A linear matrix inequality approach to optimal reset control design for a class of nonlinear systems

Shahbazzadeh, Majid; Sadati, S. Jalil; HosseinNia, S. Hassan

DOI
10.1002/rnc.7248

Publication date
2024

Document Version
Final published version

Published in
International Journal of Robust and Nonlinear Control

Received: 9 March 2023 | Revised: 26 January 2024 | Accepted: 28 January 2024

DOI: 10.1002/rnc.7248

RESEARCH ARTICLE

WILEY

A linear matrix inequality approach to optimal reset control design for a class of nonlinear systems

Majid Shahbazzadeh[1] | **S. Jalil Sadati**[1] | **S. Hassan HosseinNia**[2]

[1]Faculty of Electrical and Computer Engineering, Babol Noshirvani University of Technology, Babol, Iran

[2]Department of Precision and Microsystems Engineering, Delft University of Technology, Delft, The Netherlands

Correspondence
S. Jalil Sadati, Faculty of Electrical and Computer Engineering, Babol Noshirvani University of Technology, Babol, Iran.
Email: j.sadati@nit.ac.ir

Abstract

In this article, the problem of the optimal reset control design for Lipschitz nonlinear systems is addressed. The reset controller includes a base linear controller and a reset law that enforces resets to the controller states. The reset law design is strongly dependent on the appropriate design of the base controller. For this reason, in this article, the base controller and reset law are simultaneously designed. More precisely, an optimal dynamic output feedback is considered as the base controller which minimizes the upper bound of a quadratic performance index, and a reset law is used to improve the transient response of the closed-loop system. This design is done in a full offline procedure. The problem is transformed into a set of linear matrix inequalities (LMIs), and the reset controller is obtained by solving an offline LMI optimization problem. Finally, two examples are presented to illustrate the effectiveness and validity of the proposed method.

KEYWORDS

dynamic output feedback, linear matrix inequalities, Lipschitz condition, reset control systems, reset law

1 | INTRODUCTION

Reset controllers are a class of hybrid controllers that were first introduced by Clegg[1] to overcome the fundamental limitations of linear feedback control design. A reset controller is a linear controller equipped with a resetting mechanism that resets all or part of the controller states whenever some triggering condition is satisfied.[2,3] Reset controllers provide more flexibility in controller design, which can improve the system transient response.[4,5] During the last decades, reset control systems have attracted much attention from the control community.[6–16]

Generally, reset control design consists of two main steps: base controller design and reset law design. Over recent years, a large number of studies have only focused on the second step (i.e., reset law design). For example in Reference 17, the optimal reset law design problem was converted to a linear quadratic regulation problem and the optimal reset law was then designed by solving algebraic Riccati equations. In Reference 18, a reset law was obtained for Lipschitz uncertain systems using model predictive strategy (MPS). The problem of reset law design was addressed for linear systems under norm-bounded uncertainty in Reference 19. A discrete-time triggered reset law was proposed to adapt reset control to computer-based implementation in Reference 20. An observer-based reset law was presented for uncertain systems based on MPS in Reference 21. An event-triggered-based optimal reset law was provided for the typical head-positioning system of Hard disk drives in Reference 22. A systematic Lyapunov-based approach to design a reset law was presented for a class of nonlinear time-delay systems in Reference 23. An optimal reset law was designed for Lipschitz nonlinear systems based on guaranteed cost control approach in Reference 24.

However, in recent years, few articles have dealt with the systematic design of both stages of the reset control design (i.e., base controller and reset law design). For example in Reference 25, a reset gain-scheduling dynamic controller was provided based on MPS for polytopic linear parameter varying systems. In Reference 26, a reset dynamic output feedback controller was proposed and a genetic algorithm was applied to minimize a cost function to find the reset times. The reset control design procedure presented in Reference 26 was extended to a class of uncertain linear systems in Reference 27.

Although the base controller and reset law are designed in the methods presented in References 25–27, there are some challenges in their implementation. In fact, these methods involve offline and online steps. In the offline step, sufficient conditions for existence of base controller are obtained in terms of linear matrix inequalities (LMIs). Then, the after-reset values of the controller states are determined by solving an online optimization problem based on MPS at reset instants. The implementation of these methods demands a large computational burden due to the complex numerical optimization problem that has to be solved at each reset instant. As a result, these methods may not be useful for real-time applications with small sampling time. This motivates the current study.

In this article, the problem of reset control design for a class of nonlinear systems is addressed. In an offline design procedure, the base controller and reset law are simultaneously designed. Reset law is an additional degree of freedom in the control design, which can improve the transient response of the closed-loop system. In this article, the after-reset values are obtained based on a condition involving use of the Lyapunov function. Unlike the methods based on the Clegg integrator, the after-reset values of the controller states are not necessarily a zero vector, but the values that guarantee a negative drop in the Lyapunov function value. This problem is transferred to an offline LMI optimization problem. Thus, the reset controller is designed by solving this problem. Finally, two examples are given to illustrate the effectiveness and merits of the proposed theoretical results.

The main contributions of this article are summarized as follows:

1. In this article, the base controller and reset law are simultaneously designed for Lipschitz nonlinear systems.
2. The proposed reset controller is designed in a full offline procedure. Therefore, the computational issues do not affect its real-time implementation.
3. This problem is transformed into an LMI optimization problem, which can readily be solved via standard numerical software.

The article is organized as follows. The problem formulation is provided in Section 2. The main results are presented in Section 3. The simulation results are given in Section 4. Finally, Section 5 concludes this article.

2 | PROBLEM FORMULATION

Consider a class of nonlinear systems described as follows:

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu + Hf\left(x_p\right), \\ y &= Cx_p, \end{aligned} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the control input vector, $y \in \mathbb{R}^{n_y}$ is the output vector. $f(x_p) \in \mathbb{R}^{n_f}$ is a nonlinear function satisfying the Lipschitz condition locally on a set $\mathbb{D} \subset \mathbb{R}^{n_p}$; namely, there exists a constant matrix L_p such that

$$\|f(x_p) - f(\tilde{x}_p)\| \leq \|L_p(x_p - \tilde{x}_p)\|, \ \forall x_p, \tilde{x}_p \in \mathbb{D}, \tag{2}$$

where $f(0) = 0$.

We consider the following structure for the reset controller:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c e \quad && x_c \notin M_r \\ x_c^+ &= \rho\left(x_p, x_c\right) && x_c \in M_r \\ u &= C_c x_c, \end{aligned} \tag{3}$$

where $x_c \in \mathbb{R}^{n_c}$ and x_c^+ represent the reset controller state vector and after-reset controller state vector, respectively. The error e is defined as the difference between the reference signal r and the system output y (i.e., $e = r - y$). The after-reset value $\rho\left(x_p, x_c\right)$ is a continuous function dependent on the plant and controller states. M_r is the reset surface or jump set.

