Spherical and Cherednik-Opdam transforms of Jacobi-type polynomials

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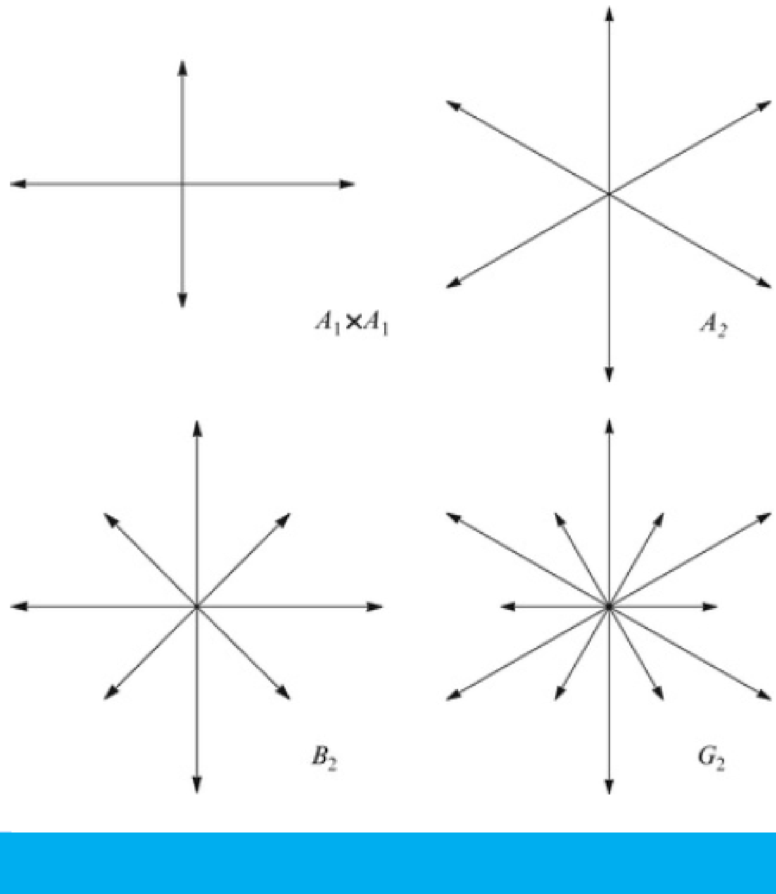

TUDelft

Spherical and Cherednik-Opdam transforms of Jacobi-type polynomials

by

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to obtain the degree of Master of Science
at the Delft University of Technology,
to be defended publicly on Wednesday June 19, 2019 at 10:00 AM.

Student number:	4304624	
Project duration:	September 5, 2018 – June 19, 2019	
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Abstract

The spherical transform maps the orthogonal basis of symmetric Jacobi-type polynomials to an orthogonal basis of (symmetric) Wilson polynomials. The spherical transform is closely related to the Cherednik-Opdam transform, as it is essentially its symmetric version. The symmetric Jacobi-type polynomials can be composed from the non-symmetric Jacobi-type polynomials. These relations, between the symmetric and non-symmetric theory, give an incentive to consider the Cherednik-Opdam transform of non-symmetric Jacobi-type polynomials. This work gives an overview of the symmetric theory about the spherical transform of Jacobi-type polynomials and lays down the groundwork for the Cherednik-Opdam transform of the non-symmetric Jacobi-type polynomials.

Acknowledgments

I would like to thank my daily supervisor Dr. ir. W. G. M. Groenevelt. During our meetings, we had several discussions about the subjects of this thesis. These discussions vastly improved my understanding of the theory. He also provided invaluable support in finishing some of the proofs of this thesis.

Contents

1
Introduction

In the paper by Koornwinder [10] the Fourier-Jacobi transform of one variable Jacobi-type functions is discussed. These Jacobi-type functions and its transform arise naturally in the work of Koornwinder on representation theory (see the article of Koornwinder [9]). The Jacobi-type functions are equal to a Gaussian function times standard Jacobi polynomials and they are a set of functions that form an orthogonal basis on some L^2-space. For example, the standard Jacobi polynomials form an orthogonal basis on the L^2 space of $[-1,1]$ with weight function $(1-x)^\alpha(1+x)^\beta$. Furthermore, the Fourier-Jacobi transforms of the Jacobi-type polynomials form an orthogonal basis on a different L^2-space and are equal to a known set of orthogonal polynomials, called Wilson polynomials.

Zhang [16] considers the spherical transform, a multivariable version of the Fourier-Jacobi transform, of multivariable Jacobi-type polynomials. This spherical transform is associated to the root system of type BC. The multivariable Jacobi-type polynomials form an orthogonal base and they are sent to an orthogonal basis of multivariable Wilson polynomials by the spherical transform. All the mentioned functions are invariant under the Weyl group of type BC. Henceforth, we shall refer to these Jacobi-type functions as symmetric. Moreover, the spherical transform is only defined for the Weyl group invariant functions.

There exists another transform which is the non-symmetric equivalent of the spherical transform. This transform is called the Cherednik-Opdam transform, given in the paper by Opdam [12], and the paper by Cherednik [2](both call it after the other person). The paper by Opdam [12] elaborately discusses the non-symmetric (multivariable) Jacobi polynomials , $E(\lambda, k)$, as well. The symmetric Jacobi polynomials can be constructed from the non-symmetric Jacobi polynomials. This leads to the question if the Cherednik-Opdam transform of non-symmetric Jacobi-type polynomials also maps to a set of orthogonal polynomials.

This thesis starts with a description of the paper of Koornwinder [10] in chapter 2, since it gives a nice introduction to the subject in one variable. The next chapter consists of an overview of the theory about the (non-symmetric) Jacobi and the Cherednik operator. The Cherednik-Opdam transform is also described in this chapter. The method of Zhang for the spherical transform of symmetric Jacobi-type functions is discussed in chapter 4. The fifth chapter gives some context for the Cherednik-Opdam transform in one variable. Furthermore, it provides the approaches to determine the Cherednik-Opdam transform of non-symmetric Jacobi-type functions in one variable and some recommendations.

2

Fourier-Jacobi transform and Jacobi polynomials

This chapter describes the two methods of Koornwinder [10] to determine the Fourier-Jacobi transform of symmetric Jacobi polynomials in one variable. The special case of the spherical transform that is treated here, is used as an introduction in the topic. The first method explains how to calculate the Fourier-Jacobi transform. The other method is less explicit and cleverly uses known properties of the standard Jacobi polynomials, although it still requires some tedious computations. This approach also offers some perspective to generalize to a method for multiple variables.

2.1. Introduction

To give some insight in mappings that send a set of orthogonal polynomial to a set of orthogonal polynomials again (not necessarily the same polynomials), one can consider the example of the Fourier transform of Hermite polynomials. The Hermite polynomials are defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}.$$

These polynomials form an orthogonal basis of the L^2 space on $\mathbb{R}$ with weight function e^{-x^2}. Applying the Fourier transform to $e^{-\frac{x^2}{2}} H_n(x)$ gives $i^n e^{-\frac{x^2}{2}} H_n(x)$. Therefore, the Hermite polynomials are eigenfunctions of the Fourier transform with eigenvalue i^n.

The next sections show that also the Fourier-Jacobi transform maps a particular set of orthogonal polynomials to in this case another set of orthogonal polynomials.

2.2. Definition of Fourier-Jacobi transform

First some important definitions and theorems are discussed, which can be found in [1] and [10].

Definition 2.1. *The gamma function* $\Gamma(x)$ *is defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

for $\operatorname{Re} x > 0$.

A nice property of the gamma function is $\Gamma(z+1) = z\Gamma(z)$. This follows from applying integration by part to the definition. The definition of the gamma function can be easily used to express the analytic continuation of $\Gamma(x)$:

$$\Gamma(x) = \int_0^1 t^{x-1} e^{-t}\, dt + \int_1^\infty t^{x-1} e^{-t}\, dt$$

$$
\begin{aligned}
&= \int_0^1 t^{x-1} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!}\, dt + \int_1^{\infty} t^{x-1} e^{-t}\, dt \\
&= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n t^{x-1+n}}{n!}\, dt + \int_1^{\infty} t^{x-1} e^{-t}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(x+n)n!} + \int_1^{\infty} t^{x-1} e^{-t}\, dt. \qquad (2.1)
\end{aligned}
$$

This last expression represents the poles of the function. The second function in (2.1) is an entire function and the first one shows that the poles are at $n = 0, 1, \ldots$. The beta function is defined next.

Definition 2.2. *The beta integral is defined for* $\mathrm{Re}\, x > 0$, $\mathrm{Re}\, y > 0$ *by*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt.$$

The beta function $B(x, y)$ *is obtained from the integral by analytic continuation.*

In the next definition the Pochhammer symbol is given.

Definition 2.3. *We define the Pochhammer symbol by*

$$(a)_n = a(a+1)\ldots(a+n-1), \quad n = 1, 2, \ldots$$

with $(a)_0 = 1$.

A useful relation between the Pochhammer symbol and gamma functions is the following formula:

$$(a)_n \Gamma(x) = \Gamma(x+n). \qquad (2.2)$$

This formula is easily derived from the fact that $\Gamma(z+1) = z\Gamma(z)$. Another use for the Pochhammer symbol is to express the definition of the hypergeometric series.

Definition 2.4. *The hypergeometric series is defined by*

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}, \quad b_j \notin \mathbb{Z}_{<0}, \text{ for } 1 \le j \le q. \qquad (2.3)$$

For computations with (2.3) it is important to know for which x the series converges. The next theorem gives when the series converges for the different relations of p and q. Note that the ratio test can be applied to the series to prove the theorem.

Theorem 2.5. *The series* ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ *converges absolutely for all* x *if* $p \le q$ *and for* $|x| < 1$ *if* $p = q+1$*, and it diverges for all* $x \ne 0$ *if* $p > q+1$ *and the series does not terminate.*

A special case of the hypergeometric series is the hypergeometric function.

Definition 2.6. *The hypergeometric function* ${}_2F_1$ *is defined by*

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \qquad (2.4)$$

for $|x| < 1$*, and elsewhere by continuation to the complex plane with branch points at* 1 *and* ∞.

Note that if either $a = -k$ or $b = -k$ with $k \in \mathbb{N}$ in the hypergeometric function the power series will be finite, since $(-k)_n = 0$ for $n > k$. That is why the following hypergeometric function is indeed a polynomial.

Definition 2.7. *The Jacobi polynomial of degree* n *is defined by*

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right). \qquad (2.5)$$

The Jacobi polynomials form an orthogonal basis on $L^2([-1,1], (1-x)^{\alpha}(1+x)^{\beta} dx)$.

Theorem 2.8. *The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are a complete set of orthogonal polynomials on the interval $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$.*

Now we can give the definition of the Jacobi function:

Definition 2.9. *The Jacobi function is defined as follows*

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\tfrac{1}{2}(\alpha+\beta+1+i\lambda), \tfrac{1}{2}(\alpha+\beta+1-i\lambda); \alpha+1; -\sinh^2(t)\right). \tag{2.6}$$

To clarify this terminology we take $\lambda = i(2n+\alpha+\beta+1)$ in (2.6)

$$\begin{aligned}\phi_{i(2n+\alpha+\beta+1)}^{(\alpha,\beta)}(i\theta) &= {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \sin^2(\theta)) \\ &= \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(\cos(2\theta)),\end{aligned} \tag{2.7}$$

which is a normalized Jacobi polynomial.

Let

$$\begin{aligned}\Delta_{\alpha,\beta}(t) &:= (2\sinh(t))^{2\alpha+1}(2\cosh(t))^{2\beta+1},\ t>0, \\ c_{\alpha,\beta}(\lambda) &:= \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\alpha+\beta+1))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}.\end{aligned}$$

The Fourier-Jacobi transform $f \mapsto g$ and its inverse are given by

$$\begin{cases} g(\lambda) &= \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt, \\ f(t) &= (2\pi)^{-1}\int_0^\infty g(\lambda)\phi_\lambda^{(\alpha,\beta)}(t)\left|c_{\alpha,\beta}(\lambda)\right|^{-2}\,d\lambda, \end{cases} \tag{2.8}$$

where $\alpha,\beta \in \mathbb{R}$, $|\beta| \le \alpha+1$. (2.8) is valid for $f \in C_c^\infty(\mathbb{R})$ and even, and the transform $f \mapsto g$ can be (uniquely) extended to an isometry of $L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)\,dt)$ to $L^2(\mathbb{R}_+;|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda)$:

$$\int_0^\infty |f(t)|^2\Delta_{\alpha,\beta}(t)\,dt = (2\pi)^{-1}\int_0^\infty |g(\lambda)|^2|c_{\alpha,\beta}|^{-2}\,d\lambda \quad \text{(Plancherel formula)}.$$

We write $\mathscr{F}f$ for the Fourier-Jacobi transform of f.

2.3. Fourier-Jacobi transform of Jacobi polynomials

Wilson polynomials are orthogonal polynomials of ${}_4F_3$-type. They form the most general family of hypergeometric orthogonal polynomials in the sense that other families of orthogonal polynomials can be obtained as limits from the Wilson polynomials. For example, the limit of $t \to \infty$ of a Wilson polynomial with certain variables gives a Jacobi polynomial (see [8]).

Definition 2.10. *Wilson polynomials are given by*

$$\begin{aligned}W_n(x^2;a,b,c,d) &:= (a+b)_n(a+c)_n(a+d)_n \\ &\cdot {}_4F_3\left(\begin{matrix}-n,\ n+a+b+c+d-1,\ a+ix,\ a-ix \\ a+b,\ a+c,\ a+d\end{matrix};1\right),\end{aligned} \tag{2.9}$$

where $n = 0,1,2,\ldots$.

The next theorem shows that the Jacobi polynomials (multiplied by some weight function) are mapped to Wilson polynomials by the Fourier-Jacobi transform. The weight functions that are used in the theorem are given by

$$f_{-2\nu}(t) = (\cosh(t))^{-2\nu}$$

with

$$2\nu = \alpha+\beta+\delta+\mu+2,$$

where $\beta,\delta,\lambda \in \mathbb{R}$, $\alpha,\delta > -1$, $\delta + \operatorname{Re}\mu > -1$.

Theorem 2.11.

$$\int_0^\infty p_n(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt$$
$$=\frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+n)}$$
$$\cdot W_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+\mu+1),\tfrac{1}{2}(\delta-\mu+1),\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha-\beta+1)),$$

where

$$p_n(t)=f_{-2\nu}(t)P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right).$$

In order to prove Theorem 2.11 a few lemmas are given. First the Jacobi polynomial in the theorem is expressed as a finite sum in terms of $\cosh^{-2}(t)$.

Lemma 2.12.

$$P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)=\frac{(-1)^n(\delta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\delta+1\\ \delta+1\end{matrix};\cosh^{-2}(t)\right).$$

Proof. First the transformation of lemma A.7 is applied to the hypergeometric function of the Jacobi polynomial:

$$\begin{aligned}P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)&=\frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\delta+1\\ \alpha+1\end{matrix};\tanh^2(t)\right)\\&=\frac{(\alpha+1)_n}{n!}\frac{(-n-\delta)_n}{(\alpha+1)_n}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\delta+1\\ \delta+1\end{matrix};\cosh^{-2}(t)\right).\end{aligned}$$

Now the term $(\alpha+1)_n$ can be eliminated in the numerator and denominator and we are only left with rewriting $(-n-\delta)_n$:

$$\begin{aligned}(-n-\delta)_n&=(-n-\delta)(-n-\delta+1)\dots(-\delta-1)\\&=(-1)^n(\delta+1)\dots(n+\delta-1)(n+\delta)=(-1)^n(\delta+1)_n,\end{aligned}$$

which concludes the proof. □

Next, the formula of the Jacobi function is rewritten to a hypergeometric function with argument $\tanh^2(t)$ instead of $-\sinh^2(t)$. To make computations with the Jacobi function for $t>0$ the hypergeometric series needs to converge on the interval $(0,\infty)$. The series with argument $\tanh^2(t)$ converges absolutely, since $|\tanh^2(t)|<1$ for $t>0$ and the absolute convergence then follows from theorem 2.5.

Lemma 2.13.

$$\phi_\lambda^{(\alpha,\beta)}(t)=(\cosh(t))^{-(\alpha+\beta+1+i\lambda)}{}_2F_1\left(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha-\beta+1+i\lambda);\alpha+1;\tanh^2(t)\right). \tag{2.10}$$

Proof. Pfaff's transformation (theorem A.4) in combination with rewriting some expressions involving hyperbolic functions gives the desired equality for the Jacobi function:

$$\begin{aligned}\phi_\lambda^{(\alpha,\beta)}(t)&={}_2F_1\left(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha+\beta+1-i\lambda);\alpha+1;-\sinh^2(t)\right)\\&=(1+\sinh^2(t))^{-\frac{1}{2}(\alpha+\beta+1+i\lambda)}\\&\quad\cdot{}_2F_1\left(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha-\beta+1+i\lambda);\alpha+1;\frac{-\sinh^2(t)}{-\sinh^2(t)-1}\right)\\&=(\cosh(t))^{-(\alpha+\beta+1+i\lambda)}{}_2F_1\left(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha-\beta+1+i\lambda);\alpha+1;\tanh^2(t)\right).\end{aligned}$$ □

The last lemma needed for the proof of theorem 2.11 calculates the Fourier-Jacobi transform $g(\lambda)$ in (2.8) with $f(t)=(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2+2j)}$.

Lemma 2.14.

$$\int_0^\infty(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2+2j)}\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt$$
$$=\frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda)+j)\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda)+j)}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+j)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+j)}.$$

Proof. Apply lemma 2.13 and recall that the series in the Jacobi function in that lemma converges absolutely, so according to Fubini's theorem the sum and integral can be interchanged:

$$\begin{aligned}
&\int_0^\infty (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2+2j)}\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt \\
&= \int_0^\infty (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2+2j)}(\cosh(t))^{-(\alpha+\beta+1+i\lambda)} \\
&\quad \cdot \sum_{n=0}^\infty \frac{(\frac{1}{2}(\alpha+\beta+1+i\lambda))_n(\frac{1}{2}(\alpha-\beta+1+i\lambda))_n}{(\alpha+1)_n\ n!}\left(\frac{\sinh^2(t)}{\cosh^2(t)}\right)^n (2\sinh(t))^{2\alpha+1}(2\cosh(t))^{2\beta+1}\,dt \\
&= 2^{2\alpha+2\beta+2}\sum_{n=0}^\infty \int_0^\infty (\sinh^2(t))^{\alpha+1+n}(\cosh^2(t))^{-(\alpha+\frac{1}{2}(\delta+\mu+2+i\lambda)+j+n)} \\
&\quad \cdot \frac{(\frac{1}{2}(\alpha+\beta+1+i\lambda))_n(\frac{1}{2}(\alpha-\beta+1+i\lambda))_n}{(\alpha+1)_n\ n!}\,dt
\end{aligned}$$

The calculation continues by using the substitution $u=\sinh^2(t)$ and applying Theorem A.1 and A.2:

$$\begin{aligned}
&= 2^{2\alpha+2\beta+1}\sum_{n=0}^\infty \int_0^\infty u^{\alpha+n}(u+1)^{-(\alpha+\frac{1}{2}(\delta+\mu+3+i\lambda)+j+n)} \\
&\quad \cdot \frac{(\frac{1}{2}(\alpha+\beta+1+i\lambda))_n(\frac{1}{2}(\alpha-\beta+1+i\lambda))_n}{(\alpha+1)_n\ n!}\,du \\
&= 2^{2\alpha+2\beta+1}\sum_{n=0}^\infty B(\alpha+n+1,\tfrac{1}{2}(\delta+\mu+1+i\lambda)+j)\frac{(\frac{1}{2}(\alpha+\beta+1+i\lambda))_n(\frac{1}{2}(\alpha-\beta+1+i\lambda))_n}{(\alpha+1)_n\ n!} \\
&= 2^{2\alpha+2\beta+1}\sum_{n=0}^\infty \frac{\Gamma(\alpha+n+1)\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda)+j)(\frac{1}{2}(\alpha+\beta+1+i\lambda))_n(\frac{1}{2}(\alpha-\beta+1+i\lambda))_n}{\Gamma(\alpha+\frac{1}{2}(\delta+\mu+3+i\lambda)+j+n)(\alpha+1)_n\ n!}.
\end{aligned}$$

Lastly, the gamma functions with the variable n in them are rewritten according to (2.2) and then Theorem A.5 gives the result:

$$\begin{aligned}
&= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+n+1)\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda)+j)}{\Gamma(\alpha+\frac{1}{2}(\delta+\mu+3+i\lambda)+j)} \\
&\quad \cdot \sum_{n=0}^\infty \frac{(\alpha+1)_n(\frac{1}{2}(\alpha+\beta+1+i\lambda))_n(\frac{1}{2}(\alpha-\beta+1+i\lambda))_n}{(\alpha+\frac{1}{2}(\delta+\mu+3+i\lambda)+j)_n(\alpha+1)_n\ n!} \\
&= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+n+1)\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda)+j)}{\Gamma(\alpha+\frac{1}{2}(\delta+\mu+3+i\lambda)+j)} \\
&\quad \cdot \frac{\Gamma(\alpha+\frac{1}{2}(\delta+\mu+3+i\lambda)+j)\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda)+j))}{\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+j)\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+j)} \\
&= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda)+j)\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda)+j)}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+j)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+j)}.
\end{aligned}$$ □

By combining the three lemmas, we can now give the proof of Theorem 2.11.

