

for each $k \in \{0, 1, \ldots\}$
is asymptotically stable. This is the case if and only if the modulus of each eigenvalue of $\boldsymbol{\Phi}_a$ is less than unity.

Referring to (7.7-6) we can also say, that the two submatrices $\boldsymbol{\Phi}-\boldsymbol{\Delta C}_\xi$ and $(\boldsymbol{I}-\boldsymbol{KM})\boldsymbol{\Phi}$ of $\boldsymbol{\Phi}_a$ must have the property that each one of their eigenvalues has a modulus which is less than unity.

$$M = I$$

$$w_1(k) = d(k) = w_2(k) = 0 \tag{7.10-11}$$

for each $k \in \{n-1, n-2, \ldots\}$

Supposing that

$$C_\xi(k) = C_\xi \tag{7.10-12}$$

for each $k \in \{n-1, n-2, \ldots\}$, where C_ξ is a given matrix, then we have

$$\sum_{k+1}^{n} \mathscr{L}(i) = \xi^{\mathrm{T}}(k) A_2(k) \xi(k) \tag{7.10-13}$$

for each $k \in \{n-1, n-2, \ldots\}$, where

$$\begin{aligned} A_2(k) = {} & \Phi'^{\mathrm{T}} Q_1 \Phi' + (\Phi'^2)^{\mathrm{T}} Q_1 \Phi'^2 + \ldots + (\Phi'^{n-k})^{\mathrm{T}} Q_1 \Phi'^{n-k} + \\ & + C_\xi^{\mathrm{T}} Q_2 C_\xi + (C_\xi \Phi')^{\mathrm{T}} Q_2 C_\xi \Phi' + \ldots + (C_\xi \Phi'^{n-k-1})^{\mathrm{T}} Q_2 C_\xi \Phi'^{n-k-1} \end{aligned} \tag{7.10-14}$$

with Φ' standing for $\Phi - \Delta C_\xi$.

From (7.10-14), it appears that $A_2(k)$, $(k = n-1, n-2, \ldots)$, converges to a limit, denoted

$$\hat{A}_2 \stackrel{\text{def}}{=} \lim_{k \to n-\infty} A_2(k) \tag{7.10-15}$$

under the sufficient condition that the matrix $(\Phi - \Delta C_\xi)$ has the property that each one of its eigenvalues has a modulus which is less than unity.

The convergency of $A_2(k)$, $(k = n-1, n-2, \ldots)$, implies the convergency of $A_2{}^*(k)$ by reason of the fact that the optimal system performs at least as well as another system, which means that

$$\xi^{\mathrm{T}}(k) A_2{}^*(k) \xi(k) \leqslant \xi^{\mathrm{T}}(k) A_2(k) \xi(k) \tag{7.10-16}$$

for each $\xi(k)$ and for each $k \in \{n-1, n-2, \ldots\}$, while the following relation holds:

$$\xi^{\mathrm{T}}(k) A_2{}^*(k) \xi(k) \geqslant \xi^{\mathrm{T}}(k) A_2{}^*(k+1) \xi(k) \tag{7.10-17}$$

for each $\xi(k)$ and for each $k \in \{n-1, n-2, \ldots\}$, which is easily verified. By (7.3-9), the convergency of $A_2{}^*(k)$ implies the convergency of $C_\xi{}^*(k)$, $(k = n-1, n-2, \ldots)$.

Sufficient conditions for the existence of a matrix C_ξ such that the modulus of each eigenvalue of $(\Phi - \Delta C_\xi)$ is less than unity, are

1. The plant is asymptotically stable; in this case $C_\xi = 0$ can be chosen.
2. The plant is completely controllable.

for each $k \in \{n-1, n-2, \ldots\}$, and

$$\lim_{j \to \infty} \hat{\boldsymbol{C}}_d^*(j) = \boldsymbol{0} \tag{7.10-27}$$

$$\lim_{j \to \infty} \hat{\boldsymbol{A}}_d^*(j) = \boldsymbol{0} \tag{7.10-28}$$

7.10.6 *Convergency of* $\boldsymbol{K}^*(k)$, $(k = 0, 1, \ldots)$

In sec. 7.2, we gave the set of equations by means of which optimal values of $\boldsymbol{K}(k)$, together with those of $\boldsymbol{P}(k)$ and $\boldsymbol{S}(k+1)$, can be computed recursively for $k = 0, 1, \ldots, n-1$.