The following structure is considered for the reset law:

$$\rho\left(x_p, x_c\right) = \Lambda x_p(t_k) + \Xi x_c(t_k), \tag{4}$$

where $\Lambda \in \mathbb{R}^{n_c \times n_p}$ and $\Xi \in \mathbb{R}^{n_c \times n_c}$ are reset map matrices, and t_k are reset times for $k = 1, 2, \ldots$.

Assumption 1. In this article, the regulation problem is investigated for the sake of simplicity, and the reference signal r is set to zero.

Substituting the reset controller (3) into the system (1) with $r = 0$, the dynamic equation of the closed-loop system is obtained as

$$\begin{aligned} \dot{\overline{x}} &= \bar{A}\overline{x} + \overline{H}f(x_p) \quad && \overline{x} \notin M \\ \overline{x}^+ &= A_R\overline{x} && \overline{x} \in M \\ y &= \overline{C}\overline{x}, \end{aligned} \tag{5}$$

where $\overline{x} = [x_p^{\mathrm{T}} \ \ x_c^{\mathrm{T}}]^{\mathrm{T}}$ with

$$\bar{A} = \begin{bmatrix} A & BC_c \\ -B_cC & A_c \end{bmatrix}, \quad A_R = \begin{bmatrix} I & 0 \\ \Lambda & \Xi \end{bmatrix}, \quad \overline{H} = \begin{bmatrix} H \\ 0 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} C & 0 \end{bmatrix},$$

and M is the reset surface defined by Beker et al.[28]

$$M = \left\{\overline{x} \in \mathbb{R}^{n_p+n_c} | e = 0 \quad \& \quad \overline{x}^+ \neq \overline{x}\right\}. \tag{6}$$

Remark 1. In practice, the reset surface M can be modified based on a discrete-time zero-crossing method.[19] That is:

$$M = \left\{\overline{x} \in \mathbb{R}^{n_p+n_c}, K \in N | e((K-1)T_s)e(KT_s) \leq 0 \quad \& \quad \overline{x}^+ \neq \overline{x}\right\}, \tag{7}$$

where T_s is the sampling time.

Assumption 2. In this article, we assume that the controller dynamic order is equal to the number of system states (i.e., $n_c = n_p$).

Remark 2. If the reset condition and after-reset values are selected improperly, beating and deadlock phenomena may occur, which can destroy the existence of solutions.[29] In order to avoid these phenomena, it is assumed that the after-reset values are not elements of the reset condition. That is:

$$\text{If } \ \overline{x}(t_k) \in M \text{ then } \overline{x}(t_k^+) \notin M. \tag{8}$$

Remark 3. The solution of a reset control system may contain an infinite number of reset actions within a compact time interval, which is called Zeno solution. To avoid this phenomenon, the resetting rate can be restricted by defining a positive constant t_ρ. Then, the reset action can only occur at least after t_ρ seconds, which is known as temporal regularization. The closed-loop reset system (5) with temporal regulation is expressed as

$$\begin{cases} \dot{\tau} = 1, & \dot{\overline{x}} = \bar{A}\overline{x} + \overline{H}f(x_p) \quad \overline{x} \notin M \quad \text{or} \quad \tau < t_\rho \\ \tau^+ = 0, & \overline{x}^+ = A_R\overline{x} \qquad\qquad\quad\; \overline{x} \in M \quad \text{and} \quad \tau \geq t_\rho, \end{cases} \tag{9}$$

where t_ρ is the minimum time between two subsequent reset times (i.e., $t_{k+1} - t_k > t_\rho$).

The goal of this article is to design a reset controller in the form of (3) which minimizes the upper bound of a quadratic performance index and improves the system transient response.

3 | MAIN RESULTS

This section is devoted to design the reset controller for Lipschitz nonlinear systems based on the minimization of the following quadratic cost function:

$$\mathcal{J} = \int_0^{\infty} (x_p^{\mathrm{T}}(t)\mathcal{Q}x_p(t) + u^{\mathrm{T}}(t)\mathcal{R}u(t))\mathrm{d}t, \quad \mathcal{Q}, \mathcal{R} > 0, \tag{10}$$

where $\mathcal{Q}$ and $\mathcal{R}$ are symmetric weight matrices.

The above equation can be written as follows:

$$\mathcal{J} = \int_0^{\infty} \bar{x}^{\mathrm{T}}(t)\hat{\mathcal{Q}}\bar{x}(t)\mathrm{d}t, \tag{11}$$

where

$$\hat{\mathcal{Q}} = \begin{bmatrix} \mathcal{Q} & 0 \\ 0 & C_c^{\mathrm{T}}\mathcal{R}C_c \end{bmatrix}.$$

First, the following proposition is borrowed from Reference 28 to prove the main results presented in this article.

Proposition 1. *Suppose there exists a positive-definite, continuously differentiable, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\begin{cases} \dot{V}(\bar{x}) < 0 & \bar{x} \notin M \quad (12) \\ \Delta V(\bar{x}) := V(\bar{x}^+) - V(\bar{x}) \leq 0 & \bar{x} \in M, \quad (13) \end{cases}$$

then, the closed-loop reset system (5) is asymptotically stable.

Theorem 1. *Consider the closed-loop reset system (5). If for a given positive scalar ε, there exist symmetric positive definite matrices $R \in \mathbb{R}^{n_p \times n_p}, S \in \mathbb{R}^{n_p \times n_p}$, matrices $K \in \mathbb{R}^{n_u \times n_p}, L \in \mathbb{R}^{n_p \times n_y}, E \in \mathbb{R}^{n_p \times n_p}, N \in \mathbb{R}^{n_p \times n_p}, F \in \mathbb{R}^{n_p \times n_p}, G \in \mathbb{R}^{n_p \times n_p}$, and a positive scalar γ such that the following LMI optimization problem is feasible:*