Proof (Theorem 2.11). Start with applying Lemma 2.12 and right after that Lemma 2.14. This is followed by using formula 2.2 twice, once for the gamma functions with j and the next time in terms of n to get the Wilson polynomial:

$$\begin{aligned}
&\int_0^\infty (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt \\
&= \frac{(-1)^n(\delta+1)_n}{n!}\sum_{j=0}^\infty \frac{(-n)_j(n+\alpha+\delta+1)_j}{(\delta+1)_j\ j!}\int_0^\infty (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2+2j)}\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^n(\delta+1)_n}{n!}\sum_{j=0}^{\infty}\frac{(-n)_j(n+\alpha+\delta+1)_j}{(\delta+1)_j\; j!} \\
&\cdot \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda)+j)\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda)+j)}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+j)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+j)} \\
&= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))(\delta+1)_n}{n!\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2))\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2))} \\
&\cdot \sum_{j=0}^{\infty}\frac{(-n)_j(n+\alpha+\delta+1)_j(\frac{1}{2}(\delta+\mu+1+i\lambda))_j(\frac{1}{2}(\delta+\mu+1-i\lambda))_j}{(\delta+1)_j(\frac{1}{2}(\alpha+\beta+\delta+\mu+2))_j(\frac{1}{2}(\alpha-\beta+\delta+\mu+2))_j\; j!} \\
&= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+n)} \\
&\cdot (\delta+1)_n(\frac{1}{2}(\alpha+\beta+\delta+\mu+2))_n(\frac{1}{2}(\alpha-\beta+\delta+\mu+2))_n \\
&\cdot {}_4F_3\left(\begin{matrix}-n,\ n+\alpha+\delta+1,\ \frac{1}{2}(\delta+\mu+1+i\lambda),\ \frac{1}{2}(\delta+\mu+1-i\lambda)\\ \delta+1,\ \frac{1}{2}(\alpha+\beta+\delta+\mu+2),\ \frac{1}{2}(\alpha-\beta+\delta+\mu+2)\end{matrix};1\right) \\
&= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+n)} \\
&\cdot W_n(\frac{1}{4}\lambda^2;\frac{1}{2}(\delta+\mu+1),\frac{1}{2}(\delta-\mu+1),\frac{1}{2}(\alpha+\beta+1),\frac{1}{2}(\alpha-\beta+1)).
\end{aligned}
$$
□

From this theorem we derive the following useful corollary.

Corollary 2.15. *The Wilson polynomials W_n with the parameters in Theorem 2.11 are orthogonal polynomials in $L^2(\mathbb{R}_+;\left|c_{\alpha,\beta}(\lambda)\right|^{-2}\,d\lambda)$ and their norm is given by*

$$\|W_n\|_{L^2(c_0|c_{\alpha,\beta}|^{-2})} = \|p_n\|_{L^2(\Delta_{\alpha,\beta})},$$

where $L^2(w)$ stands for the space $L^2(\mathbb{R}_+, w(x)dx)$ and

$$c_0 = \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+\mu+1-i\lambda))}{n!\Gamma(\frac{1}{2}(\alpha+\beta+\delta+\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+\mu+2)+n)}.$$

Proof. The functions $p_n(t)$ for $\mu\in i\mathbb{R}$, $n=0,1,2,\dots$ are a complete orthogonal system in $L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)\,dt)$ and then by the Plancherel formula the orthogonality of the Wilson polynomials in Theorem 2.11 is shown. To see that the functions $p_n(t)$ actually form an orthogonal system, in the inner product of the Jacobi polynomials $P_n^{(\alpha,\delta)}(x)$ and $P_m^{(\alpha,\delta)}(x)$ on $L^2([-1,1],(1-x)^\alpha(1+x)^\delta\,dx)$ x is replaced by $1-2\tanh^2(t)$:

$$
\begin{aligned}
&\int_{-1}^{1}P_n^{(\alpha,\delta)}(x)P_m^{(\alpha,\delta)}(x)(1-x)^\alpha(1+x)^\delta\,dx \\
&= \int_{\infty}^{0}P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)P_m^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right) \\
&\cdot(2\tanh^2(t))^\alpha(2(1-\tanh^2(t)))^\delta\cdot -4\tanh(t)\cosh^{-2}(t)\,dt \\
&= \int_{0}^{\infty}P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)P_m^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)\cdot 2^{\alpha+\delta+2} \\
&\cdot(\sinh(t))^{2\alpha}(\cosh(t))^{-2\alpha}(\cosh(t))^{-2\delta}\sinh(t)\cosh^{-1}(t)\cosh^{-2}(t)(\cosh(t))^{2\beta+1}(\cosh(t))^{-(2\beta+1)}\,dt \\
&= 2^{-\beta+\delta}\int_{0}^{\infty}P_n^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)P_m^{(\alpha,\delta)}\left(1-2\tanh^2(t)\right)\cdot(\cosh(t))^{-2(\alpha+\beta+\delta+2)}\Delta_{\alpha,\beta}(t)\,dt,
\end{aligned}
$$

which is equal to the inner product of the functions $p_n(t)$ and $p_m(t)$ except for some constant. Note that $(\cosh(t))^\mu\overline{(\cosh(t))^\mu}=1$ because $\mu\in i\mathbb{R}$, thus it does not appear in the inner product. The Jacobi polynomials $P_n^{(\alpha,\delta)}(x)$ $n=0,1,2,\dots$ form a complete orthogonal system in $L^2([-1,1],(1-x)^\alpha(1+x)^\delta\,dx)$, therefore so do the functions $p_n(t)$ in $L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)\,dt)$. □

2.4. Representation of the Jacobi function differential operator as a tridiagonal matrix

A well-known fact of orthogonal polynomials is that they satisfy a three term recurrence relation and the converse is also true. Therefore, another way to prove the orthogonality of the Wilson polynomials is by deriving the three term recurrence relation for the polynomials. In order to obtain this recurrence a tridiagonalization of the Jacobi function differential operator $\mathscr{L}_{(\alpha,\beta)}$ is given.

The Jacobi function differential operator $\mathscr{L}_{(\alpha,\beta)}$ is defined by

$$\begin{aligned}\left(\mathscr{L}_{(\alpha,\beta)}f\right)(t) &:= (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}\left(\Delta_{\alpha,\beta}(t)\frac{d}{dt}\left(f(t)\right)\right)\\ &= \frac{d^2}{dt^2}\left(f(t)\right)+\frac{\Delta'_{\alpha,\beta}(t)}{\Delta_{\alpha,\beta}(t)}\frac{d}{dt}\left(f(t)\right),\ t>0.\end{aligned} \tag{2.11}$$

Lemma 2.16. *The Jacobi functions $\phi_\lambda^{(\alpha,\beta)}$ are eigenfunctions of $\mathscr{L}_{(\alpha,\beta)}$ with eigenvalue $-(\lambda^2+(\alpha+\beta+1)^2)$.*

Proof. The hypergeometric differential equation with solution $y={}_2F_1(a,b;c;x)$ can be applied to prove this. The hypergeometric differential equation is given by

$$x(1-x)y''+(c-(a+b+1)x)y'-aby=0.$$

This equation can be written in terms of the Jacobi function, since this is a hypergeometric function with $a=\frac{1}{2}(\alpha+\beta+1+i\lambda)$, $b=\frac{1}{2}(\alpha+\beta+1-i\lambda)$, $c=\alpha+1$ and $x=-\sinh^2(t)$. After some elementary calculations the following expression is found:

$$\begin{aligned}&-\sinh^2(t)\cosh^2(t)\cdot y''(-\sinh^2(t))+(\alpha+1+(\alpha+\beta+2)\sinh^2(t))\cdot y'(-\sinh^2(t))\\ &\quad-\frac{1}{4}(\lambda^2+(\alpha+\beta+1)^2)\cdot y(-\sinh^2(t))=0.\end{aligned} \tag{2.12}$$

Note that $f'(g(t))$ is the derivative of the function f in the "point" $g(t)$. The derivatives $(f(g(t)))'$ and $\frac{d}{dt}(f(g(t)))$ will be used to denote $(g(t))'\cdot f'(g(t))$. Next the first and second derivative of the Jacobi function are given:

$$\frac{d}{dt}(\phi_\lambda^{(\alpha,\beta)}(t))=(y(-\sinh^2(t)))'=-2\sinh(t)\cosh(t)\cdot y'(-\sinh^2(t)), \tag{2.13}$$

$$\begin{aligned}\frac{d^2}{dt^2}(\phi_\lambda^{(\alpha,\beta)}(t))&=(y(-\sinh^2(t)))''\\ &=4\sinh^2(t)\cosh^2(t)\cdot y''(-\sinh^2(t))-2(\cosh^2(t)+\sinh^2(t))\cdot y'(-\sinh^2(t)).\end{aligned} \tag{2.14}$$

Before we apply the Jacobi function differential operator to the Jacobi function, the fractional $\frac{\Delta'_{\alpha,\beta}(t)}{\Delta_{\alpha,\beta}(t)}$ is calculated:

$$\frac{\Delta'_{\alpha,\beta}(t)}{\Delta_{\alpha,\beta}(t)}=(2\alpha+1)(\tanh(t))^{-1}+(2\beta+1)\tanh(t).$$

Now we can apply the differential operator to the Jacobi function:

$$\begin{aligned}\left(\mathscr{L}_{(\alpha,\beta)}\phi_\lambda^{(\alpha,\beta)}\right)(t)&=\frac{d^2}{dt^2}\left(\phi_\lambda^{(\alpha,\beta)}(t)\right)+\left((2\alpha+1)(\tanh(t))^{-1}+(2\beta+1)\tanh(t)\right)\frac{d}{dt}\left(\phi_\lambda^{(\alpha,\beta)}(t)\right)\\ &=4\sinh^2(t)\cosh^2(t)\cdot y''(-\sinh^2(t))-2(\cosh^2(t)+\sinh^2(t))\cdot y'(-\sinh^2(t))\\ &\quad-2(2\alpha+1)\cosh^2(t)\cdot y'(-\sinh^2(t))-2(2\beta+1)\sinh^2(t)\cdot y'(-\sinh^2(t))\\ &=4\sinh^2(t)\cosh^2(t)\cdot y''(-\sinh^2(t))-4(\alpha+1)\cosh^2(t)\cdot y'(-\sinh^2(t))\\ &\quad-4(\beta+1)\sinh^2(t)\cdot y'(-\sinh^2(t)).\end{aligned}$$

Substituting 2.12 in the equation gives:

$$\begin{aligned}
&4(\alpha+1+(\alpha+\beta+2)\sinh^2(t))\cdot y'(-\sinh^2(t)) - (\lambda^2+(\alpha+\beta+1)^2)\cdot y(-\sinh^2(t))\\
&\quad -4(\alpha+1)\cosh^2(t)\cdot y'(-\sinh^2(t)) - 4(\beta+1)\sinh^2(t)\cdot y'(-\sinh^2(t))\\
&= 4(\alpha+1)(\cosh^2(t)-\sinh^2(t))\cdot y'(-\sinh^2(t)) + (\alpha+\beta+2)\cdot y'(-\sinh^2(t))\\
&\quad -(\lambda^2+(\alpha+\beta+1)^2)\cdot y(-\sinh^2(t)) - 4(\alpha+1)\cosh^2(t)\cdot y'(-\sinh^2(t))\\
&\quad -4(\beta+1)\sinh^2(t)\cdot y'(-\sinh^2(t))\\
&= -(\lambda^2+(\alpha+\beta+1)^2)\cdot y(-\sinh^2(t)) = -(\lambda^2+(\alpha+\beta+1)^2)\cdot\phi_\lambda^{(\alpha,\beta)}(t).
\end{aligned}$$

□

$\mathscr{L}_{(\alpha,\beta)}$ is a symmetric operator with respect to the inner product

$$(f,g) = \int_0^\infty f(t)g(t)\Delta_{\alpha,\beta}(t)dt.$$

For $f,g\in C^2(\mathbb{R}_+)\cap L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)\,dt)$ with their first and second derivatives also in $L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)\,dt)$, we have

$$\begin{aligned}
\int_0^\infty \mathscr{L}_{(\alpha,\beta)}\left(f(t)\right)g(t)\Delta_{\alpha,\beta}(t)dt &= \int_0^\infty\left(\frac{d^2}{dt^2}\left(f(t)\right)+\frac{\Delta'_{\alpha,\beta}(t)}{\Delta_{\alpha,\beta}(t)}\frac{d}{dt}\left(f(t)\right)\right)g(t)\Delta_{\alpha,\beta}(t)dt\\
&= \int_0^\infty\frac{d^2}{dt^2}\left(f(t)\right)g(t)\Delta_{\alpha,\beta}(t)dt + \int_0^\infty\frac{d}{dt}\left(f(t)\right)g(t)\Delta'_{\alpha,\beta}(t)dt\\
&= -\int_0^\infty\frac{d}{dt}\left(f(t)\right)\frac{d}{dt}\left(g(t)\Delta_{\alpha,\beta}(t)\right)dt + \int_0^\infty\frac{d}{dt}\left(f(t)\right)g(t)\Delta'_{\alpha,\beta}(t)dt\\
&= -\int_0^\infty\frac{d}{dt}\left(f(t)\right)\frac{d}{dt}\left(g(t)\right)\Delta_{\alpha,\beta}(t)dt.
\end{aligned}$$

The third equality in the above equation is obtained by integration by parts. The boundary terms disappear, since $f'(t),g(t)\in L^2(\mathbb{R}_+;\Delta_{\alpha,\beta}(t)\,dt)$. Interchanging f and g gives the same results, so the operator $\mathscr{L}_{(\alpha,\beta)}$ is symmetric. Recall

$$p_n(t) = f_{-2\nu}(t)P_n\left(1-2\tanh^2(t)\right)$$

and set

$$q_n(t) := \mathscr{F}p_n(t).$$

Apply the Jacobi differential operator to p_n:

$$\begin{aligned}
\left(\mathscr{L}_{(\alpha,\beta)}p_n\right)(t) &= -2\left(\mu+1\right)\tanh(t)p_n'(t)\\
&\quad+\Big(\left(\alpha-\beta+\delta-\mu\right)\left(\alpha+\beta+\delta+\mu+2\right)\tanh^2(t)\\
&\quad-2\left(\alpha+1\right)\left(\alpha+\beta+\delta+\mu+2\right)-4n\left(n+\alpha+\delta+1\right)\left(1-\tanh^2(t)\right)\Big)p_n(t).
\end{aligned}\tag{2.15}$$

The above expression is found by first using the product rule for the second derivative term.

$$\left(\mathscr{L}_{(\alpha,\beta)}p_n\right)(t) = \frac{d^2}{dt^2}\left(p_n(t)\right)+\frac{\Delta'_{\alpha,\beta}(t)}{\Delta_{\alpha,\beta}(t)}\frac{d}{dt}\left(p_n(t)\right)$$

$$= \left((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\right)''P_n\left(1-2\tanh^2(t)\right)\tag{2.16}$$

$$+2\left((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\right)'\left(P_n\left(1-2\tanh^2(t)\right)\right)'\tag{2.17}$$

$$+(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\left(P_n\left(1-2\tanh^2(t)\right)\right)''\tag{2.18}$$

$$+\left((2\alpha+1)\tanh^{-1}(t)+(2\beta+1)\tanh(t)\right)p_n'(t).\tag{2.19}$$

Then in order to show 2.15 the parts 2.16, 2.17 and 2.18 are rewritten to try to eliminate 2.19 and the remaining terms result in the desired expression. The precise computations can be found in the appendix (B.1).

Applying the differential operator to p_n gives a tridiagonal expression, namely:

$$-\mathscr{L}_{(\alpha,\beta)} p_n = A_n p_{n+1} + B_n p_n + C_n p_{n-1}, \tag{2.20}$$

with

$$\begin{aligned} A_n &= \frac{(n+1)(n+\alpha+\delta+1)(2n+\alpha+\beta+\delta+\mu+2)(2n+\alpha-\beta+\delta+\mu+2)}{(2n+\alpha+\delta+1)(2n+\alpha+\delta+2)}, \\ C_n &= \frac{(n+\alpha)(n+\delta)(2n+\alpha+\beta+\delta-\mu)(2n+\alpha-\beta+\delta-\mu)}{(2n+\alpha+\delta)(2n+\alpha+\delta+1)}, \\ B_n &= -\frac{(2n+\alpha-\beta+\delta+\mu+2)(n+1)}{(2n+\alpha+\beta+\delta-\mu+2)(n+\alpha+1)} A_n - \frac{(2n+\alpha-\beta+\delta+\mu)n}{(2n+\alpha+\beta+\delta-\mu)(n+\alpha)} C_n. \end{aligned}$$

This is easily verified using the computer, for example with Maple.

Theorem 2.17. *q_n is a Wilson polynomial of the form*

$$\begin{aligned} q_n(\lambda) := F(\lambda) &\frac{(-1)^n(\alpha+1)_n(\frac{1}{2}(\alpha+\beta+\delta-\mu+2))_n}{n!(\frac{1}{2}(\alpha-\beta+\delta+\mu+2))_n} \\ &\cdot {}_4F_3\left(\begin{matrix} -n,\ \alpha+\delta+1,\ \frac{1}{2}(\alpha+\beta+1+i\lambda),\ \frac{1}{2}(\alpha+\beta+1-i\lambda) \\ \alpha+1,\ \frac{1}{2}(\alpha+\beta+\delta+\mu+2),\ \frac{1}{2}(\alpha+\beta+\delta-\mu+2) \end{matrix};1\right), \end{aligned}$$

where $F(\lambda)$ is equal to the righthandside of 2.14 with $j=0$.

Proof. Note that we have

$$\mathscr{F}(\mathscr{L}_{(\alpha,\beta)} p_n(t)) = \int_0^\infty p_n(t)\mathscr{L}_{(\alpha,\beta)}(\phi_\lambda^{(\alpha,\beta)}(t))\Delta_{\alpha,\beta}(t)\,dt = -(\lambda^2+(\alpha+\beta+1)^2)\mathscr{F}(p_n(t)). \tag{2.21}$$

Applying the Fourier-Jacobi transform on both sides of 2.20 gives a three term recurrence relation for $q_n(\lambda)$, where we use (2.21). We get:

$$(\lambda^2+(\alpha+\beta+1)^2)q_n(\lambda) = A_n q_{n+1}(\lambda) + B_n q_n(\lambda) + C_n q_{n-1}(\lambda).$$

Comparing the A_n, B_n and C_n with the constants of the three term recurrence relation for Wilson polynomials we again find that the q_n are Wilson polynomials as stated in the theorem with initial conditions $q_{-1}=0$ and $q_0 = F(\lambda)$. □

3

Cherednik operator and Jacobi polynomials

The goal of this chapter is to give an overview of the theory related to Cherednik operators and the Jacobi polynomials. Root systems and the associated Weyl group are covered and of course the Cherednik operator and the non-symmetric Jacobi polynomials are discussed extensively. Some information is given about the Cherednik-Opdam transform. Finally, the relation between the non-symmetric and symmetric Jacobi polynomials is given.

The next two sections are mainly based on the book of Hall [5] and the book of Humphreys [7].