Since we wish to make a distinction between optimal values, thus computed, and other values of these quantities, optimal values will be designated by asterisks again.

We shall now let the number of sampling instants over which $\boldsymbol{K}^*(k)$, $\boldsymbol{P}^*(k)$ and $\boldsymbol{S}^*(k+1)$, $(k = 0, 1, \ldots)$, are computed approach infinity, and we shall be concerned with the convergency of these sequences. The limits will be denoted

$$\hat{\boldsymbol{K}}^* \stackrel{\text{def}}{=} \lim_{k \to \infty} \boldsymbol{K}^*(k) \tag{7.10-29}$$

$$\hat{\boldsymbol{P}}^* \stackrel{\text{def}}{=} \lim_{k \to \infty} \boldsymbol{P}^*(k) \tag{7.10-30}$$

$$\hat{\boldsymbol{S}}^* \stackrel{\text{def}}{=} \lim_{k \to \infty} \hat{\boldsymbol{S}}^*(k) \tag{7.10-31}$$

The following statement can be made: a sufficient condition for the convergency of $\boldsymbol{K}^*(k)$, $\boldsymbol{P}^*(k)$ and $\boldsymbol{S}^*(k+1)$, $(k = 0, 1, \ldots)$, is the existence of a matrix $\boldsymbol{K}$ such that the matrix $(\boldsymbol{I}-\boldsymbol{K}\boldsymbol{M})\boldsymbol{\Phi}$ has the property that the modulus of each one of its eigenvalues is less than unity [10].

Examining (7.7-4) together with (7.7-5) through (7.7-10), we find that the quantities $\boldsymbol{\xi}(k)$ do not influence the quantities $\tilde{\boldsymbol{\xi}}(k)$, $(k = 0, 1, \ldots)$. Therefore, if we wish to study the estimator part of the control system, we only have to work with the "lower half" of (7.7-4). Supposing that

$$\boldsymbol{K}(k) = \boldsymbol{K}$$

for each $k \in \{0, 1, \ldots\}$, where $\boldsymbol{K}$ is a given matrix, we derive from (7.7-4) (7.10-32)

$$\tilde{\boldsymbol{\xi}}(k+1) = (\boldsymbol{I}-\boldsymbol{K}\boldsymbol{M})\boldsymbol{\Phi}\tilde{\boldsymbol{\xi}}(k)+(\boldsymbol{K}\boldsymbol{M}-\boldsymbol{I})\boldsymbol{w}_1(k)+\boldsymbol{K}\boldsymbol{w}_2(k+1) \tag{7.10-33}$$

for each $k \in \{0, 1, \ldots\}$

From (7.10-33), it appears that

References Chapter 7

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GLOSSARY

1 General conventions and symbols

1. Formulae, figures and tables are numbered sectionwise. They are identified by compound numbers of the form 2.3-4, where 2.3 is the section number and 4 is item number.
2. Boldface letters denote matrices and vectors.
3. $\boldsymbol{I}$ is an identity matrix, and $\mathbf{0}$ is a zero matrix or vector.
4. Superscript T denotes the transpose of a matrix or vector.
5. Superscript -1 denotes the inverse of a matrix.
6. Superscript dagger ($\dagger$) denotes the pseudo-inverse of a matrix.
7. A dot replacing a variable is used to indicate that the symbol preceding $(\cdot)$ represents a *function*. Thus if we write $y(\cdot)$ instead of $y(x)$, we stress that not a particular *value* of y is referred to, but the *rule* assigning values of y to values of x.
8. $\stackrel{\text{def}}{=}$ means "equals by definition" or "denotes".
9. p stands for "probability density".
   E stands for "expectation".
   cov stands for "covariance": if the column vectors ξ_1 and ξ_2 are random variables then
   $\operatorname{cov}(\xi_1, \xi_2) \stackrel{\text{def}}{=} \mathrm{E}\{(\xi_1 - \mathrm{E}\xi_1)(\xi_2 - \mathrm{E}\xi_2)^{\mathrm{T}}\}$
10. A vertical bar (|) in connexion with p, E or cov indicates that the probability density, the expectation or the covariance is conditional on the quantities written behind that bar.
11. E_d designates an expectation relative to the sequence of mean disturbances $\boldsymbol{d}(k)$, $(k = 0, 1, \ldots)$, treated in the stochastic manner of secs. 7.8 and 4.3.
12. A form or matrix which is said to be *positive semi-definite* may be positive definite in a special case.