$$\begin{aligned} &\min \ \gamma \\ &\text{subject to} \\ &\begin{bmatrix} \Psi_{11} & \Psi_{12} & H & \varepsilon RL_p^{\mathrm{T}} & R & K^{\mathrm{T}} \\ * & \Psi_{22} & SH & \varepsilon L_p^{\mathrm{T}} & I & 0 \\ * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & -\mathcal{Q}^{-1} & 0 \\ * & * & * & * & * & -\mathcal{R}^{-1} \end{bmatrix} < 0, \qquad (14) \\ &\Psi_{11} = AR + RA^{\mathrm{T}} + BK + K^{\mathrm{T}}B^{\mathrm{T}}, \\ &\Psi_{12} = A + E^{\mathrm{T}}, \\ &\Psi_{22} = A^{\mathrm{T}}S + SA - LC - C^{\mathrm{T}}L^{\mathrm{T}}, \\ &\begin{bmatrix} R & I \\ * & S \end{bmatrix} > 0, \qquad (15) \end{aligned}$$

$$\begin{bmatrix} -\gamma & x_p^{\mathrm{T}}(0) & x_p^{\mathrm{T}}(0)S + x_c^{\mathrm{T}}(0)N \\ * & -R & -I \\ * & * & -S \end{bmatrix} \leq 0, \tag{16}$$

$$\begin{bmatrix} -R & -I & R & G^{\mathrm{T}} \\ * & -S & I & S + F^{\mathrm{T}} \\ * & * & -R & -I \\ * & * & * & -S \end{bmatrix} \leq 0, \tag{17}$$

then, the closed-loop system (5) is asymptotically stable, and the upper bound of the cost function (10) is minimized.

In addition, the controller matrices and reset map matrices are respectively obtained as

$$\begin{cases} A_c = N^{-1}(E + LCR - SAR - SBK)M^{-\mathrm{T}} \\ B_c = N^{-1}L \\ C_c = KM^{-\mathrm{T}}, \end{cases} \tag{18}$$

and

$$\begin{cases} \Lambda = N^{-1}F \\ \Xi = N^{-1}(G - SR - FR)M^{-\mathrm{T}}. \end{cases} \tag{19}$$

Proof of Theorem. The base controller design is borrowed from Reference 30. Let us consider the following Lyapunov function candidate

$$V(\overline{x}) = \overline{x}^{\mathrm{T}}(t)P\overline{x}(t), \tag{20}$$

where P is a symmetric positive definite matrix.

Calculating the time-derivative of $V(\overline{x})$ along the solutions of (5) leads to

$$\dot{V}(\overline{x}) = \overline{x}^{\mathrm{T}}(t)\left(\bar{A}^{\mathrm{T}}P + P\bar{A}\right)\overline{x}(t) + f^{\mathrm{T}}(x_p)\overline{H}^{\mathrm{T}}P\overline{x}(t) + \overline{x}^{\mathrm{T}}(t)P\overline{H}f(x_p). \tag{21}$$

The condition $\dot{V}(\overline{x}) < 0$ is fulfilled if the following condition is satisfied:

$$\dot{V}(\overline{x}) < -(x_p^{\mathrm{T}}(t)Qx_p(t) + u^{\mathrm{T}}(t)\mathcal{R}u(t)). \tag{22}$$

The above inequality can be rewritten as

$$\overline{x}^{\mathrm{T}}(t)\left(\bar{A}^{\mathrm{T}}P + P\bar{A} + \hat{Q}\right)\overline{x}(t) + f^{\mathrm{T}}(x_p)\overline{H}^{\mathrm{T}}P\overline{x}(t) + \overline{x}^{\mathrm{T}}(t)P\overline{H}f(x_p) < 0. \tag{23}$$

The following inequality holds based on the condition (2):

$$0 \leq \varepsilon\overline{x}^{\mathrm{T}}(t)\overline{L}_p^{\mathrm{T}}\overline{L}_p\overline{x}(t) - \varepsilon f_p^{\mathrm{T}}(x_p)f_p(x_p), \tag{24}$$

where ε is an arbitrary positive scalar and

$$\overline{L}_p = \begin{bmatrix} L_p & 0 \\ 0 & 0 \end{bmatrix}.$$

Adding the inequality (24) to (23), we have

$$\dot{V}(\bar{x}) < \bar{x}^{\mathrm{T}}(t)\left(\bar{A}^{\mathrm{T}}P + P\bar{A} + \hat{Q}\right)\bar{x}(t) + f^{\mathrm{T}}(x_p)\overline{H}^{\mathrm{T}}P\bar{x}(t) \\ + \bar{x}^{\mathrm{T}}(t)P\overline{H}f(x_p) + \varepsilon x_p^{\mathrm{T}}(t)L_p^{\mathrm{T}}L_p x_p(t) - \varepsilon f^{\mathrm{T}}(x_p)f(x_p). \tag{25}$$

By defining the augmented state vector $\eta(t) = [\bar{x}^{\mathrm{T}}(t)\ f^{\mathrm{T}}(x_p)]^{\mathrm{T}}$, the above inequality can be expressed as

$$\dot{V}(\bar{x}) < \eta^{\mathrm{T}}(t)\underbrace{\begin{bmatrix} \bar{A}^{\mathrm{T}}P + P\bar{A} + \hat{Q} + \varepsilon\overline{L}_p^{\mathrm{T}}\overline{L}_p & P\overline{H} \\ * & -\varepsilon I \end{bmatrix}}_{\Sigma}\eta(t). \tag{26}$$

Hence, the condition $\dot{V}(\bar{x}) < 0$ holds if $\Sigma < 0$ is satisfied.

Now, we partition the Lyapunov matrix P as[31]

$$P = \begin{bmatrix} S_{(n_p\times n_p)} & N_{(n_p\times n_p)} \\ * & W_{(n_p\times n_p)} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} R_{(n_p\times n_p)} & M_{(n_p\times n_p)} \\ * & T_{(n_p\times n_p)} \end{bmatrix}. \tag{27}$$

Note that $PP^{-1} = I$ yields the following condition:

$$M = (I - RS)N^{-\mathrm{T}}. \tag{28}$$

Letting the matrices

$$\Gamma_1 = \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} I & S \\ 0 & N^{\mathrm{T}} \end{bmatrix}, \tag{29}$$

we can conclude that

$$P\Gamma_1 = \Gamma_2. \tag{30}$$

Therefore, if the following condition is satisfied, $P > 0$ is fulfilled:

$$\Gamma_1^{\mathrm{T}}P\Gamma_1 = \Gamma_1^{\mathrm{T}}\Gamma_2 = \begin{bmatrix} R & I \\ * & S \end{bmatrix} > 0. \tag{31}$$

Pre- and post-multiplying both sides of $\Sigma < 0$ by $\mathrm{diag}\{\Gamma_1^{\mathrm{T}}, I\}$, we have

$$\begin{bmatrix} \Gamma_1^{\mathrm{T}} & 0 \\ 0 & I \end{bmatrix}\Sigma\begin{bmatrix} \Gamma_1 & 0 \\ 0 & I \end{bmatrix} < 0. \tag{32}$$