3.1. Root systems

Let $\alpha \in \mathbb{R}^n$. The reflection in hyperplane P_α perpendicular to α for $\beta \in \mathbb{R}^n$ is given by

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha,$$

where $(\cdot,\cdot)$ is an inner product on $\mathbb{C}^n$: $(x,y) = \sum_{i=1}^n x_i \overline{y_i}$ for $x, y \in \mathbb{C}^n$. For now it would be enough to give the inner product only on $\mathbb{R}^n$, but at some point we need the inner product to be defined on $\mathbb{C}^n$ as well. From now on $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is denoted by $\langle \beta, \alpha \rangle$. An example in $\mathbb{R}^2$ of a reflection σ_α is given in Figure 3.1. Now we can define a root system.

Definition 3.1. *A subset R of the euclidean space $\mathbb{R}^n$ is called a root system in $\mathbb{R}^n$ if the following axioms are satisfied:*

1. *R is finite, spans $\mathbb{R}^n$, and does not contain 0.*
2. *If $\alpha \in R$, the reflection σ_α leaves R invariant.*
3. *If $\alpha, \beta \in R$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.*

If $\alpha \in R$ for some root system R, the vector $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ is called the coroot of α. The set of coroots $\{\alpha^\vee | \alpha \in R\}$ also forms a root system. Some more examples of root systems in $\mathbb{R}^2$ are given in Figure 3.2. For $\gamma \in \mathbb{R}^n$, we define $R_+ \subset R$ to be the set of roots $\alpha \in R$ such that $(\alpha, \gamma) > 0$. R_+ is called the set of positive roots.

The definitions for the weight and root lattice can be found in [11]: The weight lattice P of the root system R is defined by

$$P := \{\lambda \in \mathbb{R}^n \mid \langle \lambda, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in R\}.$$

We define the root lattice Q of the root system R as follows:

$$Q := \left\{ \sum_{\alpha \in R} c_\alpha \alpha | c_\alpha \in \mathbb{Z} \right\}.$$

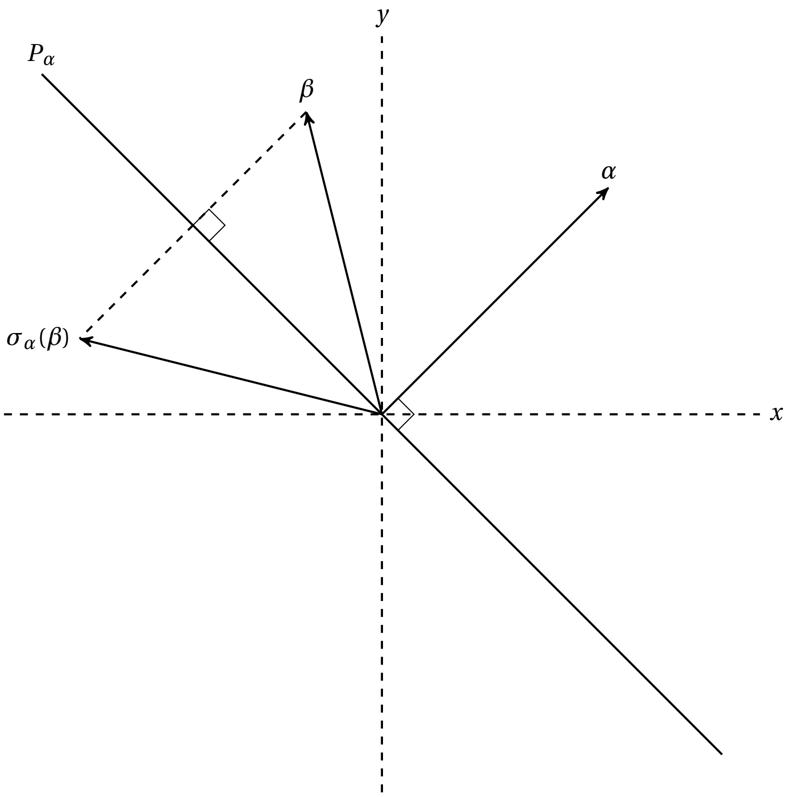

Figure 3.1: An example of the reflection σ_α of a vector β in $\mathbb{R}^2$.

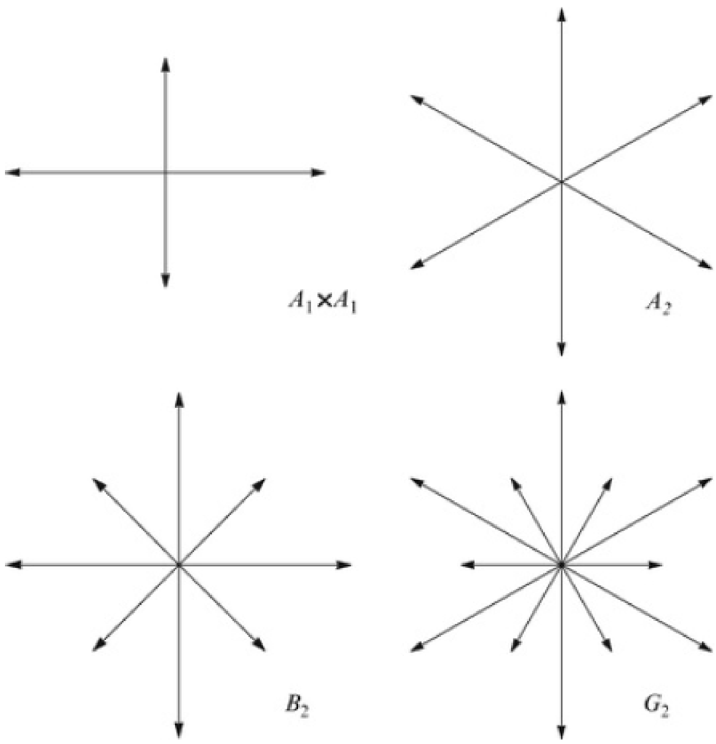

Figure 3.2: Root systems in $\mathbb{R}^2$ [5].

P_+ denotes the set P, where $\mathbb{Z}$ is replaced by $\mathbb{Z}_+$ and R by R_+:

$$P_+ := \{\lambda \in \mathbb{R}^n \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}_+ \ \forall \alpha \in R_+\}.$$

We define Q_+ in a similar way:

$$Q_+ := \left\{ \sum_{\alpha \in R_+} c_\alpha \alpha | c_\alpha \in \mathbb{Z}_+ \right\}.$$

Each root system has a base and the roots of a base are called simple roots.

Definition 3.2. *A subset Δ of R is called a base if:*

1. *Δ is a basis of $\mathbb{R}^n$,*
2. *each root β can be written as $\beta = \sum k_\alpha \alpha$ ($\alpha \in \Delta$), with $k_\alpha \in \mathbb{Z}$ all non-positive or all non-negative.*

3.2. Weyl group

Let R be a root system in $\mathbb{R}^n$. The group generated by the reflections σ_α ($\alpha \in R$) is called the Weyl group of R. The next lemma gives a nice property for elements of $GL(\mathbb{R}^n)$, the set of invertible linear transformations of $\mathbb{R}^n$. $W \subset GL(\mathbb{R}^n)$, so in particular this property holds for Weyl group elements.

Lemma 3.3. *Let R be a root system in $\mathbb{R}^n$ with Weyl group W. If $\sigma \in GL(\mathbb{R}^n)$ leaves R invariant, then $\sigma \sigma_\alpha \sigma^{-1} = \sigma_{\sigma(\alpha)}$ $\forall \alpha \in R$.*

$\mathbb{R}^n$ can be divided in finitely many regions by the hyperplanes P_α ($\alpha \in R$). The (open) Weyl chambers are the connected components of $\mathbb{R}^n \setminus \bigcup_{\alpha \in R} P_\alpha$. Let Δ be a base of R. A vector $\lambda \in P$ is dominant if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta$ and λ is strictly dominant if $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta$. For each $\lambda \in P$ there exists a unique dominant weight, λ^*, in the Weyl orbit of λ. This is a consequence of the following theorem.

Theorem 3.4. *Let C be a Weyl chamber and $\lambda \in \mathbb{R}^n$. Then there exists exactly one point in the Weyl orbit of λ that lies in the closure $\overline{C}$ of C.*

We define a partial ordering $\preceq$ on the weight lattice P as follows (from [14]):

Definition 3.5. *Let $\lambda, \mu \in P$.*

1. *We write $\lambda \leq \mu$ if $\mu - \lambda \in Q_+$, and $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$.*
2. *We write $\lambda \preceq \mu$ if $\lambda^* < \mu^*$, or if $\lambda^* = \mu^*$ and $\lambda \geq \mu$, and $\lambda \prec \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$.*

An example is given with two elements in $\mathbb{R}^2$ which shows that the ordering is indeed a partial ordering. Take $\lambda = (6,6)$ and $\mu = (8,2)$. Note that those elements are already dominant. Look at $\mu - \lambda = (2,-4) \notin Q_+$, since an element is only in Q_+ if we can construct it by a sum over positive multiples of positive roots. But also $\lambda - \mu = (-2,4) \notin Q_+$. Hence we can not compare λ and μ.

The reflections associated to the simple roots of R generate the Weyl group.

Theorem 3.6. *If Δ is a base, then W is generated by the reflections σ_α with $\alpha \in \Delta$.*

Throughout this report, one specific root system is used. Take the following root system R of type BC_n in $\mathbb{R}^n$: $R = \{\pm 2\varepsilon_i, \pm 4\varepsilon_i, \pm 2(\varepsilon_j \pm \varepsilon_k)\}$ for $1 \leq i \leq n$ and $1 \leq j < k \leq n$. We choose the set of positive roots $R_+ = \{2\varepsilon_i, 4\varepsilon_i, 2(\varepsilon_j \pm \varepsilon_k)\}$ for $1 \leq i \leq n$ and $1 \leq j < k \leq n$ and the base $\Delta = \{2(\varepsilon_i - \varepsilon_{i+1}), 2\varepsilon_n\}$ for $1 \leq i \leq n-1$. Set $\alpha_i := 2(\varepsilon_i - \varepsilon_{i+1})$ for $1 \leq i \leq n-1$ and $\alpha_n := 2\varepsilon_n$. According to theorem 3.6 the Weyl group is generated by the elements σ_{α_i} ($1 \leq i \leq n$), which from now on will also be denoted by s_i for $1 \leq i \leq n$.

The s_i's satisfy certain quadratic relations and braid relations:

$$\begin{aligned} s_i^2 &= 1 && \text{for } 1 \leq i \leq n, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \leq i \leq n-2, \\ s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i && \text{for } i = n-1, \\ s_i s_j &= s_j s_i && \text{for } |i-j| > 1. \end{aligned} \tag{3.1}$$

We proof that $s_i^2 = 1$ by working from the definition:

$$\sigma_\alpha\left(\sigma_\alpha(\beta)\right) = \sigma_\alpha\left(\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha\right)$$
$$= \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha - 2\cdot -2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\frac{(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = \beta,$$

where with the second equality linearity of the inner product is used. The other properties of s_i can be shown in similar ways.

With the s_i's we can define a length function l for all $w \in W$ by setting $l(id) = 0$ and for $w = s_{i_1}\cdots s_{i_n}$ the representation that uses the least amount of s_i's, $l(w) = n$. For $\lambda \in P_+$ we define two sets: $W_\lambda := \{w | w\lambda = \lambda\}$ and $W^\lambda := \{w | l(ww') \geq l(w)\ \forall w' \in W_\lambda\}$. Intuitively W^λ is exactly all the smallest length elements in the Weyl group that send λ to a unique element in its Weyl orbit. We can define a subset of R containing all the roots that are orthogonal to λ by $R_\lambda := \{\alpha \in R | (\lambda,\alpha) = 0\}$. The set $R_{\lambda,+}$ is R_λ with R replaced by R_+.

3.3. The Cherednik operator and non-symmetric Jacobi polynomials

This section gives the definition of the Cherednik operator and the non-symmetric Jacobi polynomials and it lists a number of properties of that operator and those functions.

We have seen that the Weyl group elements are only defined to act on $\mathbb{R}^n$, which we now expand to functions f on $\mathbb{R}^n$ by $wf(x) = f(w^{-1}x)$ for all $w \in W$. Note that the relations 3.1 are still satisfied for s_i working on functions.

Let k be a multiplicity function, i.e. a W-invariant complex function, on a root system R. We define

$$\rho_k = \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \alpha.$$

The Cherednik operator is defined as follows.

Definition 3.7. *Let $R_+ \subset R$ be a choice of positive roots, k an arbitrary multiplicity function and let $\xi \in \mathbb{R}^n$. The Cherednik operator $D_\xi = D_\xi(R_+, k)$ is the differential difference operator on $\mathbb{R}^n$ defined by*

$$D_\xi = \partial_\xi + \sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha) - (\rho_k,\xi),$$

where for all $\lambda \in \mathbb{R}^n$, e^λ is a function on $\mathbb{R}^n$ defined by $e^\lambda(x) = e^{(\lambda,x)}$ for all $x \in \mathbb{C}^n$.

The Cherednik operator will act on functions in $\mathbb{C}[P]$ as well as functions on $L^2(\mathbb{R}^n, \tau(t)dt)$, where

$$\tau(t) = \prod_{\alpha\in R_+} |2\sinh(\frac{1}{2}(\alpha,t))|^{2k_\alpha}.$$

With $\mathbb{C}[P]$ we denote the space of Laurent polynomials on the compact torus $\mathbb{T}^n = i\mathbb{R}^n/(\pi i\mathbb{Z})^n$, which exists of finite linear combinations of functions e^λ with $\lambda \in P$. Therefore $\partial_\xi e^\lambda$ and we^λ for $w \in W$, $\lambda \in P$ need to be defined: $\partial_\xi e^\lambda = (\lambda,\xi)e^\lambda$ and $we^\lambda = e^{w(\lambda)}$. The second definition is consistent with $wf(x) = f(w^{-1}x)$ if we^λ is applied to $x \in \mathbb{R}^n$. This chapter is mainly focused on the Cherednik operator on $\mathbb{C}[P]$. The next lemma gives a relation between Weyl group elements and the Cherednik operator.

Lemma 3.8. *For all $w \in W$ and $\xi \in \mathbb{R}^n$, we have*

$$w \circ D_\xi \circ w^{-1} = D_{w\xi} + \sum_{\alpha\in R_+\cap wR_-} k_\alpha(\alpha, w\xi)\sigma_\alpha.$$

Proof. We show the lemma for functions on $\mathbb{C}[P]$. Consider the operator

$$S_\xi = \partial_\xi + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}(1-\sigma_\alpha)$$

To see how S_ξ relates to D_ξ the second part of S_ξ is rewritten:

$$\frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}(1-\sigma_\alpha) = \sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha) - \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1-e^{-\alpha}}{1-e^{-\alpha}}(1-\sigma_\alpha)$$

$$= \sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha) - \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi) + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\sigma_\alpha,$$

so

$$S_\xi = D_\xi + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\sigma_\alpha.$$

Note that S_ξ is invariant with respect to the choice of R_+, since

$$k_\alpha(\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}} = k_{-\alpha}\cdot -(-\alpha,\xi)\frac{e^{-\alpha}(e^\alpha+1)}{e^{-\alpha}(e^\alpha-1)} = k_{-\alpha}(-\alpha,\xi)\frac{1+e^\alpha}{1-e^\alpha}$$

and for each $\alpha\in R$ either $\alpha\in R_+$ or $-\alpha\in R_+$. Remark also that

$$w\circ e^\alpha\circ w^{-1}\left(e^\lambda\right) = w\circ e^\alpha\left(e^{w^{-1}\lambda}\right) = w\left(e^\alpha e^{w^{-1}\lambda}\right) = e^{w\alpha}e^\lambda.$$

Also, we have

$$w\circ\partial_\xi\circ w^{-1}\left(e^\lambda\right) = w\circ\partial_\xi\left(e^{w^{-1}\lambda}\right) = w\left((w^{-1}\lambda,\xi)e^{w^{-1}\lambda}\right) = (\lambda, w\xi)e^\lambda = \partial_{w\xi}\left(e^\lambda\right),$$

and

$$w\circ\frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}(1-\sigma_\alpha)\circ w^{-1} = \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1+e^{-w\alpha}}{1-e^{-w\alpha}}(1-\sigma_{w\alpha}),$$

since $w\circ\sigma_\alpha\circ w^{-1} = \sigma_{w(\alpha)}$ by Lemma 3.3, and continuing the calculation, using that $k_{w\alpha} = k_\alpha$:

$$\begin{aligned} &= \frac{1}{2}\sum_{w^{-1}\alpha\in R_+} k_\alpha(w^{-1}\alpha,\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}(1-\sigma_\alpha) \\ &= \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,w\xi)\frac{1+e^{-\alpha}}{1-e^{-\alpha}}(1-\sigma_\alpha), \end{aligned}$$

where the last equality is a consequence of the fact that $S_{w\xi}$ is invariant with respect to the choice of R_+. Now it is clear that

$$w\circ S_\xi\circ w^{-1} = S_{w\xi}.$$

$D_{w\xi}$ is expressed in $S_{w\xi}$

$$D_{w\xi} = S_{w\xi} - \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,w\xi)\sigma_\alpha. \tag{3.2}$$

Now we have

$$\begin{aligned} w\circ D_\xi\circ w^{-1} &= S_{w\xi} - \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\sigma_{w\alpha} \\ &= S_{w\xi} - \frac{1}{2}\sum_{w^{-1}\alpha\in R_+} k_\alpha(\alpha,w\xi)\sigma_\alpha. \end{aligned} \tag{3.3}$$

We want to write the summation back to $\alpha\in R_+$ again:

$$\frac{1}{2}\sum_{w^{-1}\alpha\in R_+} k_\alpha(\alpha,w\xi)\sigma_\alpha = \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,w\xi)\sigma_\alpha - \sum_{\alpha\in R_+\cap wR_-} k_\alpha(\alpha,w\xi)\sigma_\alpha, \tag{3.4}$$

where the last summation compensates for the positive roots that are not part of $w\alpha,\ \alpha\in R_+$ (those roots are given by $w(-\alpha)$ if $w\alpha\in R_-$). Combining (3.3) and (3.4) with (3.2) proves the lemma.

$$\begin{aligned} w\circ D_\xi\circ w^{-1} &= S_{w\xi} - \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha(\alpha,w\xi)\sigma_\alpha + \sum_{\alpha\in R_+\cap wR_-} k_\alpha(\alpha,w\xi)\sigma_\alpha \\ &= D_{w\xi} + \sum_{\alpha\in R_+\cap wR_-} k_\alpha(\alpha,w\xi)\sigma_\alpha. \end{aligned}$$

□

The following result is important for showing that D_ξ has certain polynomials as eigenfunctions.

Lemma 3.9. *The operator* $\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$ *on* $\mathbb{C}[P]$ *gives a mapping from* $\mathbb{C}[P]$ *into itself. Specifically,*

$$\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)\left(e^\lambda\right) = \begin{cases} e^\lambda \sum_{n=0}^{k-1} e^{-n\alpha} & \text{if } \langle\lambda,\alpha\rangle > 0 \\ 0 & \text{if } \langle\lambda,\alpha\rangle = 0 \\ -e^{\sigma_\alpha\lambda}\sum_{n=0}^{-k-1} e^{-n\alpha} & \text{if } \langle\lambda,\alpha\rangle < 0 \end{cases},$$

where $k = \langle\lambda,\alpha\rangle$.