2 Symbols with special meaning

The numbers refer to the pages where the symbols are introduced or re-introduced.

$\boldsymbol{A}$	18	$\hat{\boldsymbol{A}}_2^*$	122	A_0	101
$\boldsymbol{A}_1$	101	$\boldsymbol{A}_d$	107	A_{21}, A_{22}, etc.	17
$\boldsymbol{A}_2$	101	$\boldsymbol{A}_d^*$	124	A'_{31}, A'_{33}	24
$\boldsymbol{A}_2^*$	122	$\hat{\boldsymbol{A}}_d^*$	124	A_l	34
$\hat{\boldsymbol{A}}_2$	123	A	18	a	104

$R^{(2)}_{44}, R^{(2)}_{55}$ 40
$R^{(3)}_{11}, R^{(3)}_{12}$, etc. 44
$R^{(\mathrm{wind})}_{11}, R^{(\mathrm{wind})}_{22}$ 67
R_C 90
R_{current} 69
R_{curv} 70
$\mathscr{R}^{\eta_\varrho}$ 98
r 19
r_0 19
r_C 19
r'_C 90
$\boldsymbol{S}$ 99
$\boldsymbol{S}^*$ 125
$\hat{\boldsymbol{S}}^*$ 125
$\boldsymbol{S}_0$ 97
$\boldsymbol{S}^{(0)}$ 39
$S^{(0)}_{11}, S^{(0)}_{12}$, etc. 39
$\mathscr{S}_\eta, \mathscr{S}_\varrho$ 98
s 43
s_1, s_2, etc. 43
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t' 32
u, u_0 15
u_{air} 34
u_{current} 35
u_{rel} 36
u_w 26
u_{wind} 28
$\mathscr{V}$ 101
v, v_0 15
v_{current} 35
$\boldsymbol{W}_1$ 109
$\boldsymbol{W}_2$ 109
$\mathscr{W}$ 41, 98
$\mathscr{W}(l)$ 41
$\mathscr{W}^{(\lambda)}$ 45
$\boldsymbol{w}_1$ 97
$\boldsymbol{w}_2$ 97
$\boldsymbol{w}^{(2)}$ 40
$\boldsymbol{w}_3$ 113
$\boldsymbol{w}^{(3)}$ 44
$\boldsymbol{w}^{(\mathrm{wind})}$ 67
$w^{(2)}_4, w^{(2)}_5$ 39
$w^{(3)}_1, w^{(3)}_2$ 44
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w_{curv} 69
$w^{(\mathrm{wind})}_1, w^{(\mathrm{wind})}_2$ 67
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$x^{(\mathrm{M})}_0$ 90
$x^{(\mathrm{P}C)}_0$ 90
$x_{\mathrm{A}}, x_{\mathrm{B}}$ 38
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x_h 93
Y 15
Y_{current} 35
Y_{curv} 20
Y_d 16
Y_{det} 43
$\hat{Y}_{\mathrm{det}}$ 60
$Y_r, Y_{\dot{r}}$ 16
$Y'_r, Y'_{\dot{r}}$ 62
Y_{tot} 16
$Y_v, Y_{\dot{v}}$ 16
$Y'_v, Y'_{\dot{v}}$ 62
Y_{waves} 28
$Y^{(p)}_{\mathrm{waves}}$ 31
$Y^{(s)}_{\mathrm{waves}}$ 30
Y_{wind} 34
Y_δ 16
Y'_δ 62
y 18
y_0 14
$y^{(\mathrm{M})}_0$ 90
$y^{(\mathrm{P}C)}_0$ 90
y_b 14
y_h 93
Z_{11}, Z_{12}, etc. 78
z_0 14
z_b 14
$\boldsymbol{\Gamma}$ 21
Γ_{11}, Γ_{12}, etc. 21
$\boldsymbol{\gamma}$ 126
$\gamma_y, \gamma_\delta, \gamma_\psi$ 80
$\boldsymbol{\Delta}$ 21, 97
Δ_1, Δ_2, etc. 21
δ 15
δ', δ'' 49
$\delta_{\max}$ 42
δ_{ref} 22
$\varepsilon_{\max}$ 22
ζ 25
$\hat{\zeta}$ 26
$\boldsymbol{\eta}$ 39, 97
η_1, η_4, η_5 39
η_x 59, 89
η_{x_0} 87
η_{x_h} 87
η_y 89
η_{y_0} 87
η_{y_h} 87
η_ψ 90
η_{ψ_0} 87
$\boldsymbol{\Theta}$ 20
Θ_{11}, Θ_{12}, etc. 21
θ 26, 108
$\varkappa$ 26
λ_w 26
$\lambda_\delta, \lambda_\varrho$ 45
μ 26
μ_a 34
$\boldsymbol{\Xi}', \boldsymbol{\Xi}''$ 121
$\boldsymbol{\Xi}'_{qs}, \boldsymbol{\Xi}''_{qs}$ 121
$\boldsymbol{\xi}$ 38, 97
$\tilde{\boldsymbol{\xi}}$ 109
$\boldsymbol{\xi}(\ldots|\ldots)$ 99
$\boldsymbol{\xi}_0$ 97
$\boldsymbol{\xi}^{(0)}$ 39
$\boldsymbol{\xi}_a$ 109
ξ_1, ξ_2, etc. 38
$\xi_1(\ldots|\ldots)$, $\xi_2(\ldots|\ldots)$, etc. 45, 46