The above matrix inequality involves the following matrices:

$$\begin{aligned} \Gamma_1^{\mathrm{T}}P\bar{A}\Gamma_1 &= \begin{bmatrix} I & 0 \\ S & N \end{bmatrix}\begin{bmatrix} A & BC_c \\ -B_cC & A_c \end{bmatrix}\begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \\ \Theta_{11} &= AR + BC_cM^{\mathrm{T}}, \\ \Theta_{12} &= A, \\ \Theta_{21} &= SAR - NB_cCR + SBC_cM^{\mathrm{T}} + NA_cM^{\mathrm{T}}, \\ \Theta_{22} &= SA - NB_cC, \end{aligned} \tag{33}$$

$$\begin{aligned}\Gamma_1^{\mathrm{T}}\hat{Q}\Gamma_1 &= \begin{bmatrix} R & M \\ I & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & C_c^{\mathrm{T}}\mathcal{R}C_c \end{bmatrix} \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} R & M \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_c^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & \mathcal{R} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_c \end{bmatrix} \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} R & MC_c^{\mathrm{T}} \\ I & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & \mathcal{R} \end{bmatrix} \begin{bmatrix} R & I \\ C_cM^{\mathrm{T}} & 0 \end{bmatrix},\end{aligned} \tag{34}$$

$$\begin{aligned}\varepsilon\Gamma_1^{\mathrm{T}}\overline{L}_p^{\mathrm{T}}\overline{L}_p\Gamma_1 &= \varepsilon \begin{bmatrix} R & M \\ I & 0 \end{bmatrix} \begin{bmatrix} L_p^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} L_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R & I \\ M^{\mathrm{T}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \varepsilon RL_p^{\mathrm{T}} \\ \varepsilon L_p^{\mathrm{T}} \end{bmatrix} \varepsilon^{-1} \begin{bmatrix} \varepsilon L_pR & \varepsilon L_p \end{bmatrix},\end{aligned} \tag{35}$$

$$\Gamma_1^{\mathrm{T}}P\overline{H} = \begin{bmatrix} I & 0 \\ S & N \end{bmatrix} \begin{bmatrix} H \\ 0 \end{bmatrix} = \begin{bmatrix} H \\ SH \end{bmatrix}. \tag{36}$$

By using the Schur complement lemma, the inequality (32) is equivalent to

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & H & \varepsilon RL_p^{\mathrm{T}} & R & MC_c^{\mathrm{T}} \\ * & \Upsilon_{22} & SH & \varepsilon L_p^{\mathrm{T}} & I & 0 \\ * & * & -\varepsilon I & 0 & 0 & 0 \\ * & * & * & -\varepsilon I & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 \\ * & * & * & * & * & -\mathcal{R}^{-1} \end{bmatrix} < 0,$$
$$\begin{aligned}\Upsilon_{11} &= RA^{\mathrm{T}} + MC_c^{\mathrm{T}}B^{\mathrm{T}} + AR + BC_cM^{\mathrm{T}}, \\ \Upsilon_{12} &= A + RA^{\mathrm{T}}S + MC_c^{\mathrm{T}}B^{\mathrm{T}}S - RC^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}} + MA_c^{\mathrm{T}}N^{\mathrm{T}}, \\ \Upsilon_{22} &= A^{\mathrm{T}}S - C^{\mathrm{T}}B_c^{\mathrm{T}}N^{\mathrm{T}} + SA - NB_cC,\end{aligned} \tag{37}$$

By performing the change of variables (18) to (37), the LMI (14) can be obtained.

Integrating both sides of (22) from 0 to ∞ leads to

$$V(\overline{x}(\infty)) - V(\overline{x}(0)) \leq -\int_0^{\infty} (x_p^{\mathrm{T}}(t)Qx_p(t) + u^{\mathrm{T}}(t)\mathcal{R}u(t))\mathrm{d}t = -\mathcal{J}. \tag{38}$$

From the condition (22), we conclude that $V(\overline{x}(\infty)) = 0$. Therefore, we obtain

$$\mathcal{J} \leq V(\overline{x}(0)) = \overline{x}^{\mathrm{T}}(0)P\overline{x}(0) \leq \gamma, \tag{39}$$

where γ is the upper bound of the cost function $\mathcal{J}$.

Applying the Schur complement lemma to the inequality (39), we have

$$\begin{bmatrix} -\gamma & \overline{x}^{\mathrm{T}}(0)P \\ * & -P \end{bmatrix} \leq 0. \tag{40}$$

Pre- and post-multiplying both sides of (40) by $\mathrm{diag}\{I, \Gamma_1^{\mathrm{T}}\}$ and its transpose yields

$$\begin{bmatrix} -\gamma & \overline{x}^{\mathrm{T}}(0)\Gamma_2 \\ * & -\Gamma_1^{\mathrm{T}}\Gamma_2 \end{bmatrix} \leq 0. \tag{41}$$

Substituting (29) into (41), we obtain the LMI condition given in (16).

From (13) and (5), we can obtain

$$\Delta V(\bar{x}) = \bar{x}^{\mathrm{T}}(t)\underbrace{\left(A_R^{\mathrm{T}} P A_R - P\right)}_{\mathcal{X}}\bar{x}(t) \leq 0. \tag{42}$$

Hence, if the condition $\mathcal{X} < 0$ holds, $\Delta V(\bar{x}) \leq 0$ is satisfied.

Applying the Schur complement lemma to $\mathcal{X} < 0$ results in

$$\begin{bmatrix} -P & A_R^{\mathrm{T}} P \\ * & -P \end{bmatrix} \leq 0. \tag{43}$$

Performing a congruence transformation to (43) by $\mathrm{diag}\{\Gamma_1^{\mathrm{T}}, I\}$ and its transpose, we have

$$\begin{aligned}
&\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ * & \Omega_{22} \end{bmatrix} \leq 0, \\
&\Omega_{11} = \Gamma_1^{\mathrm{T}} P \Gamma_1 = -\begin{bmatrix} R & I \\ * & S \end{bmatrix}, \\
&\Omega_{12} = \Gamma_1^{\mathrm{T}} A_R^{\mathrm{T}} P \Gamma_1 = \begin{bmatrix} R & M \\ I & 0 \end{bmatrix} \begin{bmatrix} I & \Lambda^{\mathrm{T}} \\ 0 & \Xi^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & S \\ 0 & N^{\mathrm{T}} \end{bmatrix} \\
&\quad = \begin{bmatrix} R & RS + R\Lambda^{\mathrm{T}} N^{\mathrm{T}} + M\Xi^{\mathrm{T}} N^{\mathrm{T}} \\ I & S + \Lambda^{\mathrm{T}} N^{\mathrm{T}} \end{bmatrix}, \\
&\Omega_{22} = -\begin{bmatrix} R & I \\ * & S \end{bmatrix}.
\end{aligned} \tag{44}$$

By applying the change of variables (19) to (44), we get the LMI (17). This concludes the proof of Theorem 1. ∎

Remark 4. According to the condition (16), if the initial conditions of the controller states are equal to zero (i.e., $x_c(0) = 0$), the decision variable matrix N can be selected as an identity matrix (i.e., $N = I$) for the sake of simplicity.