Proof. Apply $\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$ to e^λ $(\lambda \in P)$. We find

$$\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)\left(e^\lambda\right) = \frac{e^\lambda - e^{\sigma_\alpha\lambda}}{1-e^{-\alpha}} = \frac{e^\lambda\left(1-e^{-(1-\sigma_\alpha)\lambda}\right)}{1-e^{-\alpha}}. \tag{3.5}$$

If $(1-\sigma_\alpha)\lambda$ is a positive multiple of α, the last expression is equal to the sum of a geometric series.

$$(1-\sigma_\alpha)\lambda = \lambda - (\lambda - \langle\lambda,\alpha\rangle\alpha) = \langle\lambda,\alpha\rangle\alpha.$$

We know that $\langle\lambda,\alpha\rangle \in \mathbb{Z}$ and if it is also a positive number, it is possible to rewrite (3.5) in terms of a geometric series.

$$\frac{e^\lambda\left(1-e^{-(1-\sigma_\alpha)\lambda}\right)}{1-e^{-\alpha}} = e^\lambda \sum_{n=0}^{k-1} e^{-n\alpha},$$

where $k = \langle\lambda,\alpha\rangle$. There are two other cases to consider, namely $\langle\lambda,\alpha\rangle = 0$ and $\langle\lambda,\alpha\rangle < 0$. For the first case it is easily seen that e^λ is mapped to 0. The second case is obtained by noting that we can write

$$\frac{e^\lambda - e^{\sigma_\alpha\lambda}}{1-e^{-\alpha}} = \frac{-e^{\sigma_\alpha\lambda}\left(1-e^{-(\sigma_\alpha-1)\lambda}\right)}{1-e^{-\alpha}}$$

and then apply the same trick as before

$$\frac{-e^{\sigma_\alpha\lambda}\left(1-e^{-(\sigma_\alpha-1)\lambda}\right)}{1-e^{-\alpha}} = -e^{\sigma_\alpha\lambda}\sum_{n=0}^{-k-1} e^{-n\alpha}.$$

□

The resulting expression for the elements in the image of the map in the proposition above can be ordered, which is shown by the next two lemmas ([14]).

Lemma 3.10. *For all* $\mu \in P$ *we have* $\mu \le \mu^*$.

Proof. If $\mu = \mu^*$ the statement is clearly true. Note that $\mu^* \in P_+$. So let $\mu \in P\backslash P_+$. We show that $\mu < \mu^*$. There exists a root $\alpha \in R_+$ such that $\langle\mu,\alpha\rangle < 0$, since $\mu \notin P_+$. Thus we have

$$\sigma_\alpha\mu = \mu - \langle\mu,\alpha\rangle\alpha > \mu,$$

because $-\langle\mu,\alpha\rangle\alpha \in Q_+$. If $\sigma_\alpha\mu = \mu^*$, then we are done. Otherwise, we can repeat the argument from before to see that there exists $\beta \in R_+$ such that $\sigma_\beta(\sigma_\alpha\mu) > \sigma_\alpha\mu > \mu$. Note that there are finitely many Weyl chambers and μ^* is unique, so continuing the argument eventually gives $\mu < \mu^*$. □

Lemma 3.11. *Let* $\mu \in P$ *and* $\alpha \in R_+$. *If* $\langle\mu,\alpha\rangle \ge 2$, *then* $\mu - r\alpha \prec \mu$ *for* $r = 1, \ldots, \langle\mu,\alpha\rangle - 1$.

Proof. Suppose $\langle\mu,\alpha\rangle \geq 2$ and $\mu_r = \mu - r\alpha$ for $r \in \{1,\dots,\langle\mu,\alpha\rangle - 1\}$. We need to show that $\mu_r^* < \mu^*$. There exists an element $w \in W$ such that $w\mu_r = \mu_r^*$. w leaves the root system R invariant, so α is send to a positive or a negative root. Therefore $w\alpha \in Q_+$ or $-w\alpha \in Q_+$. We first show the case $w\alpha \in Q_+$. Then $\mu_r^* = w\mu - rw\alpha < w\mu \leq \mu^*$, where the last inequality follows from the previous lemma. Next the case $-w\alpha \in Q_+$. There we find $\mu_r^* = w\mu - rw\alpha < w\mu - \langle\mu,\alpha\rangle w\alpha = (w\sigma_\alpha)\mu \leq \mu^*$. □

Next we prove two useful properties of the Cherednik operator on $\mathbb{C}[P]$.

Definition 3.12. *Let δ_k be the weight function*

$$\delta_k = \prod_{\alpha\in R_+} \left| e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \right|^{2k_\alpha}.$$

For $f,g \in \mathbb{C}[P]$ we define the following inner product

$$(f,g)_k = \int_{\mathbb{T}^n} f\overline{g}\delta_k dt,$$

where dt is the Haar measure on $\mathbb{T}^n$ that is normalized by $\int_{\mathbb{T}} dt = 1$ and $\overline{g} = \sum_{\lambda\in P} \overline{c_{-\lambda}} e^\lambda$ for $g = \sum_{\lambda\in P} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{C}$ and $\overline{c_{-\lambda}}$ the complex conjugate of $c_{-\lambda}$. $\overline{g}$ is referred to as the conjugate of g.

Note that we can also represent this inner product as

$$(f,g)_k = \int_{[0,\pi]^n} f(u)\overline{g(u)}|\delta_k(u)|du, \tag{3.6}$$

where

$$\delta_k(u) = \prod_{\alpha\in R_+} \left| e^{\frac{(\alpha,iu)}{2}} - e^{\frac{(\alpha,-iu)}{2}} \right|^{2k_\alpha} = \prod_{\alpha\in R_+} \left| 2\sin\left(\frac{(\alpha,u)}{2}\right)\right|^{2k_\alpha}.$$

Before stating the next proposition, we give some remarks. δ_k is Weyl group invariant. $(w\alpha,\beta) = (\alpha, w^{-1}\beta)$, because $(s_i\alpha,\beta) = (\alpha, s_i\beta)$. This is checked easily from the definition and w can be written as a combination of s_i's.

Proposition 3.13. *The operator D_ξ is symmetric with respect to the inner product $(\cdot,\cdot)_k$ for $\xi \in \mathbb{R}^n$.*

Proof. The proof is based on the proof of Heckman [6]. Split up D_ξ in three different parts: ∂_ξ, $\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$ and (ρ_k,ξ). And determine their adjoint operators separately. Start with ∂_ξ and apply integration by parts:

$$\begin{aligned}(\partial_\xi f,g)_k &= \int_{\mathbb{T}^n} \partial_\xi(f)\overline{g}\delta_k dt \\ &= -\int_{\mathbb{T}^n} f\partial_\xi(\overline{g})\delta_k dt - \int_{\mathbb{T}^n} f\overline{g}\partial_\xi(\delta_k)dt \\ &= \int_{\mathbb{T}^n} f\overline{\partial_\xi(g)}\delta_k dt - \int_{\mathbb{T}^n} f\overline{g\left(\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{e^{-\frac{\alpha}{2}}+e^{\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}}-e^{\frac{\alpha}{2}}}\right)}\delta_k dt.\end{aligned} \tag{3.7}$$

The last step requires some explanation. The first integral is because of $\overline{\partial_\xi(g(u))} = -\partial_\xi(\overline{g(u)})$. To see this determine the derivatives of $g = \sum_{\lambda\in P} c_\lambda e^\lambda$ and $\overline{g} = \sum_{\lambda\in P} \overline{c_{-\lambda}} e^\lambda$: $\partial_\xi g = \sum_{\lambda\in P}(\lambda,\xi)c_\lambda e^\lambda$ and $\partial_\xi\overline{g} = \sum_{\lambda\in P}(\lambda,\xi)\overline{c_{-\lambda}}e^\lambda$. Moreover, $\overline{\partial_\xi g} = \sum_{\lambda\in P}\overline{(-\lambda,\xi)c_{-\lambda}}e^\lambda = -\sum_{\lambda\in P}(\lambda,\xi)\overline{c_{-\lambda}}e^\lambda$, since (λ,ξ) is real-valued. The second integral follows by writing out the derivative of δ_k and taking the conjugate. It is easier to work with a different expression for δ_k:

$$\begin{aligned}\delta_k &= \prod_{\alpha\in R_+}\left|e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right|^{2k_\alpha} = \prod_{\alpha\in R_+}\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^{k_\alpha}\overline{\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)}^{k_\alpha} \\ &= \prod_{\alpha\in R_+}\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^{k_\alpha}\left(e^{-\frac{\alpha}{2}} - e^{\frac{\alpha}{2}}\right)^{k_\alpha} = \prod_{\alpha\in R}\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^{k_\alpha}.\end{aligned}$$

The derivative of $\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^{k_\alpha}$ is

$$\partial_\xi\left(\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^{k_\alpha}\right) = k_\alpha(\frac{\alpha}{2},\xi)\left(e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}\right)\left(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}\right)^{k_\alpha-1}$$

$$= k_\alpha(\frac{\alpha}{2},\xi)\frac{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\left(e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}\right)^{k_\alpha}$$

The product rule then gives:

$$\partial_\xi(\delta_k) = \sum_{\alpha\in R} k_\alpha(\frac{\alpha}{2},\xi)\frac{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\delta_k = \sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{e^{\frac{\alpha}{2}}+e^{-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\delta_k.$$

The conjugate of the sum term is indeed equal to the one in (3.7).

Next consider $\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$. To put this operator to the other side of the inner product $\frac{1}{1-e^{-\alpha}}$ and $(1-\sigma_\alpha)$ get turned around and also the conjugate must be applied. So the operator becomes (note that the conjugate only influences $\frac{1}{1-e^{-\alpha}}$):

$$\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)(1-\sigma_\alpha)\overline{\left(\frac{1}{1-e^{-\alpha}}\right)} = \sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\left(\frac{1}{1-e^{\alpha}}-\sigma_\alpha\frac{1}{1-e^{\alpha}}\right)$$
$$= \sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\left(\frac{1}{1-e^{\alpha}}-\frac{1}{1-e^{-\alpha}}\sigma_\alpha\right).$$

The σ_α part of this equation is already correct; the other part needs to be combined with the second part of (3.7).

$$\frac{1}{1-e^{\alpha}}-\frac{e^{-\frac{\alpha}{2}}+e^{\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}}-e^{\frac{\alpha}{2}}} = \frac{e^{-\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}}-e^{\frac{\alpha}{2}}}-\frac{e^{-\frac{\alpha}{2}}+e^{\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}}-e^{\frac{\alpha}{2}}} = -\frac{e^{\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}}-e^{\frac{\alpha}{2}}} = \frac{1}{1-e^{-\alpha}},$$

so we find $\partial_\xi+\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$ as adjoint for $\partial_\xi+\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$.

(ρ_k,ξ) is a real-valued constant, thus stays the same if you put it at the other side of the inner product. Therefore we conclude that D_ξ is symmetric. □

Proposition 3.14. *The operator D_ξ is upper triangular with respect to the ordering $\preceq$, i.e. $D_\xi(e^\lambda) = \sum_{\mu\preceq\lambda} a_{\lambda,\mu}e^\mu$, for certain coefficients $a_{\lambda,\mu}$.*

Proof. Firstly, $\partial_\xi(e^\lambda) = (\lambda,\xi)e^\lambda$. Secondly, for the term $\sum_{\alpha\in R_+} k_\alpha(\alpha,\xi)\frac{1}{1-e^{-\alpha}}(1-\sigma_\alpha)$ we use Proposition 3.9. We see by lemma 3.11 that e^λ and $e^{\sigma_\alpha\lambda}$ are the highest order terms in the cases $\langle\lambda,\alpha\rangle > 0$ and $\langle\lambda,\alpha\rangle < 0$ respectively. Also, if $\langle\lambda,\alpha\rangle < 0$, then $\sigma_\alpha\lambda \preceq \lambda$: the dominant weights of $\sigma_\alpha\lambda$ and λ are clearly equal and $-\langle\lambda,\alpha\rangle\alpha \in Q_+$, so $\sigma_\alpha\lambda = \lambda-\langle\lambda,\alpha\rangle\alpha > \lambda$. Finally, $\rho(k)(\xi)$ is just a constant for a given ξ. □

The fact that D_ξ is a triangular operator gives that there are eigenfunctions of degree λ in $C[P]$ with eigenvalue $a_{\lambda,\lambda}$ from proposition 3.14.

Definition 3.15. *The non-symmetric Jacobi polynomials $E(\lambda,k)$ are defined as the unique eigenfunctions of D_ξ that are of the form:*

$$E(\lambda,k) = e^\lambda + \sum_{\mu\prec\lambda} c_{\lambda,\mu}e^\mu. \tag{3.8}$$

It is useful to have an explicit expression for the eigenvalues of the non-symmetric Jacobi polynomials.

Proposition 3.16. *Let $\lambda\in P$. The eigenvalue of $E(\lambda,k)$ is given by*

$$\gamma_\lambda = \lambda + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha\epsilon(\langle\lambda,\alpha\rangle)\alpha,$$

with

$$\epsilon(x) = \begin{cases} 1 & \textit{for } x > 0 \\ -1 & \textit{for } x \le 0 \end{cases}.$$

Proof. Apply D_ξ to e^λ and determine which parts of the operator give e^λ terms. Clearly

$$\left(\partial_\xi-(\rho_k,\xi)\right)(e^\lambda) = (\lambda,\xi)e^\lambda-(\rho_k,\xi)e^\lambda$$

$$= (\lambda, \xi)e^{\lambda} - \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha (\alpha, \xi) e^{\lambda}.$$

The remaining part of D_ξ only gives a nonzero coefficient for e^λ if $\langle \lambda, \alpha \rangle > 0$ (see 3.14). In that case the coefficient is equal to $k_\alpha(a, \xi)$. So we find

$$D_\xi(e^\lambda) = \left((\lambda, \xi) + \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \lambda, \alpha \rangle)(\alpha, \xi) \right) e^\lambda + l.o.t.$$

$(\lambda, \xi) + \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \lambda, \alpha \rangle)(\alpha, \xi) = (\gamma_\lambda, \xi)$ and since ξ is an arbitrary vector, γ_λ is indeed the eigenvalue of $E(\lambda, k)$. □

Proposition 3.17. *If $\lambda, \mu \in P$ and $\lambda \neq \mu$, then*

$$\gamma_\lambda \neq \gamma_\mu,$$

if $k_\alpha \geq 0$.

Proof. Suppose $\lambda \in P_+$. We prove that $\gamma_\lambda = w_\lambda(\lambda + \rho_k)$ with w_λ the longest of W_λ. We can describe the action of w_0 on the elements of R_+ as follows: $w_\lambda(R_{\lambda,+}) = -R_{\lambda,+}$ and $w_\lambda(R_+ \backslash R_{\lambda,+}) = R_+ \backslash R_{\lambda,+}$. Use that for applying w_λ to ρ_k:

$$w_\lambda \rho_k = \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha w_\lambda \alpha = \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \lambda, \alpha \rangle))\alpha.$$

We can also show that the jth component of ρ_k is equal to:

$$\rho_{k,j} = (k_1 + 2k_2 + (n-j)2k_3)\varepsilon_j.$$

So we have $\rho_{k,1} > \ldots > \rho_{k,n} > 0$, if $k_\alpha \geq 0$ and either $k_1 > 0$ or $k_2 > 0$. Therefore $\langle \lambda + \rho_k \rangle > 0$ and $\epsilon(\langle \lambda, \alpha \rangle) = 1$ for all $\alpha \in R_+$. Now we can derive that for $\lambda, \mu \in P_+$ with $\lambda \neq \mu$, $\lambda + \rho_k$ and $\mu + \rho_k$ are in the same open Weyl chamber. We find that γ_λ and γ_μ do not lie in the same Weyl-orbit. So $\gamma_\lambda \neq \gamma_\mu$ is true for $\lambda, \mu \in P_+$. To show the proposition also holds for $\lambda, \mu \in P \backslash P_+$, we first give the proof for the expression: $\gamma_{w\lambda^*} = w\gamma_{\lambda^*}$ for $\lambda^* \in P_+$ and $w \in W^{\lambda^*}$. Start writing out $w\gamma_{\lambda^*}$:

$$\begin{aligned} w\gamma_{\lambda^*} &= w\lambda^* + \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \lambda^*, \alpha \rangle) w\alpha \\ &= w\lambda^* + \frac{1}{2}\sum_{w^{-1}\alpha \in R_+} k_\alpha \epsilon(\langle \lambda^*, w^{-1}\alpha \rangle)\alpha. \end{aligned}$$

There are two cases in the sum for $\langle \lambda^*, w^{-1}\alpha \rangle$: $\langle \lambda^*, w^{-1}\alpha \rangle = 0$ or $\langle \lambda^*, w^{-1}\alpha \rangle > 0$. For $\langle \lambda^*, w^{-1}\alpha \rangle = 0$ we have that $\alpha \in R_+$, since saying that $w \in W^\lambda$ is equivalent to the following statement: $w\alpha \in R_+$ for all $\alpha \in R_{\lambda,+}$. For the other α it does not matter if α or $-\alpha$ is in the summation, since with $\langle \lambda^*, w^{-1}\alpha \rangle > 0$ we have:

$$\epsilon(\langle \lambda^*, w^{-1}\alpha \rangle)\alpha = -\epsilon(\langle \lambda^*, w^{-1}\alpha \rangle) \cdot (-\alpha) = \epsilon(\langle \lambda^*, w^{-1}(-\alpha) \rangle)) \cdot (-\alpha).$$

We get:

$$\begin{aligned} w\gamma_{\lambda^*} &= w\lambda^* + \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \epsilon(\langle \lambda^*, w^{-1}\alpha \rangle)\alpha \\ &= w\lambda^* + \frac{1}{2}\sum_{\alpha \in R_+} k_\alpha \epsilon(\langle w\lambda^*, \alpha \rangle)\alpha = \gamma_{w\lambda^*}. \end{aligned}$$

Take $\lambda, \mu \in P \backslash P_+$ with $\lambda \neq \mu$. We find:

$$\gamma_\lambda - \gamma_\mu = \gamma_{w\lambda^*} - \gamma_{v\mu^*} = w\gamma_{\lambda^*} - v\gamma_{\mu^*} \neq 0.$$

If $\lambda^* \neq \mu^*$ this is clear by the reasoning earlier in this proof. If $\lambda^* = \mu^*$, we have $w \neq v$ and because they are in W^{λ^*}, w and v can not send γ_{λ^*} to the same element. □

Next we mention another property of $E(\lambda, k)$.

Theorem 3.18. *The set* $\{E(\lambda,k)|\lambda \in P\}$ *forms an orthogonal basis of* $\mathbb{C}[P]$ *with respect to* $(\cdot,\cdot)$.

Proof. The set consist of polynomials of different orders, so they are independent. And by definition of $\mathbb{C}[P]$ it also spans that space. The orthogonality follows from the next calculation. For $\lambda \neq \mu \in P$, $\xi \in \mathbb{R}^n$

$$(\gamma_\lambda,\xi)(E(\lambda,k),E(\mu,k))_k = (D_\xi E(\lambda,k),E(\mu,k))_k = (E(\lambda,k),D_\xi E(\mu,k))_k = (\gamma_\mu,\xi)(E(\lambda,k),E(\mu,k))_k,$$

where the symmetry of D_ξ is used for the second equality. $\gamma_\lambda \neq \gamma_\mu$ according to Proposition 3.17, so $(\gamma_\lambda,\xi) \neq (\gamma_\mu,\xi)$ for some $\xi \in \mathbb{R}^n$. Thus, $(E(\lambda,k),E(\mu,k))_k = 0$. □

Corollary 3.19. $(E(\lambda,k),e^\mu)_k = 0$ *for* $\mu \prec \lambda$.

Proof. Write $e^\mu = E(\mu,k) + \sum_{\zeta \prec \mu} c_{\zeta,\mu} E(\zeta,k)$ by the fact that $\{E(\lambda,k)|\lambda \in P\}$ is a basis. By linearity of the inner product the corollary follows. □

Another nice property of the Cherednik operator is that D_ξ and D_η on $\mathbb{C}[P]$ are commutative.

Proposition 3.20. *The operators* D_ξ *and* D_η *are commutative on* $\mathbb{C}[P]$.

Proof. Let $f = \sum_{\mu \preceq \lambda} c_{\mu,\lambda} e^\mu$ with $c_{\mu,\lambda} \in \mathbb{C}$ and by 3.18 also $f = \sum_{\mu \preceq \lambda} b_{\mu,\lambda} E(\mu,k)$. For each $E(\mu,k)$ in that sum holds that it is an eigenfunction of D_ξ and D_η:

$$D_\xi D_\eta E(\mu,k) = D_\xi(\gamma_\mu,\eta)E(\mu,k) = (\gamma_\mu,\eta)D_\xi E(\mu,k) = (\gamma_\mu,\eta)(\gamma_\mu,\xi)E(\mu,k),$$

and similarly

$$D_\eta D_\xi E(\mu,k) = (\gamma_\mu,\xi)(\gamma_\mu,\eta)E(\mu,k).$$

This means that $\forall \mu\ D_\xi D_\eta E(\mu,k) = D_\eta D_\xi E(\mu,k)$ and thus $D_\xi D_\eta f = D_\eta D_\xi f$. □

The Cherednik operator can also be applied to functions in $L^2(\mathbb{R}^n, \tau dt)$. This space is endowed with the bilinear form $(\cdot,\cdot)_\tau$:

$$(f,g)_\tau = \int_{\mathbb{R}^n} f(t)g(-t)\tau dt. \tag{3.9}$$

D_ξ is symmetric with respect to this bilinear form.