SUMMARY

The main problem treated in this thesis is the designing, from the point of view of control theory, of a controller guiding a ship whose speed is constant along a prespecified track.

The coefficients determining the ship's behaviour are supposed to be constant and known.

It is assumed that sensors observing the ship's heading and position are on hand. These sensors in themselves are practically left out of consideration, but the limited accuracy of their observations is allowed for. In addition to being a function of the feedback signals, the controller output is also a function of the specified track and of the mean wind and the current. The last mentioned two quantities are supposed to be estimated or measured beforehand.

For the designing of the controller, some general properties of the specified track and of the deterministic and stochastic disturbing forces and moments, exerted on the ship as a result of waves, wind and current, are taken into account.

Chapter 1 is a general introduction to the problem examined.

In chapter 2 the mathematical model of the behaviour of the ship is built.

Chapter 3 offers mathematical models of the disturbing forces and moments exerted on a ship as a result of waves, wind and current.

Chapter 4 presents a procedure for the designing of a ship guidance controller.

Chapter 5 is a report of an investigation consisting of computations on a digital computer. For a particular ship, a guidance controller was designed and the performance of this controller was evaluated.

Chapter 6 deals with some practical aspects of the guidance system considered in this thesis. Particularly, a notion is given of the nature and the extent of the constituent equipment.

The purpose of chapter 7 is to present the control theory underlying the designing of the ship guidance controller. The subject of this chapter is optimal control of linear sampled-data systems with quadratic criteria, where the plant is subject to stochastic disturbances while there is noise on the plant state observations as well.

SAMENVATTING

Het hoofdprobleem dat in dit proefschrift wordt behandeld is het regeltheoretisch ontwerpen van een stuurautomaat die een schip, varend met een constante snelheid, langs een vooraf gespecificeerde baan geleidt.

De coëfficiënten die het gedrag van het schip bepalen worden constant en bekend verondersteld.

Aangenomen wordt, dat sensoren ter beschikking zijn die de koers en de positie van het schip waarnemen. Deze sensoren op zichzelf worden vrijwel buiten beschouwing gelaten, maar wel wordt de beperkte nauwkeurigheid van de waarnemingen in aanmerking genomen. Het signaal door de stuurautomaat geleverd is behalve een functie van de terugkoppelsignalen ook een functie van de vorm van de gespecificeerde baan en van de gemiddelde wind en de stroming. De laatste twee grootheden worden verondersteld tevoren te zijn gemeten of geschat.

Bij het ontwerpen van de stuurautomaat wordt rekening gehouden met enkele algemene eigenschappen van de gespecificeerde baan en van de deterministische en stochastische storende krachten en momenten, op het schip uitgeoefend als gevolg van golven, wind en stroming.

Hoofdstuk 1 is een algemene inleiding tot het behandelde probleem.

In hoofdstuk 2 wordt het in dit proefschrift gebruikte mathematische model van het gedrag van het schip opgesteld.

In hoofdstuk 3 worden mathematische modellen gegeven van de storende krachten en momenten die een schip ondervindt als gevolg van golven, wind en stroming.

In hoofdstuk 4 wordt een werkwijze gepresenteerd voor het ontwerpen van een stuurautomaat.