Remark 5. Note that the conditions provided in Theorem 1 are LMIs for a given scalar ϵ. Therefore, the optimal solution of this problem can be readily approached by solving LMI-based problems on a grid in ϵ.

Remark 6. It can be shown by the inverse of the Schur complement that not all the eigenvalues of the matrix in the condition (17) become negative, and some eigenvalues may be very small positive numbers (for example, smaller than 10^{-12}), so that the solver considers them as 0 and as a result, the message "Successfully solved" is displayed. However, the following condition can be checked in the reset instants:

$$\bar{x}^{\mathrm{T}}(t)\left(A_R^{\mathrm{T}} P A_R - P\right)\bar{x}(t) \leq 0. \tag{45}$$

If the above condition is satisfied, the controller states can jump to specified values called after-reset values.

The design conditions of the reset controller in form of (3) for linear systems are presented in the following corollary.

Corollary 1. *Consider the closed-loop reset system (5) with $f(x_p) = 0$. If there exist symmetric positive definite matrices* $R \in \mathbb{R}^{n_p \times n_p}, S \in \mathbb{R}^{n_p \times n_p}$, *matrices* $K \in \mathbb{R}^{n_u \times n_p}, L \in \mathbb{R}^{n_p \times n_y}, E \in \mathbb{R}^{n_p \times n_p}, N \in \mathbb{R}^{n_p \times n_p}, F \in \mathbb{R}^{n_p \times n_p}, G \in$

$\mathbb{R}^{n_p \times n_p}$, *and a positive scalar* γ *such that the following LMI optimization problem is feasible*:

$$\begin{aligned} &\min\ \gamma \\ &\text{subject to (15)–(17) and} \\ &\begin{bmatrix} \Psi_{11} & \Psi_{12} & R & K^{\mathrm{T}} \\ * & \Psi_{22} & I & 0 \\ * & * & -\mathcal{Q}^{-1} & 0 \\ * & * & * & -\mathcal{R}^{-1} \end{bmatrix} < 0, \end{aligned} \tag{46}$$

then, the closed-loop system (5) with $f(x_p) = 0$ *is asymptotically stable, and the upper bound of the cost function (10) is minimized. In addition, the controller matrices and reset map matrices are respectively obtained from (18) and (19).*

Remark 7. In this article, the reset controller is designed offline; therefore, the computational load does not affect the real-time implementation of the proposed control method, unlike the methods presented in References 18–23, 25–27.

4 | SIMULATION RESULTS

In this section, the performance of the designed reset controller is compared with the base controller, which is an optimal dynamic output feedback controller. For this reason, two examples are given to show the effectiveness and efficiency of the proposed method. The LMI optimization problems are solved by using the YALMIP interface[32] with the MOSEK solver.[33] In the examples, the initial conditions of the controller states are chosen to be zero, so Remark 4 is used in the controller design. In addition, the temporal regularization parameter t_ρ is set to 0.1 s.

Example 1. Consider the following nonlinear system[18]:

$$\begin{aligned} \dot{x}_p(t) &= \begin{bmatrix} -8 & 1 \\ 0 & 0 \end{bmatrix} x_p(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) + \begin{bmatrix} 0.5 \\ -1.5 \end{bmatrix} \frac{x_{p_2}(t)}{1 + x_{p_1}^2(t)}, \\ y(t) &= \begin{bmatrix} 64 & 0 \end{bmatrix} x_p(t). \end{aligned}$$

The weighting matrices are selected as $\mathcal{Q} = I_2$ and $\mathcal{R} = 0.1$.

After solving the LMI optimization problem in Theorem 1 with $\varepsilon = 2.5$, the controller and reset map matrices are respectively calculated as

$$A_c = \begin{bmatrix} -10613 & -82022 \\ 1155.9 & 8937.3 \end{bmatrix},\ B_c = \begin{bmatrix} -1.9828 \\ 0.4946 \end{bmatrix},\ C_c = \begin{bmatrix} 671.21 & 5188.5 \end{bmatrix}, \tag{47}$$

and

$$\Lambda = \begin{bmatrix} -23.873 & 3.0882 \\ 3.089 & -0.40072 \end{bmatrix},\ \Xi = \begin{bmatrix} -0.0058894 & -0.15428 \\ -0.0002331 & 0.012068 \end{bmatrix}. \tag{48}$$

Let the initial conditions be $x(0) = [0.5\ \ -0.5]^{\mathrm{T}}$. The simulation results are shown in Figures 1–3. The responses of the closed-loop system with the base controller and reset controller are presented in the Figure 1. Although the base controller is an optimal dynamic output feedback controller, the system transient response is improved by using the reset controller, as can be seen from Figure 1. The control input and the controller states are depicted in Figure 2. It is evident from the blue dash-dotted line in this figure that the reset action

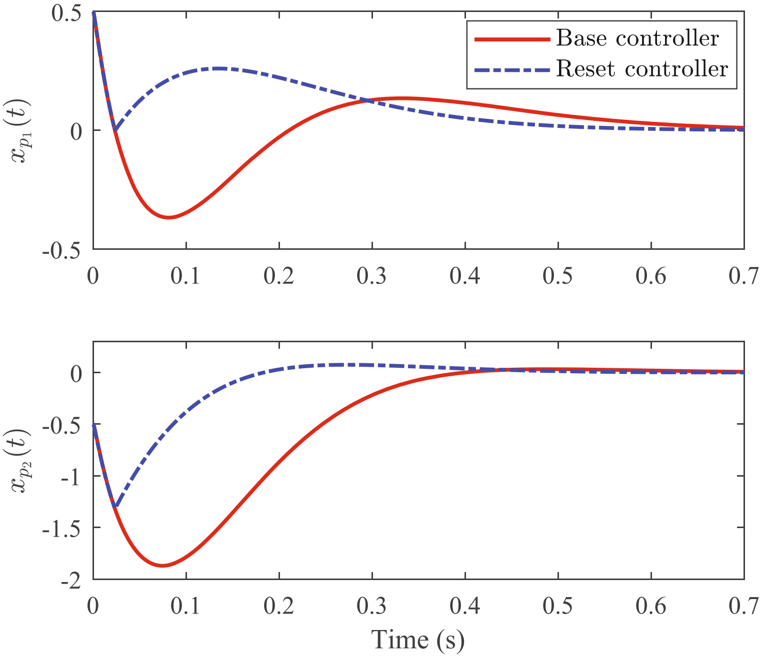

FIGURE 1 The system states $x(t)$ in Example 1.