Proposition 3.21. *The operator* D_ξ *is symmetric with respect to the bilinear form* $(\cdot,\cdot)_\tau$ *for* $\xi \in \mathbb{R}^n$.

The proof of this proposition is very similar to the proof of proposition 3.13.

3.4. The Cherednik-Opdam transform

The Cherednik-Opdam transform is the transform we want to apply to the non-symmetric Jacobi polynomials and is defined in this section.

The eigenfunctions $E(\lambda,k)$ are not the only eigenfunctions of D_ξ. According to Opdam [12] there exists an open neighborhood U of $0 \in \mathbb{R}^n$ such that there exist a holomorphic function $G(t,\lambda)$ on $(\mathbb{R}^n + iU)$ with $\lambda \in \mathbb{C}$ with the properties:

1. $G(0,\lambda) = 1$,
2. $\forall \xi \in \mathbb{C}^n$: $D_\xi G(t,\lambda) = (\lambda,\xi)G(t,\lambda)$,

and the functions also depend on the multiplicity function k. Furthermore, $G(t,\lambda)$ is bounded for $\lambda \in i\mathbb{R}^n$ (see [2]). We can use $G(t,\lambda)$ to get a W-invariant eigenfunction ϕ_λ by setting:

$$\phi_\lambda = \frac{1}{|W|} \sum_{w \in W^\lambda} wG(t,\lambda).$$

The Cherednik-Opdam transform of a function $f \in L^2(\mathbb{R}^n, \tau dt)$ is:

$$\tilde{f}(\lambda) = \int_{\mathbb{R}^n} f(t)G(-t,\lambda)\tau dt, \quad \lambda \in i\mathbb{R}^n.$$

The inverse of this transform is given by:

$$f(t) = \int_{i\mathbb{R}^n} \widetilde{f}(\lambda) G(t,\lambda) \sigma d\lambda, \quad t \in \mathbb{R}^n,$$

where

$$\sigma = (2\pi)^{-n} \prod_{\alpha \in R_+} \frac{\tilde{c}_0(\rho_k)^2}{\tilde{c}(\lambda)\tilde{c}_0(-\lambda)}, \tag{3.10}$$

with

$$\tilde{c}(\lambda) = \prod_{\alpha \in R_+} \frac{\Gamma(\langle \lambda, \alpha \rangle + k_{\frac{\alpha}{2}} + k_\alpha)}{\Gamma(\langle \lambda, \alpha \rangle + k_{\frac{\alpha}{2}})},$$

and

$$\tilde{c}_0(\lambda) = \prod_{\alpha \in R_+} \frac{\Gamma(\langle \lambda, \alpha \rangle + k_{\frac{\alpha}{2}} + k_\alpha + 1)}{\Gamma(\langle \lambda, \alpha \rangle + k_{\frac{\alpha}{2}} + 1)}.$$

Applying the Cherednik-Opdam transform to $D_j f$ with $f \in L^2(\mathbb{R}^n, d\mu)$, which acts on the variable t, leads to another operator that acts on the variable λ. We show the operator that is associated to D_j:

$$\begin{aligned} \widetilde{D_j f}(\lambda) &= \int_{\mathbb{R}^n} D_j(f(t)) G(-t,\lambda) \tau dt = \int_{\mathbb{R}^n} f(t) D_j(G(-t,\lambda)) \tau dt \\ &= \lambda_j \int_{\mathbb{R}^n} f(t) G(-t,\lambda) \tau dt = \lambda_j \widetilde{f}(\lambda). \end{aligned}$$

Here the symmetry of D_j with respect to the bilinear form is used (proposition 3.21) and the fact that $G(-t,\lambda)$ is an eigenfunction. So the multiplication operator λ_j is the operator associated to D_j.

3.5. Relation symmetric and non-symmetric Jacobi polynomials

The Jacobi polynomials defined in the paper of Zhang [16] are called symmetric Jacobi polynomials, since they are Weyl group invariant and are equal to the standard Jacobi Polynomials in the one variable case. This is shown at the end of chapter 4. A partition is a vector $\eta \in \mathbb{N}^n$ such that $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_n \geq 0$. Call the set of partitions $\mathscr{P}$. For each element $\eta \in \mathscr{P}$ we have that $2\eta \in P$. Now it is possible to define the symmetric Jacobi polynomials.

Definition 3.22. *For each $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, $\eta_1 \geq \ldots \geq \eta_n \geq 0$, P_η on $\mathbb{T}^n$ is defined as*

$$P_\eta = p_\eta^W + \sum_{\substack{\eta' \in \mathscr{P} \\ 2\eta' < 2\eta}} c_{\eta,\eta'} p_{\eta'}^W$$

such that

$$\left(P_\eta, p_{\eta'}^W\right)_k = 0,$$

where $p_\eta^W = \sum_{w \in W} w\left(e^{2(\eta_1 \varepsilon_1 + \ldots + \eta_n \varepsilon_n)}\right)$ is the Weyl group orbit sum of the power function.

These symmetric polynomials form an orthogonal basis.

Lemma 3.23. *The polynomials $\{P_\eta\}$ form an orthogonal basis for $L^2\left(\mathbb{T}, |\delta_{k^{(v)}}(s)| ds\right)^W$, L^2-space of Weyl invariant polynomials.*

The symmetric Jacobi polynomials can be described by the non-symmetric Jacobi polynomials in the following way:

Theorem 3.24.

$$P_\eta = \sum_{w \in W^\lambda} w E(2\eta, k).$$

For the proof of both the lemma and theorem we refer to [12].

4

Spherical transform of symmetric Jacobi polynomials

In the paper [16] Zhang describes a method to express the spherical transform of symmetric Jacobi polynomials and to show that the spherical transform is equal to (symmmetric) multivariable Wilson polynomials. This chapter gives an overview of Zhang's method. The chapter also includes some extra steps and calculations that are omitted in the paper.

4.1. Introduction

We start by giving a small description of the method of Zhang. The idea of the method is to write symmetric Jacobi polynomials in W-invariant polynomials of Cherednik operators, since the spherical transform of a W-invariant polynomial of Cherednik operators is easily determined. The symmetric Jacobi polynomials that form an orthogonal basis on the torus $\mathbb{T}^n$, are written as Laurent polynomials in sines. A certain transformation can be applied to get polynomials on $\mathbb{R}^n$ in variable x. After that an operator associated (and very similar) to the Cherednik operator is given in a form that can act on polynomials in x and this is used to change the Jacobi polynomials in x to polynomials in the Cherednik-like operators. To preserve the orthogonality of the Jacobi polynomials a weight function is multiplied with the Jacobi polynomial on $\mathbb{R}^n$. The spherical transform of the weight function times the Jacobi polynomial is determined resulting in a (symmetric) multivariable Wilson polynomial, which is defined at the end of this section.

The spherical transform and some necessary definitions are discussed first. Recall the root system R with $R_+ = \{2\varepsilon_i, 4\varepsilon_i, 2(\varepsilon_j \pm \varepsilon_k)\}$ for $1 \le i \le n$ and $1 \le j < k \le n$. The Cherednik operator on functions on $\mathbb{R}^n$ from the previous chapter (definition 3.7) is used in this chapter with the multiplicity function $k = (k_1, k_2, k_3)$ with $k_1 = b$, $k_2 = \frac{\iota}{2}$ and $k_3 = \frac{a}{2}$ associated to the roots of R. Define $D_i := D_{\varepsilon_i}$. For the operators D_i there exist a W-invariant function ϕ_λ that is an eigenfunction with eigenvalue $(\lambda, \varepsilon_i)^2 = \lambda_i^2$, as described in section 3.4. For a W-invariant polynomial p in the variables D_i, we have

$$p(D_1, \dots, D_n)\phi_\lambda = p(\lambda_1, \dots, \lambda_n)\phi_\lambda, \tag{4.1}$$

since p exists of combinations of D_i^2. The spherical transform of W-invariant functions $f \in L^2(\mathbb{R}^n, d\mu(t))^W$, where this L^2-space only contains $\mathbb{R}$-valued functions, is given by

$$\widetilde{f}(\lambda) = \int_{\mathbb{R}^n} f(t)\phi_\lambda(t)d\mu(t), \tag{4.2}$$

with

$$d\mu(t) := \tau(t)dt = \prod_{\alpha \in R_+} |2\sinh(\frac{1}{2}(\alpha, t))|^{2k_\alpha} dt.$$

This transform has a Plancherel formula

$$\int_{\mathbb{R}^n} |f(t)|^2 \tau dt = \int_{i\mathbb{R}} |\tilde{f}(\lambda)|^2 d\tilde{\mu}(\lambda). \tag{4.3}$$

The Plancherel measure $d\tilde{\mu}(\lambda)$ is given by

$$d\tilde{\mu}(\lambda) = \frac{(2\pi)^{-n} c_0^2}{c(\lambda)c(-\lambda)} d\lambda,$$

with

$$c(\lambda) = \prod_{j=1}^{n} \frac{\Gamma(\lambda_j + b)\Gamma(2\lambda_j)}{\Gamma(\lambda_j + b + \frac{\iota}{2})\Gamma(2\lambda_j + 2b)} \prod_{\substack{1\le j<k\le n,\\ \epsilon=\pm}} \frac{\Gamma(\lambda_j + \epsilon\lambda_k)}{\Gamma(\lambda_j + \epsilon\lambda_k + \frac{a}{2})}$$

and

$$c_0 = \prod_{j=1}^{n} \frac{\Gamma(\rho_j + b + 1)\Gamma(2\rho_j + 1)}{\Gamma(\rho_j + b + \frac{\iota}{2} + 1)\Gamma(2\rho_j + 2b + 1)} \prod_{\substack{1\le j<k\le n,\\ \epsilon=\pm}} \frac{\Gamma(\rho_j + \epsilon\rho_k + 1)}{\Gamma(\rho_j + \epsilon\rho_k + \frac{a}{2} + 1)}.$$

The inner product of $f, g \in L^2(\mathbb{R}^n, d\mu(t))^W$ is given by

$$(f, g)_{\mu(t)} = \int_{\mathbb{R}^n} f(t)g(t)d\mu(t).$$

The weight function $f_{-2\nu}$ on $\mathbb{R}^n$ is defined by

$$f_{-2\nu}(t) = \prod_{j=1}^{n} \cosh^{-2\nu}(t_j), \quad \nu \in \mathbb{R}.$$

Note that this function is W-invariant, since $\cos(-t_j) = \cos(t_j)$. For a sufficiently large ν, the spherical transform of $f_{-2\nu}$ can by made explicit.

Theorem 4.1. *For $\nu > \iota + b + a(n-1)$ the spherical transform of $f_{-2\nu}$ is given by*

$$\widetilde{f_{-2\nu}}(\lambda) = N_\nu \prod_{j=1}^{n} \prod_{\epsilon=\pm} \frac{\Gamma(\nu - \frac{1}{2}\rho_1 + \epsilon\frac{1}{2}\lambda_j)}{\Gamma(\nu - \frac{1}{2}\rho_1 + \epsilon\frac{1}{2}(\iota + b + a(j-1)))}, \quad \lambda \in i\mathbb{R}^n.$$

N_ν is a normalization constant that is given by

$$\begin{aligned} N_\nu &= \int_{\mathbb{R}^n} f_{-2\nu} d\mu(t) \\ &= 2^{n(2\iota+2b+a(n-1))} n!\, \Gamma_a\left(\frac{\iota + 1 + 2b + a(n-1)}{2}\right) \\ &\quad \cdot \frac{\Gamma_a(\nu - (\frac{a}{2}(n-1) + \iota + b))}{\Gamma_a(\nu + \frac{1-\iota}{2})} \prod_{1\le i<j\le n} \frac{\Gamma(\frac{a}{2}(j-i+1))}{\Gamma(\frac{a}{2}(j-i))}, \end{aligned}$$

where

$$\Gamma_a(\sigma) = \prod_{j=1}^{n} \Gamma(\sigma - \frac{a}{2}(j-1)).$$

For the proof, see [16, Theorem 3.2]. It consists of a lot of tedious computations.

Let S_n be the group of permutations of an element in $\mathbb{N}^n$, and call the symmetric power sum m_η for $\eta \in \mathbb{N}^n$:

$$m_\eta(y_1, \ldots, y_n) = \sum_{\zeta \in S_n \eta} y_1^{\zeta_1} \ldots y_n^{\zeta_n}.$$

We need the symmetric power sum for the definition of the Wilson polynomials.

Definition 4.2. *The (unique) polynomials q_λ that satisfy the following two conditions are called multivariable Wilson polynomials:*

1. $q_\eta = m_{\eta,W} + \sum_{\substack{\zeta\in\mathscr{P}\\ 2\zeta<2\eta}} c_{\eta,\zeta} m_{\zeta,W}, \quad c_{\eta,\zeta} \in \mathbb{C},$
2. $(q_\eta, m_{\zeta,W})_{\Delta_W} = 0$ *if* $\zeta \prec \eta$,

where $m_{\eta,W}(x) = m_\eta(x_1^2,\dots,x_n^2)$. *The weight function* Δ_W *given by*

$$\Delta_W(x) = \prod_{\substack{1\le j<k\le n,\\ \epsilon_1,\epsilon_2=\pm1}} \frac{\Gamma(\nu + i(\epsilon_1 x_j + \epsilon_2 x_k))}{\Gamma(i(\epsilon_1 x_j + \epsilon_2 x_k))} \times \prod_{\substack{1\le j\le n,\\ \epsilon=\pm1}} \frac{\Gamma(\nu_0 + i\epsilon x_j)\Gamma(\nu_1 + i\epsilon x_j)\Gamma(\nu_2 + i\epsilon x_j)\Gamma(\nu_3 + i\epsilon x_j)}{\Gamma(2i\epsilon x_j)},$$

with $\nu \ge 0$ *and* $\Re(\nu_j) > 0 \;\; (j = 0,1,2,3)$. *The inner product* $(\cdot,\cdot)_{\Delta_W}$ *is determined by*

$$(m_{\eta,W}, m_{\zeta,W})_{\Delta_W} = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} m_{\eta,W}(x)\overline{m_{\zeta,W}(x)}\Delta_W(x)dx.$$

These definitions allow us to continue with the description of Zhang's method.

4.2. Transformation of $\mathbb{T}^n$ to $\mathbb{R}^n$

The symmetric Jacobi polynomials are orthogonal with respect to a measure on $\mathbb{T}^n = i\mathbb{R}^n/(\pi i\mathbb{Z})^n$, but the spherical transform is defined on $\mathbb{R}^n$. Therefore, we need a transformation for the functions on $\mathbb{T}^n$ to maintain the orthogonality properties on $\mathbb{R}^n$. Set $y_j = e^{iu_j}$ for $1 \le j \le n$ with $u \in [0,\pi]^n$, then $y \in \mathbb{T}^n$, and $v_j = e^{t_j}$ for $1 \le j \le n$, $t \in \mathbb{R}^n$. Consider the components separately. There are many different ways to send an element $v_j \in \mathbb{R}_+$ to an element on $\mathbb{T}$, the unit circle. The transformation

$$y_j = \frac{-v_j + i}{iv_j - 1}$$

is used here. It will become clear throughout this chapter why this particular transformation is chosen. First we verify that the given transformation maps to $\mathbb{T}$. One of the characterizations of the unit circle is that the conjugate of an element is equal to its inverse. So, we check if $\overline{y} = y^{-1}$:

$$\overline{y} = \frac{-v_j - i}{-iv_j - 1} = \frac{iv_j - 1}{-v_j + i} = y^{-1}.$$

Then, after some calculations, we find

$$\sin(u_j) = \frac{y_j - y_j^{-1}}{2i} = \frac{v_j - v_j^{-1}}{v_j + v_j^{-1}} = \tanh(t_j),$$

and

$$\cos(u_j) = \frac{y_j + y_j^{-1}}{2} = -\frac{2}{v_j + v_j^{-1}} = -\cosh^{-1}(t_j).$$

The fact that $\tanh(t_j)$ is sent to $\sin(u_j)$ (or actually the other way around) is particularly important for the method of Zhang.

4.3. Orthogonal basis of symmetric Jacobi polynomials

This section describes how the symmetric Jacobi polynomials can be written in sines. Moreover, in the new sine variables the Jacobi polynomials still form an orthogonal basis.

Let $\nu > \iota + b + a(n-1)$. Choose the multiplicity function $k^{(\nu)} = (k_1^{(\nu)}, k_2^{(\nu)}, k_3^{(\nu)})$ with

$$2k_2^{(\nu)} = 2(2\nu - (1 + \iota + b + a(n-1))) + 1,\;\; 2(k_1^{(\nu)} + k_2^{(\nu)}) = \iota + 2b,\;\; 2k_3^{(\nu)} = a. \tag{4.4}$$

The symmetric Jacobi polynomials $P_{\nu,\eta}$ (definition 3.22) can be constructed with polynomials of the variables $\cos(2u_j)$, because for $l, m \in \mathbb{N}$, $\cos^m(2u_j)$ is a Laurent polynomial in $\mathbb{C}[P]$ with highest power e^{-2miu_j}, $\cos^m(2u_j)$ and $\cos^l(2u_j)$ are linearly independent in $\mathbb{C}[P]$ and $\cos^m(2u_j)$ only contains powers of e^x with the powers in $2\mathbb{Z}$. Furthermore, for each $f(u)e^{2miu_j}$, there is also $f(u)e^{-2miu_j}$ present in the polynomial.

Here $f(u)$ is some (combination of) exponential functions of u_i, $i \neq j$ and $m \in \mathbb{N}$, since the polynomial is W-invariant. Combining them leads to $2^m f(u)\cos^m(2u_j) + l.o.t.$. This argument can be applied repeatedly to replace all exponent functions by cosines.

Moreover, the Jacobi polynomials can be expressed as functions of $\sin^2(u_j)$ by the trigonometric equality $\cos(2u_j) = 1 - 2\sin^2(u_j)$. Set $x_j = \sin(u_j)$. The symmetric power sum $m_\eta(x_1^2, \ldots, x_n^2)$ is W-invariant and can be used to express $P_{\nu,\eta}$ in $x_1, \ldots, x_n$:

$$P^x_{\nu,\eta}(x_1,\ldots,x_n) = (-1)^{\eta_1+\ldots+\eta_n} 2^{2(\eta_1+\ldots+\eta_n)} m_\eta(x_1^2,\ldots,x_n^2) + \sum_{\zeta<\eta} c_{\eta,\zeta} m_\zeta(x_1^2,\ldots,x_n^2).$$

The Jacobi polynomial in the new variable x also satisfies an orthogonality property with respect to the inner product $(\cdot,\cdot)_{k^{(\nu)}}$ under change of variable:

$$(P^x_{\nu,\eta}, p^{W,x}_\zeta)_{k^{(\nu)},x} = \int_{[-1,1]^n} P^x_{\nu,\eta}(x)\overline{p^{W,x}_\zeta(x)}\delta^x_{k^{(\nu)}}(x)\,dx = 0, \quad \zeta < \eta,$$

since the changing of variable does not affect the orthogonality. The next section explains a bit more about the changing of variables in the inner product. By the same reasoning as in corollary 3.19

$$(P^x_{\nu,\eta}, m_\zeta)_{k^{(\nu)},x} = 0, \quad \zeta < \eta$$

also holds.

4.4. Change of variables in two inner products

In this chapter we work with two different inner products, namely the inner product on $\mathbb{C}[P]$ and the inner product in $L^2(\mathbb{R}^n, d\mu(t))^W$. In this section the inner products are subjected to change of variables to be able to show the equality of the norms of the Jacobi polynomials on $\mathbb{T}^n$ and the Jacobi-type polynomials on $\mathbb{R}^n$.