Hoofdstuk 5 is een verslag van een onderzoek bestaande uit berekeningen op een digitale rekenmachine. Voor een bepaald schip werd de stuurautomaat ontworpen en werden de prestaties van deze stuurautomaat geëvalueerd.

In hoofdstuk 6 worden enkele praktische aspecten behandeld van het in dit proefschrift beschouwde geleidingssysteem. In het bijzonder wordt een indruk gegeven van de aard en omvang van de apparatuur waaruit zo'n systeem zou moeten worden opgebouwd.

Hoofdstuk 7 heeft als doel de regeltheoretische principes weer te geven die bij het ontwerpen van een stuurautomaat als basis worden gebruikt. Het onderwerp van dit hoofdstuk is de optimale regeling van bemonsterde lineaire systemen met kwadratische criteria, waarbij het geregelde systeem onderhevig is aan stochastische storingen terwijl bovendien bij de waarneming van de toestand meetruis wordt toegevoegd.

STELLINGEN

1

Wil men het mathematisch model van het gedrag van het schip zoals beschreven door verg. (2.2-4) verbeteren, dan verdient het aanbeveling de verbetering ook te zoeken in een vergroting van de orde en te trachten het lineaire karakter zoveel mogelijk te bewaren.

Dit proefschrift, pag. 17.
Ref. [1] van hoofdstuk 5
Ref. [2] van hoofdstuk 2.

2

Niet alleen de benaming *energie spectrum* voor $\mathcal{P}(\omega_w, \mu)$, maar ook de wijze waarop deze grootheid gewoonlijk in een grafische voorstelling wordt geschetst is misleidend.

Dit proefschrift, pag. 28.
Ref. [3] van hoofdstuk 3, pag. 338.

3

Tegen de wijze waarop GOCLOWSKI en GELB hebben getracht het probleem van de geleiding van een schip langs een vooraf gegeven baan op te lossen zijn bezwaren in te brengen.

Ref. [3] van hoofdstuk 1.

4

De gevoeligheid gedefinieerd volgens BODE geeft minder inzicht in de invloed van een parameter-variatie op de prestaties van een regelsysteem dan dikwijls wordt gesuggereerd.

J. G. TRUXAL, Automatic feedback controlsystem synthesis. Pag. 120–127.
McGraw-Hill, New York etc., 1955.

5

Voor het beoordelen van de kwaliteit van regelsystemen verdienen in de meeste gevallen statistische criteria, eventueel met daarin de invloed van parameter-variaties verdisconteerd, de voorkeur boven criteria gebaseerd op de frequentie-responsie of de stap-responsie.

6

Bij het onderwijs in de regeltechniek is het raadzaam, vroegtijdig het stochastische karakter van de in een regelsysteem voorkomende signalen in de beschouwingen te betrekken.

7

In digitale regelsystemen worden nog te weinig sensoren en andere componenten gebruikt die door het principe van hun werking reeds digitaal zijn.

8

De opvatting dat elektrische netwerk-elementen kunnen worden verdeeld in fysisch realiseerbare en fysisch niet-realiseerbare, waarbij onder meer de nullor tot de eerste groep wordt gerekend, is aanvechtbaar.

B. D. H. TELLEGEN, On nullators and norators. IEEE Trans. on Circuit Theory, **13**-4, 1966.

9

Het verdient aanbeveling, die gemeenteambtenaren die aanzienlijke invloed kunnen uitoefenen op de gang van zaken in de gemeente (hoofden van diensten en van belangrijke afdelingen, hun naaste medewerkers) onder te brengen in een inter- of zelfs bovengemeentelijke organisatie die een uniform aanstellings- en bevorderingsbeleid en stelselmatige doorstroming waarborgt.

10

De ontwikkeling van een Europees saamhorigheidsgevoel onder de burgers moet door de overheden van de Westeuropese landen actief worden bevorderd.

"It is not just the Europe of the governments, parliaments and administrations". Commissie v. d. EG in een verklaring, 1968. Geciteerd in: A. SAMPSON, The new Europeans. Pag. 54. Hodder and Stoughton, Londen, 1968.

11

De handel in bepaalde categorieën van duurzame gebruiksgoederen dient te worden gebonden aan zekere wettelijke regels met betrekking tot het verlenen van service (onderhoud, reparatie, en levering van onderdelen en specifieke verbruiksartikelen).

J. K. Zuidweg