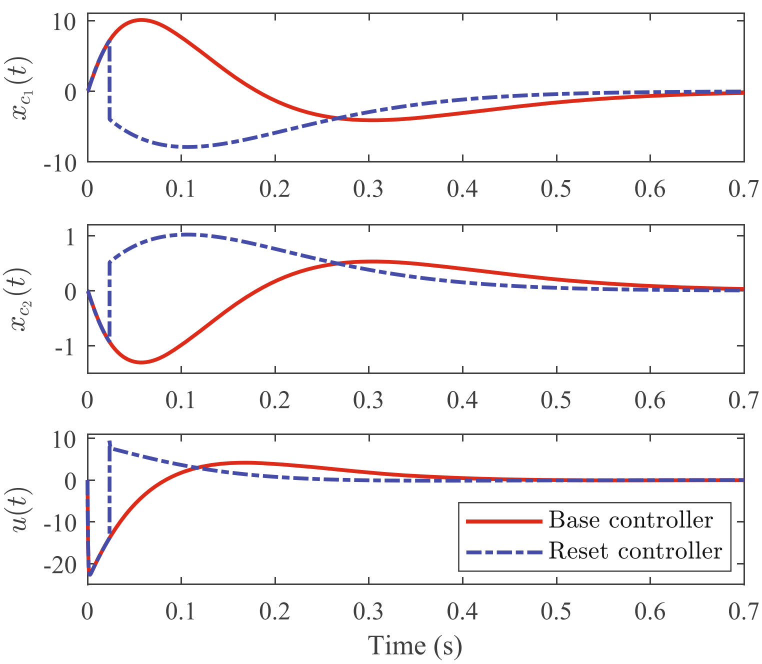

FIGURE 2 The control input $u(t)$ and the controller states $x_c(t)$ in Example 1.

occurs at 0.0235 s. According to the condition (13), the reset action causes the Lyapunov function to decrease at the reset instant as seen in Figure 3. This can improve the closed-loop system performance. For this reason, the comparative results are provided in Table 1. From this table, we can easily see that the reset controller has more satisfactory performance than the base controller.

Example 2. Consider a well-mixed continuous stirred tank reactor (CSTR) in which the following isothermal, liquid-phase, multi-component chemical reaction $A \leftrightarrows B \rightarrow C$ is being carried out. The CSTR dynamics

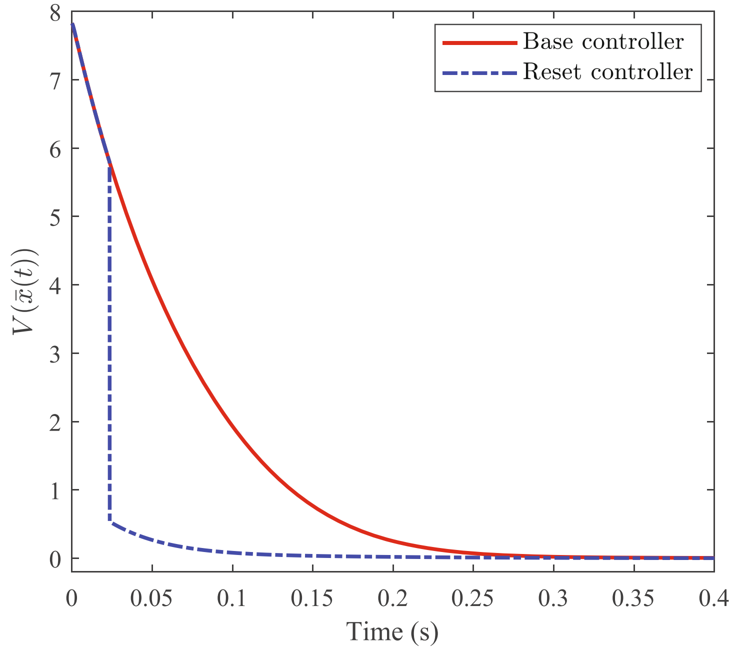

FIGURE 3 The decrement in the Lyapunov function $V(\bar{x}(t))$ value at the reset instant in Example 1.

TABLE 1 Comparison of the performance index $\mathcal{J}$ in Example 1.

	Base controller	Reset controller
$\mathcal{J}$	1.8149	1.1834

can be expressed in the following form[18,34]:

$$\dot{x}_p(t) = \begin{bmatrix} -4 & 0.8796 & 0 \\ 3 & -3.6388 & 0 \\ 0 & 1.7592 & -1 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0.5 \\ -1.5 \\ 1 \end{bmatrix} x_{p_2}^2(t),$$
$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x_p(t).$$

The initial conditions are set to be $x(0) = [0.5 \;\; -0.75 \;\; 0.1]$, and the weighting matrices $\mathcal{Q}$ and $\mathcal{R}$ are respectively chosen as I_2 and 0.001.

By solving the optimization problem in Theorem 1 with $\varepsilon = 1$, the controller and reset map matrices are computed as

$$A_c = \begin{bmatrix} 182.25 & -273.74 & 29.231 \\ 132.3 & -198.11 & 19.722 \\ 20.712 & -29.049 & -5.1521 \end{bmatrix}, \; B_c = \begin{bmatrix} -24.432 \\ -12.178 \\ 23.624 \end{bmatrix}, \; C_c = \begin{bmatrix} -16.632 & 24.55 & -2.0941 \end{bmatrix}, \tag{49}$$

and

$$\Lambda = \begin{bmatrix} -15.473 & -10.914 & -3.5167 \\ -10.914 & -7.9384 & -3.0096 \\ -3.5178 & -3.0102 & -5.1947 \end{bmatrix}, \; \Xi = 10^{-3} \begin{bmatrix} 7.1466 & -10.817 & 1.4757 \\ 2.2045 & -3.5968 & 0.32936 \\ 2.9241 & -3.966 & -0.77827 \end{bmatrix}. \tag{50}$$