We start with the inner product on $\mathbb{C}[P]$:

Lemma 4.3. *For $f, g \in \mathbb{C}[P]$ that can be written in the variables $x_j = \sin(u_j)$, we have the inner product in the variable x:*

$$(f^x, g^x)_{k^{(\nu)},x} = 2^{(2k_1^{(\nu)}+4k_2^{(\nu)}+2k_3^{(\nu)}(n-1))n} \int_{[-1,1]^n} f^x(x)\overline{g^x(x)} \prod_j |x_j|^{2k_1^{(\nu)}+2k_2^{(\nu)}} |1-x_j^2|^{k_2^{(\nu)}-\frac{1}{2}} \prod_{i<j} |x_i^2 - x_j^2|^{2k_3^{(\nu)}}\,dx.$$

Proof. The change of variables of the inner product on $\mathbb{C}[P]$ is shown by first writing the weight function in $x_j = \sin(u_j)$ as in the previous section. Recall that the weight function is $\delta_{k^{(\nu)}} = \prod_{\alpha\in R_+} \left|\sin\left(\frac{(\alpha,u)}{2}\right)\right|^{2k_\alpha^{(\nu)}}$. We get:

$$\begin{aligned}\prod_{\alpha\in R_+} \left|2\sin\left(\frac{(\alpha,u)}{2}\right)\right|^{2k_\alpha^{(\nu)}} &= \prod_j |2\sin(u_j)|^{2k_1^{(\nu)}} |2\sin(2u_j)|^{2k_2^{(\nu)}} \prod_{i<j,\epsilon=\pm} |2(\sin(u_i+\epsilon u_j))|^{2k_3^{(\nu)}} \\ &= \prod_j |2\sin(u_j)|^{2k_1^{(\nu)}} |2^2\sin(u_j)\cos(u_j)|^{2k_2^{(\nu)}} \prod_{i<j,\epsilon=\pm} |2(\sin(u_i+\epsilon u_j))|^{2k_3^{(\nu)}}. \end{aligned} \quad (4.5)$$

In the product over j the $|\cos(u_j)|$ is the same as $|1-\sin^2(u_j)|^{\frac{1}{2}}$, so we find:

$$\prod_j |2\sin(u_j)|^{2k_1^{(\nu)}} |2^2\sin(u_j)\cos(u_j)|^{2k_2^{(\nu)}} = 2^{(2k_1^{(\nu)}+4k_2^{(\nu)})n} \prod_j |x_j|^{2k_1^{(\nu)}+2k_2^{(\nu)}} |1-x_j^2|^{k_2^{(\nu)}}.$$

For the other product in (4.5) take together the following terms:

$$|2(\sin(u_i+u_j))|^{2k_3^{(\nu)}} |2(\sin(u_i-u_j))|^{2k_3^{(\nu)}} = 2^{2k_3^{(\nu)}} |\sin(u_i+u_j)\sin(u_i-u_j)|^{2k_3^{(\nu)}}.$$

Apply the angle sum and difference identities $\sin(\alpha\pm\beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$:

$$\begin{aligned}\sin(u_i+u_j)\sin(u_i-u_j) &= (\sin(u_i)\cos(u_j)+\cos(u_i)\sin(u_j))(\sin(u_i)\cos(u_j)-\cos(u_i)\sin(u_j)) \\ &= \sin^2(u_i)\cos^2(u_j) - \cos^2(u_i)\sin^2(u_j) \\ &= x_i^2(1-x_j^2) - (1-x_i^2)x_j^2 = x_i^2 - x_j^2. \end{aligned} \quad (4.6)$$

Therefore we get that:

$$\prod_{i<j,\epsilon=\pm} |2(\sin(u_i+\epsilon u_j))|^{2k_3^{(\nu)}} = 2^{2k_3^{(\nu)}} \prod_{i<j} |x_i^2 - x_j^2|^{2k_3^{(\nu)}}$$

Combining everything we find for the weight function:

$$\prod_{\alpha\in R_+} \left|2\sin\left(\frac{(\alpha,u)}{2}\right)\right|^{2k_\alpha^{(\nu)}} = 2^{(2k_1^{(\nu)}+4k_2^{(\nu)}+2k_3^{(\nu)}(n-1))n} \prod_j |x_j|^{2k_1^{(\nu)}+2k_2^{(\nu)}} |1-x_j^2|^{k_2^{(\nu)}} \prod_{i<j} |x_i^2-x_j^2|^{2k_3^{(\nu)}}.$$

Also note that the range of $\sin(u_j)$ is $[-1,1]$ and that $dx_j = \cos(u_j)du_j$, thus the inner product of $f,g\in\mathbb{C}[P]$ in the variable x becomes the inner product as stated in this lemma. □

The other inner product follows pretty much the same steps only the variable that it is changed to is different, $x_j := \tanh(t_j)$.

Lemma 4.4. *For $f,g\in L^2(\mathbb{R}^n, d\mu(t))$ that can be written in the variables $x_j = \tanh(t_j)$, we have the inner product in the variable x:*

$$(f^x,g^x)_{\mu(t),x} = 2^{(2k_1+4k_2+2k_3(n-1))n} \int_{[-1,1]^n} f^x(x) g^x(x) \prod_j |x_j|^{2k_1+2k_2} |1-x_j^2|^{-k_1-2k_2-2k_3(n-1)-1} \prod_{i<j} |x_i^2-x_j^2|^{2k_3} dx.$$

Proof. Start with the weight function τ:

$$\prod_{\alpha\in R_+} \left|2\sinh\left(\frac{(\alpha,u)}{2}\right)\right|^{2k_\alpha} = \prod_j |2\sinh(u_j)|^{2k_1} |2^2\sinh(u_j)\cosh(u_j)|^{2k_2} \prod_{i<j,\epsilon=\pm} |2(\sinh(u_i+\epsilon u_j))|^{2k_3}. \tag{4.7}$$

In the product over j we use that $\cosh^{-2}(t_j) = 1-\tanh^2(t_j)$ and we find:

$$\begin{aligned}\prod_j |2\sinh(u_j)|^{2k_1} |2^2\sinh(u_j)\cosh(u_j)|^{2k_2} &= 2^{(2k_1+4k_2)n} \prod_j \left|\frac{\sinh(u_j)}{\cosh(u_j)}\cosh(u_j)\right|^{2k_1} \left|\frac{\sinh(u_j)}{\cosh(u_j)}\cosh^2(u_j)\right|^{2k_2} \\ &= 2^{(2k_1+4k_2)n} \prod_j |x_j|^{2k_1+2k_2} |1-x_j^2|^{-k_1-2k_2}.\end{aligned}$$

For the other product in (4.7) the same angle sum and difference identities hold as for the sines in (4.6), so:

$$\begin{aligned}\prod_{i<j,\epsilon=\pm} |2(\sinh(u_i+\epsilon u_j))|^{2k_3} &= 2^{2k_3} \prod_{i<j} |\sinh^2(u_i)\cosh^2(u_j) - \cosh^2(u_i)\sinh^2(u_j)|^{2k_3} \\ &= 2^{2k_3} \prod_{i<j} \left|\frac{\sinh^2(u_i)}{\cosh^2(u_i)}\cosh^2(u_i)\cosh^2(u_j) - \cosh^2(u_i)\cosh^2(u_j)\frac{\sinh^2(u_j)}{\cosh^2(u_j)}\right|^{2k_3} \\ &= 2^{2k_3} \prod_{i<j} |x_i^2-x_j^2|^{2k_3} |\cosh^2(u_i)\cosh^2(u_j)|^{2k_3}.\end{aligned}$$

Now we still need to write the cosh in this equation in the x variable and we can even change the sum to the sum over one variable:

$$\begin{aligned}\prod_{i<j} |\cosh^2(u_i)\cosh^2(u_j)|^{2k_3} &= \prod_{i<j} |1-x_i^2|^{-2k_3} |1-x_i^2|^{-2k_3} \\ &= \prod_j |1-x_j^2|^{-2k_3(n-1)}.\end{aligned}$$

Combining all the calculations for the weight function gives:

$$\prod_{\alpha\in R_+} \left|2\sinh\left(\frac{(\alpha,u)}{2}\right)\right|^{2k_\alpha} = 2^{(2k_1+4k_2+2k_3(n-1))n} \prod_j |x_j|^{2k_1+2k_2} |1-x_j^2|^{-k_1-2k_2-2k_3(n-1)} \prod_{i<j} |x_i^2-x_j^2|^{2k_3}.$$

The inner product of $f,g\in L^2(\mathbb{R}^n, d\mu(t))$ in the variable x is as stated above, since $\tanh(t_j)$ has range $[-1,1]$ and $dx_j = \left(1-\tanh^2(t_j)\right)dt_j$. □

The inner products in the lemmas 4.3 and 4.4 look very similar. Note that adding a conjugation to g^x in 4.4 does not influence the inner product, since g^x is real valued. We have that:

$$\begin{aligned} 2k_1^{(v)} + 2k_2^{(v)} &= \iota + 2b = 2k_1 + 2k_2, \\ k_2^{(v)} - \frac{1}{2} &= 2v - (1 + \iota + b + a(n-1)) = 2v - k_1 - 2k_2 - 2k_3(n-1) - 1, \\ 2k_3^{(v)} &= a = 2k_3, \end{aligned}$$

and using the identities above we also find:

$$2^{(2k_1^{(v)} + 4k_2^{(v)} 2k_3^{(v)}(n-1))n} = 2^{2k_2^{(v)} - 2k_2} 2^{(2k_1 + 4k_2 + 2k_3(n-1))n}.$$

This means that the inner products are equal to each other except for the constant $2^{2k_2^{(v)} - 2k_2}$ and a factor $\prod_j |1 - x_j^2|^{2v}$. This factor is compensated by the function f_{-2v}, which is discussed in Section 4.6.

4.5. Symmetric power function in $\mathscr{D}$

The Cherednik operator can act on functions on $\mathbb{R}^n$. If these functions are written in the different variables $x_j = \tanh(t_j)$, it is easier to work with the Cherednik operator in the x_j variables as well. In this section the change of variables in the Cherednik operator is given. The operator that is equal to the conjugation of f_{-2v} with the Cherednik operator, $\mathscr{D}_j$, is also given in x. The $\mathscr{D}_j$'s play an important part in the calculation of the spherical transform.

We start by writing down the operator D_j explicitly:

$$\begin{aligned} D_j = \partial_j - a \sum_{i<j} \frac{1}{1 - e^{-2(t_i - t_j)}} (1 - \sigma_{\varepsilon_i - \varepsilon_j}) + a \sum_{j<k} \frac{1}{1 - e^{-2(t_j - t_k)}} (1 - \sigma_{\varepsilon_j - \varepsilon_k}) + a \sum_{i \neq j} \frac{1}{1 - e^{-2(t_i + t_j)}} (1 - \sigma_{\varepsilon_i + \varepsilon_j}) \\ + 2b \frac{1}{1 - e^{-2t_j}} (1 - \sigma_{\varepsilon_j}) + 2\iota \frac{1}{1 - e^{-4t_j}} (1 - \sigma_{\varepsilon_j}) - (\rho_k, \varepsilon_j). \end{aligned}$$

We write the operator in different variables in the next lemma.

Lemma 4.5. *Set $x_j := \tanh(t_j)$, then we have*

$$\begin{aligned} D_j = (1 - x_j^2)\partial_j - \frac{a}{2} \sum_{i<j} \frac{1 + x_i - x_j - x_i x_j}{x_i - x_j} (1 - \sigma_{\varepsilon_i - \varepsilon_j}) + \frac{a}{2} \sum_{j<k} \frac{1 + x_j - x_k - x_j x_k}{x_j - x_k} (1 - \sigma_{\varepsilon_j - \varepsilon_k}) \\ + \frac{a}{2} \sum_{i \neq j} \frac{1 + x_i + x_j + x_i x_j}{x_i + x_j} (1 - \sigma_{\varepsilon_i + \varepsilon_j}) + b(1 + \frac{1}{x_j})(1 - \sigma_{\varepsilon_j}) + \iota \left(1 + \frac{1}{2}(x_j + \frac{1}{x_j})\right)(1 - \sigma_{\varepsilon_j}) - (\rho_k, \varepsilon_j), \end{aligned}$$

on polynomials in the variables $x = (x_1, \ldots, x_n)$.

Proof. The proof consists of many tedious computations for each term in D_j. (ρ_k, ε_j) stays the same, since it is a constant. The derivative of $\tanh(t_j)$ is

$$\partial_j \tanh(t_j) = 1 - \tanh^2(t_j) = 1 - x_j^2.$$

One of the remaining terms is shown as an example. The other terms are obtained by similar calculations. We look at

$$b(1 + \frac{1}{x_j})(1 - \sigma_{\varepsilon_j}).$$

Important is that the Weyl group elements σ_α with $\alpha \in \mathbb{R}_+$ are now applied to x. The same notation is used, since $-x_j = -\tanh(t_j) = \tanh(-t_j)$ and clearly permutations of x are the same as permutations of t. We proof that

$$(1 + \frac{1}{x_j}) = \frac{2}{1 - e^{-2t_j}}.$$

Start with the lefthandside and by elementary computations the righthandside easily follows:

$$\begin{aligned} (1 + \frac{1}{x_j}) = 1 + \frac{e^{t_j} + e^{-t_j}}{e^{t_j} - e^{-t_j}} &= \frac{e^{t_j} - e^{-t_j} + e^{t_j} + e^{-t_j}}{e^{t_j} - e^{-t_j}} \\ = \frac{2e^{t_j}}{e^{t_j} - e^{-t_j}} &= \frac{2}{1 - e^{-2t_j}}. \end{aligned}$$

□

Define a new operator $\mathscr{D}_i$ by conjugation of D_i by the function $f_{-2\nu}$:

$$\mathscr{D}_i := f_{-2\nu} D_i f_{2\nu}. \tag{4.8}$$

The operator $\mathscr{D}_j$ is now easily written in the variable x with lemma 4.5. Use that $f_{-2\nu}$ is W-invariant.

$$\mathscr{D}_j = -2\nu x_j + (1-x_j^2)\partial_j - \frac{a}{2}\sum_{i<j}\frac{1+x_i-x_j-x_ix_j}{x_i-x_j}(1-\sigma_{\varepsilon_i-\varepsilon_j}) + \frac{a}{2}\sum_{j<k}\frac{1+x_j-x_k-x_jx_k}{x_j-x_k}(1-\sigma_{\varepsilon_j-\varepsilon_k})$$
$$+\frac{a}{2}\sum_{i\neq j}\frac{1+x_i+x_j+x_ix_j}{x_i+x_j}(1-\sigma_{\varepsilon_i+\varepsilon_j}) + \iota\left(1+\frac{1}{2}(x_j+\frac{1}{x_j})\right)(1-\sigma_{\varepsilon_j}) + b(1+\frac{1}{x_j})(1-\sigma_{\varepsilon_j}) - (\rho_k,\varepsilon_j).$$

The extra term $-2\nu x_j$ compared to the lemma 4.5 results from the product rule. Remark that $\mathscr{D}_j = D_j - 2\nu x_j$. The next lemma shows how the operators $\mathscr{D}_j$ act on polynomials in x. The result is somewhat similar to the upper triangularity of the operators D_j, in the sense that applying $\mathscr{D}$ does not change the ordering of the polynomial by much.

Lemma 4.6. *The action of the operators $\mathscr{D}_j$ on the monomials $x^\eta = x_1^{\eta_1}\dots x_n^{\eta_n}$ for $\eta\in\mathbb{N}^n$ is as follows,*

$$\mathscr{D}_j x^\eta = \sum_{\substack{\zeta\in\mathbb{N}^n,\\ 2\zeta^*\preceq(2\eta^j)^*}} a_{\zeta,\eta}x^\zeta,$$

where $\eta^j = (\eta_1,\dots,\eta_j+1,\dots,\eta_n)$ and the coefficient of x^{η^j} is

$$a_{\eta^j,\eta} = -2\nu - \eta_j + a\#\{i<j : \eta_i > \eta_j\} + \frac{\iota}{2}(1-(-1)^{\eta^j}).$$

Proof. The proof is elementary calculations. □

The symmetric power sum m_η in $\mathscr{D}_j^2$ acting on the monomial 1 can be expressed in a sum of symmetric powersums m_ζ in x_j^2 with $\zeta \prec \eta$.

Lemma 4.7. *The operators $m_\eta(\mathscr{D}_1^2,\dots,\mathscr{D}_n^2)$ acting on the constant monomial* 1 *gives*

$$m_\eta(\mathscr{D}_1^2,\dots,\mathscr{D}_n^2)1 = d_\eta m_\eta(x_1^2,\dots,x_n^2) + \sum_{\substack{\zeta\in\mathscr{P}\\ 2\zeta\prec 2\eta}} c_{\eta,\zeta} m_\zeta(x_1^2,\dots,x_n^2),$$

with $c_{\eta,\zeta}\in\mathbb{R}$ and the leading coefficient

$$d_\eta = \prod_{j=0}^{n}\prod_{k=0}^{\eta_j-1}(-2\nu+(n-j)a-2k)(-2\nu+(n-j)a+\iota-1-2k).$$

Proof. Apply the previous lemma successively and use that $m_\eta(\mathscr{D}_1^2,\dots,\mathscr{D}_n^2)1$ is W-invariant. □

The following statement follows directly from the lemma and is used to determine the spherical transform of the symmetric Jacobi polynomials.

Corollary 4.8.

$$m_\eta(x_1^2,\dots,x_n^2) = d_\eta^{-1} m_\eta(\mathscr{D}_1^2,\dots,\mathscr{D}_n^2)1 + \sum_{\zeta\prec\eta} c_{\zeta,\eta} m_\zeta(\mathscr{D}_1^2,\dots,\mathscr{D}_n^2)1.$$

4.6. Spherical transform of symmetric Jacobi polynomials

To determine the spherical transform of a symmetric Jacobi polynomial we need to change the variables of this polynomial. After that it is just a combination of lemmas and theorems we have seen in this chapter to show that the spherical transform results in a multivariable Wilson polynomial.

Section 4.3 gives the Jacobi polynomials in the variables $x_j = \sin(u_j)$: $P^x_{\nu,\eta}(x_1,\dots,x_n)$. Now we change the variable in $P^x_{\nu,\eta}$ such that x_j becomes $\tanh(t_j)$ corresponding with the transformation treated in section 4.2 and we define a new function on $\mathbb{R}^n$:

$$H_{\nu,\eta}(t) := f_{-2\nu}(t)P^x_{\nu,\eta}(\tanh(t_1),\dots,\tanh(t_n)).$$

These functions have the same norm as the Jacobi polynomials on $\mathbb{T}^n$ up to a multiplicative constant and even form an orthogonal basis.

Lemma 4.9. *The functions $\{H_{\nu,\eta}\}_\eta$ form an orthogonal basis for the space $L^2(\mathbb{R}^n, d\mu(t))^W$ and*

$$2^{2k_2^{(\nu)}-2k_2}(H_{\nu,\eta}, H_{\nu,\eta})_{\mu(t)} = (P^x_{\nu,\eta}, P^x_{\nu,\eta})_{k^{(\nu)}}.$$

Proof. In the section 4.4 we have seen that the inner products of the different spaces we are working on are equal to each other accept for the constant $2^{2k_2^{(\nu)}-2k_2}$ and a factor $\prod_j |1-x_j^2|^{2\nu}$. The function $f_{-2\nu}$ can be expressed in $x_j = \tanh(t_j)$:

$$f_{-2\nu}(t) = \prod_{j=1}^n \cosh^{-2\nu}(t_j) = \prod_{j=1}^n (1-x_j^2)^\nu.$$

Therefore $|f_{-2\nu}(t)|^2 = \prod_j |1-x_j^2|^{2\nu}$ adds the extra factor we need for the two inner products to be equal up to a multiplicative constant. The fact that the polynomials $\{P_{\nu,\eta}\}_\eta$ form an orthogonal basis for $\mathbb{T}^n$ now immediately gives that the functions $\{H_{\nu,\eta}\}_\eta$ form an orthogonal basis as well only for a different space: $L^2(\mathbb{R}^n, d\mu(t))^W$. □

Finally, we get to the main result, namely the spherical transform of $H_{\nu,\eta}$.

Theorem 4.10. *The spherical transforms of the Jacobi type function $H_{\nu,\eta}(t)$ on $\mathbb{R}^n$ are given by*

$$\widetilde{f_{-2\nu}}(\lambda) q_{\nu,\eta}(\lambda), \quad \lambda \in i\mathbb{R}^n,$$

where $q_{\nu,\eta}(\lambda)$ is a W-invariant polynomial. The polynomials $\{q_{\nu,\eta}(\lambda)\}_\eta$ form an orthogonal basis of the space $L^2(i\mathbb{R}^n, \widetilde{f_{-2\nu}}(\lambda)^2 d\tilde{\mu}(\lambda))^W$. Their norm in that space is given by

$$\|q_{\nu,\eta}\| = \|H_{\nu,\eta}\|_{\mu(t),x} = \|P^x_{\nu,\eta}\|_{k^{(\nu)},x},$$

and they are multiples of the multivariable Wilson polynomials.

Proof. In Section 4.3 the polynomial $P^x_{\nu,\eta}$ is written as a combination of $m_\eta(x_1,\ldots,x_n)$ and lower order terms. With Corollary 4.8 this can be changed to variables $\mathscr{D}_j$:

$$P^x_{\nu,\eta}(x_1,\ldots,x_n) = (-1)^{-(\eta_1+\ldots+\eta_n)}(2^{2(\eta_1+\ldots+\eta_n)}d_\eta)^{-1} m_\eta(\mathscr{D}_1^2,\ldots,\mathscr{D}_n^2) + \sum_{\zeta<\eta} c_{\zeta,\eta} m_\zeta(\mathscr{D}_1^2,\ldots,\mathscr{D}_n^2)1.$$

Therefore, we have

$$f_{-2\nu}(t)P^x_{\nu,\eta}(x_1,\ldots,x_n) = (-1)^{-(\eta_1+\ldots+\eta_n)}(2^{2(\eta_1+\ldots+\eta_n)}d_\eta)^{-1} m_\eta(D_1^2,\ldots,D_n^2)f_{-2\nu}(t) + \sum_{\zeta<\eta} c_{\zeta,\eta} m_\zeta(D_1^2,\ldots,D_n^2)f_{-2\nu}(t).$$

Formula (4.1) gives the spherical transform of $m_\eta(D_1^2,\ldots,D_n^2)f_{-2\nu}(t)$:

$$m_\eta(\lambda_1^2,\ldots,\lambda_n^2)\widetilde{f_{-2\nu}}(\lambda).$$

So, the spherical transform is of the form that is stated in the theorem.

Moreover, the functions $\{q_{\nu,\eta}(\lambda)\}_\eta$ satisfy the orthogonality relation by the Plancherel formula (4.3). We can take $\widetilde{f_{-2\nu}}(\lambda)^2$ together with the Plancherel measure $d\tilde{\mu}(\lambda)$ and they are equal (up to a constant) to the weight function of the Wilson polynomials, Δ_W, in definition 4.2, since $\lambda \in i\mathbb{R}^n$. The orthogonality of the $\{q_{\nu,\eta}(\lambda)\}_\eta$, and the fact that the leading term is equal to $m_\eta(\lambda_1^2,\ldots,\lambda_n^2)$, gives that they are multiples of multivariable Wilson polynomials (definition 4.2). □

4.7. Comparison method of Zhang and method of Koornwinder

The method of Zhang describes the spherical transform of multivariable Jacobi-type polynomials and the method of Koornwinder (2) gives the Fourier-Jacobi transform of a Jacobi-type polynomial in one variable. If we consider the one variable case in the method of Zhang, it should actually be the same transform as in Koornwinder. This section checks if this is indeed the case.

The method of Zhang contains a fair amount of calculations, but the paper of Koornwinder has more explicit computations. Henceforth, Zhang's method seems a bit more elegant, but this is not entirely true.

This description of Zhang's method does not contain the tedious computations of the spherical transform, while it is included in the method of Koornwinder.

Moreover, the Jacobi polynomials in Koornwinder agree with the Jacobi polynomials in one variable defined in Zhang (definition 3.15). The symmetric Jacobi-polynomials $P^x_{\nu,\eta}$ in one variable with $x = \tanh(t)$ are polynomials in $\tanh^2(t)$ and are orthogonal with respect to the weight function $(\tanh^2(t))^{k_1^{(\nu)}+k_2^{(\nu)}}(1-\tanh^2(t))^{k_2^{(\nu)}}$. The standard Jacobi polynomials $P_n^{(\alpha,\delta)}(1-2\tanh^2(t))$ are also polynomials in $\tanh^2(t)$ and are orthogonal with respect to the weight function $(\tanh^2(t))^{\alpha+\frac{1}{2}}(1-\tanh^2(t))^{\delta+1}$. The parameters of the weight functions look different, but they are actually equal to each other. $P_n^{(\alpha,\delta)}(1-2\tanh^2(t))$ and $P^x_{\nu,\eta}$ must be equal up to a multiplicative constant. This follows by Van Diejen [15], who showed that the Jacobi polynomials are the unique polynomials of $\tanh^2(t)$ that are orthogonal to the corresponding weight function. The parameters of ϕ_λ and $\phi_\lambda^{\alpha,\beta}$ can also easily be compared to see that they are the same hypergeometric functions and the same holds for the parameters of the Wilson polynomials.

Lastly, note that the inverse of the Fourier-Jacobi transform is defined on $\mathbb{R}_+$ in Koornwinder and $i\mathbb{R}$ in Zhang. This difference is explained by the fact that the functions are symmetric, so integrating over $\mathbb{R}_+$ or $\mathbb{R}$ only differs by a factor two. The formulas in the article of Koornwinder contain $i\lambda$ and the formulas in the article of Zhang only λ which clarifies the discrepancy between the $\mathbb{R}$ and $i\mathbb{R}$. Hence, the methods yield the same result.

5

Cherednik-Opdam transform of non-symmetric Jacobi polynomials in one variable

The symmetric theory about Jacobi polynomials and spherical transforms has been discussed extensively in the previous chapters. The non-symmetric Jacobi polynomials and the Cherednik-Opdam transform are already mentioned in Chapter 3, but have not really been discussed yet. This chapter covers some non-symmetric theory in one variable. Of course, the non-symmetric Jacobi polynomials and the Cherednik-Opdam transform are important subjects of the chapter. We also see the interesting operators, the intertwiners, and some different representation of a double degenerate Hecke algebra. Most of the non-symmetric theory also holds for the multivariable case, but the advantage of the one variable case is that a lot of formulas can be made explicit.

5.1. Introduction

Since this chapter is on the one variable case, the operators and functions we need are given in one variable. The root system on $\mathbb{R}$ is $R = \{\pm 2\varepsilon, \pm 4\varepsilon\}$ with $R_+ = \{2\varepsilon, 4\varepsilon\}$. The weight lattice is now $P = 2\mathbb{Z}$ and also the root lattice $Q = 2\mathbb{Z}$. The positive weight and root lattice are $P_+ = Q_+ = 2\mathbb{Z}_+$. The ordering becomes: $0 \prec 2 \prec -2 \prec 4 \prec \ldots$. The Weyl group exists of two elements: id and σ_ε. Call $s := \sigma_\varepsilon$ and $\rho = \rho_k = k_1 + 2k_2$ with $k = (k_1, k_2)$ the multiplicity function. Then, the Cherednik operator is given by

$$D := \frac{d}{dt} + 2k_1 \frac{1}{1-e^{-2t}}(1-s) + 4k_2 \frac{1}{1-e^{-4t}}(1-s) - \rho, \tag{5.1}$$

Note that for a function f, the operator s is a reflection $sf(t) = f(-t)$. The non-symmetric Jacobi polynomials are eigenfunctions of D with multiplicity function $k := k^{(v)} = (k_1^{(v)}, k_2^{(v)})$ with $2k_2^{(v)} = 2(2v - (1 + \iota + b + a(n-1))) + 1$ and $2(k_1^{(v)} + k_2^{(v)}) = \iota + 2b$. They are of the form:

$$E(\lambda, k) = e^\lambda + \sum_{\mu \prec \lambda} c_{\lambda,\mu} e^\mu,$$

and we have:

$$DE(\lambda, k) = \gamma_\lambda E(\lambda, k),$$

with

$$\gamma_\lambda = \begin{cases} \lambda + \rho & \text{for } \lambda > 0 \\ -(\lambda + \rho) & \text{for } \lambda \leq 0 \end{cases}.$$

5.2. Intertwiners

Intertwiners are operators that can be used to generate all the eigenfunctions $E(\mu, k)$ starting from $E(0, k) = 1$. Define $Xf(t) = e^{2t} f(t)$ and $U = Xs$. The operator s and D form a degenerate affine Hecke algebra, and if we add X, it becomes a double affine Hecke algebra (see [2]). In $\mathbb{R}$ we have two intertwiners: $I_0 := UD - DU$ and $I_1 := sD - Ds$.

Theorem 5.1. *The intertwiners I_1 and I_0 can be applied to the non-symmetric Jacobi polynomials to get:*

$$I_1 E(\lambda, k) = k_1 E_(- \lambda, k)$$
$$I_0 E_(- \lambda, k) = k_0 E_(2 + \lambda, k),$$

with $\lambda \in P_+$ and certain constants k_0, k_1.

Before we proof this we first give a lemma with two identities and an easier way to write D. Define

$$c(t) = \frac{4k_2^{(v)}}{1 - e^{-4t}} + \frac{2k_1^{(v)}}{1 - e^{-2t}},$$

then D can be written as

$$D = \frac{\partial}{\partial t} + c(t)(1 - s) - \rho.$$

Lemma 5.2. *For the function $c(t)$, the next identity holds:*

$$c(t) + c(-t) = 2\rho. \tag{5.2}$$

Furthermore, we have

$$e^{-2x} c(x) + e^{2x} c(-x) = -2b. \tag{5.3}$$

Proof. Remember that $\rho = k_1^{(v)} + 2k_2^{(v)}$, then the first identity follows directly from

$$\frac{1}{1 - e^{-\alpha}} + \frac{1}{1 - e^{\alpha}} = \frac{1 - e^{\alpha} + 1 - e^{-\alpha}}{1 - e^{-\alpha} - e^{\alpha} + 1} = 1.$$

To see the second identity:

$$\begin{aligned} e^{-2t} c(t) + e^{2t} c(-t) &= \frac{2be^{-2t}}{1 - e^{-2t}} + \frac{2\iota e^{-2t}}{1 - e^{-4t}} + \frac{2be^{2t}}{1 - e^{2t}} + \frac{2be^{-2t}}{1 - e^{4t}} \\ &= 2b \frac{e^{-2t} - 1 + e^{2t} - 1}{1 - e^{-2t} - e^{2t} + 1} + 2\iota \frac{e^{-2t} - e^{2t} + e^{2t} - e^{-2t}}{1 - e^{-4t} - e^{4t} + 1} = -2b. \end{aligned}$$

□

The proof of the theorem can now be given.

Proof. First take a look at the intertwiner I_1 and apply D to it. With help of identity 3.3 we see it is possible to switch the places of I_1 and D.

$$\begin{aligned} DI_1 &= DsD - DDs = (-sD - 2\rho)D - D(-sD - 2\rho) = -sD^2 + DsD \\ &= -(sD - Ds)D = -I_1 D. \end{aligned}$$

Now, using that $E(\lambda, k)$ is an eigenfunction of D:

$$DI_1 E(\lambda, k) = -I_1 DE(\lambda, k) = -\gamma_\lambda I_1 E(\lambda, k) = \gamma_{-\lambda} I_1 E(\lambda, k),$$

since $\gamma_\lambda + \gamma_{-\lambda} = 0$. Therefore, $I_1 E(\lambda, k)$ is an eigenfunction of D with eigenvalue $\gamma_{-\lambda}$ and it also means that $I_1 E(\lambda, k) = k_1 E(-\lambda, k)$ for some constant k_1. Note that $U = Xs = sX^{-1}$, so it follows that $U^2 = 1$. Using the lemma 5.2,

$$UD + DU - 2U = 2b$$

is shown step by step. UD and DU are applied to $f(t)$:

$$DUf(t) = D(e^{2t} f(-t))$$

$$= 2e^{2t}f(-t) - e^{2t}f'(-t) + c(t)\left(e^{2t}f(-t) - e^{-2t}f(t)\right) - \rho e^{2t}f(-t),$$

and

$$\begin{aligned} UDf(t) &= e^{2t}f'(-t) + e^{2t}c(-t)\left(f(-t) - f(t)\right) - \rho e^{2t}f(-t) \\ &= e^{2t}f'(-t) + c(-t)\left(e^{2t}f(-t) - e^{2}f(t)\right) - \rho e^{2t}f(-t). \end{aligned}$$

We can already see that the derivative terms cancel each other out, so those are immediately left out in the next calculation.

$$\begin{aligned} (UD + DU - 2U)f(t) &= 2e^{2t}f(-t) + e^{2t}f(-t)\left(c(t) + c(-t)\right) \\ &\quad - f(t)\left(e^{-2t}c(t) + e^{2t}c(-t)\right) - 2\rho e^{2t}f(-t) - 2e^{2t}f(-t) \\ &= 2bf(t), \end{aligned}$$

where the last step is by the identities (5.2) and (5.3). Thus $UD + DU - 2U = 2b$. This equation is necessary for the application of D to the intertwiner I_0.

$$\begin{aligned} DI_0 &= DUD - DDU = (-UD + 2U + 2b)\,D - D\,(-UD + 2U + 2b) \\ &= -UD^2 + 2UD + DUD - 2DU = -(UD - DU)D + 2I_0 \\ &= I_0\,(2 - D)\,. \end{aligned}$$

So again we find eigenfunctions for D (assume $\lambda > 0$):

$$DI_0E(-\lambda,k) = I_0\,(2-D)\,E(-\lambda,k) = \left(2 - \gamma_{-\lambda}\right)I_0E(-\lambda,k) = \gamma_{\lambda+2}I_0E(-\lambda,k).$$

This last equality follows from $2-\gamma_{-\lambda} = 2+\lambda+\rho = \gamma_{\lambda+2}$. Therefore $I_0E(-\lambda,k) = k_0E(2+\lambda,k)$ for some constant k_0.

□

5.3. Another representation of the affine Hecke algebra

The operators s and U are defined to work on x, but we can also rewrite them to get an operator working on λ. This is useful, since we can take the operators out of the inner product that depends on the variable x. The now operators we get form another representation of the double affine Hecke algebra mentioned in the previous section.

We begin with the operator s.

Proposition 5.3. *We have* $sE(0,k) = 1$ *and for* $\lambda \in P$,

$$sE(\lambda,k) = \begin{cases} E(-\lambda,k) - \frac{\rho}{\gamma_\lambda}E(\lambda,k) & \text{if } \lambda > 0 \\ -\frac{\rho}{\gamma_\lambda}E(\lambda,k) + \left(1 - \frac{\rho^2}{\gamma_\lambda^2}\right)E(-\lambda,k) & \text{if } \lambda < 0 \end{cases}. \tag{5.4}$$

Proof. This proof uses the same method as the proof in [4, Proposition 2.7]. Let $\mu \in P$ with $\mu > 0$. First we apply s to $e^{\lambda x}$: $s(e^{\lambda x}) = e^{-\lambda x}$ and recall that the set $\{E(\lambda,k)|\lambda \in P\}$ forms a basis for $C[P]$. Therefore we can write $sE(\mu,k)$ and $sE(-\mu,k)$ as a sum of $E(\lambda,k)$ with $\lambda \preceq -\mu$:

$$sE(\mu,k) = E(-\mu,k) + \sum_{\lambda \preceq \mu} a_{\mu,\lambda}E(\lambda,k) \tag{5.5}$$

and

$$sE(-\mu,k) = \sum_{\lambda \preceq -\mu} a_{-\mu,\lambda}E(\lambda,k). \tag{5.6}$$

From lemma 3.8 we get

$$sD_\varepsilon s = D_{-\varepsilon} + \left(-2k_1^{(v)} - 2k_2^{(v)}\right)s.$$

Now multiply by s on the right side and note that $D_{-\varepsilon} = -D_{\varepsilon}$. We find

$$sD + Ds = -2\rho.$$

Now we know that $(\gamma_\lambda + D)sE(\lambda, k) = -2\rho E(\lambda, k)$ and we use this to find the coefficients in (5.5) and (5.6):

$$\begin{aligned}(\gamma_\mu + D)sE(\mu, k) &= (\gamma_\mu + D)E(-\mu, k) + (\gamma_\mu + D)\sum_{\lambda \le \mu} a_{\mu,\lambda}E(\lambda, k)\\ &= (\gamma_\mu + \gamma_{-\mu})E(-\mu, k) + \sum_{\lambda \le \mu}(\gamma_\mu + \gamma_\lambda)a_{\mu,\lambda}E(\lambda, k)\\ &= (2\gamma_\mu)E(\mu, k) + \sum_{\lambda < \mu}(\gamma_\mu + \gamma_\lambda)a_{\mu,\lambda}E(\lambda, k),\end{aligned}$$

where in the last step the $E(-\mu, k)$ is removed since $\gamma_\mu + \gamma_{-\mu} = 0$. Thus the coefficients must be equal to

$$a_{\mu,\mu} = -\frac{\rho}{\gamma_\mu} \text{ and } a_{\mu,\lambda} = 0 \text{ for } \lambda < \mu.$$

Similarly, we have:

$$(\gamma_{-\mu} + D)sE(-\mu, k) = (2\gamma_{-\mu})E(-\mu, k) + \sum_{\lambda < \mu}(\gamma_{-\mu} + \gamma_\lambda)a_{-\mu,\lambda}E(\lambda, k).$$

So,

$$a_{-\mu,-\mu} = -\frac{\rho}{\gamma_{-\mu}} \text{ and } a_{-\mu,\lambda} = 0 \text{ for } \mu < \mu.$$

$a_{-\mu,\mu}$ is not determined by this formula, thus we need another way to get this coefficient. Apply s twice to $E(-\mu, k)$:

$$\begin{aligned}s^2E(-\mu, k) &= s(a_{-\mu,-\mu}E(-\mu, k) + a_{-\mu,\mu}E(\mu, k))\\ &= a_{-\mu,-\mu}(a_{-\mu,-\mu}E(-\mu, k) + a_{-\mu,\mu}E(\mu, k)) + a_{-\mu,\mu}(E(-\mu, k) + a_{\mu,\mu}E(\mu, k))\\ &= (a^2_{-\mu,-\mu} + a_{-\mu,\mu})E(-\mu, k) + a_{-\mu,\mu}(a_{-\mu,-\mu} + a_{\mu,\mu})E(\mu, k)\\ &= (a^2_{-\mu,-\mu} + a_{-\mu,\mu})E(-\mu, k).\end{aligned}$$

s^2 is the identity, therefore $a_{-\mu,\mu} = 1 - \frac{\rho^2}{\gamma^2_{-\mu}}$. □

Similarly, we have for the operator U acting on $E(\lambda, k)$:

Proposition 5.4. *We have for $\lambda \in P$,*

$$UE(\lambda, k) = \begin{cases} E(2-\lambda, k) + \frac{2k_1^{(v)}}{2\gamma_\lambda - 2}E(\lambda, k) & \text{if } \lambda > 0\\ \frac{2k_1^{(v)}}{2\gamma_\lambda - 2}E(\lambda, k) + \left(1 - \frac{(2b)^2}{(2\gamma_\lambda - 2)^2}\right)E(2-\lambda, k) & \text{if } \lambda < 0\end{cases}. \tag{5.7}$$

The proof is omitted since it is almost the same as the proof of the previous proposition.