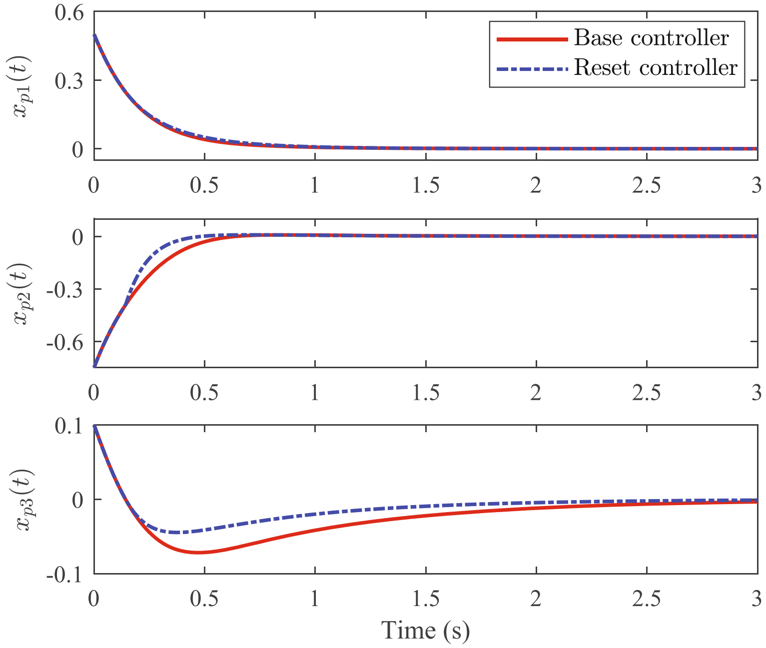

FIGURE 4 The system states $x(t)$ in Example 2.

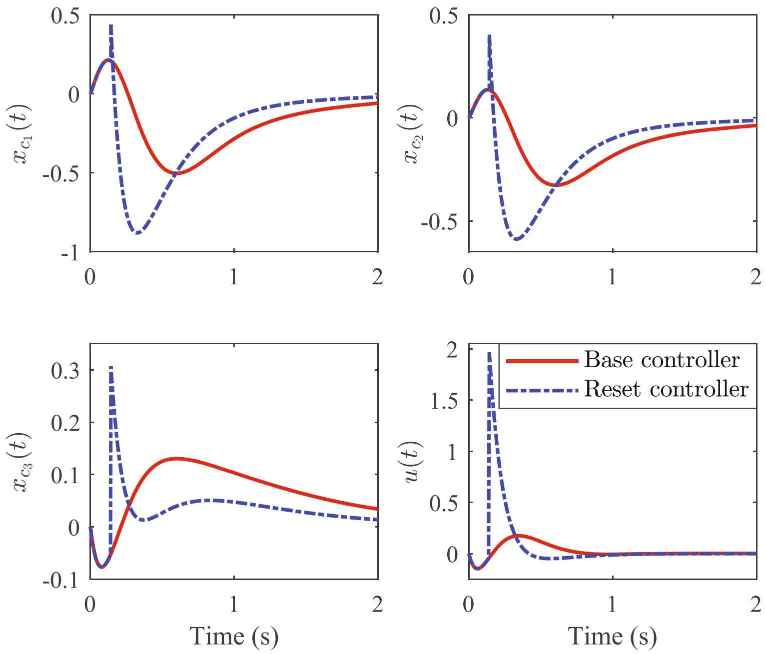

FIGURE 5 The control input $u(t)$ and the controller states $x_c(t)$ in Example 2.

It is evident from Figure 4 that although the state $x_1(t)$ remains almost unchanged, the transient response of the states $x_2(t)$ and $x_3(t)$ by using the reset controller is better than the base controller. The controller states and control input are shown in Figure 5. According to this figure, the controller states are reset at 0.1423 s. Therefore, a sharp drop in the Lyapunov function is observed at this time, as shown in Figure 6. This may improve the overall transient performance of the system. The comparison of the performance index $\mathcal{J}$ is given in Table 2. This comparison verifies the performance improvement of the reset controller compared to the base controller.

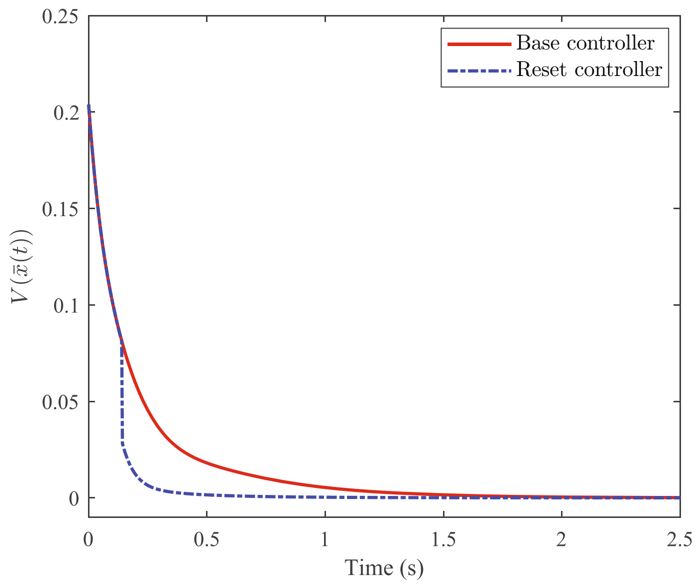

FIGURE 6 The decrement in the Lyapunov function $V(\bar{x}(t))$ value at the reset instant in Example 2.

TABLE 2 Comparison of the performance index $\mathcal{J}$ in Example 2.

	Base controller	Reset controller
$\mathcal{J}$	0.088078	0.080035

5 | CONCLUSION

The reset control design problem for a class of Lipschitz nonlinear systems is investigated in this article. In order to determine the after-reset controller states, Lyapunov theory is used in this article. The power of using Lyapunov function method in reset controller design comes from its generality, which is applicable to linear or nonlinear, finite dimensional or infinite dimensional, time-varying or time-invariant systems. In this study, the controller states are reset to values which lead to a drop in the Lyapunov function value. This may improve the overall transient performance of the closed-loop system. This problem is successfully converted into an offline LMI optimization problem, which can be easily solved by standard numerical software. The obtained results demonstrate the effectiveness and advantages of the proposed control method. In future work, the problem of reset controller design in a full offline procedure for time-delay systems will be investigated.