It would be nice if (5.4) and (5.7) can be written as a single expression for all $\lambda \in P$ without losing the properties of $E(\lambda, k)$. We try to normalize $E(\lambda, k)$ with a different constant for every $\lambda \in P$ and we call the normalized functions $F(\lambda, k)$. For $\mu \in P$ with $\mu > 0$ we assume the normalized functions look like this:

$$F(\mu, k) := \frac{E(\mu, k)}{c_\mu} \text{ and } F(-\mu, k) := \frac{E(-\mu, k)}{c_{-\mu}},\ \ c_\mu, c_{-\mu} \in \mathbb{R}.$$

Dividing (5.4) by $c_\mu c_{-\mu}$ gives expressions where $sF(\mu, k)$ and $sF(-\mu, k)$ are easily extracted from.

$$s\frac{E(\mu, k)}{c_\mu c_{-\mu}} = \frac{E(-\mu, k)}{c_\mu c_{-\mu}} - \frac{\rho}{\gamma_\mu}\frac{E(\mu, k)}{c_\mu c_{-\mu}},$$

$$s\frac{E(-\mu,k)}{c_\mu c_{-\mu}} = -\frac{\rho}{\gamma_{-\mu}}\frac{E(-\mu,k)}{c_\mu c_{-\mu}} + \left(1-\frac{\rho^2}{\gamma_{-\mu}^2}\right)\frac{E(\mu,k)}{c_\mu c_{-\mu}}.$$

Replace all the E functions by F, and multiply by $c_{-\mu}$ and c_μ respectively.

$$sF(\mu,k) = \frac{c_{-\mu}}{c_\mu}F(-\mu,k) - \frac{\rho}{\gamma_\mu}F(\mu,k),$$
$$sF(-\mu,k) = -\frac{\rho}{\gamma_{-\mu}}F(-\mu,k) + \left(1-\frac{\rho^2}{\gamma_{-\mu}^2}\right)\frac{c_\mu}{c_{-\mu}}F(\mu,k).$$

The last equations show some similarities with the operator S_j from [2]:

$$S_j = s_j^\lambda + \frac{k_j}{\lambda_{\alpha_j}}\left(s_j^\lambda - 1\right).$$

In our case the operator is assumed to work on the eigenvalues γ_λ instead of λ. Also ρ is replacing k_j. The idea is now to derive c_μ and $c_{-\mu}$ from the formulas for sF by comparing with S_j. The term without reflection in sF is already correct. S_j gives the reflection term $\left(1+\frac{\rho}{\gamma_\lambda}\right)s_j$ for $\lambda \in P$. So we get

$$\left(1+\frac{\rho}{\gamma_\mu}\right) = \frac{c_{-\mu}}{c_\mu},$$
$$\left(1+\frac{\rho}{\gamma_{-\mu}}\right) = \left(1-\frac{\rho^2}{\gamma_{-\mu}^2}\right)\frac{c_\mu}{c_{-\mu}},$$

which both lead to

$$\frac{c_{-\mu}}{c_\mu} = \left(1+\frac{\rho}{\gamma_\mu}\right).$$

Thus we find the ratio between the constants. Now it is possible to show that normalization by the value in the point 0 ($c_\lambda = E(\lambda,k)(0)$) works. Use the formula in (5.4):

$$E(-\mu,k) = sE(\mu,k) + \frac{\rho}{\gamma_\mu}E(\mu,k),$$

and evaluate in the point 0:

$$E(-\mu,k)(0) = sE(\mu,k)(0) + \frac{\rho}{\gamma_\mu}E(\mu,k)(0) = (1+\frac{\rho}{\gamma_\mu})E(\mu,k)(0).$$

$sE(\mu,k)(0) = E(\mu,k)(0)$, because s only influences terms with the variable in them and filling in 0 conveniently cancels those terms.

Repeating the same arguments for (5.7) gives the ratio:

$$\frac{c_\mu}{c_{2-\mu}} = 1 - \frac{2k_1^{(v)}}{2\gamma_{2-\mu} - 2},$$

and $c_\lambda = E(\lambda,k)(0)$ also satisfies this ratio.

Call the operator working on λ that is associated to s S^λ and the one associated to U u^λ. From section 3.4 we get that for D we have the multiplication operator λ. So we find that another representation of the double affine Hecke algebra is S^λ, λ and u^λ.

5.4. Cherednik-Opdam transform in one variable

The Cherednik-Opdam transform is given in 3.4. We now look at the transform in one variable and in particular, the function $G(t,\lambda)$ that is an important part of this transform. The one variable Cherednik-Opdam transform is given by:

$$\widetilde{f}(\lambda) = \int_{\mathbb{R}} f(t)G(-t,\lambda)\tau dt, \ \ \lambda \in i\mathbb{R}.$$

The function $G(t,\lambda)$ is an eigenfunction of the Cherednik operator D with multiplicity function $\bar{k}=(b,\frac{\iota}{2})$, and we can express it as hypergeometric functions:

$$\begin{aligned} G(t,\lambda) &= {}_2F_1\left(\frac{\lambda+\rho}{2},\frac{-\lambda+\rho}{2};\frac{1+\iota}{2}+b,-\sinh^2(t)\right) \\ &\quad + \frac{1}{\lambda-\rho}\sinh(2t){}_2F_1'\left(\frac{\lambda+\rho}{2},\frac{-\lambda+\rho}{2};\frac{1+\iota}{2}+b,-\sinh^2(t)\right). \end{aligned}$$

Note that the hypergeometric function is equal to ϕ_λ from chapter 2. Next, we prove that $G(t,\lambda)$ can be given as a function of its symmetric (or even) part and that it is equal to ϕ_λ.

Proof. First of all, take G^+ the even part and G^- the uneven part of G. We have:

$$G^{\pm} = \frac{1}{2}(G(t,\lambda)\pm G(-t,\lambda)) = \frac{1}{2}(G\pm sG)$$

and

$$G = G^+ + G^-.$$

We calculate DG^+, DG^- and D^2G^+.

$$\begin{aligned} DG^+ &= \frac{1}{2}(DG+DsG) = \frac{1}{2}(DG-(2\rho+sD)G) \\ &= \frac{1}{2}\lambda(G-sG)-\rho G = \lambda G^- - \rho G, \end{aligned} \tag{5.8}$$

where the second equality uses identity 3.3. We now find for the uneven part:

$$DG^- = DG - DG^+ = \lambda G - (\lambda G^- - \rho G) = \lambda G^+ + \rho G.$$

Use the expression for DG^+ and DG^- for D^2G^+. We find:

$$\begin{aligned} D^2G^+ &= D(\lambda G^- - \rho G) = \lambda DG^- - \lambda\rho G \\ &= \lambda^2 G^+ + \lambda\rho G - \lambda\rho G = \lambda^2 G^+. \end{aligned}$$

Use (5.8) and use that G^+ is symmetric:

$$\lambda G^- = DG^+ + \rho G = (G^+)' - \rho G^+ + \rho G.$$

Divide it by λ and fill it in for the following calculation:

$$G = G^+ + G^- = G^+ + \frac{(G^+)'}{\lambda} - \frac{\rho}{\lambda}G^+ + \frac{\rho}{\lambda}G = \left(1-\frac{\rho}{\lambda}\right)G^+ + \frac{(G^+)'}{\lambda} + \frac{\rho}{\lambda}G.$$

Thus we find:

$$G = G^+ + \frac{1}{\lambda-\rho}(G^+)'.$$

In section 3.4 ϕ_λ is given in terms of $G(t,\lambda)$ and that shows that $G^+=\phi_\lambda$. □

5.5. Determining the Cherednik-Opdam transform of the non-symmetric Jacobi polynomials

This section treats the methods that are used to determine the Cherednik-Opdam transform of the non-symmetric Jacobi polynomials. The first approach is essentially a direct calculation, the second method is adapting the method of Zhang of chapter 4 to the non-symmetric cases and the last one considers the paper of Peng and Zhang [13] .

First of all, we need the transformation of $E(\lambda,k)$ to the $\mathbb{R}^n$ domain. Unlike for the W-invariant $P_{\nu,\eta}$, powers of sines as well as powers of cosines are needed to express the non-symmetric $E(\lambda,k)$. It is still possible to apply the transform of section 4.2, and we call the transformed non-symmetric Jacobi function $E^x(\lambda,k)$.

The first method tries to express $\mu\big(\widetilde{f_{-2\nu}E^x(\lambda,k)}\big)(\mu)$ in a sum of Jacobi polynomials in different degrees. We have:

$$\begin{aligned}\mu\big(\widetilde{f_{-2\nu}E^x(\lambda,k)}\big)(\mu) &= \mu(f_{-2\nu}E^x(\lambda,k), G(x,\mu))_k = (f_{-2\nu}E^x(\lambda,k), DG(x,\mu))_k \\ &= (Df_{-2\nu}E^x(\lambda,k), G(x,\mu))_k = (f_{-2\nu}(D-2\nu x)E^x(\lambda,k), G(x,\mu))_k.\end{aligned}$$

Note that we do not know the action of D on $E^x(\lambda,k)$, but we do know how D acts on $E(\lambda,k)$. Though we have to be careful, since the D in our calculations above needs to be transformed to the $\mathbb{T}^n$ domain to be able to apply to $E(\lambda,k)$. Writing D in sines by using lemma 4.5, we find when we now set $x=\sin(u)$:

$$D_j = (1-x^2)\frac{d}{dx} + \iota\left(1+\frac{1}{2}(x+\frac{1}{x})\right)(1-s) + b(1+\frac{1}{x})(1-s) - \rho.$$

And rewrite it again to powers of e^x:

$$\begin{aligned}D_j &= \frac{d}{du} + \iota\left(1+\frac{1}{2}(\frac{e^{iu}-e^{-iu}}{2i} + \frac{2i}{e^{iu}-e^{-iu}})\right)(1-s) + b(1+\frac{2i}{e^{iu}-e^{-iu}})(1-s) - \rho \\ &= \frac{d}{du} + \iota\left(1+\frac{1}{2}(\frac{e^{iu}-e^{-iu}}{2i} + \frac{2ie^{iu}}{1-e^{-2iu}})\right)(1-s) + b(1+\frac{2ie^{iu}}{1-e^{-2iu}})(1-s) - \rho.\end{aligned}$$

Apply this operator to $E(\lambda,k)$ and we immediately see that this results in a function that does not lie in the space $\mathbb{C}[P]$. The resulting function namely contains uneven powers of e^x. Therefore this method does lead anywhere.

We continue with the method of Zhang and we take a look at the changes needed for the non-symmetric Jacobi polynomials. Most steps need to stay the same like the transformation of $\mathbb{T}^n$ to $\mathbb{R}^n$ and the changing of variables in the different inner products. What goes wrong in this approach is the fact that we cannot write our non-symmetric Jacobi function as a polynomial in the operators $\mathscr{D}$ applied to 1. This is because applying $\mathscr{D}$ to 1 only gives polynomials of $\tanh(t)$ and $E(\lambda,k)$ also contains $\cosh^{-1}(t)$. We try if applying $\mathscr{D}$ once to the function $c_0 + c_1e^{is} + c_{-1}e^{is}$ can be used to make $E(2,k)$ and $E(-2,k)$. After a lot of calculations, this turns out not to work either.

The last method we look at is the paper of Peng and Zhang [13]. They discuss the Cherednik-Opdam transform of the symmetric Jacobi polynomials and some kind of non-symmetric Jacobi polynomial in one variable. The non-symmetric polynomial in this article is just the symmetric Jacobi polynomial multiplied by $\tanh(t)$, so this polynomial is not the same non-symmetric Jacobi polynomial as defined in 3.8.

Though these methods do not give any concrete results, there are still other possibilities for determining the Cherednik-Opdam transform of the non-symmetric Jacobi-type polynomials. Some suggestions are given in the next section.

5.6. Recommendations for future research

We conclude this section by giving some recommendations for future research. In the first method we tried for calculating the Cherednik-Opdam transform, the resulting functions are not in our original space. It could work to expand the space and add the polynomials $f(t)E(\lambda,k)$, where $f(t)$ is a to be determined function consisting of combinations of e^t and e^{-t}. This idea is based on the paper of Peng and Zhang [13], where they look at the symmetric Jacobi polynomial in $\tanh(t)$ and a "non-symmetric" variant by multiplying the symmetric polynomial by $\tanh(t)$. Moreover, it might be interesting to try to expand the methods in the paper of Peng and Zhang [13] to multiple variables.

A

Useful definitions and theorems

Theorem A.1.

$$B(x,y) = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\, ds.$$

Theorem A.2.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. Use changing of variables $t = \frac{s}{s+1}$. □

Theorem A.3 (Binomial theorem)**.**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Theorem A.4 (Pfaff's transformation)**.**

$${}_2F_1(a,b;c;x) = (1-x)^{-a}{}_2F_1\left(a, c-b; c; \frac{x}{x-1}.\right)$$

Theorem A.5 (Gauss (1812))**.** *For* $\operatorname{Re}(c-a-b) > 0$*, we have*

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.1}$$

Corollary A.6 (Chu-Vandermonde)**.**

$${}_2F_1(-n,a;c;1) = \frac{(c-a)_n}{(c)_n}.$$

Lemma A.7.

$${}_2F_1(-n,b;c;x) = \frac{(c-b)_n}{(c)_n}{}_2F_1(-n,b;b+1-n-c;1-x).$$

Proof.

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}-n,\ b\\ c\end{matrix}; x\right) &= \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(c)_k\ k!}(x-1+1)^k \\
&\overset{A.3}{=} \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(c)_k\ k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!}(x-1)^j \\
&= \sum_{j=0}^{n} \frac{(x-1)^j}{j!} \sum_{k=j}^{n} \frac{(-n)_k(b)_k}{(c)_k\ (k-j)!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \frac{(-n)_j (b)_j}{(c)_j \; j!} (x-1)^j \\
&\quad \cdot \sum_{k-j=0}^{n} \frac{(-n+j)_{k-j} (b+j)_{k-j}}{(c+j)_{k-j} \; (k-j)!} \\
&\overset{A.6}{=} \sum_{j=0}^{n} \frac{(-n)_j (b)_j (c-b)_{n-j}}{(c)_j (c+j)_{n-j} \; j!} (x-1)^j \\
&= \sum_{j=0}^{n} \frac{(-n)_j (b)_j (c-b)_n}{(c)_n (c-b+n+j-1)_j \; j!} (x-1)^j \\
&= \frac{(c-b)_n}{(c)_n} \sum_{j=0}^{n} \frac{(-n)_j (b)_j}{(-c+b-n+1)_j \; j!} (1-x)^j.
\end{aligned}
$$

□

B

Proofs and calculations

B.1. Calculation in second method Koornwinder

First 2.16 and 2.17 are calculated.

$$
\begin{aligned}
&\Big((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)'' P_n\left(1-2\tanh^2(t)\right) \\
&\quad +2\Big((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)'\left(P_n\left(1-2\tanh^2(t)\right)\right)' \\
&= \Big(-\left(\alpha+\beta+\delta+\mu+2\right)\tanh(t)\Big((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)' \\
&\quad -\left(\alpha+\beta+\delta+\mu+2\right)\left(1-\tanh^2(t)\right)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)P_n\left(1-2\tanh^2(t)\right) \\
&\quad -2\left(\alpha+\beta+\delta+\mu+2\right)\tanh(t)\,(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\left(P_n\left(1-2\tanh^2(t)\right)\right)'
\end{aligned}
\tag{B.1}
$$

Next 2.18 is written out with the help of the differential equation to which the Jacobi polynomials are a solution (see [3, 10.8 (14)]).

$$
\begin{aligned}
&(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\left(P_n\left(1-2\tanh^2(t)\right)\right)'' \\
&\quad = (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big(16\tanh^2(t)\cosh^{-4}(t)P_n''\left(1-2\tanh^2(t)\right) \\
&\qquad +\left(-4\cosh^{-2}(t)+12\tanh^2(t)\right)P_n'\left(1-2\tanh^2(t)\right)\Big) \\
&\quad = (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\cdot 4\Big(2(\alpha+1)\cosh^{-2}(t) \\
&\qquad -2(\alpha+\delta+2)\tanh^2(t)\cosh^{-2}(t)P_n'\left(1-2\tanh^2(t)\right) \\
&\qquad -n(n+\alpha+\delta+1)\cosh^{-2}(t)P_n\left(1-\tanh^2(t)\right)\Big) \\
&\qquad +(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\left(-4\cosh^{-2}(t)+12\tanh^2(t)\right)P_n'\left(1-2\tanh^2(t)\right) \\
&\quad = (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\cdot 4\Big((2\alpha+1)\cosh^{-2}(t) \\
&\qquad -(2\alpha+2\delta+1)\tanh^2(t)\cosh^{-2}(t)P_n'\left(1-2\tanh^2(t)\right) \\
&\qquad -n(n+\alpha+\delta+1)\cosh^{-2}(t)P_n\left(1-\tanh^2(t)\right)\Big) \\
&\quad = (\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\cdot\Big(-(2\alpha+1)\tanh^{-1}(t) \\
&\qquad +(2\alpha+2\delta+1)\tanh(t)\Big)\left(P_n\left(1-2\tanh^2(t)\right)\right)' \\
&\qquad -4n(n+\alpha+\delta+1)\cosh^{-2}(t)\,(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\,P_n\left(1-\tanh^2(t)\right).
\end{aligned}
\tag{B.2}
$$

Then B.1 and B.2 are added to find:

$$
-\left(2\left(\mu+1\right)\tanh(t)+(2\alpha+1)\tanh^{-1}(t)+\left(2\beta+1\right)\tanh(t)\right)p_n'(t)
$$

$$
\begin{aligned}
&+\Big(-\big(\alpha-\beta+\delta-\mu-1\big)\tanh(t)\Big((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)'\\
&-\big(\alpha+\beta+\delta+\mu+2\big)\big(1-\tanh^2(t)\big)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)P_n\big(1-2\tanh^2(t)\big)\\
&+(2\alpha+1)\tanh^{-1}(t)\Big((\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)'P_n\big(1-2\tanh^2(t)\big)\\
&-4n(n+\alpha+\delta+1)\cosh^{-2}(t)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}P_n\big(1-\tanh^2(t)\big)\\
=&-\big(2\big(\mu+1\big)\tanh(t)+(2\alpha+1)\tanh^{-1}(t)+\big(2\beta+1\big)\tanh(t)\big)p_n'(t)\\
&+\Big(\big(\alpha-\beta+\delta-\mu-1\big)\big(\alpha+\beta+\delta+\mu+2\big)\tanh^2(t)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\\
&-\big(\alpha+\beta+\delta+\mu+2\big)\big(1-\tanh^2(t)\big)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}\Big)P_n\big(1-2\tanh^2(t)\big)\\
&-(2\alpha+1)\big(\alpha+\beta+\delta+\mu+2\big)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}P_n\big(1-2\tanh^2(t)\big)\\
&-4n(n+\alpha+\delta+1)\cosh^{-2}(t)(\cosh(t))^{-(\alpha+\beta+\delta+\mu+2)}P_n\big(1-\tanh^2(t)\big)\\
=&-\big(2\big(\mu+1\big)\tanh(t)+(2\alpha+1)\tanh^{-1}(t)+\big(2\beta+1\big)\tanh(t)\big)p_n'(t)\\
&+\Big(\big(\alpha-\beta+\delta-\mu\big)\big(\alpha+\beta+\delta+\mu+2\big)\tanh^2(t)\\
&-2(\alpha+1)\big(\alpha+\beta+\delta+\mu+2\big)-4n(n+\alpha+\delta+1)\cosh^{-2}(t)\Big)p_n(t).
\end{aligned}
\tag{B.3}
$$

Finally adding 2.19 and replacing $\cosh^{-2}(t)$ by $1-\tanh^2(t)$ gives 2.15.

Bibliography

[1] George E Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71. Cambridge university press, 2000.

[2] Ivan Cherednik. Inverse Harish-Chandra transform and difference operators. *International Mathematics Research Notices*, 1997(15):733–750, 1997.

[3] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, Francesco G Tricomi, and Harry Bateman. *Higher transcendental functions*, volume 2. New York McGraw-Hill, 1953.

[4] Wolter Groenevelt. Fourier transforms related to a root system of rank 1. *Transformation Groups*, 12(1): 77–116, 2007.

[5] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[6] Gerrit Heckman. *Harmonic analysis and special functions on symmetric spaces*, volume 16. Academic Press, 1995.

[7] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[8] Roelof Koekoek, Peter A Lesky, and René F Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Science & Business Media, 2010.

[9] Tom H Koornwinder. A group theoretic interpretation of Wilson polynomials. *Department of Pure Mathematics*, (R 8504), 1985.

[10] Tom H Koornwinder. Special orthogonal polynomial systems mapped onto each other by the Fourier-Jacobi transform. In *Polynômes Orthogonaux et Applications*, pages 174–183. Springer, 1985.

[11] Tom H Koornwinder. Askey-Wilson polynomials for root systems of type BC. *Contemp. Math*, 138:189–204, 1992.

[12] Eric M Opdam. Harmonic analysis for certain representations of graded Hecke algebras. *Acta Mathematica*, 175(1):75–121, 1995.

[13] Lizhong Peng and Genkai Zhang. Nonsymmetric Jacobi and Wilson-type polynomials. *International Mathematics Research Notices*, 2006, 2006.

[14] Jasper V Stokman. Lecture notes on Koornwinder polynomials. *Advances in the theory of special functions and orthogonal polynomials*, 2004.

[15] J van Diejen. Properties of some families of hypergeometric orthogonal polynomials in several variables. *Transactions of the American Mathematical Society*, 351(1):233–270, 1999.

[16] Genkai Zhang. Spherical transform and Jacobi polynomials on root systems of type BC. *International Mathematics Research Notices*, 2005(51):3169–3189, 2005.