CONFLICT OF INTEREST STATEMENT

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID

Majid Shahbazzadeh https://orcid.org/0000-0003-0877-7888
S. Jalil Sadati https://orcid.org/0000-0003-3964-7648

REFERENCES

1. Clegg J. A nonlinear integrator for servomechanisms. *Trans Am Inst Electr Eng Part II Appl Ind*. 1958;77(1):41-42.
2. Paz H, Valério D. Generalization of reset controllers to fractional orders. *Mathematics*. 2022;10(24):4630.
3. Guo Y, Chen Y. Stability analysis of delayed reset systems with distributed state resetting. *Nonlinear Anal Hybrid Syst*. 2019;31:265-274.
4. Beker O, Hollot CV, Chait Y. Plant with integrator: an example of reset control overcoming limitations of linear feedback. *IEEE Trans Automat Contr*. 2001;46(11):1797-1799.
5. Nešić D, Zaccarian L, Teel AR. Stability properties of reset systems. *Automatica*. 2008;44(8):2019-2026.

6. Guo Y, Wang Y, Xie L, Zheng J. Stability analysis and design of reset systems: theory and an application. *Automatica*. 2009;45(2):492-497.
7. Saikumar N, Sinha RK, HosseinNia SH. 'Constant in gain lead in phase' element-application in precision motion control. *IEEE/ASME Trans Mechatron*. 2019;24(3):1176-1185.
8. Bakkeheim J, Johansen TA, Smogeli ØN, Sorensen AJ. Lyapunov-based integrator resetting with application to marine thruster control. *IEEE Trans Control Syst Technol*. 2008;16(5):908-917.
9. Saikumar N, Heinen K, HosseinNia SH. Loop-shaping for reset control systems: a higher-order sinusoidal-input describing functions approach. *Control Eng Pract*. 2021;111:104808.
10. Zhao G, Nešić D, Tan Y, Hua C. Overcoming overshoot performance limitations of linear systems with reset control. *Automatica*. 2019;101:27-35.
11. Cai C, Dastjerdi AA, Saikumar N, HosseinNia SH. The optimal sequence for reset controllers. *2020 European Control Conference (ECC)*. IEEE; 2020:1826-1833.
12. Jeong S, Ban J. Robust reset controller design for switched nonlinear uncertain systems. *IEEE Access*. 2022;10:129382-129393.
13. Dastjerdi AA, Astolfi A, HosseinNia SH. A frequency-domain stability method for reset systems. *2020 59th IEEE Conference on Decision and Control (CDC)*. IEEE; 2020:5785-5791.
14. Yazdi S, Khayatian A, Asemani MH, Vafamand N. Designing an optimal reset controller for TS fuzzy systems. *IEEE Trans Syst Man Cybern Syst*. 2021;52(6):4033-4044.
15. Dastjerdi AA, Astolfi A, Saikumar N, Karbasizadeh N, Valerio D, HosseinNia SH. Closed-loop frequency analysis of reset control systems. *IEEE Trans Automat Contr*. 2022;68:1146-1153.
16. Sakthivel R, Sweetha S, Tatar N, Panneerselvam V. Delayed reset control design for uncertain fractional-order systems with actuator faults via dynamic output feedback scheme. *Chaos Solitons Fractals*. 2023;169:113313.
17. Guo Y, Wang Y, Xie L, Li H, Gui W. Optimal reset law design and its application to transient response improvement of HDD systems. *IEEE Trans Control Syst Technol*. 2010;19(5):1160-1167.
18. Ghaffari V, Karimaghaee P, Khayatian A. Reset law design based on robust model predictive strategy for uncertain systems. *J Process Control*. 2014;24(1):261-268.
19. Ghaffari V, Karimaghaee P, Khayatian A. Stability analysis and performance improvement of uncertain linear systems with designing of a suitable reset law. *IET Control Theory Appl*. 2015;9(17):2532-2540.
20. Zhao G, Wang J. Discrete-time triggered reset law design based on model predictive strategy for linear systems. *IET Control Theory Appl*. 2016;10(3):282-291.
21. Zhao G, Wang J. Observer-based reset law design for uncertain systems by using model predictive strategy. *Trans Inst Meas Control*. 2016;38(9):1064-1072.
22. Wang H, Zhu F, Tian Y. Event-triggered optimal reset control of hard disk drive head-positioning servo systems. *Proc Inst Mech Eng Part I J Syst Control Eng*. 2019;233(5):582-590.
23. Pourdehi S, Karimaghaee P. Stability analysis and design of model predictive reset control for nonlinear time-delay systems with application to a two-stage chemical reactor system. *J Process Control*. 2018;71:103-115.
24. Shahbazzadeh M, Sadati SJ, HosseinNia SH. Optimal reset law design based on guaranteed cost control method for Lipschitz nonlinear systems. *Int J Robust Nonlinear Control*. 2022;32(8):4739-4751.
25. Vafamand N, Khayatian A. Model predictive-based reset gain-scheduling dynamic control law for polytopic LPV systems. *ISA Trans*. 2018;81:132-140.
26. Yazdi S, Khayatian A. Performance improvement by optimal reset dynamic output feedback control based on model predictive strategy. *J Process Control*. 2020;88:78-85.
27. Yazdi S, Khayatian A, Asemani MH. Optimal robust model predictive reset control design for performance improvement of uncertain linear system. *ISA Trans*. 2020;107:78-89.
28. Beker O, Hollot C, Chait Y, Han H. Fundamental properties of reset control systems. *Automatica*. 2004;40(6):905-915.
29. Guo Y, Xie L, Wang Y. *Analysis and Design of Reset Control Systems*. Vol 2. IET; 2015.
30. Shahbazzadeh M, Salehifar H, Sadati SJ. Optimal dynamic output feedback control of Lipschitz nonlinear systems under input saturation. *J Vib Control*. 2023;29(1-2):158-168.
31. Scherer C, Gahinet P, Chilali M. Multiobjective output-feedback control via LMI optimization. *IEEE Trans Automat Contr*. 1997;42(7):896-911.
32. Lofberg J. YALMIP: a toolbox for modeling and optimization in MATLAB. *2004 IEEE International Conference on Robotics and Automation*. IEEE; 2004:284-289.
33. MOSEK ApS. *Mosek Optimization Toolbox for Matlab. User's Guide and Reference Manual, Version*. 2019;4:1.
34. Chen CT, Peng ST. A nonlinear control scheme for imprecisely known processes using the sliding mode and neural fuzzy techniques. *J Process Control*. 2004;14(5):501-515.

How to cite this article: Shahbazzadeh M, Sadati SJ, HosseinNia SH. A linear matrix inequality approach to optimal reset control design for a class of nonlinear systems. *Int J Robust Nonlinear Control*. 2024;1-14. doi: 10.1002/rnc.7248